

THIRTY-NINTH ANNUAL REPORT
OF THE
NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

1953

INCLUDING TECHNICAL REPORTS
NOS. 1111 to 1157



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Letter of Transmittal

To the Congress of the United States:

In compliance with the provisions of the act of March 3, 1915, as amended, establishing the National Advisory Committee for Aeronautics, I transmit herewith the Thirty-ninth Annual Report of the Committee covering the fiscal year 1953.

DWIGHT D. EISENHOWER.

THE WHITE HOUSE,
JANUARY 25, 1954.

Letter of Submittal

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

WASHINGTON, D. C., *November 24, 1953.*

DEAR MR. PRESIDENT: In compliance with the act of Congress approved March 3, 1915, as amended (U. S. C. title 50, sec. 151), I have the honor to submit herewith the Thirty-ninth Annual Report of the National Advisory Committee for Aeronautics for 1953.

This has been an historic year in aeronautics. For the first time the prototype of a military service airplane, the Air Force North American F-100 Super Sabre, flew faster than sound in level or climbing flight. Heretofore such speed has been attained only with research aircraft. The F-100 is only the first of a series of supersonic aircraft scheduled for quantity production for the military services.

In 1953 man flew at twice the speed of sound for the first time. An NACA research pilot attained a speed of 1327 miles per hour in a research airplane, the Navy Douglas D-558-II Skyrocket. This Navy plane was part of the same long-range program that produced the Air Force Bell X-1, which 6 years ago first flew at supersonic speed.

These achievements climaxing the history of the first fifty years of powered flight are tangible evidence of teamwork between science, military, and industry that promises continuing progress in American aviation. On the other hand, scientific problems associated with supersonic flight are increasing in number, complexity, and expense, and therefore many important problems cannot, within existing resources, be studied as intensively as they merit.

Respectfully submitted.

JEROME C. HUNSAKER,
Chairman.

THE PRESIDENT,
The White House, Washington, D. C.

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National Advisory Committee for Aeronautics

Headquarters, 1724 F Street NW., Washington 25, D. C.

Created by act of Congress approved March 3, 1915, for the supervision and direction of the scientific study of the problems of flight (U. S. Code, title 50, sec. 151). Its membership was increased from 12 to 15 by act approved March 2, 1929, and to 17 by act approved May 25, 1948. The members are appointed by the President and serve as such without compensation.

JEROME C. HUNSAKER, SC. D., Massachusetts Institute of Technology, *Chairman*.

DETLEV W. BRONK, PH. D., President, Rockefeller Institute for Medical Research,
Vice Chairman.

HON. JOSEPH P. ADAMS, member, Civil Aeronautics Board.

ALLEN V. ASTIN, PH. D., Director, National Bureau of Standards.

LEONARD CARMICHAEL, PH. D., Secretary, Smithsonian Institution.

LAURENCE C. CRAIGIE, Lieutenant General, United States Air Force, Deputy
Chief of Staff (Development).

JAMES H. DOOLITTLE, SC. D., Vice President, Shell Oil Company.

LLOYD HARRISON, Rear Admiral, United States Navy, Deputy and Assistant
Chief of the Bureau of Aeronautics.

RONALD M. HAZEN, B. S., Director of Engineering, Allison Division, General
Motors Corporation.

WILLIAM LITTLEWOOD, M. E., Vice President, Engineering, American Airlines, Inc.

HON. ROBERT B. MURRAY, JR., Under Secretary of Commerce for Transportation.

RALPH A. OFSTIE, Vice Admiral, United States Navy, Deputy Chief of Naval
Operations (Air).

DONALD L. PUTT, Lieutenant General, United States Air Force, Commander, Air
Research and Development Command.

ARTHUR E. RAYMOND, SC. D., Vice President, Engineering, Douglas Aircraft
Company, Inc.

FRANCIS W. REICHELDERFER, SC. D., Chief, United States Weather Bureau.

THEODORE P. WRIGHT, SC. D., Vice President for Research, Cornell University.

HUGH L. DRYDEN, PH. D., *Director*

JOHN W. CROWLEY, JR., B. S., *Associate Director for Research*

JOHN F. VICTORY, LL. D., *Executive Secretary*

EDWARD H. CHAMBERLIN, *Executive Officer*

HENRY J. E. REID, D. ENG., Director, Langley Aeronautical Laboratory, Langley
Field, Va.

SMITH J. DEFANCE, D. ENG., Director, Ames Aeronautical Laboratory, Moffett
Field, Calif.

EDWARD R. SHARP, SC. D., Director, Lewis Flight Propulsion Laboratory, Cleve-
land Airport, Cleveland, Ohio.

THIRTY-NINTH ANNUAL REPORT

OF THE

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

WASHINGTON, D. C., November 19, 1953.
To the Congress of the United States:

This is the fiftieth year since Wilbur and Orville Wright at Kitty Hawk, N. C., made their first powered flight. That airplane was a fragile and unsteady machine of no immediate utility. It flew for only a minute but it disclosed the solution of the age-old problem of human flight.

The Wrights were the first in the history of man to fly. There was no one to teach them. They had to discover principles and to learn the art by cautious and methodical experimenting. From their own research they obtained the practical information needed to design their successful flying machine.

The Wrights received no effective aid from the theoretical studies of flight made by the mathematicians of the nineteenth century. The science of aerodynamics was developed in response to the practical demands of aeronautics in the years to follow.

In 1908, the Wrights demonstrated at Fort Myer, Va., a vastly improved flyer, the first military airplane. It carried a passenger and flew for more than an hour. Following this public demonstration, the development of the airplane was taken up vigorously. At first France and Germany took the lead, then Great Britain, but the United States lagged behind in the furthering of this greatest American development of the century.

With war clouds in view in 1915, the Congress established the National Advisory Committee for Aeronautics to undertake the scientific study of the problems of flight with a view to their practical solution. President Wilson appointed the members of the first Committee, consisting of the heads of the military and civil agencies of the Government concerned with aeronautics and experts from private life.

Over the succeeding 38 years, the Congress has annually appropriated funds in support of the Committee's research. During the period between the two World Wars, the Committee supplied scientific and engineering knowledge which contributed substantially to American leadership in aeronautics and victory in the air.

The Committee has functioned as a board of directors to plan as a body and supervise the execution of research

programs conducted by a dedicated Civil Service staff of some 7,000 men and women. In this responsibility to the Nation, the Committee has had the assistance of 28 technical subcommittees recruited from governmental agencies, universities, manufacturers and the airlines. Coordination of the many interests concerned with research has been effectively accomplished within the Committee because of the interlocking character of its membership.

The intensity with which scientific research is pursued determines the rate at which new knowledge is acquired and consequently the rate of progress which can be made in improving the performance and efficiency of America's aircraft. Many problems of importance are known to exist, which cannot be studied as intensively as they merit within existing resources.

Before the end of the next fiscal year the first of the NACA Unitary Plan wind tunnels, for which the Congress appropriated \$75,000,000, will come into use. The Unitary Plan Act assigns to NACA the additional task of operating very large transonic and supersonic wind tunnels to be available to industry for developmental testing of aircraft and missiles.

Both the offensive potential of our atomic weapons, and our defense against such weapons, depend in major part on superior aircraft and missiles. By virtue of an immense effort this country holds a current position of leadership in many areas in aeronautical science. But no complacency is justified in view of the high scientific and technical capabilities disclosed by Soviet progress, including that in nuclear weapons. It cannot be assumed that the Soviets will not likewise make the advances in aeronautical science which we ourselves know to be possible.

The Committee, therefore, urges that the existing and potential capacity of the National Advisory Committee for Aeronautics be fully and intensively utilized to advance the aeronautical arts and sciences, born 50 years ago with the Wright brothers' airplane, and that the Congress provide the necessary support for this policy.

Respectfully submitted.

JEROME C. HUNSAKER,
Chairman.

Part I—TECHNICAL ACTIVITIES

THE NACA—WHAT IT IS AND HOW IT OPERATES

One of the most important functions of the National Advisory Committee for Aeronautics is that of coordinating the aeronautical research carried on in the United States. The makeup of both the Main Committee and the 28 technical subcommittees embraces the several military and civil government agencies concerned with aeronautics, and includes members from scientific institutions, and the aviation manufacturing and operating industries. Thus wasteful and costly duplication of research and development effort is avoided.

In the conduct of its business, which is scientific laboratory research in aeronautics, the NACA, since its establishment in 1915 by the Congress, has functioned to serve the needs of all departments of the Government. The 17 members of the Main Committee are appointed by and report to the President. Serving without pay, they operate like a board of directors, establishing policy and planning the research programs to be followed by the 7,000 civil-service personnel who make up the technical and administrative staff of the NACA.

The Committee is assisted in the determination and coordination of research programs by 5 major and 23 subordinate technical committees, with a total membership of more than 400. These men are selected because of their technical ability, experience, and recognized leadership in a special field. They also serve without compensation, in a personal and professional capacity. They provide material assistance in the consideration of problems related to their technological fields, review research in progress both at NACA laboratories and in other organizations, recommend research projects to be undertaken, and assist in the coordination of research programs.

Membership on the technical committees and subcommittees, as well as the Industry Consulting Committee, is listed in part II of this report, beginning on page 54.

Coordination of research is also accomplished through frequent discussions by NACA technical staff personnel, with the research organizations of the aircraft industry, educational and scientific institutions, and other aeronautical agencies. The NACA maintains a West Coast office to further liaison with the aeronautical research and engineering staffs of that geographical area.

During the 38 years since its organization as an independent Federal agency, the NACA has sought to assess the current status of development of aircraft, both civil and military; to anticipate the research needs of aeronautics; to develop the scientific staff and special research facilities required, and to acquire the needed information as rapidly as may be consistent with the national interest.

The NACA's research programs have had both the long-range, all-inclusive objective of acquiring the new scientific knowledge essential to assure American leadership in aeronautics, and the immediate objective of solving, as quickly as possible, the most pressing problems, thus to give effective support to the Nation's current aircraft construction program.

Most of the problems to be studied are assigned to NACA's research centers: the Langley Aeronautical Laboratory in Virginia, where research is conducted on aerodynamic, structures, hydrodynamic, and other problems; the Ames Aeronautical Laboratory in California, which concentrates on aerodynamic research, and the Lewis Flight Propulsion Laboratory in Ohio, which is concerned primarily with power-plant problems. Smaller NACA research installations are located at Wallops Island, off the Virginia Coast, where aerodynamic problems in the transonic and supersonic speed ranges are studied using rocket-powered models in free flight, and at Edwards Air Force Base, California, where specially designed, specially instrumented research aircraft are used in full-scale flight research on transonic and supersonic problems.

In its work, often with research tools requiring use of air under great pressure and high temperature, the NACA has had to maintain constant vigilance to assure maintenance of adequate safety for its employees. Evidence of the success of the NACA's continuing program of accident prevention was the receipt of two awards from the National Safety Council in 1953.

On August 6, 1953, the Langley Aeronautical Laboratory received the Council's highest award, the Award of Honor. This recognition of good safety was the result of the 3,300 employees at the laboratory having worked a total of 3,325,640 man-hours without a disabling injury, in the period November 8, 1952, to May 1, 1953. The Council's Award of Merit was pre-

sented to the Lewis Flight Propulsion Laboratory for its 2,400 employees having worked 1,167,588 man-hours without a disabling injury for the period February 17, 1953, to May 13, 1953.

The NACA also sponsors and finances a coordinated program of research at 20 nonprofit scientific and educational institutions, including the National Bureau of Standards and the Forest Products Laboratory. By this means, scientists and research engineers, whose skills and talents otherwise might not be available, contribute importantly to the Government's program of aeronautical research. Promising students also receive scientific training which makes them useful additions to the country's supply of technical manpower.

During the fiscal year 1953, the following institutions participated in the NACA's program of contract research:

- National Bureau of Standards
- Forest Products Laboratory
- Battelle Memorial Institute
- Polytechnic Institute of Brooklyn
- California Institute of Technology
- University of California
- University of California at Los Angeles
- Carnegie Institute of Technology
- Case Institute of Technology
- University of Chicago (NORC)
- University of Cincinnati
- Columbia University
- Cornell University
- University of Florida
- Georgia Institute of Technology
- Iowa State College
- Johns Hopkins University
- Massachusetts Institute of Technology
- University of Michigan
- University of Minnesota
- Pennsylvania State College
- University of Pittsburgh
- Purdue University
- Syracuse University
- University of Wisconsin

Proposals from such institutions are carefully screened to assure best use of the limited funds available to the NACA for sponsoring research outside its own facilities. Similarly, results from these projects are reviewed to maintain the quality of this part of the NACA program. Reports of the useful results are given the same wide distribution as other NACA publications.

During the fiscal year, most of the NACA technical subcommittees reviewed proposals for research projects from outside organizations, or gave attention to reports from completed contracts. Reports covering results of sponsored research totaled 62 during fiscal year 1953.

Research information, including that obtained in the Committee's laboratories and elsewhere under NACA sponsorship, is distributed in the form of Committee publications. Reports and technical notes, containing information that is not classified for reasons of military security, are available to the public in general. Translations of important foreign research information are published as technical memorandums.

The NACA also prepares a large number of reports containing information of classified nature. These, for reasons of national security, are closely controlled as to circulation. When it is found possible at a later date to declassify such information, these reports also may be given wider distribution.

Current announcement of NACA publications is contained in the NACA Research Abstracts. This service, in addition to telling of NACA publications, makes note of important research reports received from abroad.

In addition to other means of research information readily available, the NACA each year holds a number of technical conferences with representatives of the aviation industry, universities, and the military services. Attendance at these meetings is restricted, because of the security classification of the material presented, and the subject material is focused upon a specific field of interest. During the fiscal year 1953, two such technical conferences were held.

UNITARY WIND TUNNELS

During 1953 the prototype of the USAF F-100 Super Sabre fighter, designed and manufactured by North American Aviation, Inc., made its first flight. It since has repeatedly reached supersonic speeds in level or climbing flight.

Accomplishment of faster-than-sound velocities by an airplane now in production and soon to be assigned to tactical units of the Air Force, underlines the great progress made in aeronautics during the past decade. Widely published predictions that tomorrow's tactical airplanes will fly at least twice as fast (a speed already attained by special research aircraft) seldom attempt to detail the problems which first must be solved, or to recount the vigorous steps being taken to conquer those problems.

It is hardly 10 years since Germany was preparing to make use of four radically new weapons of war, the V-1 buzz bomb, the supersonic V-2 guided missile, the Messerschmitt 262 jet fighter, and the Messerschmitt 163 rocket interceptor. It is less than 10 years since the United States began planning the postwar steps which would be required, by way of research and development, to insure fullest exploitation of the military possibilities of supersonic aircraft and missiles.

In the immediate postwar period it became apparent that there would be required a large national investment in wind tunnel equipment to enable attaining, and maintaining, the degree of technical supremacy in aeronautics which was felt imperative. The problem was given careful and detailed consideration by the Department of Defense, the NACA, representatives of the aviation industry, and others, and a master plan was prepared to provide America's wind tunnel needs.

From those studies, beginning in 1944 and continuing into 1947, several important conclusions were reached, in addition to formulation of the master plan covering wind tunnel needs. Among these were the following:

(1) Farsighted congressional support, during the decade prior to America's entry into World War II, had enabled construction by the NACA of large high-speed, though still subsonic, wind tunnels. It was felt they would be substantially adequate for anticipated research requirements in this speed range.

(2) In the past the needs of the aircraft industry for wind tunnel facilities in which to conduct development and evaluation work had been largely satisfied by the availability of such facilities at their own plants, at educational institutions, and elsewhere.

(3) Prior to World War II, a fund of basic aeronautical information had been gathered in the United States, by the NACA and educational institutions. This information was in advance of aviation practice, and could be drawn upon by the industry. By the end of the war, the fund of aeronautical information was seriously depleted; practice was running dangerously close to the forward limits of basic knowledge.

(4) Large wind tunnels and other facilities essential for development and evaluation work in the transonic and supersonic speed ranges would be so costly as to make it impossible for the industry to acquire such equipment for its use. The cost of a single supersonic tunnel large enough for development and evaluation work was calculated to be from \$20,000,000 to \$40,000,000 each. The alternative was to provide centralized Government facilities.

In April 1946 it was agreed that the plans being promulgated by the several agencies studying America's wind tunnel needs should be assessed comprehensively, thus to eliminate nonessential facilities as well as to guard against costly duplication. The task was undertaken by a Special Panel on Supersonic Laboratory Facilities, with NACA member Arthur E. Raymond the chairman, and composed of representatives from the Army Air Corps, the Navy, the Joint Chiefs of Staff Guided Missiles Committee, the aircraft industry, the aircraft engine manufacturers, and the NACA.

The recommendations of this group, known as the Raymond Panel, were in the form of a coordinated plan for wind tunnel procurement to accommodate the requirements of the agencies and the industry which shared responsibility in the task of attaining for the United States technical supremacy in aeronautics. The estimated cost of the facilities involved was \$2,200,000,000, roughly the cost of the A-bomb.

During this period, Dr. Vannevar Bush, then a member of the NACA, used the word "unitary" in describing the national scope of the proposals. His phrasing characterized the unified approach taken to acquire a mature and informed assessment of the wind tunnel needs of the United States, and as the studies continued, the name "Unitary Wind Tunnel Plan" was adopted.

Realistic as had been the \$2,200,000,000 procurement program proposed by the Raymond Panel, it was apparent that drastic downward revision of the total cost was imperative. In June 1946, the NACA established a Special Committee on Supersonic Facilities, with Dr. J. C. Hunsaker the chairman. The report of this

committee, made the following January, proposed an integrated program of transonic and supersonic wind tunnel construction which would provide the facilities needed for both basic research and for development and evaluation work. With the necessary service facilities, the technical tools were estimated to cost \$1,063,000,000, approximately one-half the original amount.

"When, if ever, new aircraft and missiles need to be provided in great quantity, cannot be determined now, yet their design cannot be improvised when needed," the committee observed. "We do know that the art is not stationary and that immense advantages in effectiveness are possible by practical applications of now predictable advances in aeronautical science. It is proposed, in essence, that the United States proceed to discover and make practical application of the possible advances in aeronautical science, and, by so doing, maintain its position of leadership in the air."

The basic Unitary Plan, as drafted by the special committee, included 16 small tunnels to be constructed at universities. These tunnels, it was proposed, would be used in providing better training of graduate students in supersonic aerodynamics. The plan also listed new tunnels to be built at existing NACA laboratories and at a Navy facility as electric power requirements permitted. To provide the largest of the new tunnels, the plan proposed creation of a National Supersonic Research Center, to be operated by the NACA, and a new Air Force establishment for the largest of the development and evaluation facilities. The two new research centers were to be located at sites where the great quantities of electric power needed in operation of the tunnels would be available.

Following adoption by the NACA, early in 1947, of the report of its special committee, the proposals were forwarded to the National Military Establishment for consideration by what was then the Joint Research and Development Board. This action was taken so that, following concurrence, the Unitary Plan could be presented to the Congress jointly by the military services and the NACA. For practical reasons, still further reduction in the recommendations followed, resulting in agreement upon a minimum program, involving appropriations over a 5-year period which totaled about \$600,000,000.

On October 14, 1947, the Bell X-1 was piloted by then Capt. Charles E. Yeager, USAF, on a flight that attained supersonic speed. The airplane, one of several specially designed research aircraft, was the product of a joint undertaking in which the military services, the aircraft industry and the NACA were partners. The fact that, for the first time, supersonic speed had been achieved in level or climbing flight was the more impressive because the airplane's design features were, basically, so conventional. If anything, the speed of

this remarkable airplane brought into sharper focus the urgent need for the facilities envisioned by the Unitary Plan.

The Unitary Plan was studied by the President's Air Policy Board which, in its report to the President, January 1, 1948, concluded "... the United States is dangerously short of equipment for research in the transonic and supersonic speed ranges. This deficiency should be remedied as quickly as possible. We recommend that the 16 supersonic tunnels projected for the universities be authorized and installed as quickly as possible. . . . We recommend also that we proceed without delay in supplementing existing laboratory equipment with the new tunnels projected under the Unitary Plan in whatever order of priority and at whatever rate as will be recommended by the Research and Development Board."

March 1, 1948, the Congressional Aviation Policy Board, in its report, made the following recommendation concerning the Unitary Plan: "Physical facilities required for transonic and supersonic research and development of aircraft and guided missiles are so expensive they can be furnished only by Government. The NACA and the Research and Development Board are preparing a coordinated program of facilities required in the national interest. Since adequate research and development facilities are essential for continued United States aviation leadership, this plan should be expedited."

A month later, in April, the Secretary of Defense and the Chairman of the NACA forwarded jointly to the Bureau of the Budget for approval a draft of proposed legislation to authorize the Unitary Plan. Supplemental appropriation estimates to finance initial phases of the program were also presented.

Following approval by the Bureau of the Budget, the proposed Unitary Plan legislation was submitted, in March of 1949, to the 81st Congress. After House and Senate Committee hearings, followed by a Joint Conference Committee study, the final version passed both House and Senate on October 19. The Joint Bill was signed by the President October 27, 1949, and became Public Law 415, 81st Congress.

Composed of two parts, Public Law 415 authorized appropriation of \$253,000,000 for construction of transonic and supersonic facilities. Of this amount, \$10,000,000 was for tunnels at educational institutions, \$136,000,000 was for six large tunnels at the NACA Laboratories, \$7,000,000 was for a single tunnel to be operated by the Navy, and \$100,000,000 was for establishment by the Air Force of its Air Engineering Development Center, at which would be located two propulsion facilities and a gas dynamics laboratory.

The Deficiency Appropriation Act of 1950, approved June 29 of that year, included \$75,000,000 for construc-

tion of three Unitary Plan supersonic tunnels at the NACA laboratories: a 10-foot tunnel for engine work at the Lewis Flight Propulsion Laboratory, an 8-foot tunnel for aerodynamic investigations at the Ames Aeronautical Laboratory, and a 4-foot tunnel, also for aerodynamic investigations, at the Langley Aeronautical Laboratory. No funds were appropriated for supersonic tunnels at universities.

Initial appropriations for the facility to be operated by the Air Force, located at Tullahoma, Tenn., and now named the Arnold Engineering Development Center, totaled \$157,500,000. Additional funds have since been authorized, bringing the total nearly to \$170,000,000.

When in 1944 the planning began which would result in the Unitary Wind Tunnel Plan the maximum speed of airplanes was slightly over 500 miles an hour. To talk, at that time, about speeds of 1,300 or 1,400 miles an hour seemed but wishful thinking.

On November 20, 1953, NACA Research Pilot Scott Crossfield, in another of the special research airplanes, the D-558-II which Douglas had constructed under Navy contract and with NACA cooperation, flew twice the speed of sound (1327 mph at the altitude and temperature at which he was flying). As early as August, 1951, Douglas Pilot William Bridgeman had flown the Skyrocket 1238 mph. On these flights, as in the case of the Bell X-1, the research airplane was lifted to about 30,000 feet by a "mother plane" and then released to make its flight. The maximum speed was maintained for only a few seconds; the duration of the rocket engine was less than 5 minutes.

The importance of detailed transonic and supersonic experimentation, possible only under the controlled conditions of the laboratories, was not decreased by the progress made with the research airplanes; rather, it was increased as attention was focused more sharply upon the essentiality of learning how to keep aerodynamic drag as low as possible, thus to obtain maximum speed with the power plants available.

This, of course, is a problem that has been basic since the advent of the airplane, but today the possible gains, or losses, can be multiplied many times. In the past, the difference between an optimum design and one second best, might at most be only a few miles an hour. Today, the difference may be measured, literally, in hundreds of miles an hour. The "right" design may achieve the supersonic speed desired; the "wrong" design may be unable to attain higher than subsonic speeds.

The lack of mathematical methods for predicting theoretically aerodynamic behavior in the speed ranges of tomorrow's aircraft means that this kind of information must be obtained through use of experimental techniques, the most satisfactory of which requires use of the large wind tunnel of the kind being provided under the Unitary Wind Tunnel Plan.

Before the end of 1954, the first of the Unitary Plan supersonic wind tunnels will be in fruitful use as a tool in the hands of America's producers of supersonic aircraft and missiles. The planning of a decade ago began not a moment too soon.

AERODYNAMICS

The laboratories of the NACA have continued their basic and applied research programs in the field of aerodynamics to assist in the understanding of phenomena that affect the performance, stability, and control of current aircraft designs as well as to explore problems that will be faced by future designs. In addition to these efforts, test programs associated with the development of specific aircraft have been undertaken at the request of the military services. These studies have served to broaden the base of design knowledge throughout present and anticipated operating ranges of aircraft. The data have been obtained through the use of various analytical as well as experimental techniques. In many cases, special research equipment was utilized; for example, automatic computers and simulators, transonic, supersonic, and hypersonic wind tunnels, free-flight rocket and drop-body models, and full-scale research airplanes. Such problems as drag and control effectiveness at transonic and supersonic speeds, airplane tracking control, and automatic interception systems have been under intensive study.

The Committee on Aerodynamics and its subcommittees on Fluid Mechanics, High-Speed Aerodynamics, Stability and Control, Internal Flow, Propellers for Aircraft, Seaplanes, and Helicopters have continued to give guidance to the broad programs of the NACA laboratories in the aerodynamics field. This past year a special Conference on Aerodynamics of High-Speed Aircraft was held at the Ames Laboratory to assist in the early dissemination of NACA research data. The conference was attended by many representatives of the armed services and their major contractors. Their comments indicated high continued interest in this method of presenting pertinent new research information to the users.

The following paragraphs briefly describe many of the unclassified studies conducted by the NACA in the aerodynamics field during the past year.

FLUID MECHANICS

Theoretical Aerodynamics and Gas Dynamics

A method for determining the surface pressures for a family of two-dimensional airfoils in a sonic or supersonic stream has been developed. The method depends upon knowing the pressures of one member of the airfoil family. This technique is described in Technical Note 2910. For engineering purposes, this technique,

an application of the method of characteristics, may be replaced by a simple application of Prandtl-Meyer flow concepts. An explanation of the nonlinear force characteristics of two-dimensional airfoils at transonic speeds is presented on the basis of sensitivity of these flows to changes in airfoil geometry and angle of attack.

Some exact solutions of two-dimensional flows of compressible fluid have been obtained at the Johns Hopkins University under NACA sponsorship, and reported in Technical Note 2885. In this technical note a suggestion is given for classifying certain types of compressible flows. This seems to offer a convenient criterion for systematic investigation of these flows by Chaplygin's original method. The object of the paper is to present and analyze a few useful solutions of compressible potential flow with the exact gas law. These solutions include flows about convex corners and the exact solution of compressible flow through a particular contracting channel.

Several aspects of transonic flow around the forward portions of wedge profiles have been studied by means of interferometry. Measurements were made of the two kinds of flow patterns that occur at the leading edge of a wedge at an angle of attack. The growth of the supersonic region about a sharp convex corner formed by two flat surfaces was also observed. The results reported in Technical Note 2829 show the drag of a wedge of 14.5° semiangle at high subsonic Mach numbers to be consistent with that of wedges of smaller angle when plotted according to the transonic similarity law. It is demonstrated that part of the flow field around a hexagon and the wedge could be calculated rather accurately by the method of characteristics.

The effect of the blunt trailing edge on the pressure drag of rectangular and delta wings with truncated diamond-shaped airfoil sections has been studied at supersonic Mach numbers. Use is made of linearized theory to evaluate the surface pressures. Comparison is made between the drag of these wings, and similar wings with sharp trailing edges, for various aspect ratios and thickness ratios over a range of stream Mach number. The calculations of the drag characteristics for these wings show that significant drag reductions are possible at high supersonic speeds by the use of blunt trailing edges. These drag reductions are relatively independent of aspect ratio for the rectangular wings but depend considerably on aspect ratio for the delta wings; the smaller aspect ratios show the larger drag reductions. Calculations of the spanwise distri-

bution of drag are included, to compare further the effect of a blunt trailing edge on the drag for different aspect ratios. The results of this study are presented in Technical Note 2828.

In the study of unsteady flows, two methods of using the concept of linearized characteristics have been derived for the one-dimensional unsteady flow in a tube that is rotated about an axis perpendicular to the axis of the tube. One of the methods corresponds to that used by Ferri in his basic work on the subject. Solutions have been obtained by both methods for boundary conditions that allow analytic solutions. Comparison shows that both methods give the same results, but there are significant differences in their application. This work is presented in Technical Note 2794.

Steady, shock-free, transonic diffuser flow as affected by a small, short-time lowering of back pressure has been investigated analytically. A previous exploratory study by Kantrowitz (Technical Note 1225) indicated that a short-time lowering of the back pressure in this type of diffuser results in a stationary or trapped shock near the critical sonic channel throat. The new study (Technical Note 2797) considers the contribution of a higher-order term neglected in the earlier study. This more accurate approximation shows that the shock is not stationary or trapped in the diffuser unless it is supported by a negative steady-flow back pressure; thus, the result is no longer in disagreement with steady-flow solutions for stationary shocks.

A theoretical investigation to develop a procedure for calculating three-dimensional supersonic flows by the method of characteristics has been completed and is reported in Technical Note 2811. The flow was assumed to be adiabatic and inviscid, but may be rotational, and the gas may exhibit both thermal and caloric imperfections. For flows where the Mach number is very large compared to 1, an approximate method was deduced which considerably reduces the complexity of the calculations.

In view of the low temperatures and the consequent condensation processes encountered in wind-tunnel research at high supersonic speed, a program was initiated at the National Bureau of Standards, under NACA sponsorship, to determine the basic properties of liquid air and its components, to aid in the development of the theory of condensation in wind tunnels. In this study the condensation pressure of air was determined over a range of temperatures from 6° to 85° K. The experimental results reported in Technical Note 2869 are slightly higher than calculated values based on the ideal solution law. Heat of vaporization of oxygen and nitrogen was determined at four temperatures ranging from about 68° to 91° K. and 62° to 78° K., respectively.

The results of an investigation to obtain the effect of variable viscosity and thermal conductivity on high-speed slip flow between concentric cylinders are presented in Technical Note 2895. The differential equations of slip flow were first solved by Schamberg assuming that the coefficients of viscosity and heat conduction of the gas were constant. The problem is solved for variable coefficients of viscosity and thermal conductivity. The method, starting with the solution for constant coefficients, enables one to approximate the solution for variable coefficients after one or two steps. This work was conducted at the University of Washington under contract with the NACA.

Boundary Layers and Stream Mixing

Determination of the position at which the laminar boundary layer separates from a surface has been the subject of much theoretical investigation. Recently, a simplified method for estimating the separation point of a compressible boundary layer has been developed in Technical Note 2892. This method is based on Von Doenhoff's simplified solution for the incompressible case together with Stewartson's transforms which express the compressible laminar boundary layer in terms of an equivalent incompressible laminar boundary layer. Application of the method indicates that an upstream movement of the laminar separation point accompanies an increase in Mach number.

The flow over infinite wedges has been investigated theoretically to test a conclusion previously reached by the use of Schlichting's approximate method for the calculation of the laminar boundary layer. By the use of the velocity profiles determined by Hartree from a numerical solution of the boundary-layer equations for wedge flows, and by the use of Lin's rapid method for the calculation of the critical Reynolds number of a velocity profile, the result is obtained that a thick velocity profile on one wedge can be more stable than a thinner profile on a wedge of different angle although the velocity outside the boundary layer and the pressure gradient are the same for both profiles. This result confirms the conclusion reached by the use of Schlichting's approximate method. This investigation, which is reported in Technical Note 2976, also leads to the inference that the calculated effects of a change in boundary-layer thickness on the stability and on the local roughness Reynolds number should be essentially unchanged by replacing the Schlichting single-parameter family of velocity profiles by the Hartree single-parameter family of velocity profiles.

Extensive regions of laminar flow with resultant large reductions in drag may be attained at large values of Reynolds number by means of continuous suction of the boundary layer (Report 1025) provided that the

airfoil surfaces are maintained sufficiently smooth and fair. To provide quantitative information on the stabilizing effect of continuous suction in the presence of finite disturbances, an extension of the previous investigation, Technical Note 2796, was made in the Langley low-turbulence pressure tunnel on an NACA 64A010 airfoil having deliberately added two- and three-dimensional surface disturbances. Continuous suction allowed only a slight increase in the size of a small, finite surface disturbance required to cause premature boundary-layer transition as compared with that for the airfoil without suction. With or without continuous suction, the maximum size of a protuberance that will not cause premature transition is small with respect to the boundary-layer thickness.

Most available experimental values of average drag coefficient for laminar flat-plate flow have been obtained by total-pressure surveys of the boundary layer. At supersonic speeds most of these data have shown discrepancies in the range of 10 to 100 percent between theoretical and apparent experimental average flat-plate friction drag coefficients. An investigation of the causes of these discrepancies is treated in Technical Note 2891. Of the many factors investigated, only the effect of total-pressure probe size was found to be significant. A correlation describing the relation between friction-drag discrepancy and probe-tip height is presented.

For high-speed flight at very high altitudes, a knowledge of the behavior of air flow in rarefied gases is of importance. The first-order solution for the laminar compressible boundary-layer flow over a flat plate at constant wall temperature is given in Technical Note 2818. The effect of slip at the wall as well as the interaction between the boundary-layer flow and the outer stream flow are taken into consideration. The solution indicates a decrease in heat transfer and, for supersonic flow, an increase in skin friction relative to the standard continuum solution.

An investigation has been conducted to determine the average skin-friction drag coefficients of a parabolic body of revolution having a fineness ratio of 15. Average skin-friction drag coefficients were obtained from boundary-layer total-pressure measurements on the body in water at Reynolds numbers from 4.4×10^6 to 70×10^6 . The body was sting-mounted at a depth of two maximum diameters. The average skin-friction drag coefficient for the forward 69 percent of the basic body in incompressible flow is very nearly the same as that for flat plates. The distribution of boundary layer around the body is not uniform over part of the Reynolds number range, apparently being affected by a very small cross-flow component. The results of this investigation are presented in Technical Note 2854.

Systematic experiments have been conducted over the past few years to measure the magnitude of turbulent skin friction at high flight velocities. Such experiments are required in order to determine the amount of aerodynamic heating, as well as the drag of supersonic missiles and aircraft. One such investigation covered a range of Mach numbers from 0.5 to 3.6, and a range of Reynolds numbers from 4 to 32 million. A summary of the more significant results of the study has been published by the Institute of Aeronautical Sciences in the July 1953 issue of the *Journal of Aeronautical Sciences*.

The experiments showed no appreciable effect of moderate changes in pressure distribution on average skin friction. At both subsonic and supersonic velocities, the skin-friction coefficient was observed to depend only to a small extent on body fineness ratio. For each body tested, however, the effect of Mach number was found to be large, amounting to approximately a 50-percent reduction in skin-friction coefficient as the Mach number was increased. This effect of Mach number was found to be the same for all Reynolds numbers investigated.

At high subsonic speeds, the boundary layer at a given Reynolds number is thicker than that at lower speeds because of the large temperature gradient across the boundary layer. This thick boundary layer effectively distorts the body contours and thereby causes deviations from the pressure distributions predicted by theories which take no account of viscous effects. A simplified analysis has been made, based on results obtained by Busemann in 1935. Busemann indicated that the velocity profile across the boundary layer formed on an insulated flat plate is approximately linear at high Mach numbers. A comparison of results from the theoretical analysis with experimental pressures from a flat plate at a Mach number of 6.86 (presented in Technical Note 2773) shows good agreement at the low angles of attack except for the low-pressure surface, where the agreement is poor. It was found that at a Mach number of 6.86, pressures over an airfoil with a circular-arc profile can be predicted with fair accuracy over a wider range of angle of attack than is possible for the flat plate.

The behavior of the boundary layer about a cone at angles of attack is of particular interest in connection with the design of inlets and missile fuselages. In Technical Note 2844, the laminar boundary-layer flow in the plane of symmetry of a circular cone at large angles of attack in a supersonic stream has been analyzed. Beyond a certain critical angle of attack, it is found that boundary-layer flow does not exist in the plane of symmetry, thus indicating separation. This critical angle is presented as a function of Mach number and cone vertex angle.

The reflection of a weak shock wave from a boundary layer along a flat plate has been investigated by Cornell University. The problem has been simplified by dividing the flow field into a viscous layer near the wall and a supersonic potential outer flow. Ordinary linearized theory has been applied to the outer flow inasmuch as the study, presented in Technical Note 2868, has been restricted to infinitesimal compression waves and only small perturbations are encountered. The paper deals primarily with the case of laminar flow, and the boundary-layer treatment is based upon the momentum-integral equation previously derived by Howarth. A second report on the subject, Technical Note 2869, gives the rate of decay of the disturbances and the character of the disturbances upstream and downstream from the point of incidence.

In a simplified inviscid model of shock-wave boundary-layer interaction, Tsien and Finston replaced the boundary layer by a uniform subsonic stream bounded on one side by a solid wall. This model fails to simulate the separated region that generally appears in a laminar boundary layer subjected to an oblique shock-wave of moderate strength. In order to introduce a main feature of such a dead-air region, the model has been modified by replacing the solid wall by an interface with fluid at rest along which the boundary condition of constant pressure must be satisfied. The distortions of the upper and lower surfaces of the simulated boundary layer are found to be similar, except in the immediate vicinity of the incident shock, to the contour computed for the interface between supersonic flow at the same Mach number and a dead-air region. These results, reported in Technical Note 2860, support the inference that the separated region dominates the behavior in the immediate vicinity of the shock and that the upper surface of the boundary layer behaves, in that vicinity, substantially as if the entire boundary layer were replaced by a dead-air region.

Results of a flight investigation made at free-stream Mach numbers up to about 0.77, to determine the effect of laminar and turbulent boundary layers on the chordwise pressure distribution over an airfoil in the presence of shock at full-scale Reynolds numbers (up to 21×10^6), are reported in Technical Note 2765. In this study, boundary-layer and pressure-distribution measurements were made on a short-span airfoil built around the wing of a fighter airplane. The results indicate very little difference in pressure distribution, and hence in the forces and moments acting on the airfoil, with laminar and turbulent boundary layers extending up to the position of shock. These results are in contrast to those of other investigations made at low Reynolds numbers (up to 3×10^6) which indicated large pressure differences extending over an appreciable extent in the chordwise direction.

An experimental investigation has been made to study the airflow characteristics in the vicinity of a rectangular wing of low aspect ratio by means of photographs of a tuft grid located at various chordwise positions along and behind the airfoil. The results, reported in Technical Note 2790, showed that, at the foremost chordwise station considered (0.125c), the trailing vortices were distinctly visible and indicated a rapid rolling up of the trailing vortex sheet into the vortex cores. The cores appeared to leave the wing at approximately 0.125c with an initial slope somewhat less than the wing angle of attack. The slope decreased with increase in distance behind the wing. The slopes of the vortices are predicted very well by the theory of Bollay for low-aspect-ratio wings. The photographs showed no lateral displacement of the vortices with change in angle of attack. The chordwise growth of lift, and the net lift on the model, could be calculated with good accuracy from information obtained by the tuft-grid photographs.

The mean-camber surfaces for wings having uniform chordwise loading and arbitrary spanwise loading in subsonic flow have been determined and recorded in Technical Note 2908. It is shown that, for the design of such wings, the slope of the mean-camber surface at any point can be determined by a line integration around the wing boundary. By an additional line integration around the wing boundary, this method is extended to include the case where the local section lift varies with spanwise location (the chordwise loading at every section still remaining uniform).

The high landing speeds of modern aircraft have dictated a need for studies of the effects of high speeds (Mach numbers between 0.1 and 0.4) on the maximum lift. An investigation has accordingly been made to determine the effect of such Mach number variations on the maximum lift of four NACA 6-series airfoil sections. The study, reported in Technical Note 2824, was made for several values of Reynolds number and indicates that marked reductions in maximum lift may, in some cases, accompany increases in Mach number from 0.1 to 0.4.

Results of work, presented in a previous annual report, to improve the characteristics of airfoils of large thickness ratios emphasized the potential gains to be realized, for both range and maximum lift considerations, from the application of boundary-layer control as a means of eliminating wing-flow separation.

In order to evaluate the system more completely, a high-aspect-ratio wing (aspect ratio 20) having very thick root sections and equipped with a single suction slot at the 60-percent-chord station on the upper surface was investigated. The results of this study, reported in Technical Note 2980, show that discrete design of the suction slot to insure optimum spanwise suction control,

resulted in lift-drag ratios as high as 30.8 at a lift coefficient of 0.9 and a maximum lift coefficient of 4.2 could be realized with flaps deflected and full-span boundary-layer control applied. The drag equivalent of the power expended for the boundary-layer control and the effects of wing leading-edge roughness are also noted in the report.

An investigation of the low-speed aerodynamic characteristics of a two-dimensional, 10.5-percent-thick, symmetrical airfoil with area suction near the leading edge has been completed. The results of the investigation have been presented in Technical Note 2847. The maximum lift of the basic airfoil without suction was 1.3. With a suction flow coefficient of 0.0014, a maximum lift coefficient of 1.78 was obtained.

Technical Note 2887 presents the results of a study of the stability of the mixing of two parallel streams in a gas. It is shown that, when the relative speed of the two parallel streams exceeds the sum of their respective velocities of sound, subsonic oscillations cannot occur and the mixing region may be expected to be stable with respect to small disturbances. It is further shown that, when viscosity and heat conductivity are neglected, if the flow can execute a small, neutral, subsonic disturbance, it can also execute self-excited oscillations of longer wave lengths, and damped oscillations of shorter wave lengths. Additional developments of the mathematical theory of asymptotic solutions show that, at high Reynolds numbers, the damped oscillations in a strictly parallel main flow have a structure similar to that of the vorticity field in fully developed flow. This work was conducted at the Massachusetts Institute of Technology.

Turbulence

Measurements, principally with a hot-wire anemometer, were made at the National Bureau of Standards, under NACA sponsorship, in fully developed turbulent flow in a 10-inch pipe at speeds of 10 and 100 feet per second. This work, conducted under sponsorship of the NACA, is reported in Technical Note 2954. It is shown that rates of turbulent-energy production, dissipation, and diffusion have sharp maximums near the edge of the laminar sublayer and that there exists a strong movement of kinetic energy away from this point and an equally strong movement of pressure energy toward it. It is suggested that the flow field may be divided into three regions: wall proximity where turbulence, production, transfer, and viscous action are of about equal importance; the central region of the pipe where energy diffusion predominates; and the intermediate region where the local rate of change of turbulent-energy production dominates the energy received by diffusive action.

Wake development behind circular cylinders at Reynolds numbers from 40 to 10,000 was investigated by

hot-wire techniques in a low-speed wind tunnel of the California Institute of Technology, under contract with the NACA. As reported in Technical Note 2913, the Reynolds number range of periodic vortex shedding is divided into two distinct subranges. In the stable range, Reynolds numbers from 40 to 150, regular vortex streets are formed and no turbulent motion develops, the vortices decaying by viscous diffusion. The range of Reynolds numbers from 150 to 300 is a transition region to the irregular range in which turbulent velocity fluctuations accompany the periodic formation of vortices. The diffusion is turbulent and the wake becomes fully turbulent in 40 to 50 diameters. The turbulence is initiated by laminar-turbulent transition in the free layers which spring from the separation points on the cylinder. An annular vortex street was observed in the wake of a ring.

In Technical Note 2878, a linearized analysis is presented on the combined effect of a series of damping screens followed by an axisymmetric-stream convergence, or divergence, upon the longitudinal and lateral turbulence velocity fluctuations, scales, correlations, and spectra of a turbulence field described by a triple Fourier integral in the absence of viscosity. An approximate method of taking into account the effects of turbulence decay upon the mean-square fluctuations velocities thus obtained is also presented.

In measurements made with a hot-wire anemometer in a supersonic stream, where a detached bow wave stands ahead of the wire, proper interpretation requires a knowledge of the effects of convection of turbulence through the shock wave. The passage of a single representative shear wave through a plane shock is treated in Technical Note 2864. The analysis indicates the refraction and modification of the shear wave with simultaneous generation of an acoustically intense sound wave.

Aerodynamic Heating and Heat Transfer

An iteration method is presented in Technical Note 2916 for solving the laminar-boundary-layer equations for compressible flow in the absence of a pressure gradient wherein the temperature variation of all the fluid thermal properties is considered. Friction and heat-transfer characteristics have been calculated for a stream temperature of -67°F. , for Mach numbers from 1 to 10, with values of heat capacity, conductivity, and viscosity determined from experiment. Consideration of the temperature variation of all the fluid thermal properties causes the recovery factor to decrease substantially with increasing Mach number. Moreover, the heat-transfer rate is found to be proportional to the difference between an effective enthalpy, which is a function of both the surface temperature and stream Mach number, and the surface enthalpy. In contrast, the heat-transfer rate is approximately

proportional to the difference between the recovery enthalpy and the surface enthalpy for solutions which employ a constant Prandtl number. The calculated skin friction and heat-transfer rates, based upon the use of the Sutherland equation for viscosity and a Prandtl number of 0.75, however, are in excellent agreement with the results of the present analysis.

HIGH-SPEED AERODYNAMICS

Airfoils

A general theoretical method has been developed for determining the airfoil shape having the least possible drag for a variety of structural criteria. This method, described in Technical Note 2787, is based on shock-expansion theory and is, therefore, applicable at hypersonic velocities as well as supersonic velocities. It is found, in all cases, that the simpler linearized theory is adequate for determining the shape of the optimum profile, even at Mach numbers approaching infinity. However, shock-expansion theory must be used to calculate the drag.

A theoretical study has been made of the aerodynamic characteristics at small angles of attack of a thin, double-wedge profile in the range of supersonic flight speed in which the bow wave is detached. The aerodynamic characteristics of a profile of given thickness ratio are found to have little variation with free-stream Mach number as the Mach number passes through 1. As the Mach number is increased to higher values, however, the lift-curve slope rises to a pronounced maximum in the vicinity of shock attachment and then declines. These findings, reported in Technical Note 2832, are in contrast to previous results for the drag coefficient at zero angle of attack, which was found to decrease progressively as the Mach number increased above 1.

An investigation of the flow past a 12-percent-thick biconvex circular-arc profile at zero angle of attack has been conducted utilizing the interferometer technique. The purpose of the investigation was to obtain pressure distributions on the model and Mach number distributions in the field around the model with laminar and turbulent boundary layers and to study the flow conditions along and at the bases of the shock waves that occurred at the higher Mach numbers and that interacted with turbulent boundary layers. The range of Mach numbers was 0.61 to 0.89. The results presented in Technical Note 2801 show that sonic speed is first reached at an indicated Mach number of about 0.74 and the contours of constant Mach number in the supersonic region change from the symmetrical to the asymmetrical type at an indicated Mach number of about 0.78.

Wings and Bodies

Calculations have been made by means of linear theory to determine the supersonic wave drag of a non-lifting, symmetrical, double-wedge-profile, delta wing, the thickness ratio of which varies linearly in the spanwise direction. In general, it was found, from the results presented in Technical Note 2858, that a delta wing with linearly varying thickness ratio can have less wave drag than a constant thickness ratio delta wing of the same plan form, when both wings have either the same projected frontal area or the same internal volume. The thickness distributions for minimum drag and the corresponding values of the ratio of the drag of a wing with linearly varying thickness ratio to a wing with constant thickness ratio were found.

On the basis of linearized supersonic-flow theory, equations for the span load distributions resulting from constant angle of attack, steady rolling velocity, steady pitching velocity, and from constant vertical acceleration have been derived for a series of thin, sweptback, tapered wings. The results, which are valid at those supersonic speeds for which the wing leading edge is subsonic and the wing trailing edge is supersonic, are published in Technical Note 2831. Charts are presented which permit rapid estimation of the load distribution for given values of aspect ratio, taper ratio, Mach number, and leading-edge sweepback.

Few data are available to serve as a guide in the design of the aft sections of supersonic bodies or as a basis for estimating the wave drag and the associated aerodynamic loads. Available data indicate that potential theory can be used to predict these characteristics adequately for most design purposes. In Technical Note 2972 are presented the afterbody pressure distributions and wave drags calculated on the basis of a second-order theory for conical, tangent-parabolic, and secant-parabolic boattails for Mach numbers from 1.5 to 4.5, area ratios from 0.20 to 0.80, and boattail angles from 3° to 11°. For a specific Mach number, area ratio, and fineness ratio, the conical boattail has the smallest wave drag of the three types considered.

Technical Note 2944 presents zero-lift drag data of a slender body of revolution with and without stabilizing fins attached. Results from several wind tunnel studies and from free-air-flight tests are compared. These data cover a Reynolds number range from about 1×10^6 to 40×10^6 for the wind-tunnel models and 12×10^6 to 140×10^6 for the free-flight models. The Mach numbers covered include 1.5 to 2.4 in the wind tunnels and 0.85 to 2.5 in flight.

Research Equipment and Techniques

Precise measurement of low absolute pressures in supersonic wind tunnels has been a problem requiring

solution for some time. A precise, stable, rapid-response Pirani gage, described in Technical Note 2946, has been developed at Langley as one possible solution to this problem. Because of the small size of the gage it has low lag characteristics, and can be mounted close to the pressure orifices, which is an advantage for many installations. The paper describes techniques of calibration, and use of this gage. Measurements of reading errors not exceeding ± 2 percent, errors in lag not exceeding 1 second, and errors in calibration shift of 2 percent per year are shown. A description is also given of operating equipment for recording the pressures.

As the result of the need for a convenient and systematic means of selecting, designing, or redesigning a pressure-measuring system to meet the time requirements of a particular installation, a method has been obtained at Langley for the determination of time lag in pressure-measuring systems incorporating capillaries. Calculated and experimental data, presented in Technical Note 2793, show that response time in a pressure-measuring system incorporating capillaries is a function of the orifice pressure, initial pressure differential, and system volume. The study also shows that the response time is directly proportional to capillary length and to the viscosity of the gas in the capillary, and inversely proportional to the fourth power of the capillary diameter.

Hot-wire turbulence-measuring equipment has been developed to meet the stringent requirements involved in the measurement of fluctuations in flow parameters at supersonic velocities. The higher mean speed necessitates the resolution of higher frequency components than at low speed, and the relatively low turbulence level present at supersonic speed makes necessary an improved noise level for the equipment. The equipment is adaptable to all-purpose turbulence work with improved utility and accuracy over that of older types of equipment. This research was carried out by the National Bureau of Standards under the sponsorship of the NACA and is reported in Technical Note 2839. Sample measurements are given to demonstrate the performance of the equipment.

An experimental investigation has been conducted to determine feasibility of using an X-ray densitometer to measure air densities in disturbed flow fields at high supersonic speeds. It has been concluded from measurements in conical flow fields that density can be determined with sufficient accuracy at the low densities encountered to establish the instrument as a useful research tool. This work is reported in Technical Note 2845.

A low-cost interferometer that is easy to adjust and has a large field of view has been investigated. This instrument, which is based on a principle discovered by Kraushaar, uses small diffraction gratings to pro-

duce and recombine separate beams of light. The usual two-parabolic-mirror schlieren system can be converted inexpensively to a diffraction-grating interferometer. Experimental data, presented in Technical Note 2827, have been obtained which verify the ability of the instrument to provide valid and reliable measurements of air density.

A new shadowgraph technique for the observation of conical-flow phenomena in supersonic flow has been investigated. The particular advantage of this technique over conventional types of shadowgraph or schlieren systems is that it permits observation of the conical-flow phenomena in a plane normal, or nearly normal, to the axis of propagation. As presented in Technical Note 2950, the principle of the shadowgraph is utilized by superimposing a conical light field upon a conical flow field in such a way as to project the shadowgraph on a propeller screen within the test section of the tunnel. Preliminary tests with a triangular wing gave satisfactory results.

A new high-speed photographic technique has been developed which employs a variable-frequency light synchronized with a commercially available 16-millimeter high-speed motion-picture camera without appreciable alterations to the camera. The technique is described and results obtained by this technique of photographing the flow past models in a wind tunnel employing the schlieren method of flow visualization are presented in Technical Note 2949. The photographs show that the new technique, through the use of extremely short exposure times (about 4 microseconds), provides more sharply defined pictures throughout the flow field than were obtained by conventional techniques.

STABILITY AND CONTROL

Static Stability

Requirements for satisfactory high-speed performance have resulted in aircraft configurations that differ in many respects from previous designs. The low-speed longitudinal stability characteristics of wing configurations suitable for high-speed flight, including the effect of high-lift devices, have been the subject of extensive studies. The static lateral stability characteristics of wing configurations suitable for high-speed flight, including high-lift devices, have been the subject of study in the Langley stability tunnel. Recently reported were the results of a study utilizing a 45° sweptback wing in combination with a fuselage. These data, presented in Technical Note 2819, show that for moderate and high values of lift an increase in the span of a trailing-edge flap with or without a leading-edge slat generally caused an increase in the effective dihedral and directional stability. The increments in

the static-lateral-stability parameters caused by the high-lift devices could be calculated with fair accuracy by using the measured lift and drag and a simple correction for the degree of sweep.

One problem associated with the use of wing sweepback is the premature stall of the tip region. This stalling causes the aerodynamic characteristics to depart from their usual linear trends at low angles of attack. Wing twist, wing-section camber, or combinations of the two can provide a more acceptable stall pattern. Technical Note 2775 presents the results of an experimental investigation in the Langley stability tunnel to determine the effects of linear spanwise twist and a combination of camber and twist on the low-speed static and rotary characteristics of a 45° sweptback wing. It was found that a combination of twist and camber was more effective than twist alone in extending the linear range of several of the stability terms to higher values of lift and in improving the efficiency of the wing at moderate angles of attack. Twist or combined twist and camber produced only small changes in the maximum lift coefficient.

To provide additional information on the forces and rolling moment due to sideslipping at supersonic speeds, a generalized family of sweptback, tapered wings has been studied through the use of linearized theory. This analysis, reported in Technical Note 2898, is generally applicable to those supersonic speeds for which the wing trailing edge is in a supersonic flow field.

Available theories for calculating the stability characteristics for horizontal-vertical tail combinations in sideslip are rather limited. In order to check the accuracy of one such theory which utilizes a method found to be effective in predicting wing span loadings, an experimental investigation was conducted in the Langley stability tunnel. The effect of vertical location of the horizontal tail and of horizontal tail span on the aerodynamic characteristics of an unswept tail assembly were determined as a function of sideslip. The results, reported in Technical Note 2907, show that the method used does provide a simple, effective means of determining the aerodynamic characteristics of intersecting surfaces at subsonic speeds. It was found that the induced loading carried by the horizontal tail produced a rolling moment about the point of attachment to the vertical tail which was strongly influenced by horizontal-tail span and vertical location. The greatest effects of horizontal-tail span on the rolling-moment derivative of the complete tail assembly were obtained for horizontal-tail locations near the top of the vertical tail.

An experimental investigation was made at low speed in the Langley stability tunnel to determine the origin of large yawing moments which have been

found to occur for some fuselages inclined at large angles of attack. The results (Technical Note 2911) show that increases in yawing moment are caused by asymmetrical disposition of a pair of trailing vortices emanating from the nose of the fuselage. It was found that a ring or surface roughness near the fuselage nose causes a large decrease in magnitude of the yawing moments in the critical angle-of-attack range.

The magnitude of the changes in aerodynamic forces and moments resulting from propeller pitch reversal has been studied utilizing a twin-engine airplane model in the Langley 300 mph 7- by 10-foot tunnel. The results of the investigation, reported in Technical Note 2979, indicate that the lift, longitudinal-force, and pitching-moment coefficients varied almost linearly with total thrust coefficient through the negative and positive thrust ranges. The lateral-force, yawing-moment, and rolling-moment coefficients were found to vary as approximately linear functions of asymmetric-thrust coefficient. Based on an analysis of the data, a method is suggested which will give a reasonable estimate of the effects of thrust reversal on the aerodynamic characteristics of an airplane through the use of existing wind-tunnel data. For extreme asymmetric thrust conditions, the rudder and aileron control power can be marginal or inadequate.

The effects of a propeller slipstream on the aerodynamic characteristics of wing and wing-nacelle configurations at high subsonic Mach numbers have been a recurring question to aircraft designers. Tests have recently been conducted in the Langley 24-inch high-speed tunnel to determine characteristic effects of a simulated propeller slipstream on the aerodynamic characteristics of an unswept wing panel with and without nacelles. The propeller slipstream was simulated by a calibrated jet of air. Lift, drag, and pitching moment were measured at angles of attack of 0° and 3° through a range of Mach numbers from 0.30 to 0.86. The test results, reported in Technical Note 2776, show that, for Mach numbers of the propeller slipstream equal to and 10 percent greater than those of the free stream, there were no significant changes in lift and pitching-moment coefficients for the configurations investigated. However, the Mach number for drag rise near zero lift was decreased because of the propeller slipstream.

Dynamic Stability

Some present-day high-speed airplanes exhibit an undamped, low-amplitude, lateral oscillation in flight. This instability may be the result of changes in the damping of the lateral motion associated with changes in amplitude of the oscillation. The damping in yaw of a fuselage-vertical-tail combination has previously been determined by the free-oscillation technique in

which the amplitude of the motion decreased logarithmically after initial displacement. As an extension to that investigation, a similar model oscillating continuously in yaw has been studied in the Langley Stability Tunnel to determine the influence of amplitude and frequency on aerodynamic damping. A report on the results, Technical Note 2766, indicates that a reduction in damping in yaw appeared for the smallest amplitudes of oscillation. The decrease in lateral damping with reduction in frequency was slightly greater than the small variation predicted by finite-span unsteady-lift theory but not so large as the variation indicated by two-dimensional theory.

Previous analytical studies indicate that the natural frequencies of a body-tail combination in pitch and yaw will be affected by steady rolling. One of the frequencies will increase and the other decrease to a point where instability is encountered. To obtain experimental results with which to evaluate the adequacy of such analyses, tests were made with a free-flight, body-tail combination in steady roll. Results of this study, reported in Technical Note 2985, indicate that good agreement was obtained between experiment and theory for the deterioration of stability with increasing roll rate and the prediction of the roll rate at which instability would occur.

The determination of the longitudinal stability derivatives from flight data has been a relatively difficult task because the wind-tunnel technique of permitting only one variable to change at a time, while constraining the rest of the variables, cannot always be used. However, matrix methods employing the equations of motion have been found to be particularly useful in determining stability derivatives from complex flight data. In Technical Note 2902, three matrix methods involving various degrees of refinement are presented for determining the longitudinal-stability derivatives from transient flight data. The choice of methods depends on the measurements available and all indicate that good accuracy can be obtained.

Curve-fitting procedures have also been found to be applicable to the calculation of the stability parameters of an airplane from flight data. Because of the importance of knowing the errors in such methods, an analysis has been made of the errors in the parameters obtained from a curve-fitting process. Technical Note 2820 reports this study and gives an example of the process of finding the errors in the calculated stability parameters of an airplane.

Damping Derivatives

To provide additional information on the contribution of various tail configurations to the damping of the lateral oscillations of airplanes and missiles at supersonic speeds, expressions for the velocity potentials,

span loadings, and corresponding force and moment derivatives for a number of tail arrangements performing a steady rolling motion have been derived and are presented in Technical Note 2955. Illustrative variations of the rolling stability derivatives for several series of tail shapes, as well as pressure distributions and sample span loadings, are included. Some consideration is also given to the problem of wing-tail interference.

The calculation of the rolling and yawing moments due to rolling has received extensive treatment at low angles of attack where the variation of lift with angle of attack is linear. At high angles of attack where lift varies in a nonlinear manner, little attention has been given to the problem of calculating these important stability factors. The techniques for a method of calculating wing lift characteristics in the nonlinear range was applied to the calculation of the rolling and yawing-moment coefficients due to rolling for unswept wings with or without flaps or ailerons. This method, presented in Technical Note 2937, permits calculations to be made somewhat beyond maximum lift for wings employing airfoil sections which do not display large discontinuities in the lift curves. Calculations can be made up to maximum lift or wings with discontinuous twist such as that produced by partial-span flaps and/or ailerons.

In many cases, satisfactory lateral oscillatory stability cannot be obtained through reasonable geometric changes to present-day high-speed airplanes. Thus, artificial stabilization devices as means of providing satisfactory damping of the lateral oscillation are of considerable interest. An investigation has been carried out in the Langley free-flight tunnel on a free-flying dynamic airplane model equipped with an artificial stabilization device as a means of changing the model's important dynamic lateral stability terms to evaluate the effect of these artificial terms on the dynamic lateral stability and control characteristics of the airplane model. The results of this investigation, reported in Technical Note 2781, show that the only stability derivative that provided a large increase in damping of the lateral oscillation without adversely affecting other flight characteristics was the yawing moment due to yawing. Increasing the rolling moment due to rolling to moderately large negative values produced substantial increases in the damping of the lateral oscillation, but resulted in objectionable roll control. Increases in the rolling moment due to yawing and the yawing moment due to rolling in the positive direction produced an increase in damping, but caused an undesirable spiral tendency.

Stalling

Statistics indicate that a large percentage of all fatal private flying accidents in the past have occurred

because of pilot error; in more than half of these accidents the airplane stall has been involved. Accidents of this nature are referred to as "stall-spin" accidents and occur primarily in the incipient phase of a spin, that portion of the motion immediately following the stall and before a spin fully develops. A technique has been developed in the Langley spin tunnel which utilizes free dynamic models to study incipient spinning. The model is catapulted into still air in such a manner that stalled flight occurs. The ensuing motion at and beyond the stall is studied. The results of an investigation utilizing this technique on a model, typical of a present day four-place, personal-owner airplane, are presented in Technical Note 2923. These results show that soon after the model stalled, it unstalled, became inverted for a part of its motion, before again stalling and continuing toward a developed spin. The initial rates of rotation were such that, by inference, proper control movements soon after the roll-off could terminate the motion.

Flight tests have shown that when an airplane is in stalled flight and autorotative moments are present, together with violently changing burbled flow, a pilot cannot maintain satisfactory lateral control, even with special devices, such as spoilers, which give ample rolling moments. The difficulty is that the autorotative moments build up so rapidly that the pilot cannot react quickly enough to maintain the airplane at the desired lateral attitude. A flight study utilizing a typical light high-wing monoplane has been made by the Texas A and M Research Foundation, under NACA sponsorship, to furnish the designer with quantitative design information from which the proper combination of variables may be selected to insure satisfactory control near the stall. The results of the study are presented in Technical Note 2948. In the study, satisfactory lateral control was obtained even under conditions simulating extremely gusty air at angles of attack up to approximately 2° below that for maximum lift. This 2° margin was substantially the same, both with full power and with the engine throttled, throughout the range of center-of-gravity locations investigated. For a plain untwisted wing, this stall margin was obtained with widely different elevator deflections for the range of power and center-of-gravity locations considered. With the wing twisted (8° of washout), satisfactory lateral control was obtained under all conditions tested, even at angles of attack beyond that for maximum lift. The desired condition, that is, having sufficient up elevator control to accomplish three-point landings, but insufficient to exceed the angle of attack for satisfactory lateral control, was attained under limited conditions when 8° of washout or a full-span slot was utilized.

Automatic Control and Stabilization

Considerable interest is being shown in the use of automatic stabilization devices for improving the damping of the lateral oscillation of aircraft designed for flight at transonic and supersonic speeds. A method for determining the effects of the dynamic response of an autopilot on the lateral stability of an aircraft-autopilot combination is described in Technical Note 2857. The method is applied to the analysis of the lateral motion of an airplane equipped with a second-order automatic, yaw-rate damper. For any flight condition of the airplane, an optimum combination of values of autopilot natural frequency and damping ratio are shown to exist for any given value of gain or damping. A simple analytical expression is derived for obtaining a close approximation to these optimum points and is used to determine the maximum amount of damping obtainable under various flight conditions. The report illustrates a simple method of designing yaw dampers.

The addition of an autopilot to an airplane introduces many variations in the manner in which airplane motions can be controlled. As part of a general theoretical investigation of airplane automatic control systems, an analysis of a pitch-attitude control system has been made and reported in Technical Note 2882. This study shows the effects of Mach number and altitude on the transient response of a swept-wing fighter airplane equipped with an attitude control system with and without rate feedback. In this study, the dynamic characteristics of the airplane were estimated from theory and available experimental data. The frequency-response characteristics of the autopilot considered were obtained from actual tests of an autopilot. In addition to the study of autopilot characteristics which will give optimum response for each altitude and Mach number condition, the effects on airplane response of holding the autopilot characteristics constant while varying flight conditions were also determined. The results indicate satisfactory control of aircraft attitude can be obtained with this system, although the system will not provide precise control of airplane flight path.

INTERNAL FLOW

Studies of the performance characteristics of two-dimensional diffusers, proportioned to insure reasonable approximation of two-dimensional flow, have been completed at Stanford University. The diffusers had identical entrance cross sections and discharged directly into a large plenum chamber. The test program included wide variations of diffuser divergence angle and length. A few tests were made with asymmetric diffusers. Some tests were also made to determine the effects of addition of a short exit duct of uniform sec-

tion and of installation of a thin, central, longitudinal partition. This work is reported in Technical Note 2888.

PROPELLERS FOR AIRCRAFT

The aerodynamic design of a propeller is influenced considerably by structural design considerations. This matter, insofar as blade centrifugal stress is concerned, has received attention. In Technical Note 2851, a solution is reported for the distribution of cross-sectional area along the propeller blade required for constant centrifugal stress over most of the blade. The study shows that by applying a constant minimum value of thickness-chord ratio to the required radial distribution of cross-sectional area, propellers with good efficiency at high Mach numbers are realized. In a comparison of two propellers designed for a Mach number of 0.9 and an advance ratio of 2 (forward velocity divided by propeller diameter and rotational speed), with the same allowable design stress and differing only in the manner in which the area variation was applied, a blade having a constant minimum thickness-chord ratio of 2 percent was 7 percent more efficient than a rectangular-plan-form propeller with a 2-percent-thick tip.

The analytical work reported last year on the effects of wing and nacelle interference on the first-order vibratory stresses on propellers has been extended to provide a method of predicting the upwash characteristics for a wide range of propeller locations for tractor-type installations on airplanes with wings of arbitrary plan form. This information is presented in Technical Notes 2795 and 2894. In addition to these data, experimental data were obtained to verify the method of upwash prediction and to provide more detailed information on other characteristics of flow fields at the propeller plane. This latter data is reported in Technical Note 2957.

In the interests of propeller-noise abatement, charts have been prepared to permit the preliminary selection of a propeller for near-optimum cruising and take-off performance coupled with minimized propeller noise. These charts, presented in Technical Notes 2966 and 2968, apply to transport airplanes at flight Mach numbers up to 0.8 and are of sufficient scope to permit fairly rapid evaluation of the propeller performance for engine power ratings of 1,000 to 10,000 horsepower.

SEAPLANES

The increased takeoff and landing speeds of water-based aircraft and the use of hydroskis as lifting devices has emphasized a need for information on the principal planing characteristics of prismatic surfaces at high attitudes with respect to the water surface, speeds, and wetted lengths. This information is needed for performance calculations, determination of hydrodynamic balance, and prediction of impact loads.

The hydrodynamic forces and centers of pressure on prismatic surfaces have been determined for ratios of the wetted length to beam up to 7, attitudes with respect to the water as high as 30°, and speed coefficients up to 25. Data for a flat plate are presented in NACA Technical Note 2981, and data for surfaces having 20° and 40° of dead rise are presented in Technical Note 2876. Since flare at the intersection of the bottom and sides of a planing surface (chine flare) is generally desirable for spray control and for recovery of lift lost by the use of dead rise, data also were obtained for 20° and 40° deadrise surfaces with horizontal chine flare. Data for these surfaces are presented in Technical Notes 2804 and 2842, respectively.

The results of these studies show that, during high-speed steady-state planing, the planing characteristics for a given trim depend primarily on the lift coefficient (lift divided by wetted area and dynamic pressure) rather than on speed and load. Increasing the angle of dead rise from 0° to 20° and from 0° to 40° resulted in average losses in lift coefficient of approximately 27 percent and 50 percent, respectively. With horizontal chine flare, these losses in lift were reduced to 15 percent and 30 percent, respectively. In general, the ratio of the center-of-pressure location forward of the trailing edge to mean wetted length decreased with increase in dead rise. Friction drag at high attitudes was negligible, and thus the drag may be assumed to be equal to the product of the load and the tangent of the attitude angle.

HELICOPTERS

Although much progress has been made, improvement in helicopter flying qualities continues to be an important matter. Better definition of design goals and methods of relating design changes to stability characteristics are essential. In addition to the basic single-rotor case for which work has been previously reported, other important types, such as the tandem configurations, require study.

An investigation of the lateral-directional flying qualities of a tandem-rotor helicopter in forward flight was undertaken to determine desirable goals for helicopter lateral-directional flying qualities and possible methods of achieving these goals. The results of this study are presented in Technical Note 2984. Pilot opinions are included to show what considerations are important insofar as flying qualities are concerned. The conclusions are also expressed in the form of desirable flying-qualities requirements.

As an aid to understanding and analyzing the stability of a tandem-rotor helicopter as well as a single-rotor machine, a study has been made at the Georgia Institute of Technology, under NACA sponsorship, of the normal component of induced velocity in the vicinity of a lifting rotor. The tables and charts in Technical Note

2912, which reports this study, may be used to determine the interference-induced velocities arising from the second rotor of a tandem or side-by-side rotor arrangement and the induced flow angle at a horizontal tail plane.

Conditions encountered by some rotors used in convertiplane designs do not permit certain simplifying assumptions used in analysis of the stability or performance of more conventional rotors. A study was conducted at the Georgia Institute of Technology, under NACA sponsorship, to develop a blade-element analysis for lifting rotors that would avoid the approximations that blade-element inflow angle and blade angle are small, and thus be useful for convertiplane calculations. The results of this study, presented in Technical Note 2656, are in agreement with experimental results previously reported in the range of rotor operating conditions encountered by helicopters.

An experimental investigation of some factors in the problem of reducing the descending velocity of a helicopter in autorotation was conducted by Princeton University, under NACA sponsorship, and is reported in Technical Note 2870. In this study, tests were made using a model rotor to examine the effects of disk loading, rotor inertia, and amount and rate of blade-pitch change on the flare performance of the rotor. The tests were extended to ranges of the variables which would be disastrous in flight.

In order to permit greater reliability without sacrifice of useful load, particularly for less conventional rotary-wing designs, a better understanding of the loads occurring in flight is essential. As one step, an analysis has been made of load factors obtained in flight tests of two single-rotor helicopters. Some additional information obtained from military-pilot-training and commercial-airmail operations with helicopters is also incorporated in this study, which is reported in Technical Note 2990. Load factors of the order of 2.5 were found to be attainable by several different deliberate maneuvers, and this same value was also approached

under actual operating conditions. The largest flight loads, as a group, resulted from pull-ups in which both cyclic and collective control were applied with certain phasing. The assumption that flight load factors are limited to the value that would be computed by assuming all blade sections to be operating at maximum lift coefficient agreed well with flight-test results. This assumption thus provides a convenient method of estimating, for new designs, the maximum obtainable load factors for any given flight condition. It is concluded that higher speed helicopters and unorthodox configurations may be subjected to load factors materially higher than those experienced by current types.

The complex nature of the flow through a helicopter rotor, both in hovering and in forward flight, renders the prediction of aerodynamic loading on the blade very difficult. In order to provide insight into the nature and distribution of aerodynamic loading, pressure distributions were measured on a model helicopter rotor blade under hovering and simulated forward flight conditions. This study, carried out by the Massachusetts Institute of Technology under NACA sponsorship, is reported in Technical Note 2953. The work included tests of rotors with and without flapping-hinge offset. The results, showing loading distribution on the rotor disk, indicated a marked difference between the aerodynamic characteristics of the two rotors operating under identical conditions. The introduction of an appreciable amount of flapping-hinge offset resulted in a large first-harmonic aerodynamic loading in simulated forward flight. Blade-flapping measurements revealed appreciably lower values of first-harmonic flapping coefficients for the offset rotor as compared with the conventional configuration. An analysis of the angle of attack at the tip of the retreating blade, based on experimental flapping measurements, indicated that an appreciable offset flapping hinge in combination with a low blade mass constant offers a means of postponing stall on the retreating blade.

POWER PLANTS FOR AIRCRAFT

Since the achievement of supersonic flight, NACA research in the propulsion field has been intensified in obtaining greater power, better fuel economy, extended operating range of each engine component, reduction in engine weight and critical material content, and increased engine life. The results of research advanced in these areas have had a profound influence upon the future success of our military program. Today, the problems which require solution in these areas have become increasingly acute as demands for extended powerplant performance, dictated by practical supersonic flight, are made.

The high altitude and high Mach number propulsion research facilities at Lewis are the major research tools that permit an understanding of the extraordinarily difficult problems which are encountered in the supersonic propulsion system. They have made possible rapid research and development on advanced engines incorporating new basic research ideas. They have reduced the time to place into military service use those engines currently in the initial sea-level qualification test stage at manufacturers' plants.

In order to promote the early use of research results in the design and development of propulsion systems, the practice of holding technical conferences with representatives of the military services and the engine industry was continued during the past year. A technical conference on research advances in the turbojet engine for supersonic propulsion was held at the Lewis Laboratory.

In the following sections recent unclassified research for powerplants for aircraft is described.

AIRCRAFT FUELS

High-speed flight has increased the demands for greater power per pound of engine, per unit volume of engine, per pound of fuel, and per unit volume of fuel. Emphasis has therefore been placed on the development of special fuels that will materially increase the performance of a given aircraft by virtue of the fuel properties alone. In high-speed flight, fuels that provide higher thrust or fuels that reduce fuel weight or volume consumption, or both, are particularly desirable because they offer increases in aircraft speed and range.

In addition to the increased effort on specialized fuels, research on conventional fuels and hydrocarbon components of these fuels has been continued in order to more fully understand the effect of fuel structure

on the various combustion parameters such as stability limits, ignition energy, combustion efficiency, and carbon and smoke formation.

Synthesis and Analysis

The synthesis and purification of cyclopropane hydrocarbons are begun at the Lewis Laboratory in 1946 in order to provide high-purity samples of substituted cyclopropanes for an investigation of the effect of molecular structure on combustion characteristics and other properties pertinent to research on fuels for aircraft propulsion systems. The research pertaining to the synthesis of 19 hydrocarbons and 14 nonhydrocarbon derivatives of cyclopropane is summarized in Report 1112.

Dicyclopentyl, the lowest-molecular-weight dicyclic hydrocarbon, was desired for an investigation of the effects of molecular structure on combustion characteristics and other properties pertinent to research on aircraft propulsion systems. Several methods are known for the preparation of hydrocarbons which contain one cyclopropyl ring, but a search of the literature when the present work was begun did not reveal any previous attempts to prepare a hydrocarbon which contained two cyclopropyl rings. The reaction of cyclopropyl chloride with lithium was investigated as a possible approach to the synthesis of dicyclopentyl.¹

A synthesis of vinylcyclopropane was accomplished by the dehydration of methylcyclopropyl carbinol over alumina at temperatures between 265° and 300° C. and gave vinylcyclopropane in yields as high as 54 percent. By fractionating the hydrocarbon azeotropically with ethanol, vinylcyclopropane of high purity (99.9 mole percent) was obtained.¹

The preparation of a series of cyclopropylalkenes and cyclopropylalkanes, in which the cyclopropyl ring was located in the 2-position of C₃, C₄, C₅, and C₆ straight chains, was recently reported. Similar reactions were reported for the preparation of a cyclopropylalkene and a cyclopropylalkane in which the cyclopropyl ring was located in the 2-position of a branched C₄ chain.¹

The combustion process and burner-performance characteristics are ultimately the net function of physical and chemical properties of the fuel-oxidant mixture, and investigation of such fundamental fuel properties is now in progress. As a part of the overall program, the minimum energy required to ignite a fuel-air mixture was considered for investigation, not only because

¹ See Slabey paper listed on p. 52.

of its obvious implications in engine starting, but also because it may be directly related to the control of the entire combustion process. This investigation employed a controlled-duration capacitance spark as the igniting source for a quiescent combustible mixture contained in a small bomb. The minimum ignition energies for six hydrocarbon fuels are reported.

In recent research, one approach to the problem of understanding complex combustion processes in jet engines has been the study of systems under carefully controlled conditions. This approach has been used in a program to procure data on the fundamental properties of hydrocarbons which might be related to the performance of engine combustion chambers. Several phases of this program have already been published, including preliminary studies on flame speeds, minimum ignition energies, flammability limits, and quenching distances. An attempt was made to select an apparatus and a set of conditions which would yield a consistent set of reliable flammability limits. The flammability limits of 18 high-boiling hydrocarbons at reduced pressures were studied in a closed flame tube with hot-wire ignition. Characteristic two-lobe flammability limit curves were exhibited by all the hydrocarbon studies. The minimum pressure limit was not affected by the molecular weight. Correlations are presented relating the flammability limits to fuel properties.

In recent years considerable effort has been made to determine the factors and mechanisms which govern the formation of smoke during the burning of fuels, so that, eventually, methods of controlling smoke formation can be devised. Particular emphasis has been placed on the problem of preventing smoke formation during the burning of fuels in combustion chambers. Since the type of hydrocarbon present in a fuel is known to effect smoke formation, an investigation was conducted as part of the fundamental combustion program to determine the maximum rate at which various pure hydrocarbons could be burned without producing smoke. From this investigation, it was hoped that the effects on smoking of such variables as chain length, chain branching, degree of unsaturation, position of unsaturation, and ring size could be evaluated and an explanation found to account for the variations. The variations in smoking tendency among 38 gaseous and liquid pure hydrocarbon compounds when burned as diffusion flames in still air were determined.

A recent report presents the results of an analysis of the smoke and carbon obtained from turbojet-engine combustors. The characteristics of various types of carbon formation and methods of analysis which can be used to determine the various types of fuel residue are discussed. The analytical methods used were electron microscopy, carbon-hydrogen determinations in a

combustion train, and X-ray diffraction. Described also were the structures and probable methods of formation for turbojet smoke and the carbon deposits found on the dome and walls of the turbojet-combustor liners.

COMBUSTION

Fundamentals of Combustion

An investigation was performed to determine how the smoking properties of a given fuel are modified by changes in several of the variables associated with the burning of the fuel in a combustion chamber. Variations in initial gas temperature, fuel-flow rate, secondary-air-flow rate, and burner-tube diameter were studied. Fuel-air ratios were measured both at the appearance of the first evidence of carbon formation in the flame, and at the point where smoke issued from the flame. Of all the variables studied, temperature alone affected the fuel-air ratio at which the first evidence of carbon formation was visible in the flame.

It is of interest to know how these variables affect the initial formation and assimilation of smoke in a flame, for if smoke formation could be controlled in the primary flame reaction zone of a combustion chamber, the problem of burning smoke in the later stages of the combustion process would not exist.

Flame velocity by the soap-bubble method is calculated from the rate of growth of the flame sphere radius, which should be uniform, divided by the expansion ratio of the gas mixture. New work with the soap-bubble method has shown that methane in some oxygen-enriched air atmospheres and in pure oxygen at the concentration for maximum flame velocity (near stoichiometric) does not give a smoothly propagating flame. An irregular flame front develops, accompanied by an increase in the rate of flame growth.

A study of the effect of oxygen concentration on the flame velocity of isooctane-oxygen-nitrogen mixtures showed that flame velocity increased linearly with the mole fraction of oxygen. This linear relation was shown to be at variance with the predictions of the approximate solutions of two theoretical equations, one based on a thermal and the other based on a diffusion mechanism of flame propagation, either of which predicted a decreasing rate of change of flame velocity with increasing oxygen concentration. Data on the effects of the equivalence ratio and initial temperature on flame velocity have been obtained for propane and ethylene-air mixtures.

The ability to predict the flame velocities of fuels is of importance in the field of aircraft propulsion. A correlation has been found between combustion efficiency of a ram-jet burner and the laminar flame ve-

locity of the fuel. The prediction of flame velocities is difficult for three reasons: (1) There is no complete, rigorous theory which can be readily applied; (2) there are no data on the kinetics of the oxidation process under flame conditions and very few data on transport properties at high temperatures; and (3) different methods of flame-velocity measurement give different values. It is therefore difficult to compare data from different sources. The uncertainty in measurements made by a given method is of the order of 5 percent. Flame-velocity measurements for different hydrocarbons, different initial temperatures, and different compositions were used with semitheoretical and empirical methods of flame-velocity predictions to show the correspondence between the measured velocities and the predicted velocities.

Research is being conducted to study the fundamental variables affecting the ignition and the combustion of fuel-air mixtures. As part of this research, the parameters which may influence the energy required for a spark to ignite homogeneous fuel-air mixtures are being investigated. One investigation was concerned with the effect of turbulence, generated by different sizes of wire grid, on the minimum spark-ignition energy of a flowing propane-air mixture. The required ignition energy increased with the wire size of the turbulence promoter and with the gas velocity, and decreased with the distance from the promoter to the spark electrodes. The required ignition energy therefore increased with those factors that are reported to generate increased intensity of turbulence.

A study has been made obtain quantitative data on the oxidation intermediates leading to the spontaneous ignition of hydrocarbon-air mixtures. Data were obtained for isooctane- and for n-heptane-air mixtures. Marked differences in the oxidation behavior of these hydrocarbons were noted and interpreted. A preliminary comparison of the oxidation of isobutane and of 2,2,5 trimethylhexane was also made (Technical Note 2958).

There is considerable interest in heat transfer and momentum transfer for bodies submerged in a flowing fluid. These transfers occur frequently in engineering operations and may be important in combustion chambers. Heat transfer coefficients for a spherical particle heated by an induction coil in a moving air stream were determined. A comparison of the experimental values with the theoretical values for heat transfer coefficients was made and an empirical relation between the heat transfer factor and the total drag coefficient was also suggested (Technical Note 2867).

Combustion-Chamber Research

In the field of jet-engine combustor research, burner instabilities are frequently encountered. In most cases a given combustor that exhibits one or more of these

instabilities can be modified by a number of trial-and-error methods and the instabilities eliminated within the operating conditions. For example, "squealing" in afterburners is subdued by increasing the fluid velocity. Any of these instabilities can be extremely detrimental to engine performance. Types of burner instability are enumerated and the role of standing waves in burners are discussed in Technical Note 2772. The status of the problem of flame-driven standing waves is reviewed and a one-dimensional flow theory giving the mechanism whereby a flame drives or damps a standing wave is presented. In this theory, the reflection, transmission, and amplification of waves passing through a flame region were determined from the continuity and momentum equation.

The design of high-output combustors for jet-propelled aircraft and of other industrial equipment may be aided by an accurate knowledge of liquid vaporization rates. According to the type of equipment, the liquid may evaporate from spheres, cylinders, or flat surfaces. In jet engines, the fuel is frequently injected as liquid droplets at a point upstream of the combustion zone, and the concentration of vaporized fuel in the fuel-air stream entering the zone is determined by the rate of evaporation of the droplets. To determine this evaporation rate, vaporization-rate data and surface-temperature data were obtained for nine pure liquids evaporating from the surface of a porous sphere under conditions similar to those encountered in aircraft combustion systems.²

Numerous investigations have been made to determine the effect of pressure on the vaporization rate of droplets in still air. However, vaporization-rate data, which show the effect of pressure on the evaporation rate of drops in streams of air or other gases, are unavailable. Since a range of combustion-chamber pressures may be encountered in the operation of jet engines, the effect of air-stream pressure on the vaporization rate of fuel droplets may be of importance in the study of combustible fuel vapor and air mixtures for jet combustors. In order to determine the effect of pressure on vaporization rate in air streams, a range of pressure conditions encountered in aircraft combustion systems was studied (Technical Note 2850).

One of the processes in a jet-engine combustor is the exchange of radiant energy among the fuel-oxidant-flame components, combustor walls, and combustion products. A better understanding of the nature of the involved radiative processes may supply information that is useful for combustor design and for fuel specifications. The radiative processes involved in combustion were investigated to determine the present role of radiant energy transfer in combustors. It was shown that, at present, the amount of radiant energy transfer

² See Ingebo paper listed on p. 52.

from flame to fuel is quite small in a turbojet combustor. Methods of increasing the equivalent gray-body emissivities of the fuel drops and the flame, as well as the efficiency of the energy transfer itself, were examined.

LUBRICATION AND WEAR

Fundamentals of Friction and Wear

Of the known synthetic fluids, silicones have the best viscosity-temperature properties for lubricants in turbine engines. The principal characteristic that has limited consideration of silicones as lubricants for turbine engines has been that they are poor boundary lubricants for ferrous surfaces. The behavior of silicone lubricants in boundary lubrication of ferrous surfaces was studied in Technical Note 2788. In these studies a hypothesis was advanced which considered that solvents would influence the lubricating ability of silicones by affecting the molecular arrangement at the fluid-surface interface. The data obtained (with solutions of solvents blended with silicones) are considered substantiating evidence for this hypothesis. The solutions reduced friction and prevented surface failure even when the solvent as well as the silicone, was an extremely poor lubricant. Solutions of silicones and diesters were suggested as possible "practical" lubricants for current aircraft turbine engines. It was felt that this combination would provide practical low-volatility lubricants that might have viscometric properties approaching those of the pure silicone and that it would provide effective lubrication resulting from a possible solvent effect of the diesters.

Boundary lubrication and friction data on synthetic fluids in general are very limited, therefore, the lubricating effectiveness of a number of synthetic fluids was studied at high sliding velocities (Technical Note 2846). Sliding friction data and surface failure properties show that a number of synthetics, including diesters, polyethers, a silicate ester, and a phosphonate ester, as well as a silicone-diester blend, are more effective boundary lubricants at high sliding velocities than comparable petroleum oils. A silicone-diester blend, an alkyl silicate ester, and a compound diester (containing lubrication additives) were more effective boundary lubricants at high sliding velocities than the diester from which the most widely accepted synthetic lubricants are made. A diester failed to lubricate non-reactive surfaces, which indicates that the lubrication mechanism for diesters may involve chemical reaction with the lubricated surfaces.

Because of the high operating temperatures of new and projected turbine engines, an experimental study (Technical Note 2940) was conducted to learn the effect of high lubricant bulk temperatures on the boundary lubricating effectiveness of various types of synthetic

fluids. In general, under the conditions of these experiments, the upper limit of temperature for effective lubrication (effective lubrication is defined as the region where low friction coefficient is obtained and no surface damage results) was greater for synthetic lubricants than for a petroleum lubricant of similar viscosity at 100° F. The experimental results tend to substantiate the hypothesis that lubrication with esters (and possibly other fluids) results from formation of an absorbed metal soap film by free acids in the lubricants. Since oxide films are important both to soap formation and to lubrication, failure may be caused by: (1) Failure of the soap film (by melting or decomposition), or (2) failure of the lubricant to maintain the metal soap or oxide films, or both, on the surface. It appears that bulk-fluid-failure temperature is limited by either of these lubrication-failure mechanisms through the effect of temperature on the thermal stability of the bulk fluid. Thermal stability can be associated with viscosity grade within a given class. At temperatures up to its decomposition point, a silicate ester showed more promise than other lubricants studied. The phosphonate esters decomposed at high temperatures to form products that are corrosive to steel, but which prevent complete lubrication failure.

Contract research at the University of Cincinnati has been concerned with determining the spontaneous ignition temperature of organic compounds with particular emphasis on lubricants. Lubricants are known to have low ignition temperatures and, consequently, may be the source of many aircraft fires. Technical Note 2549 gives ignition temperature data for 50 pure organic compounds including paraffins, olefins, aromatic hydrocarbons, ethers, alcohols, and esters. The data indicate those molecular structural features that result in high ignition temperatures. Technical Note 2848 presents ignition data on available petroleum and synthetic lubricants and also compounds synthesized to incorporate favorable structural features reported in Technical Note 2549.

Fretting Corrosion

The minimization of the start and intensity of fretting is a pressing problem in many aircraft engine components such as bearings, bearing housing, spline shaft, and other couplings. Inasmuch as a bonded film of molybdenum disulfide (MoS_2) showed promise in delaying the occurrence of fretting, an investigation was conducted of the bonding of MoS_2 to various materials to form a solid lubricating film (Technical Note 2802). This investigation had as objectives: (1) The determination of practical methods of bonding MoS_2 to various materials, and (2) friction and endurance characteristics of films so formed. The results indicated that satisfactory solid-film lubricants can be formed on a variety of materials by brushing on the solid lubricant

mixed with a resin-forming vehicle. Under baking conditions, the liquid vehicle decomposes or polymerizes to a resin which binds the particles of MoS_2 together and to the surface to be lubricated. The choice of the resin-forming vehicle is governed by the types of application. In the use of asphalt-base varnish, for example, cleaning of the specimens was not critical because of mutual solubility of the varnish and thinner and the usual surface contaminants. Friction and endurance data obtained under severe conditions of high-sliding velocities and high surfaces stress show that solid lubricant films (between 0.0002 and 0.0005 inch thick) of MoS_2 bonded with the various resins resulted in good lubricating effectiveness.

Bearing Research

Because of the lack of information in the field, an experimental investigation (Technical Note 2841) was conducted to determine the performance characteristics of deep-groove ball bearings under radial load at high speeds. The results indicated that bearing operating temperatures were most sensitive to changes in speed and to distribution of the cooling oil. The effectiveness of lubrication depended on the quantity of oil transmitted through the bearings regardless of the method of supplying oil. At low speed, neither load nor oil flow had any appreciable effect on bearing operating temperatures.

Since a large number of present bearing failures of gas turbine engines and other high-speed bearing applications have been attributed to cage failures, it was felt a roller bearing without a cage might eliminate the source of trouble. In accordance, an investigation was conducted of conventional and special designs of cageless roller bearings at high speeds. Preliminary results at high speeds, however, indicate the greater reliability of the conventional cage-type roller bearing over the cageless roller bearing, although the cageless roller bearing may have promise under certain conditions.³

NACA sponsored research on journal bearings at Cornell University resulted in the publication of Technical Notes 2808 and 2809. Technical Note 2808 is an analytical development of a mathematical expression that can be used to predict the circumferential and axial pressure distribution in a short journal bearing. Technical Note 2809 is an experimental investigation that included the effects of bearing length, speed, load, and lubricant viscosity. The theoretical expression derived in Technical Note 2808 agrees favorably with experimental results obtained in Technical Note 2809.

COMPRESSORS AND TURBINES

Compressor Research

Compressor component research is aimed at development of compact, light-weight, efficient machines for application to aircraft propulsion systems. Theoretical and experimental studies of design limitations and flow processes are made throughout the broad operating range required for high-speed high-altitude aircraft applications in order to develop and evaluate design techniques. This research covers a wide field of axial-flow and centrifugal compressor types.

Because of compressibility effects, the stages of a high-pressure-ratio, multistage, axial-flow compressor can be properly matched at only one condition of aerodynamic speed and weight flow. The operational requirement for high-altitude high-speed flight requires operation of the compressor at low aerodynamic speeds. High-altitude climb utilizing full rated engine output, requires operation at high aerodynamic speeds. Detailed studies of a number of commercial and experimental high-pressure-ratio compressors have been made to study the effects of stage mismatching which result from these aerodynamic speed variations.

As a multistage compressor operates over a wide range of aerodynamic speeds, one or more individual stages may be expected to stall.

The flow condition defined as stall in single-stage compressors, is characterized by zones of high and low flow which rotate about the compressor axis in the direction of the rotor hub at a lower speed. These rotating stalls may be classed as either progressive stall, with multiple stall zones at the blade tip, or root-to-tip stall.

An analysis of a hypothetical compressor, with front stages having a progressive-type rotating stall and with middle and latter stages having the pressure-ratio discontinuities associated with root-to-tip stall, indicates multistage compressor performance with discontinuities in pressure-ratio weight-flow characteristics at all speeds. At low speeds, these discontinuities result from stall of the earliest stages having discontinuous characteristics, and at high speed from stall of the latter stages. The resulting discontinuities in the compressor performance map introduce problems of starting and accelerating the engine to design speed and may also cause steady-state operating problems at cruise and high-altitude flight conditions.⁴ Rotating stall may also be a potential source of vibrational excitation in multistage axial-flow compressors.

Flow-visualization techniques were employed in experimental investigations of flow processes to ascertain

³ See Macks, Nemeth, and Anderson paper listed on p. 52.

⁴ See Huppert and Benser paper listed on p. 52.

the streamline patterns of the secondary flows in the boundary layers of cascades and thereby provide a basis for more extended analyses in turbomachines. The three-dimensional deflection of the end-wall boundary layer resulted in the formation of a vortex well up in each cascade passage. The size and tightness of the vortex generated depended upon the main flow turning in the cascade passage. Once formed, a vortex resists turning in subsequent blade rows. This results in unfavorable angles of attack and possible flow disturbances in subsequent blade rows.

Two major tip-clearance effects were observed: the formation of a tip-clearance vortex and the scraping effect of a blade with relative motion past the wall boundary layer. The flow patterns indicate possible methods for improving the blade-tip loading characteristics of compressors and of low- and high-speed turbines (Technical Note 2947).

The method that was developed for predicting the steady flow process of a nonviscous compressible fluid along a stream surface between two adjacent blades was applied to a single-stage and a seven-stage axial-flow compressor (Technical Note 2961). The inlet stages, designed with a symmetrical velocity diagram at all radii, showed large radial flows as a result of the radially increasing angular momentum associated with this type of diagram. The radial distributions obtained were closely approximated by a simplified solution given in Report 955. Three-dimensional solutions have been made for rotating axial-flow passages bounded by straight blades of finite axial length. These solutions indicated that the deviation of the flow direction from the assumed orientation in two-dimensional solutions is small for typical axial-flow blade rows (Technical Note 2834).

Experimental and theoretical research was continued to explore the potentials of supersonic-type compressors. The method of characteristic coordinates was employed in obtaining a solution for the flow between blades of supersonic compressors (Technical Note 2768).

Theoretical analysis and experimental studies have been made in order to improve flow capacities and efficiencies of mixed-flow impellers and diffusers. A general design method was developed whereby two-dimensional channels could be designed to give prescribed velocity distributions.

A technique was developed and solutions for several through flow rates were made for nonviscous, three-dimensional potential flow in a rotating centrifugal-impeller passage. Comparison of these solutions with two-dimensional solutions showed that, for the type of geometry investigated, two-dimensional solutions

can be combined to describe the three-dimensional flow in rotating impellers with sufficient accuracy for general engineering analysis (Technical Note 2806).

A summary of some of the results obtained in theoretical and experimental research on centrifugal compressors has been compiled. This summary includes impeller and diffuser research as well as considerations of the general aspects of the application of centrifugal compressors to gas-turbine-type engines.

Turbine Research

Turbine research is being directed toward improved general performance and the development of means by which optimum designs can be selected for any specific application. It consists of theoretical aerodynamic studies, two- and three-dimensional cascade investigations, and single and multistage turbine research.

A rapid method for the determination of turbine-stage velocity diagrams within specified aerodynamic limits was developed. The method facilitates the selection of the number of turbine stages, the determination of the necessary proximity of staging operation to design limits, and the selection of the optimum work division between or among the stages for any given application (Technical Note 2905).

Further insight into the characteristics of highly loaded turbine stages was gained by an analysis of choked-flow turbines. This analysis indicated that the area ratios and equivalent blade speeds are the controlling factors in the design and operation of such turbines. For the usual class of turbines, increasing the equivalent blade speed of a given stage makes the internal flow conditions less critical. Six criteria are stated that will aid in establishing, from test data of multistage turbines, which blade rows are choked and which are not (Technical Note 2810).

In order to increase understanding of the origin of losses in a turbine, the secondary-flow components in the boundary layers and blade wakes of an annular cascade of turbine nozzle blades were investigated. A detailed study was made (Technical Notes 2871 and 2909) of the total-pressure contours and of the inner-wall loss cores downstream of the blades.

The inner-wall loss core associated with a blade of the turbine-nozzle cascade is largely the accumulation of low-energy fluids originating elsewhere in the cascade. This accumulation is effected by the secondary-flow mechanism which acts to transport the low-energy fluids across the channels on the walls and radially in the blade wakes and boundary layers. At one flow condition investigated, the radial transport of low-energy fluid accounted for as much as 65 percent of the inner-wall loss core.

Turbine Cooling

Better turbine performance and the reduction of critical material content are important to continued improvement of the turbojet engine. Better turbine performance is obtained by operation at higher turbine-inlet gas temperatures which results in higher thermal efficiency and higher net output. These, in turn, have a major bearing on other performance factors such as turbine size, weight, and specific fuel consumption. Since the turbine-inlet gas temperature is limited to a value which will result in a reasonable life of the turbine blades and other highly stressed members, adequate cooling of these turbine parts will permit operation at the higher gas temperatures desired.

To provide fundamental heat-transfer data through which cooled blade design may be aided, an investigation was conducted to study free-convection heat transfer in a stationary air-filled heated tube. This investigation was extended to obtain heat-transfer data under turbulent mixed-, free-, and forced-convection flow conditions similar to those encountered in cooled turbine blades. The heat-transfer limits for each type of flow were established and expressed in correlation equations (Technical Note 2974).

Revised solutions of the laminar-boundary-layer equations for cases which involved cooling at the wall combined with large pressure gradients in the main stream produced specific-weight-flow profiles which locally exceed free-stream values. Heat-transfer and friction coefficients, boundary-layer thicknesses, and velocity, temperature, and specific-weight-flow distributions resulting from the revised solutions are presented for Euler numbers of 0.5 and 1.0, stream-to-wall temperature ratios of 2 and 4, and cooling-air flow rates through porous walls designated by flow parameters of 0, -0.5, and -1 (Technical Note 2800).

An investigation has been made in Technical Note 2863 of natural convection flow for a simplified but representative case, namely the flow with and without heat sources between two long parallel plates at constant temperatures oriented parallel to the direction of the generating body force. It is found that the flow and heat transfer, in general, not only are functions of the Prandtl and Grashof numbers but also depend on a new dimensionless parameter. If this parameter is not negligibly small, the compression work and frictional heating may appreciably affect heat transfer by natural convection.

Measurements of average heat-transfer and friction factors were obtained for air flowing through a smooth, electrically heated tube with a bellmouth entrance and a length-to-diameter ratio of 15 for a range of average surface temperatures from 875° to 1735° R. and corresponding surface-to-bulk temperature ratios from 1.6 to 2.8, Reynolds numbers from 2,200 to 300,000, and heat

fluxes up to 230,000 Btu per hour per square foot of heat-transfer area. The data of this investigation correlated with data obtained in previous investigations with longer tubes on the basis that the heat-transfer coefficient varies as the -0.1 power of the length-to-diameter ratio. No effect of the length-to-diameter ratio was observed on the average friction factor.

An experimental investigation of the forced-convection heat-transfer characteristics of sodium hydroxide was made for a range of Reynolds numbers from 5,300 to 30,000, corresponding to velocities from 3.8 to 15.4 feet per second, average fluid temperatures up to 938° F., and heat-flux densities up to 226,000 Btu per hour per square foot, for both heating and cooling. Data were also obtained from the same apparatus for both water and an aqueous solution of sodium hydroxide. The results showed that sodium hydroxide heating data can be correlated by the Nusselt relation, and that the data fall slightly above the McAdams correlation line. The cooling data are, however, fairly well represented by the McAdams correlation line.

The heat transfer and fluid friction were analyzed for fully developed turbulent flow of supercritical water with variable properties in a smooth tube. A previous analysis of turbulent flow and heat transfer for air with variable properties flowing in a smooth tube is generalized in order to make it applicable to supercritical water. The generalization is necessary because all the pertinent properties of supercritical water vary markedly with temperature. The effect of variation of fluid properties across the tube on the Nusselt number and friction factor correlations can be eliminated by evaluating the properties at a reference temperature which is a function of both the wall temperature and the ratio of wall-to-bulk temperatures.

ENGINE PERFORMANCE AND OPERATION

Performance and Operating Characteristics

The problems associated with the evaluation of full scale engine performance as well as flow fluctuations within the engine are of such a nature that a knowledge of the instantaneous flow patterns is of considerable importance. The use of hot-wire anemometers is a means of obtaining this information. The results of a study of flow through compressors and burners, with the aid of hot-wire anemometers, are reported in Technical Note 2843.

The general class of unsteady flow problems is currently of increasing interest particularly in connection with stability in high speed combustion processes and the overall effects on engine performance. The effect of a shock passing through a flow field (or vice versa) is likely to be important in many applications. The propagation of a plane normal shock wave through a gas at rest as modified by the influence of a weak pat-

tern of unsteady disturbance is described in Technical Note 2879. The disturbances considered are a plane sound wave and a convected plane vorticity wave. Since sound waves may impinge on a shock either from upstream or downstream, both cases are considered.

Afterburners and Jet Exits

In wind-tunnel testing of engines the exhaust gases of the engine mix with the tunnel air. The effect of such a mixing upon the total pressure and Mach number of the combined streams is of importance for obtaining correct engine performance results. Further, accurate determination of tunnel power requirements can be made, once the effects of jet mixing on power requirement is understood. A one-dimensional flow analysis of the results of parallel-jet mixing is presented in Technical Note 2918, where the area of the streams after mixing is equal to the sum of the areas of the two streams being mixed. Changes due to burning of excess fuel downstream of the engine-exhaust station are also considered.

The results of an experimental investigation to determine the temperature profile downstream of heated air jet directed at various angles to an air stream have been incorporated in Technical Note 2855.

An increase in jet thrust over that achieved with a sonic outlet can be realized by complete expansion of the exhaust gases in a convergent-divergent nozzle. In the calculation of this process and in the evaluation of experimental nozzle data, the flow process in the nozzle is usually assumed to be adiabatic. However, for nozzles located directly downstream of combustion chambers, such as in jet powerplants, heat may be released to the working fluid during the expansion process in the divergent portion of the nozzle. Technical Note 2938 contains an analytical treatment of heat addition to a divergent stream with initially sonic flow which permits evaluation of the effect of delayed combustion on convergent-divergent nozzle performance. The analysis indicates that nozzle heat addition delays nozzle over expansion and affects the jet thrust appreciably.

Research Techniques

An investigation was conducted to determine the increase in the useful ranges of simulated flight conditions that may be obtained with a given jet-engine research facility when the choked-exhaust-nozzle technique, or exhaust jet diffuser, or both are employed. The results of this investigation describe the two methods, present the considerations involved in their application, and give typical results of their use as well as confirmation of the accuracy of the data obtained by utilization of these techniques.

Errors in the measurement of jet-engine gas temperature by thermocouples have been analyzed. Based

on this analysis, design criteria of thermocouple probes have been developed.⁵

The mechanized data-handling system currently used at the Lewis laboratory consists of engine instrumentation, digital pressure and temperature-recording instruments, tape-to-card conversion machinery, punch-card operated electronic calculator, and associated card handling equipment.⁶

A method of recording multiple pressures in digital form has been developed in which pressures are converted to the digital form at the primary transducer and subsequently handled in a numerical form.⁷

A 72-channel automatic potentiometer has been developed and put into service. This apparatus, which measures voltages in the ranges 0 to 10 and 0 to 40 millivolts to an accuracy of 0.25 percent at a speed of 1½ readings per second and records the information on a punched tape, is described in a current journal paper.⁸

An investigation has been conducted to provide a basis for the design of air-flow combination probes intended to survey static and total pressure, and direction of flow, with special reference to subsonic turbo-machine testing. Static-pressure probes, yaw-element probes, claw-type probes, and combination probes were tested. From the results of this investigation, reported in Technical Note 2830, the factors which determine the sensitivity of claw-type yaw probes were determined. Satisfactory combination survey probes for sensing static and total pressure and direction of flow in one or two planes were devised.

POWERPLANT CONTROLS

Engine controls research is directed towards defining the dynamic characteristics of the various powerplants (turbojet, ramjet, and turbine-propeller) as well as developing a control theory for engine control application.

A modification of Winer's filter theory and an application to the choice of an optimum control in a closed-loop system is described in Technical Note 2939. The approach used is applicable to all closed-loop systems, and provides an optimum control in an environment of random disturbances. The theory is particularly useful in commonly encountered situations where the environment of disturbance cannot be predicted in detail, and where the penalty in performance resulting from generous margins of safety is great.

A method for predicting the performance of hydraulic servomotors is presented in Technical Note 2767. Experimental data confirmed the theory for

⁵ See Scadron, Warshawsky, and Gettleman paper listed on p. 52.

⁶ See Rawlings paper listed on p. 52.

⁷ See Sharp, Coss and Jaffe paper listed on p. 52.

⁸ See Smith paper listed on p. 52.

servomotors in which the volume of hydraulic fluid was kept low. Servomotor design criteria derived from the theory were presented by Gold at an A. S. M. E. meeting on April 28-30, 1953.

HEAT-RESISTING MATERIALS

High Temperature Materials

The practical application of high-temperature alloys requires that a considerable amount of background information be obtained on the particular combinations of alloying elements under consideration. It is usually desirable to start with a determination of the necessary phase and equilibrium diagrams and then proceed to study variations in physical properties produced by small changes in chemistry, the effect of processing variables on final physical properties, and finally the actual performance of the alloy in the desired end application.

Studies have been conducted of portions of the iron-nickel-molybdenum and the iron-molybdenum-cobalt ternary systems at 2200° F. with emphasis placed on the face-centered cubic solid solution sections of the diagrams (Technical Note 2896). This work completed the program established at the University of Notre Dame, under NACA sponsorship, which has been reported previously. The phases occurring in the above-mentioned systems were identified by means of X-ray diffraction and by etching methods. The phase boundaries at 2200° F. were determined microscopically, using the disappearing phase method with quenched specimens. A total of 170 alloys was investigated. Of these, 8 were common to both of the systems, 105 were in the iron-molybdenum-nickel system, and 57 were in the cobalt-iron-molybdenum system.

Once the fundamental relationships in a desired alloy system are determined, it then becomes desirable to determine, insofar as possible, the influence of small changes in chemical composition on the properties of the heat-resistant alloys developed. In most high-temperature alloys, some uncertainty exists regarding the restrictions on the composition limits as used in commercial practice and the fundamental mechanisms by which the composition influences properties. For these reasons, an investigation was undertaken at the University of Michigan, under NACA sponsorship, for the purpose of providing systematic data for compositional variations in forged alloys stemming from a nominal composition of 20% chromium, 20% nickel, 20% cobalt, 3% molybdenum, 2% tungsten, and 1% columbium (Technical Note 2745). Inasmuch as processing variables (such as melting practice, forging practice, and final heat treatment) also control strength properties, every effort was made to minimize the effects of these variables. It appears that all of the alloying

elements can be varied individually between quite wide limits without significantly changing the rupture properties. This could be partially responsible for the difficulty in correlating chemical composition with physical properties at high temperatures from available data. There was no evidence to indicate that a composition variation, within the usual commercial limits, contributes materially to scatter bands in physical properties.

One of the newer materials of interest for high-temperature applications is molybdenum. This material is made commercially by both powder-sintering and arc-casting methods. Considerable data are now available on the mechanical properties of molybdenum produced by both methods, but unfortunately these data are in many instances contradictory and not reproducible. So-called ductile molybdenum is made by both methods, but frequently the ductility is low or absent. This variation in ductility at room temperature is of great interest if molybdenum is to be used in load-bearing applications. In view of this, an investigation was conducted to determine the effect of prior swaging on the transition temperature, with the objective of producing a ductile material at or near room temperature (Technical Note 2915). The material studied was sintered, wrought molybdenum, with the amount of swaging, recrystallization temperature, and stress relieving factors as the variables studied. As was expected, an increase in swaging progressively lowered the transition temperature by a maximum of 100° F. This effect of swaging is lost when molybdenum is recrystallized. No relationship could be established between the grain size of recrystallized metal and strength and ductility within the transition temperature range. However, above the transition range, for example at 200° F., the ductility increased with decreasing grain size.

Although previous research indicates that swaging increases the strength and ductility of molybdenum, it also has a detrimental effect of lowering the recrystallization temperature. This limits the use of molybdenum to less than 2000° F. and prevents brazing or the application of protective coatings at high temperatures which would recrystallize the metal. Accordingly, an investigation was undertaken to determine whether molybdenum could be produced by a slight modification of commercial fabrication methods which would have a higher recrystallization temperature without an appreciable sacrifice in strength or ductility (Technical Note 2973). The results obtained were not clear-cut since different batches of material responded in different fashions. For instance, the one-hour recrystallization temperature of one lot of metal ranged from above 2900° F. to 2300° F. for 35% to 99% swaging, whereas another two lots of metal had a constant recrystallization temperature of approximately 2075° F.

over the entire working range of 10 to 99%. No difference was detected by either chemical or X-ray diffraction analyses which might account for this variation among lots. Another interesting result was that the atmosphere used to determine the recrystallization data had a pronounced effect on the recrystallization temperature of one lot of molybdenum. This material, swaged 50%, recrystallized in one hour at 2850° F. in argon, at 2750° F. in vacuum, and at 2300° F. in hydrogen. In the same atmospheres, the recrystallization temperature of the other two batches of material, swaged 99%, varied only from 2200° F. to 2300° F.

The actual application of a high temperature alloy usually results in a number of problems which can be traced back to the processing techniques. For instance, the uniformity of life of cast alloy gas-turbine blades is generally considered unsatisfactory. This lack of uniformity results in short times to initial blade failure compared to average life and does not permit the maximum use of the potential performance of the material. Previous studies of the effects of heat treatment upon uniformity of life of small cast AMS 5385 gas-turbine blades indicate that slight, if any, improvement is obtainable in this manner. A more effective method of influencing blade life uniformity might be through the control of blade grain size. Accordingly, an investigation was conducted to compare the uniformity of life and the initial failure time of groups of experimentally cast AMS 5385 blades of several grain sizes. The second objective was to relate the individual blade life to grain size of the experimentally cast blades. The grain size was varied from 4 to 11,200 grains per blade cross-section area (0.035 square inch) and was achieved primarily by variation in pouring temperature. A total of 90 blades were investigated at a temperature of 1450° F. and a stress of 19,000 psi at the midspan of the blade. A correlation was found to exist between the number of grains per blade cross-section and blade life; usually longer life was associated with the coarser grain sizes. The coarse grain blades studied had an improved uniformity in regard to initial failure time, but lower mean life than a group of commercially cast coarse grain blades.

As a continuation of the National Bureau of Standards' investigation, under NACA sponsorship, of the mechanisms of adherence of ceramic coatings to metals, a study was made of the gases evolved during the firing of vitreous coatings (Technical Note 2865). Previous investigations have shown that hydrogen is one of the gases evolved in the firing process and it was suspected that carbon monoxide might also be present. The results showed that the principle gases produced were carbon monoxide, hydrogen, and carbon dioxide. The blistering that is often observed in the early stage of firing when vitreous coatings are applied to low-carbon

steel was found to be caused by evolution of the carbon gases formed by oxidation of the carbon in the steel.

A second portion of the National Bureau of Standards' program included an investigation of the relation between the roughness of the enamel-metal interface and the adherence of the enamel to steel (Technical Note 2934). One of the first explanations advanced for the adherence of vitreous base coats to steel was that of mechanical gripping. This hypothesis is based on the observation that when adherence is good there is usually a rough interface between the coating and the metal. The coating is thought to penetrate into the cavities or undercuts in the metal surfaces and when the coating hardens on cooling, the two materials are interlocked and thus mechanically bonded. Variations in adherence were produced by varying the amount of the metallic-oxide addition in the frit, by varying the surface roughness of the metal before application of the enamel, and by varying the firing time. It appears that there is a positive correlation between the adherence of a porcelain-enamel ground coat and the roughness of the interface. Most of the roughness which was associated with good adherence between the ground coat and the steel, developed during the firing process. Further results indicate that the roughness of the interface is necessary, but is not completely responsible for the development of good adherence.

The previous investigation concluded that roughness of the interface between an enamel and steel is an important factor in contributing to adherence. A theory was developed to explain one way in which this roughness might be produced by an electrolytic reaction between the cobalt in the enamel and the iron in the steel during the firing process (Technical Note 2935). It appears that cobalt, originally present as cobaltic-oxide in the enamel, plates out on the steel during a normal firing operation. Inasmuch as the molten enamel is an electrolyte, the mechanism involved appears to be one of a tiny galvanic cell in which the iron acts as the anode and the cobalt-plated area as the cathode. Corrosion, which is produced by galvanic attack on the iron, produces the desired surface roughness.

One phase of the problem of developing improved high-temperature alloys is the effort to gain a more detailed understanding of the nature of creep in metals. This is of importance in that almost all high-temperature alloys are designed to operate under conditions in which they will creep or stretch a definite amount in a certain time. In other words, most high temperature alloy parts are designed to fail in a finite time, usually after several thousands of hours of operation, although in special cases, such as in turbine blades, it is usually nearer 500 hours. Consequently, an understanding of how the creep process takes place and the metallurgical

factors which affect it are of importance. It has been known for some time that creep is a complex process and that, among other complications, displacement sometimes occurs along grain boundaries of metals at the same time that the grains themselves are deforming. An investigation was undertaken by the Carnegie Institute of Technology, under NACA sponsorship, to isolate and examine grain boundary displacement (Technical Note 2746). The material studied was pure aluminum, under stresses of 1 to 100 psi and temperatures from 400° F. to 1,200° F. The specimens were bicrystals pulled in tension. It was found that displacement occurs in a cyclic fashion, increasing directly with the temperature at the beginning of the cycle but subsequently obeying a chemical-rate law with a computed activation energy of 9,700 calories per mole. There is, for each temperature, a threshold stress below which displacement does not occur. A substantial thickness of crystalline metal participates in the gliding motion and becomes fragmented and extensively disoriented in the process.

Stresses Research

The subject of extrapolation of creep and stress-rupture data on materials at high temperature is of considerable current interest, particularly for applications in gas turbines. Because extensive testing cannot be conducted on all these materials, an extrapolation procedure materially assists in the identification of the more promising materials. An analysis was conducted which resulted in a parameter that yielded improved accuracy over similar extrapolation procedures (Technical Note 2890). The value of this parameter was determined in an investigation where experimentally determined constant nominal stress curves were determined for five alloys.

In general, considerable improvement in the correlation of the data was obtained with this parameter. The generalized correlation method also leads to a suggested explicit relation among stress, temperature, and rupture time. The experimental results were found to agree with this relationship over wide ranges of the variables for all five materials.

Many parts of a jet engine are subjected to high stress concentrations. An investigation of the influence of these stress concentrations on the strength of materials, when subjected to creep loading, was expanded to include additional alloys that might be used in jet engines. Sharply notched and smooth bar stress-rupture tests were carried out on a number of ferritic low-alloy steels, ferritic stainless steels, and austenitic alloys.⁹

The general time-temperature dependence of the notch effect in stress-rupture tests has been confirmed for a wide variety of materials. An analysis of the

ductility data revealed that the smooth bar reduction of area at fracture yields no definite information regarding the notch rupture sensitivity of a given alloy. The notch ductilities were found to conform with the previous hypothesis that the weakening effect of a sharp notch is associated with a retained stress concentration.

Consideration of published data for total creep on certain of the alloys indicated that in some cases design for a certain total deformation would be limited by the notch strength.

Notch and smooth bar stress-rupture tests were also carried out on the chromium-molybdenum-vanadium steel 17-22A (S) over a wide range of temperatures from 600° to 1,350° F.¹⁰ This systematic series of tests was designed to yield precise information on the time-temperature dependence of the notch effect in this low-alloy steel. The specimens possessed 50 percent 60° sharp V-notches and were tested for times up to 1,000 hours.

It was found that the trends of both the notch rupture-strength ratio and the notch ductility, when plotted against testing temperature with time as a parameter, conform with the previously advanced hypothesis that a precipitation reaction is associated with the notch sensitivity. A further analysis of these data yields an activation energy for the process which is close to that obtained for diffusion of carbon in alpha iron.

The same steel was used to investigate the influence of notch depth at 1,100° F. The results indicate a strong similarity between the behavior of a notch ductile alloy in stress-rupture tests and in room-temperature tensile tests. For the notch brittle conditions, it was found that the depth at which the onset of notch sensitivity occurred shifted to lower values with an increase in the testing time. In addition, for rupture times exceeding a certain value, it appeared that any sharp notch, however shallow, results in a weakening effect.

Available information on the behavior of brittle and ductile materials under conditions of thermal stress and thermal shock was reviewed. A simple formula relating physical properties to thermal shock resistance of brittle materials was derived and used to determine the relative significance of two indices currently used to rate materials (Technical Note 2933). The importance of simulating operating conditions in thermal-shock testing was deduced from the formula and experimentally illustrated. It was found that for ductile materials, thermal shock resistance depends upon the complex interrelationship among several metallurgical variables which seriously affect strength and ductility.

The influence of temperature gradients on the deformation and burst speeds of rotating disks was investigated experimentally and by means of plastic flow calculations (Technical Note 2803). Short-time spin

⁹ See Brown, Jones and Newman paper listed on p. 51.

¹⁰ See Newman, Jones and Brown paper listed on p. 52.

tests on parallel-sided, 10-inch-diameter disks were conducted under conditions that subjected the disks to a range of temperatures. Measured plastic strains and experimental burst speeds were compared with the strains and burst speeds calculated from the short-time tensile properties of the disk material. The agreement between the theoretical and experimental results was good over the wide range of temperature conditions investigated. Thermal gradients produced little reduction in burst speed of the high-ductility disks; however, these gradients had a strong influence on the behavior of the disk during the early stages of plastic flow. The loss in tensile properties of the material caused by the temperature of the material had a greater effect in reducing the burst speed than did the stresses set up by the thermal gradient.

Physics of Solids

Quantitative studies of the influence of the surface composition on the creep of zinc single crystals under constant loads gave results which are in accord with the theory that a continuous layer of oxide strengthens the crystal only mechanically at low loads. However, at high loads, the oxide layer must crack since creep proceeds at a rate characteristic of the pure metal rather than the oxide layer. It was further observed that an increase in creep rate occurred when the pure metal surface was treated with dilute mineral acids, an effect which had not hitherto been observed, and which is not attributable to the strengthening effect of the oxide layer. An empirical equation of a new form was developed to describe the creep curves of unoxidized crystals in air which do not follow previously existing equations.¹¹

As the first step in a study of the thermodynamic activities of copper alloys, the vapor pressure of pure copper was measured. By using the Knudsen effusion technique in its unmodified form, it has been possible to obtain values between 1242° and 1563° K. with relatively high precision.¹²

Work on order-disorder phenomena has been continued. The specific heat and the energy of transformation were measured in the range from 50° to 200° C. for the alloy Mg_3Cd using an adiabatic calorimeter. Two transformations of the second order were detected: the one took place at 153.3° C., indicating a curie point for the alloy in equilibrium; the second occurred somewhat below 100° C., being of a rate-process type. The total heat involved in these transformations was obtained.¹³

In addition, a theoretical investigation resulted in the derivation of an expression for the free energy of

a binary, cubic alloy undergoing the order-disorder transition by direct evaluation of the configurational partition function. An equation based on this expression and valid at high temperatures was developed which gives the dependence of the specific heat for such an alloy on temperature.¹⁴

Previous experiments dealing with color centers in alkali halide crystals had shown that a metastable state could be established in sodium-chloride (NaCl) crystals by electrolysis, the state being characterized by a high sensitivity for the formation of nonpermanent color centers upon irradiation with X-rays. Crystals in this state were consequently said to contain color-center precursors. To arrive at a more thorough understanding of the processes involved in setting up color-center precursors, measurements were made of the apparent free sodium content of NaCl crystals containing color centers and color-center precursors. The ratio of the number of free sodium atoms to vacancy pairs was found to be about 1 for crystals which had been subjected to electrolysis, and about 0.1 for crystals irradiated with X-rays independent of whether these crystals had been first subjected to electrolysis.¹⁵

In a further attempt to understand the effects of irradiating materials with X-rays (radiation damage), a study was made of the thermoluminescence of fluorite colored by X-rays. Fifteen emission bands were observed when the crystals were annealed. Continuous annealing curves which were obtained for the strongest emission bands indicated that the bands resulted from two distinct types of imperfections in the crystal resulting from X-ray bombardment.¹⁶

ROCKET ENGINES

Combustion instability has been encountered during various phases of the development of liquid-propellant rocket engines. This combustion instability is characterized by sustained oscillations in combustion-chamber pressure over a wide frequency range. This problem has been the subject of a number of investigations. A recent study of the spray formed by impinging jets of water indicated that upon impingement, a ruffled sheet of liquid was formed which disintegrated intermittently to form groups of drops which appear as waves propagating from the point of impingement. The effect of various injector parameters on the frequency of wave formation and on the spray pattern was investigated because of its possible relationship to combustion instability.

One form of combustion instability, termed chugging, is a cyclical low-frequency type that exhibits

¹¹ See Coffin and Weiman paper listed on p. 51.

¹² See Hersh paper listed on p. 51.

¹³ See Welber, Webeler, and Trumbore paper listed on p. 53.

¹⁴ See Schwed and Groetzinger paper listed on p. 52.

¹⁵ See Hacksaylo and Otterson paper listed on p. 51.

¹⁶ See Hill and Aron paper listed on p. 51.

chamber pressure and nozzle flow or thrust fluctuation in the frequency range of 10 to 200 cycles per second. This type of instability may reduce specific impulse and is believed to be accompanied by changes in combustion efficiency. Results of experimental measurements of low-frequency combustion instability of a 300-pound-thrust acid-heptane rocket engine were compared with the trend predicted by an analysis of combustion instability in a rocket engine with a pressurized-gas propellant pumping system. (Technical Note 2936.)

For the past several years, considerable interest has been shown in film cooling as a promising method of cooling rocket engines. Film cooling is accomplished by establishing a film or layer of liquid between the hot gases and the confining wall, the film of liquid thus shielding the wall from the hot-gas stream. The overall flow system therefore consists of the simultaneous flow of a gas and a liquid with the liquid flowing as an annulus or film along the duct wall, and the gas flows at high velocities through the central core. This flow system is termed annular liquid flow with cocurrent gas flow and is one of several types of two-phase, two-component flow systems. An understanding of the annular liquid-flow system is therefore desirable for its application to film cooling. Visual observations and flow analysis were made of annular liquid flow with

cocurrent air flow in horizontal tubes. The effect of liquid properties and air-stream conditions on the characteristics of the liquid flow were observed. The results were presented in the form of shadowgraph pictures of the liquid flow. Analysis of the flow system related the characteristics of the liquid flow to the flow conditions.¹⁷

An experimental investigation of internal-liquid-film cooling was conducted in 2- and 4-inch diameter straight metal tubes with air flows at 600° to 2,000° F. and Reynolds numbers from 2.2 to 14×10^5 . The coolant was water at flows of 0.8 to 12 percent of the air flow. Visual observation of liquid-film flows were made in transparent tubes with air flows at 80° and 800° F. and diameter Reynolds numbers from 4.1 to 29×10^5 . Flows of water, water-detergent solutions, and aqueous ethylene glycol solutions were investigated (Report 1087).

Nitric acid is an important oxidant in rocket propulsion. This use requires data on the physical properties of the concentrated acid. Consequently, a literature survey and a supplemental experimental study were made to obtain information on the following physical properties: thermal conductivity, dynamic viscosity, specific heat, density, and vapor pressures (Technical Note 2970).

¹⁷ See Abramson paper listed on p. 51.

AIRCRAFT CONSTRUCTION

The need for increased research in the field of aircraft construction continues. More and more it is being appreciated that significant advances in aeronautics are going to be dependent on our knowledge of the behavior of aircraft structures under the ever expanding environment of flight. The so-called thermal barrier resulting from aerodynamic heating at high speeds looms as an important obstacle to the designer of new airplanes and missiles. The inter-relation of this problem with problems in aeroelasticity and flutter add serious complications to the design of new structures and the selection of new materials. Fatigue of aircraft structures also continues to be of major importance not only to transport type aircraft but also to aircraft subjected to lower numbers of extremely high loads and to aircraft where there is a cyclic load pattern as in a helicopter rotor mechanism.

In order to promote the early use of research results in the design and development of airplanes and missiles, the practice of holding technical conferences with representatives of the military services and the aircraft industry was continued during the past year. A technical conference on structures, loads, and flutter problems was held at the Langley Laboratory in March 1953.

As in the past, a portion of NACA's research in this field was performed under contract at universities and other nonprofit scientific institutions. A description of the Committee's recent unclassified research on aircraft construction is given on the following pages and is divided into four sections: Aircraft structures, aircraft loads, vibration and flutter, and aircraft structural materials.

AIRCRAFT STRUCTURES

Static Properties

In order to support the severe loads encountered in supersonic flight and to obtain adequate torsional stiffness, wing structures of thick plate construction are being used. Recent structures research by the NACA has placed emphasis on methods of analysis for such thick-plate construction.

If the form of the thick-plate structure is the same as that of a familiar thin-sheet structure, some extension of previous methods of analysis may be adequate to take into account the use of thicker material. For example, the experimental investigation reported in Technical Note 2930 showed that the methods of beam-web strength analysis presented in the last annual

report (Technical Note 2661) are applicable to beams with thick webs, provided that previously neglected portal-frame action associated with the thick webs is taken into account.

More frequently, thick-plate construction is most economically incorporated in a new constructional form. For wings, the use of thick skins permits a reduction in the number of stiffening elements required to prevent skin buckling. Thus, with thick skins, pure monocoque, multiweb wings, and multiweb wings with various combinations of stiffeners between webs and stiffeners supported by posts become more suitable than the old thin skin-stiffener construction. The new forms of wing construction have been studied in a series of investigations, both theoretically and experimentally.

In Technical Note 2875 an analysis was made of the bending strength of a pure monocoque wing and it was shown that failure of such a wing will occur by a flattening action not encountered in previous types of wing construction.

The use of a number of shear webs to support wing skins has been investigated both experimentally and theoretically. The factors which influence the strength of multiweb construction have been determined from the experimental studies. In Technical Note 2987 these factors are analytically evaluated and charts are presented for the evaluation of the strength of beams for a wide range of types of multiweb construction.

The use of stiffeners between shear webs for multiweb wings is, in some cases, economical from a production cost and from a structural weight standpoint. This form of construction can be analyzed by the methods given in the previously cited Technical Note 2987, provided the properties of the stiffeners are correctly evaluated. The properties of stiffeners attached to one side of a sheet or plate are analyzed in Technical Note 2873 and this analysis may be used to evaluate the properties of stiffeners between shear webs.

The use of stiffeners between shear webs with the stiffeners supported at intervals by posts provides a wing internal structure of greater accessibility than can be achieved with solid shear webs alone. An analysis comprising a series of computations performed on the National Bureau of Standards SEAC digital computer, together with test results, delineates the regions in which such post-stiffener construction can be successfully substituted for multiweb construction.

The large root chord of delta-plan-form wings reduces the intensity of loading on both tension and com-

pression covers and makes a thinner skin more adequate for delta plan forms than for those with smaller root chords. Hence, thin skin-stiffener construction may still be suitable for much of the structure in delta wings. In previous years the NACA has provided extensive information on basic column behavior as well as on the compressive strength of skin-stiffener panels tested as columns. Further information on these subjects is provided in Technical Note 2792 and Technical Note 2872. The latter paper shows graphically the effect of variations in stress-strain properties, column proportions, and initial curvature on column strength. The former confirms results previously obtained in column tests that hat-section stiffeners are not structurally superior to Z-section stiffeners. These results had been questioned on the basis that column tests are not sufficiently representative of actual conditions for which the skin-stiffener panel is part of a box beam. Box beam tests reported in Technical Note 2792, however, gave results similar to those previously found for column tests.

In recent years a considerable amount of progress has been made in the development of applications of electric-circuit analogies to problems in structural analysis. In order that structural engineers may better understand the process of designing such electric circuits, California Institute of Technology has described in Technical Note 2785 the analogy between electrical and mechanical systems to calculate the stresses and deflections of beams. This method of analysis has been applied to thin multicell wings with rather thick skin and no stringers. The analysis which is presented in Technical Note 2786 shows that such a wing deflects in the manner of a plate rather than as a beam and that the internal stress distributions may be considerably different from those given by beam theory.

The accurate calculation of the maximum strength of aircraft structures usually requires stress analysis in the plastic range. In Technical Note 2886, a procedure is given for the analysis of statically indeterminate trusses having members stressed beyond the elastic limit. The method, based on the principle of minimum complementary energy, involves setting up simultaneous nonlinear equations that can be readily solved by iteration.

When a thin-curved beam of small curvature is subjected to lateral loads acting toward the center of curvature, the axial thrust induced by the bending of the arch may cause the arch to buckle so that the curvature becomes suddenly reversed. This problem has been investigated by California Institute of Technology and a general solution is presented in Technical Note 2840.

The ability to calculate accurately the distortion of aircraft wings and tails is essential for aeroelastic analysis. The distortion of delta wings having carry-through bays smaller in chord than the wing root chord

are dealt with in Technical Note 2927. Experimental deflection patterns are given for solid metal specimens of uniform and tapered thickness and fair correlation is achieved with a theoretical analysis based on plate bending theory.

Dynamic Properties

For very sudden loadings, such as those that may occur in hard landings, the usual methods of stress analysis of wings and fuselages based on elementary-beam theory are not adequate. The beam equations of Timoshenko, taking into account shear deformations and rotary inertia, are more accurate and lend themselves to a step-by-step analysis of the stresses as they propagate through the beam. In Technical Note 2874, a basic theoretical study of traveling waves in beams is reported; the mathematical implications of Timoshenko's equations are discussed in detail, and the solutions to several problems are obtained by various methods.

Fatigue Properties

The fatigue problem in aircraft structures has been sharply accentuated as a result of a number of major accidents, in this country and abroad, that can be attributed to fatigue. Although many fatigue tests of small laboratory specimens have been made to supply basic knowledge and engineering information about fatigue, such tests do not reproduce the complex conditions that exist in a full-scale built-up structure. Therefore, full-scale C-46 "Commando" airplane wings were subjected to fatigue tests at a level of $1g \pm 0.625g$ or about $1g \pm 14$ percent of the design ultimate load. The results of these tests are presented in Technical Note 2920. Information was obtained on effective stress concentration factors, the spread in lifetime for various locations of fatigue failures, and on the rate of growth of fatigue cracks. It was also noted that no change occurred in the natural frequency or damping characteristics of the wings prior to development of a fatigue crack.

Thermal Properties

Nonuniform heating induces thermal stresses in the structure of an aircraft. If the stresses are sufficiently large, the induced stresses may cause buckling of the skin of the aircraft. In Technical Note 2769 an experimental and theoretical study was made to assess the suitability of wire strain gages for measuring the thermal strains induced under moderate nonuniform temperatures, up to 300°F. , and to develop simple methods of thermal stress analysis. It was found that, with adequate calibration, the strains measured by wire gages agreed within 5 percent with those predicted by theory. It was also found that moderate temperature

gradients induced in a plate (less than a change of 150° F. in 12 inches) were sufficient to buckle the plate and produce large bending deflections. In Technical Note 2771 an approximate method, based on large-deflection plate theory, was developed to predict the depth of the buckles.

Two studies of a preliminary nature have been completed to determine ways by which available room-temperature data on structural strength may be used to determine the strength at elevated temperatures. A survey has been made of the problem of predicting structural strength at elevated temperatures. In Technical Note 2963 a study is made of the effect of variation in rivet strength on the strength of skin-stiffener panels to provide a basis for the determination of the effect of riveting on the strength of structures at elevated temperatures.

Because high temperatures can develop in the structures of supersonic aircraft, creep may have to be considered in the structural design of such aircraft, especially those intended for sustained or repeated flights. The creep problem in supersonic aircraft has an important aspect not generally encountered in other high-temperature equipment, inasmuch as a large part of the aircraft structure consists of plate compression elements which are relatively thin and which unavoidably have slight initial crookedness. Creep in such elements will tend to magnify the crookedness continuously and eventually lead to collapse. As an initial step in the study of creep strength of columns an analysis was made of a slightly bent H-section column under constant load. An extension of this work to include the effect of the actual distribution of material over a rectangular cross section is presented in Technical Note 2956. This analysis revealed theoretical parameters in terms of which test data on column creep life should be plotted in order to permit the maximum generalization of the data.

AIRCRAFT LOADS

Research Techniques

A general method has been developed for calibrating installations of strain gages on aircraft structures to permit the measurement in flight of the shear or lift, the bending moment, and the torque or pitching moment on the principal lifting or control surfaces. The results of this investigation are presented in Technical Note 2993. Although the stress in a structural member may not be a simple function of the shear, bending moment, and torque, a straightforward procedure is given for numerically combining the outputs of several strain-gage bridges in such a way that the loads may be obtained. In addition to the basic procedures, extensions of the method are described which, by elec-

trical combination of the strain-gage bridges, permit compromises between strain-gage installation time, availability of recording instruments, and data reduction time. The basic principles of strain-gage calibration procedures are illustrated by reference to the data for two aircraft structures of typical construction, one, a straight, and the other, a swept horizontal stabilizer.

Steady Flight Loads

A considerable amount of load data useful for the structural and aerodynamic design of aircraft have been obtained from pressure-distribution tests of a 45° swept wing of aspect ratio 8. Wing loads produced by trailing-edge flaps of various spans at various spanwise positions are reported in Technical Note 2983. The experimental incremental lift, bending moments, pitching moments, and centers of pressure are compared with calculated values. This comparison indicates that there is a need for improved methods for calculating these parameters.

It has been shown that a satisfactory mathematical model for a wing-fuselage combination can be obtained with a simple vortex image system. By using this scheme, a method of calculating the lift on the fuselage in the presence of the wing has been derived. This mathematical model can also be used for calculating the downwash behind the wing and for calculating the spanwise lift on the wing in the presence of the fuselage.

A simplified analysis has been made of the factors affecting the loss in lift and the shift in aerodynamic center of a swept wing due to its distortion under aerodynamic load. The manner in which these aeroelastic effects influence the loads and longitudinal stability of a complete airplane is considered. Results of this analysis, presented in Technical Note 2901, show that the most important geometric parameters of a wing with regard to their influence on aeroelastic effects are aspect ratio, sweep angle, and thickness ratio. With the thickness ratios being contemplated today, aeroelastic considerations would appear to restrict the wing aspect ratios to low values for large sweep angles and, conversely, restrict sweep angles to low values for high aspect ratios. Although the effects of wing torsion, in general, tend to alleviate the effects of wing bending, the magnitudes of the torsional distortion are usually small compared with those due to bending except for wings with low sweep or aspect ratio. Some alleviation of aeroelastic effects occurs in maneuvers because of the inertia of the wing. In addition, the effect of wing-aerodynamic-center shift on the longitudinal stability of an airplane may be compensated for by the associated loss in wing lift provided the percentage reduction in wing-lift-curve slope is greater than that for the tail.

Maneuver Loads

During the past year attention has been given to the problem of devising a simple and rational method for computing the design maneuver load on the horizontal tail. A damped sine wave elevator motion simulating the Air Force pull-up push-down maneuver is suggested in Technical Note 2877 as an alternate procedure for computing the design load on the horizontal tail. The damped sine-wave motion is not only more representative of that actually applied in flight but it permits a simple short solution for the maneuvering tail load using operational methods.

An investigation to determine the effect of horizontal-tail span and vertical position on the aerodynamic characteristics of an unswept tail-stub fuselage assembly in sideslip is reported in Technical Note 2907. The results indicate that the induced load carried by the horizontal tail and the net load on the vertical tail were greatly influenced by the horizontal-tail span and vertical position, the greatest effects occurring when the horizontal tail was placed at either the root or tip of the vertical tail. It was found that loads on the individual tail surfaces, as well as on the entire assembly, could be calculated fairly well by use of the finite-step horseshoe-vortex method, wherein the tail surfaces are represented by a finite number of horseshoe vortices.

Gust Loads

Methods used in the description of atmospheric turbulence and the calculation of gust loads have for simplicity been restricted to the case of a single discrete gust. The limitations of this approach have, in recent years, encouraged the development of more generally applicable techniques. In Technical Note 2853 the techniques of generalized harmonic analyses have been used to extend the scope of the analyses beyond the discrete-gust case to the more realistic case of continuous rough air. The concept of a power spectrum has been used to describe continuous rough air and the general relation for linear systems between the power spectrums of a random disturbance and an output response used to relate the spectrum of load to the spectrum of atmospheric gust velocity. The results obtained show that the spectrum of loads provides a simple measure of the load intensity in terms of the root-mean-square of the loads. Experimental evidence indicates that in some cases the distribution of loads in rough air is approximately normal and is thus completely described by the standard deviation. The results obtained in an application of this approach in a series of calculations indicate that the airplane short-period damping characteristics may have an important effect on the loads in continuous rough air.

In order to provide for uniformity of gust-load calculations, a revised gust-load formula with a new gust

factor has been derived to replace the previous gust-load formula and alleviation factor widely used in gust studies. The revised formula utilizes the same principles and retains the same simple form of the original formula but provides a more appropriate and acceptable basis for gust-load calculations. The gust factor is calculated on the basis of a one-minus-cosine gust shape and is presented in Technical Note 2964 as a function of a mass-ratio parameter in contrast to the ramp gust shape and wing loading formerly used for the alleviation factor.

One of the problems that concerns the aeronautical engineer in the formulation of his design relative to critical airplane loading is the effect of Mach number on the maximum value of the load induced on an airplane by its sudden entry into a gust of given structure. This effect has been estimated in Technical Note 2925, on the basis of theoretical calculations, for rectangular wings having an aspect ratio greater than 2 and flying at any Mach number between 0 and 2. The airplane wing is considered to be rigid and to undergo a negligible angle of pitch in the time required for the maximum gust load to develop. In general it was shown that decreasing the wing aspect ratio and the wing-to-air density ratio decreased the maximum gust load, and the reductions were most pronounced at high subsonic Mach numbers.

Normal acceleration and airspeed data obtained for one type of twin-engine transport airplane in commercial operation over a northern transcontinental route have been analyzed to determine the gust and gust-load experiences of the airplane. The results are reported in Technical Note 2833. The data covered a period of about $1\frac{1}{2}$ years and consisted of 388 records for about 39,000 flight hours. These data are almost a complete history of this type of airplane for the period covered inasmuch as the 24 airplanes instrumented constituted nearly all the airplanes of this type that were built. The acceleration increments experienced equaled or exceeded the limit-gust-load factor, on the average, twice (once positive and once negative) in about 7.5×10^6 flight miles, and an effective gust velocity of 30 feet per second was equaled or exceeded twice in about 1×10^6 flight miles.

The results of an analysis of 48,187 hours of normal-acceleration and airspeed data, obtained on a 4-engine type of transport airplane operating at altitudes up to 20,000 feet in commercial operation on an eastern United States route from November 1947 to February 1950 are reported in Technical Note 2965. The analysis indicates that the gusts encountered were less severe than those previously encountered in operations below 10,000 feet although about as severe in terms of percent of limit load factor. The analysis also indicates that high airspeeds were attained more frequently. The

gust and gust-load experience during the summer (April through September) was found to be approximately 20 percent more severe than during the winter (October through March).

An analytical evaluation of the gust-response characteristics of three twin-engine transports and a 4-engine bomber with wing bending flexibility included, together with a limited correlation of some of the calculated results with flight data is presented in Technical Notes 2763 and 2897. The effect of gust-gradient distance, gust shape, spanwise mass distribution, forward velocity, altitude, compressibility corrections, and aspect-ratio corrections on dynamics were investigated. The results indicate that three of the airplanes have rather appreciable elastic-body dynamic-overshoot effects, but one does not.

The analyses of results obtained during flight studies of transient wing response on a 4-engine bomber airplane are reported in Technical Notes 2780 and 2951. Acceleration and strain measurements at a number of spanwise stations were made during flights through clear-air turbulence for two speeds and two weight conditions. The amplification of strain due to flexibility effects was determined by comparing the strains developed for a unit normal acceleration when the airplane encounters a gust to the strains developed for a unit normal acceleration in a pull-up. The results indicated that the strains at a station near the wing root were on the average 31 percent greater for the gust case than for the corresponding pull-up. The amplification was also found to vary with spanwise location, diminishing slightly at successive outboard stations except at the most outboard station where it increased. This is in contrast to previous results for a two-engine transport where an amplification of 20 percent was indicated for each of the measuring stations.

A gust-tunnel investigation to determine the effects of center-of-gravity position on the gust loads on a delta-wing model with the leading edge swept back 60° and for a range of center-of-gravity positions from 4 percent ahead of to 11 percent behind the leading edge of the mean geometric chord indicated that a 1 percent rearward movement of the center-of-gravity position increased the acceleration increment in the sharp-edge gust by 0.5 percent. In a gust with a gradient distance of 6.5 chords, the acceleration increment was increased by approximately 2 percent for the same movement of the center of gravity. Comparison of these results with those for a conventional airplane model indicates that the change in load for both configurations would be nearly the same in a sharp-edge gust, but in a gust with a gradient distance of 6.5 chords, the change in load for a given change in center-of-gravity position would be approximately twice as great for the delta-wing model as for the conventional airplane model.

Landing Loads

Static force-deflection characteristics of six aircraft tires were determined for the following conditions: Vertical loading, combined vertical and side loading, combined vertical and torsional loading, and, for one tire at several inflation pressures, combined vertical and fore-and-aft loading. The results of this investigation are presented in Technical Note 2926. The lateral spring constants for all the tire specimens tested decreased with increasing vertical tire deflection; whereas, the torsional and fore-and-aft spring constants increased with increasing vertical deflection. The lateral and fore-and-aft shifts of the center of pressure of the vertical reaction were found to average 75 percent and 25 percent, respectively, of the side and fore-and-aft tire deflections.

Measurements have been obtained with a specially built motion-picture camera of landings of transport airplanes during routine operations at the Washington National Airport in clear-air daylight conditions. From these measurements, sinking speeds, roll attitude angles, and rolling velocities have been evaluated. The equipment and technique employed for the measurements have been found to be accurate and practical, at least for daytime operations. Continued operation of the project, in which horizontal velocities will also be determined, is expected to yield information on the influences of various factors such as airplane characteristics and weather conditions (turbulence, lowered ceilings, reduced visibility, and so forth). An airborne vertical landing-speed recorder has also been developed and tested in controlled drop tests with forward velocity as well as in flight on a small trainer-type airplane. A description of the operation of the instrument together with collected data is presented in Technical Note 2906. Evaluation of the data shows that the instrument is both accurate and practical for obtaining vertical velocities at the instant of wheel contact.

The general equations governing the fixed-trim water landing of a straight-keeled seaplane of arbitrary constant cross section are presented in Technical Note 2814 in such a form that the landing motions and loads are expressed in terms of the steady planing characteristics of the seaplane. Solutions for a rectangular flat plate are compared with experimental impact data and are shown to be in good agreement. Experimental impact load data for a V-step flat-bottom model having a high beam loading presented in Technical Note 2932 show results similar to those for transverse steps, and comparisons of time histories show good agreement between experimental results and results computed by the method of Technical Note 2814. The effect of partial wing lift upon the loads and motions of wide prismatic V-bottom seaplanes landing with the resultant velocity normal to the keel is treated theoretically

in Technical Note 2815. The increase in vertical hydrodynamic load factor is shown to be about 133 percent of the decrease in air load. Additional development in hydrodynamic impact theory is presented in Technical Note 2817 where calculated side-force, rolling moment, yawing moment, and pressure distribution for yawed landings and planing of seaplanes are compared with experimental data for a prismatic wedge having a deadrise of 22.5° and are shown to be in reasonable agreement.

Water-pressure distributions on a prismatic float having an angle of dead rise of $22\frac{1}{2}^\circ$ are given in Technical Note 2816 along with velocity, draft, and acceleration data. These data are for beam-loading coefficients of 0.48 and 0.97, fixed trims from 0.2° to 30.3° , and flight-path angles from 4.6° to 25.9° .

An approximate method for computing water loads and pressure distributions of lightly loaded elliptical cylinders during oblique water impacts is presented in Technical Note 2889. The method, of special interest for the case of water landings of helicopters, makes use of theory developed and checked for landing impacts of V-bottom seaplanes. Comparison of results computed by this method with limited experimental data obtained during drops of a circular cylinder at 0° trim and 90° flight-path angle shows rough agreement.

VIBRATION AND FLUTTER

Dynamic Model Technique

There are so many unknown factors, both structural and aerodynamic, involved in the flutter analysis of an unusual airplane configuration flown at sonic and supersonic speeds that recourse is being made to the testing of models having scaled down dynamic properties. Many of these models are tested in flight using the rocket model technique. The testing of such models in a wind tunnel is under consideration and effort is being directed toward the development of a method of supporting the model in the tunnel which will allow the model the degrees of freedom which are required to simulate the flutter conditions of the full scale airplane.

Fuel Motion in Wing Tip Tanks

When an airplane wing is vibrated or shaken due to flutter, buffeting, or rough air the sloshing motion of the fuel serves to damp the wing motion. In Technical Note 2789 are reported the results of an exploratory investigation of the dynamic effects of fluid sloshing in tip tanks using two simplified model beam-tank systems. It was concluded that the damping effect of the fluid motion and the effective mass of the fluid were unaffected by changes in viscosity of the fluid but were strongly affected by changes in density of the fluid and the amount of fluid in the tank.

AIRCRAFT STRUCTURAL MATERIALS

Fatigue

Fatigue failure due to variable or constant cyclic loading is an important phase in the design of aircraft structures and structural elements. Efficient use of available materials was once dependent only upon the knowledge of such properties of the materials as strength, ductility, stiffness, etc. Knowledge of the fatigue properties of the various materials has been recognized for some time as an addition to these structural properties. The NACA, therefore, instituted a long-range program to determine the fatigue properties of currently used aluminum-alloy sheet materials. Axial load fatigue data were obtained on 24S-T3 and 75S-T6 aluminum alloy sheet at both the Langley Laboratory and Battelle Memorial Institute. A third set of data was obtained from tests on cylindrical specimens by the Aluminum Company of America. The data from the three investigations are presented and compared in Technical Note 2928.

The direct application of fatigue data obtained in laboratory tests of small specimens to the design of large parts can lead to large unconservative errors for several reasons. One of these reasons is the fact that the fatigue stress concentration factors, for geometrically similar parts, tend to increase as the size of the part increases. An engineering rule for predicting the effect of the size of parts on the stress concentration factors has been developed. A relation originally developed by Neuber in Germany and involving a new material constant is used to convert the theoretical stress concentration factor, which does not depend on absolute size, into the actual fatigue factor. A large number of fatigue tests of steels tested under completely reversed load have been correlated reasonably well by the method. This work is presented in Technical Note 2805.

The simplest type of fatigue test is usually made for constant levels of maximum and minimum stress. Many practical cyclic loading conditions, such as the one found in aircraft wings, occur at varying amplitudes of cycles of stress. The prediction of the life expectancy of such parts when subjected to irregularly varying stresses from constant-level data requires the knowledge of the cumulative damaging effect of the successive application of such stress cycles. An experimental investigation was conducted in which 24S-T4 aluminum-alloy rotating-beam specimens were subjected to cycles of stresses of constant amplitude and of two distributions of continuously varying amplitudes. The results and a preliminary analysis, based on an engineering method of predicting cumulative effects, are presented in Technical Note 2798. The results of the analysis were found to be influenced by the distribution of continuously varying amplitudes of cyclic stress.

The effects of cyclic loading on the mechanical behavior of 24S-T4 and 75S-T6 aluminum alloys and SAE 4130 steel were studied at the Massachusetts Institute of Technology. Constant level stress tests were made on specimens subjected to a number of cycles above and below the endurance limits. Special slow-bend tests were then performed at constant deflection rates and temperatures to determine the effects of prior cycles of fatigue stresses on the transition temperature to brittle fracture of SAE 4130, and on the energy absorption capacity of the aluminum alloys. These tests showed an increase in the transition temperature for the steel when fatigued above the endurance limit, and a decrease in energy absorption for the aluminum alloys as the number of fatigue cycles increased. In tests conducted at low temperatures (-320° F.) on V-notched specimens, 75S-T6 showed considerably less energy absorption than 24S-T4. These results are presented in Technical Note 2812.

Creep

The properties and behavior of materials subjected to long-term loading at elevated temperatures must be known before efficient design of many structural elements can be accomplished. Creep of materials is one of the basic phenomena associated with elevated temperatures. The program instituted at the Battelle Memorial Institute to study the fundamentals of creep of materials is continuing. That portion of the program reported in Technical Note 2945 covers the creep of single crystals of aluminum from room temperature to 400° F. during constant load creep tests and increasing load tensile tests. Observation of the strain markings on the specimens indicates that slip plays an important role in deformation during constant load creep tests and also during increasing load tensile tests.

The frequency distribution of the angular orientation of slip lines within separate grains on the surface of a plastically deformed polycrystal provides a basis

for assessing the accuracy of assumptions regarding the detailed mechanism of plastic deformation. In Technical Note 2777 this frequency distribution has been calculated theoretically for a polycrystal composed of aluminum crystals, each of which has twelve possible modes of slip. The results are compared with experiment and also with those previously reported in Technical Note 2577 (reported in the last annual report) for a polycrystal composed of crystals which have but a single mode of slip. The comparisons between the two theories show that the differences are small at low stress levels but become large as the stress level is increased; the comparison with experiment is considerably better for the twelve-mode theory than for the single-mode theory.

In an investigation carried out at the Pennsylvania State College and reported in Technical Note 2737 an attempt was made to determine the validity of several theories of plasticity and the correctness of the various assumptions made in these theories. Plastic stress-strain relations for biaxial tension-compression stresses were determined for 14S-T4 aluminum alloy. Combined tension and compression principal stresses were produced by subjecting a thin-walled tube to combined axial tension and torsion.

Plastics

Transparent cabin enclosures used in the modern pressurized military or commercial airplanes are subject to failure by crazing or impact. The National Bureau of Standards is continuing a project to determine means of improving craze resistance of plastics and has biaxially stretched polymethyl methacrylate to 100 percent and 150 percent. Ultimate tensile strengths were increased slightly. The threshold stress for stress-solvent crazing was approximately three-fourths of ultimate stress and close to the ultimate stress for the 100 percent and 150 percent biaxially stretched material, respectively.

OPERATING PROBLEMS

During the past year, aircraft have been flown at higher speeds, higher altitudes and with greater frequency than ever before. This ever increasing frequency of flight under these conditions has accentuated some of the operating problems that are pertinent to safety and to the successful attainment of all-weather flight; namely, aircraft fire, icing, meteorology, crash survival, and ditching. Another very important operating problem currently being investigated intensively is that of the noise generated by the high-powered engines being employed in these high performance aircraft. Although the NACA has established the basic principles of aircraft engine and propeller noise suppression for conventional aircraft of the recent past, the noise generated by the turbo-jet and turbo-prop engines now in use and coming into use is of substantially greater magnitude and requires new techniques for its suppression. The gravity of this situation led to the creation of a Special Subcommittee on Aircraft Noise in 1952. Under the guidance of this subcommittee, research in this field has been intensified. This subcommittee has been very active and, in view of the current broad research and development activity within the noise field, has maintained close liaison with other organizations. To accelerate progress in this field a series of informal conferences have been initiated to bring together the various workers both in this country and abroad to exchange information and assist in pinpointing the most urgent and desirable research to be undertaken.

Although the magnitude of the noise problem is well understood and a comprehensive program of research is vigorously being pursued, the problem is of such a difficult nature that early solutions are not to be expected.

Other NACA groups that have helped guide the research which has been carried out in the three NACA laboratories and under contract with nonprofit research organizations include, in addition to the parent Committee on Operating Problems, the Subcommittee on Icing Problems, the Subcommittee on Meteorological Problems, and the Subcommittee on Aircraft Fire Prevention.

The following is a résumé of the various unclassified research accomplishments in Operating Problems during the past year.

Airspeed Measurement at High Speeds and Altitudes

Detailed measurements of temperature and static pressure at altitudes approaching the tropopause were

taken in flight to determine whether the atmospheric conditions necessary for accurate use of the temperature method of airspeed calibration were present. The results of the investigation (Technical Note 2807) indicated that the requirements for constancy of temperature over small intervals of time and horizontal distance and for a temperature lapse rate less than the NACA standard atmosphere were not met during the tests. The temperature method of airspeed calibration therefore was considered impractical at high subsonic speeds in the altitude range just below the tropopause.

Ditching

An experimental investigation has been conducted to study the effects of various rear fuselage shapes on aircraft ditching behavior. The results (Technical Note 2929) indicate that, at low landing speeds, a flattened rear-fuselage cross section is advantageous except where there is no longitudinal bottom curvature. At the higher landing speeds, it was noted that a rounded cross section tends to avoid undesirable skipping. In addition, the fuselage with the highest fineness ratio was more moderate in behavior.

Investigations of the ditching behavior of specific airplane configurations have been continued utilizing scale models. The tests were made in calm and rough water to determine the effects of various wing planforms and nacelle locations of new aircraft. Structural damage was simulated to show the behavior of the models following ditching impact.

Aircraft Accident Survival

Full-scale light-airplane crashes simulating stall-spin accidents were conducted to determine the decelerations to which occupants are exposed and the resulting harness forces encountered in this type of accident (Technical Note 2991). In crashes where the impact speeds varied from 42 to 60 miles per hour, crumpling of the forward structure prevented the maximum deceleration at the rear seat location from exceeding 26 to 33 *g*. Restraining forces in the seat-belt-shoulder-harness combination reached 5,800 pounds. It is concluded from this investigation that the rear seat occupant can survive crashes of the type studied at impact speeds up to 60 miles per hour if body movement is properly restrained.

During the conduct of the NACA Full-Scale Crash Fire Program, data were obtained concerning the factors which affect the survival of human beings in airplane accidents. The time intervals during which

occupants could escape from burning aircraft were determined from the data provided by these studies. This was accomplished by using the time histories of cabin temperatures and toxic gas concentrations in conjunction with data that define the environmental conditions which can be tolerated by human beings. Survival times as limited by pain or skin burning ranged from 50 to 300 seconds. Other hazardous factors, such as flying detached airplane parts, explosions, and crushing of the airplane structure, were also studied.

AERONAUTICAL METEOROLOGY

Atmospheric Turbulence

A cooperative investigation among the NACA, Air Weather Service, the U. S. Weather Bureau, and the Brookhaven National Laboratory has been made to gather and compare turbulence measurements obtained by a suitably instrumented airplane flying in the vicinity of a tower with data from wind vanes on the tower. The results of the study indicate a good correlation between the vertical gusts measured by the airplane and the horizontal gusts evaluated from the vane motions, provided that the gusts selected during these simultaneous measurements have nearly the same wave length.

To gain more data on the frequency of occurrence and the degree of severity of high-altitude clear-air turbulence, a program for the collection of pilots' reports of the phenomenon encountered in normal civil and military operations was undertaken in 1949. During a 2-year period, several hundred encounters were reported and a summary of these data indicate that turbulence at high altitudes is a factor which should be considered in both the design and operation of aircraft.

Physics of the Icing Cloud

The meteorological parameters that define an icing cloud, so that accurate prediction of the severity of icing which may be experienced by various aircraft components in flight, include liquid water content and droplet size distribution. Although research toward the development of new principles to provide the basis for reliable instrumentation to measure these quantities accurately in flight has resulted in significant accomplishments in research instrumentation, the time delay between its use as a research tool and its development into an operational piece of equipment may consume several years, depending upon the required further development by industry and its acceptance by the aircraft operator. Development of aircraft instrumentation which will define the icing cloud along the aircraft flight path and which will feed this intelligence to the ice-protection system controls is at present a goal which

may not be reached for many years, but which is certainly worthy of achievement if all-weather flight is to be attained with a minimum of operational and performance losses for optimum ice protection. Application of principles to measure the icing parameters have been previously reported in various NACA publications and include the cloud camera, the charge-droplet cloud analyzer, the rotating disk, the rotating multicylinder, the pressure-type icing-rate meter, the hot-rod, and the hot-wire icing severity meter.

Despite its disadvantages, perhaps the most widely used instrument is the rotating multicylinder, particularly in the field of flight testing of aircraft in icing conditions. Because of its continued use in this field, the NACA initiated a study of the phenomena of water-droplet impingement and related information on cylinders. The NACA water-droplet-trajectory analogue has been used to determine the impingement of water droplets on a cylinder in incompressible and compressible flow fields. The results of these studies (Technical Notes 2903 and 2904) have been used as a basis for evaluation of the sensitivity and limitations of the multicylinder method, especially as applied to droplet-size distribution. It has been shown that on the basis of droplet impingement alone, with other factors such as blow-off and Ludlam effect not considered, large errors in the measurement of drop size of the order of 70 percent can exist.

The NACA-Airline-Air Force icing data program which is utilizing about 100 installations of the NACA pressure-type icing rate meter to collect icing data on a world-wide basis has just completed its second icing season. One preliminary report of the analysis of data collected during the first year of the program has been published and preparations for the final data collection season for 1953-54 have been completed.

ICING PROBLEMS

The degree of ice protection required for any aircraft component is dependent upon many variables. The ideal solution is to provide enough protection to insure no compromise with safety and at the same time minimize the aircraft performance penalties associated with system weight, aerodynamic losses due to ice accretion, and system energy requirements. Recent NACA research has been directed toward study of ice protection design requirements for high performance aircraft of the high-speed turbojet category. The rather large increase in the operating speeds and altitudes of these jet aircraft over the more conventional type has dictated the need for more efficient and effective types of ice protection systems. The thin swept and unswept wings now being employed on these high-speed aircraft have also accentuated the requirement for data that will help the designer determine accurately, without expen-

sive flight studies, the areas requiring protection. Other aircraft components requiring special studies include leading-edge slots, porous leading-edge wings, radomes, helicopter rotor blades, and air inlets and scoops.

The NACA droplet trajectory analogue has been used to determine the area, rate, and extent of impingement of water droplets on an NACA 65₁-208 airfoil for a range of altitude and icing conditions up to a speed equivalent to the flight critical Mach number of the airfoil. Variation of the impingement characteristics of this airfoil have been compared with those of an NACA 65₁-212 airfoil and the results are published in Technical Note 2952. A method for applying this type of impingement data to a sweptback airfoil has also been developed (Technical Note 2931). Impingement of water droplets on wedges and diamond shaped airfoils has been studied at supersonic speeds and the results are presented in Technical Note 2971 for a range of meteorological conditions.

Aircraft icing at high-flight speeds does not always present a problem since the component surface temperature increases with flight speed, due to aerodynamic heating, and when this temperature reaches 32° F. ice will no longer form. This phenomenon is known as the icing limit. The results of an analytical study of a general method for the rapid calculation of the icing limits of any given body in both subsonic and supersonic flow have been presented in Technical Note 2914, covering Mach numbers up to 1.8 and altitudes up to 45,000 feet. However, although the icing limit may be reached on the forward portion of an airfoil, the results of an analytical investigation of a diamond airfoil at transonic speeds (Technical Note 2861) show that the liquid water that runs back past the maximum thickness is subject to freezing in the reduced pressure area which indicates that areas behind this point may be susceptible to icing up to a Mach number of 1.4.

In the design of ice-protection systems, the solutions of equations of heat transfer and evaporation from wetted surfaces during an anti-icing process can be very laborious. These equations have been simplified and presented so as to permit solution by graphical means (Technical Note 2799). In connection with the design of ice protection systems, an analysis of the problems involved in providing icing protection for a turbojet transport aircraft has been made (Technical Note 2866). The optimum icing protection system for any particular aircraft cannot be generally specified since the choice of the optimum system is dependent upon the specific characteristics of airplane and engine, the flight plan, the probable icing conditions, and the performance requirements of the aircraft. However, this study presents all the ramifications associated with the heating requirements for various methods of protection (both

cyclic and continuous) and the assessment of performance penalties associated with providing this protection from various energy sources.

For many years the degree of ice protection required for aircraft wings has been considered from the point of view that the wing should be clean after heat has been applied, whether it be involved in a cyclic or continuous type of system. The fact that most systems on operational aircraft do not afford clean wings under all meteorological conditions and that the resulting residual ice accumulated appears to affect the performance of the aircraft in varying degrees, a general study of the aerodynamic penalties associated with ice accretions has been initiated. As a first attempt to assess the performance penalties associated with ice accretions, an experimental investigation has been made in the Lewis 6- by 9-foot icing tunnel to show the effects on section drag characteristics of those accretions associated with conditions of cyclic de-icing, continuous anti-icing, and no protection on an 8-foot chord airfoil section for several angles of attack. Results of this investigation are encouraging and additional data, including lift and drag on other airfoil sections, should result in significant data from which a more realistic and general interpretation of the problem may be possible.

Other published information in the field of ice protection during the past year include (1) experimental studies of cyclic de-icing of a gas-heated airfoil with an evaluation of the effects of various parting-strips on system performance, (2) an investigation of the performance characteristics of three electrical de-icing boots designed for fighter-type jet aircraft, (3) an experimental investigation of an alternate carburetor air inlet configuration, (4) an experimental study of the dry-air heat transfer from a body of revolution, (5) an experimental study of radome heat requirements, and (6) an investigation to provide design data for the ice-protection of hollow-steel propeller blades utilizing electrical heaters.

AIRCRAFT FIRE PREVENTION

To aid in the reduction of the crash-fire hazards for reciprocating engine type aircraft, general distribution has been made of the very important results of the NACA Full-Scale Crash-Fire Program in the forms of a technical report and a technical motion picture in color and sound. This latter form of information distribution was necessitated because of the dynamics involved in the study of crash fires and the aircraft industry demand for a detailed account of these results for the instruction of their employees who are directly concerned with aircraft and equipment design for crash safety. Pilots are also being thoroughly briefed by this

film and will therefore receive the background of the research and development leading to the equipment and piloting techniques that they will utilize in the event of a crash.

The NACA is continuing the study of the crash-fire hazard as it is related to the turbo-jet aircraft. Results to date indicate that many of the principles learned in the reciprocating engine type aircraft can be successfully applied to the turbo-jet type aircraft, even though the character of some of the ignition sources are significantly different.

In order to minimize the crash-fire hazard, the obvious advantage of jettisoning fuel prior to landing has been investigated to determine the fuel concentrations that may exist when the dumped fuel approaches the ground. The results of a study of the free-fall and evaporation of *n*-octane droplets in the atmosphere with reference to the jettisoning of gasoline at altitudes up to 11,000 feet over a droplet size range from 6 to 2,000 microns and for temperatures from -37° to $+30^{\circ}$ C. have been published. It was concluded that use of atomizing devices causing production of droplets *less* than 200 microns in diameter would allow gasoline jettisoning without ground contamination at ground clearances in excess of 250 feet at temperatures above -37° C. Ground clearances of 1,000 feet should suffice even when the largest (0.2 cm.) droplets are present, that is, when no atomizing nozzles are used at temperatures up to $+30^{\circ}$ C.

AIRCRAFT NOISE

Propeller-Noise Charts for Transport Aircraft

The rotational and vortex noise levels, at 300 feet distance, for a number of propellers in static operation have been calculated for engine ratings of 1,000 to 10,000 horsepower. Charts and tables for the rapid estimation of the noise levels and spectrums for a range of tip Mach numbers from 0.40 to 1.00 (Technical Note 2968) are useful for design purposes. The results indicate that, if noise reductions are to be obtained or if present noise levels are to be maintained for higher power ratings, future propellers should operate at lower tip speeds than those now in use. In addition, single rather than dual-rotating propellers were found to generate the lowest over-all noise level for a given number of blades.

Muffler Research Program

An extensive theoretical and experimental muffler research program was undertaken to aid in establishing a rational basis for the design of mufflers for internal-combustion-type aircraft engines. A large number of muffler sizes and types were evaluated (Technical Notes 2893 and 2943) to determine the validity of equations developed from acoustical theories under certain simplifying assumptions. The results demonstrated that effective muffling was possible, but that the theoretically computed attenuation could not be realized.

RESEARCH PUBLICATIONS

REPORTS ¹

- No.
1059. A Biharmonic Relaxation Method for Calculating Thermal Stress in Cooled Irregular Cylinders. By Arthur G. Holms.
 1060. Detailed Computational Procedure for Design of Cascade Blades With Prescribed Velocity Distributions in Compressible Potential Flows. By George R. Costello, Robert L. Cummings, and John T. Sinnette, Jr.
 1061. Effect of Initial Mixture Temperature on Flame Speed of Methane-Air, Propane-Air and Ethylene-Air Mixtures. By Gordon L. Dugger.
 1062. Investigation of Wear and Friction Properties Under Sliding Conditions of Some Materials Suitable for Cages of Rolling-Contact Bearings. By Robert L. Johnson, Max A. Swikert, and Edmond E. Bisson.
 1063. Airfoil Profiles for Minimum Pressure Drag at Supersonic Velocities—General Analysis With Application to Linearized Supersonic Flow. By Dean R. Chapman.
 1064. Lubrication and Cooling Studies of Cylindrical-Roller Bearings at High Speeds. By E. Fred Macks and Zolton N. Nemeth.
 1065. Correlation of Physical Properties With Molecular Structures for Some Dicyclic Hydrocarbons Having High Thermal-Energy Release Per Unit Volume—2-Alkylbiphenyl and the Two Isomeric 2-Alkylbicyclohexyl Series. By Irving A. Goodman and Paul H. Wise.
 1066. Analysis of Temperature Distribution in Liquid-Cooled Turbine Blades. By John N. B. Livingood and W. Byron Brown.
 1067. Generalization of Boundary-Layer Momentum-Integral Equations to Three-Dimensional Flows Including Those of Rotating System. By Artur Mager.
 1068. Automatic Control Systems Satisfying Certain General Criteria on Transient Behavior. By Aaron S. Boksenbom and Richard Hood.
 1069. On a Solution of the Nonlinear Differential Equation for Transonic Flow Past a Wave-Shaped Wall. By Carl Kaplan.
 1070. Matrix Method of Determining the Longitudinal-Stability Coefficients and Frequency Response of an Aircraft From Transient Flight Data. By James J. Donegan and Henry A. Pearson.
 1071. Theoretical Symmetric Span Loading Due to Flap Deflection for Wings of Arbitrary Plan Form at Subsonic Speeds. By John DeYoung.
 1072. Inelastic Column Behavior. By John E. Duberg and Thomas W. Wilder III.
 1073. An Iterative Transformation Procedure for Numerical Solution of Flutter and Similar Characteristic-Value Problems. By Myron L. Gossard.
 1074. Hydrodynamic Impact of a System With a Single Elastic Mode. I—Theory and Generalized Solution With an Application to an Elastic Airframe. By Wilbur L. Mayo.

¹ The missing numbers in the series of Reports were released before or after the period covered by this report.

- No.
1075. Hydrodynamic Impact of a System With a Single Elastic Mode. II—Comparison of Experimental Force and Response With Theory. By Robert W. Miller and Kenneth F. Merten.
 1076. Effects on Longitudinal Stability and Control Characteristics of a Boeing B-29 Airplane of Variations in Stick-Force and Control-Rate Characteristics Obtained Through Use of a Booster in the Elevator-Control System. By Charles W. Mathews, Donald B. Talmage, and James B. Whitten.
 1077. Two- and Three-Dimensional Unsteady Lift Problems in High-Speed Flight. By Harvard Lomax, Max. A. Heaslet, Franklyn B. Fuller, and Loma Sluder.
 1078. Effects of Compressibility on the Performance of Two Full-Scale Helicopter Rotors. By Paul J. Carpenter.
 1079. Sound from a Two-Blade Propeller at Supersonic Tip Speeds. By Harvey H. Hubbard and Leslie W. Lassiter.
 1080. A Theoretical Analysis of the Effects of Fuel Motion on Airplane Dynamics. By Albert A. Schy.
 1081. A Study of Second-Order Supersonic Flow Theory. By Milton D. Van Dyke.
 1082. Method of Analysis for Compressible Flow Through Mixed-Flow Centrifugal Impellers of Arbitrary Design. By Joseph T. Hamrick, Ambrose Ginsburg, and Walter M. Osborn.
 1083. Axisymmetric Supersonic Flow in Rotating Impellers. By Arthur W. Goldstein.
 1084. Comparison of High-Speed Operating Characteristics of Size 215 Cylindrical-Roller Bearings as Determined in Turbojet Engine and in Laboratory Test Rig. By E. Fred Macks and Zolton N. Nemeth.
 1085. Discussion of Boundary-Layer Characteristics Near the Wall of an Axial-Flow Compressor. By Artur Mager, John J. Mahoney, and Ray E. Budinger.
 1086. Analysis of the Effects of Wing Interference on the Tail Contributions to the Rolling Derivatives. By William H. Michael, Jr.
 1087. Internal-Liquid-Film-Cooling Experiments With Air-Stream Temperature to 2,000° F. in 2- and 4-Inch-Diameter Horizontal Tubes. By George R. Kinney, Andrew E. Abramson, and John L. Sloop.
 1088. Theoretical Damping Roll and Rolling Moment Due to Differential Wing Incidence for Slender Cruciform Wings and Wing-Body Combinations. By Gaynor J. Adams and Duane W. Dugan.
 1089. Single-Degree-of-Freedom-Flutter Calculations for a Wing in Subsonic Potential Flow and Comparison with an Experiment. By Harry L. Runyan.
 1090. Method for Calculating Lift Distributions for Unswept Wings With Flaps or Ailerons by Use of Nonlinear Section Lift Data. By James C. Sivells and Gertrude C. Westrick.
 1091. Effect of Aspect Ratio on the Low-Speed Lateral Control Characteristics of Untapered Low-Aspect-Ratio Wings Equipped With Flap and With Retractable Ailerons. By Jack Fischel, Rodger L. Naeseth, John R. Hagerman, and William M. O'Hare.

- No.
1092. Flight Investigation of the Effect of Control Centering Springs on the Apparent Spiral Stability of a Personal-Owner Airplane. By John P. Campbell, Paul A. Hunter, Donald E. Hewes and James B. Whitten.
1093. Heat Transfer to Bodies in a High-Speed Rarefied-Gas Stream. By Jackson R. Stalder, Glen Goodwin, and Marcus O. Creager.
1094. An Experimental Investigation of Transonic Flow Past Two-Dimensional Wedge and Circular-Arc Sections Using a Mach-Zehnder Interferometer. By Arthur Earl Bryson, Jr.
1095. Transonic Flow Past a Wedge Profile With Detached Bow Wave. By Walter G. Vincenti and Cleo B. Wagoner.
1096. Experimental Determination of the Effect of Horizontal-Tail Size, Tail Length, and Vertical Location on Low-Speed Static Longitudinal Stability and Damping in Pitch of a Model Having 45° Sweptback Wing and Tail Surfaces. By Jacob Lichtenstein.
1097. Stresses in a Two-Bay Noncircular Cylinder Under Transverse Loads. By George E. Griffith.
1098. Summary of Methods for Calculating Dynamic Lateral Stability and Response and for Estimating Lateral Stability Derivatives. By John P. Campbell and Marion O. McKinney.
1099. Air Forces and Moments on Triangular and Related Wings With Subsonic Leading Edges Oscillating in Supersonic Potential Flow. By Charles E. Watkins and Julian H. Berman.
1100. On Reflection of Shock Waves From Boundary Layers. By H. W. Liepmann, A. Roshko, and S. Dhawan.
1101. Flight Investigation of a Mechanical Feel Device in an Irreversible Elevator Control System of a Large Airplane. By B. Porter Brown, Robert G. Chilton, and James B. Whitten.
1102. The Linearized Characteristics Method and Its Application to Practical Nonlinear Supersonic Problems. By Antonio Ferri.
1103. Generalized Theory for Seaplane Impact. By Benjamin Milwitzky.
1104. Preliminary Investigation of a New Type of Supersonic Inlet. By Antonio Ferri and Louis M. Nucci.
1105. Chordwise and Compressibility Corrections to Slender-Wing Theory. By Harvard Lomax and Loma Sluder.
1106. The Langley Annular Transonic Tunnel. By Louis W. Habel, James H. Henderson, and Mason F. Miller.
1107. An Empirically Derived Basis for Calculating the Area, Rate, and Distribution of Water-Drop Impingement on Airfoils. By Norman R. Bergun.
1108. Experimental Aerodynamic Derivatives of a Sinusoidally Oscillating Airfoil in Two-Dimensional Flow. By Robert L. Halfman.
1109. Experimental Investigation of Base Pressure on Blunt-Trailing-Edge Wings at Supersonic Velocities. By Dean R. Chapman, William R. Wimbrow, and Robert H. Kester.
1110. Some Features of Artificially Thickened Fully Developed Turbulent Boundary Layers With Zero Pressure Gradient. By P. S. Klebanoff and Z. W. Diehl.
- No.
2577. On the Angular Distribution of Slip Lines in Polycrystalline Aluminum Alloy. By John M. Hedgepeth, S. B. Batdorf, and J. Lyell Sanders, Jr.
2643. Span Load Distributions Resulting From Angle of Attack, Rolling, and Pitching, for Tapered Sweptback Wings With Streamwise Tips. Supersonic Leading and Trailing Edges. By John C. Martin and Isabella Jeffreys.
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Part II—COMMITTEE ORGANIZATION AND MEMBERSHIP

The National Advisory Committee for Aeronautics was established by Act of Congress approved March 3, 1915 (U. S. Code Supplement IV, title 50, sec. 151). The membership, appointed by the President, includes two representatives each of the Department of the Air Force, the Department of the Navy, and the Civil Aeronautics Authority, and one representative each of the Smithsonian Institution, the United States Weather Bureau, and the National Bureau of Standards. In addition seven members are appointed for five-year terms from persons "acquainted with the needs of aeronautical science, either civil or military, or skilled in aeronautical engineering or its allied sciences." The representatives of the Government organizations serve for indefinite periods, and all members serve as such without compensation.

The following changes in the membership of the Committee have taken place during the past year:

Honorable Joseph P. Adams, member of the Civil Aeronautics Board, was appointed a member of the Committee on November 21, 1952, succeeding Honorable Donald W. Nyrop, who resigned as chairman of the CAB and member of the NACA October 31, 1952.

On January 14, 1953, Dr. Leonard Carmichael, newly appointed secretary of the Smithsonian Institution, was named a member of the NACA to succeed Dr. Alexander Wetmore, his predecessor as secretary of the Smithsonian, who retired December 31, 1952. Dr. Wetmore was succeeded as vice chairman of the NACA by Dr. Detlev W. Bronk, who was elected to that office on January 23, 1953.

Vice Admiral Ralph A. Ofstie, USN, Deputy Chief of Naval Operations (Air), was appointed a member of the Committee March 30, 1953, succeeding Vice Admiral Matthias B. Gardner, who had just been detached from the same Navy post and assigned to other duty.

On July 8, 1953, the President appointed Rear Admiral Lloyd Harrison, USN, Deputy and Assistant Chief of the Bureau of Aeronautics, a member of the NACA succeeding Vice Admiral (then Rear Admiral) Thomas S. Combs, who was detached June 26 as Chief of the Bureau of Aeronautics and assigned to sea duty.

Honorable Robert B. Murray, Jr., Under Secretary of Commerce for Transportation, was appointed to the NACA on July 10, 1953, as successor to Honorable Thomas W. S. Davis, who resigned as Assistant Secretary of Commerce and NACA member January 20, 1953.

The abolition of the Research and Development Board, effective June 30, 1953, in accordance with Reorganization Plan No. 6, resulted in the termination of

the membership on the NACA of Honorable Walter G. Whitman, who had been serving as chairman of that Board.

In accordance with the regulations of the Committee as approved by the President, the chairman and vice chairman and the chairman and vice chairman of the executive committee are elected annually.

On October 23, 1953, Dr. Jerome C. Hunsaker was re-elected chairman of the NACA and of the executive committee, Dr. Detlev W. Bronk, vice chairman of the NACA, and Dr. Francis W. Reichelderfer, vice chairman of the executive committee.

The Committee membership is as follows:

Dr. Jerome C. Hunsaker, Massachusetts Institute of Technology, Chairman.

Dr. Detlev W. Bronk, President, Rockefeller Institute for Medical Research, Vice Chairman.

Honorable Joseph P. Adams, member, Civil Aeronautics Board.

Dr. Allen V. Astin, Director, National Bureau of Standards.

Dr. Leonard Carmichael, Secretary, Smithsonian Institution.

Lieutenant General Laurence C. Craigie, USAF, Deputy Chief of Staff, Development.

Dr. James H. Doolittle, Vice President, Shell Oil Company.

Rear Admiral Lloyd Harrison, USN, Deputy and Assistant Chief of the Bureau of Aeronautics.

Mr. Ronald M. Hazen, Director of Engineering, Allison Division, General Motors Corporation.

Mr. William Littlewood, Vice President, Engineering, American Airlines, Inc.

Honorable Robert B. Murray, Jr., Under Secretary of Commerce for Transportation.

Vice Admiral Ralph A. Ofstie, USN, Deputy Chief of Naval Operations (Air).

Lieutenant General Donald L. Putt, USAF, Commander, Air Research and Development Command.

Dr. Arthur E. Raymond, Vice President, Engineering, Douglas Aircraft Company, Inc.

Dr. Francis W. Reichelderfer, Chief, U. S. Weather Bureau.

Dr. Theodore P. Wright, Vice President for Research, Cornell University.

Assisting the Committee in its coordination of aeronautical research and the formulation of its research programs are four technical committees: Aerodynamics, Power Plants for Aircraft, Aircraft Construction, and Operating Problems. Each of these committees is aided by from four to eight technical subcommittees. The Committee is advised on matters of policy affecting the aircraft industry by an Industry Consulting Committee.

Membership of the committees, with the subcommittees listed under the technical committees having cognizance, is as follows:

COMMITTEE ON AERODYNAMICS

Dr. Theodore P. Wright, Cornell University, Chairman.
 Capt. Walter S. Diehl, U. S. N. (Ret.), Vice Chairman.
 Dr. Albert E. Lombard, Jr., Directorate of Research and Development, U. S. Air Force.
 Col. Robert G. Ruegg, U. S. A. F., Wright Air Development Center.
 Mr. F. A. Loudon, Bureau of Aeronautics, Department of the Navy.
 Capt. M. R. Kelley, U. S. N. (Ret.), Bureau of Ordnance.
 Maj. Gen. Leslie E. Simon, U. S. A., Chief, Ordnance Research and Development Division.
 Mr. Harold D. Hoekstra, Civil Aeronautics Administration.
 Dr. Hugh L. Dryden (ex officio).
 Mr. Floyd L. Thompson, NACA Langley Aeronautical Laboratory.
 Mr. Russell G. Robinson, NACA Ames Aeronautical Laboratory.
 Prof. Emerson W. Conlon, Fairchild Engine and Airplane Corp.
 Mr. Alexander H. Flax, Cornell Aeronautical Laboratory, Inc.
 Mr. Edward J. Horkey, Pastushin Aviation Corp.
 Mr. Clarence L. Johnson, Lockheed Aircraft Corp.
 Dr. Clark B. Millikan, California Institute of Technology.
 Dr. W. Bailey Oswald, Douglas Aircraft Co., Inc.
 Dr. Allen E. Puckett, Hughes Aircraft Co.
 Mr. George S. Schairer, Boeing Airplane Co.
 Mr. E. G. Stout, Consolidated Vultee Aircraft Corp.
 Prof. E. S. Taylor, Massachusetts Institute of Technology.
 Mr. R. H. Widmer, Consolidated Vultee Aircraft Corp.
 Mr. Robert J. Woods, Bell Aircraft Corp.

Mr. Milton B. Ames, Jr., Secretary

Subcommittee on Fluid Mechanics

Dr. Clark B. Millikan, California Institute of Technology, Chairman.
 Dr. Theodore Theodorsen, Air Research and Development Command, U. S. Air Force.
 Major Michael Zubon, U. S. A. F., Air Research and Development Command.
 Mr. Phillip Eisenberg, Office of Naval Research, Department of the Navy.
 Comdr. L. G. Pooler, U. S. N., Bureau of Ordnance.
 Mr. Joseph Sternberg, Ballistic Research Laboratories, Aberdeen Proving Ground.
 Dr. G. B. Schubauer, National Bureau of Standards.
 Dr. Carl Kaplan, NACA Langley Aeronautical Laboratory.
 Mr. John Stack, NACA Langley Aeronautical Laboratory.
 Mr. Robert T. Jones, NACA Ames Aeronautical Laboratory.
 Mr. Walter G. Vincenti, NACA Ames Aeronautical Laboratory.
 Dr. John C. Evvard, NACA Lewis Flight Propulsion Laboratory.
 Prof. Walker Bleakney, Princeton University.
 Dr. Francis H. Clauser, The Johns Hopkins University.
 Dr. Antonio Ferri, Polytechnic Institute of Brooklyn.
 Dr. Arthur T. Ippen, Massachusetts Institute of Technology.
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 Mr. H. L. Anderson, Wright Air Development Center, U. S. Air Force.

Mr. F. A. Loudon, Bureau of Aeronautics, Department of the Navy.
 Dr. H. H. Kurzweg, Naval Ordnance Laboratory.
 Mr. C. L. Poor III, Ballistic Research Laboratories, Aberdeen Proving Ground.
 Mr. Robert R. Gilruth, NACA Langley Aeronautical Laboratory.
 Mr. John Stack, NACA Langley Aeronautical Laboratory.
 Mr. H. Julian Allen, NACA Ames Aeronautical Laboratory.
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 Mr. Benedict Cohn, Boeing Airplane Co.
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 Mr. David S. Lewis, Jr., McDonnell Aircraft Corp.
 Prof. John R. Markham, Massachusetts Institute of Technology.
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 Mr. William N. Harrison, National Bureau of Standards.
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 Mr. Howard C. Cross, Battelle Memorial Institute.
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 Mr. Robert Rosenbaum, Civil Aeronautics Administration.
 Mr. I. E. Garrick, NACA Langley Aeronautical Laboratory.
 Mr. Albert Erickson, NACA Ames Aeronautical Laboratory.
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Part III—FINANCIAL REPORT

Appropriations for the fiscal year 1953.—Funds in the following amounts were appropriated for the Committee for the fiscal year 1953 in the Independent Offices Appropriation Act, 1953, approved July 5, 1952:

Salaries and expenses.....	\$48,586,100
Construction and equipment of laboratory facilities:	

Funds to continue financing of the fiscal year 1951 program:

Langley Aeronautical Laboratory.....	\$516,324
Ames Aeronautical Laboratory.....	483,676

1,000,000

Funds to completely finance the fiscal year 1953 program:

Langley Aeronautical Laboratory.....	11,115,000
Lewis Flight Propulsion Laboratory.....	5,585,000

16,700,000

Total appropriated funds, fiscal year 1953.....	66,286,100
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Obligations incurred against the fiscal year 1953 appropriated funds are listed below, together with the unobligated balances remaining on June 30, 1953. The figures shown for salaries and expenses include the costs for personal services, travel, transportation, communication, utility services, printing and reproduction, contractual services, supplies, equipment, and taxes and assessments.

Salaries and expenses:

NACA Headquarters.....	\$1,185,835
Langley Aeronautical Laboratory.....	19,218,601
Pilotless Aircraft Station.....	593,496
High-Speed Flight Station.....	1,368,065
Ames Aeronautical Laboratory.....	7,782,541
Western Coordination Office.....	17,330
Lewis Flight Propulsion Laboratory.....	17,276,417
Wright-Patterson Coordination Office.....	12,346
Research contracts—educational institutions.....	739,543
Services performed by National Bureau of Standards and Forest Products Laboratory.....	204,800
Unobligated balance.....	187,117

48,586,100

Construction and equipment of laboratory facilities:

Funds to continue financing of the fiscal year 1951 program:

Langley Aeronautical Laboratory.....	\$512,325
Ames Aeronautical Laboratory.....	483,676
Unobligated balance.....	3,999

1,000,000

Funds to completely finance the fiscal year 1953 program:

Langley Aeronautical Laboratory.....	\$1,817,214
Lewis Flight Propulsion Laboratory.....	4,536,230

Construction and equipment of laboratory facilities—Continued

Funds to completely finance the fiscal year 1953 program—Con.

Reserve for transfer to the fiscal year 1954 program.....	\$1,450,000
Reserve for transfer to the fiscal year 1955 program.....	70,000
Unobligated balance.....	¹ 8,826,556

\$16,700,000

Total appropriated funds, fiscal year 1953.....

66,286,100

¹ This unobligated balance remains available for obligation until expended.

Appropriation for the Unitary Wind Tunnel Plan Act.—Funds in the amount of \$75,000,000 were appropriated in the Deficiency Appropriation Act, 1950, approved June 29, 1950, for the construction of wind tunnels authorized in the Unitary Wind Tunnel Plan Act of 1949 (Public Law 415, 81st Congress, approved October 27, 1949). These funds are available until expended. Allotments and obligations as of June 30, 1953, are as follows:

	Allotments	Obligations as of June 30, 1953
Langley Aeronautical Laboratory..	\$15,150,000	\$14,625,966
Ames Aeronautical Laboratory.....	26,994,000	25,953,021
Lewis Flight Propulsion Laboratory..	32,856,000	28,448,991
Total	75,000,000	69,027,978

Appropriations for the fiscal year 1954.—The major allotments of the funds appropriated for the Committee for the fiscal year 1954 in the First Independent Offices Appropriation Act, 1954, approved July 31, 1953, are as follows:

Salaries and expenses.....	\$50,000,000
Budget reserve.....	1,000,000
Construction and equipment of laboratory facilities:	

Funds to complete financing of the fiscal year 1951 program:

Langley Aeronautical Laboratory.....	\$550,000
Ames Aeronautical Laboratory.....	3,650,000

4,200,000

Funds to finance the fiscal year 1954 program:

Langley Aeronautical Laboratory.....	² 3,939,200
Ames Aeronautical Laboratory.....	990,700
Lewis Flight Propulsion Laboratory.....	10
Budget reserve.....	1,999,080
Reserve for transfer to the fiscal year 1955 program.....	310,000

7,239,000

Total appropriated funds, fiscal year 1954.....

62,439,000

² The fiscal year 1953 reserve of \$1,450,000 for transfer to the fiscal year 1954 program will be used to complete the financing of the Langley fiscal year 1954 projects.

REPORT 1111

AN ANALYSIS OF LAMINAR FREE-CONVECTION FLOW AND HEAT TRANSFER ABOUT A FLAT PLATE PARALLEL TO THE DIRECTION OF THE GENERATING BODY FORCE ¹

By SIMON OSTRACH

SUMMARY

The free-convection flow and heat transfer (generated by a body force) about a flat plate parallel to the direction of the body force are formally analyzed and the type of flow is found to be dependent on the Grashof number alone. For large Grashof numbers (which are of interest in aeronautics), the flow is of the boundary-layer type and the problem is reduced in a formal manner, which is analogous to Prandtl's forced-flow boundary-layer theory, to the simultaneous solution of two ordinary differential equations subject to the proper boundary conditions.

Velocity and temperature distributions for Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000 are computed, and it is shown that velocities and Nusselt numbers of the order of magnitude of those encountered in forced-convection flows may be obtained in free-convection flows. The theoretical and experimental velocity and temperature distributions are in good agreement.

A flow and a heat-transfer parameter, from which the important physical quantities such as shear stress and heat-transfer rate can be computed, are derived as functions of Prandtl number alone. Comparison of theoretically computed values of the heat-transfer parameter with values obtained from an approximate calculation and experiments yielded good agreement over a large range of Prandtl number. Agreement between the theoretical values and those obtained from a frequently used semiempirical heat-transfer law was good only in restricted Prandtl number ranges (depending on an arbitrary constant).

INTRODUCTION

Two important types of fluid flow problems involving heat transfer are those of forced and those of free convection. By forced-convection flow is meant flows maintained mechanically as, for example, by a pressure drop or an agitator. Free-convection flow, on the other hand, results from the action of body forces on the fluid, that is, forces which are proportional to the mass or the density of the fluid. The flow is generally produced in the following manner: Consider, for example, a fixed object (such as a plate) in a quiescent fluid subject to a body force. When the plate is at the same temperature as the surrounding fluid, the body forces acting on the fluid are in equilibrium with the hydrostatic pressure and no flow ensues in the steady state. If a temperature gradient normal to the body force is imposed by heating (or cooling) the plate, there will exist a defect (or excess) of

body force because of the decreased (or increased) density, with the fluid closer to the plate having the greater defect (or excess) than that away from the plate. This unbalance of the forces causes the fluid to be accelerated with the particles nearer the plate moving more rapidly than those farther from the plate. Free-convection flow has usually been considered to be generated in a gravitational field where the previously mentioned defect or excess of body force was the Archimedian (buoyancy) force. However, since centrifugal forces are also proportional to the fluid density, free-convection flows can also be set up by the action of such forces. (See ref. 1 for a more explicit discussion of the development of free-convection flows by centrifugal forces.)

Free-convection flows produced by centrifugal forces are now of practical importance in aeronautics because many aircraft propulsion systems contain components (such as gas turbines and helicopter ram jets) which rotate at high speeds and in which heat is being transferred. The method of free-convection cooling of gas-turbine rotor blades where the centrifugal forces create a free-convection flow of the coolant in the blade passages is an example of a practical application of the free-convection phenomenon in aeronautics. Also, free-convection flow due to centrifugal force is superimposed on the flow through helicopter ram jets and on the flow of cooling air in hollow rotor blades of air-cooled turbines and, under proper conditions, can appreciably influence the resultant flow and heat transfer.

As a simplification of the many free-convection problems which are now of some consequence in aeronautics, consideration is here given to the special case of free-convection flow about a flat plate parallel to the direction of the generating body force. The experimental and theoretical considerations of Schmidt and Beckmann (ref. 2) concerning the free-convection flow of air subject to the gravitational force about a vertical flat plate constitute the most complete treatment of this subject up to the present time. Eckert (ref. 1) as well as others has further verified and extended the experimental results of Schmidt and Beckmann, and Schuh (ref. 3) has extended the numerical calculations by computing the velocity and temperature distributions for several Prandtl numbers different from that for air. However, all the theoretical work in these references is based on the incompressible equations in which the density (or temperature) variation is introduced in the buoyancy term alone.

¹ Supersedes NACA TN 2635, "An Analysis of Laminar Free-Convection Flow and Heat Transfer about a Flat Plate Parallel to the Direction of the Generating Body Force" by Simon Ostrach, 1952.

Various terms are omitted from the equations at the start on the basis of either intuitive arguments or no arguments at all. Although a theoretical development made in such a manner led to good final results, the significance of all the important factors associated with the free-convection flow phenomenon is not obtained from such an analysis.

The problem of free-convection flow as produced by a body force about a flat plate in the direction of the body force was studied at the NACA Lewis laboratory during 1951 and is treated in a formal and more general manner herein. The method used is somewhat similar to that used in reference 4 wherein consideration was given to the free-convection flow at high Grashof numbers in a horizontal cylinder which had a variable surface-temperature distribution. The application of this method to the present problem leads to a development which is analogous to Prandtl's treatment of high Reynolds number forced-convection flows. Although the final equations obtained by this method are the same as those of Schmidt and Beckmann, this more general approach not only clearly demonstrates the significance of all the important parameters and assumptions and hence leads to a better understanding of this type of flow but also indicates the quantitative limitations of the theory. In addition, the numerical solutions of references 2 and 3 are herein extended to cover a more complete range of parameters. The new calculations yield information on the free-convection flow for Prandtl numbers corresponding to those of liquid metals, gases, liquids, and very viscous fluids.

ANALYSIS

STATEMENT OF PROBLEM AND BASIC EQUATIONS

The steady-state equations expressing the conservation of mass, momentum, and energy for a compressible, viscous, and heat-conducting fluid subject to a body force together with an equation of state govern the flow and associated temperature distribution about the plate. These equations in Cartesian tensor notation are (see ref. 5), respectively,

$$\frac{\partial}{\partial X_j} (\rho U_j) = 0 \quad (1)$$

$$\rho U_j \frac{\partial U_i}{\partial X_j} = \rho f_i + \frac{\partial}{\partial X_j} \left[\mu \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right) \right] - \frac{2}{3} \frac{\partial}{\partial X_i} \left(\mu \frac{\partial U_j}{\partial X_j} \right) - \frac{\partial P}{\partial X_i} \quad (2)$$

$$\rho c_p U_j \frac{\partial T}{\partial X_j} = U_j \frac{\partial P}{\partial X_j} + \frac{\partial}{\partial X_j} \left(k \frac{\partial T}{\partial X_j} \right) + \mu \left[\frac{\partial U_i}{\partial X_j} \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right) - \frac{2}{3} \left(\frac{\partial U_j}{\partial X_j} \right)^2 \right] \quad (3)$$

$$\rho = \rho(P, T) \quad (4)$$

(A complete list of the symbols used herein is given in appendix A.) For the two-dimensional case, equations (1) to (4) represent a system of five equations in the five dependent variables U_1 , U_2 , ρ , P , and T . For later use, equation (4) can be written

$$d\rho = \rho(K dP - \beta dT) \quad (4a)$$

where K and β are the coefficients of isothermal compressibility and volumetric expansion, respectively (see ref. 6). In addition to a general state equation, such as is given in equations (4) or (4a), it will be convenient at times in the discussion to refer to some specific state equation. To this end, the equation of state for an ideal gas

$$P = \rho R T \quad (4b)$$

will be used.

Particular consideration is here given to the two-dimensional free-convection flow about a semi-infinite vertical flat plate. The X_1 -axis of the coordinate system is taken along the plate and the X_2 -axis, normal to it. No distinction is made as to the specific type of body force acting, for example, gravitational or centrifugal, but the force is assumed to be acting in the vertical direction only (that is, parallel to the plate). Centrifugal and Coriolis forces which are connected with flows on curved paths and with rotating systems generally vary with position and velocity. However, in order not to make the analysis unduly complicated, the body force is taken to be constant.

In order to define the problem clearly, a choice must still be made of the position of the origin of the coordinate system. Before making a definite decision on this point, note that for constant plate temperatures there are four permutations of the body-force direction (either upward or downward) and the plate thermal condition (either heated or cooled) which will lead to free-convection flows. Once the position of the edge of the plate, which is also to be the origin of the coordinate system, is decided, there are two combinations of the body-force direction and plate thermal condition that will yield flows which proceed away from the edge. It is this type of flow that is amenable to the type of analysis to be made here. This point will be more fully discussed subsequently. If the edge of the plate (recall that a semi-infinite plate has but one edge) is taken at the bottom of the plate (that is, the plate extends to $+\infty$ in the X_1 -direction), the two combinations leading to flows in the proper direction (upward in this case) are, respectively, the body force acting downward with a heated plate and the body force acting upward with a cooled plate. The equations developed for one of the cases reduce directly to those for the other. The remaining two permutations, namely, the body force acting downward with a cooled plate and the body force acting upward with a heated plate, would yield flows which proceed downward or toward the edge of the plate if this edge were taken at the bottom of the plate. This type of flow would violate a physical condition of the problem which states that the flow starts at the plate edge. The latter combinations hence will not be considered further.

Because the two acceptable configurations can be reduced essentially to one, for the development to be given here, the origin of the coordinate system will be taken at the bottom of a heated plate, with the body force acting downward. The assumption is now made that the viscosity and thermal-conductivity coefficients are functions of the temperature alone and obey the following laws:

$$\begin{aligned}\mu &= \mu_{\infty} \left(\frac{T}{T_{\infty}} \right)^m \\ k &= k_{\infty} \left(\frac{T}{T_{\infty}} \right)^n\end{aligned}\quad (5)$$

The choice of the body-force direction together with equations (5) alters equations (2) and (3) so that they become

$$\rho U_j \frac{\partial U_i}{\partial X_j} = -\rho(-f_i) + \frac{\mu_{\infty}}{T_{\infty}^m} \frac{\partial}{\partial X_j} \left[T^m \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right) \right] - \frac{2}{3} \frac{\mu_{\infty}}{T_{\infty}^m} \frac{\partial}{\partial X_i} \left(T^m \frac{\partial U_j}{\partial X_j} \right) - \frac{\partial P}{\partial X_i} \quad (6)$$

$$\begin{aligned}\rho c_p U_j \frac{\partial T}{\partial X_j} &= U_j \frac{\partial P}{\partial X_j} + \frac{k_{\infty}}{T_{\infty}^n} \frac{\partial}{\partial X_j} \left(T^n \frac{\partial T}{\partial X_j} \right) + \\ &\frac{\mu_{\infty}}{T_{\infty}^m} T^m \left[\frac{\partial U_i}{\partial X_j} \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right) - \frac{2}{3} \left(\frac{\partial U_j}{\partial X_j} \right)^2 \right]\end{aligned}\quad (7)$$

Note that the only nonzero component of the body force is the X_1 -component.

BOUNDARY CONDITIONS

The boundary conditions associated with the given problem are that:

(a) The fluid must adhere to the plate (the no-slip condition of viscous flows) and the plate must be a streamline, or mathematically,

$$U_1(X_1, 0) = U_2(X_1, 0) = 0 \quad (8)$$

(b) The temperature of the fluid at the plate must be equal to the plate temperature, that is,

$$T(X_1, 0) = T_0 \quad (9)$$

(c) The velocity U_1 at large distances from the plate must be undisturbed, or

$$U_1(X_1, \infty) = 0 \quad (10)$$

(d) The temperature at large distances from the plate must be equal to the undisturbed fluid temperature, or

$$T(X_1, \infty) = T_{\infty} \quad (11)$$

SIMPLIFICATION OF EQUATIONS

Let a small quantity ϵ now be defined as

$$\epsilon = \beta(T_0 - T_{\infty}) \quad (12)$$

which is a measure of the magnitude of temperature variation in the flow field. The coefficient of volumetric expansion β is generally of the order of magnitude between 10^{-2} and 10^{-4} (see table 15 of ref. 7, for example) and for gases, $\beta = 1/T$. (Thus, for gases, if β is taken to be constant, $\epsilon = (T_0 - T_{\infty})/T_{\infty}$; that is, ϵ is the relative temperature difference.) The coefficient β will be assumed constant. Because in the steady state flow ensues only when there is a temperature

variation in the fluid, the free-convection velocity should then depend directly on ϵ , and the variations in pressure and density (from the static, $\epsilon = 0$, case) due to the temperature differences should also depend on ϵ . Thus

$$U_1 = \epsilon \left(\frac{\rho_{\infty} f_X l^2}{\mu_{\infty}} \right) u_i \quad (13)$$

$$P = P_s + P_{\infty} \epsilon \sigma \quad (14)$$

$$\rho = \rho_s + \rho_{\infty} \epsilon \varphi \quad (15)$$

$$T = T_{\infty} (1 + \epsilon \theta) \quad (16)$$

where $-f_X$ denotes the X_1 -component of the body force per unit mass, u_i , σ , φ , and θ denote dimensionless functions (which, in general, can be functions of ϵ), l is some characteristic length (for example, the distance from the edge of the plate to the point of interest), P_s and ρ_s are the pressure and the density, respectively, for the static case ($U_i = 0$ or $\epsilon = 0$), and P_{∞} and ρ_{∞} denote constant values of the pressure and the density (that is, the values if no force field were present) defined by the state equation (in the case of a gas, in particular, $P_{\infty} = \rho_{\infty} R T_{\infty}$). Because there is no characteristic velocity associated with the type of flow under consideration, the velocity is dimensionalized by the factor given in parentheses on the right side of equation (13).

In order to determine the static quantities, it will at first be convenient to consider the particular case of a gas. The problem is then considered with the temperature uniform throughout the flow field at the value T_{∞} (therefore there will be no flow and $U_i = 0$). For this situation, equations (4b) and (6) become

$$P_s = \rho_s R T_{\infty} \quad (17)$$

and

$$\left. \begin{aligned} \frac{\partial P_s}{\partial X_1} + \rho_s f_X &= 0 \\ \frac{\partial P_s}{\partial X_2} &= 0 \end{aligned} \right\} \quad (18)$$

(It should be noted that eq. (18) expresses the physical fact previously stated that the body force and hydrostatic pressure are in equilibrium for the static case.) Substitution of equation (17) into equation (18) leads to

$$P_s = P_{\infty} \exp \left(-\frac{f_X}{R T_{\infty}} X_1 \right) \quad (19)$$

and equation (19) together with equation (17) and the equation defining P_{∞} and ρ_{∞} yields

$$\rho_s = \rho_{\infty} \exp \left(-\frac{f_X}{R T_{\infty}} X_1 \right) = \rho_{\infty} \exp \left(-\frac{f_X \rho_{\infty}}{P_{\infty}} X_1 \right) \quad (20)$$

If the exponential in equation (20) is expressed in terms of its series expansion, that equation becomes

$$\rho_s = \rho_{\infty} \left(1 - \frac{\rho_{\infty} f_X}{P_{\infty}} X_1 + \dots \right) \quad (21)$$

A computation of the second term in the parentheses of equation (21) for the case of air under normal conditions with $f_x = g$ and the fact that X_1 is of the order of magnitude l show that $\rho_\infty g l / P_\infty \sim 10^{-5} l / \text{foot}$. For the type of problem under consideration, l will always be of unit order of magnitude so that even if the body force f_x represents a centrifugal force many times that of gravity, the inequality $\rho_\infty f_x X_1 / P_\infty \ll 1$ may still be satisfied. Thus, in the subsequent development it will be assumed that $\rho_s \cong \rho_\infty$. This assumption, which was justified by the computation for the case of a gas, is expected to be reasonable for other fluids as well. The physical interpretation of this assumption is that under static conditions ($\epsilon = 0$), the density (or pressure) is not affected by the force field.

In order that all quantities in the following equations be dimensionless, it is further necessary to define $x_i = X_i / l$, where the x_i are now dimensionless space coordinates. Substituting these new coordinates along with equations (13) to (16) into equations (1), (6), (7), and (4a) and noting equations (18) and that $\rho_s \cong \rho_\infty$ yield, on neglect of terms of higher order in ϵ compared with those of order ϵ ,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \quad (22)$$

$$Gr \left(u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} \right) = \Delta u_1 - N Gr \frac{\partial \sigma}{\partial x_1} - \varphi \quad (23)$$

$$Gr \left(u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} \right) = \Delta u_2 - N Gr \frac{\partial \sigma}{\partial x_2} \quad (24)$$

$$Gr Pr \left(u_1 \frac{\partial \theta}{\partial x_1} + u_2 \frac{\partial \theta}{\partial x_2} \right) = \frac{\gamma}{\gamma - 1} Gr Pr \left(u_1 \frac{\partial \sigma}{\partial x_1} + u_2 \frac{\partial \sigma}{\partial x_2} \right) + \Delta \theta \quad (25)$$

$$d\varphi = K P_\infty d\sigma - \beta T_\infty d\theta \quad (26)$$

where $P_\infty / \rho_\infty f_x l = N Gr$ and the Grashof number Gr and the Prandtl number Pr are defined as

$$Gr = \frac{\rho_\infty^2 f_x l^3 \epsilon}{\mu_\infty^2}$$

and

$$Pr = \frac{c_p \mu_\infty}{k_\infty}$$

Physically speaking, the Grashof number represents the ratio of the body forces to the viscous forces. The free-convection flows of interest here are those associated with large Grashof numbers. The factors K and β in equation (26) may well be taken to be constants (see ref. 6).

The boundary conditions (eqs. (8) to (11)) in terms of the new dimensionless variables are

$$u_1(x_1, 0) = u_2(x_1, 0) = 0 \quad (27)$$

$$\theta(x_1, 0) = \frac{1}{\beta T_\infty} \quad (28)$$

$$u_1(x_1, \infty) = 0 \quad (29)$$

$$\theta(x_1, \infty) = 0 \quad (30)$$

Thus, to a first approximation, equations (22) to (26) together with the boundary conditions replace the original equations and boundary conditions. (Note that for gases, $\beta T_\infty = 1$.)

The prime assumption made in this analysis is that the higher-order terms in ϵ are negligible, which implies that ϵ is small, and consequently, that the temperature difference or β is moderately small. It is a consequence of this assumption alone that the basic equations were simplified to equations (22) to (26) wherein the viscosity term in the energy equation is neglected and the only coupling of the momentum and energy equations occurs by means of the body-force term in equation (23). As a result of this assumption, the variations of the viscosity and heat-conductivity coefficients with temperature are also negligible. Without any discussion, the authors of reference 2 start directly from simplified equations of the same form wherein the pressure terms in the energy equation were also neglected. In reference 3 some intuitive arguments are given to justify the simplified equations.

It is now convenient to revert to the more familiar notation where $x = x_1$, $y = x_2$, $u = u_1$, and $v = u_2$. Equation (22) implies the existence of a stream function ψ such that

$$\left. \begin{aligned} u &= \frac{\partial \psi}{\partial y} = \psi_y \\ v &= -\frac{\partial \psi}{\partial x} = -\psi_x \end{aligned} \right\} \quad \text{and} \quad (31)$$

where subscripts denote differentiation. Applying equation (31) to equations (23) to (25) yields, respectively,

$$\frac{1}{Gr} (\Delta \psi_y - \varphi) = \psi_y \psi_{xy} - \psi_x \psi_{yy} + N \sigma_x \quad (32)$$

$$-\frac{1}{Gr} (\Delta \psi_x) = -\psi_y \psi_{xx} + \psi_x \psi_{xy} + N \sigma_y \quad (33)$$

$$\frac{1}{Gr Pr} (\Delta \theta) = \psi_y \theta_x - \psi_x \theta_y - \frac{\gamma}{\gamma - 1} (\psi_y \sigma_x - \psi_x \sigma_y) \quad (34)$$

The boundary conditions (eqs. (27) to (30)) become

$$\psi_y(x, 0) = \psi_x(x, 0) = 0 \quad (35)$$

$$\theta(x, 0) = \frac{1}{\beta T_\infty} \quad (36)$$

$$\psi_y(x, \infty) = \theta(x, \infty) = 0 \quad (37)$$

Equations (32) to (34) and equation (26) form the system of equations for the four unknown functions ψ , θ , φ , and σ of the problem. The system is nonlinear, and therefore

further simplification of the equations would be desirable. Just as in the case of forced-convection flows where the Reynolds number determines the type of flow or, in mathematical terms, the type of solution, the Grashof number is the prime factor for free-convection flows. For the case of small Grashof number, it can be seen from equations (32) to (34) that a perturbation in the small parameter Gr will yield a system of linear equations. For Grashof numbers of unit order of magnitude, no further important simplification can be made and the solutions would have to be obtained numerically. For the other limiting case, that of large Grashof numbers (which is the case under consideration herein), it would, at first thought, appear that some simplification could be obtained by performing a perturbation in the small parameter $1/Gr$. However, this would then imply that the term containing the highest-order derivatives (the left term in equations (32) to (34)) could, among others, be neglected. (This argument would also imply that the body-force term φ in equation (32), which is essentially causing the flow, could also be neglected.) The omission of the highest-order derivatives from consideration, however, would lead to solutions which would not satisfy all the boundary conditions. Problems of this type are referred to as singular perturbation problems. For further discussions of singular perturbation problems, see references 8 and 9.

Equations in which a small parameter multiplies the highest-order terms are said to be of the boundary-layer type, because in order for solutions which satisfy all the boundary conditions to be obtained, the highest-order terms must be considered near the boundary. This fact implies the existence of a thin region, called the boundary layer, wherein the functions vary rapidly from the value at the boundary to that in the flow outside this layer. The conclusion to be drawn from the preceding discussion is that for large Grashof numbers the flow is of the boundary-layer type. Schmidt and Beckmann (ref. 2) also made the boundary-layer assumptions in their theoretical development, and these assumptions were justified on the basis of their experimental observations. The Grashof numbers for their experiments were of the order of 8×10^6 .

In view of the fact, previously discussed, that highest-order derivatives of each dependent variable as well as of those terms of physical importance (as, for example, the body-force term) must be retained in the boundary layer, it is convenient to make both sides of each of the equations of the same order in Gr . In this way, as will be shown, the equations will be further simplified. It is thus convenient to make the following transformations in the system of equations (32) to (34) and (26) and then to retain only the dominant parts (that is, those multiplied by Gr to the highest power) of each individual term.

Let $\bar{y} = Gr^r y$, $\bar{\psi} = Gr^s \psi$, $\bar{\sigma} = Gr^t \sigma$, $\bar{\varphi} = \varphi$, and $\bar{\theta} = \theta$. Then equations (32) to (34) and (26) become

$$Gr^{s+3r-1} \bar{\psi}_{\bar{y}\bar{y}\bar{y}} - Gr^{-1} \bar{\varphi} = Gr^{2s+2r} (\bar{\psi}_{\bar{y}} \bar{\psi}_{\bar{y}\bar{y}} - \bar{\psi}_{\bar{z}} \bar{\psi}_{\bar{y}\bar{z}}) + N Gr^t \bar{\sigma}_z \quad (38)$$

$$-Gr^{s+2r-1} \bar{\psi}_{\bar{y}\bar{y}\bar{z}} = Gr^{2s+r} (-\bar{\psi}_{\bar{y}} \bar{\psi}_{\bar{z}\bar{z}} + \bar{\psi}_{\bar{z}} \bar{\psi}_{\bar{y}\bar{z}}) + N Gr^{t+r} \bar{\sigma}_{\bar{y}} \quad (39)$$

$$\frac{Gr^{2r-1}}{Pr} \bar{\theta}_{\bar{y}\bar{y}} = Gr^{s+r} (\bar{\psi}_{\bar{y}} \bar{\theta}_{\bar{z}} - \bar{\psi}_{\bar{z}} \bar{\theta}_{\bar{y}}) - \frac{\gamma}{\gamma-1} Gr^{t+s+r} (\bar{\psi}_{\bar{y}} \bar{\sigma}_{\bar{z}} - \bar{\psi}_{\bar{z}} \bar{\sigma}_{\bar{y}}) \quad (40)$$

$$d\bar{\varphi} = KP_{\infty} Gr^t d\bar{\sigma} - \beta T_{\infty} d\bar{\theta} \quad (41)$$

It now can be seen that by proper choice of r , s , and t a transformation of the type given provides a means for making the important terms in the differential equations of the same order in Gr . Thus if $r = 1/4$, $s = -3/4$, and $t = -1$, equations (38) to (41) become

$$\bar{\psi}_{\bar{y}\bar{y}\bar{y}} - \bar{\varphi} = \bar{\psi}_{\bar{y}} \bar{\psi}_{\bar{z}\bar{z}} - \bar{\psi}_{\bar{z}} \bar{\psi}_{\bar{y}\bar{z}} + N \bar{\sigma}_z \quad (42)$$

$$N \bar{\sigma}_{\bar{y}} = O(Gr^{-1/2}) \cong 0 \quad (43)$$

$$\bar{\theta}_{\bar{y}\bar{y}} = Pr (\bar{\psi}_{\bar{y}} \bar{\theta}_{\bar{z}} - \bar{\psi}_{\bar{z}} \bar{\theta}_{\bar{y}}) \quad (44)$$

$$d\bar{\varphi} + \beta T_{\infty} d\bar{\theta} = 0 \quad (45)$$

More generally, if N is very much different from unit order of magnitude, a value of t can always be chosen (depending on N) such that equations (42) to (45) are obtained. (For any negative t less than -1 , the last term of eq. (42) will also disappear.)

There are now several important points to be discussed concerning the transformation just made and the resulting simplified equations. First, it should be noted that the transformation is merely a formal expression of the boundary-layer assumptions first made by Prandtl and hence the solutions will be asymptotic for large Gr . Second, the second equation of motion here also reduces to state that the pressure across the boundary layer is constant. Third, the pressure terms in the energy and state equations are here found to be negligible. This fact verifies a priori assumptions made by others from the physics of the problem. Finally, note that integration of the general state equation (independent of pressure) as now given by equation (45) leads to

$$\bar{\varphi} + \beta T_{\infty} \bar{\theta} = 0 \quad (46)$$

where the constant of integration has been taken as zero without any loss of generality. For the particular case of a gas, $\beta = 1/T_{\infty}$ so that equation (46) becomes

$$\bar{\varphi} + \bar{\theta} = 0$$

The boundary conditions (eqs. (35) to (37)) now can be written

$$\bar{\psi}_{\bar{y}}(x, 0) = \bar{\psi}_{\bar{z}}(x, 0) = 0 \quad (47)$$

$$\bar{\theta}(x, 0) = \frac{1}{\beta T_{\infty}} \quad (48)$$

$$\bar{\psi}_{\bar{y}}(x, \infty) = \bar{\theta}(x, \infty) = 0 \quad (49)$$

If now it is assumed that $\bar{\sigma}_z = 0$ in equation (42) since consideration is here being given to a flat plate, and if $\bar{\varphi}$ is eliminated from equation (42) by use of equation (46), there results the system of equations

$$\bar{\psi}_{vv} + \beta T_{\infty} \bar{\theta} = \bar{\psi}_v \bar{\psi}_{xv} - \bar{\psi}_x \bar{\psi}_{vv} \quad (50)$$

$$\bar{\theta}_{vv} = Pr(\bar{\psi}_v \bar{\theta}_x - \bar{\psi}_x \bar{\theta}_v) \quad (51)$$

Thus the problem has been reduced to the solution of the two simultaneous partial differential equations (eqs. (50) and (51)) subject to the boundary conditions (eqs. (47) to (49)).

Final simplification of the equations is made by application of the so-called similarity transformation of boundary-layer theory. Thus, let

$$\eta = \frac{\bar{y}}{(4x)^{\frac{1}{4}}} \quad (52)$$

and

$$\bar{\psi} = (4x)^{\frac{3}{4}} F(\eta) \quad (53)$$

$$\bar{\theta} = \frac{H(\eta)}{\beta T_{\infty}} \quad (54)$$

Then equations (50) and (51) are reduced to the following ordinary differential equations:

$$F''' + 3FF'' - 2F'^2 + H = 0 \quad (55)$$

$$H'' + 2Pr F H' = 0 \quad (56)$$

where the primes denote differentiations with respect to η . The reciprocal one-fourth power similarity as given in equation (52) is characteristic of free-convection flows just as the reciprocal square-root type is characteristic of the forced-convection flows. The boundary conditions become

$$F'(0) = F(0) = 0 \quad (57)$$

$$H(0) = 1 \quad (58)$$

$$F'(\infty) = H(\infty) = 0 \quad (59)$$

The use of a transformation like equation (52) essentially specifies an additional boundary condition, namely, that the conditions to be satisfied at $y = \infty$ (or $\eta = \infty$) should also be satisfied at $x = 0$. It is for this reason that the flows previously discussed which would flow toward the edge (downward) are not amenable to this type of analysis, for such flows would violate this additional condition, which essentially states that the boundary-layer development starts at the edge of the plate.

SOLUTION OF THE BOUNDARY-VALUE PROBLEM

The solutions of the simplified equations (55) and (56), satisfying the boundary conditions as given by equations (57) to (59), were obtained by use of an IBM Card-Programmed Electronic Calculator. A detailed account of the procedure followed in the determination of the unknown functions is presented in appendix B by Dr. Lynn U. Albers. The functions F and H together with their derivatives are given in table I for Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000. Even though the Prandtl number for air is taken as 0.72 in this report, the solutions for $Pr = 0.733$ were also computed and are presented as a check with the

Schmidt-Beckmann calculations wherein the value of Prandtl number of 0.733 was used. The particular values of the Prandtl numbers given were chosen to correspond to those for liquid metals, gases, liquids (such as water and oil), and very viscous liquids (such as glycerin or oils at very low temperatures).

RESULTS

VELOCITY AND TEMPERATURE DISTRIBUTIONS

By means of the various transformations made in the analysis it can easily be verified that

$$\frac{U}{2\sqrt{\beta(T_0 - T_{\infty})f_x X}} = \frac{\frac{UX}{v_{\infty}}}{2\sqrt{Gr_x}} = F'(\eta) \quad (60)$$

and

$$\left(\frac{T - T_{\infty}}{T_0 - T_{\infty}}\right) = H(\eta) \quad (61)$$

where

$$\eta = \left[\frac{f_x(T_0 - T_{\infty})\beta}{4v_{\infty}^2 X}\right]^{\frac{1}{4}} Y = \left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{Y}{X} \quad (62)$$

Equations (60) to (62) relate the physical quantities to the dimensionless functions F and H which are now known. The dimensionless velocity and temperature distributions as given by equations (60) and (61) are presented in figures 1 and 2, respectively, as functions of η for the various values of Prandtl number. The computations made here agree with those for $Pr = 0.733$ as given in reference 2 up to the third significant figure. For $Pr = 10, 100$, and 1000, the present results agree in general with those of reference 3. Since only curves are presented in reference 3, the precision of the agreement cannot be stated.

The maximum values of the dimensionless velocity distributions occur at larger values of the argument η as the Prandtl number decreases and the velocities decrease with increasing Pr . It should also be noted that the dynamic and thermal boundary-layer thicknesses can be estimated from the abscissas of figures 1 and 2, respectively, and that for $Pr \gg 1$ the velocity boundary layer is much thicker than the thermal boundary layer.

The occurrence of f_x (or Gr_x as given by eq. (60)), which may be very large for flows generated by centrifugal forces, in the denominator of the ordinate implies that velocities of appreciable magnitude can be associated with such free-convection flows. In particular, if $f_x = 10^6$ feet per second squared, which is a reasonable conservative figure for present-day rotating systems, $\epsilon = 0.2$ (which is within the limits of the theory presented herein), and arbitrarily $X = 0.25$ foot, then the maximum velocity attained at a Prandtl number of 0.72 is approximately 125 feet per second. This value of the maximum velocity could, of course, be doubled or even tripled under the proper conditions. One limitation to a calculation of this sort, as can be seen by comparison of the denominators of the left and middle terms of equation (60), should be kept in mind; namely, the limiting Grashof number for laminar flows. In lieu of a complete stability

analysis on this type of flow, this limiting value is taken to be 10^9 , as indicated in reference 10. Consideration of this limitation then implies (see eq. (60)) that for large laminar velocities either v_∞ must be large or X must be small.

COMPARISON WITH EXPERIMENTS

Careful experiments of free-convection flows (as generated by gravitational forces) about vertical flat plates were made by Schmidt and Beckmann (ref. 2) in which velocity measurements at various points along the plate were made by means of a quartz-filament anemometer and the temperature measurements were obtained by means of manganese-constantan thermocouples. Eckert (ref. 1) performed similar experiments in which the measurements were made by means of a

Zehnder-Mach interferometer. The results of both sets of experiments are in good agreement, but since the data presented in reference 2 by Schmidt and Beckmann appear in more detail, these data will be used for comparison with the theory.

The experiments of reference 2 were performed on two different (in that the edges were smoothed either symmetrically or not) 12- by 25-centimeter plates and on one 50- by 50-centimeter plate. It should here be pointed out that the results for the two smaller plates were almost identical and that the flow was entirely laminar except near the outer edge of the boundary layer where the slight turbulence of the room air disturbed the measurements somewhat. (This effect was also observed by Eckert.) Large periodic oscillations of the flow near the downstream edge of the larger plate were observed in addition to the slight turbulence near the

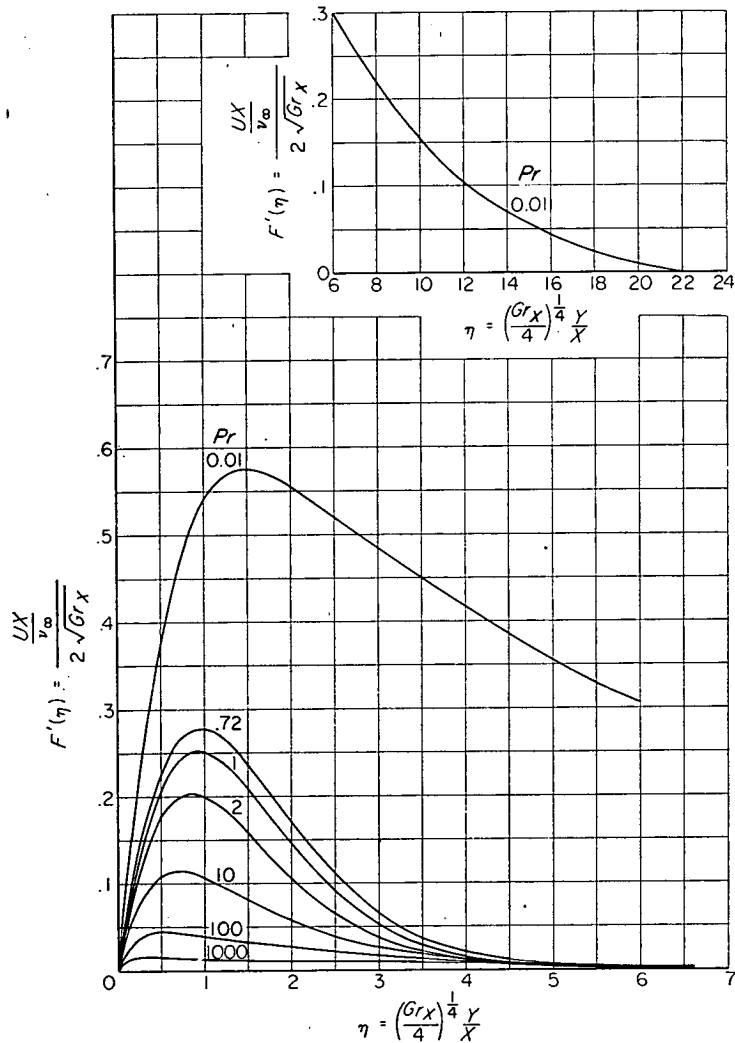


FIGURE 1.—Dimensionless velocity distributions for various Prandtl numbers.

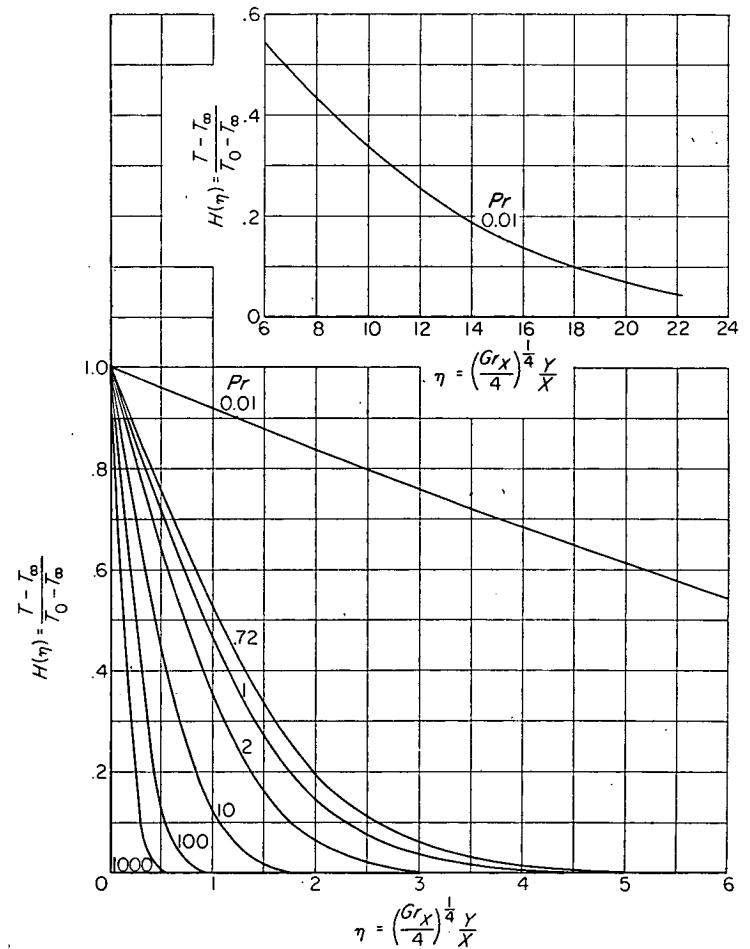


FIGURE 2.—Dimensionless temperature distributions for various Prandtl numbers.

outer edge of the boundary layer. Hence the data from the larger plate should not be expected to yield completely satisfactory agreement with the laminar theory as presented here.

Since the physical quantities can be expressed in terms of a single variable as in equations (60) and (61), it is to be expected that the data taken at the various points along the plates should all lie on a single line if the data are correlated according to equations (60) and (61). Thus for the smaller plates where $(T_0 - T_\infty) = 95.22^\circ \text{ R}$ and $T_\infty = 518.68^\circ \text{ R}$, equations (60) to (62) become

$$\frac{U}{4.862\sqrt{X}} = F'(\eta) \quad (63)$$

$$\frac{T - 518.68}{95.22} = H(\eta) \quad (64)$$

$$\eta = 88.26 \frac{Y}{X^{\frac{1}{4}}} \quad (65)$$

The velocity and temperature distributions are so plotted in figures 3 and 4, respectively, as are the curves computed theoretically for $Pr = 0.72$. It can be seen that the agreement is in general very good for small values of η and somewhat less satisfactory though still rather good for the larger values of η . The scatter in the range of the larger values of η is believed to be caused by the previously discussed room

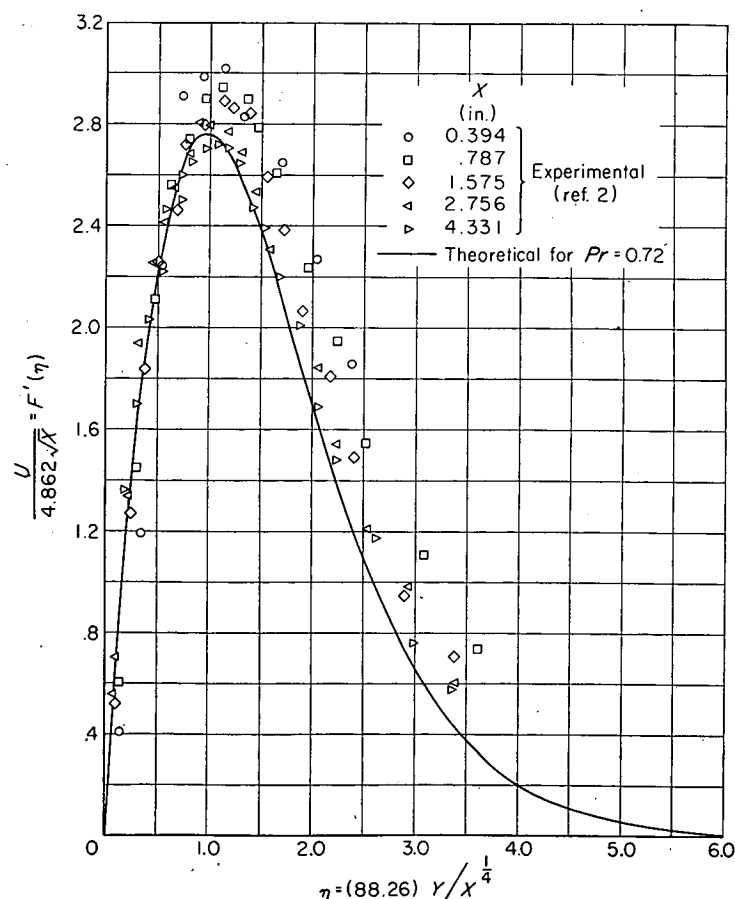


FIGURE 3.—Comparison of small plate experimental and theoretical velocity distributions for Prandtl number of 0.72.

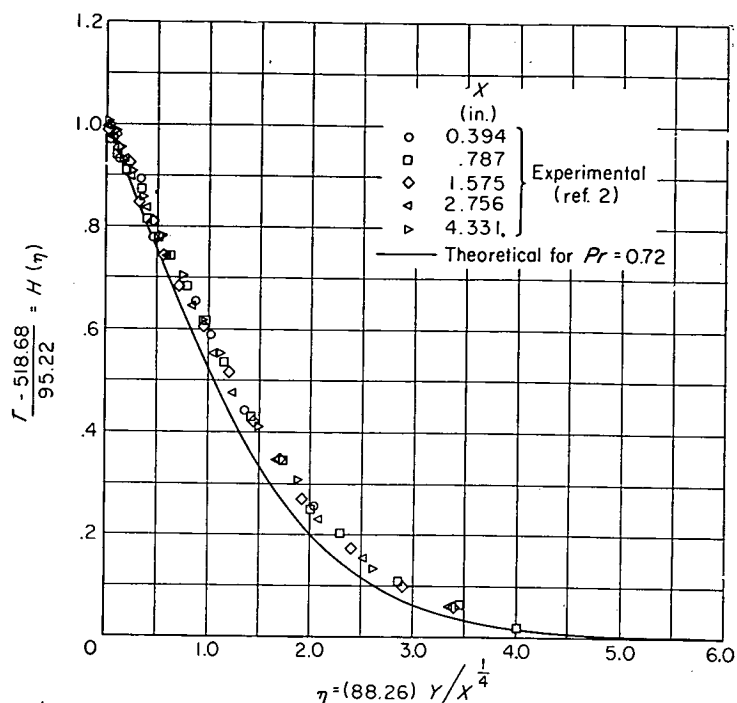


FIGURE 4.—Comparison of small plate experimental and theoretical temperature distributions for Prandtl number of 0.72.

turbulence. It should also be noted that the points farthest away from the theoretical are those measured near the leading edge. These points should not, of course, be expected to agree too well with the theory since the boundary-layer assumptions made in the theoretical development imply that the distance along the plate is large as compared with the boundary-layer thickness. Hence, this assumption is invalid near the leading edge. Schmidt and Beckmann obtained closer agreement between the theory and the experiments for the temperature data and poorer agreement for the velocity data by basing the kinematic viscosity coefficient in equation (62) on the plate temperature rather than on the undisturbed stream temperature as was done here.

For the larger plate, $(T_0 - T_\infty) = 83.7^\circ \text{ R}$ and $T_\infty = 527.14^\circ \text{ R}$, so that equations (60) to (62) become

$$\frac{U}{4.522\sqrt{X}} = F'(\eta) \quad (66)$$

$$\frac{T - 527.14}{83.7} = H(\eta) \quad (67)$$

$$\eta = 83.93 \frac{Y}{X^{\frac{1}{4}}} \quad (68)$$

The velocity and temperature distributions for this experiment are plotted in figures 5 and 6, respectively, and again the theoretical curves for $Pr = 0.72$ are included. In figure 5 it can be seen that for large η the agreement is rather poor, particularly for the data for both small and large values of X . The poor agreement for small values of X is again due to the theory limitation near the edge of the plate and for large values of X , to the fact that the flow was becoming turbulent there.

FLOW AND HEAT-TRANSFER PARAMETERS

In addition to the velocity and temperature distributions, it is often desirable to compute other physically important quantities (such as shear stress, drag, heat-transfer rate, and heat-transfer coefficient) associated with the free-convection flow. To this end, two parameters, a flow parameter and a heat-transfer parameter, are derived in appendixes C and D, respectively.

The flow parameter

$$\frac{\tau}{(4Gr_x^3)^{1/4}(\nu_\infty\mu_0/X^2)} = F''(0)$$

is presented as a function of Prandtl number in figure 7. Thus, the various flow quantities for a given set of conditions can easily be computed by application of figure 7.

The local heat-transfer parameter

$$\frac{Nu}{(Gr_x/4)^{1/4}} = -H'(0)$$

as determined here is given as a function of Prandtl number in figure 8. A calculation of the local Nusselt number from this equation for $Pr=0.72$, and $Gr_x=10^9$ yields a value of 63.5, which indicates that large heat-transfer coefficients can also be obtained with free-convection flows.

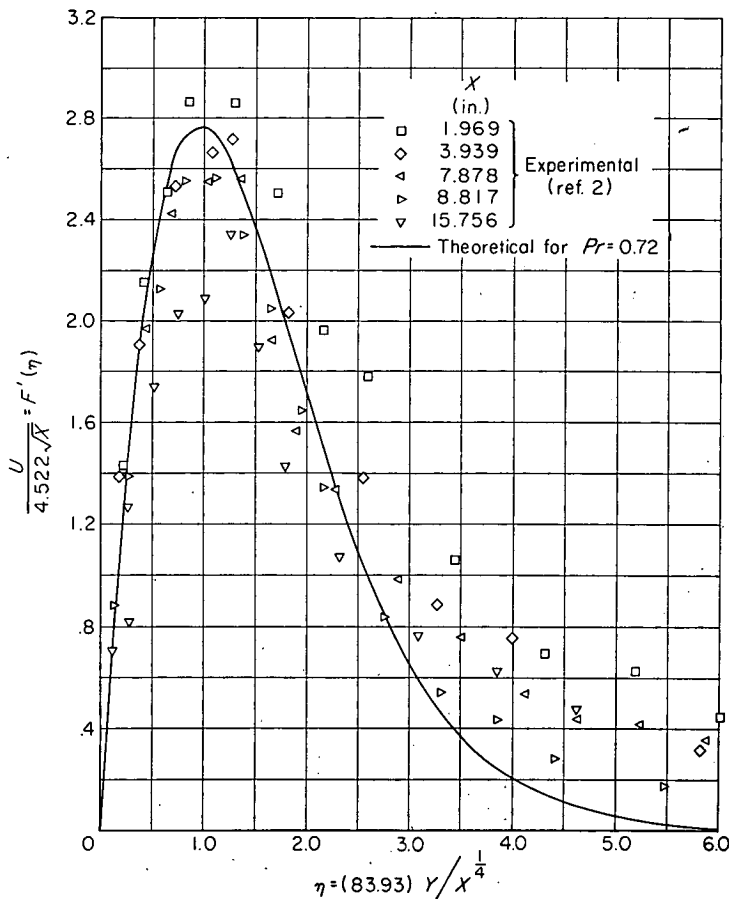


FIGURE 5.—Comparison of large plate experimental and theoretical velocity distributions for Prandtl number of 0.72.

On the basis of a simplified theory (that is, by use of integrated momentum and energy equations and assumed velocity and temperature distributions), Eckert (see p. 162 of ref. 7) obtained the approximate relation

$$\frac{Nu}{(Gr_x/4)^{1/4}} = \frac{0.718(Pr)^{1/2}}{(0.952 + Pr)^{1/4}}$$

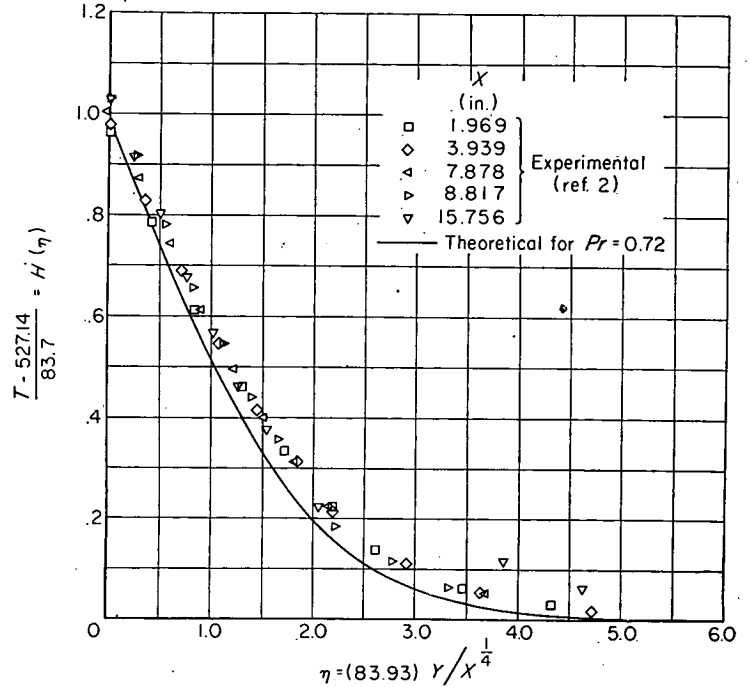


FIGURE 6.—Comparison of large plate experimental and theoretical temperature distributions for Prandtl number of 0.72.

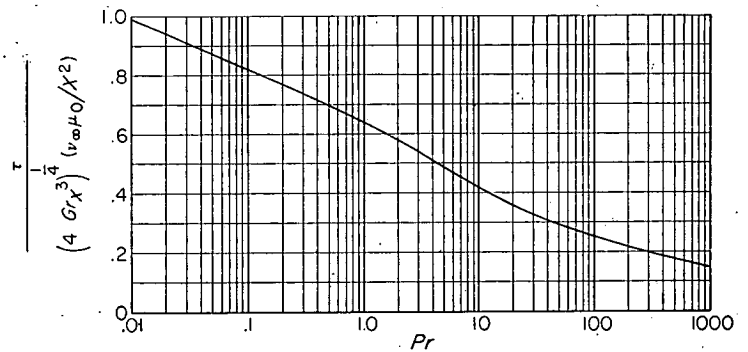


FIGURE 7.—Dimensionless flow parameter as function of Prandtl number.

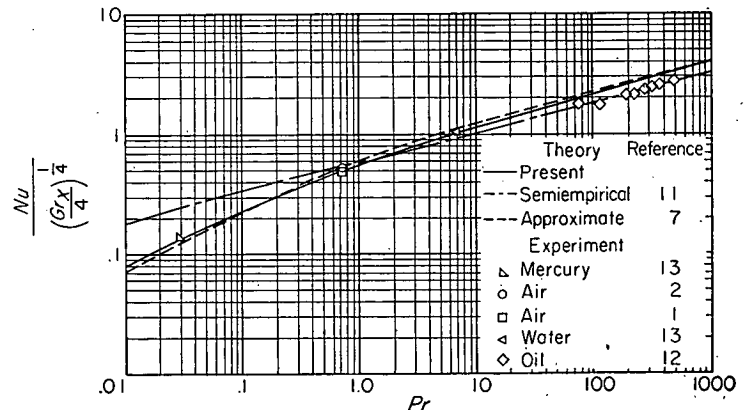


FIGURE 8.—Dimensionless heat-transfer parameter as function of Prandtl number.

The curve representing this equation is also presented in figure 8, and it closely approximates (to within about 10 percent) the curve determined by the more exact considerations of this report over the entire Prandtl number range. A semiempirical equation as given in reference 11 relating the average (over the length X) Nusselt number to the Prandtl and Grashof numbers which has been used in the heat-transfer calculations up to the present is

$$Nu_{av} = 0.548 [(Pr)Gr]^{\frac{1}{4}}$$

The constant 0.548 pertains specifically to air; for oil it should be 0.555 (see ref. 12) and for mercury, approximately 0.33 (see ref. 13). In order to obtain local values from the average ones given by the last equation, it is merely necessary to multiply the average values by 0.75. (The determination of this reduction factor of 0.75 is discussed in appendix D.) Thus in terms of the local quantities the semiempirical relation becomes

$$Nu = 0.411 [(Pr)Gr_x]^{\frac{1}{4}}$$

or

$$\frac{Nu}{(Gr_x/4)^{\frac{1}{4}}} = 0.581 (Pr)^{\frac{1}{4}}$$

The curve given by this equation is also presented in figure 8 and the agreement with the theoretical curve is very good for Prandtl numbers near unity, not so good for large Prandtl numbers, and very poor for the small Prandtl numbers. Of course, changes of the constants in the semiempirical relation as previously discussed for the large or small Prandtl number cases (oil and mercury, respectively, for example) would cause the semiempirical curve to approximate the theoretical curve more closely. The values of the heat-transfer parameter obtained experimentally for mercury ($Pr=0.03$), air ($Pr=0.72$), water ($Pr=7$), and oil ($Pr=75.5, 115, 190, 224, 275, 318, 368, \text{ and } 442$) are here reduced by the factor 0.75 from the average values reported. The value for mercury is an average taken of four readings from a curve, since this experiment was the only one not reported in tabular form. From figure 8 it can be seen that all of the experimental values except those for the oil experiments are in very good agreement with the theoretically computed values. The

data from the oil experiments, though not so good, show reasonable agreement (maximum error of approximately 20 percent) with the theoretical curve and good agreement, as is to be expected, with the semiempirical curve. The difference between the theoretical values and the oil experiment results can possibly be due to the fact that the viscosity changes in oil are large even for small temperature differences or due to the end effects in the measurements.

CONCLUSIONS

An analysis was made of the free-convection flow about a flat plate oriented in a direction parallel to that of the generating body force under the prime assumption that the relative temperature difference is small. It was found that the Grashof number was the principal factor determining the type of flow and that for large Grashof numbers the flow was of the boundary-layer type. The theoretical development was then continued to consider only the cases of large Grashof number because these are of most importance in aeronautics.

Velocity and temperature profiles for Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000 were computed on the basis of a constant body force and plate temperature and agreement with experiments where the fluid was air (Prandtl number of 0.72) was good. It was also demonstrated that velocities and Nusselt numbers of the order of magnitude of those obtained in forced-convection could be obtained in free-convection flows.

A flow parameter and a heat-transfer parameter which are functions of the Prandtl number alone were derived. Calculations of the important physical quantities such as shear stress, heat-transfer rate, and the like can be computed from these parameters. Values of the heat-transfer parameter obtained from an approximate theoretical development and from experiments compared with values computed from the present development showed good agreement over a wide range of Prandtl number (0.01 to 1000). It is shown that the commonly used semiempirical relation for the heat-transfer coefficient will yield good results only in restricted Prandtl number ranges.

LEWIS FLIGHT PROPULSION LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

CLEVELAND, OHIO, October 3, 1951

APPENDIX A

SYMBOLS

The following notation is used in this report:

$A_{ij}^{(n)}, B_j^{(n)},$	coefficients in numerical differentiation and
$C^{(n)}, D_i^{(n)}$	integration formulas
c_p	specific heat at constant pressure
F	dimensionless velocity function
f_i	components of body force per unit mass, $i=1, 2, 3$
f_x	negative of X -component of body force per unit mass

Gr	Grashof number, $\frac{\rho_\infty^2 \int_X l^3 \epsilon}{\mu_\infty^2}$
Gr_x	Grashof number based on X
g	gravitational force per unit mass (or acceleration due to gravity)
H	dimensionless temperature function
h	heat-transfer coefficient
K	isothermal compressibility coefficient, $-\rho \left[\frac{\partial (1/\rho)}{\partial P} \right]_T$

k	thermal-conductivity coefficient	β	coefficient of volumetric expansion, $\rho \left[\frac{\partial (1/\rho)}{\partial T} \right]_p$
l	characteristic length	γ	ratio of specific heats
m, n	arbitrary exponents	Δ	Laplacian operator
N	a number, defined following equation (26)	ϵ	relative temperature difference, $\beta(T_0 - T_\infty)$
Nu	Nusselt number, hX/k	η	similarity variable
Nu_{av}	average Nusselt number	θ	dimensionless temperature function
P	pressure	κ	step size used in numerical calculations
Pr	Prandtl number	μ	absolute viscosity
R	gas constant	ν	kinematic viscosity
r, s, t	arbitrary exponents	ρ	density
T	absolute temperature	σ	dimensionless pressure function
U_i	velocity components, $i=1, 2, 3$	τ	shear stress
u_i	dimensionless velocity components, $i=1, 2, 3$	φ	dimensionless density function
u	dimensionless velocity component in x -direction	ψ	stream function
v	dimensionless velocity component in y -direction	Subscripts:	
X_i	Cartesian coordinates, $i=1, 2, 3$	i, j	Cartesian tensor and summation subscripts
x_i	dimensionless Cartesian coordinates, $i=1, 2, 3$	s	denotes evaluation at static conditions ($\epsilon=0$)
x	dimensionless Cartesian coordinate	0	denotes evaluation at plate surface
Y	Cartesian coordinate	∞	denotes evaluation at undisturbed conditions
y	dimensionless Cartesian coordinate	Subscript notation is used to denote partial differentiation.	
		Superscripts:	
		Primes denote ordinary differentiation.	
		Bars (as $\bar{\sigma}$ or $\bar{y}$) denote transformed dimensionless quantities.	

APPENDIX B

NUMERICAL SOLUTION OF SIMPLIFIED BOUNDARY-VALUE PROBLEM

By LYNN U. ALBERS

The method is presented herein by which solutions to the boundary-value problem

$$F''' + 3FF'' - 2F'^2 + H = 0 \quad (B1)$$

$$H'' + 3PrFH' = 0 \quad (B2)$$

$$F(0) = F'(0) = 0 \quad H(0) = 1$$

$$F'(\infty) = H(\infty) = 0$$

were obtained for the cases of Pr equal to 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000. This discussion will enable the results to be clearly evaluated and will perhaps serve as a guide in the numerical solution of similar problems.

Each of the cases of the problem has a solution for a particular set of values for $F''(0)$ and $H'(0)$, hereinafter called eigenvalues. The basic approach to the problem was to estimate the eigenvalues and to integrate out from zero, obtaining functions which satisfied equations (B1) and (B2) at each step. The integration was continued until the functions F' and H behaved in a fashion inconsistent with the boundary values at infinity; for example, when they became negative or diverged to infinity. Improved estimates of the eigenvalues were then made on the basis of the results of preceding runs and the process was repeated successively until a solution was obtained.

Modifications required to overcome specific obstacles will be discussed after sufficient details of the basic procedure have been given. Then an evaluation of the accuracy of the numerical results will be made.

The integration process consists of two parts, a starting phase and an extension phase. The starting phase begins with an estimate of the eigenvalues $F''(0)$ and $H'(0)$ and a decision on the step size κ to be used. It continues with an iterative process of alternately computing F''' and H'' at the first four points and integrating them by five-point formulas. This process and that in the extension phases are so arranged that the differential equations are satisfied at each integral multiple of the step size.

The extension phase used preceding data to integrate step by step beyond the fourth point. Diagrams of both phases will be given after a few preliminary explanations.

All integration formulas used are based on the same idea. If a function, for example, F''' , is known at five points, there is a unique fourth-degree polynomial which agrees with it at these five points. Moreover, if the successive antiderivatives (integrals) F'' , F' , and F of F''' are known at one point, there are unique fifth-, sixth-, and seventh-degree polynomials which are successive antiderivatives of this fourth-degree polynomial and which agree with F'' , F' , and F , respectively, at the one point. It is then a simple algebra problem to deduce from the values of F''' at five points and

F , F' , and F'' at a single point the values of any of these four polynomials at any point. These results will approximate the functions F , F' , F'' , and F''' to a degree dependent on step size, the relative positions of the points in question, and the magnitude of the fifth derivative of F''' in the neighborhood of these points.

The preceding algebra problem can be presolved in all situations that arise in the starting and extension phases of the present problem and specific integration formulas may be deduced. These formulas are discussed in the next paragraph.

Let F''' be denoted at five successive points by F_0''' , F_1''' , F_2''' , F_3''' , and F_4''' . In the starting phase, these points are 0, κ , 2κ , 3κ , and 4κ , and F_0 , F_0' , and F_0'' are also known. Then the five sets of formulas required in the starting phase are

$$F_i'' = F_0'' + \frac{\kappa}{D_i^{(1)}} \sum_{j=0}^4 A_{ij}^{(1)} F_j''' \quad i=1, 2, 3, 4 \quad (B3)$$

$$H_i' = H_0' + \frac{\kappa}{D_i^{(1)}} \sum_{j=0}^4 A_{ij}^{(1)} H_j'' \quad i=1, 2, 3, 4 \quad (B4)$$

$$F_i' = F_0' + i\kappa F_0'' + \frac{\kappa^2}{D_i^{(2)}} \sum_{j=0}^4 A_{ij}^{(2)} F_j''' \quad i=1, 2, 3, 4 \quad (B5)$$

$$H_i = H_0 + i\kappa H_0' + \frac{\kappa^2}{D_i^{(2)}} \sum_{j=0}^4 A_{ij}^{(2)} H_j'' \quad i=1, 2, 3, 4 \quad (B6)$$

$$F_i = F_0 + i\kappa F_0' + \frac{i^2 \kappa^2}{2} F_0'' + \frac{\kappa^3}{D_i^{(3)}} \sum_{j=0}^4 A_{ij}^{(3)} F_j''' \quad i=1, 2, 3, 4 \quad (B7)$$

where the superscripts on the $A_{ij}^{(n)}$ and $D_i^{(n)}$ refer to the order of integration.

The constants $A_{ij}^{(n)}$ and $D_i^{(n)}$ may be read from the following tables:

For $A_{ij}^{(1)}$ and $D_i^{(1)}$:

$A_{ij}^{(1)}$						$D_i^{(1)}$
$i \backslash j$	0	1	2	3	4	
1	251	646	-264	106	-19	720
2	29	124	24	4	-1	90
3	27	102	72	42	-3	80
4	14	64	24	64	14	45

For $A_{ij}^{(2)}$ and $D_i^{(2)}$:

$A_{ij}^{(2)}$						$D_i^{(2)}$
$i \backslash j$	0	1	2	3	4	
1	367	540	-282	116	-21	1440
2	53	144	-30	16	-3	90
3	441	1404	162	180	-27	480
4	56	192	48	64	0	45

For $A_{ij}^{(3)}$ and $D_i^{(3)}$:

$A_{ij}^{(3)}$						$D_i^{(3)}$
$i \backslash j$	0	1	2	3	4	
1	1017	1070	-618	258	-47	10080
2	331	664	-240	104	-19	630
3	1431	3726	-486	450	-81	1120
4	744	2176	96	384	-40	315

It is now possible to diagram the steps of the starting phase of the integration. If each bar above a function denotes an improved estimate of it, and the first estimates of F_1''' , F_2''' , F_3''' , and F_4''' are all equal to F_0''' , and similarly for the H'' , then the starting phase diagrams are

(1) $(F_0, F_0', F_0'', F_0''', F_1''', F_2''', F_3''', F_4''') \rightarrow F_1, F_1', F_1''$
(This diagram means that the values in parentheses are used with appropriate integration formulas from (B3) to (B7) to obtain F , F' , and F'' at $\eta=\kappa$.)

(2) $(H_0, H_0', H_0'', H_1''', H_2''', H_3''', H_4''') \rightarrow H_1, H_1'$

(3) $(F_1, F_1', F_1'', H_1, H_1') \rightarrow \bar{F}_1''', \bar{H}_1''$

(The preceding diagram means that the values in parentheses are substituted in the differential equations (B1) and (B2) to obtain F''' and H'' at $\eta=\kappa$.)

(4) $(F_0, F_0', F_0'', F_0''', \bar{F}_1''', F_2''', F_3''', F_4''') \rightarrow F_2, F_2', F_2''$

(5) $(H_0, H_0', H_0'', \bar{H}_1'', H_2''', H_3''', H_4''') \rightarrow H_2, H_2'$

(6) $(F_2, F_2', F_2'', H_2, H_2') \rightarrow \bar{F}_2''', \bar{H}_2''$

(7) $(F_0, F_0', F_0'', F_0''', \bar{F}_1''', \bar{F}_2''', F_3''', F_4''') \rightarrow F_3, F_3', F_3''$

(8) $(H_0, H_0', H_0'', \bar{H}_1'', \bar{H}_2'', H_3''', H_4''') \rightarrow H_3, H_3'$

(9) $(F_3, F_3', F_3'', H_3, H_3') \rightarrow \bar{F}_3''', \bar{H}_3''$

(10) $(F_0, F_0', F_0'', F_0''', \bar{F}_1''', \bar{F}_2''', \bar{F}_3''', F_4''') \rightarrow F_4, F_4', F_4''$

(11) $(H_0, H_0', H_0'', \bar{H}_1'', \bar{H}_2'', \bar{H}_3'', H_4''') \rightarrow H_4, H_4'$

(12) $(F_4, F_4', F_4'', H_4, H_4') \rightarrow \bar{F}_4''', \bar{H}_4''$

It may be noted here that all four values of F''' and H'' have been improved, and further improvement will require iteration of steps 1 to 12. The start of the second iteration is diagrammed as follows:

(13) $(F_0, F_0', F_0'', F_0''', \bar{F}_1''', \bar{F}_2''', \bar{F}_3''', \bar{F}_4''') \rightarrow \bar{F}_1, \bar{F}_1', \bar{F}_1''$

(14) $(H_0, H_0', H_0'', \bar{H}_1'', \bar{H}_2'', \bar{H}_3'', \bar{H}_4'') \rightarrow \bar{H}_1, \bar{H}_1'$

(15) $(\bar{F}_1, \bar{F}_1', \bar{F}_1'', \bar{H}_1, \bar{H}_1') \rightarrow \bar{\bar{F}}_1''', \bar{\bar{H}}_1''$

(16) $(F_0, F_0', F_0'', F_0''', \bar{F}_1''', \bar{F}_2''', \bar{F}_3''', \bar{F}_4''') \rightarrow \bar{F}_2, \bar{F}_2', \bar{F}_2''$

Successive sets of 12 steps are performed until the values of F_1''' and H_1'' no longer change.

On the IBM Card-Programmed Electronic Calculator, a deck of punched cards 2 inches thick sufficed to perform steps 1 to 12. Three runs of this starter deck at 3 minutes per run accomplished complete convergence in most cases. At the end of the starting process there have been computed and stored F_4 , F_4' , F_4'' , H_4 , and H_4' , and final estimates of F_1''' , F_2''' , F_3''' , F_4''' , H_1'' , H_2'' , H_3'' , and H_4'' .

The extension phase has now been reached. It used a different set of integration formulas based on the same general ideas as equations (B3) to (B7). If F_0''' , F_1''' , F_2''' , F_3''' , and F_4''' now designate F''' at any five successive points, and the subscript 5 denotes the next point,

$$F_5'' = F_4'' + \frac{\kappa}{C^{(1)}} \sum_{j=0}^4 B_j^{(1)} F_j''' \quad (\text{B8})$$

$$H_5' = H_4' + \frac{\kappa}{C^{(1)}} \sum_{j=0}^4 B_j^{(1)} H_j'' \quad (\text{B9})$$

$$F_5' = F_4' + \kappa F_4'' + \frac{\kappa^2}{C^{(2)}} \sum_{j=0}^4 B_j^{(2)} F_j''' \quad (\text{B10})$$

$$H_5 = H_4 + \kappa H_4' + \frac{\kappa^2}{C^{(2)}} \sum_{j=0}^4 B_j^{(2)} H_j'' \quad (\text{B11})$$

$$F_5 = F_4 + \kappa F_4' + \frac{\kappa^2}{2} F_4'' + \frac{\kappa^3}{C^{(3)}} \sum_{j=0}^4 B_j^{(3)} F_j''' \quad (\text{B12})$$

where the $B_j^{(n)}$ and $C^{(n)}$ are given in the following table:

$B_j^{(n)}$						$C^{(n)}$
$n \backslash j$	0	1	2	3	4	
1	251	-1274	2616	-2774	1901	720
2	135	-692	1446	-1596	1427	1440
3	410	-2116	4476	-5084	5674	20160

The extension phase may then be diagrammed simply as follows:

- (1) $(F_4, F_4', F_4'', F_0''', F_1''', F_2''', F_3''', F_4''') \rightarrow F_5, F_5', F_5''$
- (2) $(H_4, H_4', H_0'', H_1'', H_2'', H_3'', H_4'') \rightarrow H_5, H_5'$
- (3) $(F_5, F_5', F_5'', H_5, H_5') \rightarrow F_5''', H_5'''$

The values of the functions at the next point are computed in similar manner, where the latest sets of five values of F''' and H'' are used. This process advances step by step toward infinity.

The extended deck of punched cards was about 3 inches thick and took a little over 3 minutes per run. For $Pr=0.72$, a step size of 0.1 was used, the starting phase took 10 minutes, and the extension phase, about 30 minutes. When it is realized that about 11,000 operations were performed in the 40 minutes per run, it may be seen that solution of the present problem would have been prohibitively difficult on desk-type calculators. Simplifications in method would have sacrificed accuracy or required smaller step size.

In two-point boundary-value problems where one point is infinity, some problems of judgment are involved as to where infinity is, and as to when a satisfactory approximation to a solution has been obtained. In most cases this question was settled for the present problem by calling a run satisfactory when it fell between two runs for which F' and H did not differ at important points in the fourth decimal place, and

for which F' and H flattened out at zero, correct to four decimal places.

Certain difficulties were met in the attempt to use the basic procedure previously discussed. These necessitated certain modifications.

For $Pr=2, 10, 100$, and 1000 , H would settle down to zero at an early stage; but while F' was still coming down, H'' would begin to oscillate and these oscillations increased and fed back into all other functions. This trouble was avoided by the following modifications: It is a consequence of equation (B2) that

$$H'(\eta) = H'(0) \exp\left(-3Pr \int_0^\eta F(t) dt\right) \\ = H'(\eta - \kappa) \exp\left(-3Pr \int_{\eta-\kappa}^\eta F(t) dt\right) \quad (\text{B13})$$

The extension phase was modified to require the additional integration formulas

$$\int_{\eta-\kappa}^\eta F(t) dt = \frac{\kappa}{720} \sum_{i=1}^5 A_i F_i \quad (\text{B14})$$

$$H_5 = H_4 + \frac{\kappa}{720} \sum_{i=1}^5 A_i H_i' \quad (\text{B15})$$

where $A_1 = -19$, $A_2 = 106$, $A_3 = -264$, $A_4 = 646$, and $A_5 = 251$.

These formulas were used along with equations (B8) to (B10) according to the following diagram:

- (1) $(F_4, F_4', F_4'', F_0''', F_1''', F_2''', F_3''', F_4''') \rightarrow F_5, F_5', F_5''$
- (2) $(F_1, F_2, F_3, F_4, F_5, H_4') \rightarrow H'$ by means of (B14) and (B13)
- (3) $(H_1', H_2', H_3', H_4', H_5', H_4) \rightarrow H_5$ by means of (B15)
- (4) $(F_5, F_5', F_5'', H_5) \rightarrow F_5'''$

The value F_0''' is discarded and F''' at the last five points is used to repeat the whole process again and again ad infinitum. As long as F stays positive, H' is guaranteed to approach zero and H will flatten out to some value and not oscillate.

For $Pr=0.01, 0.72, 0.733$, and 1 , the F''' began to oscillate at an advanced point and these oscillations grew and fed into the other functions. For all cases but $Pr=0.01$, the oscillations appeared very late, near the end of the run, and a suitable halving of step size when oscillation was detected in the fourth differences of F''' was sufficient to avoid the difficulty. But for the 0.01 case, oscillations of F''' appeared early in the run, namely, soon after the peak in F . These oscillations were found to be step-size connected, so that reduction of the step to 0.02 avoided them. Even then oscillations in F''' would begin to appear every 25 steps or so, and these were smoothed out regularly by repeated runs of a deck similar to the starter deck. Each run under these conditions took about 16 hours, making this the most difficult case to solve.

APPENDIX C

DERIVATION OF FLOW PARAMETER

By definition the shear stress is given by

$$\tau = \mu_0 \left(\frac{\partial U}{\partial Y} \right)_0 \quad (C1)$$

To express $(\partial U / \partial Y)_0$ in terms of the known function $F(\eta)$, use can be made of equations (60) and (62). Then

$$\frac{\partial U}{\partial Y} = (4Gr_x^3)^{\frac{1}{4}} \frac{\nu_\infty}{X^2}$$

Substitution of this expression into equation (C1) yields the flow parameter

$$\frac{\tau}{(4Gr_x^3)^{\frac{1}{4}}(\nu_\infty\mu_0/X^2)} = F''(0)$$

Note that from the general derivation, the flow parameter contains the viscosity evaluated at two different points. Recall, however, that the analysis has shown that to a first approximation the variation of viscosity with temperature can be neglected. Thus the viscosity can be taken as constant in the entire flow field.

APPENDIX D

DERIVATION OF HEAT-TRANSFER PARAMETER

The local Nusselt number is defined as

$$Nu = \frac{hX}{k} = \frac{-X}{(T_0 - T_\infty)} \left(\frac{\partial T}{\partial Y} \right)_0 \quad (D1)$$

To express $(\partial T / \partial Y)_0$ in terms of the known function $H(\eta)$, use is made of equations (61) and (62). Thus

$$\frac{\partial T}{\partial Y} = \frac{(T_0 - T_\infty)}{X} \left(\frac{Gr_x}{4} \right)^{\frac{1}{4}} H'(\eta)$$

Substitution of this expression into equation (D1) yields the heat-transfer parameter

$$\frac{Nu}{(Gr_x/4)^{\frac{1}{4}}} = -H'(0) \quad (D2)$$

The heat-transfer parameter as given by equation (D2) is, as was previously stated, a local parameter. It is often desired to compute the average (over the length X) value of this parameter. To this end, the Nusselt number (as given in equation (D1)) must be defined in terms of an average heat-transfer coefficient and the quantity thus obtained must then be integrated over the length X and divided by X . This procedure yields the result

$$Nu = \frac{3}{4} (Nu)_{av}$$

It is from this last equation that the 0.75 reduction factor previously discussed was obtained.

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TABLE I.—FUNCTIONS F AND H AND DERIVATIVES FOR VARIOUS PRANDTL NUMBERS

(a) Prandtl number, 0.01

η	F	F'	F''	H	H'	η	F	F'	F''	H	H'
0	0.0000	0.0000	0.9862	1.0000	-0.0812	4.16	1.9319	0.4037	-0.0615	0.6741	-0.0723
.1	.0048	.0936	.8868	.9919	-.0812	4.18	1.9400	.4024	-.0614	.6726	-.0722
.2	.0184	.1774	.7891	.9838	-.0812	4.20	1.9480	.4012	-.0612	.6712	-.0721
.3	.0399	.2516	.6942	.9756	-.0812	4.24	1.9640	.3988	-.0609	.6683	-.0720
.4	.0684	.3164	.6030	.9675	-.0811	4.28	1.9799	.3964	-.0605	.6654	-.0718
.5	.1029	.3723	.5163	.9594	-.0811	4.32	1.9957	.3939	-.0602	.6626	-.0716
.6	.1426	.4198	.4349	.9513	-.0811	4.36	2.0114	.3915	-.0599	.6597	-.0714
.7	.1866	.4595	.3595	.9432	-.0811	4.40	2.0270	.3891	-.0595	.6568	-.0713
.8	.2342	.4919	.2906	.9351	-.0810	4.50	2.0657	.3832	-.0587	.6497	-.0708
.9	.2848	.5178	.2285	.9270	-.0809	4.60	2.1037	.3774	-.0579	.6427	-.0704
1.0	.3376	.5379	.1734	.9189	-.0809	4.70	2.1411	.3716	-.0572	.6357	-.0699
1.1	.3922	.5527	.1253	.9108	-.0808	4.80	2.1780	.3660	-.0564	.6287	-.0695
1.2	.4480	.5631	.0839	.9028	-.0807	4.90	2.2143	.3604	-.0556	.6218	-.0690
1.3	.5047	.5697	.0489	.8947	-.0806	5.00	2.2501	.3548	-.0549	.6149	-.0686
1.4	.5618	.5731	.0198	.8866	-.0804	5.10	2.2853	.3494	-.0541	.6081	-.0681
1.5	.6192	.5739	-.0039	.8786	-.0803	5.20	2.3200	.3440	-.0534	.6013	-.0676
1.6	.6765	.5725	-.0229	.8706	-.0801	5.40	2.3877	.3335	-.0520	.5878	-.0667
1.7	.7336	.5694	-.0379	.8626	-.0800	5.60	2.4534	.3232	-.0506	.5746	-.0657
1.8	.7904	.5650	-.0493	.8546	-.0798	5.80	2.5170	.3132	-.0493	.5615	-.0648
1.9	.8466	.5596	-.0579	.8466	-.0796	6.00	2.5787	.3035	-.0479	.5487	-.0638
2.0	.9023	.5535	-.0641	.8387	-.0794	6.20	2.6384	.2940	-.0467	.5360	-.0628
2.1	.9573	.5469	-.0685	.8307	-.0792	6.40	2.6963	.2849	-.0454	.5236	-.0618
2.2	1.0117	.5399	-.0714	.8228	-.0789	6.60	2.7524	.2759	-.0442	.5113	-.0608
2.3	1.0653	.5326	-.0732	.8150	-.0787	6.80	2.8067	.2671	-.0430	.4993	-.0598
2.4	1.1182	.5253	-.0742	.8071	-.0784	7.00	2.8593	.2586	-.0419	.4874	-.0588
2.5	1.1703	.5178	-.0746	.7993	-.0782	7.40	2.9594	.2423	-.0397	.4643	-.0568
2.6	1.2217	.5104	-.0745	.7915	-.0779	7.80	3.0532	.2269	-.0376	.4420	-.0547
2.7	1.2724	.5029	-.0741	.7837	-.0776	8.20	3.1411	.2123	-.0356	.4205	-.0527
2.8	1.3223	.4956	-.0735	.7760	-.0773	8.60	3.2232	.1984	-.0337	.3998	-.0508
2.9	1.3715	.4882	-.0728	.7682	-.0770	9.00	3.2999	.1853	-.0319	.3799	-.0489
3.0	1.4200	.4810	-.0719	.7606	-.0766	9.40	3.3715	.1729	-.0302	.3607	-.0470
3.1	1.4677	.4739	-.0711	.7529	-.0763	9.80	3.4383	.1612	-.0285	.3423	-.0451
3.2	1.5148	.4668	-.0701	.7453	-.0760	10.20	3.5005	.1501	-.0270	.3246	-.0432
3.3	1.5611	.4598	-.0692	.7377	-.0756	10.60	3.5584	.1396	-.0254	.3069	-.0413
3.4	1.6067	.4530	-.0683	.7302	-.0753	11.00	3.6123	.1297	-.0241	.2915	-.0397
3.45	1.6293	.4496	-.0678	.7264	-.0751	11.40	3.6622	.1203	-.0228	.2759	-.0380
3.5	1.6517	.4462	-.0674	.7227	-.0749	11.80	3.7086	.1114	-.0215	.2610	-.0364
3.55	1.6739	.4428	-.0669	.7189	-.0747	12.20	3.7514	.1031	-.0203	.2468	-.0348
3.6	1.6960	.4395	-.0664	.7152	-.0745	12.60	3.7911	.0952	-.0192	.2332	-.0332
3.65	1.7179	.4362	-.0660	.7115	-.0743	13.00	3.8276	.0877	-.0181	.2202	-.0317
3.7	1.7396	.4329	-.0655	.7078	-.0741	13.60	3.8771	.0773	-.0165	.2018	-.0296
3.75	1.7611	.4296	-.0651	.7041	-.0739	14.20	3.9206	.0679	-.0151	.1847	-.0276
3.8	1.7825	.4264	-.0646	.7004	-.0738	14.80	3.9587	.0592	-.0138	.1687	-.0257
4.85	1.8308	.4232	-.0642	.6967	-.0736	15.40	3.9918	.0513	-.0126	.1538	-.0239
3.9	1.8249	.4200	-.0638	.6930	-.0734	16.00	4.0204	.0441	-.0114	.1399	-.0223
3.95	1.8458	.4168	-.0633	.6893	-.0732	17.00	4.0591	.0355	-.0097	.1189	-.0197
4.00	1.8665	.4136	-.0629	.6857	-.0729	18.00	4.0880	.0246	-.0082	.1003	-.0175
4.05	1.8871	.4105	-.0625	.6821	-.0727	19.00	4.1088	.0171	-.0069	.0839	-.0154
4.10	1.9076	.4074	-.0620	.6784	-.0725	20.00	4.1226	.0108	-.0057	.0694	-.0137
4.12	1.9157	.4061	-.0619	.6770	-.0725	21.00	4.1308	.0057	-.0046	.0565	-.0121
4.14	1.9238	.4049	-.0617	.6755	-.0724	22.00	4.1343	.0015	-.0037	.0452	-.0107

(b) Prandtl number, 0.72

η	F	F'	F''	H	H'
0	0.0000	0.0000	0.6760	1.0000	-0.5046
.1	.0032	.0627	.5785	.9495	-.5045
.2	.0122	.1159	.4866	.8991	-.5037
.3	.0261	.1602	.4007	.8488	-.5016
.4	.0440	.1962	.3210	.7989	-.4979
.5	.0651	.2246	.2479	.7493	-.4921
.6	.0887	.2460	.1817	.7005	-.4840
.7	.1141	.2612	.1224	.6526	-.4735
.8	.1407	.2708	.0700	.6059	-.4607
.9	.1681	.2754	.0246	.5606	-.4456
1.0	.1957	.2759	-.0143	.5168	-.4284
1.1	.2231	.2728	-.0468	.4749	-.4095
1.2	.2501	.2667	-.0734	.4350	-.3891
1.3	.2764	.2583	-.0945	.3972	-.3676
1.4	.3017	.2480	-.1106	.3615	-.3453
1.5	.3260	.2363	-.1224	.3281	-.3227
1.6	.3490	.2236	-.1302	.2970	-.3000
1.7	.3707	.2104	-.1347	.2681	-.2775
1.8	.3910	.1968	-.1363	.2415	-.2556
1.9	.4100	.1832	-.1356	.2170	-.2344
2.0	.4277	.1697	-.1331	.1945	-.2141
2.1	.4440	.1566	-.1290	.1741	-.1949
2.2	.4590	.1440	-.1238	.1555	-.1768
2.3	.4728	.1319	-.1178	.1387	-.1598
2.4	.4854	.1204	-.1113	.1235	-.1441
2.5	.4969	.1097	-.1044	.1098	-.1296
2.6	.5074	.0996	-.0973	.0976	-.1163
2.7	.5168	.0902	-.0903	.0865	-.1041
2.8	.5254	.0815	-.0834	.0767	-.0930
2.9	.5332	.0735	-.0767	.0679	-.0830
3.0	.5401	.0661	-.0703	.0601	-.0739
3.1	.5464	.0594	-.0642	.0531	-.0657
3.2	.5520	.0533	-.0585	.0469	-.0584
3.3	.5571	.0477	-.0531	.0414	-.0518
3.4	.5616	.0427	-.0481	.0365	-.0459
3.6	.5692	.0339	-.0392	.0284	-.0359
3.8	.5735	.0269	-.0317	.0220	-.0281
4.0	.5801	.0212	-.0255	.0170	-.0219
4.2	.5838	.0166	-.0204	.0132	-.0170
4.4	.5868	.0130	-.0162	.0102	-.0132
4.6	.5891	.0101	-.0128	.0078	-.0102
4.8	.5908	.0078	-.0102	.0060	-.0080
5.0	.5922	.0060	-.0079	.0046	-.0061
5.2	.5933	.0046	-.0063	.0035	-.0048
5.4	.5941	.0034	-.0048	.0027	-.0037
5.8	.5951	.0019	-.0029	.0015	-.0022
6.2	.5957	.0010	-.0018	.0009	-.0013
6.8	.5960	.0003	-.0008	.0003	-.0006
7.3	.5961	.0000	-.0004	.0001	-.0003

(c) Prandtl number, 0.733

η	F	F'	F''	H	H'
0	0.0000	0.0000	0.6741	1.0000	-0.5080
.1	.0032	.0625	.5767	.9492	-.5079
.2	.0122	.1155	.4849	.8984	-.5071
.3	.0260	.1597	.3990	.8478	-.5050
.4	.0438	.1955	.3194	.7975	-.5012
.5	.0649	.2238	.2465	.7477	-.4953
.6	.0884	.2451	.1804	.6985	-.4870
.7	.1137	.2601	.1212	.6503	-.4763
.8	.1402	.2695	.0691	.6033	-.4632
.9	.1674	.2741	.0237	.5578	-.4478
1.0	.1949	.2745	-.0149	.5139	-.4303
1.1	.2222	.2713	-.0473	.4718	-.4110
1.2	.2490	.2652	-.0737	.4317	-.3902
1.3	.2752	.2568	-.0946	.3938	-.3684
1.4	.3003	.2465	-.1106	.3581	-.3458
1.5	.3244	.2348	-.1222	.3246	-.3228
1.6	.3473	.2222	-.1299	.2935	-.2998
1.7	.3688	.2090	-.1342	.2647	-.2771
1.8	.3891	.1954	-.1358	.2381	-.2549
1.9	.4079	.1819	-.1350	.2136	-.2335
2.0	.4254	.1685	-.1324	.1913	-.2130
2.1	.4416	.1554	-.1283	.1710	-.1937
2.2	.4565	.1429	-.1230	.1526	-.1754
2.3	.4702	.1309	-.1170	.1359	-.1584
2.4	.4827	.1195	-.1104	.1208	-.1427
2.5	.4941	.1088	-.1035	.1073	-.1281
2.6	.5045	.0988	-.0965	.0952	-.1148
2.7	.5139	.0895	-.0895	.0843	-.1026
2.8	.5224	.0809	-.0826	.0746	-.0916
2.9	.5301	.0729	-.0759	.0660	-.0816
3.0	.5370	.0657	-.0695	.0583	-.0725
3.1	.5433	.0590	-.0635	.0514	-.0644
3.2	.5489	.0530	-.0578	.0454	-.0571
3.3	.5539	.0475	-.0524	.0400	-.0506
3.4	.5584	.0425	-.0474	.0352	-.0448
3.6	.5660	.0339	-.0386	.0273	-.0350
3.8	.5720	.0269	-.0312	.0211	-.0272
4.0	.5769	.0213	-.0250	.0163	-.0211
4.2	.5807	.0169	-.0200	.0126	-.0164
4.4	.5837	.0133	-.0159	.0097	-.0127
4.6	.5860	.0105	-.0125	.0074	-.0098
4.8	.5879	.0082	-.0099	.0057	-.0076
5.0	.5893	.0065	-.0077	.0044	-.0058
5.2	.5905	.0051	-.0061	.0033	-.0045
5.4	.5914	.0040	-.0047	.0025	-.0035
5.6	.5921	.0032	-.0036	.0019	-.0027
5.8	.5927	.0025	-.0028	.0015	-.0021
6.0	.5932	.0021	-.0022	.0011	-.0016
6.4	.5938	.0014	-.0013	.0006	-.0009
6.8	.5943	.0010	-.0008	.0003	-.0006
7.2	.5946	.0007	-.0004	.0001	-.0003
7.6	.5949	.0006	-.0002	.0000	-.0002
8.0	.5951	.0005	-.0001	.0000	-.0001

TABLE I.—FUNCTIONS F AND H AND DERIVATIVES FOR VARIOUS PRANDTL NUMBERS—Continued

(d) Prandtl number, 1

η	F	F'	F''	H	H'
0	0.0000	0.0000	0.6421	1.0000	-0.5671
.1	.0030	.0593	.5450	.9433	-.5669
.2	.0115	.1092	.4540	.8867	-.5657
.3	.0246	.1503	.3694	.8302	-.5627
.4	.0413	.1833	.2916	.7742	-.5572
.5	.0610	.2089	.2208	.7189	-.5488
.6	.0829	.2277	.1572	.6645	-.5371
.7	.1064	.2406	.1008	.6116	-.5221
.8	.1308	.2481	.0516	.5602	-.5038
.9	.1558	.2511	.0093	.5109	-.4826
1.0	.1809	.2502	-.0263	.4638	-.4589
1.1	.2058	.2461	-.0557	.4192	-.4330
1.2	.2300	.2393	-.0793	.3772	-.4056
1.3	.2535	.2304	-.0975	.3381	-.3772
1.4	.2761	.2199	-.1110	.3018	-.3484
1.5	.2975	.2083	-.1203	.2684	-.3197
1.6	.3177	.1960	-.1260	.2379	-.2915
1.7	.3367	.1832	-.1287	.2101	-.2642
1.8	.3543	.1708	-.1288	.1850	-.2382
1.9	.3707	.1575	-.1268	.1624	-.2136
2.0	.3859	.1450	-.1233	.1422	-.1907
2.1	.3997	.1329	-.1185	.1242	-.1695
2.2	.4125	.1213	-.1127	.1082	-.1501
2.3	.4240	.1104	-.1064	.0941	-.1324
2.4	.4346	.1001	-.0997	.0817	-.1164
2.5	.4441	.0904	-.0928	.0708	-.1020
2.6	.4527	.0815	-.0859	.0613	-.0892
2.7	.4604	.0733	-.0791	.0529	-.0777
2.8	.4673	.0657	-.0725	.0457	-.0676
2.9	.4736	.0588	-.0662	.0392	-.0587
3.0	.4791	.0524	-.0602	.0339	-.0509
3.1	.4841	.0467	-.0546	.0291	-.0441
3.2	.4885	.0415	-.0493	.0250	-.0381
3.3	.4924	.0368	-.0444	.0215	-.0329
3.4	.4959	.0326	-.0399	.0185	-.0283
3.6	.5016	.0254	-.0321	.0136	-.0210
3.8	.5061	.0197	-.0255	.0099	-.0155
4.0	.5096	.0151	-.0202	.0072	-.0115
4.2	.5122	.0116	-.0158	.0053	-.0084
4.4	.5143	.0087	-.0124	.0038	-.0062
4.6	.5158	.0066	-.0096	.0027	-.0045
4.8	.5169	.0049	-.0075	.0020	-.0034
5.0	.5177	.0035	-.0057	.0014	-.0024
5.2	.5183	.0025	-.0044	.0010	-.0018
5.5	.5189	.0014	-.0029	.0006	-.0011
6.0	.5194	.0004	-.0014	.0002	-.0005
6.25	.5194	.0000	-.0010	.0000	-.0004

(e) Prandtl number, 2

η	F	F'	F''	H	H'
0	0.0000	0.0000	0.5713	1.0000	-0.7165
.1	.0027	.0523	.4749	.9284	-.7161
.2	.0101	.0952	.3861	.8569	-.7135
.3	.0215	.1297	.3049	.7858	-.7069
.4	.0358	.1565	.2318	.7157	-.6949
.5	.0525	.1763	.1666	.6470	-.6768
.6	.0709	.1901	.1095	.5805	-.6523
.7	.0904	.1985	.0602	.5168	-.6215
.8	.1104	.2024	.0185	.4564	-.5852
.9	.1307	.2024	-.0161	.3999	-.5443
1.0	.1508	.1994	-.0440	.3476	-.5002
1.1	.1705	.1938	-.0659	.2999	-.4543
1.2	.1895	.1864	-.0824	.2568	-.4077
1.3	.2077	.1775	-.0942	.2183	-.3619
1.4	.2250	.1677	-.1020	.1844	-.3178
1.5	.2412	.1572	-.1063	.1547	-.2763
1.6	.2564	.1465	-.1078	.1290	-.2380
1.7	.2706	.1357	-.1071	.1069	-.2032
1.8	.2836	.1252	-.1046	.0882	-.1721
1.9	.2956	.1149	-.1008	.0724	-.1446
2.0	.3066	.1050	-.0960	.0592	-.1207
2.1	.3166	.0957	-.0906	.0482	-.1001
2.2	.3257	.0869	-.0849	.0391	-.0826
2.3	.3340	.0787	-.0789	.0316	-.0677
2.4	.3415	.0711	-.0729	.0254	-.0553
2.5	.3483	.0641	-.0671	.0204	-.0450
2.6	.3543	.0577	-.0614	.0164	-.0364
2.7	.3598	.0518	-.0560	.0131	-.0294
2.8	.3647	.0465	-.0509	.0105	-.0237
2.9	.3691	.0417	-.0461	.0083	-.0190
3.0	.3731	.0373	-.0416	.0066	-.0152
3.1	.3766	.0333	-.0375	.0053	-.0121
3.2	.3798	.0298	-.0338	.0042	-.0097
3.3	.3826	.0266	-.0303	.0033	-.0077
3.4	.3851	.0237	-.0272	.0026	-.0061
3.6	.3893	.0188	-.0218	.0017	-.0038
3.8	.3927	.0149	-.0174	.0010	-.0024
4.0	.3953	.0018	-.0138	.0007	-.0015
4.2	.3974	.0094	-.0109	.0004	-.0009
4.4	.3991	.0074	-.0086	.0003	-.0006
4.6	.4004	.0059	-.0068	.0002	-.0004
4.8	.4015	.0047	-.0054	.0001	-.0002
5.0	.4023	.0037	-.0042	.0001	-.0001
5.4	.4035	.0024	-.0026	.0001	-.0001
5.8	.4043	.0015	-.0016	.0000	.0000
6.4	.4049	.0008	-.0008	.0000	.0000
7.0	.4053	.0005	-.0004	.0000	.0000
8.0	.4056	.0002	-.0001	.0000	.0000
9.0	.4058	.0001	.0000	.0000	.0000
10.0	.4059	.0001	.0000	.0000	.0000
11.0	.4059	.0001	.0000	.0000	.0000

(f) Prandtl number, 10

η	F	F'	F''	H	H'
0	0.0000	0.0000	0.4192	1.0000	-1.1694
.1	.0019	.0371	.3251	.8832	-1.1671
.2	.0071	.0654	.2428	.7670	-1.1521
.3	.0147	.0861	.1723	.6534	-1.1155
.4	.0241	.1003	.1134	.5448	-1.0526
.5	.0346	.1091	.0655	.4437	-.9640
.6	.0458	.1137	.0279	.3527	-.8545
.7	.0573	.1150	-.0011	.2733	-.7322
.8	.0687	.1137	-.0221	.2064	-.6061
.9	.0800	.1107	-.0367	.1519	-.4849
1.0	.0908	.1066	-.0462	.1090	-.3753
1.1	.1012	.1016	-.0518	.0763	-.2813
1.2	.1111	.0963	-.0545	.0522	-.2045
1.3	.1205	.0908	-.0552	.0349	-.1445
1.4	.1293	.0853	-.0544	.0228	-.0993
1.5	.1376	.0799	-.0527	.0146	-.0665
1.6	.1453	.0748	-.0505	.0092	-.0435
1.7	.1525	.0699	-.0480	.0056	-.0278
1.8	.1593	.0652	-.0453	.0034	-.0174
1.9	.1656	.0608	-.0427	.0020	-.0107
2.0	.1714	.0567	-.0400	.0012	-.0065
2.1	.1769	.0528	-.0375	.0007	-.0038
2.2	.1820	.0491	-.0351	.0004	-.0022
2.3	.1868	.0458	-.0328	.0002	-.0013
2.4	.1912	.0426	-.0306	.0001	-.0007
2.5	.1953	.0396	-.0286	.0001	-.0004
2.6	.1991	.0369	-.0267	.0000	-.0002
2.7	.2027	.0343	-.0249	.0000	-.0001
2.8	.2060	.0319	-.0232	.0000	-.0001
2.9	.2090	.0297	-.0216	.0000	.0000
3.0	.2119	.0276	-.0201	.0000	.0000
3.1	.2146	.0256	-.0187	.0000	.0000
3.2	.2170	.0238	-.0174	.0000	.0000
3.3	.2193	.0221	-.0162	.0000	.0000
3.4	.2215	.0206	-.0151	.0000	.0000
3.6	.2253	.0178	-.0131	.0000	.0000
3.8	.2286	.0153	-.0113	.0000	.0000
4.0	.2314	.0132	-.0098	.0000	.0000
4.2	.2339	.0114	-.0084	.0000	.0000
4.4	.2360	.0098	-.0073	.0000	.0000
4.6	.2379	.0085	-.0063	.0000	.0000
4.8	.2394	.0073	-.0054	.0000	.0000
5.0	.2408	.0063	-.0047	.0000	.0000
5.4	.2430	.0047	-.0035	.0000	.0000
5.8	.2446	.0035	-.0026	.0000	.0000
6.2	.2458	.0026	-.0019	.0000	.0000
7.0	.2474	.0014	-.0011	.0000	.0000
8.0	.2484	.0007	-.0005	.0000	.0000
9.0	.2489	.0003	-.0002	.0000	.0000
10.0	.2491	.0002	-.0001	.0000	.0000

(g) Prandtl number, 100

η	F	F'	F''	H	H'
0.000	0.0000	0.0000	0.2517	1.0000	-2.191
.025	.0001	.0060	.2274	.9452	-2.191
.050	.0003	.0114	.2044	.8905	-2.188
.075	.0006	.0162	.1828	.8359	-2.180
.100	.0011	.0205	.1626	.7815	-2.166
.125	.0017	.0244	.1438	.7276	-2.144
.150	.0023	.0277	.1263	.6744	-2.113
.175	.0031	.0307	.1101	.6221	-2.071
.200	.0039	.0332	.0952	.5709	-2.018
.225	.0047	.0355	.0816	.5213	-1.954
.250	.0056	.0373	.0692	.4733	-1.880
.275	.0066	.0389	.0580	.4273	-1.786
.300	.0076	.0402	.0479	.3836	-1.704
.325	.0086	.0413	.0389	.3422	-1.604
.350	.0096	.0422	.0309	.3034	-1.498
.375	.0107	.0429	.0238	.2674	-1.388
.400	.0118	.0434	.0176	.2341	-1.276
.425	.0129	.0438	.0123	.2036	-1.163
.450	.0140	.0440	.0076	.1759	-1.052
.475	.0151	.0442	.0036	.1510	-.9430
.500	.0162	.0442	.0002	.1287	-.8393
.525	.0173	.0442	-.0026	.1089	-.7404
.550	.0184	.0441	-.0050	.0916	-.6478
.575	.0195	.0439	-.0070	.0765	-.5621
.600	.0206	.0437	-.0087	.0634	-.4837
.65	.0227	.0432	-.0111	.0427	-.3496
.70	.0249	.0426	-.0126	.0280	-.2440
.75	.0270	.0420	-.0135	.0178	-.1657
.80	.0291	.0413	-.0140	.0111	-.1088
.85	.0311	.0406	-.0142	.0067	-.0693
.90	.0331	.0399	-.0142	.0039	-.0428
.95	.0351	.0392	-.0141	.0022	-.0256
1.00	.0371	.0385	-.0140	.0012	-.0149
1.10	.0408	.0371	-.0136	.0004	-.0046
1.20	.0445	.0357	-.0132	.0001	-.0013
1.30	.0480	.0344	-.0128	.0000	-.0003
1.40	.0514	.0332	-.0124	.0000	-.0001
1.50	.0546	.0320	-.0119	.0000	.0000
1.60	.0578	.0308	-.0116	.0000	.0000
1.70	.0608	.0297	-.0112	.0000	.0000
1.80	.0637	.0286	-.0108	.0000	.0000
1.90	.0665	.0275	-.0104	.0000	.0000
2.0	.0692	.0265	-.0101	.0000	.0000
2.1	.0718	.0255	-.0097	.0000	.0000
2.2	.0743	.0245	-.0094	.0000	.0000
2.3	.0767	.0236	-.0091	.0000	.0000
2.4	.0790	.0227	-.0088	.0000	.0000
2.6	.0834	.0210	-.0082	.0000	.0000
2.8	.0874	.0195	-.0076	.0000	.0000
3.0	.0912	.0180	-.0071	.0000	.0000

TABLE I.—FUNCTIONS F AND H AND DERIVATIVES FOR VARIOUS PRANDTL NUMBERS—Concluded

(g) Prandtl number, 100—Concluded

η	F	F'	F''	H	H'
3.2	0.0947	0.0166	-0.0066	0.0000	0.0000
3.4	.0979	.0154	-.0061	.0000	.0000
3.6	.1008	.0142	-.0056	.0000	.0000
3.8	.1035	.0131	-.0052	.0000	.0000
4.0	.1061	.0121	-.0049	.0000	.0000
4.4	.1105	.0103	-.0042	.0000	.0000
4.8	.1143	.0088	-.0036	.0000	.0000
5.2	.1176	.0074	-.0031	.0000	.0000
5.6	.1203	.0063	-.0026	.0000	.0000
6.0	.1226	.0053	-.0022	.0000	.0000
6.6	.1254	.0041	-.0018	.0000	.0000
7.2	.1276	.0032	-.0014	.0000	.0000
8.0	.1297	.0022	-.0010	.0000	.0000
9.0	.1315	.0014	-.0007	.0000	.0000
10.0	.1326	.0008	-.0005	.0000	.0000
11.0	.1332	.0004	-.0003	.0000	.0000
12.0	.1335	.0002	-.0002	.0000	.0000
13.0	.1336	.0000	-.0001	.0000	.0000

(h) Prandtl number, 1000

η	F	F'	F''	H	H'
0.	0.0000	0.0000	0.1450	1.0000	-3.966
.025	.0000	.0033	.1212	.9009	-3.962
.050	.0002	.0061	.0999	.8021	-3.933
.075	.0003	.0083	.0811	.7046	-3.861
.100	.0006	.0102	.0647	.6096	-3.731
.125	.0008	.0116	.0506	.5186	-3.538
.150	.0012	.0127	.0387	.4332	-3.283
.175	.0015	.0135	.0289	.3549	-2.975
.200	.0018	.0142	.0209	.2847	-2.628
.225	.0022	.0146	.0146	.2236	-2.261
.250	.0026	.0149	.0096	.1717	-1.893
.275	.0029	.0151	.0059	.1288	-1.541
.300	.0033	.0152	.0032	.0944	-1.220
.325	.0037	.0153	.0012	.0675	-.9381
.350	.0041	.0153	-.0003	.0471	-.7012
.375	.0045	.0152	-.0012	.0321	-.5093
.400	.0048	.0152	-.0019	.0213	-.3596
.425	.0052	.0152	-.0023	.0138	-.2467
.450	.0056	.0151	-.0026	.0087	-.1645
.475	.0060	.0150	-.0027	.0053	-.1066
.500	.0063	.0150	-.0028	.0032	-.0672
.525	.0067	.0149	-.0029	.0019	-.0412
.550	.0071	.0148	-.0029	.0011	-.0245
.575	.0075	.0147	-.0029	.0006	-.0142
.600	.0078	.0147	-.0029	.0003	-.0080
.625	.0082	.0146	-.0029	.0002	-.0044
.650	.0107	.0141	-.0028	.0000	.0000
.675	.0135	.0136	-.0027	.0000	.0000
.700	.0167	.0125	-.0025	.0000	.0000
.725	.0205	.0115	-.0023	.0000	.0000
.750	.0250	.0106	-.0022	.0000	.0000
.775	.0300	.0098	-.0020	.0000	.0000
.800	.0355	.0090	-.0019	.0000	.0000
.825	.0410	.0080	-.0017	.0000	.0000
.850	.0464	.0070	-.0015	.0000	.0000
.875	.0505	.0060	-.0012	.0000	.0000
.900	.0549	.0050	-.0011	.0000	.0000
.925	.0593	.0039	-.0008	.0000	.0000
.950	.0638	.0032	-.0007	.0000	.0000
.975	.0691	.0022	-.0004	.0000	.0000
1.00	.0727	.0015	-.0003	.0000	.0000
1.10	.0752	.0011	-.0002	.0000	.0000
1.20	.0771	.0008	-.0001	.0000	.0000
1.40	.0786	.0007	.0000	.0000	.0000
1.60	.0798	.0006	.0000	.0000	.0000
1.80	.0809	.0005	.0000	.0000	.0000
2.00	.0816	.0005	.0000	.0000	.0000

REPORT 1112

HYDROCARBON AND NONHYDROCARBON DERIVATIVES OF CYCLOPROPANE

By VERNON A. SLABEY, PAUL H. WISE, and LOUIS C. GIBBONS

SUMMARY

The methods used to prepare and purify 19 hydrocarbon derivatives of cyclopropane are discussed. Of these hydrocarbons, 13 were synthesized for the first time. In addition to the hydrocarbons, six cyclopropylcarbinols, five alkyl cyclopropyl ketones, three cyclopropyl chlorides, and one cyclopropanedicarboxylate were prepared as synthesis intermediates.

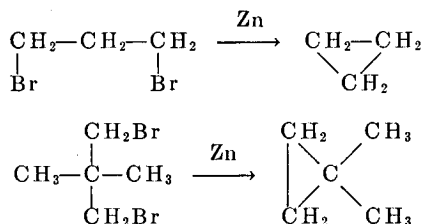
The melting points, boiling points, refractive indices, densities, and, in some instances, heats of combustion of both the hydrocarbon and nonhydrocarbon derivatives of cyclopropane were determined. These data and the infrared spectrum of each of the 34 cyclopropane compounds are presented herein.

The infrared absorption bands characteristic of the cyclopropyl ring are discussed, and some observations are made on the contribution of the cyclopropyl ring to the molecular refractions of cyclopropane compounds.

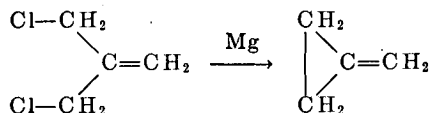
INTRODUCTION

The synthesis and purification of cyclopropane hydrocarbons was begun at the NACA Lewis laboratory in 1944 in order to provide high-purity samples of substituted cyclopropanes for an investigation of the effect of molecular structure on combustion characteristics and other properties pertinent to research on fuels for aircraft propulsion systems. The present report summarizes the research pertaining to the synthesis of 19 hydrocarbon and 15 nonhydrocarbon derivatives of cyclopropane.

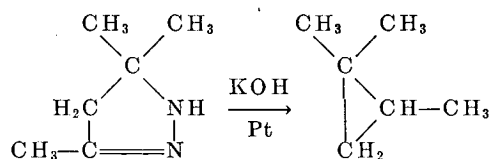
Few general methods for preparing hydrocarbons which contain the cyclopropyl ring are known. The method of Gustavson, which involves the reaction of α,γ -dibromides with zinc dust in a protonic solvent, has frequently been used (refs. 1 to 6):



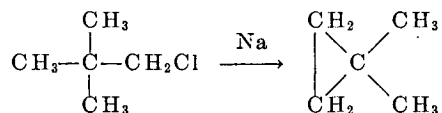
A modification of the Gustavson reaction, in which magnesium reacted in tetrahydrofuran with an α,γ -dichloride, was recently used to prepare methylenecyclopropane (ref. 7):



The pyrolysis of pyrazolines has also found limited use for the preparation of certain cyclopropanes (refs. 8 to 11):

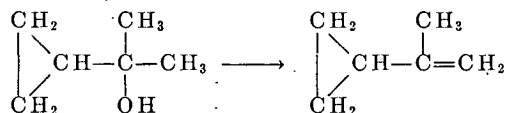


A method reported by Whitmore and co-workers (refs. 12 to 15) involves the removal of hydrogen halide from alkyl halides in which the halogen atom is one carbon removed from a quaternary carbon atom:

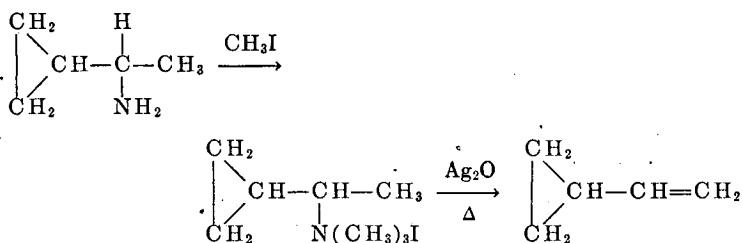


In each of these methods the cyclopropane ring is formed during the reaction.

Another approach to the synthesis of cyclopropane hydrocarbons is that which involves the conversion of nonhydrocarbon derivatives of cyclopropane to the corresponding hydrocarbon derivatives. For example, some of the carbinol derivatives of cyclopropane have been dehydrated with acidic catalysts to the corresponding cyclopropylalkenes (refs. 16 to 20):

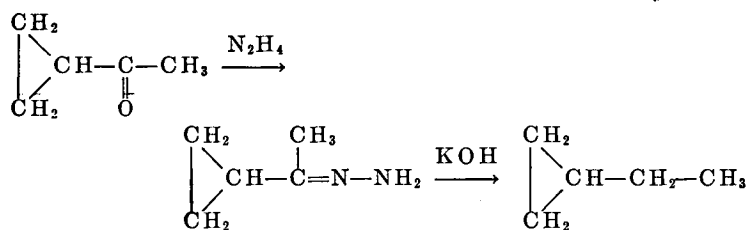


Exhaustive methylation of cyclopropylcarbinylamines also has been reported to yield cyclopropylalkenes (ref. 21):



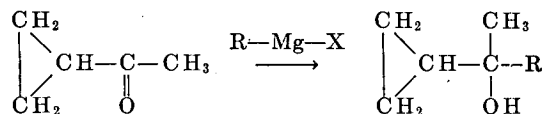
In an analogous manner, cyclopropene has been prepared from cyclopropylamine (ref. 22). For the preparation of

certain cyclopropylalkanes, the Wolff-Kishner reduction of cyclopropyl ketones has been used (refs. 23 to 25):



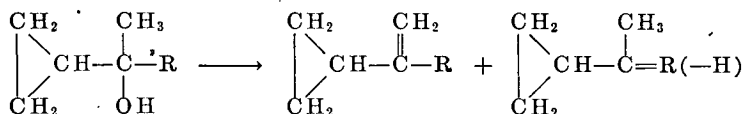
The usefulness of all the reactions described is limited by the availability of starting materials. With few exceptions, the α,γ -dibromides for the Gustavson reaction, the pyrazolines, the halides suitable for hydrogen halide elimination reactions, and the nonhydrocarbon derivatives of cyclopropane are not commercially available. Consequently, cyclopropane hydrocarbons generally have been prepared only in limited research quantities. With the exception of cyclopropane itself, which has been used as an anesthetic, none of the cyclopropane hydrocarbons are available commercially.

The announcement during World War II of the commercial availability of methyl cyclopropyl ketone coincided with the interest of the Lewis laboratory in the preparation of cyclopropane hydrocarbons. It was believed that this nonhydrocarbon derivative of cyclopropane could be used to prepare a series of cyclopropane hydrocarbons in the following manner: The ketone was known to react with Grignard reagents to give methylalkylcyclopropylcarbinols (refs. 26 and 27):

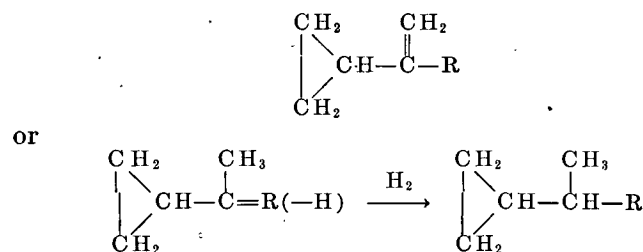


[R, hydrocarbon radical; X, halogen atom]

By using different R—X compounds for the preparation of R—Mg—X, the length and the degree of branching of the hydrocarbon chain could be varied. The methylalkylcyclopropylcarbinols were known to dehydrate in the presence of acid catalysts to the cyclopropylalkenes (refs. 16 to 20), although the practicability of the reaction as a synthesis method had not been established:



The selective hydrogenation of the cyclopropylalkenes to the corresponding cyclopropylalkanes had not been investigated; but it was believed that by proper selection of catalyst, temperature, and pressure, the desired hydrogenation could be accomplished:



(Concurrently with the present research, the authors of refs. 28 and 29 attempted the catalytic hydrogenation of 2-cyclopropylpropene and vinylcyclopropane. The catalyst and the reaction conditions employed by them yielded the corresponding cyclopropylalkanes but also gave considerable amounts of paraffinic hydrocarbons.)

A total of 12 cyclopropylalkenes and 5 cyclopropylalkanes were prepared in this manner from methyl cyclopropyl ketone:

By dehydration of alkylcyclopropylcarbinols.—

Vinylcyclopropane
2-Cyclopropylpropene
2-Cyclopropyl-1-butene
2-Cyclopropyl-1-pentene
2-Cyclopropyl-1-hexene
2-Cyclopropyl-3-methyl-1-butene
2-Cyclopropyl-2-butene (l. b.)
2-Cyclopropyl-2-butene (h. b.)
2-Cyclopropyl-2-pentene (l. b.)
2-Cyclopropyl-2-pentene (h. b.)
2-Cyclopropyl-2-hexene (l. b.)
2-Cyclopropyl-2-hexene (h. b.)

(The abbreviations l. b. and h. b. denote the low-boiling and the high-boiling geometrical isomers, respectively.)

By hydrogenation of cyclopropylalkenes.—

2-Cyclopropylpropane
2-Cyclopropylbutane
2-Cyclopropylpentane
2-Cyclopropylhexane
2-Cyclopropyl-3-methylbutane

Of these 17 hydrocarbons, 12 were prepared for the first time.

In addition to the hydrocarbons, six cyclopropylcarbinols were obtained as synthesis intermediates, and five cyclopropyl ketones were obtained from the ozonization of the 2-cyclopropyl-1-alkenes:

Cyclopropylcarbinols from Grignard reactions.—

Dimethylcyclopropylcarbinol
Methylethylcyclopropylcarbinol
Methylpropylcyclopropylcarbinol
Methylisopropylcyclopropylcarbinol
Methylbutylcyclopropylcarbinol

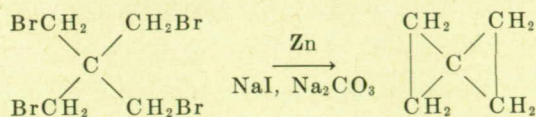
Cyclopropyl ketones from ozonization of 2-cyclopropyl-1-alkenes.—

Methyl cyclopropyl ketone
Ethyl cyclopropyl ketone
Propyl cyclopropyl ketone
Isopropyl cyclopropyl ketone
Butyl cyclopropyl ketone

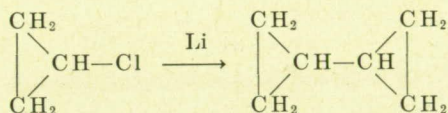
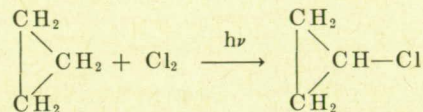
Cyclopropylcarbinol from reduction of cyclopropyl ketone.—

Methylethylcyclopropylcarbinol

Two other cyclopropane hydrocarbons, spiropentane and the smallest of the dicyclic hydrocarbons, dicyclopropyl, were prepared in the present investigation. Spiropentane was obtained from the debromination of pentaerythrityl tetrabromide, which was accomplished in a manner similar to that described by Hass and co-workers for preparing cyclopropane (ref. 30):



The preparation of dicyclopropyl involved the photochemical chlorination of cyclopropane (ref. 31) and the reaction of one of the chlorination products, cyclopropyl chloride, with lithium in ether:



From the photochemical chlorination of cyclopropane, two other cyclopropyl chlorides were isolated, namely, 1,1-dichlorocyclopropane and *trans*-1,2-dichlorocyclopropane.

Melting points, boiling points, refractive indices, densities, and, in some instances, heats of combustion of the hydrocarbon and nonhydrocarbon derivatives of cyclopropane which were either prepared for the first time or isolated in a higher state of purity than heretofore were determined. The infrared spectra of the 34 cyclopropane compounds were also determined, and each of the spectra is presented herein. Some observations on the molecular refraction of cyclopropane compounds are included in the report.

APPARATUS

The Grignard reactions were carried out in either a 10- or a 30-gallon glass-lined reactor which was double-walled so that the reaction temperatures could be controlled by passing steam or cold or hot water between the inner and outer walls. The reactors were equipped with efficient motor-driven stirrers, high-capacity reflux condensers, and stainless steel tanks from which liquid reactants were fed by gravity into the reactors.

The column used in dehydrating some of the cyclopropylcarbinols consisted of a 2.5- by 90-centimeter pyrex tube which was filled with 8 to 14 mesh alumina and heated by resistance-element tube furnaces. Temperatures of the furnaces were controlled by manual adjustment of variable transformers between the furnaces and the laboratory power supply. Column temperatures were read from an indicating potentiometer connected to thermocouples which were held in place against the outer walls of the pyrex dehydration tube by means of copper strips. A bellows-type pump was used to force the liquid carbinols into the top of the column; the vaporized products issuing from the bottom of the column were condensed by a water-cooled spiral condenser and collected in a flask at room temperature. The flask in turn was connected to a trap chilled with solid carbon dioxide and acetone, so that products volatile at room temperature

could also be collected. A sketch of the assembly is shown in figure 1.

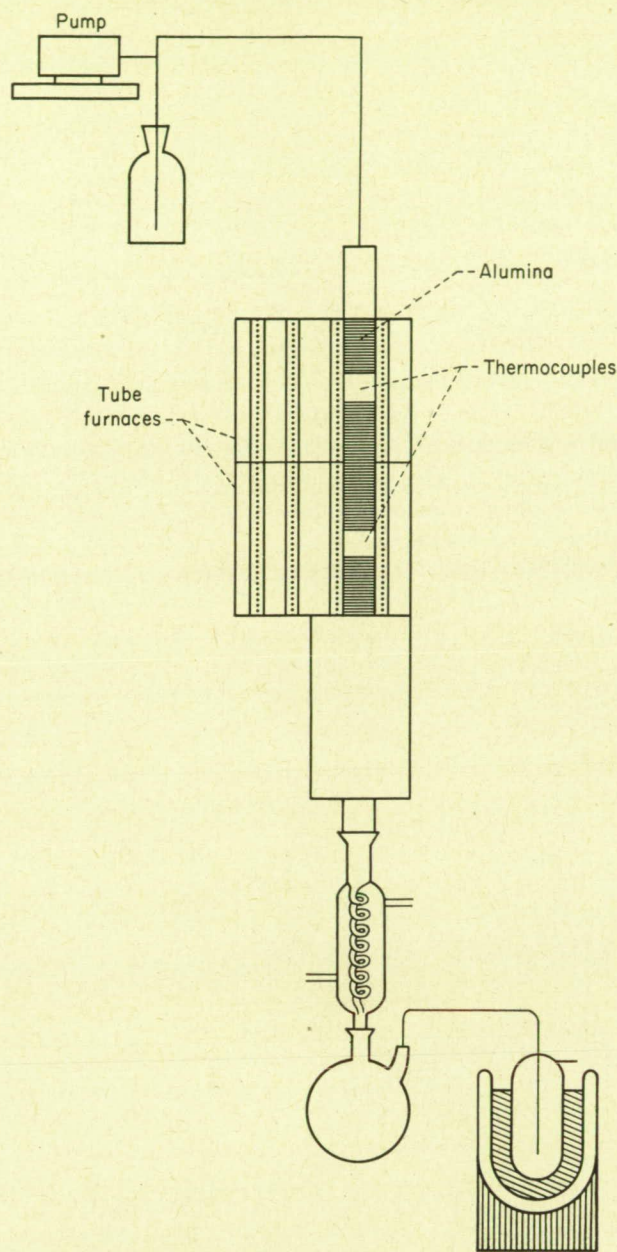


FIGURE 1.—Apparatus used in dehydrating cyclopropylcarbinols over alumina.

The hydrogenation reactions were conducted in high-pressure steel autoclaves of 1-, 3.4-, and 4.4-liter capacities. The autoclaves were equipped with rocker-type shaking mechanisms, resistance heaters, thermocouples and potentiometers for determining reaction temperatures, and appropriate valves, pressure gages, and high-pressure lines for introducing hydrogen into the vessels.

Fractional distillation columns were used in the purification of intermediates and final products. For fractionations at reduced pressures, 2.5- by 180-centimeter pyrex columns, which were packed with $\frac{3}{16}$ -inch single-turn glass helices, were used. Columns 2.2 by 180 centimeters, packed with $\frac{1}{8}$ -inch single-turn glass helices, were used for fractionations of those intermediates which could be distilled at atmospheric pressure and for the initial fractionations of the hydrocarbons.

For final purification of the hydrocarbons, 2.5- by 180 centimeter Podbielniak columns, operated at efficiencies in excess of 150 theoretical plates, were employed.

The ozonization apparatus, used in the identification of cyclopropylalkenes, was similar to that described in reference 32. A 0.25 kilovolt-ampere transformer with an input of 120 volts and an output of 25,000 volts was used to supply the necessary potential to the electrodes of the ozonizer tube.

The apparatus for the photochemical chlorination of cyclopropane was essentially the same as that described in reference 31 except that the recycling system was eliminated. The reactor was constructed of 0.7-centimeter pyrex tubing which was bent to form a planar grid; total length of the tubing was 470 centimeters. The reactor was illuminated by two G. E. type-RS sun lamps placed directly in front of the grid and mounted so that their distance from the grid could be varied. The scrubbing towers for removing the hydrogen chloride and the chlorine from the reaction products were constructed of 0.45- by 122-centimeter pyrex tubing and were packed with $\frac{1}{4}$ -inch Berl saddles to increase the contact area. The flowmeters in the system were used primarily to check the constancy of gas flows; quantities of reactants were measured by loss in weight of the gas cylinders. A sketch of the apparatus is shown in figure 2.

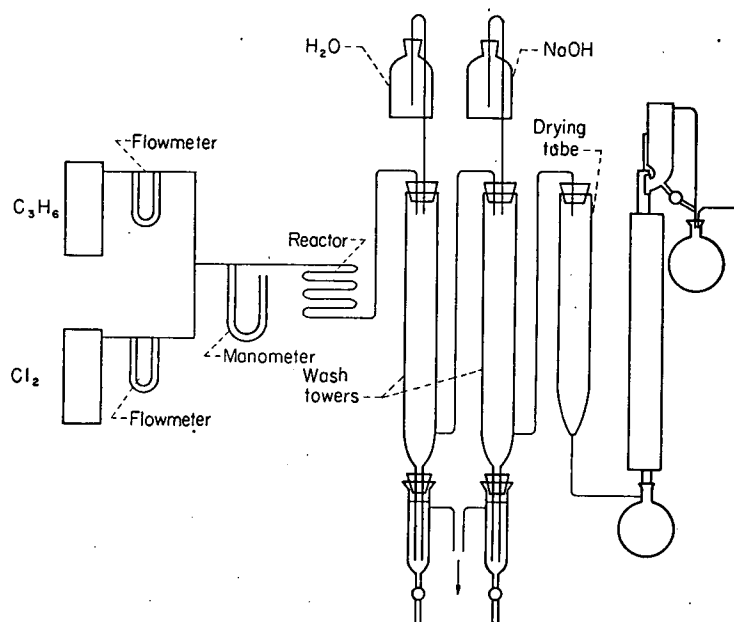


FIGURE 2.—Apparatus used for photochlorination of cyclopropane.

The infrared spectrophotometer used in determining the infrared spectra of the hydrocarbon and nonhydrocarbon derivatives of cyclopropane was a Baird Associates double-beam recording spectrophotometer equipped with a sodium chloride prism. Liquid sample cells of 0.1-millimeter thickness were used, and the spectra of undiluted samples and also of samples diluted with either carbon tetrachloride or carbon disulfide were obtained.

DISCUSSION OF SYNTHESSES

Descriptions of the syntheses have been generalized in the following discussion; for detailed descriptions of each syn-

thesis, the reader is referred to the synthesis reports previously published (refs. 33 to 39).

ALKYLCYCLOPROPYLCARBINOLS

Methylcyclopropylcarbinol.—Five-mole quantities of methyl cyclopropyl ketone were reduced by four methods: (1) with sodium metal in 75-percent ethyl alcohol, (2) with lithium aluminum hydride in ether, (3) with hydrogen in the presence of Raney nickel catalyst, and (4) with hydrogen in the presence of copper chromite catalyst. The reactions and products are summarized in the following table (ref. 33):

Reducing agent	Moles of reducing agent	Reaction temperature, °C	Moles of ketone recovered	Yield of carbinol, percent	Yield of pentanol-2, percent
Na and aqueous alcohol	14	10-15	None	42	0
LiAlH ₄	2	35-45	None	76	0
H ₂ -Raney Ni	Theoretical	90-125	0.55	34	30
H ₂ -copper chromite	do	100	None	90	0
Do	do	120	None	87	0
Do	do	150	None	76	ca. 10

These data show that of the methods investigated, the hydrogenation of methyl cyclopropyl ketone in the presence of copper chromite catalyst was the most satisfactory for preparing methylcyclopropylcarbinol. The reduction with lithium aluminum hydride was also acceptable for preparing small quantities of the carbinol, although the yields were not so high as those obtained catalytically with copper chromite. Neither the reduction with sodium nor that with Raney nickel was satisfactory because of the low yields of carbinol obtained and also, in the latter method, because of the formation of a close-boiling impurity, pentanol-2.

Methylalkylcyclopropylcarbinols.—The methylalkylcyclopropylcarbinols were prepared in ether by the reaction of methyl cyclopropyl ketone with the Grignard reagents of appropriate alkyl halides. In general, the Grignard reagent was prepared (in 2 to 5 percent excess) by adding the alkyl halide to a suspension of magnesium metal in ether, then adding the ketone to the Grignard reagent, and finally hydrolyzing the products of the reaction with saturated aqueous ammonium chloride solution. The quantities of reactants and the yields of the carbinols are summarized in the following table:

Alkyl halide	Moles of ketone	Cyclopropylcarbinol	Yield, percent	Reference
Methyl chloride	150	Dimethyl	64	34
Ethyl bromide	20	Methylethyl	77	34
Propyl bromide	150	Methylpropyl	51	34
Butyl bromide	97	Methylbutyl	52	34
Isopropyl bromide	120	Methylisopropyl	51	35

Because no attempt was made to determine the reaction conditions necessary for optimum yields of the carbinols, any relations existing between yields of the carbinols and their structures could not be deduced.

Halogenated impurities were found in all the carbinols. It has been suggested (ref. 28) that the use of an excess of ammonium chloride in the hydrolysis of the Grignard product is responsible for the halogenated impurities.

Although ammonium chloride may affect the extent of the side reaction which yields the halogenated impurities, a simple metathesis of the carbinols and the ammonium chloride does not appear to be the source of the impurities. For example, in the preparation of methylethylcyclopropyl-

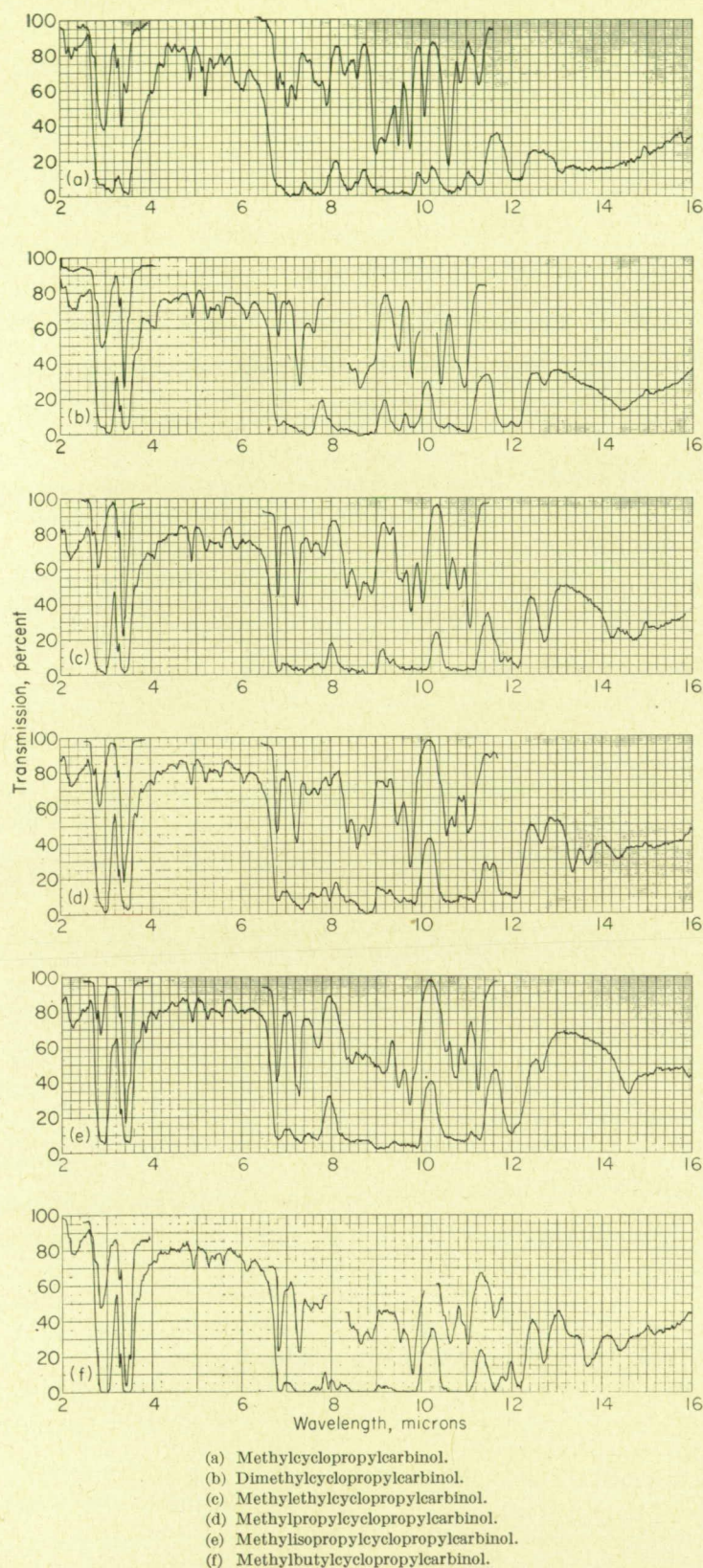


FIGURE 3.—Infrared spectra of cyclopropylcarbinols. Liquid phase; 0.1-millimeter cell. Upper trace, diluted 1:10 with carbon tetrachloride; lower trace, undiluted.

carbinol, the halogenated impurity was isolated and the elemental analysis of it corresponded not to a chloride, but to a bromide, $C_7H_{13}Br$. The infrared spectrum of the impurity indicated the basic structure to be that of a type IV olefin ($R,R'C=CHR''$) rather than a cyclopropane derivative. The impurity was presumed to be 1-bromo-4-methyl-3-hexene; a similar structure was obtained when dimethylcyclopropylcarbinol was treated with hydrogen halide (ref. 40). The halogenated impurities were removed from the carbinols by refluxing them with alcoholic sodium hydroxide, removing the precipitated sodium halide by extraction with water, and fractionating the carbinol at reduced pressure.

The physical properties of the cyclopropylcarbinols are compared in the following table with those properties previously reported:

Cyclopropylcarbinol	Melting point, °C	Boiling point, °C	n_D^{20}	d_4^{20} , g/ml	Reference
Methyl-----	-31.04	122.5	1.4316	0.8886	33
Literature-----	-32.1	123.5	1.4316	.8893	29
Dimethyl-----	(b)	123.7	1.4335	.8789	34
Literature-----	-43.0	123.4	1.4337	.8842	28
Methylethyl-----	(c)	142.3	1.4412	.8807	34
Literature-----		143.4-143.8 (776 mm)	1.44163	.88565	26
Methylpropyl-----	(c)	163.5	1.4438	.8763	34
Literature-----		163.5-163.9 (761 mm)	1.44344	.87721	26
Methylbutyl-----	(c)	183.7	1.4466	.8735	34
Literature-----		182-183 (752 mm)	1.44514	.87447	26
Methylisopropyl-----	(c)	160.3	1.4465	.8856	35

^a Pressure, 760 mm Hg unless otherwise noted.

^b Equilibrium melting curves could not be obtained.

^c Formed glasses.

The infrared spectra of the cyclopropylcarbinols are shown in figure 3; characteristic absorption for the hydroxyl group is observed between 2.8 and 3.0 microns and for the cyclopropyl group, between 9.75 and 9.80 microns.

CYCLOPROPYLALKENES

The cyclopropylalkenes were prepared by dehydrating the appropriate alkyl- or methylalkylcyclopropylcarbinols. Two methods of dehydrating the carbinols were investigated:

Alumina.—The carbinol (in some cases dissolved in toluene) was passed at a rate of 5 to 10 milliliters per minute through a 2.5- by 120-centimeter pyrex tube which was packed with 8 to 14 mesh alumina and heated to between 200° and 300° C. A sketch of the apparatus is shown in figure 1.

The yields of products and the reaction conditions for the dehydration of methylcyclopropylcarbinol are summarized in the following table (ref. 36):

Column temperature, °C	Rate, ml/min	Yield of products, percent				
		Propene	Isoprene	Vinylcyclopropane	1,3-Pentadienes	2-Methyltetrahydrofuran
265-280-----	10	1	1	52	2	4
285-300-----	10	4	3	54	5	4
285-300-----	5	6	3	38	8	3

In each of these experiments the carbinol was dissolved in an approximately equal volume of toluene, and the solution was passed through the dehydration column.

It can be seen from the data presented in the table that although increasing the temperature from the range 265° to 280° C to the range 285° to 300° C increased the yields of products to some extent, the change in rate of introducing the solution of carbinol into the dehydration column had a much greater effect on the yields of products. At the lower rate the yield of vinylcyclopropane was significantly reduced, and the yields of other products except the tetrahydrofuran and isoprene were increased. The decomposition and isomerization of vinylcyclopropane in the presence of alumina has not been investigated; therefore, it is not known whether the increase in the yields of propene and 1,3-pentadienes at the lower rate is caused by decomposition and isomerization of vinylcyclopropane or by isomerization of the methylcyclopropylcarbinol and subsequent dehydration of the resultant carbinol.

Several methylalkylcyclopropylcarbinols were also dehydrated by passing the pure carbinol or, in the case of dimethylcyclopropylcarbinol, a toluene solution of the carbinol, over alumina at a rate of 5 milliliters per minute and at temperatures between 200° and 250° C. With the exception of dimethylcyclopropylcarbinol, the dehydration of all

the methylalkylcyclopropylcarbinols investigated gave mixtures of 2-cyclopropyl-1- and 2-alkenes. Dimethylcyclopropylcarbinol gave in addition to 2-cyclopropylpropene small amounts of methylpentadienes and 2,2-dimethyltetrahydrofuran. The yields of 2-cyclopropylalkenes from each of the carbinols are given in the following table:

Cyclopropyl-carbinol	Column temperature, °C	Cyclopropyl-1-alkene	Yield, percent	Cyclopropyl-2-alkene	Yield, percent	Reference
Dimethyl.....	200-250	2-Cyclopropylpropene.	57	-----	-----	34
Methylethyl...	200-250	2-Cyclopropyl-1-butene.	32	2-Cyclopropyl-2-butene.	47	34
Methylpropyl...	225-250	2-Cyclopropyl-1-pentene.	39	2-Cyclopropyl-2-pentene.	38	34
Methylbutyl...	225-250	2-Cyclopropyl-1-hexene.	41	2-Cyclopropyl-2-hexene.	37	34
Methylisopropyl-	200-250	2-Cyclopropyl-3-methyl-1-butene.	64	2-Cyclopropyl-3-methyl-2-butene.	0	35

These data indicate that in general the yields of the 2-cyclopropyl-1-alkenes increase as the length of the alkyl chain increases, while the yields of the 2-cyclopropyl-2-alkenes decrease. The presence of branching at the carbon adjacent to the hydroxyl group also increases the yield of the 1-alkene and decreases the yield of the 2-alkene.

Sulfuric acid.—In the dehydrations of the methylalkylcyclopropylcarbinols with concentrated sulfuric acid, 6 to 10 moles of the carbinol with 0.4 to 0.8 milliliters of acid was heated to reflux in a flask attached to a 2.2- by 160-centimeter fractionating column which was packed with 1/8-inch glass helices. The dehydration products were removed through a distilling head at the top of the column as they formed. The only products obtained by this method were the 2-cyclopropyl-1- and 2-alkenes. The quantities of reactants and the yields of products are summarized in the following table (ref. 34):

Cyclopropyl-carbinol	Carbinol, moles	Acid, ml	2-Cyclopropyl-1-alkene	Yield, percent	2-Cyclopropyl-2-alkene	Yield, percent
Dimethyl.....	6	0.4	-propene.....	69	-----	-----
Methylethyl.....	8	.8	-butene.....	23	-butene.....	56
Methylpropyl.....	10	.4	-pentene.....	19	-pentene.....	56
Methylbutyl.....	10	.8	-hexene.....	21	-hexene.....	57

From these data the length of the alkyl chain does not appear to have a significant effect on the yields of the 2-cyclopropyl-1- and 2-alkenes.

A dependence of the yields of products on the method of dehydrating the methylalkylcyclopropylcarbinols is illustrated by the distillation data presented in figure 4. In the case of dimethylcyclopropylcarbinol, the dehydration with sulfuric acid yielded only the 2-cyclopropylpropene, whereas the dehydration with alumina also gave small quantities of a product of higher refractive index (probably methylpentadienes) and a product of lower refractive index (2,2-dimethyltetrahydrofuran). In the case of the other carbinols, the dehydration with sulfuric acid did not give so large a proportion of the 2-cyclopropyl-1-alkene as did the dehydration with alumina.

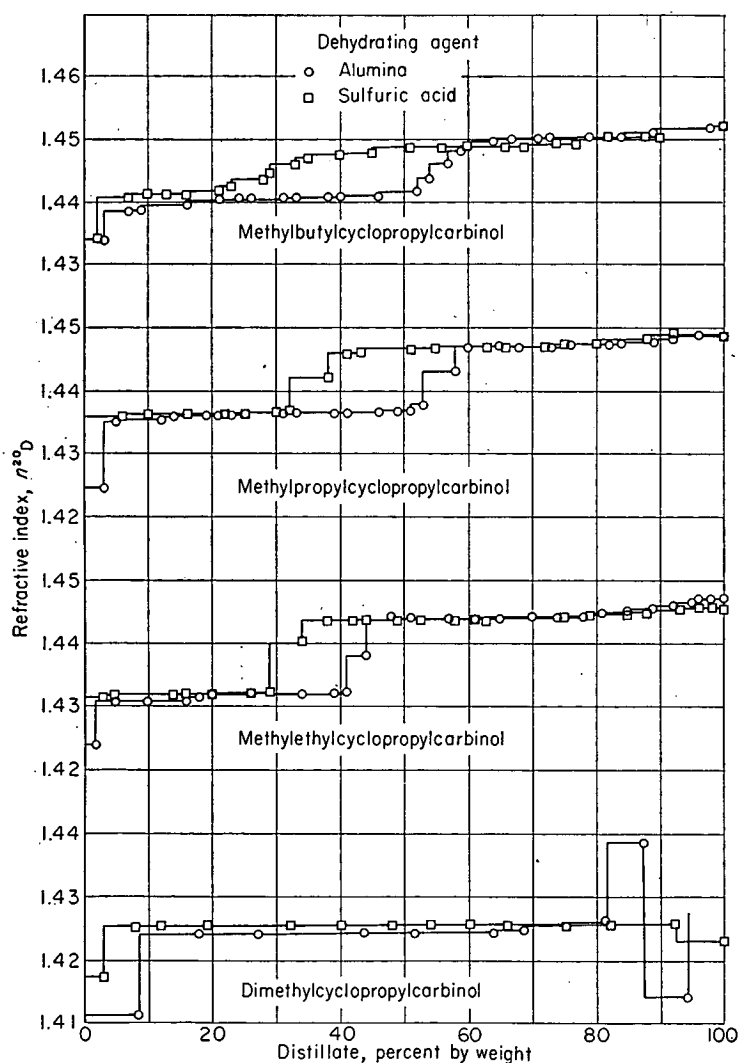


FIGURE 4.—Distillation of products from dehydration of methylalkylcyclopropylcarbinols.

Distillation data are also presented for the products from the dehydration of methylcyclopropylcarbinol (fig. 5) and for the dehydration of methylisopropylcyclopropylcarbinol (fig. 6) with alumina. From methylcyclopropylcarbinol, propene and isoprene were obtained at the beginning of the fractionation and 1,3-pentadienes and 2-methyltetrahydrofuran at the end of the fractionation. From methylisopropylcyclopropylcarbinol, only the 1-alkene (2-cyclopropyl-3-methyl-1-butene) and a high-boiling residue were obtained.

With the exception of 2-cyclopropylpropene and 2-cyclopropyl-3-methyl-1-butene, the cyclopropylalkenes were purified by azeotropic fractionations with appropriate entrainers in the Podbielniak columns after preliminary fractionations at efficiencies of 50 to 60 theoretical plates. Azeotropic fractionation was found to be necessary in order to separate geometrical isomers and close-boiling impurities from the cyclopropylalkenes. In the following table pertinent physical properties of the hydrocarbons, the entrainers, and the azeotropes are given:

Cyclopropylalkene	n_D^{20}	Entrainer	n_D^{20}	Azeotrope	
				n_D^{20}	Boiling point, °C
Vinylcyclopropane.....	1.4138	Ethanol.....	1.3614	1.4105	39
2-Cyclopropyl-1-butene.....	1.4319	do.....	1.3614	1.3975	73
2-Cyclopropyl-2-butene (i. b.).....	1.4428	do.....	1.3614	1.3950	75
2-Cyclopropyl-2-butene (h. b.).....	1.4474	do.....	1.3614	1.3957	76
2-Cyclopropyl-1-pentene.....	1.4362	Propanol.....	1.3854	1.4036	95
2-Cyclopropyl-2-pentene (i. b.).....	1.4458	do.....	1.3854	1.4029	96
2-Cyclopropyl-2-pentene (h. b.).....	1.4502	do.....	1.3854	1.4023	96
2-Cyclopropyl-1-hexene.....	1.4403	Cellosolve.....	1.4079	1.4215	131
2-Cyclopropyl-2-hexene (i. b.).....	1.4486	do.....	1.4079	1.4239	132
2-Cyclopropyl-2-hexene (h. b.).....	1.4529	do.....	1.4079	1.4249	132

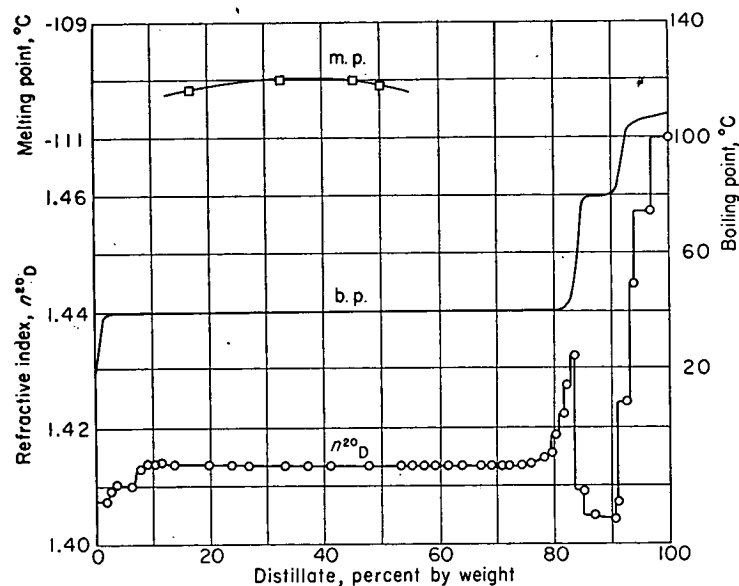


FIGURE 5.—Distillation of products from dehydration of methylcyclopropylcarbinol over alumina at 265° to 280° C.

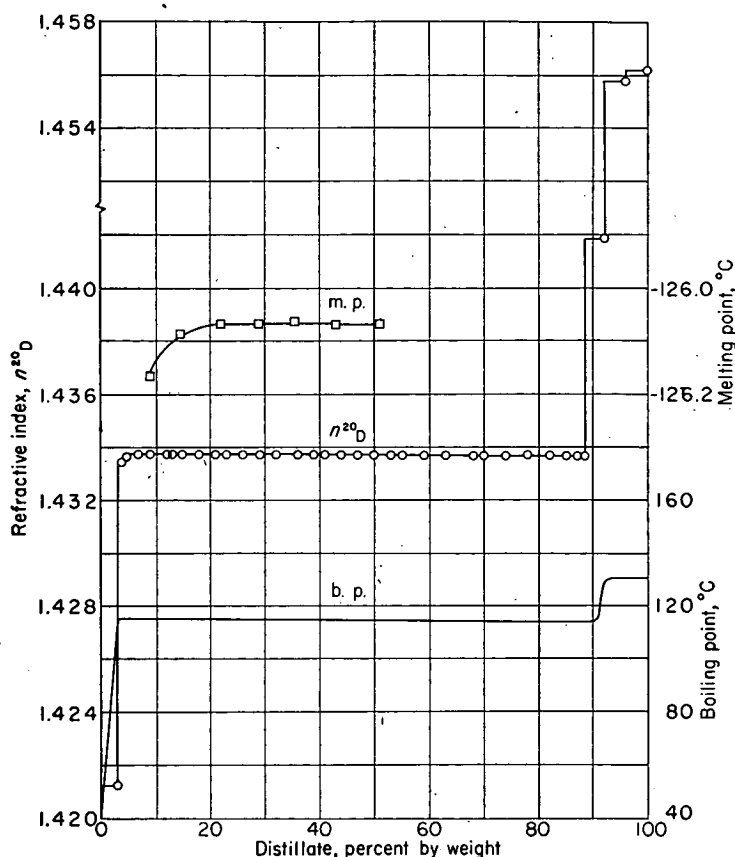


FIGURE 6.—Distillation of products from dehydration of methylisopropylcyclopropylcarbinol over alumina. Distillate, 1120 grams.

The physical properties of the cyclopropylalkenes are given in the following table. In those instances in which the compound has been previously reported, reference is made to the properties obtained by other investigators.

Cyclopropylalkene	Melting point, °C	Boiling point ^a , °C	n_D^{20}	d_4^{20} , g/ml	Net heat of combustion, kcal/mole
Vinylcyclopropane.....	-109.82	40.19	1.4138	0.72105	720
Ref. 29.....	-112.6	40.41	1.4156	.7160	-----
Ref. 21.....	-----	40.0-40.2 (755 mm)	1.4172	.723(18°C)	-----
2-Cyclopropylpropene.....	-102.34	70.33	1.4255	.75153	875
Ref. 28.....	-102.36	70.41	1.4254	.7514	-----
Ref. 20.....	-----	69.5-70.0 (751 mm)	1.42524	.74999	-----
2-Cyclopropyl-1-butene.....	^b -119.55	98.57	1.4319	.76820	1,025
Ref. 20.....	-121.94	103.5-103.8	1.43901	.7772	-----
2-Cyclopropyl-2-butene (i. b.).....	-97.83	106.55	1.4428	.78100	1,020
2-Cyclopropyl-2-butene (h. b.).....	-74.07	107.46	1.4474	.78745	-----
Ref. 20.....	-----	105.5-106	1.44253	.7804	-----
2-Cyclopropyl-1-pentene.....	-113.87	123.94	1.4362	.77660	1,165
2-Cyclopropyl-2-pentene (i. b.).....	-113.65	128.61	1.4458	.78402	1,165
2-Cyclopropyl-2-pentene (h. b.).....	-107.61	129.98	1.4502	.79077	-----
2-Cyclopropyl-1-hexene.....	-106.16	148.89	1.4403	.78447	1,320
2-Cyclopropyl-2-hexene (i. b.).....	Glass	152.08	1.4486	.79051	-----
2-Cyclopropyl-2-hexene (h. b.).....	-97.40	153.08	1.4529	.79666	1,305
2-Cyclopropyl-3-methyl-1-butene.....	-126.06	116.81	1.4337	.77395	1,160

^aPressure, 760 mm Hg unless otherwise stated.

^bTwo crystalline modifications were observed.

The estimated melting points for zero impurity, the depressions in melting points per mole percent of added impurity, and the calculated purities of some of the cyclopropylalkenes are given in the following table:

Hydrocarbon	Observed melting point, °C	Estimated melting point for zero impurity, °C	Change in melting point per mole percent of added impurity, °C	Calculated purity, mole percent
Vinylcyclopropane.....	-109.82	-109.81	b 0.28	99.9
2-Cyclopropylpropene.....	-102.34	-102.33	c. 25	99.9
2-Cyclopropyl-1-butene.....	-119.55	-119.50		
2-Cyclopropyl-2-butene (l. b.).....	-97.83	-97.62		
2-Cyclopropyl-2-butene (h. b.).....	-74.07	-74.03		
2-Cyclopropyl-1-pentene.....	-113.87	-113.83		
2-Cyclopropyl-2-pentene (l. b.).....	-113.65	-113.55		
2-Cyclopropyl-2-pentene (h. b.).....	-107.61	-107.43		
2-Cyclopropyl-3-methyl-1-butene.....	-126.06	-125.98		
2-Cyclopropyl-1-hexene.....	-106.16	-106.09		

a Determined by the geometrical construction of reference 41.

b Determined by adding known amounts of 1-*trans*-3-pentadiene.

c Value taken from reference 28.

Although the purities of only the first two cyclopropylalkenes were calculated, it is believed that the purities of the others listed in the table are better than 99 mole percent, because the differences in the estimated melting points for zero impurity and the observed melting points are small. The geometrical isomers of 2-cyclopropyl-2-hexene are not included in the table, because the lower-boiling isomer could not be crystallized and the melting curve of the higher-boiling isomer was not of sufficient duration to make a reliable estimate of the melting point for zero impurity.

The infrared spectra of the cyclopropylalkenes from 2 to 16 microns are shown in figure 7. It can be seen that for those molecules having a terminal C=C, the absorption for the double-bond stretching frequencies occurs at 6.1 ± 0.02 microns; whereas for those molecules having an internal C=C, the absorption occurs at a lower wavelength, 6.02 ± 0.02 microns. The intensity of the absorption is much greater for the terminal double bond than for the internal double bond. Characteristic absorption for the cyclopropyl ring occurs at 9.78 microns in those molecules having a terminal double bond; in those molecules having an internal double bond the absorption for the cyclopropyl ring occurs at a slightly higher wavelength, 9.81 microns.

ALKYL CYCLOPROPYL KETONES

The alkyl cyclopropyl ketones were obtained as fragmentation products from the ozonolysis of the 2-cyclopropyl-1-alkenes. The methods of ozonolysis and hydrogenation of the ozonide have been described in reference 32. In general, the olefin was dissolved in 100 to 150 milliliters of absolute alcohol and an oxygen-ozone mixture containing from 5 to 10 percent ozone was passed through the solution until all the olefin had been converted to ozonide. The ozonide solution was then transferred to a low-pressure hydrogenation apparatus, and the ozonide decomposed with hydrogen in the presence of a palladium catalyst to give the alkyl cyclopropyl ketone and formaldehyde. The alkyl cyclopropyl ketones were identified by their physical properties and by analysis of their 2, 4-dinitrophenylhydrazone derivatives. The quantities of olefin used, the yields of the alkyl cyclopropyl ketones,

and the melting points of the 2,4-dinitrophenylhydrazone derivatives are given in the following table:

2-Cyclopropyl-1-alkene a	Olefin, moles	Alkyl cyclopropyl ketone	Yield, percent	Melting point of 2,4-dinitrophenylhydrazone, °C	Reference
-propene.....	0.3	Methyl.....	(b)	148.6-149.2	34
-butene.....	.28	Ethyl.....	39	160.4-160.5	34
-pentene.....	.48	Propyl.....	43	165.0-165.5	34
-hexene.....	.3	Butyl.....	75	114.5-115.0	34
-3-methyl-1-butene.....	.3	Isopropyl.....	42	187.5-188.0	35

a The structures of the 2-cyclopropyl-2-alkenes were also proved by ozonolysis; only those alkenes yielding cyclopropyl ketones are listed here.

b Paraformaldehyde sublimed at the boiling temperature of the ketone making it difficult to determine the yield of the ketone.

The infrared spectra of the alkyl cyclopropyl ketones are shown in figure 8. All show strong absorption at 5.9 microns, which is characteristic of the carbonyl group. With the exception of methyl cyclopropyl ketone, all show characteristic absorption for the cyclopropyl ring between 9.75 and 9.80 microns; cyclopropyl ring absorption in methyl cyclopropyl ketone occurs at a lower wavelength, 9.69 microns.

2-CYCLOPROPYLALKANES

The 2-cyclopropylalkanes were prepared by hydrogenating the 2-cyclopropyl-1- and -2-alkenes in the presence of a barium-promoted copper chromite catalyst. The 1-alkenes and 2-alkenes were hydrogenated separately, because the position of the double bond was found to affect the ease of hydrogenation and the yields of products. In general, the olefin, an equal volume of ethanol, and a quantity of the catalyst equal to 10 percent of the weight of the olefin were put into the hydrogenator, and hydrogen was admitted to between 1500 and 1800 pounds per square inch gage. The hydrogenator rocking mechanism and the resistance heaters were turned on, and the vessel was heated to 100° C. The 2-cyclopropyl-1-alkenes hydrogenated readily at this temperature, the heat of reaction generally raising the temperature to between 120° and 130° C. The 2-cyclopropyl-2-alkenes hydrogenated sluggishly even at 130° C, and, in some instances, the reaction temperature was increased to as high as 175° C in order to obtain more rapid hydrogenation. The reaction conditions and the yields of products are summarized in the following table:

Olefin	Reaction conditions		Estimated composition of products				
	Temperature, °C	Initial H ₂ pressure, lb/sq in. gage	2-Cyclopropylalkane	Weight, percent	Methylalkane	Weight, percent	Reference
2-Cyclopropylalkene							
-propene.....	100-130	1,500	-propane....	98	2-Methylpentane	2	37
-1-butene.....	100-130	1,600	-butane.....	99	3-Methylhexane	1	37
-2-butene.....	110-130	1,500	do.....	72	do.....	a 15	37
-1-pentene.....	100-110	1,500	-pentane.....	99	4-Methylheptane	1	37
-2-pentene.....	120-130	1,750	do.....	71	do.....	a 17	37
-1-hexene.....	100-120	1,500	-hexane.....	99	4-Methyloctane	1	37
-2-hexene.....	150-175	1,500	do.....	79	do.....	a 16	37
-3-methyl-1-butene.....	100-140	1,850	-3-methylbutane.....	99	2,3-Dimethylhexane.....	1	35

a Unhydrogenated olefin was found in the product from each of the 2-alkenes.

These data and the distillation curves shown in figure 9 indicate that the hydrogenation of the 2-cyclopropyl-1-alkenes yields the corresponding 2-cyclopropylalkanes in

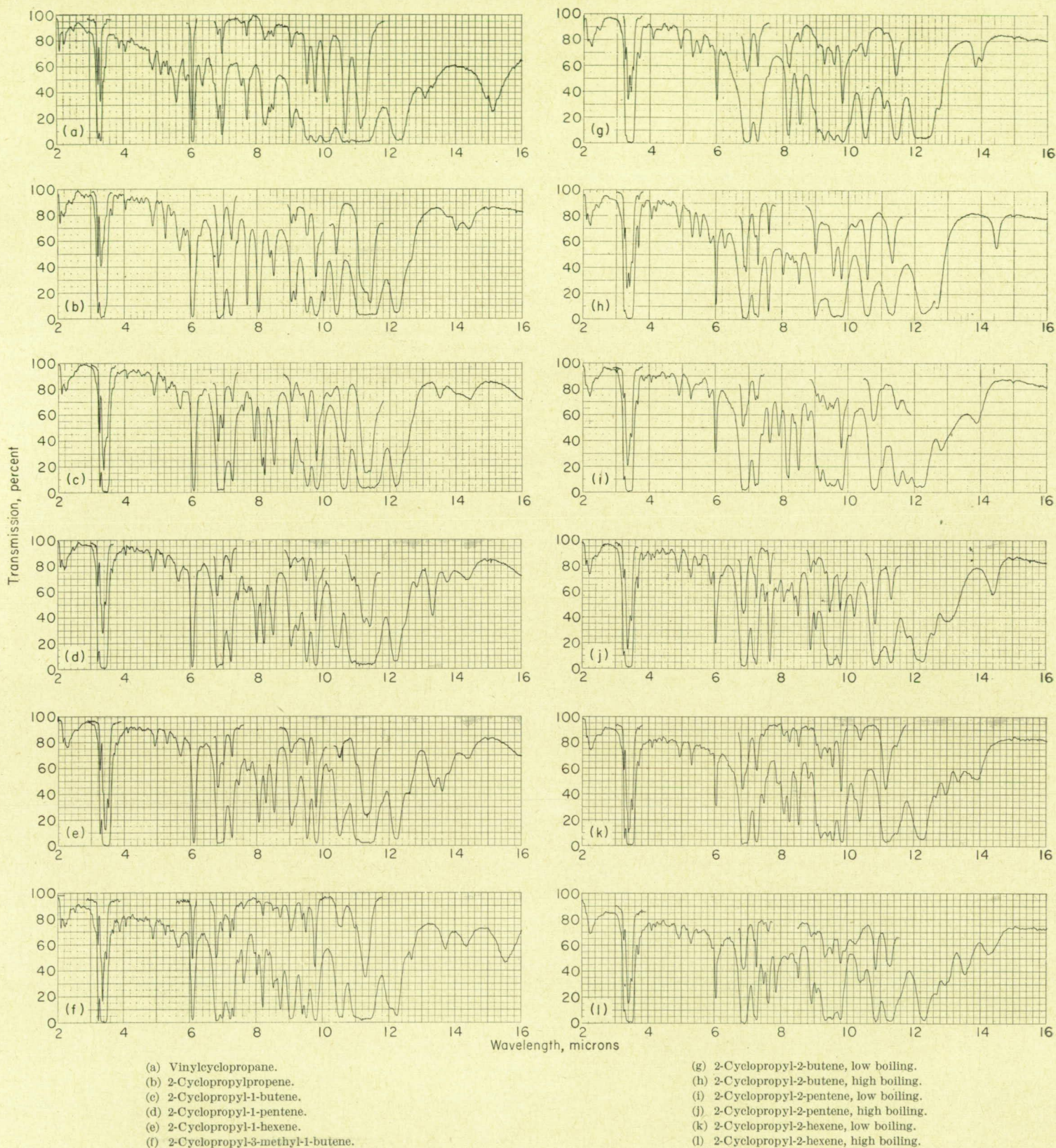


FIGURE 7.—Infrared spectra of cyclopropylalkenes. Liquid phase; 0.1-millimeter cell. Upper trace, diluted 1:10 with carbon tetrachloride; lower trace, undiluted.

purities between 98 and 99 percent, whereas the hydrogenation of the 2-cyclopropyl-2-alkenes yields also significant quantities of paraffinic hydrocarbons. The paraffinic hydrocarbons do not appear to result from hydrogenolysis of the cyclopropyl ring, because if this were the case, larger amounts

of the paraffinic hydrocarbons would be obtained also from the hydrogenations of the 2-cyclopropyl-1-alkenes. The data support the proposal made in reference 28 that systems in which a cyclopropyl ring is conjugated with a double bond can add hydrogen by either a 1,2- or a 1,4-mechanism.

Purification of the 2-cyclopropylalkanes was difficult in those experiments in which methylalkanes were formed, because the boiling points of the paraffinic hydrocarbons

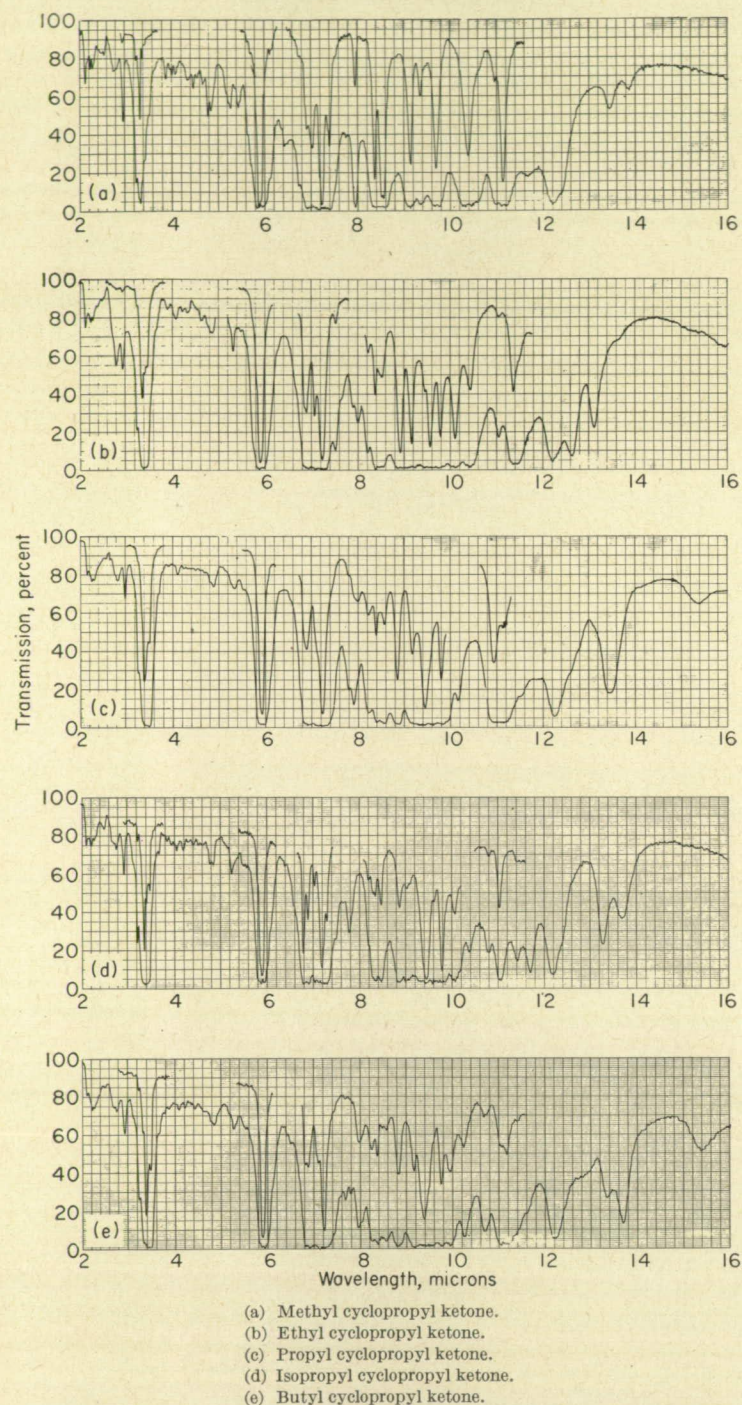


FIGURE 8.—Infrared spectra of alkyl cyclopropyl ketones. Liquid phase; 0.1-millimeter cell. Upper trace, diluted 1:10 with carbon tetrachloride; lower trace, undiluted.

were within 2° C of the corresponding cyclopropylalkanes. Azeotropic fractionations with appropriate entrainers were found to be effective for separating the close-boiling mixtures. Pertinent data for the azeotropic mixtures are given in the following table:

Cyclopropylalkane	n_D^{20}	Entrainer	n_D^{20}	Azeotrope	
				n_D^{20}	Boiling point, °C
2-Cyclopropylbutane.....	1.4024	Ethanol.....	1.3614	1.3862	70
2-Cyclopropylpentane.....	1.4112	Propanol.....	1.3854	1.3973	93
2-Cyclopropylhexane.....	1.4178	Cellosolve.....	1.4079	1.4117	129
2-Cyclopropylhexane.....	1.4178	Butanol.....	1.3992	1.4056	116
2-Cyclopropyl-3-methylbutane.....	1.4140	Propanol.....	1.3854	1.3999	92

Purifications of the 2-cyclopropylalkanes were considered complete when (a) the melting curves of selected samples were interpreted to be indicative of purities of better than

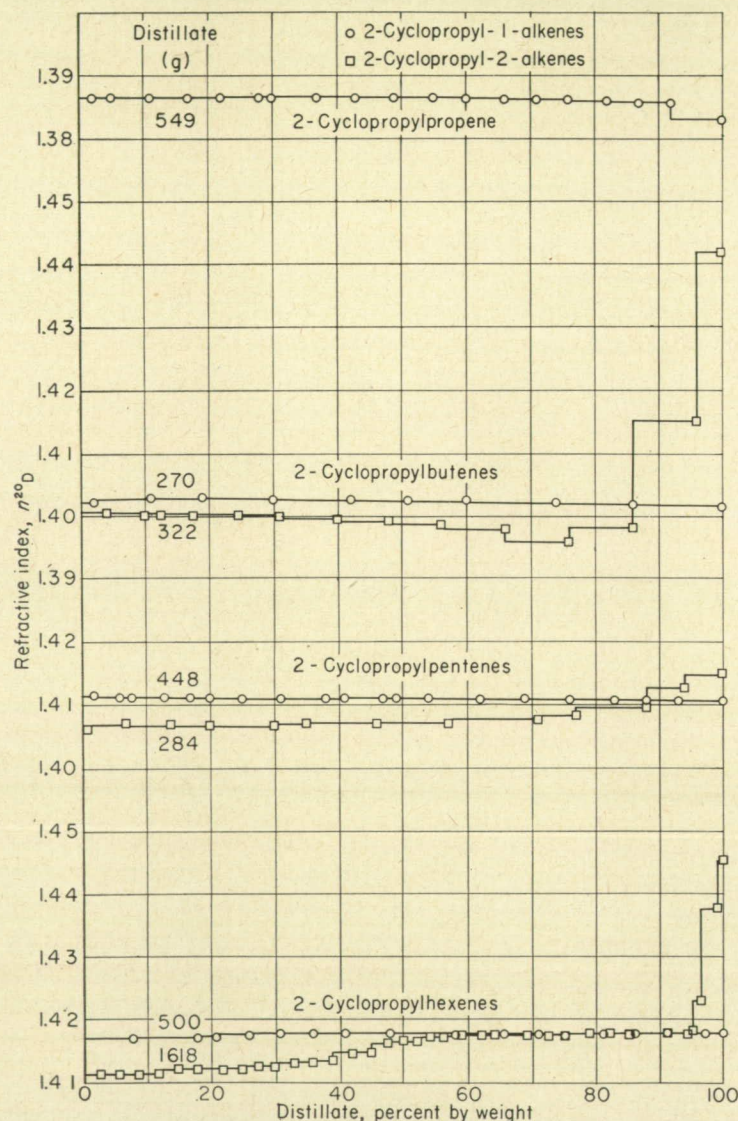


FIGURE 9.—Distillation of products from hydrogenations of 2-cyclopropylalkenes.

99 mole percent, or (b) repeated fractionations and azeotropic fractionations through columns rated at better than 150 theoretical plates gave no significant changes in refractive index and density of the hydrocarbons.

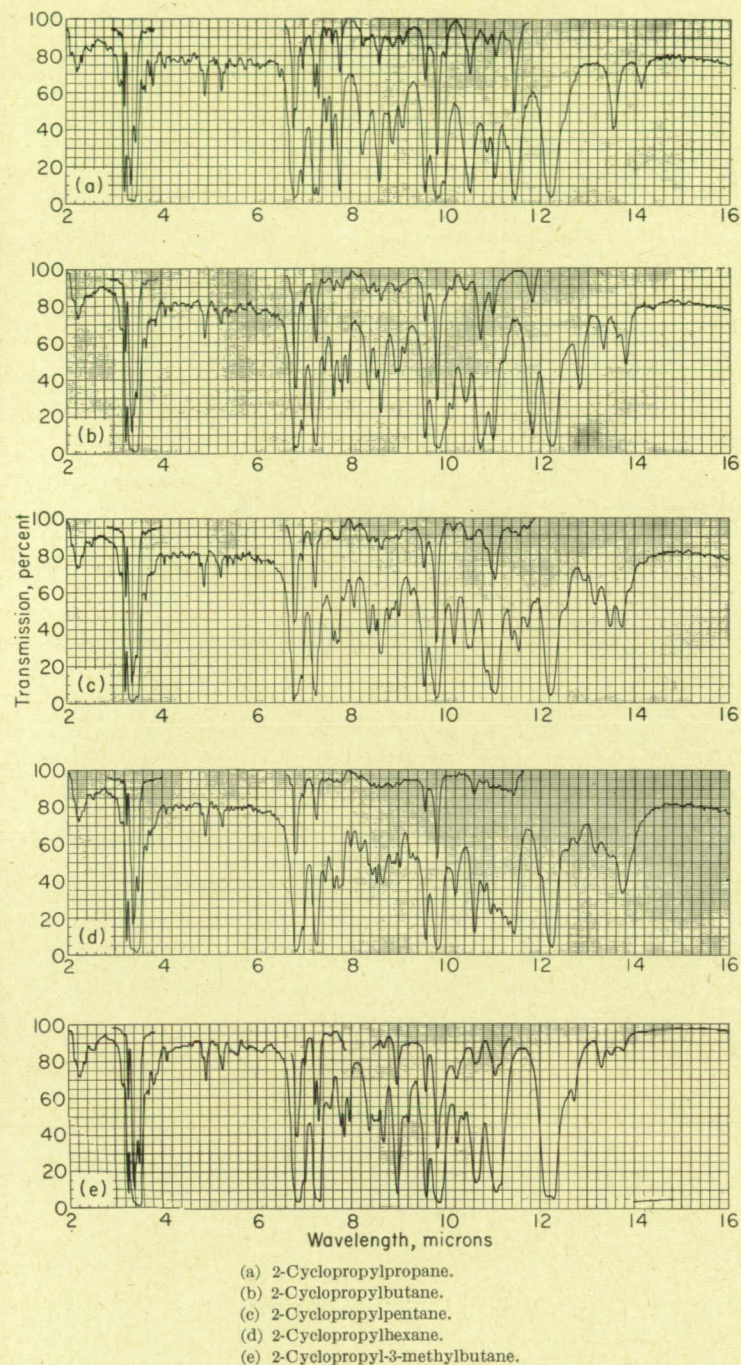


FIGURE 10.—Infrared spectra of 2-cyclopropylalkanes. Liquid phase; 0.1-millimeter cell. Upper trace, diluted 1:10 with carbon tetrachloride; lower trace, undiluted.

The physical properties of the 2-cyclopropylalkanes are presented in the following table; previously reported properties are also referenced.

Although the purities of only two of the 2-cyclopropylalkanes could be calculated, it is believed that the other three hydrocarbons are of similar purities.

The infrared spectra of the 2-cyclopropylalkanes are shown in figure 10. All the spectra show characteristic cyclopropyl ring absorption at 9.82 microns.

2-Cyclopropylalkane	Melting point, °C	Melting point for zero impurity, °C	Change in melting point per mole percent of impurity, °C	Estimated purity, mole percent	Boiling point at 760 mm Hg, °C	n_D^{20}	d_4^{20} , g/ml	Net heat of combustion, kcal/mole
-propane...	-112.97	^a -112.93	^b 0.28	99.9	58.31	1.3865	0.69858	900
Ref. 28...	-118.3	-115.24	.20	85	58.7	1.3833	.6889	
-butane...	Glass	-----	-----	-----	90.98	1.4024	.72830	1,045
-pentane...	-118.46	(^c)	-----	-----	117.74	1.4111	.74294	1,200
-hexane...	-98.04	^a -97.99	^d .20	99.7	142.95	1.4178	.75438	1,350
-3-methylbutane	Glass	-----	-----	-----	115.49	1.4140	.75026	1,200

^a Determined by geometrical method of reference 41.

^b Determined by adding known amounts of 2-methylpentane.

^c Melting curve was not of sufficient duration to make calculation valid.

^d Determined by adding known amounts of 4-methyloctane.

CYCLOPROPYL CHLORIDES

Cyclopropyl chloride, 1,1-dichlorocyclopropane, and *trans*-1,2-dichlorocyclopropane were isolated from the reaction products of the photochemical chlorination of cyclopropane. A sketch of the photochlorination apparatus is shown in figure 2. In a typical reaction the source of illumination (two sun lamps) was placed 11.5 centimeters from the reaction chamber and the cyclopropane was passed through the reaction chamber at a rate of about 0.12 mole per minute. Chlorine was then added at a rate of about 0.046 mole per minute. The gases were passed through the reaction chamber simultaneously for 6 hours, and at the end of this period the amount of cyclopropane used was 44.4 moles (1866 g), and the amount of chlorine 16.2 moles (1148 g). The excess cyclopropane was distilled from the products, and the products were then combined with the products from eleven similar chlorination experiments. Distillation of the combined products gave the data plotted in figure 11.

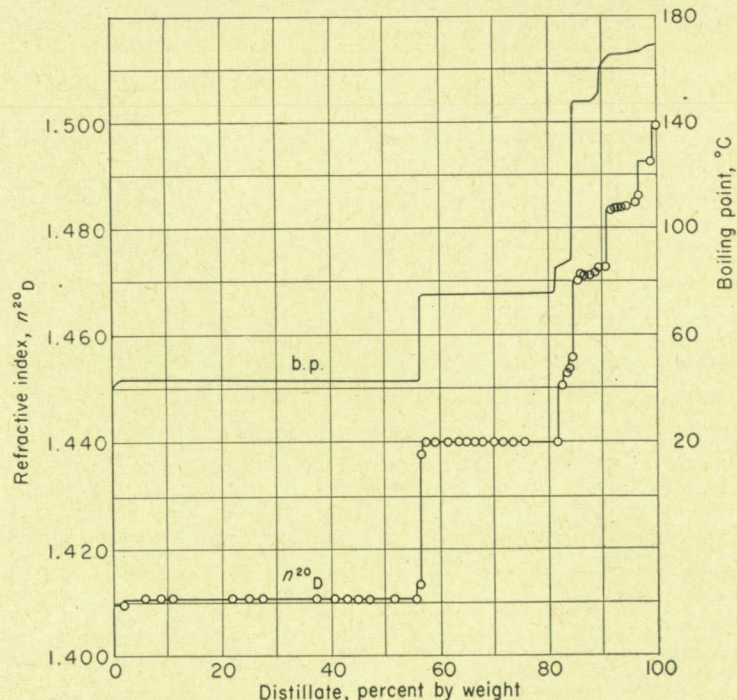


FIGURE 11.—Distillation of products from photochlorination of cyclopropane. Total distillate, 8155 grams; residue, 615 grams.

From these data and subsequent fractional distillation data, the following composition of the chlorination products was estimated: 52 weight percent cyclopropyl chloride, 24 weight percent 1,1-dichlorocyclopropane, 2 weight percent *trans*-1,2-dichlorocyclopropane, 2 weight percent of a compound believed from its physical properties to be 1,1,3-trichloropropane, and 20 weight percent of a complex mixture of polyhalides which could not be separated by fractional distillation.

The cyclopropyl halides were purified by refractionation at efficiencies of 50 to 60 theoretical plates. The physical properties of the purified halides are compared in the following table with those values previously reported:

Cyclopropyl halide	Melting point, °C	Boiling point at 760 mm Hg, °C	n_D^{20}	d_4^{20} , g/ml
Cyclopropyl chloride.....	a -97.68	43.43	1.4108	0.9962
(ref. 42)		43	b 1.4079	b .9899
1,1-Dichlorocyclopropane.....	a -37.47	75.55	1.4400	1.2158
(ref. 43)		75.0	b 1.4377	b 1.2178
<i>trans</i> -1,2-Dichlorocyclopropane.....	-18.74	87.94	1.4526	1.2459
(ref. 43)	-19.5	87.2	b 1.4502	b 1.2492

a Freezing point, °C.
b At 25° C.

The infrared spectra of the three chlorinated cyclopropanes are shown in figure 12; the absorption band characteristic of

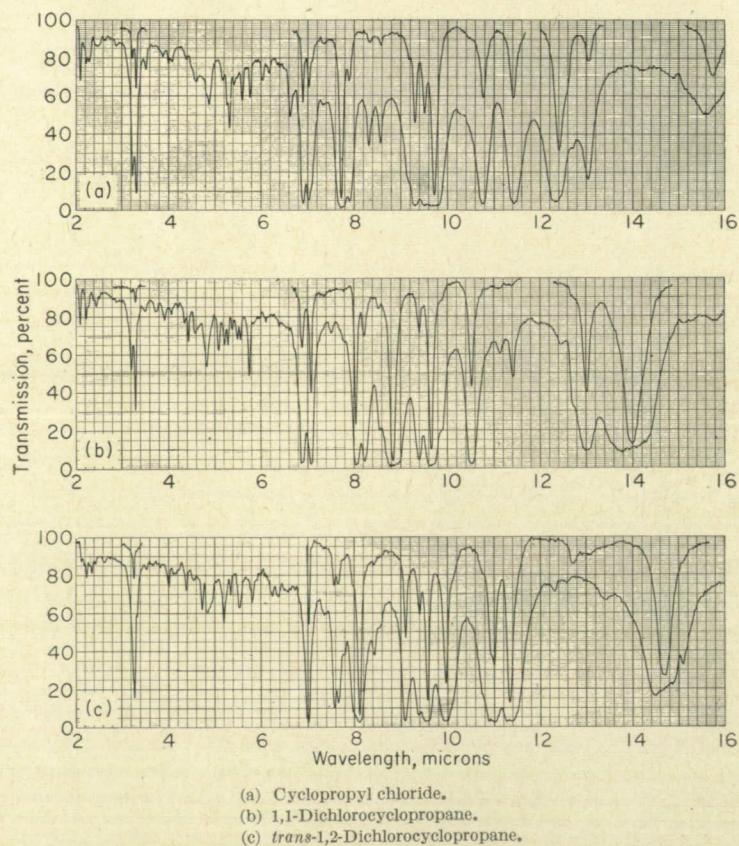


FIGURE 12.—Infrared spectra of cyclopropyl chlorides. Liquid phase; 0.1-millimeter cell. Upper trace, diluted 1:10 with carbon disulfide; lower trace, undiluted.

the cyclopropyl ring is seen to shift from 9.73 microns (cyclopropyl chloride) to 9.66 microns (1,1-dichlorocyclopropane) as the number of chlorine atoms is increased. The absorption shifts to even lower frequency when the two

chlorine atoms are on different carbon atoms (9.58 microns in *trans*-1,2-dichlorocyclopropane).

DICYCLOPROPYL

The reaction of cyclopropyl chloride with lithium or sodium was used to prepare dicyclopropyl. The reaction was tried with lithium both in methycyclohexane and in ether, and with sodium in ether. In general, the chloride was dissolved in the reaction solvent and added to a suspension of the metal in the solvent contained in a flask equipped with a stirrer, a reflux condenser, and an addition funnel. All reactions were carried out at 25° to 35° C. Other reaction conditions and yields of products are summarized in the following table (ref. 38):

Chloride, moles	Alkali metal	Metal, gram-atoms	Solvent	Reaction time, days	Yields of products, percent	
					Cyclopropane	Dicyclopropyl
2	Sodium.....	2	Ether.....	5	40	10
20	Lithium.....	20	do.....	10	42	12
4	do.....	4	Methycyclohexane.....	7	None	None

These data indicate that the reaction with sodium or lithium in ether gives approximately the same yields of the two products listed in the table. The fact that no reaction was obtained in methycyclohexane solution clearly indicates that the solvent in some manner influences the course of the reaction.

Although cyclopropane and dicyclopropyl were the principal reaction products, trace quantities of two other hydrocarbons were detected in the crude reaction product by means of infrared spectra. The infrared evidence indicated that one of these hydrocarbons was 1-hexyne; the other was not identified.

Dicyclopropyl was purified by extracting the hydrocarbon with an ice-cold saturated solution of silver nitrate to remove the unsaturated impurities, passing the extracted hydrocarbon through silica gel, and, finally, azeotropically fractionating the hydrocarbon with ethanol through a Podbielniak column. The purified hydrocarbon had the following physical properties:

Melting point, °C.....	-82.62
Boiling point at 760 mm Hg, °C.....	76.10
Refractive index, n_D^{20}	1.4239
Density, g/ml.....	0.78979
Net heat of combustion, kcal/mole.....	885

The infrared spectrum of dicyclopropyl, shown in figure 13, has a strong absorption band at 9.82 microns, which is characteristic of the cyclopropyl ring.

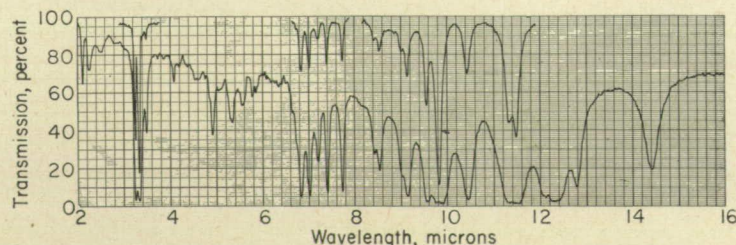


FIGURE 13.—Infrared spectrum of dicyclopropyl. Liquid phase; 0.1-millimeter cell. Upper trace diluted 1:10 with carbon tetrachloride; lower trace, undiluted.

SPIROPENTANE

Spiropentane was prepared by the debromination of pentaerythrityl tetrabromide (ref. 44) with zinc dust in the presence of sodium carbonate and sodium iodide. The reaction was conducted by adding small portions of the powdered tetrabromide to a slurry of the zinc dust, sodium carbonate, and sodium iodide either in molten acetamide or in 75 percent ethanol (ref. 39). The hydrocarbon product was distilled from the reaction mixture as it formed, and was collected in receivers chilled with a mixture of solid carbon dioxide and acetone. In addition to spiropentane, two other hydrocarbons were obtained from the reaction, namely, methylenecyclobutane and 2-methyl-1-butene. Some typical experiments are summarized in the following table (ref. 39):

Solvent	Reactants, moles				Yields of products, percent		
	Pentaerythrityl tetrabromide	Zinc	Sodium carbonate	Sodium iodide	Spiropentane	Methylenecyclobutane	2-Methyl-1-butene
Ethanol.....	5	20	5	0.83	21	43	12
Do.....	5	20	5	.83	22	44	12
Acetamide.....	1	6	1.2	.16	22	4	11

These data indicate that the reduction in either solvent gives nearly identical yields of spiropentane, but that the reduction in ethanol gives ten times the quantity of methylenecyclobutane as the reduction in acetamide. For preparations of sufficient size so that precise fractional distillations can be employed to separate the products, reduction in ethanol is the preferable method, especially if methylenecyclobutane as well as spiropentane is desired. For small preparations of spiropentane, the reduction in acetamide may be more suitable (ref. 45).

The physical properties of spiropentane are given in the following table with those values previously obtained:

Hydrocarbon	Freezing point, °C	Boiling point		n_D^{20}	d_4^{20} , g/ml
		Temperature, °C	Pressure, mm Hg		
Spiropentane.....	-107.05	38.84	760	1.4122	0.7551
Ref. 45.....	-----	38.3-38.5	750	1.4117	.755

The infrared spectrum of spiropentane is shown in figure 14. The absorption band characteristic of the cyclopropyl ring is located at 9.54 microns, a shorter wavelength than in

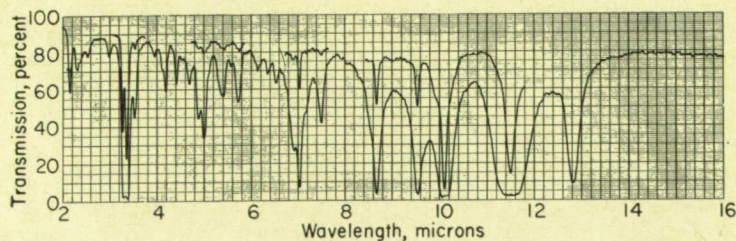


FIGURE 14.—Infrared spectrum of spiropentane. Liquid phase; 0.1-millimeter cell. Upper trace, diluted 1:10 with carbon tetrachloride; lower trace, undiluted.

any of the other compounds prepared in the present investigation. Such a shift in the absorption is not entirely unexpected, because the force constants in the spiro structure would be different from those in the other cyclopropanes.

DIETHYL CYCLOPROPANE-1,1-DICARBOXYLATE

The diethyl cyclopropane-1,1-dicarboxylate was prepared by the method of reference 46 in which a solution of sodium ethoxide, obtained by dissolving 365 grams (16.5 g-atoms) of sodium metal in 5500 milliliters of absolute ethanol, was added dropwise to a mixture of 1275 grams (8 moles) of ethyl malonate and 1550 grams (7.2 moles) of ethylene bromide. The products of the reaction were distilled at 10 millimeters of mercury pressure, and the material boiling between 80° and 120° C was collected for subsequent fractionation at about 50 theoretical plate efficiency at atmospheric pressure. The yield of diethyl cyclopropane-1,1-dicarboxylate was 495 grams (33 percent). Several fractions of constant refractive index were combined and passed through silica gel in order to obtain the sample used for the determination of physical properties and infrared spectrum. The physical properties are compared in the following table with values previously reported:

	Melting point, °C	Boiling point		n_D^{20}	d_4^{20} , g/ml
		Temperature, °C	Pressure, mm Hg		
Diethyl cyclopropane-1,1-dicarboxylate, Ref. 47.....	-47.05	217.4	760	1.4340	1.0595
-----	-----	114	22	1.43310	1.0615

The infrared spectrum of diethyl cyclopropane-1,1-dicarboxylate is shown in figure 15; strong absorption is

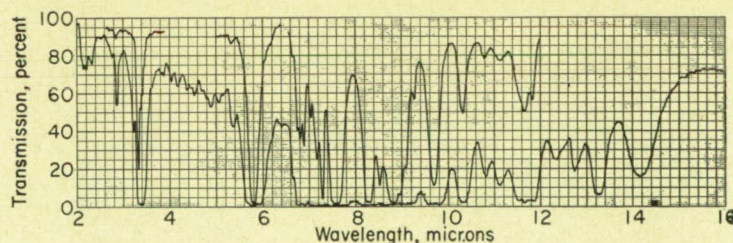


FIGURE 15.—Infrared spectrum of diethylcyclopropane-1,1-dicarboxylate. Liquid phase; 0.1-millimeter cell. Upper trace, diluted 1:10 with carbon tetrachloride; lower trace, undiluted.

observed at 5.8 microns, the region in which carbonyl groups absorb strongly, and also at 9.70 microns. The 9.70-micron band is undoubtedly that band characteristic of the cyclopropyl ring.

INFRARED ABSORPTION CHARACTERISTIC OF CYCLOPROPYL RING

The difficulty in establishing the presence of the cyclopropyl ring in organic molecules by chemical means has promoted interest in identifying the presence of the ring by infrared techniques. Infrared absorption bands in three regions of the spectrum have been proposed to offer a means of identifying the cyclopropyl ring. In reference 4 bands at 9.75 and 11.55 microns were employed. The authors of reference 48 found that in the spectra of 14 cyclopropane

hydrocarbons a strong band was present between 9.8 and 10.0 microns, but that strong absorption did not occur consistently at 11.6 microns. It was reported recently in reference 49 that a study of the 3- to 4-micron region with a lithium fluoride prism disclosed that all the cyclopropane derivatives investigated had absorption bands at 3.23 and 3.32 microns which were characteristic of the C-H vibrations of the cyclopropyl ring system.

The infrared spectra of the 34 cyclopropane derivatives reported herein have been examined in each of the three regions proposed previously. Unfortunately, a critical examination of the 3- to 4-micron region could not be made in the present work, because the sodium chloride prism, employed in determining the spectra, does not give sufficient resolution to separate both the 3.23- and 3.32-micron cyclopropyl ring C-H bands from other carbon-hydrogen bands in this region. An examination of the spectra of the diluted samples does disclose that many of the cyclopropane derivatives have bands at 3.23 ± 0.02 microns and at 3.33 ± 0.02 microns. In compounds such as vinylcyclopropane, dicyclopropyl, cyclopropyl chloride, and spiro-pentane, in which there is little interference from carbon-hydrogen bands other than those of the ring, the 3.23- and 3.32-micron bands show up clearly (table I).

Nearly all the cyclopropane derivatives show strong absorption between 11 and 12 microns; however, the position of the absorption which might be characteristic of the cyclopropyl ring is difficult to determine because of the broadness of the absorption. Interfering absorption from certain olefinic structures also occurs in this region.

The absorption which appears to be most promising for determining the presence of the cyclopropyl ring is that used in reference 48. In the spectra of all the cyclopropane compounds prepared in the present investigation, a strong band was observed at 9.7 to 9.8 microns (table II). In only a few instances was the band shifted appreciably from this region, and in these instances, for example, in spiro-pentane and in *trans*-1,2-dichlorocyclopropane, the shift in absorption is not entirely unexpected. Not only is the absorption strong in the undiluted spectra, but it is persistent also upon dilution.

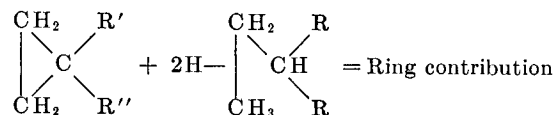
MOLECULAR REFRACTION OF CYCLOPROPANE DERIVATIVES

The experimentally observed molecular refractions of cyclopropane compounds are generally higher than those calculated from the atomic and group refractivities. This difference in observed and calculated values was interpreted by Tschugaeff (ref. 50) to result from a contribution to the refraction by the three-carbon ring structure. From a comparison of observed and calculated refractions of three cyclopropane derivatives and several bicyclic terpenes which had three-carbon rings in their structures, Tschugaeff estimated the magnitude of the ring contribution to be about 0.7. A more comprehensive investigation by Östling (ref. 51) yielded a value which agreed well with that of reference 50. Subsequent investigations, however, showed that the difference between observed and calculated molec-

ular refractions varied considerably among different classes of cyclopropane derivatives and even among members of the same class (refs. 26, 27, and 52). Data obtained in the present work (table III) also showed this variation in ΔM_R .

The lack of agreement among the ΔM_R values from various classes of cyclopropane derivatives and even among members of an homologous series is not surprising. It must be assumed in determining ring contribution by this procedure that the atomic and group refractivities are truly constant and additive. Atomic and group refractivities, however, are constant and additive only within limitations (ref. 53), and as the character of the atoms or groups in the structure is varied, the atomic and group refractivities also vary. Consequently, ΔM_R reflects not only ring contribution, but also deviations from constancy of the atomic and group refractivities used to determine the calculated molecular refractions.

In a recent investigation, Jeffery and Vogel (ref. 47) employed a procedure for determining ring contribution that reduces the effect of inconstancy of atomic and group refractivities. The observed molecular refraction of a structurally similar acyclic compound is subtracted from the sum of the observed molecular refraction of the cyclopropane compound and two hydrogen atomic refractivities:



From data obtained principally with alkyl cyclopropane mono- and dicarboxylates, the ring contribution to the molecular refraction was calculated to be 0.614 (ref. 47).

Vogel's procedure has been used in the present work to determine the ring contribution in the cyclopropylalkanes and in some of the cyclopropylalkenes (table IV). A nearly constant value for ring contribution among the cyclopropylalkanes is obtained by this procedure in contrast to the inconstant ΔM_R in table III for the same compounds. Although Vogel's procedure minimizes the effect of environment on the atomic refractivities of carbon and hydrogen, the discrepancy between the ring contribution obtained with the cyclopropylalkanes and that obtained with the cyclopropane carboxylates, 0.44 and 0.614, respectively, indicates that the nature of the ring substituents also influences the ring contribution. While it may be said that the compounds used by Vogel are capable of conjugation between the cyclopropyl ring and the carbonyl double bond, and that the value 0.614 includes exaltation due to conjugation, nevertheless, environmental factors influence the combined effects of ring contribution and conjugation as indicated by the data in table IV for the cyclopropylalkenes, methyl cyclopropyl ketone, and diethyl cyclopropane-1,1-dicarboxylate.

Because of the varying influence of the substituent groups on the cyclopropyl ring, it is doubtful that any of the proposed values will adequately express for all structures the contribution of the cyclopropyl ring to the molecular refraction. The calculation of molecular refraction of cyclopropane derivatives can, therefore, at best give only an approximation of the observed refraction.

CONCLUDING REMARKS

The methods of synthesizing and purifying 34 hydrocarbon and nonhydrocarbon derivatives of cyclopropane were discussed, and the physical properties and the infrared spectra of each of the compounds were presented.

It was found that a series of cyclopropylalkenes could be prepared by dehydrating appropriate alkyl- or methylalkyl-cyclopropylcarbinols, either with concentrated sulfuric acid or with alumina at temperatures between 200° and 300° C. Furthermore, in the presence of a barium-promoted copper chromite catalyst, the cyclopropylalkenes were catalytically hydrogenated to the corresponding cyclopropylalkanes. The position of the double bond was found to greatly influence both the ease of hydrogenation and the yield of products. The cyclopropyl-1-alkenes hydrogenated readily at 100° C to give the corresponding cyclopropylalkane and only 1 to 2 percent of paraffinic product, whereas the cyclopropyl-2-alkenes hydrogenated sluggishly even at higher temperatures to give 15 to 17 percent of paraffinic product in addition to the cyclopropylalkanes. In the hydrogenation experiments, additional evidence was found to support the proposal that conjugated cyclopropylalkenes can add hydrogen by both a 1,2- and a 1,4-mechanism.

Dicyclopentyl, the smallest of the dicyclic hydrocarbons, was prepared for the first time. Spiropentane, the smallest of the spiro hydrocarbons, was also prepared.

Infrared absorption bands characteristic of the cyclopropyl ring were discussed, and some observations were made on the contribution of the cyclopropyl ring to the molecular refractions of cyclopropane derivatives.

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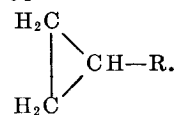
CLEVELAND, OHIO, August 15, 1952

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TABLE I.—CARBON-HYDROGEN ABSORPTION BANDS IN INFRARED SPECTRA OF CYCLOPROPANE DERIVATIVES

Cyclopropane derivatives	Wavelength, microns						
	3.21-3.25	3.26-3.30	3.31-3.35	3.36-3.40	3.41-3.45	3.48-3.50	3.51-3.55
2-Cyclopropylpropane.....	3.24	-----	3.32s	3.37	-----	3.48	-----
2-Cyclopropylbutane.....	3.25	-----	-----	3.38	3.42s	3.48	-----
2-Cyclopropylpentane.....	3.25	-----	-----	3.38	3.42s	3.49	-----
2-Cyclopropylhexane.....	-----	3.26	-----	3.38	3.42s	3.49	-----
2-Cyclopropyl-3-methylbutane.....	-----	3.26	-----	3.40	3.43s	-----	3.51
Vinylcyclopropane.....	3.25	-----	3.32	-----	-----	-----	-----
2-Cyclopropylpropene.....	3.22	3.30	-----	3.38s	-----	3.47s	-----
2-Cyclopropyl-1-butene.....	3.25	-----	3.33s	3.37	3.42s	3.48s	-----
2-Cyclopropyl-1-pentene.....	3.22	* 3.30s	-----	3.37	3.45	-----	-----
2-Cyclopropyl-1-hexene.....	-----	3.27	3.35s	-----	3.44	-----	3.52
2-Cyclopropyl-3-methyl-1-butene.....	3.24	-----	3.33s	3.37	3.41s	3.48	-----
2-Cyclopropyl-2-butene (l. b.).....	3.25	-----	3.34	-----	3.43	3.49	-----
2-Cyclopropyl-2-pentene (l. b.).....	3.24	-----	3.35	-----	-----	3.47	-----
2-Cyclopropyl-2-hexene (l. b.).....	3.25	-----	3.34s	3.39	3.43s	-----	3.52
2-Cyclopropyl-2-butene (h. b.).....	3.24	-----	3.32	-----	3.41	3.49	-----
2-Cyclopropyl-2-pentene (h. b.).....	3.24	-----	-----	3.37	-----	3.48	-----
2-Cyclopropyl-2-hexene (h. b.).....	-----	3.27	-----	3.40	-----	-----	3.51
Spiropentane.....	-----	3.28	-----	3.36	-----	-----	3.53
Dicyclopentyl.....	3.24	-----	3.33	3.40	-----	3.49	-----
Methylcyclopropylcarbinol.....	3.25	-----	-----	3.37	-----	3.48	-----
Dimethylcyclopropylcarbinol.....	-----	3.30	-----	-----	3.42	3.50s	-----
Methylethylcyclopropylcarbinol.....	3.25	-----	-----	3.37	3.42s	-----	-----
Methylpropylcyclopropylcarbinol.....	-----	3.27	-----	3.40	-----	-----	-----
Methylbutylcyclopropylcarbinol.....	-----	3.27	-----	-----	3.42	-----	3.52
Methylisopropylcyclopropylcarbinol.....	-----	3.28	-----	-----	3.41	3.50s	-----
Cyclopropyl chloride.....	3.23	-----	3.32	-----	-----	-----	-----
1, 1-Dichlorocyclopropane.....	3.21	3.29	-----	-----	-----	-----	-----
trans-1, 2-Dichlorocyclopropane.....	3.25s	3.27	3.34s	-----	-----	-----	-----
Methyl cyclopropyl ketone.....	3.24	-----	3.34	3.38	3.44s	-----	-----
Ethyl cyclopropyl ketone.....	-----	3.26s	-----	3.36	3.43s	3.48s	-----
Propyl cyclopropyl ketone.....	-----	-----	-----	3.40	-----	-----	3.51s
Butyl cyclopropyl ketone.....	3.24	-----	3.32	3.38	-----	3.48	-----
Isopropyl cyclopropyl ketone.....	3.22	-----	-----	3.36, 3.40	-----	3.48	-----
Diethyl cyclopropane-1, 1-dicarboxylate.....	-----	-----	3.33	3.38s	3.42	-----	-----

* The symbol s denotes band which shows only as a shoulder on a stronger absorption.

TABLE II.—CHARACTERISTIC ABSORPTION IN INFRARED FOR CYCLOPROPYL RING

Cyclopropane derivative	Wavelength, microns
2-Cyclopropylpropane.....	9.82
2-Cyclopropylbutane.....	9.82
2-Cyclopropylpentane.....	9.82
2-Cyclopropylhexane.....	9.82
2-Cyclopropyl-3-methylbutane.....	9.83
Vinylcyclopropane.....	9.77
2-Cyclopropylpropene.....	9.77
2-Cyclopropyl-1-butene.....	9.78
2-Cyclopropyl-1-pentene.....	9.78
2-Cyclopropyl-1-hexene.....	9.79
2-Cyclopropyl-3-methyl-1-butene.....	9.78
2-Cyclopropyl-2-butene (l. b.).....	9.82
2-Cyclopropyl-2-pentene (l. b.).....	9.80
2-Cyclopropyl-2-hexene (l. b.).....	9.82
2-Cyclopropyl-2-butene (h. b.).....	9.80
2-Cyclopropyl-2-pentene (h. b.).....	9.80
2-Cyclopropyl-2-hexene (h. b.).....	9.80
Spiropentane.....	9.54
Dicyclopentyl.....	9.82
Methylcyclopropylcarbinol.....	9.75
Dimethylcyclopropylcarbinol.....	9.78
Methylethylcyclopropylcarbinol.....	9.77
Methylpropylcyclopropylcarbinol.....	9.76
Methylbutylcyclopropylcarbinol.....	9.80
Methylisopropylcyclopropylcarbinol.....	9.75
Cyclopropyl chloride.....	9.73
1,1-Dichlorocyclopropane.....	9.66
trans-1,2-Dichlorocyclopropane.....	9.58
Methyl cyclopropyl ketone.....	9.69
Ethyl cyclopropyl ketone.....	9.78
Propyl cyclopropyl ketone.....	9.80
Butyl cyclopropyl ketone.....	9.75
Isopropyl cyclopropyl ketone.....	9.78
Diethyl cyclopropane-1,1-dicarboxylate.....	9.70

TABLE III.—MOLECULAR REFRACTIONS OF CYCLOPROPANE DERIVATIVES

Cyclopropane derivative	M_R^b (calc.)	M_R^b (obs.)	ΔM_R
2-Cyclopropylpropane.....	27.88	28.33	0.45
2-Cyclopropylbutane.....	32.53	32.85	.32
2-Cyclopropylpentane.....	37.18	37.51	.33
2-Cyclopropylhexane.....	41.82	42.15	.33
2-Cyclopropyl-3-methylbutane.....	37.18	37.37	.19
Vinylcyclopropane.....	22.75	23.60	0.85
2-Cyclopropylpropene.....	27.40	27.98	.58
2-Cyclopropyl-1-butene.....	32.05	32.46	.41
2-Cyclopropyl-1-pentene.....	36.70	37.12	.42
2-Cyclopropyl-1-hexene.....	41.34	41.75	.41
2-Cyclopropyl-3-methyl-1-butene.....	36.70	37.06	.36
2-Cyclopropyl-2-butene (l. b.).....	32.05	32.63	0.58
2-Cyclopropyl-2-pentene (l. b.).....	36.70	37.47	.77
2-Cyclopropyl-2-hexene (l. b.).....	41.34	42.11	.77
2-Cyclopropyl-2-butene (h. b.).....	32.05	32.66	0.61
2-Cyclopropyl-2-pentene (h. b.).....	36.70	37.45	.75
2-Cyclopropyl-2-hexene (h. b.).....	41.34	42.13	.79
Spiropentane.....	21.18	22.45	1.27
Dicyclopentyl.....	25.83	26.53	.70
Methylcyclopropylcarbinol.....	24.75	25.12	0.37
Dimethylcyclopropylcarbinol.....	29.40	29.65	.25
Methylethylcyclopropylcarbinol.....	34.05	34.25	.20
Methylpropylcyclopropylcarbinol.....	38.69	38.85	.16
Methylbutylcyclopropylcarbinol.....	43.34	43.48	.14
Methylisopropylcyclopropylcarbinol.....	38.69	38.64	-.05
Cyclopropyl chloride.....	18.76	19.07	0.31
1,1-Dichlorocyclopropane.....	23.57	24.06	.49
trans-1,2-Dichlorocyclopropane.....	23.57	24.06	.49
Methyl cyclopropyl ketone.....	23.34	23.94	0.60
Diethyl cyclopropane-1,1-dicarboxylate.....	44.93	45.77	0.84

^a Atomic and group refractivities of reference 54 were used.

^b $\frac{n^2-1}{n^2+2} \times \frac{m}{d} = M_R$ (observed).

TABLE IV.—CONTRIBUTION OF CYCLOPROPYL RING TO MOLECULAR REFRACTION

$\text{Ring contribution} = \begin{array}{c} \text{CH}_2 \quad \text{R}' \\ \diagup \quad \diagdown \\ \text{C} \\ \diagdown \quad \diagup \\ \text{CH}_2 \quad \text{R}'' \end{array} + 2\text{H} - \begin{array}{c} \text{CH}_2 \quad \text{R}' \\ \diagup \quad \diagdown \\ \text{CH} \\ \diagdown \quad \diagup \\ \text{CH}_3 \quad \text{R}'' \end{array}$				
Cyclopropane derivatives	M_R (obs.)	Aliphatic compounds	M_R^b (obs.)	Ring contribution
2-Cyclopropylpropane.....	28.33	2-Methylpentane.....	29.85	0.44
2-Cyclopropylbutane.....	32.85	3-Methylhexane.....	34.47	.44
2-Cyclopropylpentane.....	37.51	4-Methylheptane.....	39.12	.45
2-Cyclopropyl-3-methylbutane.....	37.37	2, 3-Dimethylhexane.....	38.99	.44
2-Cyclopropylhexane.....	42.14	4-Methyloctane.....	43.77	.43
Dicyclopentyl.....	26.53	n-Hexane.....	29.91	.37
Vinylcyclopropane.....	23.60	1-Pentene.....	24.83	^c 0.83
2-Cyclopropylpropene.....	27.98	2-Methyl-1-pentene.....	29.42	^c .62
2-Cyclopropyl-1-butene.....	32.46	2-Ethyl-1-pentene.....	34.0	^c .5
Methyl cyclopropyl ketone.....	23.94	Methyl propyl ketone.....	^d 25.25	^e 0.75
Diethyl cyclopropane-1, 1-dicarboxylate.....	45.77	Diethyl ethylmalonate.....	^e 47.07	^e 0.77

^a Atomic refraction for hydrogen taken from reference 54.

^b From reference 55.

^c Includes ring contribution plus any exaltation accompanying conjugation.

^d From reference 56.

^e From reference 57.

REPORT 1113

SPECTRUM OF TURBULENCE IN A CONTRACTING STREAM¹

By H. S. RIBNER and M. TUCKER

SUMMARY

The spectrum concept is employed to study the selective effect of a stream contraction on the longitudinal and lateral turbulent velocity fluctuations of the stream. By a consideration of the effect of the stream contraction on a single plane sinusoidal disturbance wave, mathematically not dissimilar to a triply periodic disturbance treated by G. I. Taylor, the effect on the spectrum tensor of the turbulence and hence on the correlation tensor is determined. Lack of interference between waves follows from the postulation of a low level of turbulence; this and the assumption of an inviscid fluid imply neglect of decay effects. The compressibility of the main stream is taken into account, but the density fluctuations associated with the turbulence are assumed to be negligible; this would be the case if the turbulence originated from wakes and boundary layers in the very low-speed portion of the flow. For an axisymmetric contraction and a particular isotropic initial turbulence, some explicit results are obtained. The one-dimensional longitudinal spectrum is found to be distorted (as well as reduced in amplitude) with its peak shifted well to the right of the initial position above the zero of the wave-number scale. The selective effect of the contraction on the mean square longitudinal and lateral components of turbulent velocity is found to be given uniquely when the initial turbulence is isotropic, regardless of the details of the spectrum. If the initial spectrum is anisotropic, as, for instance, that produced by a damping screen, then the selective effect is altered.

In a crude extension, decay effects outside the scope of the theory are allowed for in first approximation. With this extension, a comparison with experiment is made of the selective effect on turbulent intensity where the estimated decay effects are comparable with the contraction effects. The agreement is good for the longitudinal component, very poor for the lateral component, the experimental data themselves being in conflict.

INTRODUCTION

The generation of a wind-tunnel flow is always accompanied by a certain amount of turbulence; this is one respect in which the flow fails to simulate free-flight conditions. Measurements in the tunnel, particularly those sensitive to boundary-layer behavior, are known to be affected by this turbulence. Accordingly, the tunnel designer attempts to reduce the intensity to the lowest practicable level. The

use of honeycombs and damping screens in a large low-speed section (settling chamber) followed by a sharp contraction to the much-higher-speed working section is known to be effective. The honeycombs and screens located in a low-speed section reduce the absolute level of the turbulence with little drag penalty; then the relative level is greatly reduced by the large gain in tunnel speed through the contraction, aside from any effect of the contraction on the absolute level.

Once the characteristics of honeycombs and screens are known, the further quantitative estimate of the reduction in turbulence involves a knowledge of the effect of the tunnel contraction² on the turbulence. It is known that the longitudinal component of the turbulence is greatly reduced (in absolute value) by the contraction; the behavior of the lateral component appears, on the other hand, to vary from no change to a substantial increase. Prandtl (ref. 1) obtained a quantitative estimate of the first effect by considering the conservation of energy for a perturbed longitudinal filament: if the initial stream speed is U , the filament speed $U+u$, with $u \ll U$, and the final stream speed is lU , then the final filament speed must be $lU+l^{-1}u$; that is, the contraction reduces the longitudinal perturbation velocity u by the factor l^{-1} . For the lateral effect, Prandtl applied conservation of momentum to a small rotating cylinder of the fluid, with its axis cross stream, as the fluid traversed the tunnel contraction. He concluded that the lateral perturbation velocity v is increased by a factor $\sqrt{l}$.

Prandtl's considerations on the effect of a stream contraction were limited, as has been noted, to particular idealized "eddies." G. I. Taylor (ref. 2) later attempted more realism by treating a mathematically defined model of turbulence which amounted to vortices in parallelepiped partitions arranged in a regular three-dimensional array. The changes in vorticity on traversing the contraction were determined from a theorem based on conservation of circulation for an inviscid fluid; the corresponding altered turbulent velocity pattern was then calculated. The final result of the analysis consisted in expressions for the root-mean-square longitudinal and lateral turbulent velocity components u' and v' downstream of the contraction expressed as ratios of the corresponding values upstream.

¹ Supersedes NACA TN 2606, "Spectrum of Turbulence in a Contracting Stream," by H. S. Ribner and M. Tucker, 1952.

² The considerations of this paper are not limited to a wind-tunnel contraction: they may be applied to any stream tube of varying cross section large compared with the scale of the turbulence.

The initial condition of isotropic turbulence (mean values unaffected by rotation or reflection of axes) was approximated by specifying the vortex partitions to be cubical. For this case the reduction in the longitudinal component u' was found to vary more nearly like $1.5 l^{-1}$ than the value l^{-1} suggested by Prandtl. No explicit result was found for the variation of the lateral component, however: the calculations contained a free parameter.

Taylor's results for the longitudinal component agreed fairly well with the experimental data then available, but it is now considered that the measurements were made too close behind the screens for the screen-produced turbulence to have been isotropic. On theoretical grounds, the objection to Taylor's theory is threefold: first, the decay processes of turbulent mixing and viscous dissipation, which result in a reduction of the mean intensity with axial distance in the wind tunnel, are neglected; second, the assumed model of turbulence fails to exhibit the spatial and temporal randomness of actual turbulence; third, no choice of the parameters in Taylor's model corresponds to isotropy. In a sense all three objections apply likewise to Prandtl's results: no model was employed in his considerations, and hence no distinctions between the effects of isotropy and anisotropy were made.

The second and third objections can be removed by working, not with a model of turbulence, but instead with a Fourier integral representation of a random turbulent field. The integral can be interpreted as a superposition of plane transverse sinusoidal waves of all wave lengths and with apparently random orientations and phases. This aggregate of plane waves constitutes the (three-dimensional) spectrum of the turbulence. Only the statistical aspects of this spectrum will be known, not, for example, the detailed phase relations. Mean-square velocity components may be obtained by an integral of certain spectrum functions in which the phase relations are suppressed; these functions are included in the 'spectrum tensor' (ref. 3).

Taylor's concepts may be applied to find the effect of a stream contraction on a single plane wave. The effect under the assumptions is linear; therefore, the superposition implied by the Fourier integral may be employed to obtain the contraction effect on a field of turbulence. In particular, if the initial spectrum tensor is known, the final spectrum tensor is determined; the initial and final mean-square velocity components then result from quadratures. Indeed, from the same information, changes due to the stream contraction in correlations of velocity at different points may be calculated: use is made of the fact that the correlation tensor is an inverse Fourier transform of the spectrum tensor (ref. 3).

Accordingly, the injection of the spectrum point of view into Taylor's original concept of the contraction effect makes possible a more realistic calculation of the changes in mean-square velocity components. In addition, it provides much more detailed information concerning changes in the statistical properties of the turbulence; that

is, it provides the changes in the spectrum tensor and in the correlation tensor.

The ideas just outlined are developed in the present paper. The first section is devoted to an account of turbulent spectrum analysis in a form specially adapted to the analysis of the contraction effect. In this account, which is a generalization of a development in reference 4, the role of the spectrum tensor is subordinated to that of the individual Fourier components (plane waves) in contradistinction to the customary treatment. This approach has perhaps an auxiliary merit in providing some better physical insight into the significance of the spectrum tensor.

Next the effect of a stream contraction on a single plane wave is calculated by an application of Taylor's concepts. The treatment is slightly more general in that compressibility of the main stream is allowed for. The density fluctuations associated with the turbulence are assumed to be negligible; this would be the case if the turbulence originated entirely from boundary layers and wakes in the very-low-speed portion of the flow. Following Taylor, the problem is linearized by postulating a sufficiently weak turbulence so that the self-distortion of the turbulent eddies is small compared with the distortion imposed by the contraction of the main stream; this together with the assumption of an inviscid fluid implies neglect of the decay of the turbulence.

In succeeding sections the spectrum and correlation tensors downstream of the contraction are expressed in terms of the corresponding initial tensors. For the special case of an axisymmetric contraction and isotropic initial turbulence, the ratios of the root-mean-square longitudinal and lateral velocity fluctuations downstream and upstream are obtained explicitly in terms of the parameters defining the contraction. For a particular subcase where the initial isotropic spectrum tensor is specified, the corresponding 'one-dimensional' spectrums (as would be recorded by stationary hot-wire probes) upstream and downstream of the contraction are calculated; the specification is such that the upstream one-dimensional spectrum corresponds to experiment (ref. 5, p. 35) in a number of cases of isotropic turbulence.

Most of the calculated contraction effects are amenable to experimental checks either directly or indirectly. The available experimental data, however, are limited to the changes in the root-mean-square velocities. A comparison with these experimental data is given with an estimated allowance for decay effects outside the scope of the theory. Design curves of the changes in the root-mean-square velocity components neglecting decay are included for engineering purposes.

SPECTRUM ANALYSIS

Representation of turbulence by superposition of plane sinusoidal waves.—Suppose q_1, q_2, q_3 represent the components of velocity in a turbulent field; that is q_1, q_2 , and q_3 vary in an apparently random manner in space and time, and the mean values $\bar{q}_1 = \bar{q}_2 = \bar{q}_3 = 0$. Subject to certain conditions, a snapshot of this field at any instant can be

represented as a set of three-dimensional Fourier integrals

$$q_\alpha(x_1, x_2, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_\alpha(k_1, k_2, k_3) e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} dk_1 dk_2 dk_3$$

where $\alpha=1, 2$, or 3 and the significance of k_1, k_2 , and k_3 will be brought out later. A continuous representation of the turbulent field is obtained by allowing the Q_α to vary with time.

It will be convenient to abbreviate the Fourier integral to

$$q_\alpha(\underline{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_\alpha(\underline{k}) e^{i \underline{k} \cdot \underline{x}} d\tau(\underline{k}) \quad (1a)$$

and to introduce the companion equation

$$Q_\alpha(\underline{k}) = 8\pi^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q_\alpha(\underline{x}) e^{-i \underline{k} \cdot \underline{x}} d\tau(\underline{x}) \quad (1b)$$

where $\underline{k}=k_1, k_2, k_3$; $\underline{x}=x_1, x_2, x_3$; $d\tau(\underline{x})=dx_1 dx_2 dx_3$. The second equation allows, in principle at least, the coefficients $Q_\alpha(\underline{k})$ to be calculated. Mathematically, $q_\alpha(\underline{x})$ and $Q_\alpha(\underline{k})$ are termed three-dimensional Fourier transforms of each other; use will be made of this relation later.

The velocity components q_α are connected by the condition of continuity. In many cases of practical interest, these turbulent velocities originate from boundary layers and the wakes of obstacles in flows of low subsonic speed, so that associated density fluctuations may be ignored; this is still permissible when the turbulence so produced is transported by a high-speed stream. Thus the incompressible form of the continuity equation may be used and the result is

$$Q_1 k_1 + Q_2 k_2 + Q_3 k_3 = 0$$

This relation may be written more compactly as

$$\sum_{\alpha} Q_\alpha k_\alpha = 0 \quad (2)$$

Physical interpretation.—The amplitude components Q_α are complex in general. According to equation (1b), then, the requirement that the velocity components q_α be real implies that $Q_\alpha(-\underline{k})$ is the complex conjugate of $Q_\alpha(\underline{k})$. If corresponding terms for $\underline{k}$ and $-\underline{k}$ in equation (1a) are paired, their sum is thus equal to the real quantity

$$2(\text{Re } Q_\alpha) \cos(\underline{k} \cdot \underline{x}) - 2(\text{Im } Q_\alpha) \sin(\underline{k} \cdot \underline{x}) \quad (3)$$

The imaginary parts cancel in the pairing, which implies that they contribute nothing to the integral. Expression (3) represents a pair of plane standing waves, a cosine wave and a sine wave, with normals in the direction $\underline{k}=(k_1, k_2, k_3)$, where $\underline{x}=(x_1, x_2, x_3)$ is the radius vector to any point. The vector $\underline{k}$ is termed the wave-number vector and its magnitude k simply the wave number; the corresponding wave length is 2π divided by the wave number. Since $\underline{k}$ is perpendicular to

the wave front, it is sometimes referred to herein as the 'wave normal.'

The continuity condition, equation (2), states that both the real part ($\text{Re } Q_\alpha$) and the imaginary part ($\text{Im } Q_\alpha$) of the amplitude vector $\underline{Q}=Q_\alpha=(Q_1, Q_2, Q_3)$ are perpendicular to the wave normal $\underline{k}$; that is, both waves of expression (3) are transverse. For each wave any one of the parallel planes containing both the local velocity vector $\underline{q}$ and the wave-number vector $\underline{k}$ (which is perpendicular to $\underline{Q}$ and hence to $\underline{q}$) is called the plane of polarization. The cosine wave (real part) and sine wave (imaginary part) may be polarized in different planes in general; the necessary and sufficient condition that they be polarized in the same plane is

$$\frac{\text{Re } Q_1}{\text{Im } Q_1} = \frac{\text{Re } Q_2}{\text{Im } Q_2} = \frac{\text{Re } Q_3}{\text{Im } Q_3} \quad (4)$$

Equations (1) are now seen to represent a superposition of plane sinusoidal waves (Fourier components) with all orientations of the wave fronts (all directions of the wave normal $\underline{k}$) and all wave lengths (all wave numbers k). Each wave is transverse, and all planes of polarization are permitted. For each value of $\underline{k}$ there exist a cosine wave and a sine wave; their respective amplitudes and planes of polarization are different in general. The complex amplitude components $Q_\alpha(\underline{k})$ express, in their real and imaginary parts, how the respective amplitudes and planes of polarization vary with the wave-front orientation and the wave number.

Mean values: correlation tensor.—Consider the spatial³ mean value of the product of the velocity component q_α at $\underline{x}$ and the velocity component q_β at $\underline{x}'=\underline{x}+\underline{r}$ as $\underline{x}$ varies but the separation $\underline{r}$ of the two points remains fixed during the averaging process; this mean value is called a velocity correlation and is given the symbol $R_{\alpha\beta}(\underline{r})$. There are nine such correlations, corresponding to $\alpha=1, 2, 3$, and $\beta=1, 2, 3$. The form $R_{\alpha\beta}(\underline{r})$ has been shown to transform like a second-order tensor and has been designated (sometimes divided by $\overline{q^2}$) as the 'correlation tensor' (ref. 6).

Evaluation of correlation tensor in terms of spectral quantities.—The basis of the idea that the correlation tensor might be expressed in terms of individual-wave parameters is drawn from reference 4. In that paper mean-square values of q_1, q_2, q_3 in terms of such parameters are discussed; these constitute the diagonal terms of $R_{\alpha\beta}(\underline{r})$ evaluated at $\underline{r}=0$. The following derivation of $R_{\alpha\beta}(\underline{r})$ amounts to a generalization of the derivation of $R_{11}(0)$ given in that reference.

Assume that the turbulence is confined to a large, but not infinite, parallelepiped of edges $2D_1, 2D_2, 2D_3$ and vanishes everywhere outside.⁴ The space average $R_{\alpha\beta}(\underline{r})$ is to be

³ If the statistical properties of the turbulence are independent of position (homogeneous turbulence) and time-independent, an average at a given time over all space equals an average at a given point (or pair of correlated points) over all time; a proof is given in appendix B. If the statistical properties vary slowly with time, the space average will still approximate a time average over an interval just long enough to smooth out the fluctuations.

⁴ The space integral of the square of any component velocity is thereby bounded, which is a prerequisite for the existence of a Fourier integral, equations (1); in other words, equation (1b) shows that Q_α must depend on the volume $8D_1 D_2 D_3$ of the region within which q_α differs from zero: if this volume is infinite Q_α is infinite, and the Fourier integral, equation (1a), does not exist.

taken for all points $\underline{x}$ in the interior, the separation $\underline{r}$ remaining constant and very small compared with $D_{1,2,3}$. The average is

$$R_{\alpha\beta}(\underline{r}) = \overline{q_\alpha(\underline{x})q_\beta(\underline{x}+\underline{r})} = \frac{1}{8D_1D_2D_3} \int_{-D_3}^{D_3} \int_{-D_2}^{D_2} \int_{-D_1}^{D_1} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_\alpha(\underline{k}) Q_\beta(\underline{k}') e^{i[\underline{k} \cdot \underline{x} + \underline{k}' \cdot (\underline{x} + \underline{r})]} d\tau(\underline{k}) d\tau(\underline{k}') \right\} dx_1 dx_2 dx_3$$

or, interchanging the order of integration,

$$= \frac{1}{D_1D_2D_3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_\alpha(\underline{k}) Q_\beta(\underline{k}') e^{i\underline{k}' \cdot \underline{r}} \frac{\sin(k_1 + k_1')D_1}{k_1 + k_1'} \frac{\sin(k_2 + k_2')D_2}{k_2 + k_2'} \frac{\sin(k_3 + k_3')D_3}{k_3 + k_3'} d\tau(\underline{k}) d\tau(\underline{k}')$$

Put $K_1 = k_1 + k_1'$, $K_2 = k_2 + k_2'$, and $K_3 = k_3 + k_3'$. Then recalling that $\underline{k} = (k_1, k_2, k_3)$, $d\tau(\underline{k}) = dk_1 dk_2 dk_3$, etc., and $\underline{k}' \cdot \underline{r} = k_1' r_1 + k_2' r_2 + k_3' r_3$

$$R_{\alpha\beta}(\underline{r}) = \frac{1}{D_1D_2D_3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_\alpha(k_1, k_2, k_3) \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_\beta(K_1 - k_1, K_2 - k_2, K_3 - k_3) e^{i[(K_1 - k_1)r_1 + (K_2 - k_2)r_2 + (K_3 - k_3)r_3]} \frac{\sin K_1 D_1}{K_1} \frac{\sin K_2 D_2}{K_2} \frac{\sin K_3 D_3}{K_3} dK_1 dK_2 dK_3 \right\} dk_1 dk_2 dk_3$$

The inner integral approaches a limit as the parallelepiped edges $2D_1$, $2D_2$, $2D_3$ become indefinitely large. For this limiting case the correlation tensor is then

$$R_{\alpha\beta}(\underline{r}) = \lim_{D \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\pi^3}{D_1 D_2 D_3} Q_\alpha(k_1, k_2, k_3) Q_\beta(-k_1, -k_2, -k_3) e^{-i\underline{k} \cdot \underline{r}} dk_1 dk_2 dk_3 = \lim_{\tau \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{8\pi^3}{\tau} Q_\alpha(\underline{k}) Q_\beta^*(\underline{k}) e^{-i\underline{k} \cdot \underline{r}} d\tau(\underline{k}) \quad (5)$$

where $Q_\beta^*(\underline{k}) [= Q_\beta(-\underline{k})]$ is the complex conjugate of $Q_\beta(\underline{k})$ and τ is the volume $8D_1D_2D_3$ of the parallelepiped.

Identification of $\lim_{\tau \rightarrow \infty} \frac{8\pi^3}{\tau} Q_\alpha Q_\beta^*$ with spectrum tensor.—

If the field of turbulence is homogeneous, the correlations should be independent of the volume τ averaged over when this volume is sufficiently large; thus $\lim_{\tau \rightarrow \infty} \frac{8\pi^3}{\tau} Q_\alpha Q_\beta^*$ should exist.

From the mathematical standpoint, equation (5) shows that

$R_{\alpha\beta}(\underline{r})$ is the Fourier transform of $\lim_{\tau \rightarrow \infty} \frac{8\pi^3}{\tau} Q_\alpha(\underline{k}) Q_\beta^*(\underline{k})$, and

conversely; since it is known that the integral of $|R_{\alpha\beta}(\underline{r})|$ over all $\underline{r}$ is finite, the inverse Fourier transform relation ensures that $\lim_{\tau \rightarrow \infty} \frac{8\pi^3}{\tau} Q_\alpha Q_\beta^*$ exists. Finally, since the Fourier

transform of a function is unique, $\lim_{\tau \rightarrow \infty} \frac{8\pi^3}{\tau} Q_\alpha(\underline{k}) Q_\beta^*(\underline{k})$ may be identified with the form $\Gamma_{\alpha\beta}(\underline{k})$ defined by Batchelor (ref. 3) as the Fourier transform of $R_{\alpha\beta}(\underline{r})$. The form $\Gamma_{\alpha\beta}(\underline{k})$ is known as the spectrum tensor. The Fourier transform relations connecting the spectrum tensor and the correlation tensor are summarized as

$$\Gamma_{\alpha\beta}(\underline{k}) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\alpha\beta}(\underline{r}) e^{i\underline{k} \cdot \underline{r}} d\tau(\underline{r}) \quad (6a)$$

$$R_{\alpha\beta}(\underline{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{\alpha\beta}(\underline{k}) e^{-i\underline{k} \cdot \underline{r}} d\tau(\underline{k}) \quad (6b)$$

By use of the Fourier transform relation, Batchelor demonstrated that $\Gamma_{\alpha\beta}$ is a second-order tensor and obtained a number of its properties. Thus, for example, $\Gamma_{\alpha\beta}$ is complex, in general, with $\Gamma_{\beta\alpha} = \Gamma_{\alpha\beta}^*$, and the diagonal elements $\Gamma_{\alpha\alpha}$ are real; also, $\Gamma_{\alpha\beta}(-\underline{k}) = \Gamma_{\beta\alpha}(\underline{k})$. It is of interest to observe

that these same properties result immediately from the identification of $\lim_{\tau \rightarrow \infty} \frac{8\pi^3}{\tau} Q_\alpha Q_\beta^*$ with $\Gamma_{\alpha\beta}$. Thus $Q_\alpha Q_\beta^*$ is complex, in general; $Q_\beta Q_\alpha^*$ equals $[Q_\alpha Q_\beta^*]^*$; and $Q_\alpha Q_\alpha^*$ is, of course, real. Furthermore, since $Q_\alpha(-\underline{k}) = Q_\alpha^*(\underline{k})$, $Q_\alpha(-\underline{k}) Q_\beta^*(-\underline{k})$ equals $Q_\alpha^*(\underline{k}) Q_\beta(\underline{k})$; hence $\Gamma_{\alpha\beta}(-\underline{k}) = \Gamma_{\beta\alpha}(\underline{k})$.

The distinction between cases where $\Gamma_{\alpha\beta}$ is real and cases where it is complex may be given a physical interpretation. The product $Q_\alpha Q_\beta^*$, and correspondingly $\Gamma_{\alpha\beta}$, is seen to be real when the condition equation (4) is satisfied. This implies that the cosine wave and sine wave associated with wave number $\underline{k}$ are polarized in the same plane. The alternative condition where $Q_\alpha Q_\beta^*$, and hence $\Gamma_{\alpha\beta}$, is complex implies polarization of corresponding cosine and sine waves in different planes. The velocity pattern of such a pair of waves is quite interesting: successive velocity vectors along a line in the direction of the wave normal $\underline{k}$ turn progressively about this line in spiral fashion; the tips of the vectors trace out a helical curve on a cylinder of oval cross section.

Energy spectral density.—Each of the diagonal elements Γ_{11} , Γ_{22} , and Γ_{33} of the spectrum tensor $\Gamma_{\alpha\beta}$ may be interpreted as an energy spectral density. Thus, according to equation (6b)

$$R_{11}(0) = \overline{u_1^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{11}(\underline{k}) d\tau(\underline{k})$$

Therefore, the differential $\frac{1}{2} \Gamma_{11} d\tau(\underline{k})$ represents the contribution to the kinetic energy component $\frac{1}{2} \overline{u_1^2}$ per unit mass made by waves with wave number within the range $d\tau(\underline{k})$.

One-dimensional spectrum.—The elements of the three-dimensional spectrum tensor are not directly measurable; they may be obtained by taking the Fourier transform of the measured correlation tensor. A hot-wire probe placed in the moving stream will, however, develop a fluctuating output voltage whose (one-dimensional) frequency spectrum (ref. 7) is related to a diagonal element of the three-dimensional spectrum tensor. Thus by equation (6b) the contribution to the mean-square velocity component $\overline{q_\alpha^2} (=R_{\alpha\alpha}(0))$ from all waves with wave-number components in the direction of the x_1 -axis between $|k_1|$ and $|k_1| + |dk_1|$ is

$$F_\alpha(k_1)dk_1 \equiv 2 \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{\alpha\alpha} dk_2 dk_3 \right) dk_1 \quad (7)$$

the factor of 2 accounting for suppression of negative values of k_1 . The function $F_\alpha(k_1)$ is the one-dimensional spectrum corresponding to the velocity component q_α ; the values $\alpha=1, 2, 3$ correspond respectively to the longitudinal and two lateral spectrums. The particular spectrum obtained depends on the arrangement of the hot-wire probe elements.

EFFECT OF STREAM CONTRACTION

Consider now that the turbulent velocity pattern q_1, q_2, q_3 is carried along by an inviscid general stream with velocity $U(x_1)$ in the x_1 -direction. Consider also that q_1, q_2, q_3 are so small that their effect on the streamlines may be neglected as the flow traverses a wind-tunnel contraction. The contraction will, however, distort the shape of fluid elements. (See fig. 1.) The vorticity distribution will be forced to alter accordingly to conserve the circulation about each element. The net result will be an altered pattern of turbulence. Each plane wave (Fourier component) $Q_1, Q_2, Q_3 e^{ik \cdot x}$ will, in fact, be altered independently under the linearizing assumption to be made; the over-all effect on q_1, q_2, q_3 can be obtained by the summation expressed by equation (1a). It thus suffices to consider the effect of the contraction on a single representative plane wave.

EFFECT OF CONTRACTION ON REPRESENTATIVE PLANE WAVE

Velocity and vorticity at upstream station.—Designate by A a reference station upstream of the contraction and by B a reference station downstream of the contraction. (See fig. 1.) Let a typical Fourier component (plane wave) of the turbulent field $q_\alpha (\alpha=1, 2, 3)$ at station A be represented at time $t=0$ by

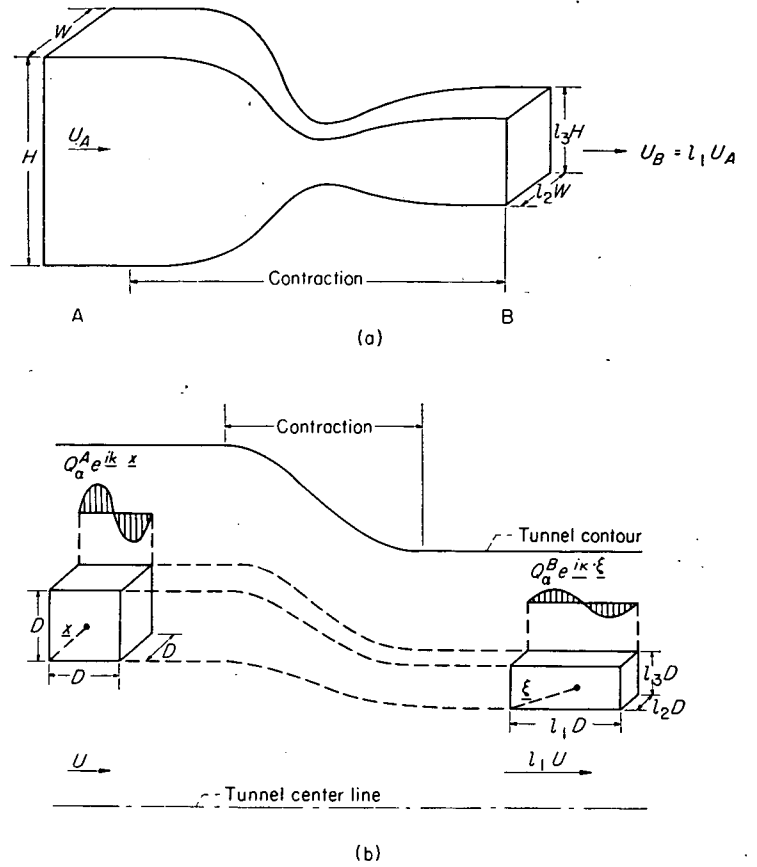
$$\tilde{q}_\alpha^A = \tilde{Q}_\alpha^A e^{i(k \cdot x)} \quad (8)$$

This wave, equation (8), is supposed to be carried along by the main stream with velocity U .

The vorticity $\tilde{\omega}_\alpha$ is obtained from the curl of equation (8) as

$$\tilde{\omega}_\alpha^A = i \sum_{\beta, \gamma} \epsilon_{\alpha\beta\gamma} k_\beta \tilde{Q}_\gamma^A e^{i(k \cdot x)} \quad (9)$$

where $\epsilon_{\alpha\beta\gamma} = \begin{cases} 0, & \text{if any pair of subscripts are equal} \\ 1, & \text{if } \alpha\beta\gamma \text{ are in cyclic order} \\ -1, & \text{if } \alpha\beta\gamma \text{ are in anticyclic order} \end{cases}$



(a) Tunnel geometry.
(b) Stream tube geometry.
FIGURE 1.—Schematic representation of flow contraction parameters.

Distortion of fluid element in passage through contraction.—Suppose the contraction is such that the stream velocity U is increased by a factor l_1 between stations A and B while the breadth and the height of the tunnel are reduced by factors l_2 and l_3 , respectively. (See fig. 1 (a).) In traveling from A to B, an initially cubical element of fluid of edge D will be distorted into a parallelepiped of edges $l_1 D, l_2 D, l_3 D$ (see fig. 1 (b)); a particle in the element originally ($t=0$) a vector distance x from a corner particle will finally ($t=t$) be found a distance ξ from the corner particle, where ξ is related to x by

$$\left. \begin{aligned} \xi_1 &= l_1 x_1 \\ \xi_2 &= l_2 x_2 \\ \xi_3 &= l_3 x_3 \end{aligned} \right\} \quad (10)$$

In this argument the modification of the streamlines due to the turbulent velocity fluctuations has been neglected. This implies that the relative displacement of two adjacent particles due to the superposed turbulent motion is small compared with the displacement due to the tunnel contraction. This key assumption, due to Taylor (ref. 2), linearizes and vastly simplifies the problem. The limitations imposed by the assumption are discussed later under **DECAY CONSIDERATIONS**.

The velocity ratio l_1 and the lateral and vertical contraction ratios l_2 and l_3 are related by the continuity condition

$$l_1 l_2 l_3 = 1$$

where σ is the ratio of stream densities at stations B and A; the density is considered uniform at each station in accordance with the initial assumption of negligible turbulent density fluctuations.

Vorticity at downstream station.—The vorticity is carried along by the flow, the fluid elements undergoing the distortion pictured in figure 1 (b), to the approximation used. During the motion the strength changes in such a way as to maintain the constancy of circulation of the fluid elements.⁵ The changes are expressed by the equations for the transport of vorticity in the Lagrangian form, due to Cauchy (ref. 8, p. 205).

$$\tilde{\omega}_\alpha^B = \sigma \sum_B \tilde{\omega}_\beta^A \frac{\partial \xi_\alpha}{\partial x_\beta}$$

where σ is the density ratio between stations B and A, and the derivatives $\partial \xi_\alpha / \partial x_\beta$ express the effect of the fluid distortion.⁶ Evaluation by means of the distortion equations (10) yields simply

$$\left. \begin{aligned} \tilde{\omega}_\alpha^B &= \sigma l_\alpha \tilde{\omega}_\alpha^A \\ \tilde{\omega}_1^B &= \sigma l_1 \tilde{\omega}_1^A \\ \tilde{\omega}_2^B &= \sigma l_2 \tilde{\omega}_2^A \\ \tilde{\omega}_3^B &= \sigma l_3 \tilde{\omega}_3^A \end{aligned} \right\} \quad (11a)$$

These equations relating downstream and upstream vorticity embody the entire dynamics of the contraction effect. The equations are not limited to the plane sinusoidal waves discussed earlier, but apply to any (weak) vorticity distribution whatsoever.

The above derivation of the vorticity changes is substantially in the form given originally by G. I. Taylor (ref. 2) for the case $\sigma=1$ (incompressible flow). In order to assess the influence of the simplifying assumptions a more general derivation based on the Navier-Stokes equations is given in the section entitled **DECAY CONSIDERATIONS**.

By virtue of equation (9) as applied to equation (11a), the vorticity at station B is obtained explicitly as

$$\tilde{\omega}_\alpha^B = i \sigma l_\alpha \sum_{\beta, \gamma} \epsilon_{\alpha\beta\gamma} k_\beta \tilde{Q}_\gamma^A e^{i \mathbf{k} \cdot \mathbf{x}} \quad (11b)$$

where, it will be remembered, $\mathbf{x}$ is the radius vector to a fluid particle at time $t=0$ when the fluid element is at station A in the moving coordinate system of figure 1 (b). The corresponding vector to the particle at time $t=t$, when the fluid element is at station B, is ξ in that figure. When equations (10) are used to express $\mathbf{x}$ in terms of ξ , the exponential $\mathbf{k} \cdot \mathbf{x}$ becomes, in expanded form,

$$\mathbf{k} \cdot \mathbf{x} = \frac{k_1 \xi_1}{l_1} + \frac{k_2 \xi_2}{l_2} + \frac{k_3 \xi_3}{l_3}$$

⁵ This statement is exact for the postulated inviscid fluid. The modification produced by the diffusive effect of viscosity, in the case of a gas, becomes appreciable for the smaller eddies or higher wave numbers; for this analysis a criterion for neglect of viscous effects is $\nu k^2 \ll |dU/dx|$. (See DECAY CONSIDERATIONS, equation (43).)

⁶ These equations refer to axes moving with some fluid particle rather than axes fixed as in reference 8; the form of the equations is unaffected.

The right-hand side may be expressed as $\kappa \cdot \xi$, where

$$\kappa = \frac{k_1}{l_1}, \frac{k_2}{l_2}, \frac{k_3}{l_3}$$

defines a new wave-number vector.

Velocity at downstream station, general case.—The velocity distribution whose curl in the ξ_1, ξ_2, ξ_3 system is given by equation (11b) and which satisfies continuity is found to be expressible in the form

$$\tilde{Q}_\alpha^B = \tilde{Q}_\alpha^A e^{i \mathbf{k} \cdot \xi} \quad (12)$$

with

$$\tilde{Q}_\alpha^B = \frac{1}{l_\alpha} \left(\tilde{Q}_\alpha^A - \sum_\beta \frac{\tilde{Q}_\beta^A k_\beta k_\alpha}{l_\beta^2 k^2} \right) \quad (13)$$

where κ is the magnitude of the wave-number vector. This result is the general solution for the contraction effect on a single plane wave.

Equations (12) and (13) admit of a simple (but not obvious) geometrical interpretation: *traversal of the stream contraction alters the initial plane wave, equation (8), so that its wave-number vector $\mathbf{k} = (k_1, k_2, k_3)$ is transformed into $\kappa = \frac{k_1}{l_1}, \frac{k_2}{l_2}, \frac{k_3}{l_3}$ and its amplitude vector $(\tilde{Q}_1^A, \tilde{Q}_2^A, \tilde{Q}_3^A)$ is transformed into the projection of $(\tilde{Q}_1^A/l_1, \tilde{Q}_2^A/l_2, \tilde{Q}_3^A/l_3)$ on a plane normal to the new wave-number vector κ .*

Velocity at downstream station, axisymmetric contraction.—In case the stream contraction is axisymmetric,⁷ a considerable simplification results. The condition for axisymmetry $l_2=l_3$, with use of the continuity equation (2), reduces equations (13) to

$$\left. \begin{aligned} \tilde{Q}_1^B &= \frac{\tilde{Q}_1^A}{l_1} \frac{k_1^2 + k_2^2 + k_3^2}{\epsilon k_1^2 + k_2^2 + k_3^2} \\ \tilde{Q}_2^B &= \frac{1}{l_2} \left[\tilde{Q}_2^A + \frac{\tilde{Q}_1^A k_1 k_2 (1-\epsilon)}{\epsilon k_1^2 + k_2^2 + k_3^2} \right] \\ \tilde{Q}_3^B &= \frac{1}{l_2} \left[\tilde{Q}_3^A + \frac{\tilde{Q}_1^A k_1 k_3 (1-\epsilon)}{\epsilon k_1^2 + k_2^2 + k_3^2} \right] \end{aligned} \right\} \quad (14)$$

where $\epsilon = l_2^2/l_1^2$. The considerably greater complexity of equation (13) is perhaps obscured by the purposely expanded form of equations (14).

If the initial wave normal $\mathbf{k}$ is perpendicular to the (longitudinal) x_1 -axis, the component k_1 vanishes and equations (14) reduce further to

$$\left. \begin{aligned} \tilde{Q}_1^B &= \tilde{Q}_1^A/l_1 \\ \tilde{Q}_2^B &= \tilde{Q}_2^A/l_2 \\ \tilde{Q}_3^B &= \tilde{Q}_3^A/l_2 \end{aligned} \right\} \quad (15)$$

The same equations result when $\tilde{Q}_3^A$ may be neglected in comparison with $\tilde{Q}_2^A$ and $\tilde{Q}_3^A$, that is, when the amplitude vector is substantially normal to the x_1 -axis. *Equations (15) state that an axisymmetric contraction defined by l_1, l_2 alters*

⁷ A contraction such that all cross sections of the tunnel are similar, whence $l_2=l_3$, is termed *axisymmetric*; the sections need not be circular.

these waves by a factor of $1/l_1$ in the longitudinal velocity component and a factor of $1/l_2$ in the lateral velocity components. These equations apply only to particular types of wave; yet when the contraction effect is later integrated over the random aggregation of waves representing isotropic turbulence, the over-all results are found not to differ greatly from the simple factors $1/l_1$ and $1/l_2$, respectively.

The same factors were obtained by Prandtl (ref. 1) for other special disturbances: the factor $1/l_1$ from energy considerations for purely longitudinal disturbance velocities, and the factor $1/l_2$ from momentum considerations for a rotating cylindrical element of fluid with axis normal to the stream.

EFFECT OF CONTRACTION ON SPECTRUM AND CORRELATION TENSORS

Effect of contraction on correlation tensor.—The analysis herein leads first to the changes in the spectrum tensor $\Gamma_{\alpha\beta}^A(\underline{k}) \rightarrow \Gamma_{\alpha\beta}^B(\underline{\kappa})$. Then the corresponding changes in the correlation tensor may be obtained from the Fourier transform relation, equation (6b):

$$R_{\alpha\beta}^A(\underline{r}) = \int \int_{-\infty}^{\infty} \Gamma_{\alpha\beta}^A(\underline{k}) e^{-i\underline{k} \cdot \underline{r}} d\tau(\underline{k}) \quad (16a)$$

$$R_{\alpha\beta}^B(\underline{r}) = \int \int_{-\infty}^{\infty} \Gamma_{\alpha\beta}^B(\underline{\kappa}) e^{i\underline{\kappa} \cdot \underline{r}} d\tau(\underline{\kappa}) \quad (16b)$$

In succeeding paragraphs $\Gamma_{\alpha\beta}^B(\underline{\kappa})$ will be determined in terms of the initial spectrum tensor $\Gamma_{\alpha\beta}^A(\underline{k})$ for various cases.

Spectrum tensors at upstream and downstream stations in terms of Q_α .—In an earlier discussion, the Fourier coefficients Q_α were chosen so as to define a field of turbulence confined to a large parallelepiped of volume τ , and vanishing everywhere outside; for this case $\lim_{\tau \rightarrow \infty} \frac{8\pi^3}{\tau} Q_\alpha Q_\beta^*$ was to be identified with the correlation tensor $\Gamma_{\alpha\beta}$. For station A upstream of the contraction it will be convenient to specialize this parallelepiped to a cube of edge D . Such a cube will, however, be distorted into a parallelepiped of edges $l_1 D, l_2 D, l_3 D$ by the stream contraction by the time it reaches station B downstream. (See fig. 1(b).) The spectrum tensors for stations A and B, respectively, are therefore,

$$\left. \begin{aligned} \Gamma_{\alpha\beta}^A(\underline{k}) &= \lim_{D \rightarrow \infty} \frac{8\pi^3}{D^3} Q_\alpha^A(\underline{k}) Q_\beta^{A*}(\underline{k}) \\ \Gamma_{\alpha\beta}^B(\underline{\kappa}) &= \lim_{D \rightarrow \infty} \frac{8\pi^3}{l_1 l_2 l_3 D^3} Q_\alpha^B(\underline{\kappa}) Q_\beta^{B*}(\underline{\kappa}) \end{aligned} \right\} \quad (17)$$

Evaluation of spectrum tensor at downstream station, general case.—The identifications made in the last paragraph allow the spectrum tensor to be evaluated at station B in terms of the spectrum tensor at station A and the parameters l_1, l_2, l_3 defining the stream contraction between stations A and B. For a single plane wave $\tilde{q}_\alpha^A = \tilde{Q}_\alpha^A e^{i\underline{k} \cdot \underline{z}}$, which is transformed by the contraction into $\tilde{q}_\alpha^B = \tilde{Q}_\alpha^B e^{i\underline{\kappa} \cdot \underline{z}}$, equations (13) give

$$\tilde{Q}_\alpha^B = \frac{1}{l_\alpha} \left(\tilde{Q}_\alpha^A - \sum_\beta \frac{\tilde{Q}_\beta^A k_\beta k_\alpha}{l_\beta^2 k^2} \right)$$

In the Fourier integral representation $\tilde{q}_\alpha^A$ is to be interpreted as dq_α^A , $\tilde{q}_\alpha^B$ as dq_α^B , $\tilde{Q}_\alpha^A$ as $Q_\alpha^A d\tau(\underline{k})$, and $\tilde{Q}_\alpha^B$ as $Q_\alpha^B d\tau(\underline{\kappa})$. Accordingly,

$$Q_\alpha^B = \frac{l_1 l_2 l_3}{l_\alpha} \left(Q_\alpha^A - \sum_\beta \frac{Q_\beta^A k_\beta k_\alpha}{l_\beta^2 k^2} \right) \quad (18)$$

since $l_1 l_2 l_3 = d\tau(\underline{k})/d\tau(\underline{\kappa})$. Thus

$$Q_\alpha^B Q_\beta^{B*} = \frac{l_1^2 l_2^2 l_3^2}{l_\alpha l_\beta} \left[Q_\alpha^A Q_\beta^{A*} + \sum_{\gamma, \delta} \left(-\frac{Q_\alpha^A Q_\gamma^{A*} k_\gamma k_\beta}{l_\gamma^2 k^2} - \frac{Q_\beta^{A*} Q_\gamma^A k_\gamma k_\alpha}{l_\gamma^2 k^2} + \frac{Q_\beta^{A*} Q_\delta^A k_\gamma k_\alpha k_\delta k_\beta}{l_\gamma^2 l_\delta^2 k^4} \right) \right]$$

The corresponding relation between the postcontraction and precontraction spectrum tensors is, by virtue of equations (17),

$$\Gamma_{\alpha\beta}^B(\underline{\kappa}) = \frac{l_1 l_2 l_3}{l_\alpha l_\beta} \left[\Gamma_{\alpha\beta}^A(\underline{k}) + \sum_{\gamma, \delta} \left(-\frac{\Gamma_{\alpha\gamma}^A(\underline{k}) k_\gamma k_\beta}{l_\gamma^2 k^2} - \frac{\Gamma_{\gamma\beta}^A(\underline{k}) k_\gamma k_\alpha}{l_\gamma^2 k^2} + \frac{\Gamma_{\gamma\delta}^A(\underline{k}) k_\gamma k_\delta k_\alpha k_\beta}{l_\gamma^2 l_\delta^2 k^4} \right) \right] \quad (19)$$

where $\underline{k}$ is related to $\underline{\kappa}$ by

$$k_1, k_2, k_3 = l_1 \kappa_1, l_2 \kappa_2, l_3 \kappa_3 \quad (20)$$

Special case: axisymmetric contraction but arbitrary initial spectrum.—When the contraction is axisymmetric ($l_2 = l_3$), the equation of continuity in the form

$$\sum_\gamma k_\gamma \Gamma_{\gamma\delta}^A = 0 \quad (21)$$

may be used to simplify equation (19). The result may be written

$$\Gamma_{\alpha\beta}^B(\underline{\kappa}) = \frac{l_1 l_2^2}{l_\alpha l_\beta} \left[\Gamma_{\alpha\beta}^A(\underline{k}) + \frac{(\Gamma_{\alpha 1}^A k_\beta + \Gamma_{1\beta}^A k_\alpha) k_1 (1 - \epsilon)}{\epsilon k_1^2 + k_2^2 + k_3^2} + \frac{\Gamma_{11}^A k_1^2 k_\alpha k_\beta (1 - \epsilon)^2}{(\epsilon k_1^2 + k_2^2 + k_3^2)^2} \right] \quad (22)$$

where the ratio $\epsilon = l_2^2/l_1^2$; for a large speed gain in the contraction $\epsilon \ll 1$.

Special case: axisymmetric contraction, isotropic initial spectrum.—A further simplification occurs when the turbulence at station A is isotropic. In that case, the spectrum tensor $\Gamma_{\alpha\beta}^B(\underline{\kappa})$ downstream of the contraction can be expressed explicitly to within an unknown multiplicative factor $G(k)$. This results from the fact (ref. 3) that $\Gamma_{\alpha\beta}^A(\underline{k})$ must then be an isotropic second-order tensor; the isotropic property together with the continuity condition, equation (21), requires $\Gamma_{\alpha\beta}^A(\underline{k})$ to be of the form

$$\Gamma_{\alpha\beta}^A(\underline{k}) = G(k) (k^2 \delta_{\alpha\beta} - k_\alpha k_\beta) \quad (23)$$

where

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta \end{cases}$$

The right-hand side of equation (22) may be evaluated by means of equation (23). The diagonal terms reduce to relatively simple forms:

$$\Gamma_{11}^B(\kappa) = \frac{l_2^2}{l_1} G(k) \frac{(k^2 - k_1^2)k^4}{(\epsilon k_1^2 + k_2^2 + k_3^2)^2} \quad (24)$$

$$\Gamma_{22}^B(\kappa) = l_1 G(k) \left[k^2 - k_2^2 - \frac{2k_1^2 k_2^2 (1 - \epsilon)}{\epsilon k_1^2 + k_2^2 + k_3^2} + \frac{k_1^2 k_2^2 (k^2 - k_1^2) (1 - \epsilon)^2}{(\epsilon k_1^2 + k_2^2 + k_3^2)^2} \right] \quad (25)$$

and $\Gamma_{33}^B(\kappa)$ is obtained from $\Gamma_{22}^B(\kappa)$ by replacing k_2 by k_3 and vice versa. The relation, equation (20), between $\underline{k}$ and $\underline{\kappa}$ applies here.

One-dimensional longitudinal spectrums.—If the form of the initial spectrum tensor $\Gamma_{\alpha\beta}^A(\underline{k})$ is known, the corresponding one-dimensional spectrums $F_{\alpha}^A(k_1)$ can be calculated, according to the defining equation (7) as applied at the upstream station A:

$$F_{\alpha}^A = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{\alpha\alpha}^A(k_1, k_2, k_3) dk_2 dk_3 \quad (26)$$

A particular case of isotropic turbulence is of special interest (ref. 4): In equation (23) for $\Gamma_{\alpha\beta}^A(\underline{k})$ the function $G(k)$ is taken to be $N(k^2 + \gamma^2)^{-3}$, where N, γ are constants. Then

$$\Gamma_{11}^A = \frac{N(k_2^2 + k_3^2)}{(k_1^2 + k_2^2 + k_3^2 + \gamma^2)^3}$$

and after integration

$$F_1^A(k_1) = \frac{\pi N}{(k_1^2 + \gamma^2)}$$

This one-dimensional longitudinal spectrum is of the same form as an empirical relation obtained in reference 5 for that of isotropic turbulence in the initial period; this agreement is the special virtue of the form assumed for $G(k)$.

The one-dimensional lateral spectrum functions corresponding to the same $G(k)$ are readily evaluated; they are

$$F_2^A(k_1) = F_3^A(k_1) = \frac{\pi N(3k_1^2 + \gamma^2)}{2(k_1^2 + \gamma^2)^2}$$

The equality of the F_2 and the F_3 functions results, of course, from the isotropy of the turbulence.

The effect of the stream contraction on these one-dimensional spectrums is found by employing the postcontraction value of $\Gamma_{\alpha\alpha}$, that is, the value $\Gamma_{\alpha\alpha}^B$ appropriate to the downstream station B. Since $\Gamma_{\alpha\alpha}^B$ is a function of the local wave-number vector $\underline{\kappa}$ at station B, the equation corresponding to (26) is

$$F_{\alpha}^B = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{\alpha\alpha}^B(\kappa_1, \kappa_2, \kappa_3) d\kappa_2 d\kappa_3$$

which is a function of κ_1 .

For performing the integration and making later comparisons of spectrums, it is convenient to transform from $\kappa_1, \kappa_2, \kappa_3$ to k_1, k_2, k_3 , where

$$\kappa_1 = k_1/l_1$$

$$\kappa_2 = k_2/l_2$$

$$\kappa_3 = k_3/l_3$$

and to define $F_{\alpha}^B(k_1) = l_1^{-1} F_{\alpha}^B$, such that $\int_0^{\infty} F_{\alpha}^B(k_1) dk_1 = \int_0^{\infty} F_{\alpha}^B d\kappa_1$; thus

$$F_{\alpha}^B(k_1) = \frac{2}{l_1 l_2 l_3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{\alpha\alpha}^B\left(\frac{k_1}{l_1}, \frac{k_2}{l_2}, \frac{k_3}{l_3}\right) dk_2 dk_3 \quad (27)$$

The spectrum tensor elements $\Gamma_{\alpha\alpha}^B$ following an axisymmetric contraction have been evaluated in equations (24) and (25). With these values inserted and $G(k)$ specified as before, the integrations of equation (25) are best effected in polar coordinates. The results are expressed most simply in terms of a 'normalized' longitudinal wave number k_1/γ as incorporated in the two parameters

$$s \equiv \frac{k_1^2}{\gamma^2} + 1$$

$$t \equiv \frac{\epsilon k_1^2 - k_1^2}{\gamma^2} - 1$$

The final result for the one-dimensional longitudinal spectrum following an axisymmetric contraction ($l_2 = l_3$) is

$$F_1^B(k_1) = \frac{2\pi N}{l_1^2 \gamma^2 t^3} \left[3 + 4t + t^2 + \frac{t}{2s} + \left(2 + 4s + 2t + st + \frac{3s}{t} \right) \ln \left(\frac{s}{s+t} \right) \right] \quad (28)$$

The corresponding result for the one-dimensional lateral spectrums following an axisymmetric contraction is

$$F_2^B(k_1) = F_3^B(k_1) = \frac{\pi N}{2\gamma^2 l_2^2 t^2} \left(\frac{(3s-2)t^2}{s^2} - 4(1-\epsilon)(s-1) \left\{ \frac{2s+t}{2s} + \frac{s+t}{t} \ln \left(\frac{s}{s+t} \right) + \frac{1-\epsilon}{2t} \left[\frac{6s+5t}{2} + \frac{(s+t)(3s+t)}{t} \ln \left(\frac{s}{s+t} \right) \right] \right\} \right) \quad (29)$$

and for $\epsilon \ll 1$ (large speed gain) a simple but very close approximation is

$$F_2^B(k_1) = \frac{\pi N}{2l_2^2 \gamma^2} \frac{1 + 2k_1^2/\gamma^2}{(1 + k_1^2/\gamma^2)^2} + 0(\epsilon, \epsilon \ln \epsilon)$$

(The corresponding approximation for $F_1^B(k_1)$ is not simple enough to warrant noting.) The parameters $l_1, l_2 = l_3$, and $\epsilon = l_2^2/l_1^2$ are related to the initial and final Mach numbers of the main stream, if the fluid is air, by the equations

$$\left. \begin{aligned} l_1^2 &= \left(\frac{M_B}{M_A} \right)^2 \left(\frac{5 + M_A^2}{5 + M_B^2} \right) \\ l_2^2 &= \frac{M_A}{M_B} \left(\frac{5 + M_B^2}{5 + M_A^2} \right)^3 \\ \epsilon &= \left(\frac{M_A}{M_B} \right)^3 \left(\frac{5 + M_B^2}{5 + M_A^2} \right)^4 \end{aligned} \right\} \quad (30a)$$

For incompressible flow ($M_B, M_A \rightarrow 0$) these reduce to

$$\left. \begin{aligned} l_1 &= U_B/U_A \\ l_2 &= l_1^{-1/2} \\ \epsilon &= l_1^{-3} \end{aligned} \right\} \quad (30b)$$

These postcontraction spectrums, equations (28) and (29), are compared with the initial spectrums in figures 2 and 3, respectively; the comparison is based on an assumed initial stream Mach number of 0.05 (station A) followed by an axisymmetric contraction such that the final Mach number is 2.0 (station B); the corresponding parameters are $l_1=29.8$, $l_2=0.382$, $\epsilon=0.00016$. Consider first the longitudinal spectrums, figure 2. Normalizing factors are incorporated such that the areas under the two curves, if replotted on a linear scale, would be the same; this normalization serves to differentiate changes in shape from changes in amplitude. The figure exhibits a rather striking distortion of the spectrum after traversing the stream contraction: the peak spectral density is shifted from zero wave number to $k_1/\gamma=1.4$ along with a general shift of density to the higher wave numbers. Associated with this change in shape is a reduction in amplitude by the factor $\overline{u_B^2}/\overline{u_A^2}$, $\overline{u_A^2}$ and $\overline{u_B^2}$ being the respective integrals of the spectral density curves. These integrals are evaluated in a later section.

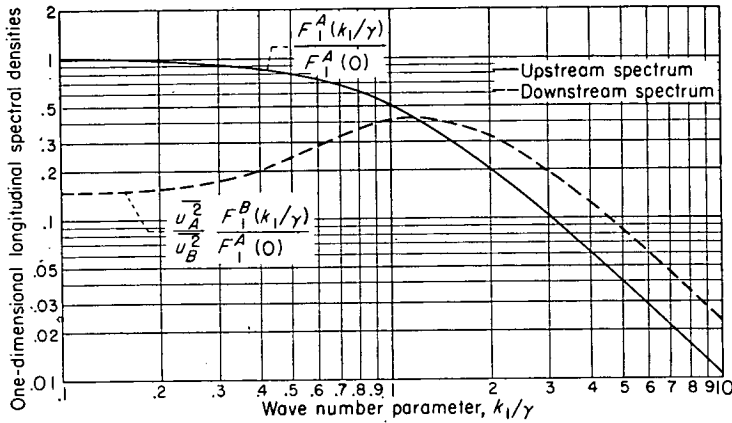


FIGURE 2.—Comparison of one-dimensional longitudinal spectrum upstream (F_1^A) and downstream (F_1^B) of axisymmetric contraction. Curves normalized to same area. Isotropic initial turbulence with $G(k)=N(k^2+\gamma^2)^{-3}$, $M_A=0.05$, $M_B=2.0$, corresponding to $l_1=29.821$, $\epsilon=0.0001637$.

The corresponding comparison for the lateral one-dimensional spectrums is made in figure 3. In this case the axisymmetric contraction has made very little distortion in the spectrum. There is again a change in magnitude (this time an increase) in the ratio $\overline{v_B^2}/\overline{v_A^2}$.

The changes in magnitude (that is, the changes in area under the spectral density curves) correspond to the changes $\overline{u_B^2}/\overline{u_A^2}$ and $\overline{v_B^2}/\overline{v_A^2}$ in the mean-square components of turbulence and are, at least qualitatively, well-known. The predicted changes in the shape of the spectrum curves are apparently new.

In the above comparisons both precontraction and postcontraction spectrums have been expressed in terms of the precontraction longitudinal wave number k_1 , whereas the local postcontraction wave number is $\kappa_1=k_1/l_1$. Consider, however, a representative longitudinal wave which has the form $\cos k_1 x$ at station A and $\cos \kappa_1 \xi$ at station B. If x and ξ

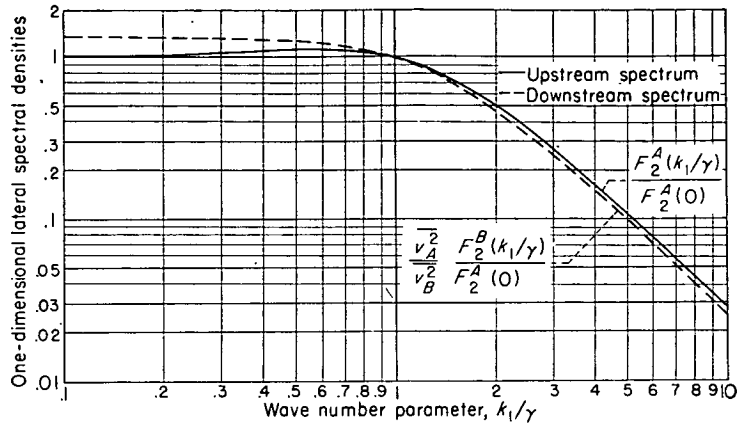


FIGURE 3.—Comparison of one-dimensional lateral spectrum upstream (F_2^A) and downstream (F_2^B) of axisymmetric contraction. Curves normalized to same area. Isotropic initial turbulence with $G(k)=N(k^2+\gamma^2)^{-3}$, $M_A=0.05$, $M_B=2.0$, corresponding to $l_1=29.821$, $\epsilon=0.0001637$.

are identified with the respective distances swept in time t by the moving waves over stationary hot-wire probes at stations A and B, respectively, then $k_1 x = k_1 U_A t$ and $\kappa_1 \xi = \kappa_1 U_B t = \frac{k_1}{l_1} U_A t$. Thus the (temporal) frequency seen by the hot wire in both cases is $k_1 U_A / 2\pi$. The comparison based on k_1 therefore constitutes, in effect, a comparison of the time spectrums that would be seen by stationary hot-wire probes, in contradistinction to the space spectrums discussed in the earlier parts of the paper.

EFFECT OF CONTRACTION ON MEAN-SQUARE VELOCITY COMPONENTS FOR ISOTROPIC TURBULENCE

The mean-square velocity components of the turbulent field may be identified as the diagonal terms of the correlation tensor $R_{\alpha\beta}(\underline{r})$ with $\underline{r}$ set equal to zero. Thus

$$\overline{u^2} = R_{11}(0)$$

$$\overline{v^2} = R_{22}(0)$$

$$\overline{w^2} = R_{33}(0)$$

where u, v, w have been written for q_1, q_2, q_3 , respectively. The evaluation of these means is much less laborious than the evaluation of the general correlation tensor. In particular, the evaluation of the ratio of the means $\overline{u_B^2}/\overline{u_A^2}$, etc., may be made when the initial turbulence is specified to be isotropic but no further details of its spectrum are known. These ratios will be calculated in the following paragraphs.

Evaluations of $\overline{u^2}$ and $\overline{v^2}$ at upstream station.—According to equation (6)

$$R_{11}^A(0) = \overline{u_A^2} = \int \int \int_{-\infty}^{\infty} \Gamma_{11}^A(\underline{k}) d\tau(\underline{k})$$

For isotropic turbulence $\Gamma_{\alpha\beta}$ has the form specified in equation (23), whence

$$\overline{u_A^2} = \int \int \int_{-\infty}^{\infty} G(k)(k^2 - k_1^2) d\tau(\underline{k})$$

where, $G(k)$ is an arbitrary function. It is convenient to transform to spherical polar coordinates:

$$\left. \begin{aligned} k_1 &= k \cos \theta \\ k_2 &= k \sin \theta \cos \varphi \\ k_3 &= k \sin \theta \sin \varphi \\ d\tau(k) &= k^2 \sin \theta \, d\theta \, d\varphi \, dk \end{aligned} \right\} \quad (31)$$

Then

$$\overline{u_A^2} = \int_0^\infty k^4 G(k) dk \int_0^{2\pi} d\varphi \int_0^\pi \sin^3 \theta \, d\theta \quad (32)$$

For the present purpose the function $G(k)$, which, together with the condition of isotropy, defines the turbulence, may be left unspecified; the integral involving $G(k)$ will cancel out in forming the ratio $\overline{u_B^2}/\overline{u_A^2}$. Let this integral have the value H ; then

$$\overline{u_A^2} = \frac{8}{3} \pi H$$

By virtue of the assumed isotropy

$$\overline{v_A^2} = \overline{w_A^2} = \frac{8}{3} \pi H$$

Evaluation of ratio of $\overline{u^2}$ at downstream station to $\overline{u^2}$ at upstream station.—The mean value $\overline{u_B^2}$ is obtained from an integration involving the spectrum tensor after the latter has been transformed by passage of the flow through the tunnel contraction; according to equation (16b)

$$R_{11}^B(0) = \overline{u_B^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{11}^B(\underline{k}) d\tau(\underline{k})$$

For the present case, where the spectrum tensor at station A is assumed isotropic and the contraction is axisymmetric, the transformed tensor $\Gamma_{11}^B(\underline{k})$ has been determined in equation (24). Thus

$$\overline{u_B^2} = \frac{l_2^2}{l_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{G(k) k^4 (k^2 - k_1^2) d\tau(\underline{k})}{(\epsilon k_1^2 + k_2^2 + k_3^2)^2}$$

Because of the unspecified function $G(k)$, it is convenient to change the variables of integration from the components of $\underline{k}$ to the components of k . In other words, a transformation is made from the "wave-number space" of station B to the "wave-number space" of station A. The transformation follows from the Cartesian relations

$$\left. \begin{aligned} d\tau(\underline{k}) &= dk_1 dk_2 dk_3 \\ d\tau(\underline{k}) &= d\kappa_1 d\kappa_2 d\kappa_3 \\ &= \frac{dk_1}{l_1} \frac{dk_2}{l_2} \frac{dk_3}{l_3} \end{aligned} \right\} \quad (33)$$

together with $l_2 = l_3$ for an axisymmetric contraction, whence

$$\overline{u_B^2} = \frac{1}{l_1^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{k^4 G(k) (k^2 - k_1^2) d\tau(\underline{k})}{(\epsilon k_1^2 + k_2^2 + k_3^2)^2}$$

Again the polar-coordinate transformation (equation (31)) is made, with the result

$$\overline{u_B^2} = \frac{1}{l_1^2} \int_0^\infty k^4 G(k) dk \int_0^{2\pi} d\varphi \int_0^\pi \frac{\sin^3 \theta \, d\theta}{(\epsilon \cos^2 \theta + \sin^2 \theta)^2}$$

The first two integrals occur also in $\overline{u_A^2}$ (equation (32)), and they cancel in obtaining the ratio $\overline{u_B^2}/\overline{u_A^2}$; thus

$$\frac{\overline{u_B^2}}{\overline{u_A^2}} = \frac{3}{4l_1^2} \int_0^\pi \frac{\sin^3 \theta \, d\theta}{(\epsilon \cos^2 \theta + \sin^2 \theta)^2}$$

The final result may be written

$$\frac{\overline{u_B^2}}{\overline{u_A^2}} = \frac{3}{4l_1^2} \left[\frac{-1}{1-\epsilon} + \frac{2-\epsilon}{(1-\epsilon)^{3/2}} \tanh^{-1} \sqrt{1-\epsilon} \right] \quad (34)$$

and an asymptotic expansion for small ϵ is

$$\frac{\overline{u_B^2}}{\overline{u_A^2}} = -\frac{3}{4l_1^2} \left[1 + \frac{3}{2} \epsilon + (1+\epsilon) \ln \frac{\epsilon}{4} + O(\epsilon^2 \ln \epsilon) \right]$$

Equation (34) gives the ratio of the mean-square longitudinal velocity fluctuation downstream of an axisymmetric tunnel contraction to the corresponding mean square upstream of the contraction, when the initial turbulence is isotropic. The contraction is characterized by an increase in the stream speed in the specified ratio l_1 and a decrease in the lateral dimensions in the specified ratio l_2 ; the parameters l_1 , l_2 , and $\epsilon \equiv l_2^2/l_1^2$ are completely defined by the initial and final Mach numbers of the stream according to equations (30a) and (30b).

The variation of $\sqrt{\overline{u_B^2}/\overline{u_A^2}}$ with the speed ratio l_1 is plotted in figure 4 for two examples: in the first the flow is assumed compressible with a Mach number 0.05 at the start of the contraction; in the second the flow is assumed incompressible ($M_A, M_B \rightarrow 0$). The Mach number scale at the bottom applies only to the compressible case, the l_1 scale to both cases. The salient characteristic of the curves is the marked reduction in the longitudinal component of turbulence with increasing speed ratio l_1 .

Compressibility is seen to have but a secondary effect, which is appreciable only at supersonic speeds. Note (equations (30) and (34)) that with l_1 as the independent variable, the effect of compressibility appears only in the parameter ϵ . The physical significance of ϵ follows from the definition of l_1 as the speed ratio provided by the contraction and l_2^2 as the area ratio of the contraction (in the axisymmetric case considered), with $\epsilon \equiv l_2^2/l_1^2$. For supersonic final speeds it is more proper to speak of a converging-diverging nozzle than a contraction, the term 'contraction' having been retained herein primarily for reasons of past usage.

The basis of the compressibility effect may be summed up in the following way. The influence of an axisymmetric stream contraction arises from distortion of the fluid elements, as described by the parameters l_1 and l_2 . (See fig. 1 (b).)

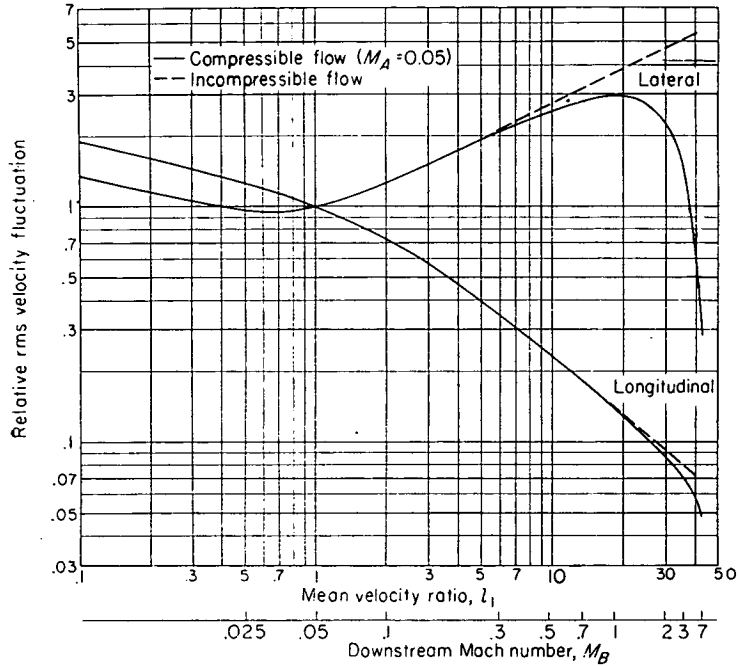


FIGURE 4.—Typical examples of selective effect of axisymmetric stream contraction without decay on components of turbulent intensity, showing influence of compressibility. Isotropic turbulence at $l_1=1$.

These parameters are related by the continuity condition $\sigma l_1 l_2^2 = 1$, where σ is the density ratio. Thus compressibility, in allowing σ to deviate from unity, changes the relation between l_1 and l_2 somewhat, and consequently modifies the contraction effect.

The graph of equation (34) in figure 4 is primarily for illustrative purposes; a form more useful for engineering applications is given in figure 5. The single curve provides the variation of $\sqrt{u_B^2/u_A^2}$ with both l_1 and ϵ ; l_1 and ϵ may be determined from the initial and final Mach numbers by means of the simple relations (30a).

Evaluation of ratio of $\bar{v}^2$ at downstream station to $\bar{v}^2$ at upstream station.—The value of $\bar{v}_B^2$ results from an integration involving the transformed spectrum tensor, according to equation (16b)

$$R_{22}^B(0) = \bar{v}_B^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_{22}^B(\underline{k}) d\tau(\underline{k})$$

For isotropic initial turbulence and an axisymmetric contraction, the transformed spectrum tensor $\Gamma_{22}^B(\underline{k})$ has been evaluated in equation (25). Thus

$$\bar{v}_B^2 = l_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k) \left[k^2 - k_2^2 - \frac{2k_1^2 k_2^2 (1-\epsilon)}{\epsilon k_1^2 + k_2^2 + k_3^2} + \frac{k_1^2 k_2^2 (k^2 - k_1^2) (1-\epsilon)^2}{(\epsilon k_1^2 + k_2^2 + k_3^2)^2} \right] d\tau(\underline{k})$$

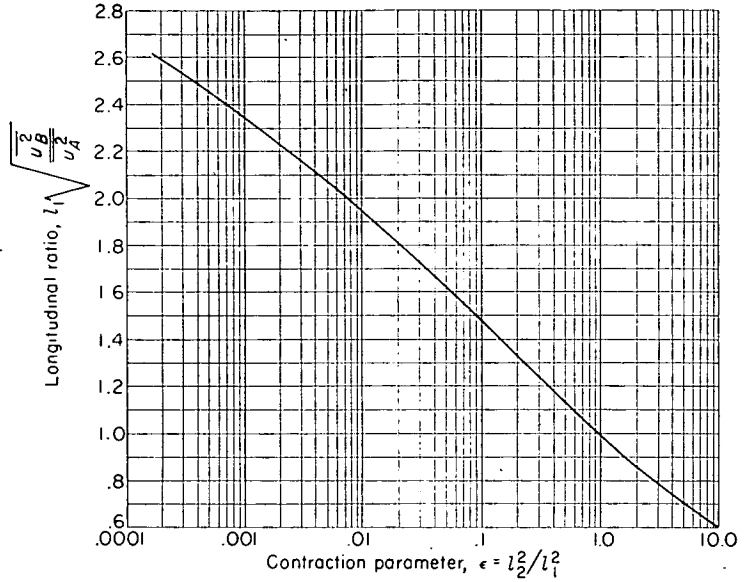


FIGURE 5.—Variation of relative root-mean-square longitudinal velocity fluctuation with both speed ratio l_1 and area ratio l_2^2 for axisymmetric contraction and isotropic turbulence at $l_1=1$.

Again it is convenient to transform from $\underline{k}$ -space to $\underline{k}$ -space (equations (33)) and to introduce polar coordinates k , φ , and θ (equations (31)). The integrations with respect to k and φ are readily disposed of, with the result

$$\frac{\bar{v}_B^2}{\bar{v}_A^2} = \frac{H}{l_2^2} \left[2\pi \int_0^\pi \sin^3 \theta d\theta - 2\pi(1-\epsilon) \int_0^\pi \frac{\sin^3 \theta \cos^2 \theta d\theta}{\sin^2 \theta + \epsilon \cos^2 \theta} + \pi(1-\epsilon)^2 \int_0^\pi \frac{\sin^5 \theta \cos^2 \theta d\theta}{(\sin^2 \theta + \epsilon \cos^2 \theta)^2} \right]$$

where $H \equiv \int_0^\infty k^4 G(k) dk$, as before. Upon carrying out the integration and dividing by $\bar{v}_A^2 = \frac{8}{3} \pi H$, there is obtained finally

$$\frac{\bar{v}_B^2}{\bar{v}_A^2} = \frac{3}{8l_2^2} \left[\frac{2-\epsilon}{1-\epsilon} - \frac{\epsilon^2}{(1-\epsilon)^{3/2}} \tanh^{-1} \sqrt{1-\epsilon} \right] \quad (35)$$

For small ϵ this has the asymptotic expansion

$$\frac{\bar{v}_B^2}{\bar{v}_A^2} = \frac{3}{8l_2^2} \left\{ \left[2 + \epsilon + \frac{1}{2} \epsilon^2 \ln \epsilon \right] + O(\epsilon^2) \right\}$$

Equation (35) gives the ratio of the mean-square lateral velocity fluctuation downstream of an axisymmetric tunnel contraction to the corresponding mean square upstream of the contraction, where the turbulence has been assumed to be isotropic. The variation of $\sqrt{\bar{v}_B^2/\bar{v}_A^2}$ with the speed ratio l_1 is plotted in figure 4, which already contains the graph of $\sqrt{u_B^2/u_A^2}$ discussed earlier; again the two cases are incompressible flow and compressible flow with an initial Mach number of 0.05. For $l_1 \geq 1$ and incompressible flow, the lateral component of turbulence is seen to increase steadily with l_1 , in marked contrast to the decrease exhibited by the

longitudinal component. The curve (of the lateral component) for compressible flow begins to differ sensibly from the curve for incompressible flow for downstream Mach numbers above 0.3; above sonic speed compressibility is seen to effect a complete reversal of the curve. The over-all effect of compressibility on the contraction effect is thus much greater for the lateral than for the longitudinal component of the turbulence.

The graph of equation (35) in figure 4 is primarily illustrative; a form more useful for engineering applications is given in figure 6. The single curve provides the variation of $\sqrt{v_B^2/v_A^2}$ with both l_1 and ϵ ; l_1 and ϵ may be determined from the initial and final Mach numbers by means of equations (30a).

DECAY CONSIDERATIONS

CRITERIA FOR NEGLIGIBLY SMALL DECAY

The basis of the present analysis of the contraction effect is embodied in equations (11a) relating the precontraction and postcontraction vorticity distributions. The simplicity of this result and its derivation arises from the neglect of the turbulent decay; by decay is meant the viscous dissipation and all the (nonlinear) intermixing processes of the eddies which together cause the mean turbulent intensity to diminish with time. The postulation of an inviscid fluid eliminated the viscous dissipation, and the limitation to very weak turbulence eliminated the intermixing processes. (While there can be no dissipation in an inviscid fluid, the intermixing processes ordinarily associated with decay will occur.) In order to assess the influence of these assumptions, equations (11a) will now be derived in a more general fashion with the Navier-Stokes equations as the starting point. For simplicity the fluid is taken to be incompressible, since the major conclusions are unaffected thereby.

General formulation of changes in vorticity.—By rearrangement and cross differentiation to eliminate the pressure term (ref. 8, p. 578), the Navier-Stokes equations can be transformed into

$$\frac{D\omega_1}{Dt} = \omega_1 \frac{\partial q_1'}{\partial x_1} + \omega_2 \frac{\partial q_1'}{\partial x_2} + \omega_3 \frac{\partial q_1'}{\partial x_3} + \nu \nabla^2 \omega_1 \quad (36)$$

and two similar equations, where $\omega = \omega_1, \omega_2, \omega_3$ is the vorticity and $q' = q_1', q_2', q_3'$ is the resultant velocity. Now let q' be the sum of a stream velocity U, V, W and a turbulent velocity field $q = q_1, q_2, q_3$; also, let $\text{curl } U, V, W = 0$, so that ω is just $\text{curl } q$. Then equation (36) becomes (in tensor notation)

$$\frac{D\omega_1}{Dt} = \underbrace{\omega_\beta \frac{\partial U}{\partial x_\beta}}_{\text{Contraction}} + \underbrace{\omega_\beta \frac{\partial q_1}{\partial x_\beta}}_{\text{Decay}} + \nu \nabla^2 \omega_1 \quad (37)$$

and there are two similar equations. The first set of terms on the right-hand side is identified as the contraction effect, the second set as the decay effect. First the decay terms will be neglected in an attempt to recover equations (11a);

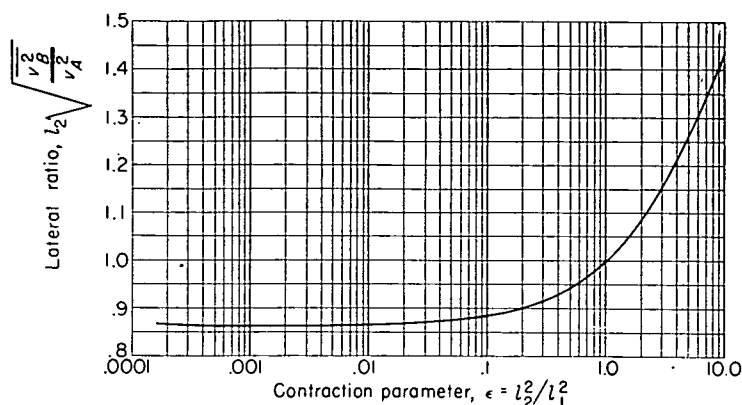


FIGURE 6.—Variation of relative root-mean-square lateral velocity fluctuation with both speed ratio l_1 and area ratio l_2^2 for axisymmetric contraction and isotropic turbulence at $l_1=1$.

then the neglected decay terms will be examined and criteria for their neglect arrived at.

Neglect of decay terms.—Equation (37) minus the viscous terms reads, in expanded form,

$$\frac{D\omega_1}{Dt} = \omega_1 \frac{\partial(U+q_1)}{\partial x_1} + \omega_2 \frac{\partial(U+q_1)}{\partial x_2} + \omega_3 \frac{\partial(U+q_1)}{\partial x_3} \quad (38)$$

In this and the earlier equations $\frac{D}{Dt}$ is the 'Lagrangian' derivative following the fluid motion. Now consider a line segment $\delta x_1, \delta x_2, \delta x_3$ following the fluid motion: its Lagrangian derivative can be shown to be

$$\frac{D\delta x_1}{Dt} = \delta x_1 \frac{\partial(U+q_1)}{\partial x_1} + \delta x_2 \frac{\partial(U+q_1)}{\partial x_2} + \delta x_3 \frac{\partial(U+q_1)}{\partial x_3} \quad (39)$$

and two similar equations. It can be seen that a solution of equations (38) and (39), together with their companion equations, is given by

$$\omega_1, \omega_2, \omega_3 \sim \delta x_1, \delta x_2, \delta x_3$$

for all time t ; this result is well known. Now complete the neglect of the decay terms by omitting the terms in q_1 in equation (38) and correspondingly in equation (39). By this neglect the turbulent perturbations of the flow streamlines have been suppressed: this can be inferred from the revised equation (39). If the particles are at station A at a time $t=0$ and reach station B at time $t=t$, there results

$$\frac{\omega_1^B}{\omega_1^A} = \frac{\delta x_1^B}{\delta x_1^A}$$

and two similar equations. But $\frac{\delta x_1^B}{\delta x_1^A}$ is just l_1 , $\frac{\delta x_2^B}{\delta x_2^A}$ is l_2 , and $\frac{\delta x_3^B}{\delta x_3^A}$ is l_3 .

Therefore equations (11a) have been recovered for the incompressible case (density ratio $\sigma=1$).

Consideration of inertial decay terms.—In equations (37) to (39) the decay terms not involving ν are the inertial or

intermixing terms. These are seen to be nonlinear. The condition for their neglect is evidently

$$\left| \omega_\beta \frac{\partial q_1}{\partial x_\beta} \right| \ll \left| \omega_\beta \frac{\partial U}{\partial x_\beta} \right| \quad (40)$$

and two similar conditions between q_2 and V , q_3 and W , respectively. In a contraction like that of a wind tunnel, the dominant velocity gradients will be $\frac{\partial U}{\partial x_1}$, $\frac{\partial V}{\partial x_2}$, $\frac{\partial W}{\partial x_3}$, and these will be of the same order of absolute magnitude. A sufficient condition to replace expression (40) is therefore

$$\left| \frac{\partial q_\alpha}{\partial x_\beta} \right| \ll \left| \frac{\partial U}{\partial x_1} \right| \quad (41)$$

that is, all of the turbulent velocity gradients are very much less than the axial gradient of the stream velocity. This is essentially the assumption underlying the distortion equations (10), which led directly to the vorticity changes (11a) in Taylor's method.

In statistical terms an approximate inference from equation (41) is, for isotropic turbulence,

$$\sqrt{\left(\frac{\partial u}{\partial x_1} \right)^2} \ll \left| \frac{\partial U}{\partial x_1} \right|$$

But by definition of λ this may be written

$$\frac{\sqrt{\overline{u^2}}}{\lambda} \ll \left| \frac{\partial U}{\partial x_1} \right| \quad (42)$$

The "microscale" λ may be interpreted as a sort of average eddy diameter weighted in favor of the smaller eddies. Equation (42) may be accepted as a practical criterion for the neglect of the inertial decay terms, equivalent to one of the two assumptions underlying equation (10). The other assumption, neglect of viscosity, is considered next.

Consideration of viscous decay term.—The viscous decay term in equation (37) is the term containing ν . This term is linear and so will affect individual plane waves separately without mutual interference. The magnitude of the term may be estimated to a sufficient approximation by considering a wave carried along by the contracting stream and neglecting (for this term only) the distortion of the wave imposed by the contraction. Thus a component of the wave may be written

$$\omega_1 = \Omega_1 e^{i(k \cdot x - k_1 U t)}$$

Then, if the inertial decay terms of equation (37) are negligible, the equation reads, with $\left| \frac{\partial U}{\partial x_2} \right| = \left| \frac{\partial U}{\partial x_3} \right| \ll \left| \frac{\partial U}{\partial x_1} \right|$

$$\begin{aligned} \frac{D\omega_1}{Dt} &= \omega_1 \frac{\partial U}{\partial x_1} + \nu \nabla^2 \omega_1 \\ &= \omega_1 \frac{\partial U}{\partial x_1} + \nu(k_1^2 + k_2^2 + k_3^2)\omega_1 \\ &= \omega_1 \left(\frac{\partial U}{\partial x_1} + \nu k^2 \right) \end{aligned}$$

Accordingly, viscosity may be neglected for that portion of the spectrum which satisfies the inequality

$$\nu k^2 \ll \left| \frac{\partial U}{\partial x_1} \right| \quad (43)$$

In the "initial period" of decay, if the inertial decay criterion (42) is satisfied, the major part of the spectrum will satisfy condition (43).

ROUGH ESTIMATION OF MUTUAL EFFECTS OF DECAY AND CONTRACTION

When the decay effects are not negligible compared with the contraction effects (see criterion developed in the last section) the theoretical basis of the present theory of the contraction effect is violated. Because negligible decay is more the exception than the rule, there is considerable incentive to attempt to apply the theory outside the valid range by means of assumptions concerning the simultaneous effects of decay and contraction.

Suppose, now, the decay and the contraction are considered to occur alternatively in small steps, starting from isotropic turbulence. Each stream tube is considered to contract stepwise: between steps there is decay without contraction; at each step there is a sudden contraction without decay. Let the change in speed ratio per step be dl_1 , the reduction in $\overline{u^2}$ due to decay be $(d\overline{u^2})_D$, and the reduction in $\overline{u^2}$ due to contraction be $(d\overline{u^2})_C$. Express the effect of decay in the absence of contraction in the form

$$\left(\frac{\overline{u^2}}{\overline{u_A^2}} \right)_D = D(l_1) \quad (44)$$

where l_1 is a function of the time of travel (decay time) t , and the effect of contraction in the absence of decay in the form

$$\left(\frac{\overline{u^2}}{\overline{u_A^2}} \right)_C = C(l_1) \quad (45)$$

The corresponding differential forms are

$$\frac{(d\overline{u^2})_D}{\overline{u^2}} = \frac{D'(l_1)}{D(l_1)} dl_1 \quad (46)$$

$$\frac{(d\overline{u^2})_C}{\overline{u^2}} = \frac{C'(l_1)}{C(l_1)} dl_1 \quad (47)$$

The assumption is now made that equation (46) applies to the decay effect per step and equation (47) to the contraction effect per step, the only interaction being in the common $\overline{u^2}$.⁸ This assumption neglects the tendency of the decay process to counteract the anisotropy introduced by the

⁸ It is known that in the 'initial' period of decay $\frac{d\overline{u^2}}{dt} \sim -\overline{u^2} \frac{d}{dt}$. Equation (46) amounts to replacing the $-\overline{u^2}$ on the right-hand side by $-(\overline{u^2})_{\text{decay only}}$; some defense may be made of this approximation, considering the progressive deviation from isotropy.

contraction effect. The total effect per step is then

$$\frac{d\bar{u}^2}{u^2} = \left[\frac{C'(l_1)}{C(l_1)} + \frac{D'(l_1)}{D(l_1)} \right] dl_1$$

whence upon integration the over-all effect is

$$\frac{\bar{u}_B^2}{\bar{u}_A^2} = C(l_1)D(l_1) \quad (48)$$

That is, if the effect of contraction alone is expressed by $C(l_1)$ (equation (45)) and the effect of decay alone by $D(l_1)$ (equation (44)), then the joint effect under the assumption is expressed by the product $C(l_1)D(l_1)$.

Equation (48) is intended to provide a very rough adjustment of the theoretical contraction effect $C(l_1)$ to account for decay. This adjustment will be made in the attempt to compare the theory with experimental results in which the decay effects are of the same order as the contraction effects.

Equation (48) refers to the longitudinal velocity component u ; an equation of the same form is obtained for the lateral component v . For both cases the function $D(l_1)$ is taken to be the right-hand side of the empirical decay law (ref. 9)

$$\left(\frac{\bar{u}_B^2}{\bar{u}_A^2} \right)_D = \frac{1}{1 + 0.58 \frac{\sqrt{\bar{u}_A^2}}{L_A} t(l_1)} = D(l_1) \quad (49)$$

for isotropic turbulence in the initial period. The decay time $t(l_1)$ in the formula is the time required by a particle of the main stream to pass through the contraction, the initial velocity being U_A and the final velocity $l_1 U_A$.

COMPARISON WITH EXPERIMENT

There appear to have been no experimental investigations with which to compare the predicted changes imposed by a stream contraction on the spectrum of the turbulence, or on the correlation tensor of the turbulence. The available experimental data seem to be limited to measurements bearing on the changes in the root-mean-square velocity components. These data apply, moreover, to conditions outside the proper scope of the present theory in that large decay effects are present. The experimental data are therefore compared with a crude extension of the theory in which the decay is allowed for in first approximation. (See preceding section.)

The most extensive data are those of MacPhail (ref. 10), which in effect cover a range of contraction ratios from $l_1=1$ to $l_1=9.65$ inasmuch as measurements were made at various stations along the contraction. Isolated points for particular contraction ratios were obtained from investigations made for other purposes by Dryden and Schubauer (ref. 11) and by Hall (ref. 12). Only those points were chosen for which the initial turbulence was indicated to be approximately isotropic.

In table I are listed, for the three experimental arrangements, the parameters used in the estimation of the decay factor (equations (48) and (49)). In reference 10 the initial stream velocity U_A and scale of turbulence L_A were given. In reference 11 the value of U_A was given, and the value of L_A was taken to be 0.05 foot, the only scale mentioned; it was not clear, however, whether this value of scale applied with or without screens. In reference 12 the value of U_A was inferred from collateral information and is somewhat uncertain; the scale L_A was estimated from the dimensions of the honeycomb. In all three experimental arrangements the initial relative levels of turbulence were specified. The decay time t of the turbulence was computed as the time for a particle to traverse the contraction; the value arrived at for Hall's data (ref. 12) reflects the uncertainty in the assumed U_A .

Root-mean-square longitudinal velocity components.—The comparison of the theory, including estimated decay, with experiment for the longitudinal component of turbulence is given in figure 7. The theoretical curve, in each instance, is the product of a value computed for contraction alone, neglecting decay, (obtainable from fig. 5) and a value estimated for decay alone neglecting contraction. (See equations (48) and (49).) The agreement with MacPhail's data and with Hall's single point can be considered good. The agreement with the Dryden-Schubauer point, on the other hand, is poor; a slight improvement would result on correction for the spurious contribution of the noise background.

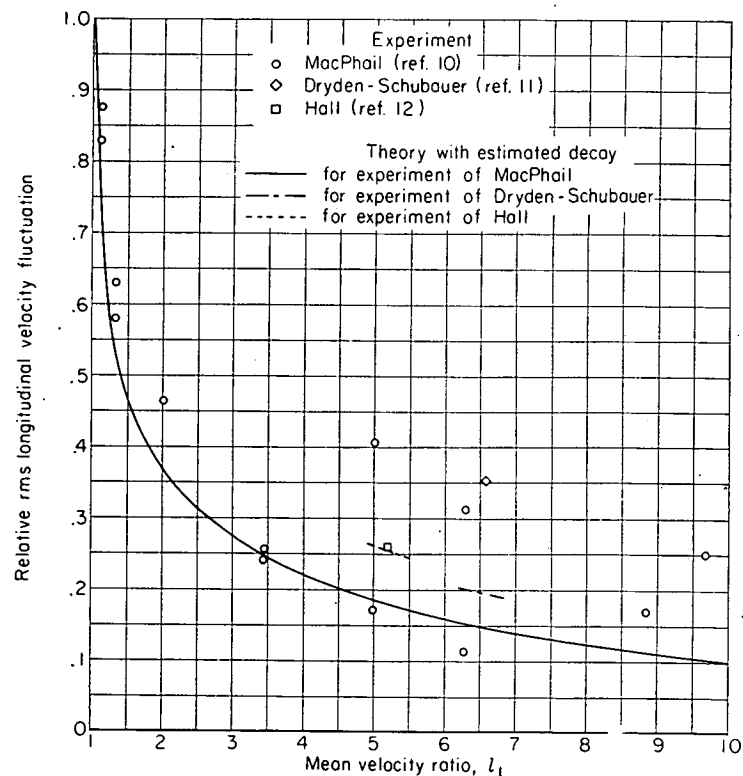


FIGURE 7.—Comparison of predicted axisymmetric contraction effect with experiment for longitudinal component of turbulence, with decay allowed for in first approximation. Initial isotropic turbulence assumed.

Root-mean-square lateral velocity components.—Comparison of the theory, again including estimated decay, with experiment for the lateral component of turbulence is given in figure 8. There is complete disagreement with MacPhail's data and Hall's single point, and on the other hand, good agreement with the Dryden-Schubauer single point. Thus there is the curious result that MacPhail's and Hall's data agree well with theory for the longitudinal component and disagree entirely for the lateral component, whereas the converse is true for the Dryden-Schubauer data.

Discussion.—The uncertainty both in the manner of estimating the decay effect and in the data (table I) on which the estimate was based is still far from sufficient to account for the discrepancies between theory and experiment for the lateral component of turbulence. The very large amplification found by MacPhail is particularly hard to explain. On the other hand, the experimental data of the several observers show considerable disagreement, especially when differences in decay are allowed for. This disagreement would tend to cast doubt on the validity of some of the data;

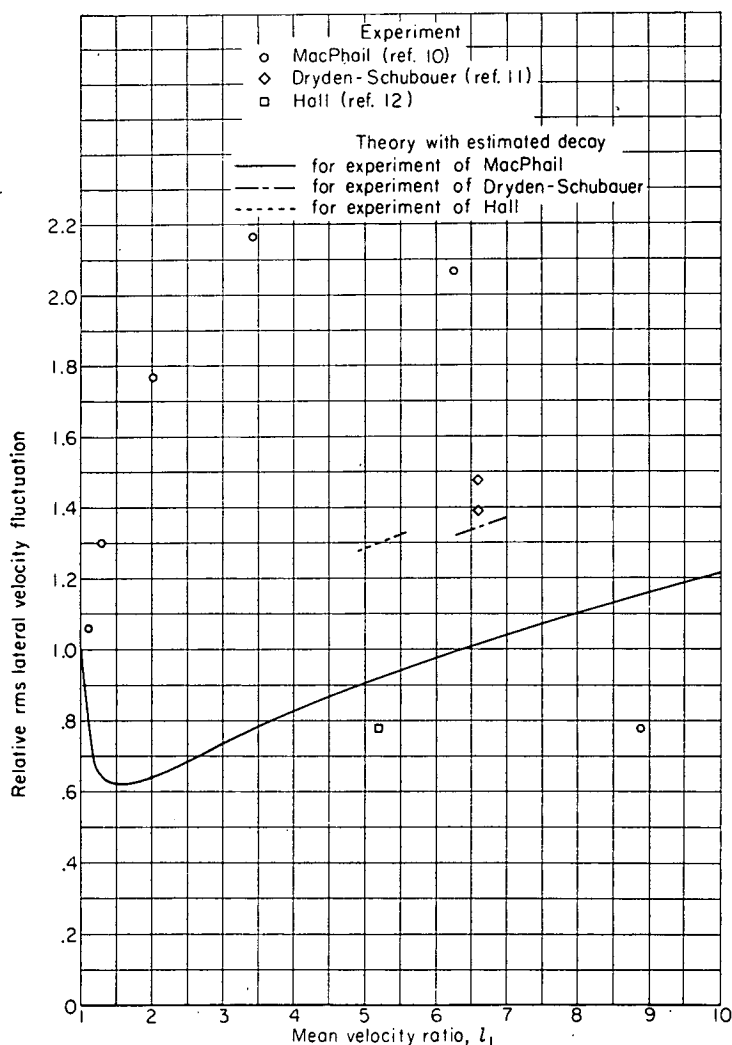


FIGURE 8.—Comparison of predicted axisymmetric contraction effect with experiment for lateral component of turbulence, with decay allowed for in first approximation. Initial isotropic turbulence assumed.

the disagreement may also be in part a consequence, predicted by the theory, of possible differences of the initial spectrums from each other and from isotropy.

CONCLUDING REMARKS

The original aim of this paper was to provide a quantitative explanation of the observed changes in the root-mean-square velocity components of the turbulence of a wind-tunnel stream after passing through the tunnel contraction. The simplifying assumption of negligible decay was made to make the analysis tractable, although the decay and contraction effects are ordinarily comparable. The analysis on this basis disclosed, in addition to the above integrated effects, pronounced changes in the spectrum of the turbulence. The changes in the shape of the spectral density curves, as distinguished from over-all changes in amplitude, would appear to be considerably less sensitive to modification by decay than would the mean-square velocity components. For this reason, and because such spectral changes have not previously been discussed, the emphasis of the present paper has been placed most heavily on these spectral effects.

In particular, it has been found that the one-dimensional longitudinal spectrum for isotropic turbulence exhibits a rather interesting change in shape downstream of the contraction; the center of gravity of the curve of spectral density versus longitudinal wave number is shifted substantially to higher wave numbers, the resulting distortion moving the peak of the curve well to the right of its initial position above the origin. The distortion is quite pronounced and would appear to be readily amenable to experimental observation.

The restrictive assumption of negligible decay largely defeats the original aim of the paper. Nevertheless, for practical reasons an attempt has been made to provide a crude extension to the theory in which decay is allowed for in first approximation. With this approximation the theory has been compared with experimental values of the contraction effect on the longitudinal and lateral component root-mean-square velocity fluctuations. The agreement for the longitudinal component is good, whereas there appears to be almost complete disagreement for the lateral component, the experimental data themselves being in conflict. It is perhaps premature to attempt any general conclusion. For the present, the theory as augmented by the estimated decay effect may be useful in wind-tunnel-design applications.

It is clear that the tunnel contraction effect on the components of turbulent intensity cannot be represented by fixed fractional changes independent of the character of the initial turbulence. Instead the separate factors for the longitudinal and lateral components depend markedly on the spectrum of the turbulence. For initial isotropy, however, unique factors are predicted that, when decay is neglected, are independent of the details of the spectrum.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, August 30, 1951

APPENDIX A

SYMBOLS

The following notation is used in this report:

The subscripts 1, 2, 3 refer to a rectangular coordinate system with the 1-axis aligned with the axis of the main flow and directed downstream, the 2-axis directed horizontally, and the 3-axis directed vertically. Separate systems are used with the origins at stations A and B, respectively. (See fig. 1.) Vector and tensor notations are used interchangeably; for example, $\underline{k} = k_\alpha = (k_1, k_2, k_3)$, where $\alpha = 1, 2$, or 3 designates a vector with components k_1, k_2 , and k_3 .

$C(l_1)$	function defined in equation (45)
$D(l_1)$	function defined in equation (44)
D	edge length of cube within which turbulent field is defined
e	base of natural logarithms
$F_\alpha =$	F_1, F_2 , or F_3
F_1	one-dimensional longitudinal spectral density (see equation (7))
F_2, F_3	one-dimensional lateral spectral densities (see equation (7))
$G(k)$	function appearing in isotropic spectrum tensor
H	constant $\left(\int_0^\infty k^4 G(k) dk\right)$
Im	imaginary part of
$i =$	$\sqrt{-1}$
K_1, K_2, K_3	$k_1 + k_1', k_2 + k_2', k_3 + k_3'$, respectively
k	amplitude of $\underline{k} (= \sqrt{k_1^2 + k_2^2 + k_3^2})$
$\underline{k} =$	$k_\alpha = (k_1, k_2, k_3)$ wave number vector (station A)
L	lateral scale
l_α	l_1, l_2, l_3
l_1	stream velocity at station B divided by stream velocity at station A (see fig. 1)
l_2	stream breadth at station B divided by stream breadth at station A (see fig. 1)
l_3	stream height at station B divided by stream height at station A (see fig. 1)
M	Mach number of main stream
N	amplitude of special isotropic spectrum tensor (see following equation (26))
$\underline{Q} =$	$Q_\alpha = (Q_1, Q_2, Q_3)$ disturbance wave amplitude vector
$\underline{q} =$	$q_\alpha = (q_1, q_2, q_3)$ disturbance velocity vector
$R_{\alpha\beta}(\underline{r})$	correlation tensor (ref. 6)
Re	real part of
r	magnitude of $\underline{r} = \sqrt{r_1^2 + r_2^2 + r_3^2}$
$\underline{r} =$	$r_\alpha = (r_1, r_2, r_3)$ separation vector of two correlated points
s	parameter in equation (28) $\left(= \frac{k_1^2}{\gamma^2} + 1\right)$

t	parameter in equation (28) $\left(= \frac{(\epsilon - 1)k_1^2}{\gamma^2} - 1\right)$
t	time
U	main-stream velocity
$u, v, w =$	q_1, q_2, q_3 disturbance velocity components
X	length of wind-tunnel contraction (distance between stations A and B)
x	used occasionally in place of x_1
$\underline{x} =$	$x_\alpha = (x_1, x_2, x_3)$ position vector (station A)
$\Gamma_{\alpha\beta}(\underline{k})$	spectrum tensor (ref. 3)
γ	constant in special isotropic spectrum tensor (see following equation (26)) ($\gamma = 1/L$)
ϵ	contraction parameter $(= l_2^2/l_1^2)$; see fig. 1
$\epsilon_{\alpha\beta\gamma}$	alternating tensor defined after equation (9)
θ	polar angle (equation (31))
κ	magnitude of $\underline{\kappa} (= \sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2})$
$\underline{\kappa} =$	$\kappa_\alpha = (\kappa_1, \kappa_2, \kappa_3)$ transformed wave number vector (station B); ($\kappa_\alpha = k_\alpha/l_\alpha$)
ν	kinematic viscosity
ξ	magnitude of $\underline{\xi} (= \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2})$
$\underline{\xi} =$	$\xi_\alpha = (\xi_1, \xi_2, \xi_3)$ transformed position vector (station B) (see equation (10) and fig. 1(b))
$\sum_\alpha$	summation over α for $\alpha = 1, 2, 3$
σ	stream density at station B divided by stream density at station A
τ	a volume
φ	azimuth angle (equation (31))
$\underline{\Omega} =$	$\Omega_\alpha = (\Omega_1, \Omega_2, \Omega_3)$ vector amplitude of vorticity wave
$\underline{\omega} =$	$\omega_\alpha = (\omega_1, \omega_2, \omega_3)$ vorticity vector

Superscripts:

A	measured in vicinity of station A, upstream of contraction
B	measured in vicinity of station B, downstream of contraction
$*$	complex conjugate

Subscripts:

A	measured in vicinity of station A, upstream of contraction
B	measured in vicinity of station B, downstream of contraction
$\alpha, \beta, \gamma, \delta$	take on values 1, 2, or 3 and designate tensor quantities
1, 2, 3	specific values of α, β, γ , or δ

A symbol with the mark $\sim$ above it refers to a single plane wave. A bar over a symbol designates an average (usually a spatial average); a bar under a symbol designates a vector.

APPENDIX B

EQUIVALENCE OF SPACE AND TIME AVERAGES IN STATISTICALLY STEADY, HOMOGENEOUS TURBULENCE

The definitions of statistical homogeneity and statistical time independence will first be made precise. Let $F(x, y, z, t)$ be some property of a turbulent field that varies in time and from point to point; thus F may be the pressure, or any of the velocity components, or a correlation of velocity components at two points of fixed separation, (x, y, z) being one of the two points. If, for all choices of the property F , (a) the average of F over a time $T \rightarrow \infty$ is independent of (x, y, z) , the turbulence is defined to be statistically homogeneous; if (b) the average of F over a volume $V \rightarrow \infty$ is independent of t , the turbulence is defined, in the sense used herein, to be statistically steady or time-independent. The respective averages are supposed to be approached uniformly, in the mathematical sense, as T or V , respectively, approaches infinity. (A statistically steady or "stationary" condition is defined differently in the theory of random processes.)

It will now be proved that if the turbulence satisfies the two conditions (a) and (b), the time and space averages defined therein are equal. In this proof, no resort will be made to the "ergodic hypothesis" of statistical mechanics, which leads to the equivalence of the time average and the "ensemble" average. The possibility of the joint existence of the conditions (a) and (b) probably amounts, however, to just as fundamental an assumption.

The space average will be made over a parallelepiped of edges a, b, c and the time average over a time T , and then a limiting process will be applied. The average of F over both space and time is thus

$$\bar{F}_{s,t} = \lim_{a,b,c,T \rightarrow \infty} \frac{1}{abcT} \int_0^T \int_0^c \int_0^b \int_0^a F dx dy dz dt \quad (B1)$$

Any order of integration is permissible, since the integration limits are constants. If the time integration is performed first the expression may be written

$$\bar{F}_{s,t} = \lim_{a,b,c,T \rightarrow \infty} \frac{1}{abcT} \int_0^c \int_0^b \int_0^a \left(\int_0^T F dt \right) dx dy dz$$

By virtue of the postulated uniform convergence of the time and space averages, the operation $\lim_{T \rightarrow \infty}$ may be brought under the integral sign:

$$\begin{aligned} \bar{F}_{s,t} &= \lim_{a,b,c \rightarrow \infty} \frac{1}{abc} \int_0^c \int_0^b \int_0^a \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F dt \right) dx dy dz \\ &= \lim_{a,b,c \rightarrow \infty} \frac{1}{abc} \int_0^c \int_0^b \int_0^a \bar{F}_t dx dy dz \end{aligned} \quad (B2)$$

where $\bar{F}_t$ is the time average of F . But, by condition (a), $\bar{F}_t$ is independent of x, y , and z . Therefore

$$\bar{F}_{s,t} = \bar{F}_t \quad (B3)$$

Alternatively, the space integration and limiting process may be performed first:

$$\begin{aligned} \bar{F}_{s,t} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\lim_{a,b,c \rightarrow \infty} \frac{1}{abc} \int_0^c \int_0^b \int_0^a F dx dy dz \right) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \bar{F}_s dt \end{aligned} \quad (B4)$$

where $\bar{F}_s$ is the space average of F . By condition (b), $\bar{F}_s$ is independent of t ; therefore

$$\bar{F}_{s,t} = \bar{F}_s \quad (B5)$$

Equations (B3) and (B5) together state that

$$\bar{F}_s = \bar{F}_t = \bar{F}_{s,t} \quad (B6)$$

or the *space average*, the *time average*, and the *space-time average* are all equal.

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TABLE 1.—DATA FOR ESTIMATION OF DECAY

Source	h	U_A , ft/sec	L_A , ft	t , sec	$\sqrt{u_A^2}$, ft/sec
MacPhail (ref. 10)	1.20	3.55	0.012	0.19	0.149
	1.60	3.55	.012	.34	-----
	2.55	3.55	.012	.45	-----
	4.90	3.55	.012	.51	-----
	9.65	3.55	.012	.57	-----
Dryden-Schubauer (ref. 11)	6.6	6.86	.05	1.31	.114
Hall (ref. 12)	5.2	1.54	.025	1.22	.046

REPORT 1114

A THERMODYNAMIC STUDY OF THE TURBINE-PROPELLER ENGINE¹

By BENJAMIN PINKEL and IRVING M. KARP

SUMMARY

Equations and charts are presented for computing the thrust, the power output, the fuel consumption, and other performance parameters of a turbine-propeller engine for any given set of operating conditions and component efficiencies. Included are the effects of the pressure losses in the inlet duct and the combustion chamber, the variation of the physical properties of the gas as it passes through the system, and the change in mass flow of the gas by the addition of fuel.

In order to illustrate some of the turbine-propeller-system performance characteristics, the total thrust horsepower per unit mass rate of air flow and the specific fuel consumption are presented for a wide range of flight and engine-design operating conditions and a given set of design component efficiencies.

The performance of a turbine-propeller engine containing a matched set of components is presented for a range of engine operating conditions. The influence of the characteristics of the individual components on off-design-point performance is shown.

The flexibility of operation of two turbine-propeller engines is discussed; one engine has a divided turbine system in which the first turbine drives only the compressor and the second turbine independently drives the propeller, and the other engine has a connected turbine system which drives both the compressor and the propeller.

INTRODUCTION

Various thermodynamic analyses have been prepared for the purpose of studying the many aspects of the performance of turbine-propeller engines. The charts presented in reference 1, for example, permit step-by-step calculation of the turbine-propeller cycle; also presented therein are some performance characteristics of the basic turbine-propeller system and systems incorporating intercooling, reheat, and regeneration at design-point conditions. Reference 2 presents a general comparison of part-load performance characteristics of a large variety of both simple and complex turbine-propeller-engine configurations. It also discusses briefly the way in which component characteristics affect the efficiency of each engine and limit the part-load operation of each engine.

In the present report, charts (developed from an extension of the analysis given in ref. 3) are presented which permit determination of the performance parameters of the engine directly from component efficiencies and operating conditions.

These charts eliminate much of the step-by-step cycle calculation and apply particularly when the engine over-all performance rather than the details of the cycle is of major interest. In order to illustrate some of the turbine-propeller-engine design-point performance characteristics, the thrust horsepower per unit mass rate of air flow and specific fuel consumption are presented for a wide range of design combustion-chamber-outlet temperatures and compressor pressure ratios. These results are presented for constant component efficiencies and a range of flight speeds and altitude conditions.

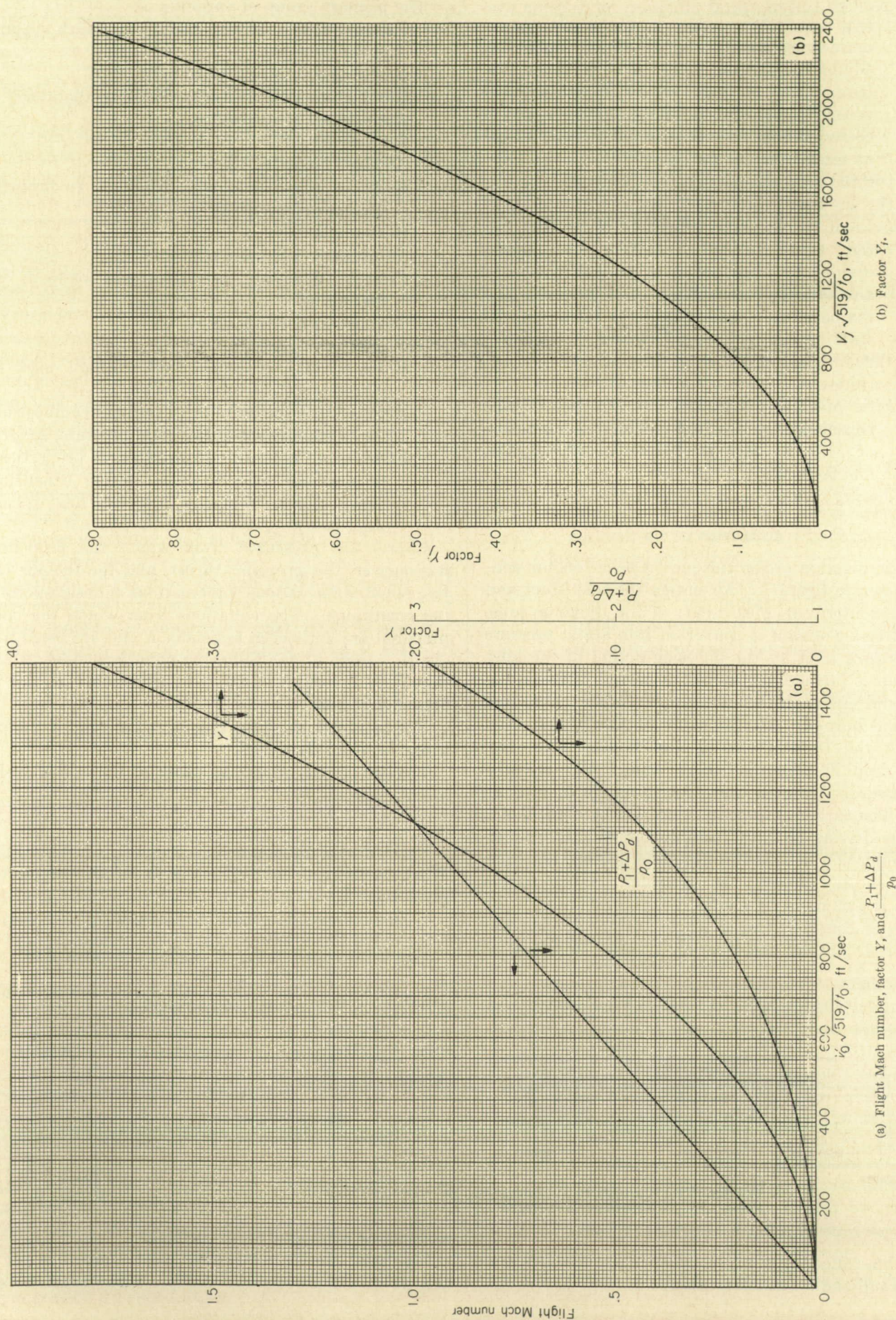
The report also presents a detailed discussion of the off-design-point performance of two turbine-propeller-engine configurations each having a given set of components. The performance of a given turbine-propeller engine is a complex function of the individual characteristics of the compressor, the turbine, and the propeller. The limitations in operating range and performance imposed by these component characteristics, the interrelation between components, some of the problems involved in matching the components, and the method for evaluating and presenting engine performance are discussed for the two engine configurations. One engine has a divided turbine system consisting of two independent turbines; the first turbine drives the compressor and the second turbine drives the propeller through reduction gears. The other engine has the two turbines connected to provide a single rotating system. The flexibility in operation provided by a variable-area exhaust nozzle is also discussed for both configurations. This analysis was made at the NACA Lewis laboratory.

SYMBOLS

The significance of the symbols appearing in the charts and in the subsequent discussion is as follows:

- | | |
|-------|---|
| A_n | effective exhaust-nozzle area, sq ft (For isentropic expansion in exhaust nozzle, flow through area A_n is equal to actual mass flow through nozzle.) |
| a | correction factor that accounts for total-pressure drop in inlet diffuser |
| b | correction factor that accounts for total-pressure drop in combustion chamber |
| c | correction factor that accounts for difference in physical properties of hot exhaust gases and cold air, involved in computation of work from expansion process |

¹ Supersedes NACA TN 2653, "A Thermodynamic Study of the Turbine-Propeller Engine," by Benjamin Pinkel and Irving M. Karp, 1952.



(a) Flight Mach number, factor Y , and $\frac{P_1 + \Delta P_d}{\rho_0}$.

(b) Factor Y_i .

FIGURE 2.—Charts for determining flight Mach number, compressor-inlet total pressure, and factors Y and Y_i .

For the engine operating at a given set of conditions, an optimum division of power between the exhaust jet and the propeller exists for which the total thrust horsepower and efficiency of the system are maximum. The jet velocity for this optimum condition is given very closely by

$$V_{j,opt} = \frac{C_v^2}{\eta_t \eta_p} V_0 \quad (4a)$$

This expression is derived in appendix A. The jet velocity can vary appreciably from this optimum value with only a small effect on the total thrust horsepower and engine efficiency.

At zero flight speed ($V_0=0$ and $\eta_p=0$) the expression for $V_{j,opt}$ is indeterminate. When the factor α is introduced where α is the pounds of thrust produced by the propeller per propeller-shaft horsepower input, the expression for $V_{j,opt}$ becomes

$$V_{j,opt} = \frac{550 C_v^2}{\eta_t \alpha} \quad (4b)$$

The thrust developed by the propeller at any plane speed V_0 ($V_0 \neq 0$) is obtained from the propeller-thrust horsepower by the equation

$$F_p = th p_p \frac{550}{V_0} \quad (5)$$

For the case of $V_0=0$, the factor F_p/α is determined from the charts.

The compressor-shaft horsepower is given by

$$hp_c = M_a J c_{p,a} t_0 Z / 550 = 5675 M_a Z t_0 / 519 \quad (6)$$

The compressor-inlet total temperature is

$$T_1 = t_0(1 + Y) \quad (7)$$

The turbine-shaft horsepower is

$$hp_t = \frac{th p_p}{\eta_p} + hp_c \quad (8a)$$

At zero flight speed ($th p_p=0$ and $\eta_p=0$), the turbine power is

$$hp_t = \frac{F_p}{\alpha} + hp_c \quad (8b)$$

The fuel consumption per unit mass rate of air flow is determined from the fuel-air ratio by the relation

$$W_f/M_a = 115,900 f \quad (9)$$

DISCUSSION OF CHARTS

By means of equations (1) to (9) and the curves of figures 2 to 7, the performance of the turbine-propeller engine can be readily evaluated. The curves are presented in a form that shows the effects of the variables on performance. In figures 2 to 4 are shown curves for evaluating some of the primary parameters that are used in the principal performance chart (fig. 5) from which the thrust horsepower of the propeller is determined. The fuel-air ratio is evaluated with the use of figures 6 and 7.

Curves for obtaining the flight Mach number, the compressor-inlet total pressure, and the factor Y for various

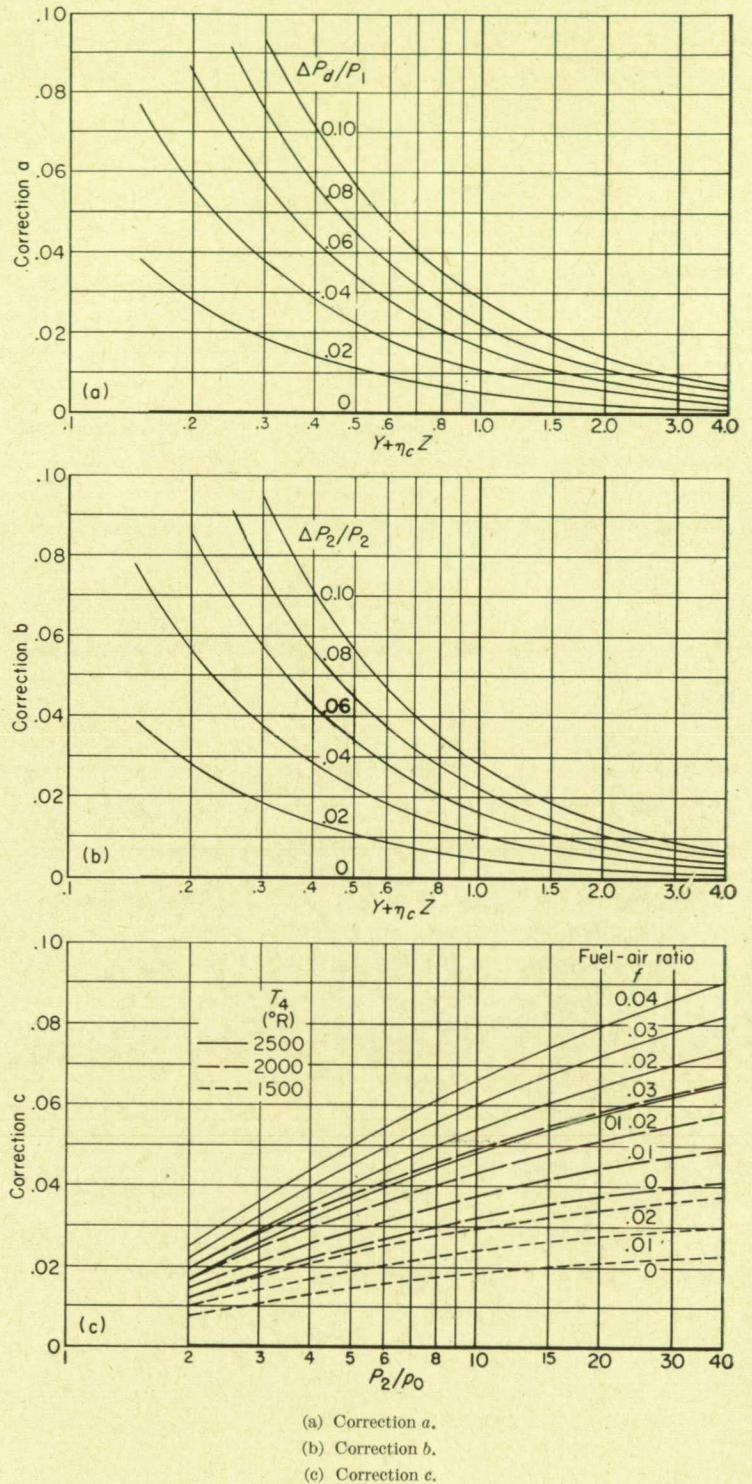


FIGURE 3.—Chart for determining factor ϵ . ($\epsilon = 1 - a - b + c$)

values of the factor $V_0 \sqrt{519/t_0}$ are shown in figure 2 (a). Values of Y_j plotted against the factor $V_j \sqrt{519/t_0}$ are shown in figure 2 (b).

The value of ϵ , which accounts for the effect of the secondary group of variables, is obtained from figure 3. The quantity ϵ is given by the relation

$$\epsilon = 1 - a - b + c$$

Correction a , which gives the effect of total-pressure drop through the inlet duct ΔP_d , is shown in figure 3 (a). Cor-

rection b , which measures the effect of total-pressure drop through the combustion chamber ΔP_2 , is introduced in figure 3 (b). Correction c , which corrects for the difference between the physical properties of the hot exhaust gases and the cold air involved in the computation of the expansion processes through the turbine and exhaust nozzle, is given in figure 3 (c). In general, the value of ϵ is close to unity and can be taken as equal to unity when a rapid approximation is desired.

The compressor total-pressure ratio is plotted against the quantity $\eta_c Z/(1+Y)$ in figure 4. The compressor-shaft horsepower is computed from equation (6) and the value of Z . The effect of the variation in specific heat of air during compression is neglected in this plot; the maximum error in Z introduced is about 1 percent for the range of compressor pressure ratios shown in figure 4 and for compressor-inlet temperatures up to 550° R.

The value of $(P_2/P_1)_{ref}$ plotted against the factor $\eta_c \eta_t \epsilon \frac{T_4}{t_0} \left(\frac{1}{1+Y} \right)^2$ is also shown in figure 4. The quantity $(P_2/P_1)_{ref}$ is useful in that it is the compressor pressure ratio for maximum power per unit mass rate of air flow for any given values of $\eta_c \eta_t \epsilon T_4/t_0$ and Y , provided that the change in component efficiencies and ϵ with change in pressure ratio is negligible. When the change in ϵ with P_2/P_1 is appreciable, $(P_2/P_1)_{ref}$ differs somewhat from the compressor pressure ratio giving maximum power per unit mass rate of air flow. Even in this case, the power per unit mass rate of air flow corresponding to $(P_2/P_1)_{ref}$ is generally within 1 percent of the true maximum. The actual compressor pressure ratio P_2/P_1 divided by the quantity $(P_2/P_1)_{ref}$ defines the value of X used in figure 5.

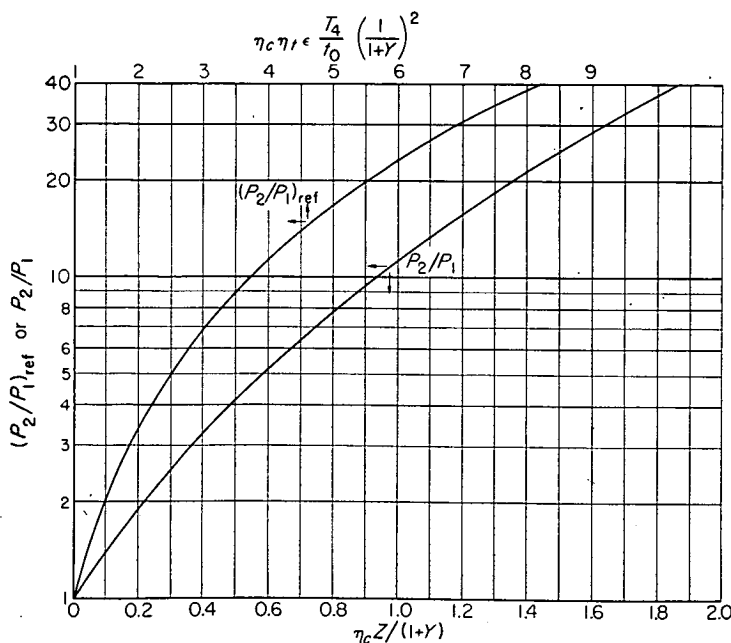


FIGURE 4.—Chart for determining $(P_2/P_1)_{ref}$ for various values of $\eta_c \eta_t \epsilon \frac{T_4}{t_0} \left(\frac{1}{1+Y} \right)^2$ and for determining $\frac{\eta_c Z}{1+Y}$ for various values of P_2/P_1 . (A 17- by 19-in. print of this chart is available from NACA upon request.)

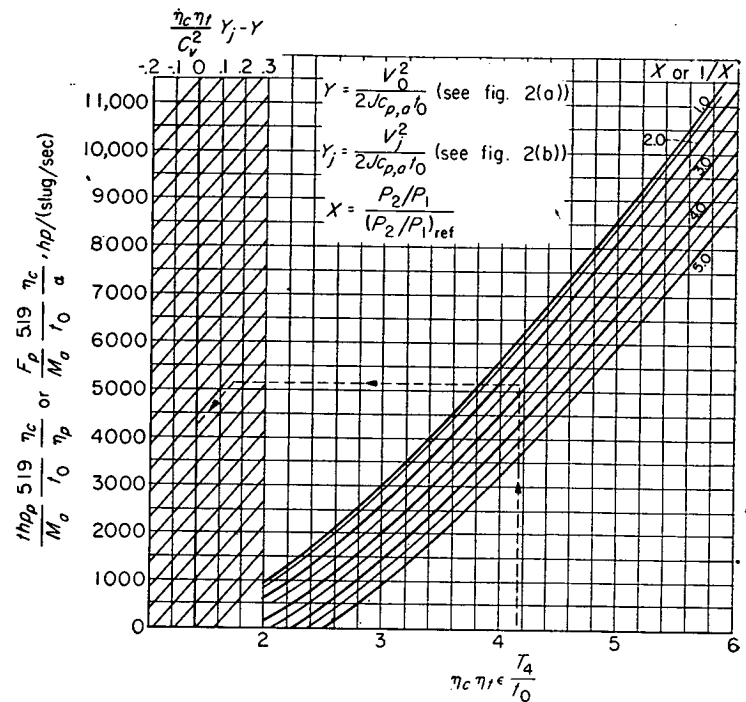


FIGURE 5.—Chart for determining thrust-horsepower or shaft-horsepower factors. (A 17- by 21-in. print of this chart (in two sections) is available from NACA upon request.)

The curves in figure 5 are used to determine the propeller-horsepower factors $\frac{thp_p}{M_a} \frac{519}{t_0} \frac{\eta_c}{\eta_p}$ or $\frac{F_p}{M_a} \frac{519}{t_0} \frac{\eta_c}{\alpha}$ for various values of the parameters $\eta_c \eta_t \epsilon T_4/t_0$, $(\eta_c \eta_t / C_p^2) Y_j - Y$, and X or $1/X$. When X is less than unity, the value of $1/X$ is used; the reason is apparent from an examination of equation (A34) (see appendix B) for figure 5. Corresponding to the values of $\eta_c \eta_t \epsilon T_4/t_0$ and X or $1/X$, a point on the right-hand set of curves is determined; then progressing horizontally across the chart to the desired value of $(\eta_c \eta_t / C_p^2) Y_j - Y$, a second point is located. Moving from this second point parallel to the left-hand set of direction lines until the $(\eta_c \eta_t / C_p^2) Y_j - Y = 0$ line is intercepted, the value of the propeller-thrust-horsepower factor or propeller-shaft-horsepower factor is then read at the intercept.

The compressor-outlet total temperature T_2 plotted against the factor $t_0(1+Y+Z)$ is shown in figure 6. This curve includes the variation in specific heat of air during compression and was computed from reference 4. The variation in specific heat is accounted for in this case; whereas it is neglected in figure 4, because the error introduced in the evaluation of the temperature rise during compression by the assumption of a constant value of specific heat is greater than the error introduced by the same assumption in the evaluation of the compressor power.

The fuel-air-ratio factor $\eta_b f$ is plotted in figure 7 against $T_4 - T_2$ (total-temperature rise in combustion chamber) for various values of T_4 . These curves are based on information given in reference 5 and are for a fuel having a lower heating value h of 18,900 Btu per pound and a hydrogen-carbon ratio of 0.185. For fuels having other values of h , the value of f given in figure 7 is corrected accurately by multiplying it by

the factor $18,900/h$. The effect of hydrogen-carbon ratio of fuel on f is generally small; and, for a range of hydrogen-carbon ratios from 0.16 to 0.21, the error due to the deviation from the value of 0.185 is less than 0.5 percent. The fuel consumption per unit mass rate of air flow is obtained from the value of f and equation (9).

In the preceding discussion of the charts, the effect of the mass of injected fuel was not mentioned. It is shown in appendix A that the effect of the added fuel can be taken into account by substituting the product of turbine total efficiency η_t and $(1+f)$ for the value of η_t in the charts. This adjustment occurs in figure 4 in the factor $\eta_c \eta_t \epsilon \frac{T_4}{t_0} \left(\frac{1}{1+Y} \right)^2$

(which is used in determining $(P_2/P_1)_{ref}$) and in the factors $\eta_c \eta_t \epsilon T_4/t_0$ and $(\eta_c \eta_t / C_v^2) Y_j - Y$ of figure 5. The value of the jet-thrust horsepower is evaluated from the jet velocity by means of equations (1b) and (2). Equations (4a) and (4b), which are used to determine the optimum jet velocity, do not require this adjustment in the value of η_t .

EXAMPLES OF USE OF CHARTS

As an example of the use of figures 2, 4, and 5 and equations (1) and (2) for a rapid approximate computation of the thrust horsepower per unit mass rate of air flow thp/M_a , a case is considered in which the following conditions are given:

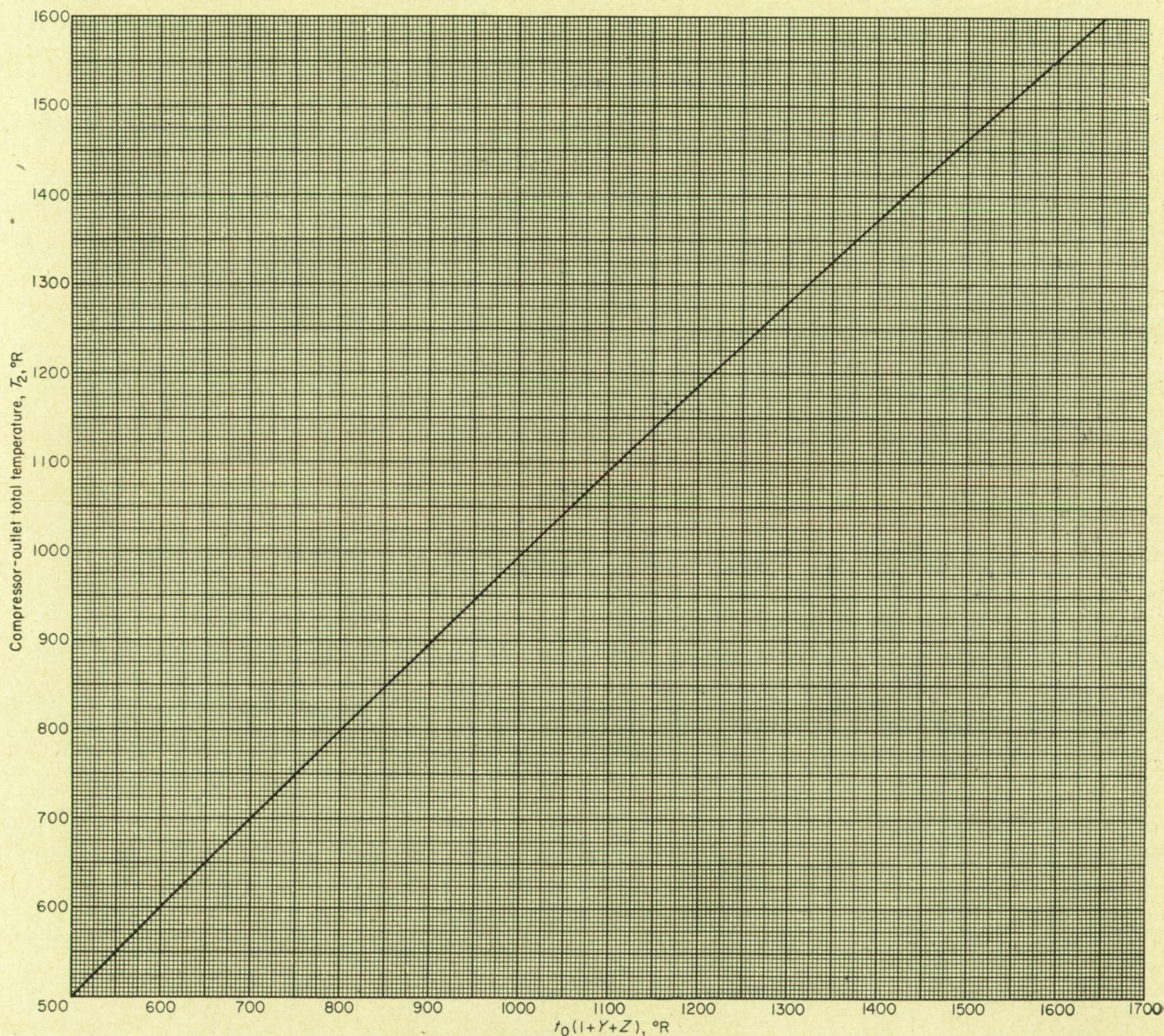


FIGURE 6.—Chart for determining T_2 for various values of $t_0(1+Y+Z)$.

Compressor adiabatic efficiency, η_c -----	0.80
Turbine total efficiency, η_t -----	0.90
Propeller efficiency, η_p -----	0.85
Exhaust-nozzle velocity coefficient, C_v -----	0.96
Airplane velocity, V_0 , ft/sec-----	733
Jet velocity, V_j , ft/sec-----	1000
Compressor pressure ratio, P_2/P_1 -----	6
Ambient-air temperature, t_0 , °R-----	519
Combustion-chamber-outlet temperature, T_4 , °R-----	1960

From the assumption that ϵ is equal to 1, thp/M_a is then evaluated with these given quantities as follows:

$V_0\sqrt{519/t_0}$, ft/sec-----	733
Y (from fig. 2 (a))-----	0.086
$V_j\sqrt{519/t_0}$, ft/sec-----	1000
Y_j (from fig. 2 (b))-----	0.160
$\eta_c\eta_t\epsilon T_4/t_0$ -----	2.719
$\eta_c\eta_t\epsilon \frac{T_4}{t_0} \left(\frac{1}{1+Y} \right)^2$ -----	2.305
$(P_2/P_1)_{ref}$ (from fig. 4)-----	4.31
$X = (P_2/P_1)/(P_2/P_1)_{ref}$ -----	1.39
$(\eta_c\eta_t/C_v^2) Y_j - Y$ -----	0.039
$\frac{thp_p \eta_c}{M_a \eta_p} \frac{519}{t_0}$, hp/(slug/sec) (from fig. 5)-----	2085
thp_p/M_a , hp/(slug/sec)-----	2215
thp_j/M_a , hp/(slug/sec) (from eqs. (1) and (2))-----	355
thp/M_a , hp/(slug/sec)-----	2570

The use of figures 2 to 7 and equations (1) to (9), which permit evaluation of such performance values as compressor-shaft power, fuel consumption, thrust horsepower, and specific fuel consumption with a high degree of accuracy, is illustrated in detail in the following example. The effects of the secondary parameters are now accurately evaluated and the method of accounting for added fuel mass is also shown. The example is based on the engine having the following design conditions:

(1) Compressor adiabatic efficiency, η_c -----	0.80
(2) Turbine total efficiency, η_t -----	0.90
(3) Combustion efficiency, η_b -----	0.97
(4) Propeller efficiency, η_p -----	0.85
(5) Exhaust-nozzle velocity coefficient, C_v -----	0.96
(6) Airplane velocity, V_0 , ft/sec-----	733
(7) Jet velocity, V_j , ft/sec-----	1000
(8) Compressor total-pressure ratio, P_2/P_1 -----	6
(9) Ambient-air temperature, t_0 , °R-----	519
(10) Combustion-chamber-outlet total temperature, T_4 , °R-----	1960
(11) Ambient-air pressure, p_0 , lb/sq in.-----	14.7
(12) Total-pressure drop through inlet duct, ΔP_d , lb/sq in.-----	0.25
(13) Total-pressure drop through combustion chamber, ΔP_c , lb/sq in.-----	1.5
(14) Lower heating value of fuel, h , Btu/lb-----	18,500

DETERMINATION OF Y AND FLIGHT MACH NUMBER

From items (6) and (9):

(15) $V_0\sqrt{519/t_0}$, ft/sec-----	733
From item (15) and figure 2 (a):	
(16) Y -----	0.086
(17) Flight Mach number-----	0.66

DETERMINATION OF Z AND hp_c/M_a

Item (8) read on figure 4 determines

(18) $\eta_c Z/(1+Y)$ -----	0.669
From items (18), (16), and (1):	
(19) Z -----	0.908

Using items (19) and (9) in equation (6) gives

(20) hp_c/M_a , hp/(slug/sec)-----	5153
--------------------------------------	------

DETERMINATION OF f AND W_f/M_a

From items (9), (16), and (19):

(21) $t_0(1+Y+Z)$, °R-----	1035
-----------------------------	------

From item (21) read on figure 6:

(22) T_2 , °R-----	1025
----------------------	------

From items (22) and (10):

(23) $T_4 - T_2$, °R-----	935
----------------------------	-----

From items (23) and (10) and figure 7:

(24) $\eta_b f$ -----	0.01372
-----------------------	---------

Items (24) and (3) give

(25) f -----	0.01414
----------------	---------

Because the lower heating value of the fuel is equal to 18,500 Btu per pound (item (14)), item (25) must be multiplied by the factor 18,900/18,500, and the adjusted value is

(26) f -----	0.01445
----------------	---------

Using item (26) in equation (9) gives

(27) W_f/M_a , (lb/hr)/(slug/sec)-----	1675
--	------

DETERMINATION OF FACTOR ϵ

From items (11), (12), and (13):

(28) $\Delta P_d/p_0$ -----	0.017
-----------------------------	-------

(29) $\Delta P_c/p_0$ -----	0.10
-----------------------------	------

From figure 2 and item (15):

(30) $(P_1 + \Delta P_d)/p_0$ -----	1.335
-------------------------------------	-------

and using item (28) with item (30) gives

(31) P_1/p_0 -----	1.318
----------------------	-------

Using items (31) and (8) yields

(32) P_2/p_0 -----	7.91
----------------------	------

From items (28) and (31):

(33) $\Delta P_d/P_1$ -----	0.013
-----------------------------	-------

whereas from items (29) and (32):

(34) $\Delta P_c/P_2$ -----	0.013
-----------------------------	-------

From items (16) and (18):

(35) $Y + \eta_c Z$ -----	0.812
---------------------------	-------

Items (33) and (35) in figure 3 (a) give

(36) a -----	0.005
----------------	-------

while items (34) and (35) in figure 3 (b) give

(37) b -----	0.005
----------------	-------

When items (10), (26), and (32) are used in figure 3 (c),

(38) c -----	0.035
----------------	-------

From items (36), (37), and (38):

(39) $\epsilon = 1 - 0.005 - 0.005 + 0.035$ -----	1.025
---	-------

DETERMINATION OF $(P_2/P_1)_{ref}$ AND X

Using items (1), (2), (39), (10), (9), and (16) gives

(40) $\eta_c\eta_t\epsilon \frac{T_4}{t_0} \left(\frac{1}{1+Y} \right)^2$ -----	2.363
--	-------

From item (40) read on figure 4:

(41) $(P_2/P_1)_{ref}$ -----	4.50
------------------------------	------

From items (8) and (41) and the definition of the parameter X ,

(42) X -----	1.333
----------------	-------

DETERMINATION OF thp/M_a , SPECIFIC FUEL CONSUMPTION, AND OTHER PERFORMANCE PARAMETERS

Using items (1), (2), (39), (10), and (9) gives

(43) $\eta_c\eta_t\epsilon T_4/t_0$ -----	2.787
---	-------

From items (7) and (9):

(44) $V_j\sqrt{519/t_0}$, ft/sec-----	1000
--	------

and from item (44) read on figure 2 (b):

(45) Y_j -----	0.160
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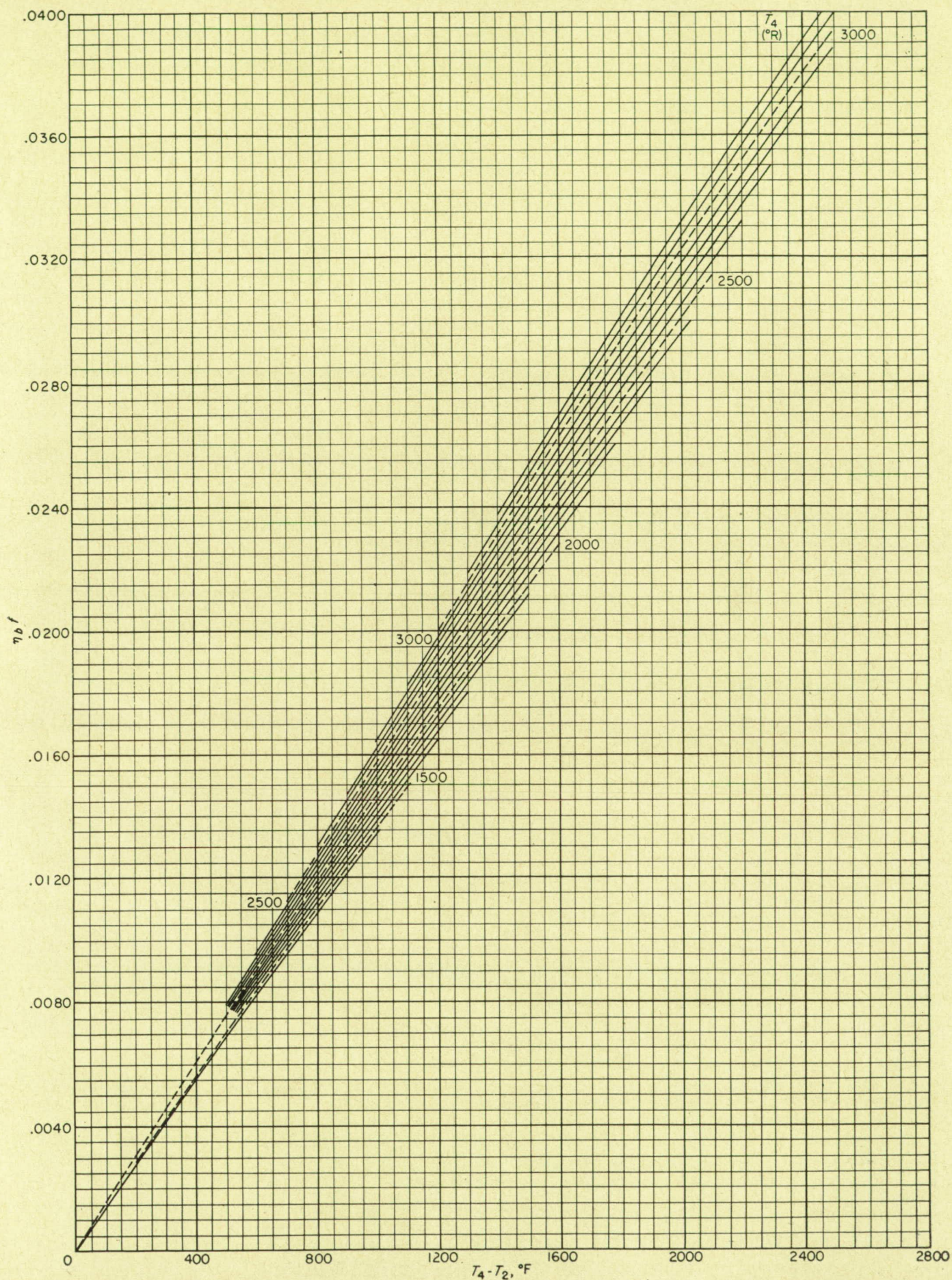


FIGURE 7.—Chart for determining f for various values of $T_4 - T_2$ and T_4 ; h , 18,900 Btu per pound; hydrogen-carbon ratio, 0.185. (A 15- by 21-in. print of this chart is available from NACA upon request.)

From items (1), (2), (5), (45), and (16):

$$(46) \eta_c \eta_i Y_i / C_2^2 - Y \dots\dots\dots 0.039$$

From items (46), (43), and (42), read on figure 5:

$$(47) \frac{thp_p}{M_a} \frac{\eta_c}{\eta_p} \frac{519}{t_0}, \text{ hp/(slug/sec)} \dots\dots\dots 2257$$

The thrust horsepower per unit mass rate of air flow developed by the propeller is obtained by using items (47), (1), (4), and (9):

$$(48) thp_p / M_a, \text{ hp/(slug/sec)} \dots\dots\dots 2399$$

Items (48) and (6) are substituted in equation (5); the thrust per unit mass rate of air flow developed by the propeller is

$$(49) F_p / M_a, \text{ lb/(slug/sec)} \dots\dots\dots 1799$$

Using items (6) and (7) in equation (1a) gives

$$(50) F_i / M_a, \text{ lb/(slug/sec)} \dots\dots\dots 267$$

and then items (50) and (6) in equation (2) give

$$(51) thp_i / M_a, \text{ hp/(slug/sec)} \dots\dots\dots 356$$

Now, from items (48) and (51):

$$(52) thp / M_a, \text{ hp/(slug/sec)} \dots\dots\dots 2755$$

and from items (49) and (50):

$$(53) F / M_a, \text{ lb/(slug/sec)} \dots\dots\dots 2066$$

The specific fuel consumption is evaluated from items (27) and (52):

$$(54) W_f / thp, \text{ lb/hp-hr} \dots\dots\dots 0.608$$

EFFECT OF MASS OF ADDED FUEL ON F/M_a AND thp/M_a

When more accurate results are desired, the effect of the mass of fuel introduced is accounted for in the calculations. The effect of the added fuel is handled by substituting the product of the turbine total efficiency η_t and $(1+f)$ for the value of η_i , which will now be done for the case just considered.

From item (26) the adjusted value of item (40) becomes

$$(55) \eta_c \eta_t \epsilon \frac{T_4}{t_0} \left(\frac{1}{1+Y} \right)^2 \dots\dots\dots 2.396$$

From figure 4 is obtained the corresponding

$$(56) (P_2/P_1)_{ref} \dots\dots\dots 4.61$$

From items (56) and (8):

$$(57) X \dots\dots\dots 1.30$$

By use of the modified value of η_i , item (43) becomes

$$(58) \eta_c \eta_t \epsilon T_4 / t_0 \dots\dots\dots 2.827$$

and item (46) becomes

$$(59) \eta_c \eta_t Y_i / C_2^2 - Y \dots\dots\dots 0.0410$$

Using items (57), (58), and (59) in figure 5 gives

$$(60) \frac{thp_p}{M_a} \frac{\eta_c}{\eta_p} \frac{519}{t_0}, \text{ hp/(slug/sec)} \dots\dots\dots 2351$$

or

$$(61) thp_p / M_a, \text{ hp/(slug/sec)} \dots\dots\dots 2497$$

Evaluating the propeller thrust from items (61) and (6) and from equation (5) gives

$$(62) F_p / M_a, \text{ lb/(slug/sec)} \dots\dots\dots 1874$$

In evaluating the jet thrust from items (6) and (7), equation (1b) is now used and gives

$$(63) F_i / M_a, \text{ lb/(slug/sec)} \dots\dots\dots 281$$

From items (63) and (6) and from equation (2):

$$(64) thp_i / M_a, \text{ hp/(slug/sec)} \dots\dots\dots 375$$

The total thrust horsepower per unit mass rate of air flow from items (61) and (64) is

$$(65) thp / M_a, \text{ hp/(slug/sec)} \dots\dots\dots 2872$$

From items (62) and (63):

$$(66) F / M_a, \text{ lb/(slug/sec)} \dots\dots\dots 2155$$

and from items (65) and (27):

$$(67) W_f / thp, \text{ lb/thp-hr} \dots\dots\dots 0.583$$

ACCURACY OF METHOD

The final equations for thrust horsepower (eqs. (A24) and (A34)), from which figure 5 is plotted and which are derived with the aid of several simplifying assumptions, give values of thrust horsepower which check very closely with values obtained from very accurate step-by-step evaluation of conditions throughout the cycle. Over a wide range of operating and flight conditions, the maximum error in the value of thrust horsepower obtained from the charts is less than 0.5 percent.

The results of the examples used to illustrate the use of the charts give an indication of the effect of ϵ on the thrust horsepower. For the conditions of the example, choosing an approximate value of $\epsilon=1.0$ results in a value of thrust horsepower about 7 percent different from the value of thrust horsepower evaluated from the more accurate value of $\epsilon=1.025$. In general, the percentage error in thrust horsepower will range from two to four times the percentage error in ϵ . In cases where the combination of conditions is such as to result in low values of thrust horsepower, an error in ϵ has a much greater effect on thrust horsepower.

Also, for the set of conditions chosen, the thrust power is about 4 percent greater and the specific fuel consumption about 4 percent lower when the added mass of fuel is taken into account. In general, if good accuracy of results is desired from the charts, it is necessary to include the effect of added fuel.

TURBINE-PROPELLER-ENGINE PERFORMANCE

In order to illustrate the performance and some of the characteristics of the turbine-propeller engine, several cases are presented. First, the performance of the turbine-propeller system over a range of flight and engine-design-point operating conditions and fixed design-point component efficiencies is discussed. Each set of design-point operating conditions and component efficiencies represents a different engine. Second, the off-design-point performance of two specific turbine-propeller engines (with given sets of matched components) is discussed. For this case a method is presented for matching components; also the interrelation between component characteristics and over-all engine performance is shown.

DESIGN-POINT ENGINES

For the purpose of illustrating the manner in which the thrust horsepower per unit mass rate of air flow and specific fuel consumption are influenced by compressor pressure ratio, combustion-chamber-outlet temperature, flight speed, and ambient-air temperature, the following fixed parameters are assumed:

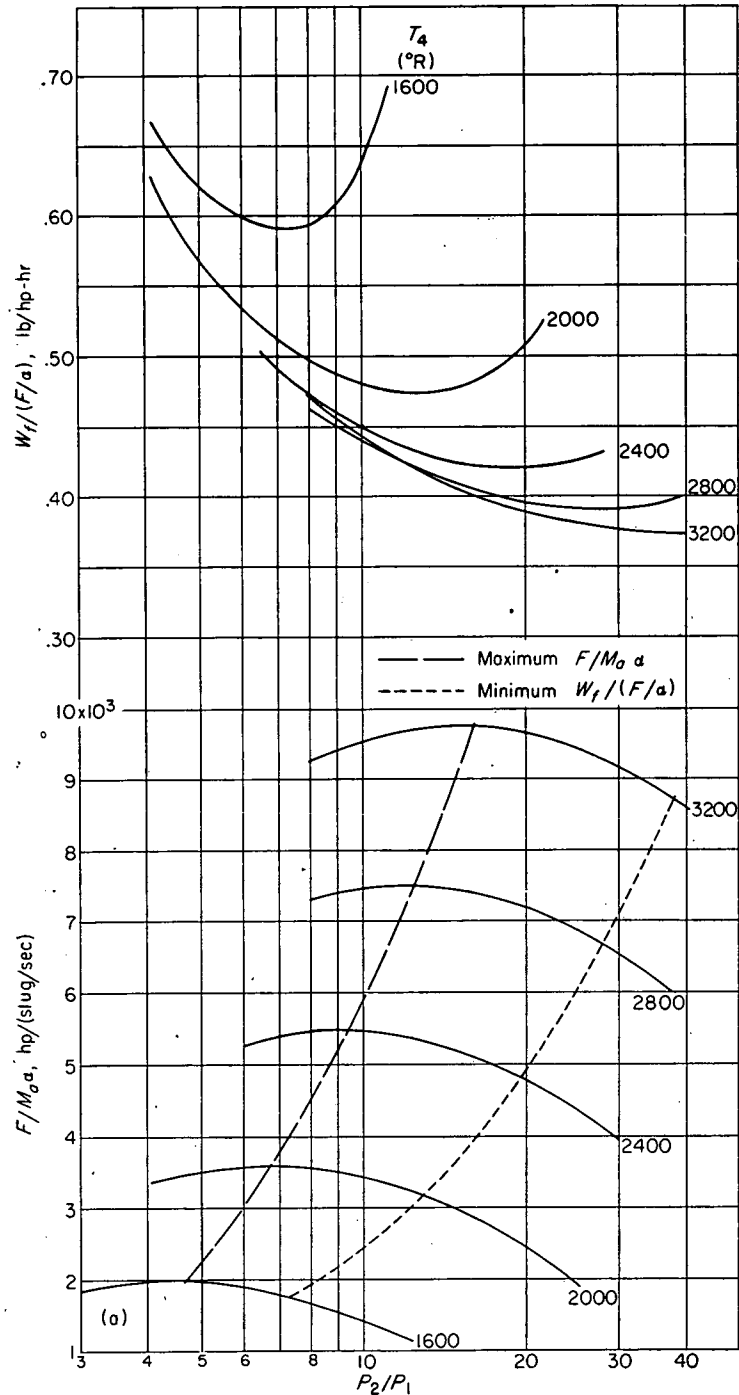
Compressor adiabatic efficiency, η_c	0.85
Turbine total efficiency, η_t	0.90
Combustion efficiency, η_b	0.96
Propeller efficiency, η_p	0.85
Exhaust-nozzle velocity coefficient, C_v	0.97
Lower heating value of fuel, h , Btu/lb.....	18,900
Correction factor, ϵ	1.00

The jet velocities in all cases considered were the optimum jet velocities evaluated from equations (4a) and (4b).

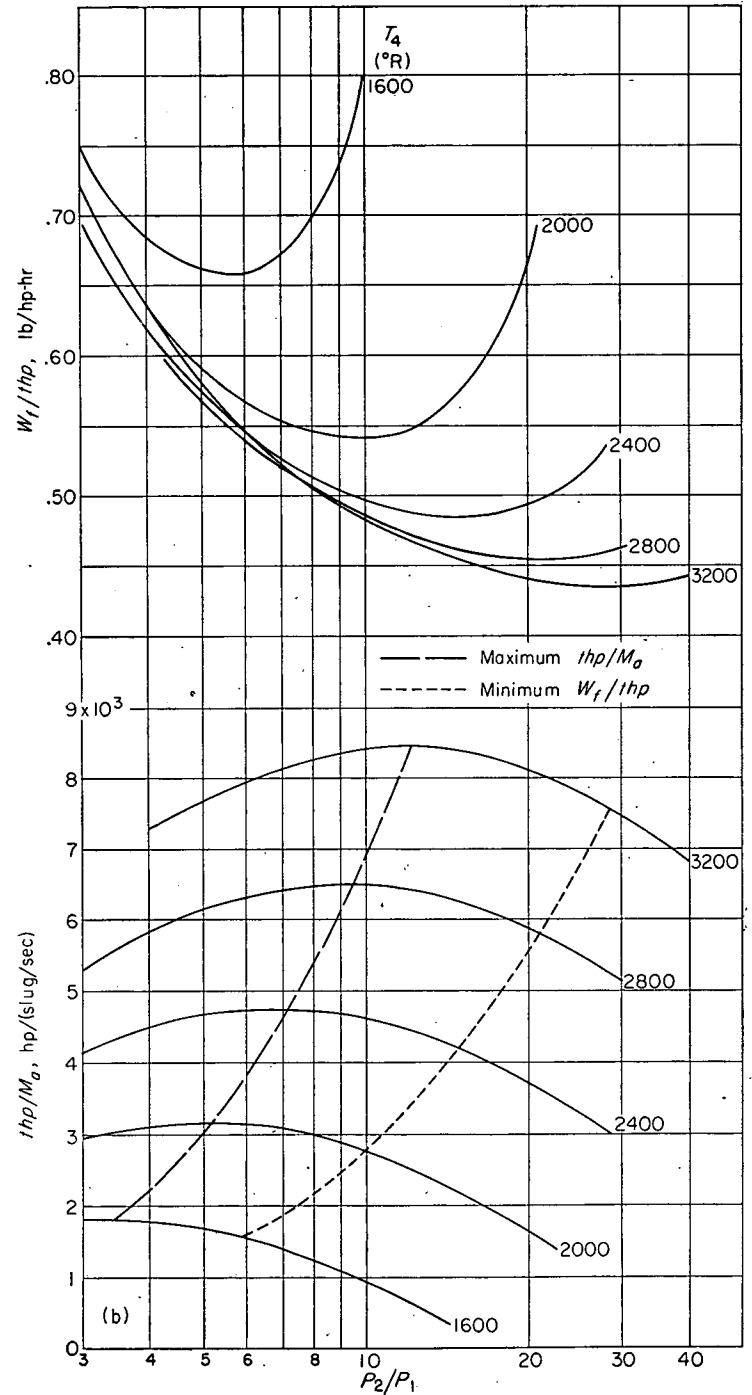
The effect of the mass of fuel added is included in these performance calculations.

The values of component efficiencies and ϵ at the design point will tend to vary with change in design-point operating conditions. In the present computations, the component efficiencies and ϵ were assumed constant at the values listed. The method illustrated in the examples for using the charts was followed in evaluating the performance.

The performance of the system is presented in figure 8; the thrust horsepower per unit mass rate of air flow (or equivalent propeller shaft horsepower per unit mass rate of



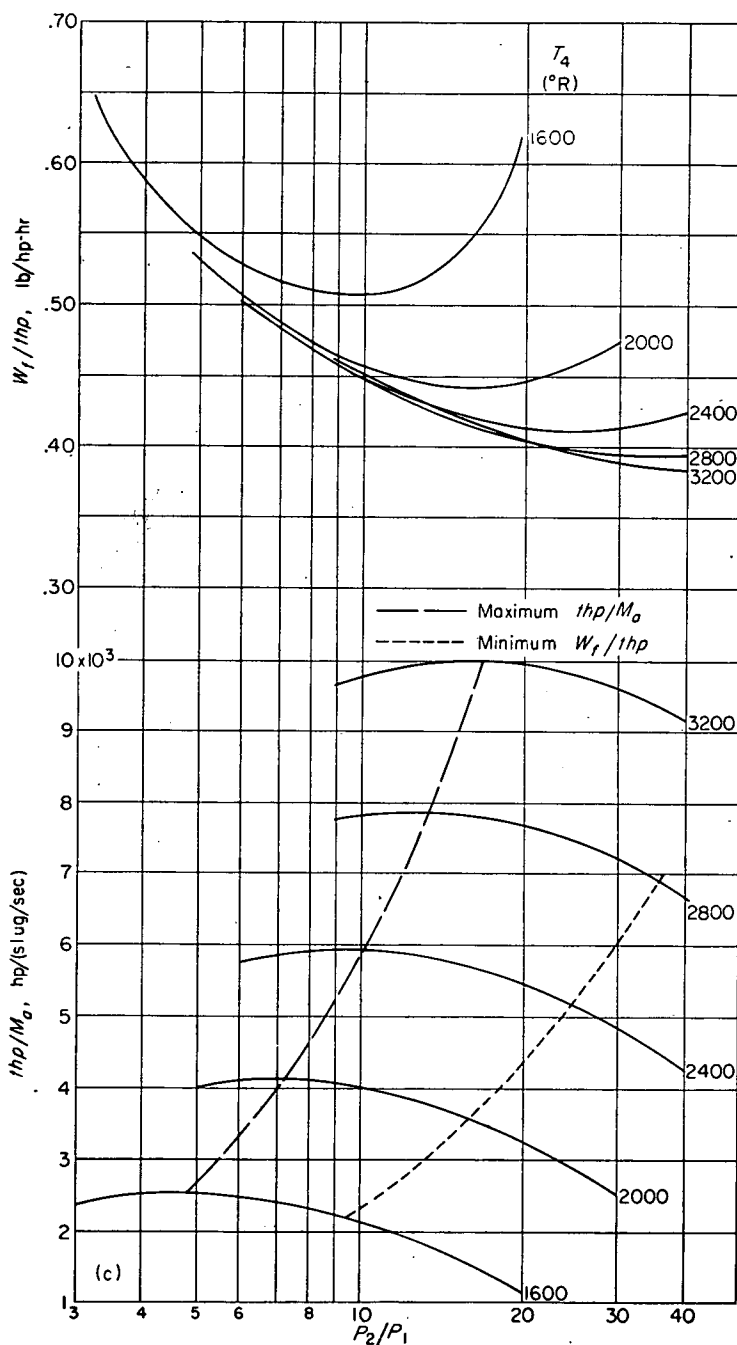
(a) V_0 , 0; t_0 , 519° R; V_j , 144 feet per second; α , 4 pounds thrust per horsepower (to convert jet thrust to equivalent shaft horsepower).



(b) V_0 , 733 feet per second; t_0 , 519° R; V_j , 902 feet per second; η_p , 0.85.

FIGURE 8.—Performance of turbine-propeller engines for range of design-point flight and engine operating conditions; η_c , 0.85; η_t , 0.90; η_b , 0.96; C_v , 0.97; h , 18,900 Btu per pound; ϵ , 1.00.

air flow for the static case, $V_0=0$) and the specific fuel consumption are plotted against compressor pressure ratio for various values of combustion-chamber-outlet temperature at several combinations of airplane velocity and ambient-air temperature. The range of T_4 investigated was from 1600° to 3200° R, and the compressor pressure ratios ranged up to 40. Lines for compressor pressure ratios giving maximum thrust horsepower per unit mass rate of air flow ($X=1$) and for minimum specific fuel consumption at any temperature T_4 are included in the figure.



(c) V_0 , 733 feet per second; t_0 , 412° R; V_1 , 902 feet per second; η_p , 0.85.

FIGURE 8.—Concluded. Performance of turbine-propeller engines for range of design-point flight and engine operating conditions; η_c , 0.85; η_t , 0.90; η_b , 0.96; C_t , 0.97; h , 18,900 Btu per pound; ϵ , 1.00.

Performance curves at $V_0=0$ and $t_0=519^\circ$ R are shown in figure 8 (a). For this case where the thrust horsepower is zero, the equivalent total shaft power per unit mass rate of air flow $F/M_a\alpha$ and specific fuel consumption based on equivalent shaft power are plotted against compressor pressure ratio. The factor α was assumed equal to 4 pounds per horsepower in order to convert the jet thrust into equivalent shaft power. The optimum jet velocity calculated from equation (4b) for this case is 144 feet per second.

In figures 8 (b) and 8 (c) are presented curves of thrust horsepower per unit mass rate of air flow and of specific fuel consumption plotted against compressor pressure ratio. Figure 8 (b) is for the case of $V_0=733$ feet per second and $t_0=519^\circ$ R; and figure 8 (c), for $V_0=733$ feet per second and $t_0=412^\circ$ R. The optimum jet velocity for both these cases, computed from equation (4a), is 902 feet per second.

The curves of figure 8 show that with no limitation on compressor pressure ratio, higher thp/M_a (or equivalent total shaft power per unit mass rate of air flow when the system is at rest) and lower specific fuel consumption can be obtained by increasing the T_4 . At any given T_4 there is an optimum P_2/P_1 for maximum thp/M_a and an optimum P_2/P_1 for minimum W_f/thp . The compressor pressure ratio for minimum specific fuel consumption is greater than that required for maximum thp/M_a .

The effects of flight speed and ambient-air temperature on the performance of the turbine-propeller system at a given combustion-chamber-outlet temperature of 1960° R are shown in figure 9. In figure 9 (a), the thp/M_a and W_f/thp are plotted against ambient-air temperatures at V_0 of 367 and 733 feet per second for the following cases:

- Compressor pressure ratio chosen to give maximum thp/M_a ($X=1$)
- Compressor pressure ratio chosen to give minimum W_f/thp

At each flight speed, the corresponding optimum jet velocity is used.

This figure shows that t_0 has an important effect on the performance values; the thp/M_a decreases and the specific fuel consumption increases appreciably as t_0 increases. For the given conditions, the thp/M_a decreases about 30 percent for both cases (a) and (b) as t_0 increases from 393° to 519° R (the 393° R temperature corresponds to an NACA standard altitude of 35,332 ft, the start of the tropopause). The values of thp/M_a for case (a) are about 13 percent greater than those for case (b) over the range of t_0 investigated.

The specific fuel consumption increases about 25 percent for both cases (a) and (b) as the ambient-air temperature increases from 393° to 519° R. The values of W_f/thp for case (a) are about 10 percent higher than those for case (b). It is also evident from figure 9 that increasing flight speed results in only slightly increased values of thp/M_a and slightly decreased values of W_f/thp for both cases (a) and (b). Over the range of ambient-air temperatures considered, increasing the flight speed from 367 to 733 feet per second results in changes in thp/M_a and W_f/thp of about 2 percent.

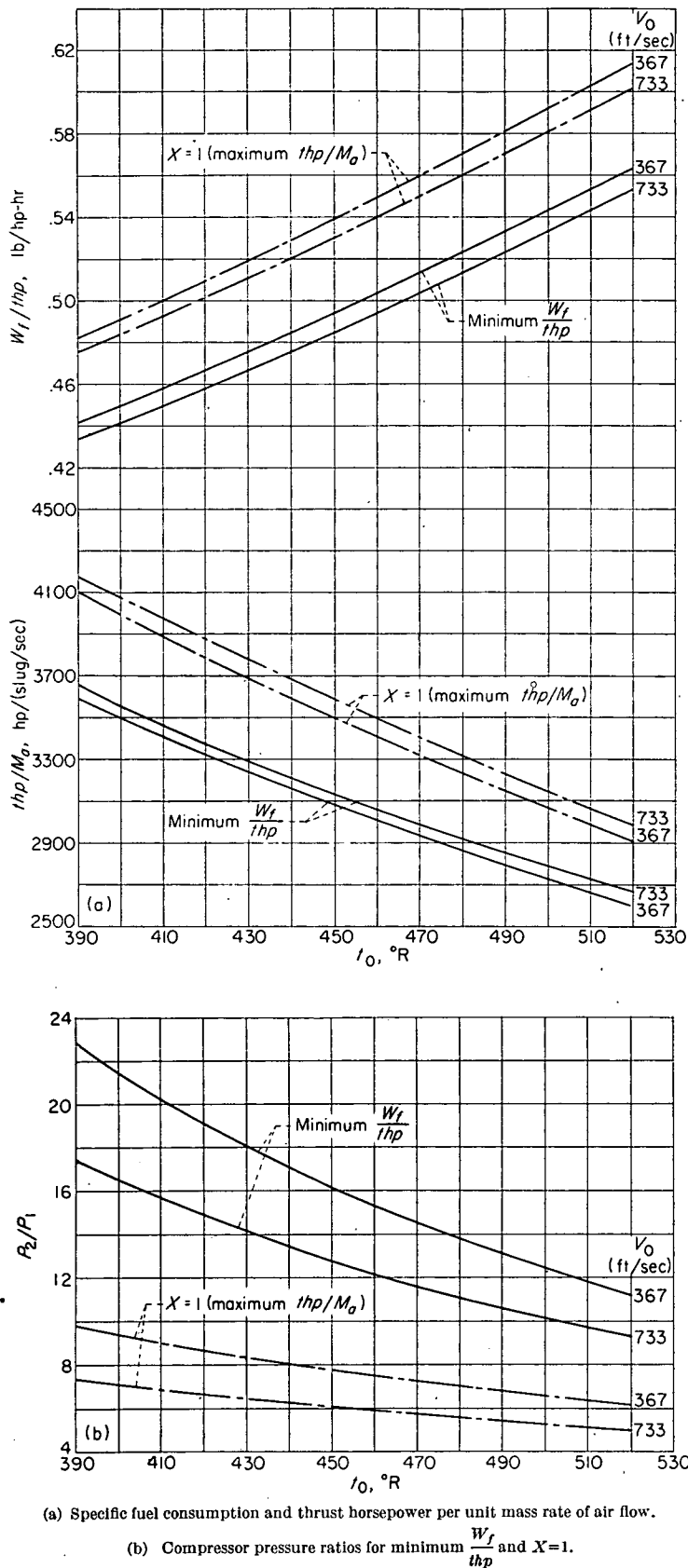


FIGURE 9.—Effects of flight speed and ambient-air temperature on performance at compressor pressure ratios for minimum specific fuel consumption and for maximum thrust horsepower per unit mass rate of air flow; T_1 , 1960° R; η_c , 0.85; η_t , 0.90; η_p , 0.85; η_b , 0.96; C_s , 0.97; h , 18,900 Btu per pound; ϵ , 1.00.

The compressor pressure ratios associated with the performance values given in figure 9 (a) are presented in figure 9 (b). The large increase in required pressure ratio from the condition of $X=1$ to the condition of minimum specific fuel consumption is to be noted.

In figures 8 and 9, it was assumed that the compressor adiabatic efficiency remains constant at 0.85 regardless of the pressure ratio. As the desired pressure ratio is increased, however, it becomes increasingly difficult to design a compressor to maintain a high adiabatic efficiency. A reduction in the compressor adiabatic efficiency with increase in pressure ratio reduces the gains derived from increase in pressure ratio and hence reduces the value of the optimum pressure ratios for maximum thrust horsepower per unit mass rate of air flow and for minimum specific fuel consumption.

This condition is illustrated in figure 10 in which a turbine-propeller engine equipped with a multistage axial-flow compressor having a polytropic efficiency $\eta_{c,p}$ of 0.88 is considered. The other parameters of the engine are the same

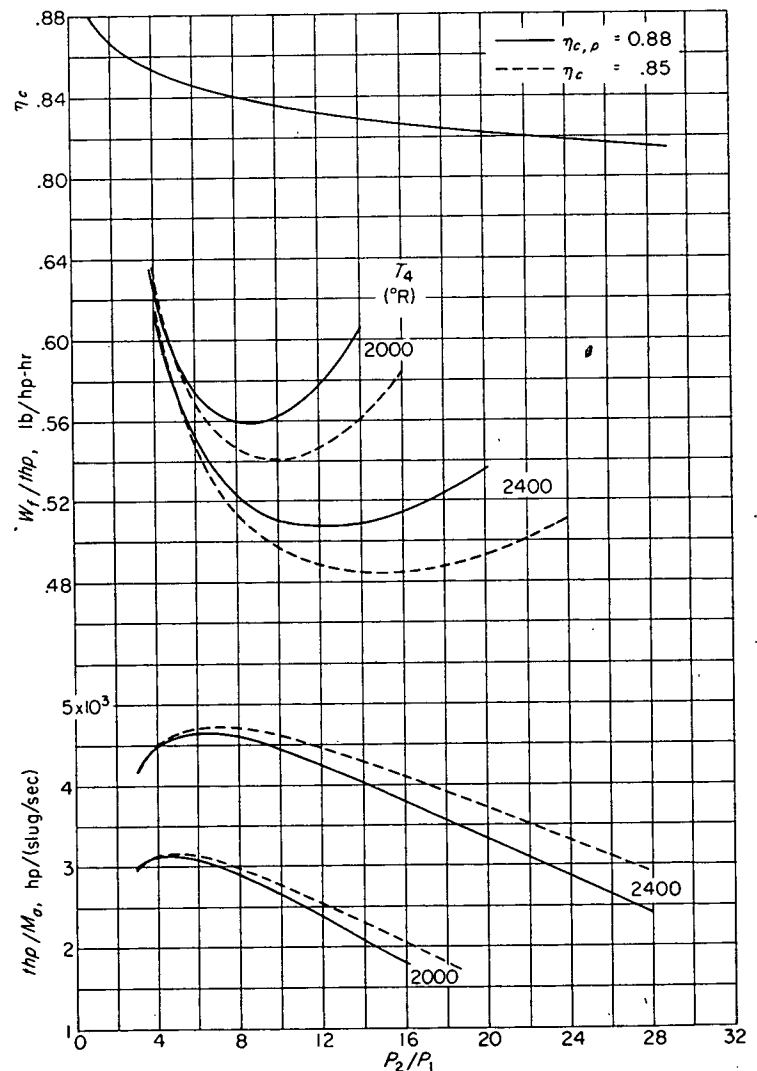


FIGURE 10.—Comparison of performance with constant η_c and with constant $\eta_{c,p}$ at various compressor pressure ratios; V_0 , 733 feet per second; V_1 , 902 feet per second; t_0 , 519° R; η_p , 0.85; η_t , 0.90; η_b , 0.96; C_s , 0.97; h , 16,900 Btu per pound; ϵ , 1.00.

as for figure 8 (b). Figure 10 shows the over-all adiabatic efficiency of the compressor, the specific fuel consumption, and the thrust horsepower per unit mass rate of air flow for a range of pressure ratios. The pressure ratio is increased by adding stages to the compressor. Although the polytropic efficiency is held constant, the over-all compressor adiabatic efficiency decreases with increase in pressure ratio. At a pressure ratio of 5, the compressor adiabatic efficiency is 0.85, the value used in the computations for figures 8 and 9. For the range of T_4 shown, the values of P_2/P_1 for maximum thp/M_a and minimum W_f/thp are lower for the case when the reduction in compressor adiabatic efficiency with increased pressure ratio is considered than those for the case of constant η_c of 0.85. This change in P_2/P_1 for maximum thp/M_a and minimum W_f/thp is more pronounced at the higher value of T_4 .

The effect of increasing pressure ratio across the turbine on turbine efficiency is not easily predicted. In the design of a turbine for greater pressure ratios, the number of turbine stages is usually increased and the pressure ratio per stage is increased, in order to economize on the size and weight of the turbine. Increasing the number of turbine stages tends to improve the over-all turbine efficiency (provided that the efficiency per stage remains the same). However, increasing the pressure ratio per stage may result in some reduction in stage efficiency, which will offset the gains obtained by the increased number of stages. The net effect on the over-all turbine efficiency depends on the compromise between pressure ratio per stage and number of stages.

ENGINE WITH GIVEN SET OF MATCHED COMPONENTS

The points on the previous performance curves pertain to a series of turbine-propeller engines in which the components are changed to provide the desired characteristics at each point. It is of interest to examine over a variety of operating conditions the characteristics of a turbine-propeller engine having given components.

Two engines are considered in this study. One engine has a propeller, an axial-flow compressor, and two independent turbines (divided turbine system); one turbine drives only the compressor and the other drives only the propeller. The second engine contains the same components, except that the two turbines are connected to provide a single rotating system (connected turbine system). Components chosen have performance characteristics fairly representative of their type and are not to be interpreted as being the best available. Although the performance of each engine depends on the characteristics of the particular components chosen, some general trends may be demonstrated by considering the illustrative case. Plots of the characteristics of the components and the performance of the turbine-propeller engines incorporating these components are presented in figures 11 to 18. Calculations of engine performance were simplified by neglecting the mass of fuel in evaluating turbine output, by neglecting any losses in transmitting the shaft power from turbine to propeller, and

by assuming the drop in total pressure through the combustion chamber proportional to the combustion-chamber-inlet total pressure. The errors introduced by these simplifications are too small to influence the basic trends illustrated. In the computation of the performance of the turbine-propeller engine, the following parameters are assumed:

Combustion efficiency, η_b	0.96
Exhaust-nozzle velocity coefficient, C_v	0.97
Lower heating value of fuel, h , Btu/lb.....	18,900

In order to simplify the specific fuel consumption plots, the specific heat at constant pressure of the gases during combustion was assumed to be a function only of T_4/T_1 .

Compressor characteristics.—The conventional presentation of the performance curves for an illustrative 8-stage axial-flow compressor is given in figure 11. Values of P_2/P_1 , η_c , and K_c are plotted against the $M_a\sqrt{\theta_1/\delta_1}$ for various values of $U_c/\sqrt{\theta_1}$. The slip factor is defined as

$$K_c = \frac{550 h p_c}{M_a U_c^2} \quad (10)$$

At a given $U_c/\sqrt{\theta_1}$, reducing the $M_a\sqrt{\theta_1/\delta_1}$ by throttling the compressor outlet first results in a very rapid increase in pressure ratio and efficiency and then a more gradual increase in these parameters to peak values. Excessive throttling to the position indicated by the surge line results

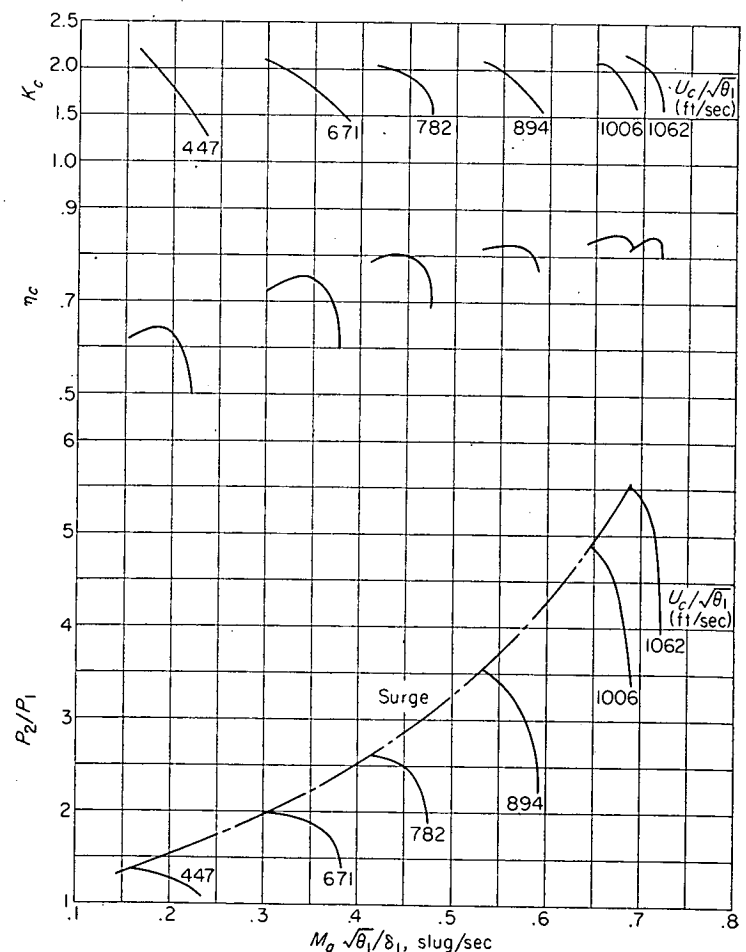


FIGURE 11.—Characteristics of axial-flow compressor.

in stalling of the compressor, which is accompanied by surging of the air flow. It is to be noted that operation at higher tip speeds is limited to narrower ranges of mass flow.

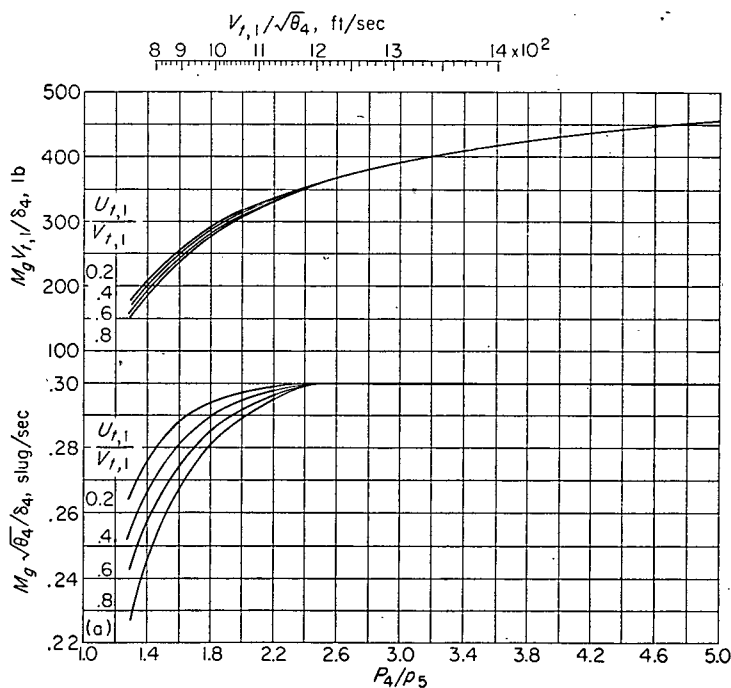
Turbine characteristics.—The performance characteristics of a typical single-stage turbine with moderate reaction are shown in figure 12. The mass-flow factor of the gas through the turbine is plotted in figure 12 (a) against P_4/p_5 for various values of $U_{t,1}/V_{t,1}$. The turbine jet velocity $V_{t,1}$ is defined as the theoretical jet velocity developed by the gas expanding isentropically through the turbine nozzle from turbine-inlet total temperature and pressure to turbine-outlet static pressure. The values of the upper abscissa $V_{t,1}/\sqrt{\theta_4}$ corresponding to the values of P_4/p_5 are obtained from the velocity equation

$$\frac{V_{t,1}}{\sqrt{\theta_4}} = \sqrt{2Jc_{p,g}519 \left[1 - \left(\frac{p_5}{P_4} \right)^{\frac{\gamma_g-1}{\gamma_g}} \right]}$$

The values of the upper ordinate $M_g V_{t,1}/\delta_4$ are obtained from the product of $M_g \sqrt{\theta_4}/\delta_4$ and $V_{t,1}/\sqrt{\theta_4}$. For pressure ratios across the turbine greater than 2.54, the value of $M_g \sqrt{\theta_4}/\delta_4$ is constant (that is, choking occurs at the turbine nozzle).

The turbine total efficiency $\eta_{t,1}$ is principally a function of $U_{t,1}/V_{t,1}$ and, to a lesser extent, a function of the pressure ratio across the turbine and the Reynolds number; $\eta_{t,1}$ is defined as

$$\eta_{t,1} = \frac{550 h p_{t,1}}{\frac{M_g}{2} V_{t,1}^2 - \frac{M_g}{2} V_5^2} \quad (11a)$$



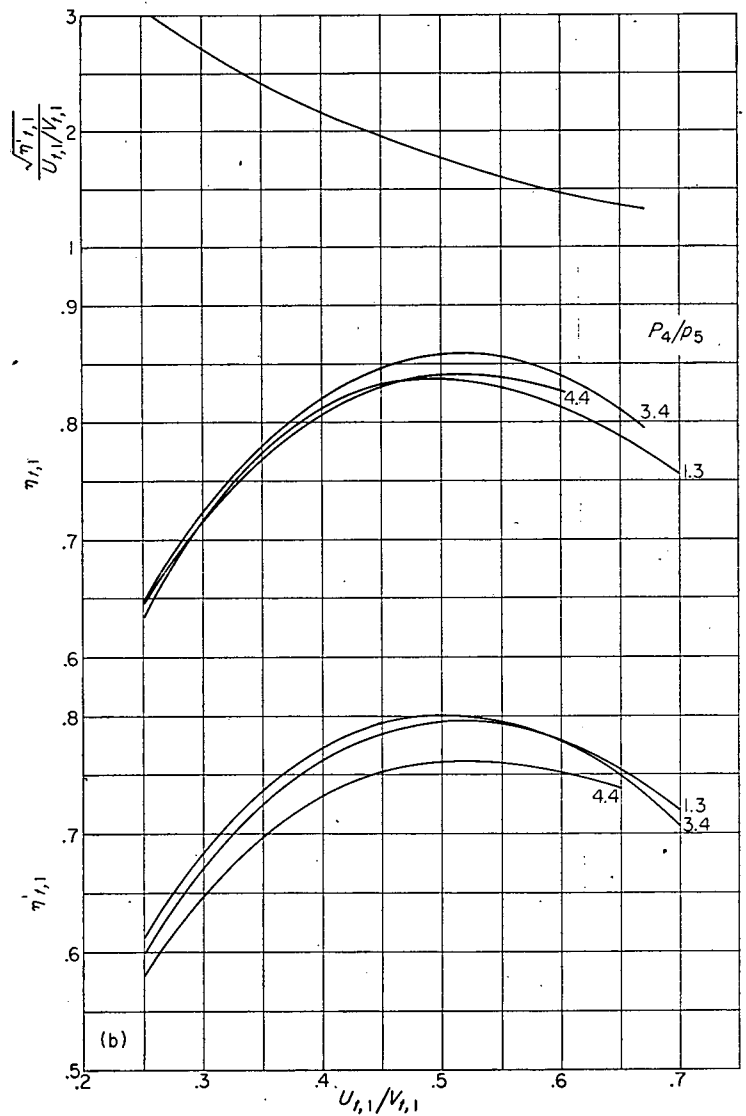
(a) Mass-flow characteristics.

FIGURE 12.—Characteristics of first single-stage turbine.

The relation between total efficiency, blade-to-jet speed ratio, and pressure ratio is given in figure 12 (b); the Reynolds number effect is omitted in this analysis. The turbine-shaft efficiency $\eta'_{t,1}$, also shown in figure 12 (b), is defined as

$$\eta'_{t,1} = \frac{550 h p_{t,1}}{\frac{1}{2} M_g V_{t,1}^2} \quad (11b)$$

In this definition the turbine is not credited with kinetic power corresponding to the average axial velocity of the gas at the turbine discharge. In the plot of $\frac{\sqrt{\eta'_{t,1}}}{U_{t,1}/V_{t,1}}$ against $U_{t,1}/V_{t,1}$ in figure 12 (b), the effect of P_4/p_5 is so slight that only a single curve is shown. The parameters $\frac{\sqrt{\eta'_{t,1}}}{U_{t,1}/V_{t,1}}$ and $M_g V_{t,1}/\delta_4$ are useful in evaluating turbine operating conditions for given turbine work output and compressor operating conditions.



(b) Efficiency characteristics.

FIGURE 12.—Concluded. Characteristics of first single-stage turbine.

Gas generator.—The combination of compressor, combustor, and turbine which drives the compressor is referred to as the gas generator. It converts the heat energy from the fuel to energy available in high-temperature, pressurized gases. These gases are expanded in a second turbine which drives the propeller and are further expanded in the exhaust nozzle to provide jet power.

When the compressor and the turbine of the gas generator are matched, it is necessary that the mass flows through the components be consistent (that is, the gas flow through the turbine equal the sum of the air flow through the compressor and the fuel flow) and that the compressor work required equal the turbine work output. The compressor and the turbine sizes are so adjusted that desired mass-flow factors through the components are obtained when the compressor is operating at its design point and design turbine-inlet to compressor-inlet temperature ratio T_4/T_1 is maintained. The turbine of figure 12 is matched with the compressor of figure 11 for a design-point temperature ratio T_4/T_1 of 4.23

and a design compressor pressure ratio P_2/P_1 of 5.1 at a tip speed factor $U_c/\sqrt{\theta_1}$ of 1062 feet per second. These conditions permit operation sufficiently far from the compressor surge line to ensure stable operation of the gas generator over a wide range of conditions off the design point.

Lines of constant temperature ratio T_4/T_1 for the matched turbine and compressor are shown superimposed on compressor characteristics in figure 13 (a). If the difference between M_a and M_g (due to added fuel) is neglected, the following identity can be written:

$$\frac{M_a \sqrt{T_1}}{P_1} = \frac{P_4}{P_1} \sqrt{\frac{T_1}{T_4}} \frac{M_g \sqrt{T_4}}{P_4}$$

If r represents the ratio of the drop in combustion-chamber total pressure to the compressor-outlet total pressure, then

$$\Delta P_2 = P_2 - P_4 = r P_2$$

or

$$P_4 = (1-r) P_2$$

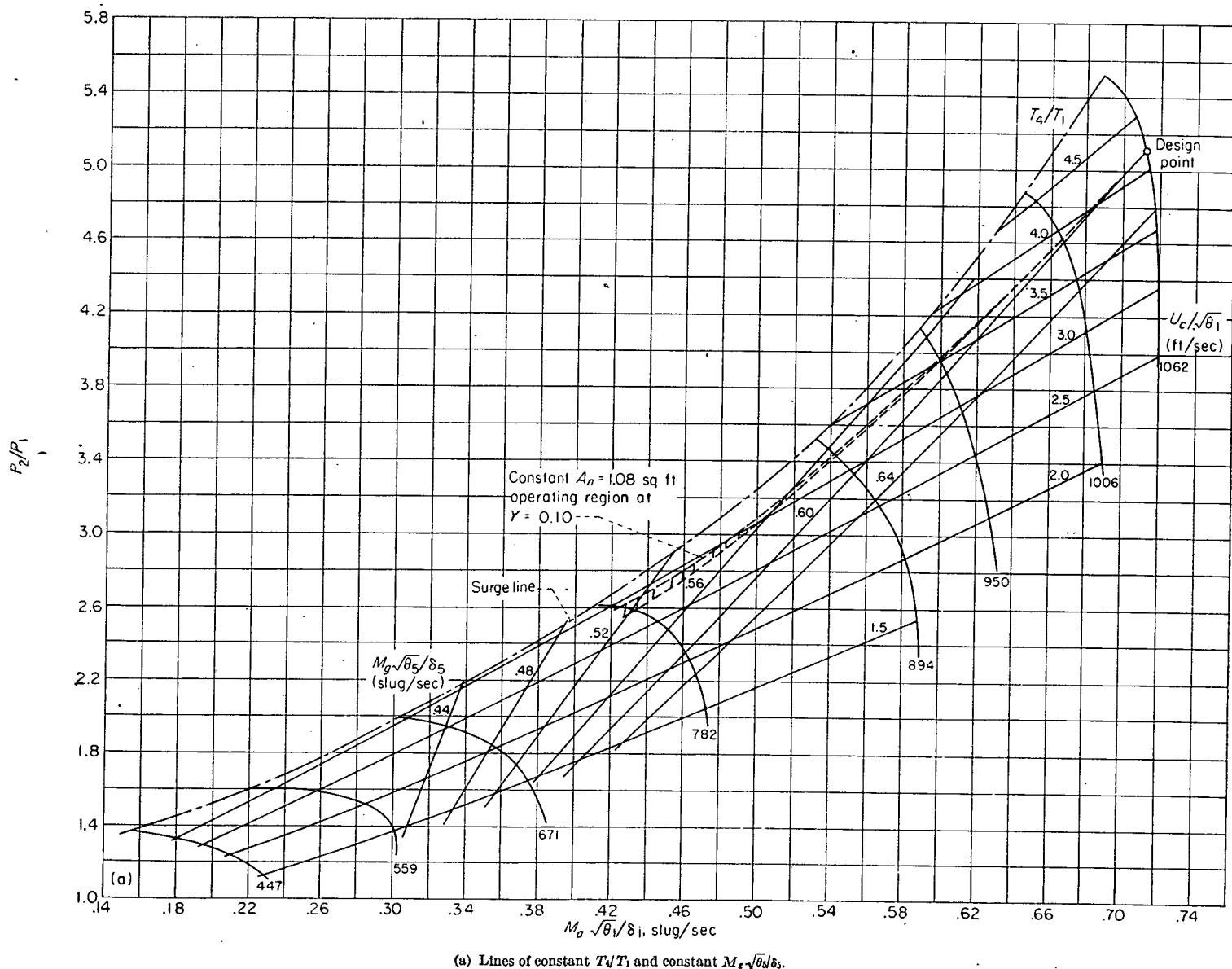
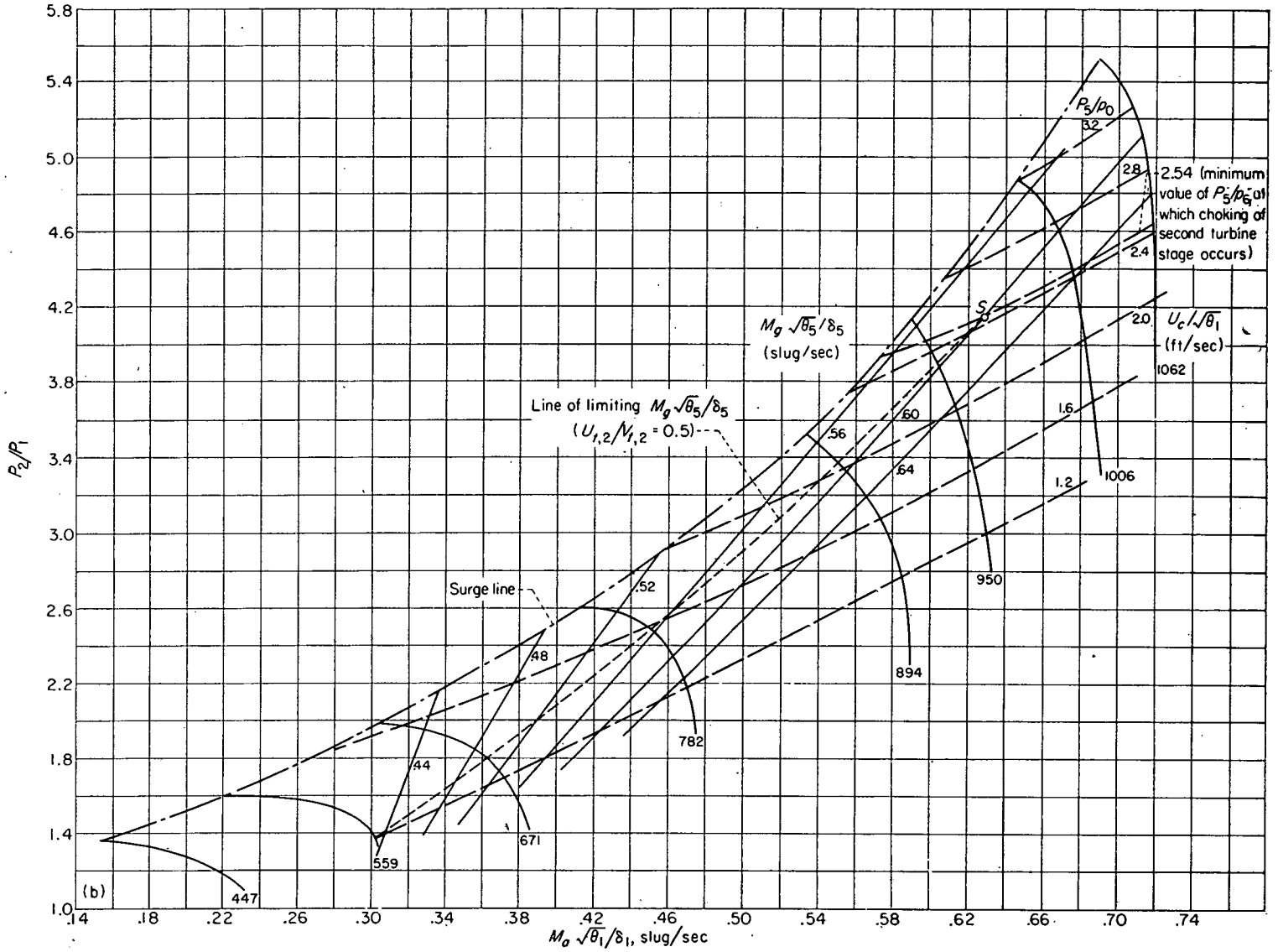


FIGURE 13.—Operating characteristics of gas generator consisting of matched compressor and turbine which drives compressor; $\Delta P_2/P_2, 0.05$; $U_c/U_{c1}, 1.0$



(b) Lines of constant P_4/p_5 and constant $M_g\sqrt{\theta_5}/\delta_5$; Y , 0.10 (flight Mach number, 0.71).
 FIGURE 13.—Continued. Operating characteristics of gas generator consisting of matched compressor and turbine which drives compressor; $\Delta P_2/P_2$, 0.05; $U_c/U_{t,1}$, 1.0.

Hence,

$$\frac{M_a\sqrt{T_1}}{P_1} = (1-r) \frac{P_2}{P_1} \sqrt{\frac{T_1}{T_4}} \frac{M_a\sqrt{T_4}}{P_4}$$

or

$$\frac{M_a\sqrt{\theta_1}}{\delta_1} = (1-r) \frac{P_2}{P_1} \sqrt{\frac{T_1}{T_4}} \frac{M_g\sqrt{\theta_4}}{\delta_4} \quad (12)$$

In operating regions where choking of the flow occurs at the turbine nozzle, the value of $M_g\sqrt{\theta_4}/\delta_4$ becomes constant (for example, for pressure ratios P_4/p_5 greater than 2.54 for the turbine shown in fig. 12 (a)). When this constant value of $M_g\sqrt{\theta_4}/\delta_4$ is substituted into equation (12) and a value is assumed for r , it is possible to compute the value T_4/T_1 at any value of $M_a\sqrt{\theta_1}/\delta_1$ and the corresponding P_2/P_1 . In the nonchoking region, the value of $M_g\sqrt{\theta_4}/\delta_4$ is not so easily determined, and the more general method described in appendix C is used.

It is evident from the lines of constant T_4/T_1 that, at any given rotational speed and compressor-inlet temperature

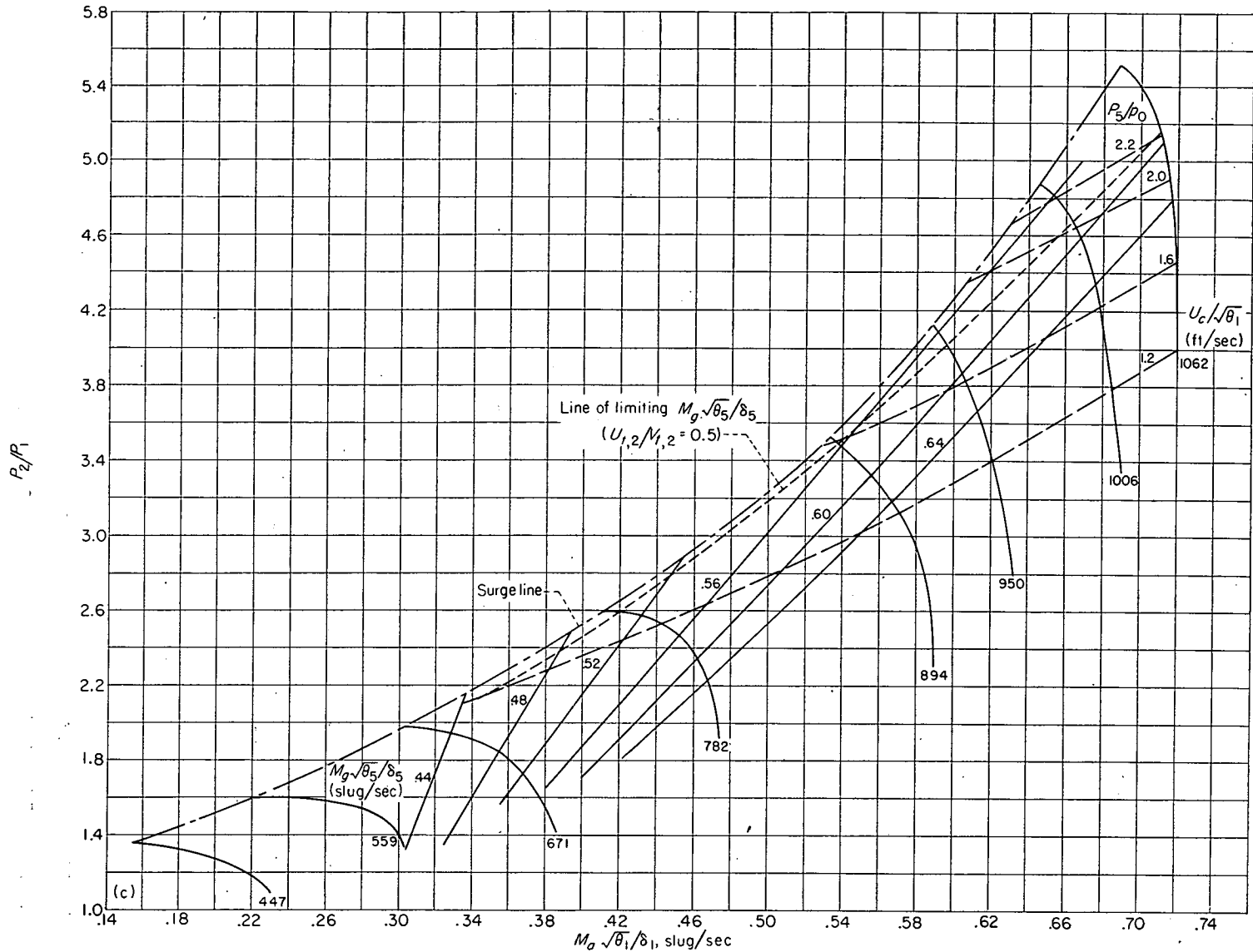
T_1 , increasing the combustion-chamber-outlet temperature T_4 is equivalent to throttling the compressor. This increase in T_4 causes an increase in compressor pressure ratio and decreases the mass-flow factor. Excessive combustion-chamber-outlet temperature may carry operation into the surge zone.

Lines of constant $M_g\sqrt{\theta_5}/\delta_5$ are also plotted on the gas-generator operating curves (fig. 13) because these curves link the operation of the second turbine with that of the gas generator. If the difference between M_a and M_g is again neglected, the following identity can be written:

$$\frac{M_g\sqrt{\theta_5}}{\delta_5} = \frac{M_a\sqrt{\theta_1}}{\delta_1} \sqrt{\frac{T_5}{T_4} \frac{T_4}{T_1}} \left(\frac{P_1}{P_4} \frac{P_4}{P_5} \right) \quad (13)$$

At any operating point, the quantities $M_a\sqrt{\theta_1}/\delta_1$, T_4/T_1 , and P_2/P_1 can be read and P_4/P_1 computed from

$$P_4/P_1 = (1-r)P_2/P_1$$



(c) Lines of constant P_5/p_0 and constant $M_c\sqrt{\theta_5/\delta_5}$; $Y, 0$ (flight Mach number, 0).
 FIGURE 13.—Concluded. Operating characteristics of gas generator consisting of matched compressor and turbine which drives compressor; $\Delta P_2/P_2, 0.05$; $U_c/U_{t,1}, 1.0$.

The corresponding values of P_4/P_5 and T_5/T_4 are determined by the method described in appendix C, and the corresponding $M_g\sqrt{\theta_5/\delta_5}$ is evaluated from equation (13).

For the gas-generator combination, the compressor power is equal to the turbine power; hence, from equations (10) and (11) and again from the assumption that M_a and M_g are equal,

$$K_c U_c^2 = \frac{1}{2} \eta'_{t,1} V_{t,1}^2 \quad (14)$$

and

$$K_c \left(\frac{U_c}{U_{t,1}} \right)^2 = \frac{1}{2} \left(\frac{V_{t,1}}{U_{t,1}} \right)^2 \eta'_{t,1} \quad (15)$$

The ratio of compressor tip speed U_c to the turbine-blade speed $U_{t,1}$ is a constant for any given engine. Thus, any value of K_c determines the value of $\eta'_{t,1} (V_{t,1}/U_{t,1})^2$ and, from figure 12 (b), determines the value of $U_{t,1}/V_{t,1}$ from which the values of $\eta_{t,1}$ and $\eta'_{t,1}$ can be obtained, when the effect of

pressure ratio across the turbine is neglected. A value of $U_c/U_{t,1}$ equal to 1.0 was chosen for the engine. For the compressor shown in figure 11, the variation in the value of K_c over the entire operating range was from 1.5 to 2.1. The corresponding variation in $U_{t,1}/V_{t,1}$ was small enough that the turbine operated at nearly constant efficiency over the entire operating range of the gas generator.

Matching second turbine.—For operation at a given flight Mach number, a characteristic of the gas generator is that along a given $U_c/\sqrt{\theta_1}$ line the pressure ratio available for further expansion at the inlet to the second turbine P_5/p_0 decreases as $M_g\sqrt{\theta_5/\delta_5}$ increases. This trend is shown in figures 13 (b) (for $Y=0.10$) and 13 (c) (for $Y=0$), which are plots of the gas-generator operating characteristics showing lines of $M_g\sqrt{\theta_5/\delta_5}$ (which are independent of flight speed) and lines of P_5/p_0 (which are a function of flight speed). The possible operating range on figure 13 is located between the surge line and the line of limiting values of $M_g\sqrt{\theta_5/\delta_5}$ permitted

by the second turbine. Any limiting value of $M_g\sqrt{\theta_5/\delta_5}$ is the value of the mass-flow factor through the second turbine when the pressure ratio across the turbine is equal to the maximum pressure ratio available P_5/p_0 . It is evident that the second turbine should be designed to permit limiting values of $M_g\sqrt{\theta_5/\delta_5}$ sufficiently large to provide the engine with a reasonable range of operation on the gas-generator plot. The increase in $M_g\sqrt{\theta_5/\delta_5}$ is accomplished by designing a larger second-turbine nozzle area. Excessive increase in design turbine-nozzle area should be avoided, however, because, for a given $M_g\sqrt{\theta_5/\delta_5}$ (as set by the gas generator), there results a reduction in pressure ratio required across the second turbine, which is accompanied by a reduction in the power obtainable from this turbine. An inefficient distribution of power between the propeller and the exhaust jet may result. The second-turbine nozzle area chosen for design operating conditions should be the best compromise between extent of operation and efficiency of power distribution between the propeller and the exhaust jet.

The characteristics of the second turbine which drives the propeller are shown in figure 14. They are similar to those of the turbine driving the compressor (fig. 12), except that the maximum values of the mass-flow factor are different.

Matching propeller.—The problem of propeller matching involves mainly the selection of proper propeller size and proper relation between propeller and turbine speeds, so that the propeller will have a high efficiency at engine design conditions and maintain high efficiency over a wide range of off-design engine operation. A value of $U_{t,2}/N_p$, based on reasonable estimates of turbine pitch-line diameter and gear reduction ratio between the turbine and the propeller, is chosen. This value of $U_{t,2}/N_p$, when once fixed for a given engine, determines the value of $N_p/\sqrt{\theta_1}$ for any engine operating condition ($U_{t,2}/V_{t,2}$ and $V_{t,2}/\sqrt{\theta_1}$ being known for that operating condition) and affects the manner in which propeller efficiency varies with changing engine operating conditions. At any operating condition of the engine, the flight speed factor $V_0/\sqrt{\theta_1}$ is known, and the propeller-shaft horsepower factor $hp_p/\sqrt{\theta_1\delta_1}$ can be evaluated. Fixing the values

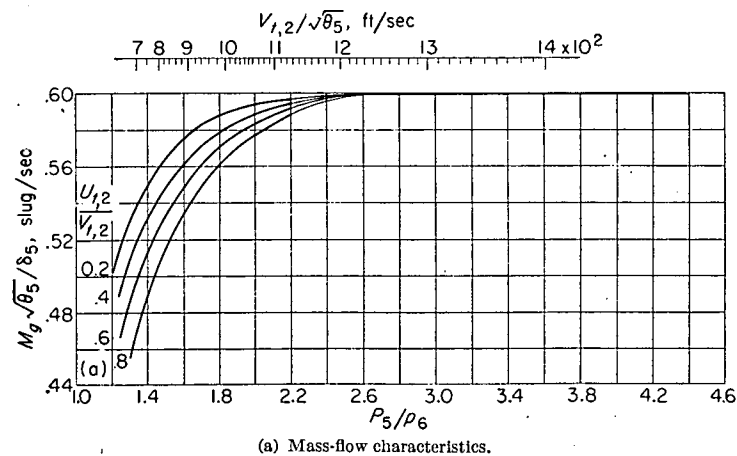


FIGURE 14.—Characteristics of second single-stage turbine.

of propeller diameter D_p and $U_{t,2}/N_p$ (and hence, $N_p/\sqrt{\theta_1}$) determines the values of power coefficient C_p and advance ratio $V_0/N_p D_p$ at every operating point; these values in turn determine the propeller efficiency. The variation of propeller efficiency with engine operating conditions depends on the values of D_p and $U_{t,2}/N_p$ selected. Therefore, the propeller diameter D_p and, to some extent, the value of $U_{t,2}/N_p$ are adjusted by trial until a good compromise of propeller efficiency and engine operating range is obtained. Reference 6 presents a similar and more detailed discussion of matching a propeller to a turbine for the same type engine considered here. Figure 15 shows the characteristics of a high-efficiency four-bladed propeller at flight Mach numbers of 0.55 and 0.71. Values of the power coefficient C_p are plotted against the advance ratio $V_0/N_p D_p$ for various values of propeller efficiency and blade angle. The propeller efficiency is largely dependent on tip Mach number and drops off rapidly when a value of tip Mach number slightly greater than 1 is exceeded. The maximum efficiencies range from

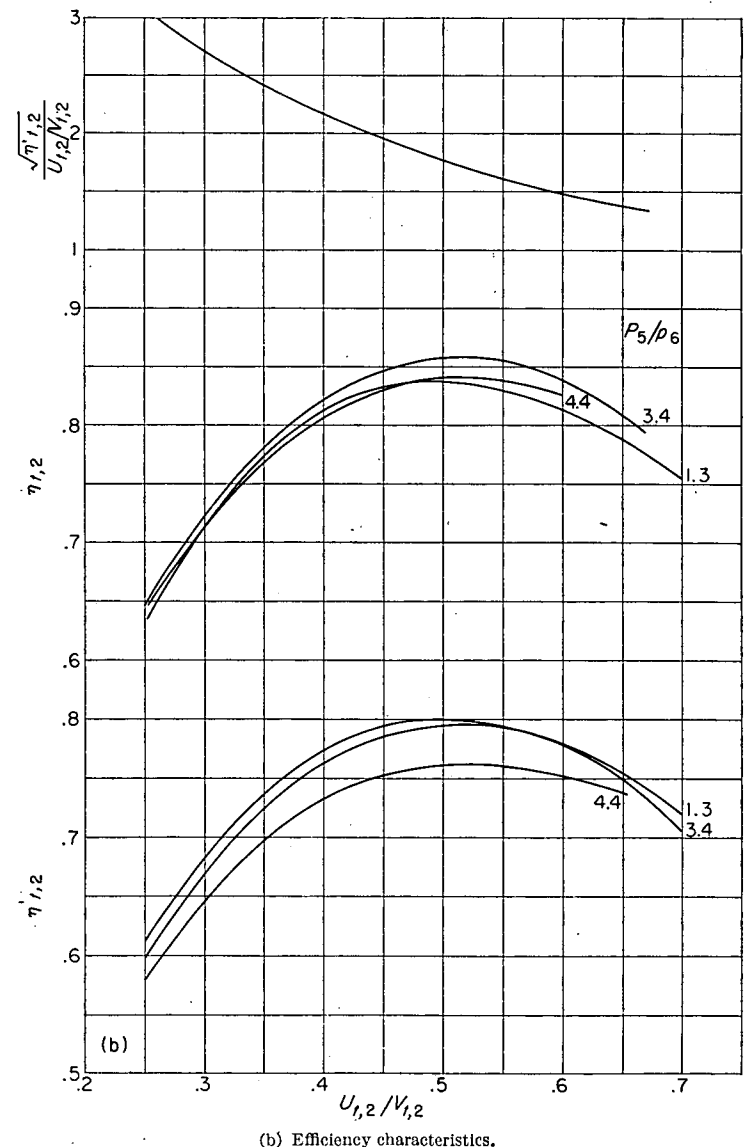
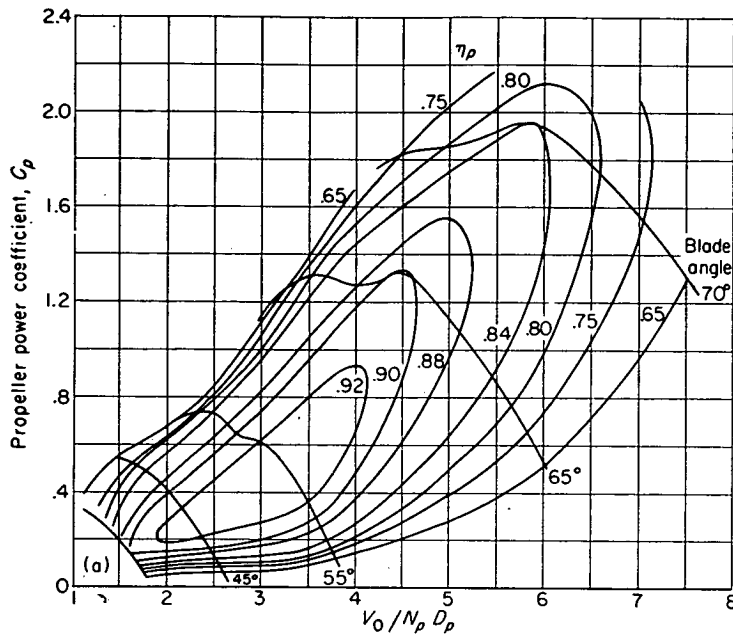


FIGURE 14.—Concluded. Characteristics of second single-stage turbine.



(a) $Y, 0.06$ (flight Mach number, 0.55).
FIGURE 15.—Characteristics of high-speed propeller

about 0.93 at flight Mach number of 0.55 to about 0.88 at flight Mach number of 0.71. At higher flight Mach numbers, because tip Mach numbers greater than 1 are almost always encountered, lower efficiencies exist. The values of propeller efficiency presented in figure 15 are based on actual shaft power input to the propeller and do not include losses involved in the transmission of power from turbine to propeller.

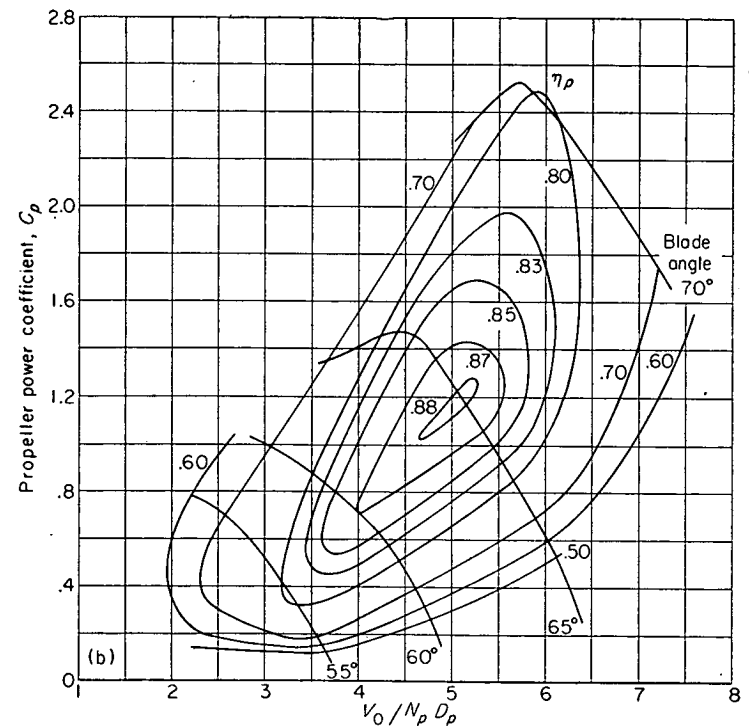
Performance of engine with divided turbine system.—Any point on the plot in figure 13 (a) represents the gas generator operating at a given set of conditions such as $U_c/\sqrt{\theta_1}$, $M_a\sqrt{\theta_1}/\delta_1$, T_4/T_1 , P_2/P_1 , and $M_g\sqrt{\theta_5}/\delta_5$. The $M_g\sqrt{\theta_5}/\delta_5$ at any point is fixed; it is therefore evident from figure 14 (a) that the second turbine, if unchoked, can operate only over a range of definite combinations of $U_{t,2}/V_{t,2}$ and P_5/p_6 . Hence, for every point on the gas-generator curve the second turbine can operate over an extent of pressure ratio P_5/p_6 and corresponding $U_{t,2}/V_{t,2}$; the power output of the turbine, the turbine efficiency, the jet power of the engine, the propeller speed, and the propeller efficiency all vary accordingly.

When the mass-flow factor $M_g\sqrt{\theta_5}/\delta_5$ is the maximum value that the second turbine can attain (in other words the flow through the turbine is choked), the pressure ratio across the turbine is independent of $U_{t,2}/V_{t,2}$ (see fig. 14 (a)). Thus any point on the gas-generator plot along this choking $M_g\sqrt{\theta_5}/\delta_5$ line corresponds to operation at any combination of $U_{t,2}/V_{t,2}$ and P_5/p_6 . (The quantity P_5/p_6 may have any value equal to or greater than that required for choking but not appreciably greater than the available P_5/p_0 at the operating point. The case in which p_6 is less than p_0 is possible; however, it necessitates a diffuser to discharge the gas from the turbine outlet to the atmosphere and is generally not a desirable condition.)

The value of $M_g\sqrt{\theta_5}/\delta_5$ which gives choked flow in the second turbine sets one of the limits on the possible operating region in figure 13 (a). For the turbine under consideration (fig. 14), the maximum value of $M_g\sqrt{\theta_5}/\delta_5$ is 0.60 slug per second, and hence all points in figure 13 (a) at values of $M_g\sqrt{\theta_5}/\delta_5$ greater than 0.60 slug per second are unattainable. Furthermore, in order to attain this choked flow, the available P_5/p_0 must be equal to or greater than the choking pressure ratio 2.54 given in figure 14 (a). Figure 13 (b) shows the pressure ratios P_5/p_0 corresponding to figure 13 (a) and $Y=0.10$ (Mach number, 0.71). The point S marks the intersection of the line of $M_g\sqrt{\theta_5}/\delta_5=0.60$ slug per second with the line of $P_5/p_0=2.54$. All points on the line of $M_g\sqrt{\theta_5}/\delta_5=0.60$ slug per second to the right of point S have values of P_5/p_0 greater than 2.54 and hence are attainable conditions.

To the left of point S in figure 13 (b), the line of $M_g\sqrt{\theta_5}/\delta_5=0.60$ slug per second intersects values of P_5/p_0 less than 2.54 and hence is an unattainable condition (if p_6 is limited to values equal to or greater than p_0). Figure 14 (a) indicates that in this region the attainable value of $M_g\sqrt{\theta_5}/\delta_5$ depends on the values of P_5/p_6 and $U_{t,2}/V_{t,2}$. The dashed curve in figure 13 (b) gives the maximum attainable value of $M_g\sqrt{\theta_5}/\delta_5$ corresponding to the condition $U_{t,2}/V_{t,2}=0.5$ (at which the efficiency of the second turbine is a maximum). The region in figure 13 (b) between the line of maximum attainable $M_g\sqrt{\theta_5}/\delta_5$ and the surge line is the possible operating range for the combination of turbines chosen in this illustration at $Y=0.10$. The dashed curve can be shifted somewhat by choosing a different value of $U_{t,2}/V_{t,2}$.

The conditions of figure 13 (c) correspond to those of figure 13 (a) and zero flight speed ($Y=0$). In this case all values of



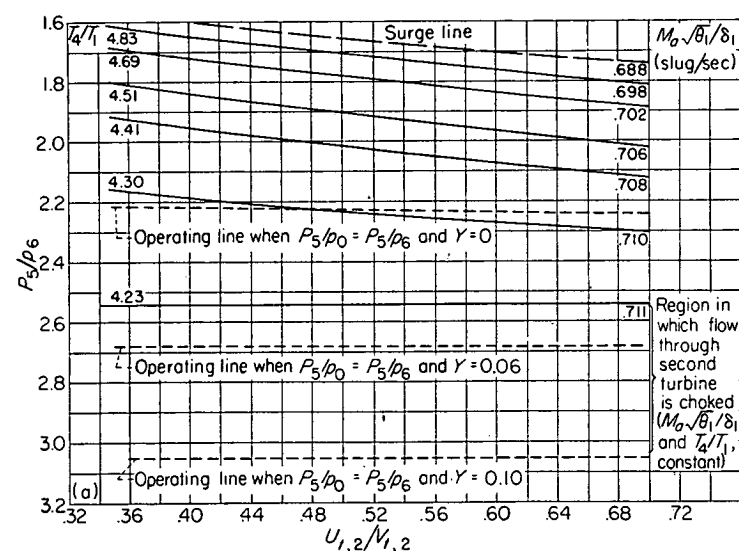
(b) $Y, 0.10$ (flight Mach number, 0.71).
FIGURE 15.—Concluded. Characteristics of high-speed propeller.

P_5/p_0 shown are less than 2.54, and the choked condition in the second turbine is not attained. The dashed curve is again the upper limit on the value of $M_g\sqrt{\theta_5/\delta_5}$ for $U_{t,2}/V_{t,2} = 0.50$, and operation for this flight condition is limited to the narrow strip between the dashed line and the surge line.

The permissible operational region discussed in connection with figure 13 will now be discussed in greater detail.

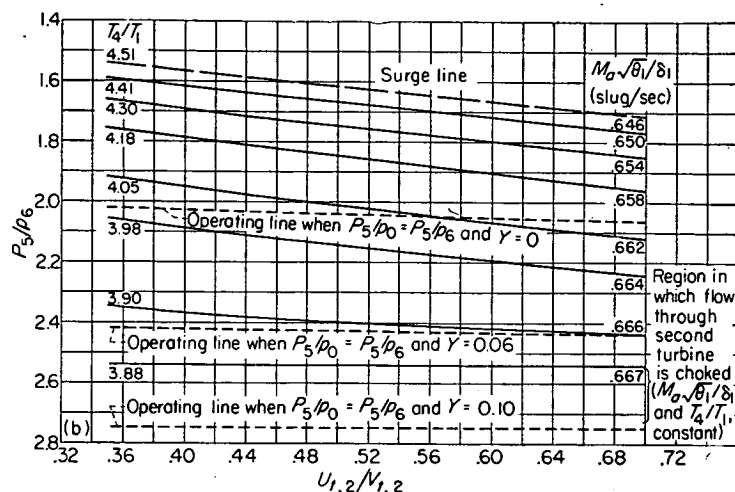
A plot showing the relation between operating parameters of the second turbine and the gas generator is shown in figure 16. The blade-to-jet speed ratio $U_{t,2}/V_{t,2}$ is plotted against P_5/p_6 for lines of constant T_4/T_1 or $M_g\sqrt{\theta_1/\delta_1}$. Figure 16 (a) is for a constant tip-speed factor of 1062 feet per second. Every point on the gas-generator plot (fig. 13) at this value of tip-speed factor has a unique value of $M_g\sqrt{\theta_1/\delta_1}$ and T_4/T_1 and is represented by a line of operation on the engine operating plot of figure 16 (a). This characteristic is evident from the previous discussion, in which it was shown that there is a series of combinations of $U_{t,2}/V_{t,2}$ and P_5/p_6 corresponding to every point on the gas-generator plot. The region in figure 16 (a) at pressure ratios P_5/p_6 of 2.54 and greater (in which the second turbine is choked) corresponds to operation at the point in figure 13 on the $U_c/\sqrt{\theta_1} = 1062$ feet per second line where $M_g\sqrt{\theta_5/\delta_5} = 0.60$ slug per second.

The operation-limit lines at which the entire P_5/p_0 available equals P_5/p_6 , for flight Mach numbers equal to 0 ($Y=0$), 0.55 ($Y=0.06$), and 0.71 ($Y=0.10$), are also shown on figure 16 (a); operation is possible between the surge line and these limit lines for the specified Mach number. It is noted that for zero flight speed ($Y=0$) the P_5/p_0 available is never great enough to obtain choked flow through the second turbine. Figure 16 (b) is similar to figure 16 (a), except that the compressor tip-speed factor is 1006 feet per second. Because the available P_5/p_0 is lower at this lower $U_c/\sqrt{\theta_1}$, the range of operation of the turbine at the same flight Mach numbers is smaller.



(a) $U_c/\sqrt{\theta_1}$, 1062 feet per second.

FIGURE 16.—Relation between operating parameters of second turbine and gas generator.



(b) $U_c/\sqrt{\theta_1}$, 1006 feet per second.

FIGURE 16.—Concluded. Relation between operating parameters of second turbine and gas generator.

The over-all performance values of the engine which have been superimposed on figure 16 are shown in figure 17. The values of the total-thrust-horsepower factor $thp/\sqrt{\theta_1\delta_1}$, the specific fuel consumption, the propeller efficiency η_p , and second-turbine efficiency $\eta_{t,2}$ at any values of $U_{t,2}/V_{t,2}$ and P_5/p_6 are shown in figure 17 (a) for $Y=0.10$ (Mach number, 0.71) and compressor tip-speed factor of 1062 feet per second. Figure 17 (b) is for $U_c/\sqrt{\theta_1} = 1062$ feet per second and $Y=0.06$ (Mach number, 0.55); figure 17 (c), for $U_c/\sqrt{\theta_1} = 1006$ feet per second and $Y=0.10$; and figure 17 (d), for $U_c/\sqrt{\theta_1} = 1006$ feet per second and $Y=0.06$. The curves of $\eta_{t,2}$, which are functions of $U_{t,2}/V_{t,2}$ and P_5/p_6 only, are the same for all plots of figure 17 and hence are given only in figure 17 (a). In order to evaluate η_p , values of $D_p = 13$ feet and $U_{t,2}/N_p = 80$ (ft/sec)/rps were selected for the engine.

These figures show the effect of varying P_5/p_6 and $U_{t,2}/V_{t,2}$ on the over-all performance of the engine. In the range of values of $U_{t,2}/V_{t,2}$ from 0.45 to 0.60 and over the entire range of values of P_5/p_6 shown in figure 17, the maximum variation in $thp/\sqrt{\theta_1\delta_1}$ is about 15 percent and the maximum variation in specific fuel consumption about 20 percent. In this range of values of $U_{t,2}/V_{t,2}$ (0.45 to 0.60) and P_5/p_6 , the values of turbine efficiency and propeller efficiency do not vary appreciably. At low values of P_5/p_6 , the power output is large, mainly because the operating T_4/T_1 is high. However, the percentage of propeller-shaft power to total power available as useful work is much less than the optimum value. Increasing P_5/p_6 improves the efficiency of the power distribution between the propeller and the jet, which together with an accompanying improvement in compressor efficiency leads to decreased specific fuel consumption of the engine.

The effect of flight speed on performance factors is slight at both tip-speed factors shown. The total-thrust-horsepower factor increases slightly, and the specific fuel consumption decreases slightly with increased flight speed. It should be noted that, inasmuch as the factors θ_1 and δ_1 both increase with flight speed, the values of thrust horsepower increase much more than the values of $thp/\sqrt{\theta_1\delta_1}$. At a given

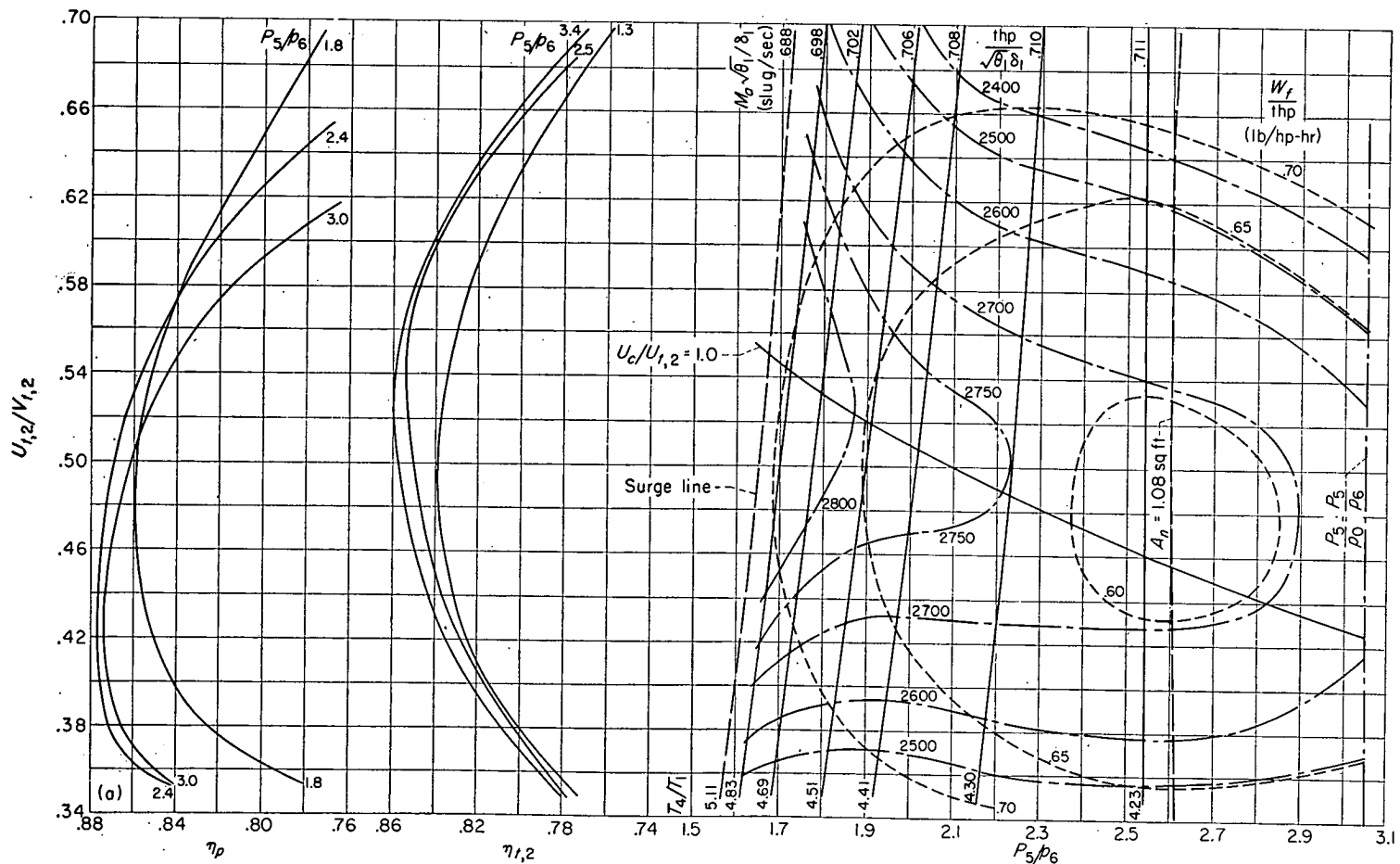
$M_a\sqrt{\theta_1/\delta_1}$ or T_4/T_1 , the specific fuel consumption decreases as flight speed increases mainly because of operation at higher values of T_4 . At the lower flight speed, the propeller efficiency is less sensitive to a change in operating conditions and remains at a higher value over a large part of the operating range shown in figure 17.

The thrust-horsepower factor drops off rapidly as compressor tip-speed factor is reduced from 1062 to 1006 feet per second. This decrease is due to operation in a region of lower T_4/T_1 , lower mass-flow factors, and lower compressor pressure ratios. The specific fuel consumption also suffers when the tip-speed factor is reduced, mainly because of the accompanying decreases in compressor pressure ratio and T_4/T_1 . At a tip-speed factor of 1062 feet per second and $Y=0.10$, an optimum specific fuel consumption of about 0.60 pound per thrust-horsepower-hour is obtained, and the corresponding thrust-horsepower factor is about 2700 horsepower. At $U_c/\sqrt{\theta_1}$ of 1006 feet per second and $Y=0.10$, the optimum specific fuel consumption is about 0.64 pound per thrust-horsepower-hour, and the corresponding thrust-horsepower factor about 2200 horsepower.

The rapid drop in thrust-horsepower factor with compressor tip-speed factor is even more pronounced at the lower tip-speed factors, because the propeller is unsuited to handle efficiently the low shaft powers that accompany the lower tip-speed factors (this characteristic is not evident from

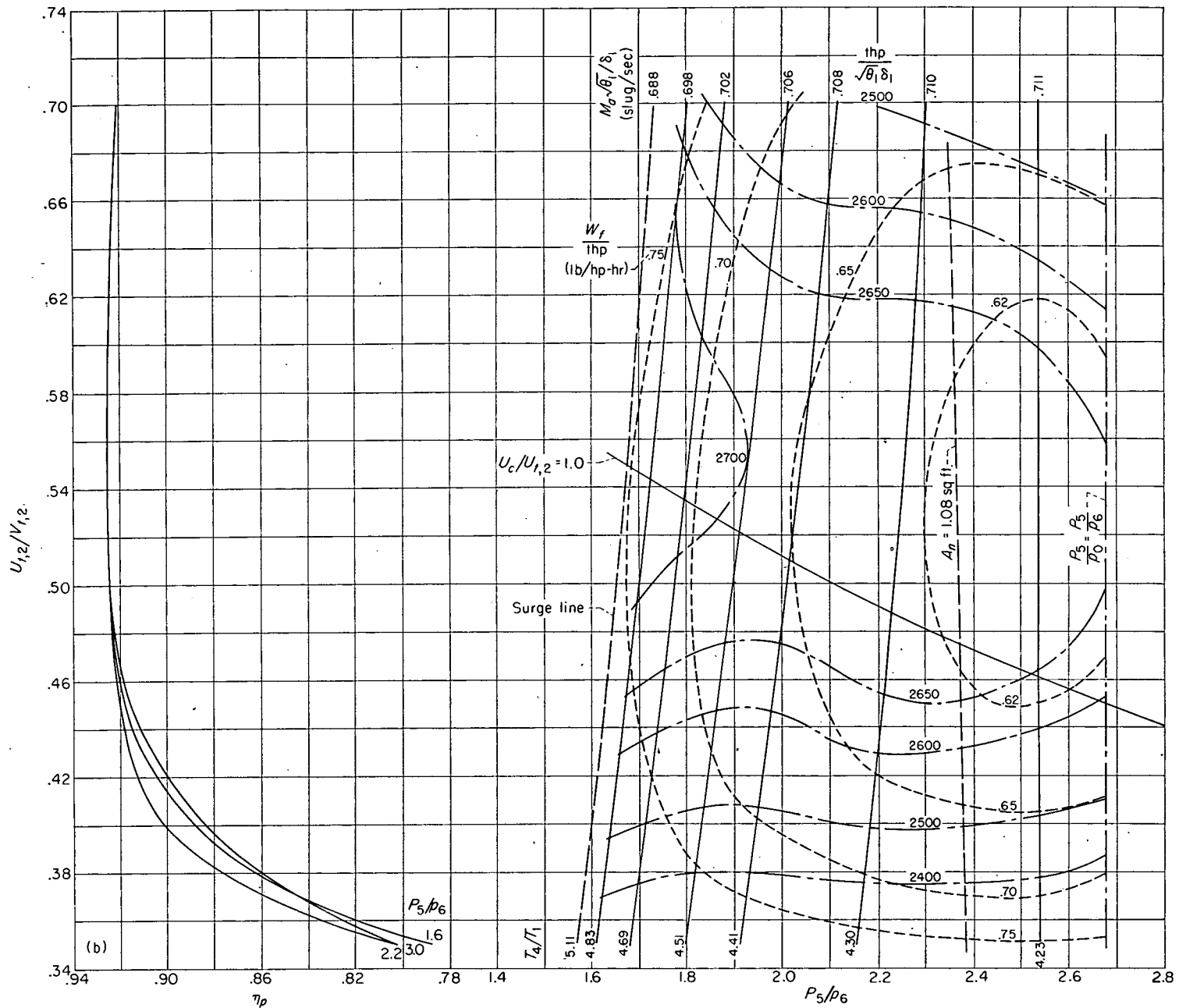
engine-performance figures which are presented only for values of $U_c/\sqrt{\theta_1}$ of 1062 and 1006 ft/sec). This observation emphasized the fact that this type of engine is very largely a maximum compressor tip-speed engine. The use of a free-wheeling turbine to drive the propeller permits a wider range of propeller operating parameters for given engine operating conditions than does the single-turbine engine, because the speed of the second turbine is independent of compressor speed. Better propeller performance can thereby be obtained.

In order to obtain the operating ranges shown in the plots of figures 13 and 17, a variable-area exhaust nozzle is required. The effective exhaust-nozzle area corresponding to design conditions ($U_c/\sqrt{\theta_1}=1062$ ft/sec, $T_4/T_1=4.23$, $P_2/P_1=5.1$, $U_{t,2}/V_{t,2}=0.50$, and $P_5/p_6=2.6$ at $Y=0.10$) is 1.08 square feet (evaluated by the method described in appendix D). This constant-area line of operation is included on figure 17. At any given flight Mach number, each point on the gas-generator plot has a small range of exhaust-nozzle areas associated with it. Conversely, at any flight Mach number each exhaust-nozzle area corresponds to a small range of operation along any $U_c/\sqrt{\theta_1}$ line on figure 13. The entire region of operation for constant exhaust-nozzle area $A_n=1.08$ square feet over a range of compressor tip-speed factors is included between the dashed curves in figure 13 (a) for $Y=0.10$. Figure 17 shows that the design nozzle area



(a) $U_c/\sqrt{\theta_1}$, 1062 feet per second; Y , 0.10 (flight Mach number, 0.71).

FIGURE 17.—Total-thrust-horsepower factor and specific fuel consumption of matched turbine-propeller engine; η_p , 0.96; C_s , 0.97; h , 18,900 Btu per pound; $\Delta P_4/P_1$, 0.03; $\Delta P_2/P_2$, 0.05; $U_c/U_{t,1}$, 1.0.



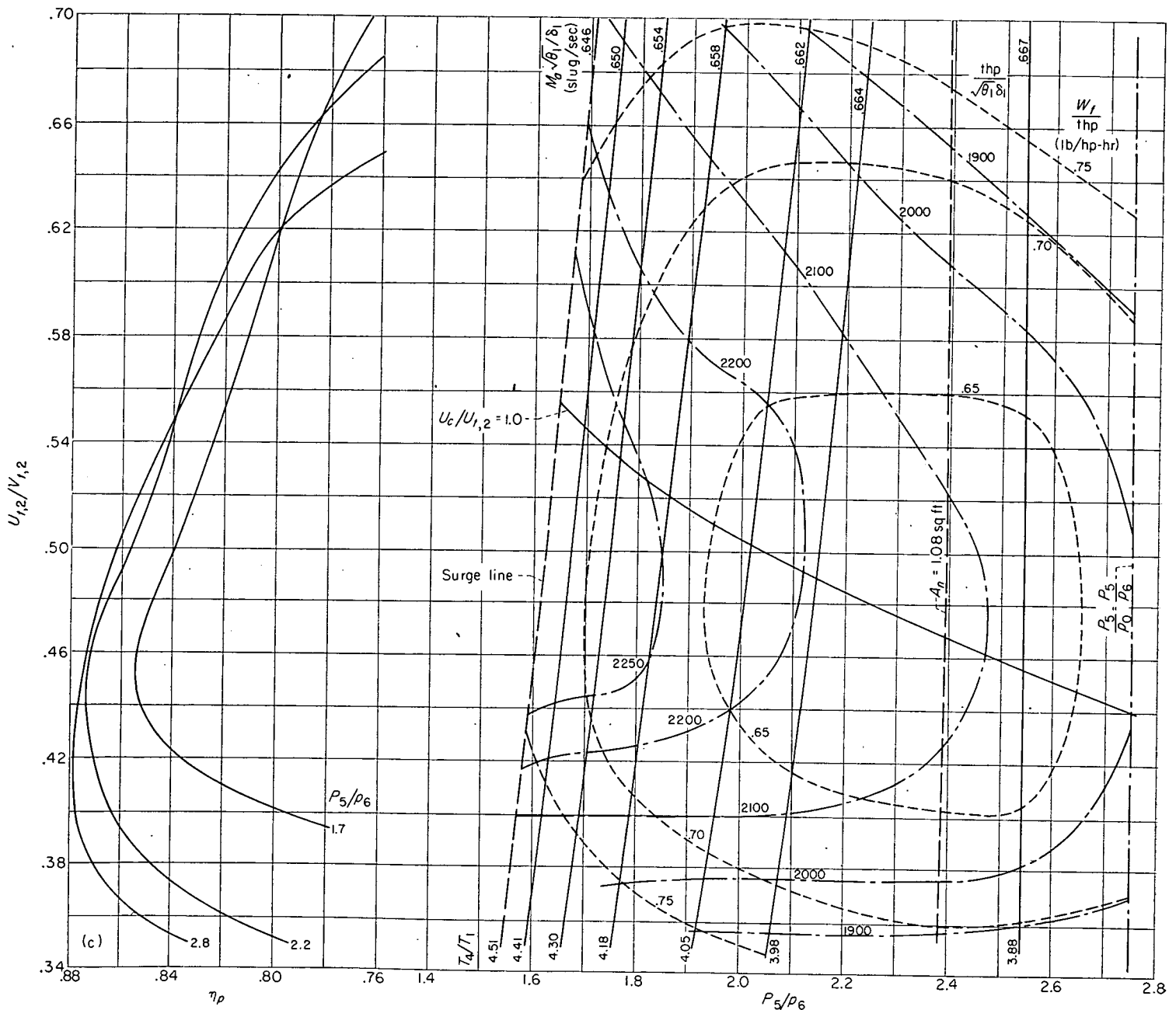
limitation due to excessive stresses diminishes, particularly for the first turbine, which has a pitch-line velocity directly proportional to $U_c/\sqrt{\theta_1}$.

Performance of engine with connected turbine system.—

The engine with both turbines connected is restricted to a fixed relation between turbine pitch-line velocity $U_{t,2}$ and compressor tip speed U_c . However, the power out of the first stage (the first turbine in the engine with divided turbine system) is no longer limited to being equal to compressor power. Some characteristics associated with the removal of the limit on turbine power distribution will now be discussed.

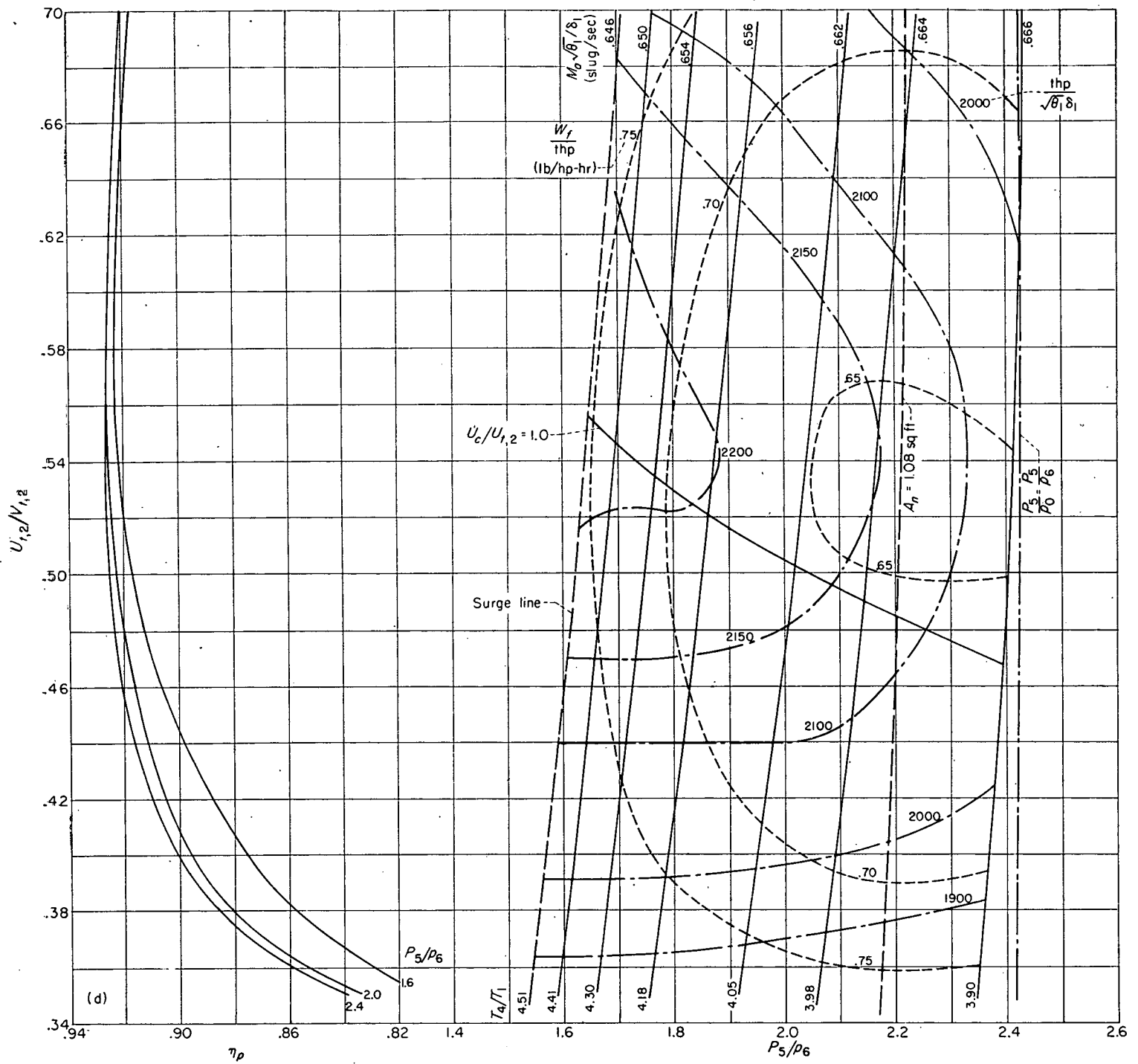
At any operating point on the compressor map (such as

fig. 13), the corresponding values of T_4/T_1 , $M_g\sqrt{\theta_4/\delta_4}$, and $M_g\sqrt{\theta_5/\delta_5}$ depend on the power output of the first turbine. When this power output is prescribed as in the case of the engine with divided turbine system (where the power output is always equal to the compressor power), these factors are all determined at every point. At a given $M_g\sqrt{\theta_5/\delta_5}$, the power output of the second turbine (or second stage) is limited by the range of P_5/p_6 possible at this mass-flow factor (see fig. 14). When the engine with a divided turbine system is operating at a given compressor point, it is thus limited to a range of power output of the second turbine, which depends on the corresponding value of $M_g\sqrt{\theta_5/\delta_5}$. The adjustment of the distribution of the available power



(c) $U_c/\sqrt{\theta_1}$, 1006 feet per second; Y , 0.10 (flight Mach number, 0.71).

FIGURE 17.—Continued. Total-thrust-horsepower factor and specific fuel consumption of matched turbine-propeller engine; η_b , 0.96; C_e , 0.97; h , 18,900 Btu per pound; $\Delta P_d/P_1$, 0.03; $\Delta P_1/P_2$, 0.05; $U_c/U_{t,1}$, 1.0.



(d) $U_d/\sqrt{\theta_1}$, 1006 feet per second; Y , 0.06 (flight Mach number, 0.55).
 FIGURE 17.—Concluded. Total-thrust-horsepower factor and specific fuel consumption of matched turbine-propeller engine: η_p , 0.96; C_w , 0.97; h , 18,900 Btu per pound; $\Delta P_d/P_1$, 0.03; $\Delta P_2/P_2$, 0.05; $U_d/U_{t,1}$, 1.0.

between the propeller and the exhaust-nozzle jet is thus often limited, and a low over-all engine efficiency may result at some off-design-point conditions. On the other hand, when the engine with connected turbines operates at a given compressor point, the values of T_4/T_1 and P_4/P_5 can be varied to give variable power output from the first turbine stage by an adjustment in fuel flow, exhaust-nozzle area, and propeller blade angle. This variation also leads to a

range of values of $M_0\sqrt{\theta_1}/\delta_1$ that correspond to the given compressor operating point; hence, a larger range of power output may be obtained from the second-turbine stage. Power extraction from both turbines may then be adjusted to give more economical distribution of available power between propeller and exhaust jet than is possible with the divided turbine system, particularly at off-design operation.

Included in figure 17 are lines of operation for constant

$U_c/U_{t,2}=1.0$. This condition pertains to the operating line of an engine with the given connected turbine system having a fixed relation between $U_{t,2}$ and U_c and with the power output of the first-stage turbine equal to the compressor power. Figures 18 (a) and 18 (b) are performance plots of the engine similar to that of figure 17 (a), except that the power output of the first turbine is 3 and 7 percent greater, respectively, than the compressor power. These increases

in power output of the first-stage turbine are obtained by increasing the combustion-chamber-outlet temperature T_4 and adjusting the exhaust-nozzle area and the propeller pitch; the range of values of the increase in turbine power is limited in order to avoid exceeding appreciably the design value of T_4 . Operating conditions for these plots are the same as for figure 17 (a), namely $Y=0.10$ and $U_c/\sqrt{\theta_1}=1062$ feet per second. Lines of operation at

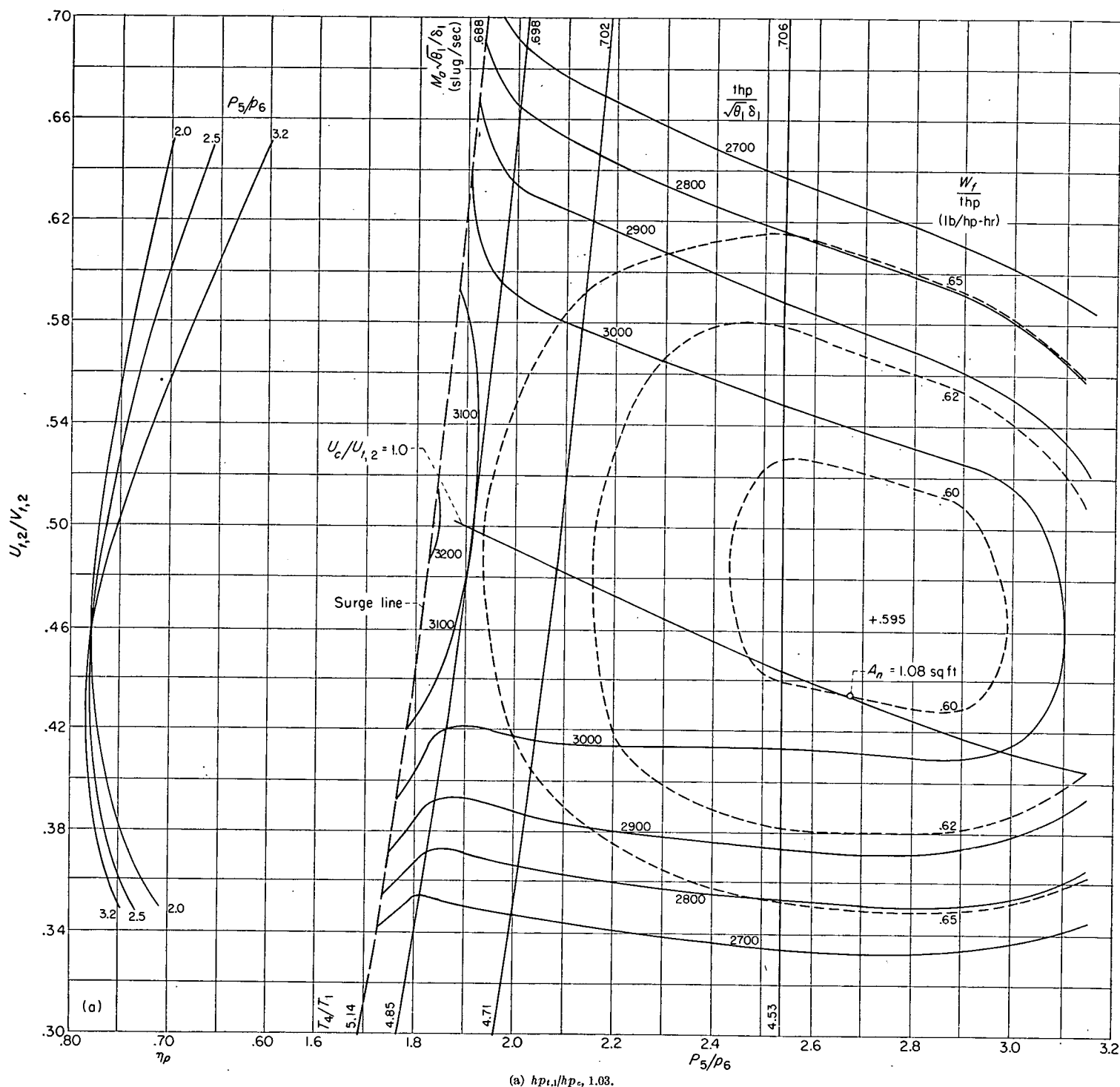


FIGURE 18.—Performance plots of matched turbine-propeller engine for various values of power output of first turbine to compressor power; $U_c/\sqrt{\theta_1}$, 1062 feet per second; Y , 0.10 (flight Mach number, 0.71); η_p , 0.96; C_π , 0.97; h , 18,900 Btu per pound; $\Delta P_d/P_1$, 0.03; $\Delta P_2/P_2$, 0.05; $U_c/U_{t,1}$, 1.0.

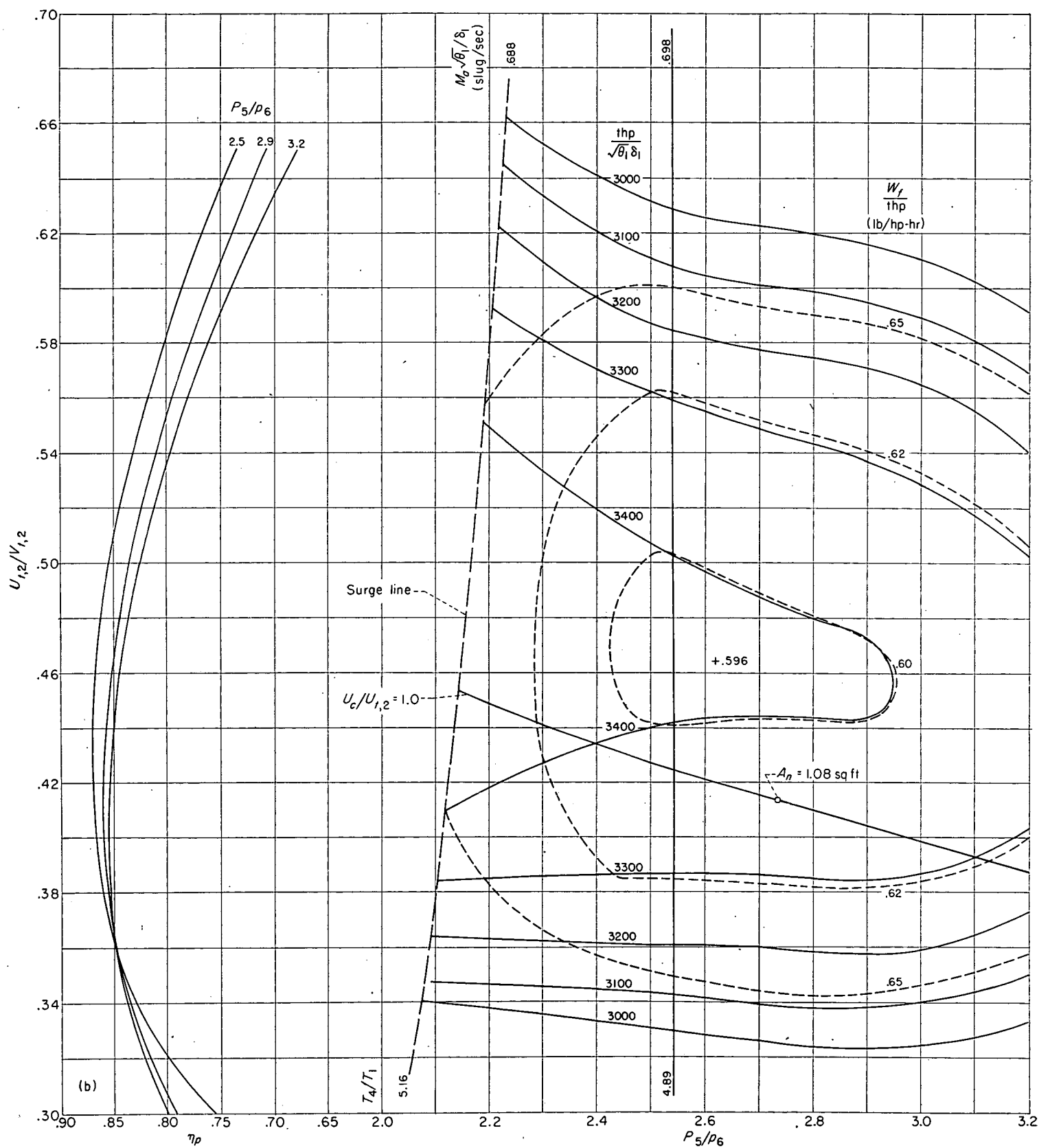


FIGURE 18.—Continued. Performance plots of matched turbine-propeller engine for various values of power output of first turbine to compressor power; $U_d \sqrt{\gamma_1}$, 1062 feet per second; Y , 0.10 (flight Mach number, 0.71); η_b , 0.96; C_π , 0.97; h , 18,900 Btu per pound; $\Delta P_d/P_1$, 0.03; $\Delta P_2/P_2$, 0.05; $U_d/U_{t,3}$, 1.0.

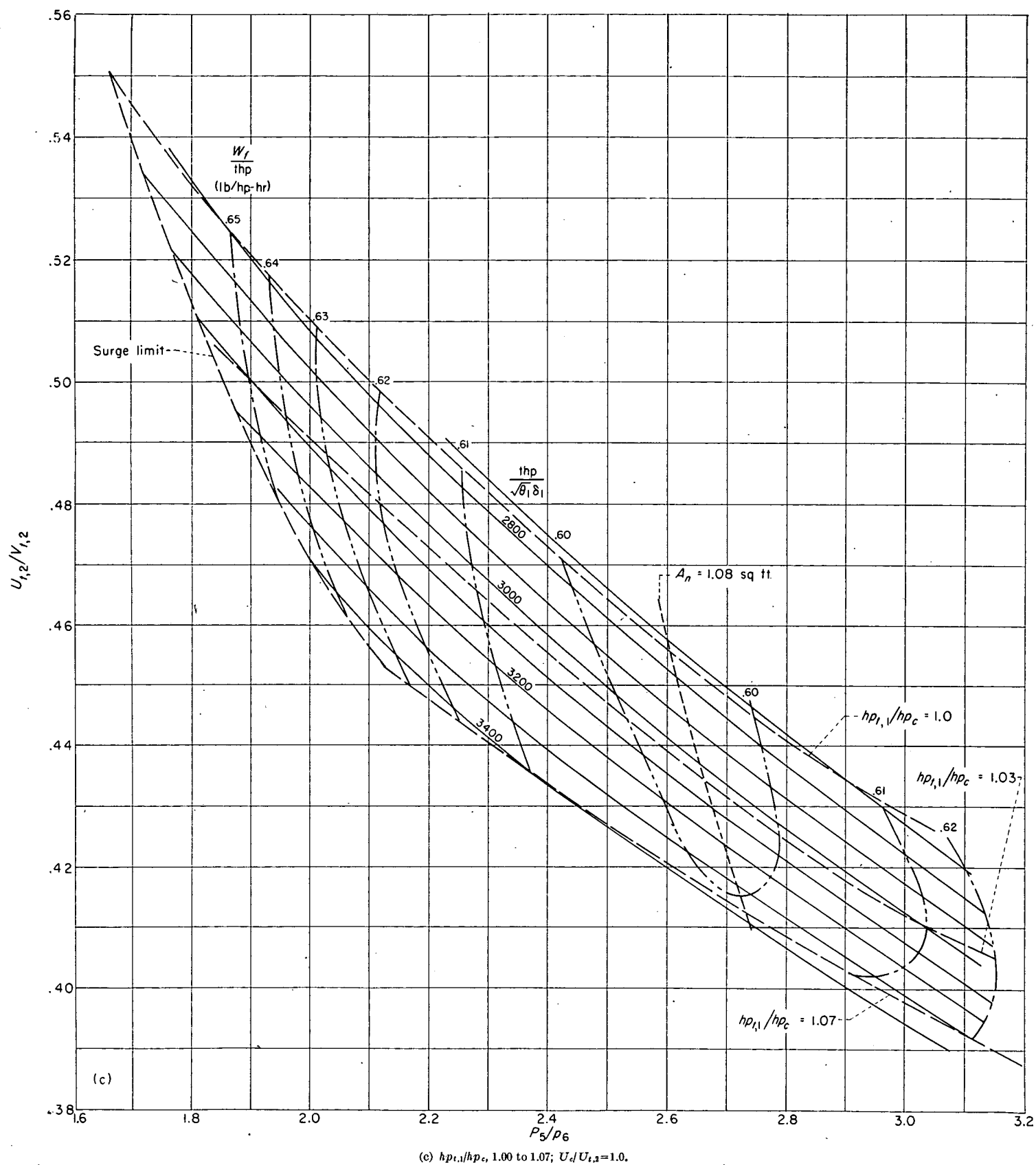


FIGURE 18.—Concluded. Performance plots of matched turbine-propeller engine for various values of power output of first turbine to compressor power; $U_d/\sqrt{\theta_1}$, 1062 feet per second; Y , 0.10 (flight Mach number, 0.71); η_b , 0.96; C_v , 0.97; h , 18,900 Btu per pound; $\Delta P_d/P_1$, 0.03; $\Delta P_2/P_2$, 0.05; $U_d/U_{t,1}$, 1.0.

$U_c/U_{t,2}=1.0$ are again located on these plots and represent the case of the engine with connected turbines under consideration.

The operating lines for $U_c/U_{t,2}=1.0$ of figures 17 (a), 18 (a), and 18 (b) and the corresponding engine performance are shown on a single plot in figure 18 (c). This figure shows the effect on total power and specific fuel consumption due to varying the values of the ratio of the power output of the first turbine to the compressor power $hp_{t,1}/hp_c$ from 1.00 to 1.07. Operation over this range of values of $hp_{t,1}/hp_c$ is possible for the engine with connected turbines, whereas the divided turbine system limits engine operation to a value of $hp_{t,1}/hp_c=1.00$. Figures 17 (a) and 18 show that the engine with connected turbine system can achieve high power increases and maintain good economy by increasing the value of T_4/T_1 . This desirable performance characteristic is possible because efficient distribution of power between propeller and jet is obtained at higher temperature ratios by extracting more power from the first turbine stage, which (by causing a higher value of $M_\infty\sqrt{\theta_5}/\delta_5$, which is accompanied by an increase in pressure differential across the second turbine) permits greater power output from the second turbine. For the engine with connected turbines illustrated, approximately a 20-percent increase in the total power output is obtained by increasing the T_4/T_1 from 4.23 (the design-point value) to 4.89 and increasing the value of $hp_{t,1}/hp_c$ from 1 to 1.07 with an increase in specific fuel consumption of only 1 percent. On the other hand, the engine with a divided turbine achieves an increase in total power of about 5 percent for this same change in T_4/T_1 (see fig. 17 (a)), while the specific fuel consumption increases about 17 percent.

For the conditions shown in figures 17 (a) and 18, the propeller and turbine efficiencies for the engine with the connected turbines are not appreciably decreased by restrictions imposed by the fixed relation among U_c , $U_{t,1}$, and $U_{t,2}$ for the interesting operating range. (The turbine efficiencies in fig. 17 (a) apply to fig. 18.)

Operation over the entire region shown in figure 18 (c) requires a variable-area exhaust nozzle. The line of constant A_n of 1.08 square feet is included in this figure. For the particular operating conditions covered in this figure, it is seen that this constant exhaust-nozzle area permits operation to be maintained at best efficiency for a range of powers including maximum power. With A_n of 1.08 square feet, the engine with a connected turbine can operate at $U_c/\sqrt{\theta_1}=1062$ feet per second from the surge line to the limiting-line

at which $M_\infty\sqrt{\theta_5}/\delta_5=0.60$ slug per second. At other values of compressor tip-speed factor, the region of operation on the compressor map with constant-area exhaust nozzle is much greater for the engine with a connected turbine than for the engine with a divided turbine.

SUMMARY OF RESULTS

For a series of turbine-propeller engines in which the appropriate components are used to give the design-point conditions and efficiencies, the following results were obtained:

1. At a given set of operating conditions, maximum thrust per unit mass rate of air flow and maximum thrust horsepower per unit mass rate of air flow occurred at a lower compressor pressure ratio than that at which minimum specific fuel consumption occurred.

2. An increase in combustion-chamber-outlet temperature caused an increase in power output per unit mass rate of air flow. Improved specific fuel consumption was obtained with increased combustion-chamber-outlet temperature, provided that the compressor pressure ratio was correspondingly increased.

3. An optimum jet velocity existed which gave best distribution between propeller-shaft power and jet power for maximum thrust horsepower and efficiency.

The following results were obtained from a study of two turbine-propeller engines, one engine having a divided turbine system in which the first turbine drove only the compressor and the second turbine independently drove the propeller, and the other engine having a connected turbine system which drove both the compressor and the propeller:

1. The connected turbine system offered greater adjustment of distribution of available power to the propeller shaft and the exhaust-nozzle jet, thereby permitting more efficient power distribution at off-design-point operation.

2. The divided turbine system offered more flexible control of component rotational speeds, which enabled speed adjustment giving best combination of turbine and propeller efficiencies.

3. When equipped with a constant-area exhaust nozzle, the engine with the divided turbine system was limited to a very narrow region of operation on the compressor operating diagram, whereas the engine with the connected turbine system was capable of a wide range of compressor operation.

LEWIS FLIGHT PROPULSION LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

CLEVELAND, OHIO, August 2, 1951

APPENDIX A

DERIVATION OF PERFORMANCE EQUATIONS AND MISCELLANEOUS EXPRESSIONS

In addition to those symbols previously listed, the following symbols are used in these derivations:

- $\bar{c}_p$ average specific heat at constant pressure of gases during combustion process, Btu/(slug)(°F) (This term, when used with the temperature change during combustion, is used to determine fuel consumption.)
- H_0 enthalpy of air corresponding to ambient-air temperature t_0 , Btu/slug
- H_2 enthalpy of air corresponding to compressor-outlet total temperature T_2 , Btu/slug
- P_0 free-stream total pressure, lb/sq ft abs
- R_a gas constant of air, 1716 ft-lb/(slug)(°F)
- R_g gas constant of exhaust gas, ft-lb/(slug)(°F)
- T_6 total temperature at outlet of second turbine, °R
- X' factor equal to $\left[\frac{P_2/P_1}{(P_2/P_1)_{ref}} \right]^{\frac{\gamma_a-1}{\gamma_a}}$ or $(X)^{\frac{\gamma_a-1}{\gamma_a}}$

W_{th} ideal work for adiabatic process, ft-lb/slug

The jet velocity is given by

$$\frac{V_j^2}{C_v^2} = 2Jc_{p,g}T_4 \left[1 - \left(\frac{p_0}{P_4} \right)^{\frac{\gamma_g-1}{\gamma_g}} \right] - \frac{550hp_t}{\frac{1}{2}M_a(1+f)\eta_t} \quad (A1)$$

$$\left(1 - \frac{p_0}{P_4} \right)^{\frac{\gamma_g-1}{\gamma_g}} = 1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_g-1}{\gamma_g}} \left(1 - \frac{\Delta P_2}{P_2} \right)^{-\frac{\gamma_g-1}{\gamma_g}} \quad (A2)$$

and when the last term is expanded into a series,

$$\left(1 - \frac{\Delta P_2}{P_2} \right)^{-\frac{\gamma_g-1}{\gamma_g}} = 1 + \frac{\gamma_g-1}{\gamma_g} \frac{\Delta P_2}{P_2} \quad (A3)$$

for small $\frac{\Delta P_2}{P_2}$.

By the use of equations (A2) and (A3),

$$c_{p,g} \left[1 - \left(\frac{p_0}{P_4} \right)^{\frac{\gamma_g-1}{\gamma_g}} \right] = c_{p,g} \left[1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_g-1}{\gamma_g}} - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_g-1}{\gamma_g}} \left(\frac{\gamma_g-1}{\gamma_g} \right) \frac{\Delta P_2}{P_2} \right] \quad (A4)$$

Let

$$K = \frac{c_{p,g} \left[1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_g-1}{\gamma_g}} \right]}{c_{p,a} \left[1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a-1}{\gamma_a}} \right]} \quad (A5)$$

and

$$K' = \frac{\left(\frac{p_0}{P_2} \right)^{\frac{\gamma_g-1}{\gamma_g}} \left(\frac{\gamma_g-1}{\gamma_g} \right) c_{p,g}}{\left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a-1}{\gamma_a}} \left(\frac{\gamma_a-1}{\gamma_a} \right) c_{p,a}} \quad (A6)$$

When equations (A4) to (A6) are used in equation (A1),

$$\frac{V_j^2}{C_v^2} = 2Jc_{p,g}T_4 \left\{ \left[1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a-1}{\gamma_a}} \right] K - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a-1}{\gamma_a}} \left(\frac{\gamma_a-1}{\gamma_a} \right) \frac{\Delta P_2}{P_2} K' \right\} - \frac{550hp_t}{\frac{1}{2}M_a(1+f)\eta_t} \quad (A7)$$

Define

$$Y = \frac{V_0^2}{2Jc_{p,a}t_0} \quad (A8)$$

The total temperature at the compressor inlet (which is equal to the total temperature of the inlet air) is

$$T_1 = t_0 + \frac{V_0^2}{2Jc_{p,a}} = t_0(1+Y) \quad (A9)$$

The ratio of the ideal stagnation pressure P_0 to the ambient-air pressure is

$$\frac{P_0}{p_0} = \left(\frac{T_1}{t_0} \right)^{\frac{\gamma_a}{\gamma_a-1}} = (1+Y)^{\frac{\gamma_a}{\gamma_a-1}} \quad (A10)$$

The ratio of the actual total pressure at compressor inlet to the ambient-air pressure is

$$\frac{P_1}{p_0} = \frac{P_0 - \Delta P_d}{p_0} = (1+Y)^{\frac{\gamma_a}{\gamma_a-1}} - \left(\frac{\Delta P_d}{P_1} \right) \left(\frac{P_1}{p_0} \right) \quad (A11)$$

from which

$$\frac{P_1}{p_0} = \frac{(1+Y)^{\frac{\gamma_a}{\gamma_a-1}}}{1 + \frac{\Delta P_d}{P_1}} \quad (A12)$$

The compressor-shaft horsepower expressed as a function of compressor-inlet temperature and pressure ratio is

$$hp_c = \frac{M_a J c_{p,a} T_1}{550 \eta_c} \left[\left(\frac{P_2}{P_1} \right)^{\frac{\gamma_a - 1}{\gamma_a}} - 1 \right] \quad (A13)$$

where the specific heats of air during the compression process are assumed constant. Because of this assumption, the value of the compressor-shaft power calculated from equation (A13) for a given pressure ratio, inlet temperature, and efficiency is slightly greater than the actual compressor power. The deviation increases with increasing pressure ratio and inlet temperature. For values of T_1 up to 550°R and P_2/P_1 up to 40, the maximum error in compressor power is about 1 percent.

Define

$$Z = \frac{550 hp_c}{M_a J c_{p,a} t_0} \quad (A14)$$

so that substituting equations (A9) and (A14) into equation

(A13) results in

$$\frac{P_2}{P_1} = \left(1 + \frac{\eta_c Z}{1 + Y} \right)^{\frac{\gamma_a}{\gamma_a - 1}} = \left(\frac{1 + Y + \eta_c Z}{1 + Y} \right)^{\frac{\gamma_a}{\gamma_a - 1}} \quad (A15)$$

Equations (A12) and (A15) are combined so that

$$\left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a - 1}{\gamma_a}} = \frac{\left(1 + \frac{\Delta P_d}{P_1} \right)^{\frac{\gamma_a - 1}{\gamma_a}}}{1 + Y + \eta_c Z} \quad (A16)$$

and expanding gives

$$\left(1 + \frac{\Delta P_d}{P_1} \right)^{\frac{\gamma_a - 1}{\gamma_a}} = 1 + \frac{\gamma_a - 1}{\gamma_a} \frac{\Delta P_d}{P_1} \quad (A17)$$

which is accurate for small values of $\Delta P_d/P_1$.

From equations (A16) and (A17)

$$1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a - 1}{\gamma_a}} = \frac{Y + \eta_c Z - \left(\frac{\gamma_a - 1}{\gamma_a} \right) \frac{\Delta P_d}{P_1}}{1 + Y + \eta_c Z} \quad (A18)$$

When equations (A16) to (A18) are substituted into equation (A7),

$$\frac{V_i^2}{C_v^2} = 2 J c_{p,a} T_4 \left[K \left(\frac{Y + \eta_c Z - \frac{\gamma_a - 1}{\gamma_a} \frac{\Delta P_d}{P_1}}{1 + Y + \eta_c Z} \right) - K' \left(\frac{\gamma_a - 1}{\gamma_a} \right) \frac{\Delta P_2}{P_2} \left(\frac{1 + \frac{\gamma_a - 1}{\gamma_a} \frac{\Delta P_d}{P_1}}{1 + Y + \eta_c Z} \right) \right] - \frac{550 hp_t}{\frac{1}{2} M_a (1 + f) \eta_t} \quad (A19)$$

The term involving the product of the pressure-drop ratios $\frac{\Delta P_d}{P_1} \frac{\Delta P_2}{P_2}$ can be neglected, so that equation (A19) becomes

$$\frac{V_i^2}{C_v^2} = \frac{2 J c_{p,a} T_4}{1 + Y + \eta_c Z} \left[K \left(Y + \eta_c Z - \frac{\gamma_a - 1}{\gamma_a} \frac{\Delta P_d}{P_1} \right) - K' \frac{\gamma_a - 1}{\gamma_a} \frac{\Delta P_2}{P_2} \right] - \frac{550 hp_t}{\frac{1}{2} M_a (1 + f) \eta_t} \quad (A20)$$

The factor ϵ is defined by the equation

$$\frac{V_i^2}{C_v^2} = 2 J c_{p,a} T_4 \frac{Y + \eta_c Z}{1 + Y + \eta_c Z} \epsilon - \frac{550 hp_t}{\frac{1}{2} M_a (1 + f) \eta_t} \quad (A21)$$

from which

$$\epsilon = K - K' \left(\frac{\gamma_a - 1}{\gamma_a} \right) \left(\frac{\Delta P_d}{P_1} \right) \left(\frac{1}{Y + \eta_c Z} \right) - K' \left(\frac{\gamma_a - 1}{\gamma_a} \right) \left(\frac{\Delta P_2}{P_2} \right) \left(\frac{1}{Y + \eta_c Z} \right) \quad (A22)$$

Equation (A21) can be rewritten

$$\frac{hp_t}{M_a} = \left(\frac{T_4}{t_0} \epsilon \frac{Y + \eta_c Z}{1 + Y + \eta_c Z} - \frac{V_i^2}{2 C_v^2 J c_{p,a} t_0} \right) \frac{\eta_t (1 + f) J c_{p,a} t_0}{550} \quad (A23)$$

The thrust horsepower developed by the propeller is

$$\frac{thp_p}{M_a} = \eta_p \left(\frac{hp_t}{M_a} - \frac{hp_c}{M_a} \right) \quad (A24)$$

and substituting equations (A14) and (A23) in (A24) yields

$$\frac{thp_p}{M_a} = \frac{\eta_p J c_{p,a} t_0}{550} \left[\eta_t (1 + f) \frac{T_4}{t_0} \epsilon \frac{Y + \eta_c Z}{1 + Y + \eta_c Z} - \frac{V_i^2 \eta_t (1 + f)}{2 C_v^2 J c_{p,a} t_0} - Z \right] \quad (A25)$$

The thrust horsepower developed by the jet is

$$\frac{thp_j}{M_a} = [V_j (1 + f) - V_0] \frac{V_0}{550} \quad (A26)$$

The total thrust horsepower of the engine is the sum of the propeller-thrust horsepower and jet-thrust horsepower; thus, from equations (A25) and (A26)

$$\frac{thp}{M_a} = \frac{\eta_p J c_{p,a} t_0}{550} \left[\eta_i (1+f) \frac{T_4}{t_0} \epsilon \frac{Y + \eta_c Z}{1 + Y + \eta_c Z} - \frac{V_j^2 \eta_i (1+f)}{2 C_v^2 J c_{p,a} t_0} - \frac{\eta_c Z}{\eta_c} \right] + [V_j (1+f) - V_0] \frac{V_0}{550} \quad (A27)$$

OPTIMUM JET VELOCITY AND $\eta_c Z$ FOR MAXIMUM thp/M_a

For given values of V_0 , η_p , η_i , η_c , T_4 , t_0 , and ϵ and with the component efficiencies and ϵ assumed to remain constant as $\eta_c Z$ and V_j vary, optimum values of $\eta_c Z$ and V_j are obtained from

$$\frac{\partial(thp/M_a)}{\partial(\eta_c Z)} = \frac{J c_{p,a} t_0 \eta_p}{550} \left[\eta_i (1+f) \frac{T_4}{t_0} \epsilon \frac{(1 + Y + \eta_c Z) - (Y + \eta_c Z)}{(1 + Y + \eta_c Z)^2} - \frac{1}{\eta_c} \right] = 0 \quad (A28)$$

and

$$\frac{\partial(thp/M_a)}{\partial(V_j)} = \frac{J c_{p,a} t_0 \eta_p}{550} \left[-\frac{2 V_j \eta_i (1+f)}{2 C_v^2 J c_{p,a} t_0} \right] + \frac{V_0}{550} (1+f) = 0 \quad (A29)$$

From equation (A29)

$$V_{j,opt} = V_0 \frac{C_v^2}{\eta_i \eta_p} \quad (A30)$$

and from equation (A28)

$$1 + Y + (\eta_c Z)_{ref} = \sqrt{\eta_c \eta_i (1+f) \epsilon \frac{T_4}{t_0}} \quad (A31)$$

where the term $(\eta_c Z)_{ref}$ is used to designate the value of $\eta_c Z$ for which the thp/M_a given by equation (A27) is a maximum. Define

$$X' = \frac{1 + Y + \eta_c Z}{\sqrt{\eta_c \eta_i (1+f) \epsilon \frac{T_4}{t_0}}} \quad (A32)$$

and

$$Y_j = \frac{V_j^2}{2 J c_{p,a} t_0} \quad (A33)$$

When equations (A32) and (A33) are substituted into equation (A25),

$$\frac{thp_p}{M_a} \frac{\eta_c}{\eta_p} \frac{519}{t_0} = \frac{519 J c_{p,a}}{550} \left[\eta_c \eta_i (1+f) \epsilon \frac{T_4}{t_0} - \left(X' + \frac{1}{X'} \right) \sqrt{\eta_c \eta_i (1+f) \epsilon \frac{T_4}{t_0}} + 1 + Y - Y_j \frac{\eta_i (1+f) \eta_c}{C_v^2} \right] \quad (A34)$$

The thrust produced by the propeller is

$$\frac{F_p}{M_a} = \frac{550}{V_0} \frac{thp_p}{M_a} \quad (A35)$$

Define the factor α as the ratio of the pounds of thrust produced by the propeller to the excess of turbine horsepower output over compressor horsepower input; then,

$$\alpha = \frac{F_p}{hp_t - hp_c} \quad (A36)$$

When equations (A35), (A36), and (A24) are combined,

$$\frac{F_p}{M_a} \left(\frac{519}{t_0} \right) \left(\frac{\eta_c}{\alpha} \right) = \frac{thp_p}{M_a} \left(\frac{519}{t_0} \right) \left(\frac{\eta_c}{\eta_p} \right) \quad (A37)$$

This equation permits the propeller thrust to be evaluated from equation (A34) when $V_0=0$ (at which velocity $\eta_p=0$ and $thp_p=0$).

For the case of $V_0=0$, the expression for optimum jet velocity (for optimum thrust) similarly reduces to

$$V_{j,opt} = \frac{C_v^2}{\eta_i} \frac{550}{\alpha} \quad (A38)$$

when equations (A35), (A36), (A24), and (A30) are combined.

EVALUATION OF CORRECTION FACTOR ϵ

The factors a and b are defined as

$$a = \frac{\Delta P_a}{P_1} \frac{\gamma_a - 1}{\gamma_a} \frac{1}{Y + \eta_c Z} \quad (A93)$$

and

$$b = \frac{\Delta P_2}{P_2} \frac{\gamma_a - 1}{\gamma_a} \frac{1}{Y + \eta_c Z} \quad (\text{A40})$$

When equations (A39) and (A40) are substituted into equation (A22),

$$\epsilon = K - Ka - K'b \quad (\text{A41})$$

The terms K and K' are close to unity in value, whereas the values of a and b are small in comparison with unity; therefore only a small error is introduced by letting

$$\epsilon = K - a - b \quad (\text{A42})$$

The quantity c is defined as

$$c = K - 1 \quad (\text{A43})$$

then

$$\epsilon = 1 - a - b + c \quad (\text{A44})$$

From equation (A5) and reference 9,

$$c = \frac{\frac{W_{th}}{T_4}}{Jc_{p,a} \left[1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a - 1}{\gamma_a}} \right]} - 1 \quad (\text{A45})$$

where the values of W_{th}/T_4 are obtained from references 4 and 9 and unpublished data extending figure 9 of reference 9. These values correspond to the temperature T_4 and pressure ratio P_2/p_0 .

FUEL CONSUMPTION

The expression for the fuel-air-ratio factor to obtain a rise in total temperature in the combustion chamber from T_2 to T_4 is

$$\eta_b f = \frac{\bar{c}_p (T_4 - T_2)}{32.2 h} \quad (\text{A46})$$

where values of $\bar{c}_p$ are determined from reference 5.

From the conservation of energy,

$$H_2 = H_0 + \frac{V_0^2}{2J} + 550 \frac{h p_c}{M_a J} \quad (\text{A47})$$

where H_2 is the enthalpy of the air corresponding to the compressor-outlet total temperature T_2 in Btu per slug. (Zero enthalpy is arbitrarily fixed at absolute zero temperature.) The symbol H_0 is the enthalpy of air corresponding to the ambient-air temperature t_0 in Btu per slug and is given by

$$H_0 = c_{p,a} t_0 \quad (\text{A48})$$

If equations (A48), (A8), and (A14) are used in equation (A47),

$$H_2 = c_{p,a} t_0 (1 + Y + Z) \quad (\text{A49})$$

Since T_2 is a function only of H_2 ,

$$T_2 = \phi(H_2) = \phi[c_{p,a} t_0 (1 + Y + Z)] \quad (\text{A50})$$

and the T_2 corresponding to the enthalpy H_2 is obtained from reference 4.

DERIVATION OF MISCELLANEOUS EXPRESSIONS

The pressure ratio $(P_2/P_1)_{ref}$ corresponding to any values of Y and $(\eta_c Z)_{ref}$ is, from equation (A15),

$$\left(\frac{P_2}{P_1} \right)_{ref} = \left[\frac{1 + Y + (\eta_c Z)_{ref}}{1 + Y} \right]^{\frac{\gamma_a}{\gamma_a - 1}} \quad (\text{A51})$$

or, substituting equation (A31) in equation (A51) gives

$$\left(\frac{P_2}{P_1} \right)_{ref} = \left[\eta_c \eta_t (1 + f) \epsilon \frac{T_4}{t_0} \left(\frac{1}{1 + Y} \right)^2 \right]^{\frac{\gamma_a}{2(\gamma_a - 1)}} \quad (\text{A52})$$

From equations (A15), (A31), (A32), and (A51) it is seen that

$$X' = \left[\frac{P_2/P_1}{(P_2/P_1)_{ref}} \right]^{\frac{\gamma_a - 1}{\gamma_a}} \quad (\text{A53})$$

Define

$$X = \frac{P_2/P_1}{(P_2/P_1)_{ref}} \quad (\text{A54})$$

then

$$X = (X')^{\frac{\gamma_a}{\gamma_a - 1}} \quad (\text{A55})$$

APPENDIX B

EQUATIONS FOR PERFORMANCE CHARTS

(The equation numbers correspond to those in the derivation given in appendix A.)

Figure 2 (a).—

$$Y = \frac{V_0^2}{2Jc_{p,a}t_0} = \frac{1}{2Jc_{p,a}519} \left(V_0 \sqrt{\frac{519}{t_0}} \right)^2 \quad (\text{A8})$$

$$\frac{P_0}{p_0} = \frac{P_1 + \Delta P_d}{p_0} = (1 + Y)^{\frac{\gamma_a}{\gamma_a - 1}} = \left[1 + \frac{1}{2Jc_{p,a}519} \left(V_0 \sqrt{\frac{519}{t_0}} \right)^2 \right]^{\frac{\gamma_a}{\gamma_a - 1}} \quad (\text{A10})$$

$$\text{Flight Mach number} = \frac{V_0}{\sqrt{\gamma_a R_a t_0}} = \frac{1}{\sqrt{\gamma_a R_a 519}} \left(V_0 \sqrt{\frac{519}{t_0}} \right)$$

Figure 2 (b).—

$$Y_j = \frac{V_j^2}{2Jc_{p,a}t_0} = \frac{1}{2Jc_{p,a}519} \left(V_j \sqrt{\frac{519}{t_0}} \right)^2 \quad (\text{A33})$$

Figure 3 (a).—

$$a = \frac{\Delta P_d}{P_1} \frac{\gamma_a - 1}{\gamma_a} \frac{1}{Y + \eta_c Z} \quad (\text{A39})$$

Figure 3 (b).—

$$b = \frac{\Delta P_2}{P_2} \frac{\gamma_a - 1}{\gamma_a} \frac{1}{Y + \eta_c Z} \quad (\text{A40})$$

Figure 3 (c).—

$$c = \frac{\frac{W_{th}}{T_4}}{Jc_{p,a} \left[1 - \left(\frac{p_0}{P_2} \right)^{\frac{\gamma_a - 1}{\gamma_a}} \right]} - 1 \quad (\text{A45})$$

where values of W_{th}/T_4 are obtained from references 4 and 9 and unpublished data extending data in reference 9.

Figure 4.—

$$\frac{P_2}{P_1} = \left(1 + \frac{\eta_c Z}{1+Y}\right)^{\frac{\gamma_a}{\gamma_a-1}} \quad (\text{A51})$$

$$\left(\frac{P_2}{P_1}\right)_{ref} = \left[\eta_c \eta_t \epsilon \frac{T_4}{t_0} \left(\frac{1}{1+Y}\right)^2\right]^{\frac{\gamma_a}{2(\gamma_a-1)}} \quad (\text{A52})$$

In order to include the effect of added mass of fuel, the value of η_t used in equation (A52) should be the product $\eta_t(1+f)$.

Figure 5.—

$$\frac{th p_p}{M_a} \frac{\eta_c}{\eta_p} \frac{519}{t_0} = \frac{519 J c_{p,g}}{550} \left[\eta_c \eta_t \epsilon \frac{T_4}{t_0} - \left(X' + \frac{1}{X'}\right) \sqrt{\eta_c \eta_t \epsilon \frac{T_4}{t_0}} + 1 + Y - Y_1 \frac{\eta_t \eta_c}{C_p^2} \right] \quad (\text{A34})$$

where

$$X' = (X)^{\frac{\gamma_a-1}{\gamma_a}}$$

In order to include effect of added mass of fuel, the value of η_t used in equation (A34) should be the product $\eta_t(1+f)$.

Figure 6.—

$$H_2 = c_{p,a} t_0 (1 + Y + Z) \quad (\text{A49})$$

The T_2 corresponding to the enthalpy H_2 is obtained from reference 4.

Figure 7.—

$$\eta_b f = \frac{\bar{c}_p (T_4 - T_2)}{32.2 h} \quad (\text{A46})$$

where $\bar{c}_p$ is determined from reference 5.

APPENDIX C

METHOD FOR DETERMINING OPERATING LINES OF CONSTANT T_4/T_1 AND CONSTANT $M_g \sqrt{\theta_5/\delta_5}$ FOR MATCHED SET OF TURBINE-PROPELLER-ENGINE COMPONENTS CONSISTING OF ONE TURBINE DRIVING ONLY COMPRESSOR AND SECOND TURBINE DRIVING ONLY PROPELLER

The procedure for plotting lines of constant T_4/T_1 and $M_g \sqrt{\theta_5/\delta_5}$ for the gas-generator plot of figure 13 (a) is as follows:

Equation (14), based on compressor power being equal to the power of first turbine, can be converted to

$$K_c \frac{U_c^2}{\theta_1} = \frac{1}{2} \frac{V_{t,1}^2}{\theta_4} \eta'_{t,1} \frac{T_4}{T_1} \quad (\text{C1})$$

When T_4/T_1 is eliminated between equations (12) and (C1),

$$\sqrt{K_c} \frac{M_a \sqrt{\theta_1}}{\delta_1} \frac{U_c}{\sqrt{\theta_1}} = (1-r) \frac{P_2}{P_1} \frac{M_g V_{t,1}}{\delta_4} \sqrt{\frac{\eta'_{t,1}}{2}} \quad (\text{C2})$$

(1) A point on figure 11 is selected at which the value of T_4/T_1 is desired. The values of $U_c/\sqrt{\theta_1}$, P_2/P_1 , $M_a \sqrt{\theta_1/\delta_1}$, and K_c corresponding to the point are read on the figure.

(2) A value of $U_c/U_{t,1}$ is chosen based on sizes of turbine and compressor used; then, from the values of $U_c/U_{t,1}$ and K_c , $\sqrt{\eta'_{t,1}}/(U_{t,1}/V_{t,1})$ is evaluated from equation (15). A probable value of P_4/p_5 is assumed, and the value of $\sqrt{\eta'_{t,1}}/(U_{t,1}/V_{t,1})$ is used to read the values of $\eta_{t,1}$, $\eta'_{t,1}$, and $U_{t,1}/V_{t,1}$ from figure 12 (b).

(3) Equation (C2) is used to evaluate $M_g V_{t,1}/\delta_4$.

(4) Corresponding to the values of $M_g V_{t,1}/\delta_4$ and $U_{t,1}/V_{t,1}$, the values of $M_g \sqrt{\theta_4/\delta_4}$, $V_{t,1}/\sqrt{\theta_4}$, and P_4/p_5 are read on figure 12 (a). This value of P_4/p_5 should be used to check the values of η_t and $\eta'_{t,1}$ determined in step (2). If any appreciable differences in values of the efficiencies result, the revised value of $\eta'_{t,1}$ is used in step (3).

(5) The value of T_4/T_1 is now evaluated from equation (12).

In order to illustrate this method, the point is picked on figure 11 at which T_4/T_1 is to be evaluated.

(1) The point is selected where $U_c/\sqrt{\theta_1} = 1006$ ft/sec, $P_2/P_1 = 4.74$, $M_a \sqrt{\theta_1/\delta_1} = 0.660$ slug/sec, and $K_c = 2.05$.

(2) A value of $U_c/U_{t,1} = 1.0$ is used for this engine. When $U_c/U_{t,1}$ and K_c are used in equation (15), a value of $\sqrt{\eta'_{t,1}}/(U_{t,1}/V_{t,1}) = 2.02$ is obtained. This value is used in

figure 12 (b), a value of P_4/P_5 of 2.0 is assumed, and the following values are read: $U_{t,1}/V_{t,1} = 0.435$, $\eta_{t,1} = 0.835$, and $\eta'_{t,1} = 0.782$.

(3) From equation (C2) and an assumed value of $r = 0.05$, a value of $M_g V_{t,1}/\delta_4 = 338$ ft/sec is calculated.

(4) When the values of $M_g V_{t,1}/\delta_4$ and $U_{t,1}/V_{t,1}$ are used in figure 12 (a), the following values are read: $V_{t,1}/\sqrt{\theta_4} = 1136$ ft/sec, $M_g \sqrt{\theta_4/\delta_4} = 0.298$ slug/sec, and $P_4/p_5 = 2.23$. This value of P_4/p_5 does not cause any appreciable change in values of turbine efficiencies from those determined in step (2) with an assumed value of P_4/p_5 of 2.0.

(5) The value of T_4/T_1 evaluated from equation (12) is 4.13.

In order to determine the total pressure at the turbine outlet, the following approximation of the turbine total efficiency is made:

$$\eta_{t,1} = \frac{550 h p_{t,1}}{M_g J c_{p,g} T_4 \left[1 - \left(\frac{p_5}{P_4}\right)^{\frac{\gamma_g-1}{\gamma_g}} \right] - \frac{1}{2} M_g V_{t,1}^2}$$

$$= \frac{550 h p_{t,1}}{M_g J c_{p,g} T_4 \left[1 - \left(\frac{P_5}{P_4}\right)^{\frac{\gamma_g-1}{\gamma_g}} \right]}$$

so that

$$\left. \begin{aligned} \eta_{t,1} J c_{p,g} T_4 \left[1 - \left(\frac{P_5}{P_4}\right)^{\frac{\gamma_g-1}{\gamma_g}} \right] &= \eta'_{t,1} \frac{V_{t,1}^2}{2} \\ \text{or} \quad 2 J c_{p,g} 519 \left[1 - \left(\frac{P_5}{P_4}\right)^{\frac{\gamma_g-1}{\gamma_g}} \right] &= \frac{\eta'_{t,1}}{\eta_{t,1}} \frac{V_{t,1}^2}{\theta_4} \end{aligned} \right\} \quad (\text{C3})$$

On figure 12 (a), the value of the upper abscissa $V_{t,1}/\sqrt{\theta_4}$ is related to the value of the lower abscissa P_4/p_5 according to the equation

$$2 J c_{p,g} 519 \left[1 - \left(\frac{p_5}{P_4}\right)^{\frac{\gamma_g-1}{\gamma_g}} \right] = \frac{V_{t,1}^2}{\theta_4}$$

Figure 12 (a) can similarly be applied to equation (C3) by using the value of $\sqrt{\eta_{t,1}/\eta_{t,1}} V_{t,1}/\sqrt{\theta_4}$ on the upper abscissa instead of $V_{t,1}/\sqrt{\theta_4}$ and reading the value of P_4/P_5 instead of P_4/p_5 on the lower abscissa.

The total-temperature ratio T_5/T_4 is

$$\frac{T_5}{T_4} = 1 - \frac{550 h p_{t,1}}{J c_{p,g} T_4 M_g} = 1 - \frac{\eta'_{t,1}}{2} \frac{V_{t,1}^2}{\theta_4} \frac{1}{J c_{p,g} 519} \quad (C4)$$

The mass-flow factor $M_g \sqrt{\theta_5}/\delta_5$ can now be obtained from equation (13).

APPENDIX D

METHOD FOR DETERMINING EXHAUST-NOZZLE AREA OF MATCHED SET OF TURBINE-PROPELLER-ENGINE COMPONENTS CONSISTING OF ONE TURBINE DRIVING ONLY COMPRESSOR AND SECOND TURBINE DRIVING ONLY PROPELLER

Useful equations for determining the values of several parameters needed in this method are as follows:

From the definition of effective exhaust-nozzle area,

$$M_g = A_n \left(\frac{p_0}{P_6} \right)^{\frac{1}{\gamma_t}} \rho_6 \sqrt{2 J c_{p,g} T_6 \left[1 - \left(\frac{p_0}{P_6} \right)^{\frac{\gamma_t-1}{\gamma_t}} \right]}$$

where ρ_6 is the stagnation density at the second-turbine outlet and T_6 , the total temperature at the outlet of the second turbine. Thus,

$$\frac{M_g \sqrt{\theta_6}}{A_n \delta_6} = \frac{2116}{\sqrt{519} R_g} \left(\frac{p_0}{P_6} \right)^{\frac{1}{\gamma_t}} \sqrt{2 J c_{p,g} \left[1 - \left(\frac{p_0}{P_6} \right)^{\frac{\gamma_t-1}{\gamma_t}} \right]} \quad (D1a)$$

Equation (D1a) is used until the critical pressure ratio is reached. The value of $M_g \sqrt{\theta_6}/A_n \delta_6$ remains constant thereafter, as p_0/P_6 becomes less than the critical pressure ratio. The mass-flow-per-unit-area factor for critical flow is

$$\frac{M_g \sqrt{\theta_6}}{A_n \delta_6} = \frac{2116}{\sqrt{519} R_g} \sqrt{\frac{2 \gamma_g}{\gamma_g + 1} \left(\frac{2}{\gamma_g + 1} \right)^{\frac{2}{\gamma_g - 1}}} \quad (D1b)$$

Also, from energy considerations,

$$T_6 = T_5 - \frac{\eta'_{t,2} V_{t,2}^2}{2 J c_{p,g}}$$

from which

$$\frac{T_6}{T_5} = 1 - \frac{\eta'_{t,2}}{2 J c_{p,g} 519} \left(\frac{V_{t,2}}{\sqrt{\theta_5}} \right)^2 \quad (D2)$$

The procedure for determining the exhaust-nozzle area is as follows:

(1) For a given operating point, all conditions in the engine up to and including those at the second turbine are obtained by the method described in appendix C.

(2) For a given $M_g \sqrt{\theta_5}/\delta_5$, there can be a range of $U_{t,2}/V_{t,2}$ over which the second turbine can operate. At any $U_{t,2}/V_{t,2}$, the value of P_5/p_6 can be obtained from figure 14 (a), and $\eta'_{t,2}$ and $\eta_{t,2}$, from figure 14 (b).

(3) The total-pressure ratio P_5/P_6 is found by a method similar to that described in appendix C. From the value of P_5/p_6 , the corresponding $V_{t,2}/\sqrt{\theta_5}$ is obtained on figure 14 (a);

$\frac{V_{t,2}}{\sqrt{\theta_5}} \sqrt{\frac{\eta'_{t,2}}{\eta_{t,2}}}$ is then calculated and used to evaluate P_5/P_6 .

In order to illustrate the calculations involved:

(6) From the values of $V_{t,1}/\sqrt{\theta_4}$, $\eta_{t,1}$, and $\eta'_{t,1}$ previously determined, $\frac{V_{t,1}}{\sqrt{\theta_4}} \sqrt{\frac{\eta'_{t,1}}{\eta_{t,1}}} = 1100$ ft/sec. Corresponding to this

value, $P_4/P_5 = 2.10$ is obtained.

(7) From equation (C4), a value of T_5/T_4 of 0.859 is obtained when a value of $c_{p,g} = 8.9$ Btu/(slug)(°F) is assumed.

(8) Finally, from equation (13) a value of $M_g \sqrt{\theta_5}/\delta_5 = 0.581$ is determined.

(4) The pressure ratio $\frac{P_6}{p_0} = \left(\frac{P_6}{P_5} \right) \left(\frac{P_5}{P_4} \right) \left(\frac{P_4}{P_1} \right) \left(\frac{P_1}{p_0} \right)$ is evaluated, where P_1/p_0 is a function of flight Mach number and pressure loss in inlet duct (see equation (A12)).

(5) With suitable values for γ_g and $c_{p,g}$, the value of P_6/p_0 is used to calculate $M_g \sqrt{\theta_6}/A_n \delta_6$ from equation (D1a); equation (D1b) is used to calculate $M_g \sqrt{\theta_6}/A_n \delta_6$ if flow through the exhaust nozzle is choked.

(6) The values of $\eta'_{t,2}$ and $V_{t,2}/\sqrt{\theta_5}$ and an appropriate value of $c_{p,g}$ are used in equation (D2) to evaluate T_5/T_6 .

(7) The value of A_n is then calculated from the identity

$$A_n = \frac{\frac{M_g \sqrt{\theta_5}}{\delta_5} \sqrt{\frac{T_6}{T_5} \frac{P_5}{P_6}}}{\frac{M_g \sqrt{\theta_6}}{A_n \delta_6}}$$

This method will be illustrated for the same operating point used in the illustration of appendix C:

(1) Some of the conditions corresponding to the given point of operation are $U_c/\sqrt{\theta_1} = 1006$ ft/sec, $P_2/P_1 = 4.74$, $P_4/P_5 = 2.10$, and $M_g \sqrt{\theta_5}/\delta_5 = 0.581$ slug/sec.

(2) From the value of $M_g \sqrt{\theta_5}/\delta_5$ and a chosen value of 0.5 for $U_{t,2}/V_{t,2}$, the corresponding $P_5/p_6 = 1.91$ is obtained on figure 14 (a). Then from figure 14 (b) $\eta_{t,2} = 0.850$ and $\eta'_{t,2} = 0.796$.

(3) Corresponding to $P_5/p_6 = 1.91$, $V_{t,2}/\sqrt{\theta_5} = 1032$ ft/sec is read on figure 14 (a). The parameter $\frac{V_{t,2}}{\sqrt{\theta_5}} \sqrt{\frac{\eta'_{t,2}}{\eta_{t,2}}}$ equal to 1000 ft/sec is evaluated, and the corresponding P_5/P_6 of 1.83 is read on figure 14 (a).

(4) The pressure ratio P_1/p_0 at a value of $Y = 0.10$ and inlet pressure loss $\Delta P_d/P_1 = 0.03$ is 1.356 by equation (A12). Thus, $P_6/p_0 = 1.589$ is evaluated from the values of P_5/P_6 , P_5/P_4 , P_4/P_1 , and P_1/p_0 .

(5) From equation (D1a), values of $\gamma_g = 1.35$ and R_g of 1720 ft-lb/(slug)(°F), and the value of P_6/p_0 previously determined, $M_g \sqrt{\theta_6}/A_n \delta_6 = 1.484$ (slug/sec)/sq ft is evaluated.

(6) The temperature ratio $T_6/T_5 = 0.877$ is obtained from $V_{t,2}/\sqrt{\theta_5}$, $\eta'_{t,2}$, and a value of $c_{p,g} = 8.6$ Btu/(slug)(°F) in equation (D2).

(7) The value of the effective exhaust-nozzle area obtained by use of the identity given in step (7) is $A_n = 0.67$ sq ft.

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REPORT 1115

DESIGN OF TWO-DIMENSIONAL CHANNELS WITH PRESCRIBED VELOCITY DISTRIBUTIONS ALONG THE CHANNEL WALLS ¹

By JOHN D. STANITZ

SUMMARY

A general method of design is developed for two-dimensional unbranched channels with prescribed velocities as a function of arc length along the channel walls. The method is developed for both compressible and incompressible, irrotational, nonviscous flow and applies to the design of elbows, diffusers, nozzles, and so forth. Two types of compressible flow are considered: the general type, with the ratio of specific heats γ equal to 1.4, for example, and the linearized type, in which γ is -1.0 . Two methods of solution are used: In part I solutions are obtained by relaxation methods; in part II solutions are obtained by a Green's function.

Five numerical examples are given in part I including three elbow designs with the same prescribed velocity as a function of arc length along the channel walls but with incompressible, linearized compressible, and compressible flow. It is concluded that if a nonviscous gas with arbitrary γ (1.4, for example) were to flow through a channel designed for linearized compressible flow ($\gamma = -1.0$), the resulting velocity distribution along the channel walls would be nearly the velocity distribution prescribed for the linearized compressible flow.

One numerical example is presented in part II for an accelerating elbow with linearized compressible flow, and the time required for the solution by a Green's function in part II was considerably less than the time required for the same solution by relaxation methods in part I.

INTRODUCTION

There are two general types of theoretical problem in two-dimensional fluid motion: (1) the direct problem, in which the distribution of velocity is determined for a prescribed shape of boundary, and (2) the inverse problem, in which the shape of boundary is determined for a prescribed distribution of velocity along the boundary. The direct problem is an analysis problem; the inverse problem is a design problem. This report is concerned with the inverse, or design, problem for two-dimensional, irrotational flow in unbranched channels with prescribed velocities as a function of arc length along the channel walls.

The design of channels with prescribed velocities is important because: (1) Boundary-layer separation losses can be avoided by prescribed velocities that do not decelerate rapidly enough to cause separation, (2) shock losses in com-

pressible flow and cavitation in incompressible flow can be avoided by prescribed velocities that do not exceed certain maximum values dictated by these phenomena, and (3) for compressible flow the desired flow rate can be assured by prescribed velocities that do not result in "choke flow" conditions.

Several methods of channel design have been developed for particular application (refs. 1 and 2, for example). In reference 1 a design method is developed for accelerating elbows in which the velocity increases monotonically along the channel walls. The method is developed for incompressible and linearized ($\gamma = -1.0$) compressible flow. The velocity distribution along the channel walls is not arbitrary and the design method applies to elbows only. In reference 2 a design method is developed for straight, symmetrical channels with contracting or expanding walls. The method is developed for incompressible flow and the velocities are prescribed not as a function of arc length along the channel walls but as a function of circle angle in the transformed circle plane. A more general design is suggested in reference 3, but no attempt is made to develop and apply the method.

In the present report a general method of design is developed for two-dimensional, unbranched channels with prescribed velocities as a function of arc length along the channel walls. The method is developed for both compressible and incompressible, irrotational, nonviscous flow and applies to the design of elbows, diffusers, nozzles, and so forth. Two types of compressible flow are considered: the general type with arbitrary value of γ (1.4, for example) and the linearized type with γ equal to -1.0 . In general, if the prescribed velocity along one channel wall differs from that along the other, the channel turns so that the downstream flow direction is different from the upstream direction. This change in flow direction cannot be arbitrarily chosen but depends on the prescribed velocity distribution along the walls. Equations are developed for computing this change in flow direction for an arbitrary prescribed velocity distribution with incompressible or linearized compressible flow. Two methods of solution have been developed for the design method and are presented in separate parts of this report. In part I solutions are obtained by relaxation methods (ref. 4). This method of solution results in complete information concerning the distribution of flow conditions throughout the

¹ Supersedes NACA TN 2593, "Design of Two-Dimensional Channels with Prescribed Velocity Distributions Along the Channel Walls. I—Relaxation Solutions" by John D. Stanitz, 1952, and NACA TN 2595, "Design of Two-Dimensional Channels with Prescribed Velocity Distributions Along the Channel Walls. II—Solution by Green's Function" by John D. Stanitz, 1952.

channel and, in addition, can be used to obtain nonlinear solutions for compressible flow with arbitrary values of γ . In part II solutions are obtained by means of a Green's function. This method of solution is limited to incompressible and linearized ($\gamma = -1.0$) compressible flow, but the method is more rapid than relaxation methods, provided information within the channel is not required.

The design method reported herein was developed at the NACA Lewis laboratory during 1950 and is part of a doctoral thesis conducted with the advice of Professor Ascher H. Shapiro of the Massachusetts Institute of Technology.

PART I

GENERAL THEORY AND SOLUTION BY RELAXATION METHODS

A general method of design is developed for two-dimensional, unbranched channels with prescribed velocities as functions of arc length along the channel walls. The method is developed for both incompressible and compressible, irrotational, nonviscous flow. Two types of compressible flow are considered: the general type with arbitrary value for the ratio of specific heats γ (1.4, for example), and the linearized type in which γ is equal to -1.0 . The solutions in part I of this report are obtained by relaxation methods and give complete information concerning the flow throughout the channel. Five numerical examples are given, including three elbow designs with the same prescribed velocity as a function of arc length along the channel walls but with incompressible, linearized compressible, and compressible flow.

THEORY OF DESIGN METHOD

The design method is developed for two-dimensional channels with prescribed velocities along the channel walls. The prescribed velocity is arbitrary except that stagnation points cannot be prescribed. This exception limits the design method to unbranched channels.

PRELIMINARY CONSIDERATIONS

Assumptions.—The fluid is assumed to be nonviscous and either compressible or incompressible. The flow is assumed to be two dimensional and irrotational.

The assumption of two-dimensional, nonviscous, irrotational motion limits the design method in practice to channels with thin (negligible) boundary layers, such as exist near the entrance to the channel or after a rapid acceleration of the flow through a contraction in the channel. Even if the boundary layer is thin, the design method is limited to (and finds its most useful application for) prescribed velocity distributions that, from boundary-layer theory, do not decelerate fast enough to result in separation of the boundary layer, which separation alters the "effective" shape of the channel and completely changes the character of the flow.

In some channels with fully developed turbulent boundary layers, the design method might be expected to yield results that are satisfactory, although approximate, because for this

type of flow the rotational motion occurs primarily in regions close to the channel walls. In channel walls with thick or fully developed laminar boundary layers the design method cannot be used, because not only is the rotation of the flow important in most of the channel but, if the channel bends, important secondary flows develop that are not considered by the two-dimensional design method.

Flow field.—The flow field of the two-dimensional channel is considered to lie in the physical xy -plane where x and y are Cartesian coordinates expressed as ratios of a characteristic length equal to the constant channel width downstream at infinity. (All symbols are defined in appendix A.)

At each point in the channel (fig. 1) the velocity vector has a magnitude Q and a direction θ where Q is the fluid velocity expressed as the ratio of a characteristic velocity equal to the constant channel velocity downstream at infinity. For convenience, the velocity Q is related to a velocity q by

$$q = Qq_a \quad (1)$$

where q is the velocity expressed as a ratio of the stagnation speed of sound and the subscript d refers to conditions downstream at infinity.

The flow direction θ at each point in the channel is measured counterclockwise from the positive x -axis. From figure 1

$$dx = ds \cos \theta \quad (2a)$$

$$dy = ds \sin \theta \quad (2b)$$

where ds is a differential distance in the direction of Q , that is, along a streamline.

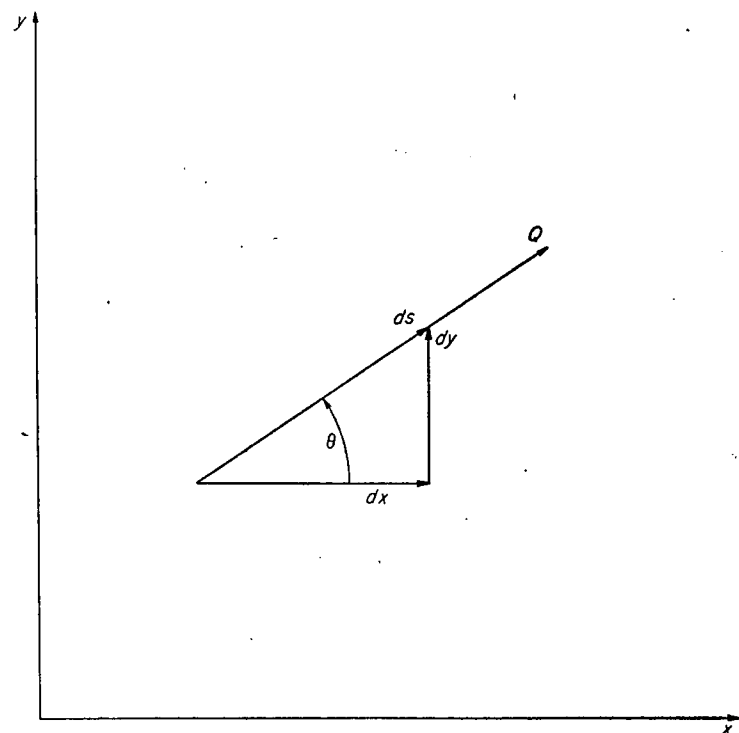


FIGURE 1.—Magnitude and direction of velocity at point in xy -plane.

Stream function and velocity potential.—If the condition of continuity is satisfied, a stream function ψ can be defined such that

$$d\psi = \rho Q \, dn \quad (3)$$

where ρ is the fluid density expressed as the ratio of a characteristic density equal to the stagnation density and where dn is a differential distance measured normal to the direction of Q , that is, normal to a streamline. Along a streamline, dn is zero so that from equation (3) the stream function ψ is constant.

If the condition of irrotational fluid motion is satisfied, a velocity potential φ can be defined such that

$$d\varphi = Q \, ds \quad (4)$$

Normal to a streamline, ds is zero so that from equation (4) the velocity potential φ is constant. Thus lines of constant φ and ψ are orthogonal in the physical xy -plane.

Outline of method.—Solutions of two-dimensional flow depend on known conditions imposed along the boundaries of the problem. In the inverse problem of channel design, the geometry of the channel walls in the physical xy -plane is unknown. This unknown geometry apparently precludes the possibility of solving the problem in the physical plane and necessitates the use of some new set of coordinates, that is, a transformed plane, in which to solve the problem. These new coordinates must be such that the geometric boundaries along which the velocities are prescribed are known in the transformed plane. It is also desirable, for mathematical simplicity, that the coordinate system in the transformed plane be orthogonal in the physical plane. A set of coordinates that satisfies these requirements is provided by φ and ψ , which are orthogonal in the physical xy -plane and for which the geometric boundaries are known constant values of ψ in the transformed $\varphi\psi$ -plane. The distribution of velocity as a function of φ along these boundaries of constant ψ is known because, if

$$Q = Q(s)$$

is prescribed, equation (4) integrates to give

$$\varphi = \varphi(s)$$

from which equations,

$$Q = Q(\varphi)$$

The technique of the proposed method of channel design is therefore to obtain a differential equation for the distribution of velocity in the $\varphi\psi$ -plane. The velocity distribution obtained from the solution of this equation is then used to obtain the distribution of flow direction, from which distribution the channel walls in the physical xy -plane are obtained directly. The differential equation for the distribution of velocity in the $\varphi\psi$ -plane is nonlinear (for compressible flow with γ other than -1.0) and is solved by numerical methods (relaxation methods).

DIFFERENTIAL EQUATION FOR DISTRIBUTION OF VELOCITY IN TRANSFORMED $\varphi\psi$ -PLANE

The differential equation for the distribution of velocity in the transformed $\varphi\psi$ -plane is obtained from the equations for

continuity and irrotational fluid motion expressed in terms of the transformed coordinates φ and ψ .

Continuity.—The continuity equation expressed in terms of φ and ψ becomes (appendix B):

$$\frac{1}{\rho} \left(\frac{\partial \log_e \rho}{\partial \varphi} + \frac{\partial \log_e Q}{\partial \varphi} \right) + \frac{\partial \theta}{\partial \psi} = 0 \quad (5)$$

Irrotational fluid motion.—The equation for irrotational fluid motion, expressed in terms of φ and ψ , becomes (appendix B):

$$\rho \frac{\partial \log_e Q}{\partial \psi} - \frac{\partial \theta}{\partial \varphi} = 0 \quad (6)$$

Differential equation for distribution of velocity.—The second-order partial differential equation for the distribution of $\log_e Q$ in the transformed $\varphi\psi$ -plane is obtained by differentiating equations (5) and (6) with respect to φ and ψ , respectively, and combining to eliminate $\frac{\partial^2 \theta}{\partial \varphi \partial \psi}$. Thus,

$$\begin{aligned} \frac{\partial^2 \log_e \rho}{\partial \varphi^2} + \frac{\partial^2 \log_e Q}{\partial \varphi^2} - \frac{\partial \log_e \rho}{\partial \varphi} \left(\frac{\partial \log_e \rho}{\partial \varphi} + \frac{\partial \log_e Q}{\partial \varphi} \right) + \\ \rho^2 \frac{\partial \log_e Q}{\partial \psi} \frac{\partial \log_e \rho}{\partial \psi} + \rho^2 \frac{\partial^2 \log_e Q}{\partial \psi^2} = 0 \end{aligned} \quad (7)$$

Equation (7), together with a relation between ρ , Q , and q_a , determines the distribution of $\log_e Q$ in the $\varphi\psi$ -plane for compressible flow with a given value of q_a and for arbitrarily prescribed variations in $\log_e Q$ along the boundaries of constant ψ .

Density.—The density ρ is related to the velocity q by (ref. 5, p. 26, for example)

$$\rho = \left(1 - \frac{\gamma-1}{2} q^2 \right)^{\frac{1}{\gamma-1}} \quad (8a)$$

which, from equation (1), becomes

$$\rho = \left(1 - \frac{\gamma-1}{2} Q^2 q_a^2 \right)^{\frac{1}{\gamma-1}} \quad (8b)$$

Equation (8b) relates the density ρ to the velocity Q for a given value of q_a .

Incompressible flow.—For incompressible flow ρ is constant and equal to 1.0 so that equation (7) becomes

$$\frac{\partial^2 \log_e Q}{\partial \varphi^2} + \frac{\partial^2 \log_e Q}{\partial \psi^2} = 0 \quad (9)$$

Equation (9) determines the distribution of $\log_e Q$ in the $\varphi\psi$ -plane for incompressible flow.

CHANNEL-WALL GEOMETRY

After equation (7) or (9) has been solved to obtain the distribution of $\log_e Q$ in the transformed $\varphi\psi$ -plane (for the arbitrary specified variations in $\log_e Q$ with φ along the boundaries of constant ψ), the geometry of the channel walls in the physical xy -plane can be determined from the resulting distribution of flow direction θ .

Flow direction θ .—The distribution of flow direction θ along a streamline (constant ψ) is obtained from equation (6), which integrates to give

$$\theta = \int_{\psi} \rho \frac{\partial \log_e Q}{\partial \psi} d\psi \quad (10a)$$

where the subscript ψ indicates that the integration is taken along a line of constant ψ and where the constant of integration is selected to give a known value of θ at one value of φ along each streamline. The integrand in equation (10a) is obtained from the distribution of $\log_e Q$, which is known from the solution of equation (7) or (9).

The distribution of flow direction θ along a velocity-potential line (constant φ) is obtained from equation (5), which integrates to give

$$\theta = - \int_{\varphi} \frac{1}{\rho} \left(\frac{\partial \log_e \rho}{\partial \varphi} + \frac{\partial \log_e Q}{\partial \varphi} \right) d\varphi \quad (10b)$$

where the subscript φ indicates that the integration is taken along a line of constant φ and where the constant of integration is selected to give a known value of θ at one value of ψ along each velocity-potential line. As for equation (10a), the integrand in equation (10b) is known from the distribution of $\log_e Q$ obtained from the solution of equation (7) or (9).

Channel-wall coordinates.—The variation in x along a line of constant ψ in the $\varphi\psi$ -plane is given by

$$\frac{\partial x}{\partial \varphi} = \left(\frac{dx}{ds} \frac{ds}{d\varphi} \right)_{\psi}$$

which, combined with equations (2a) and (4), integrates to give

$$x = \int_{\psi} \frac{\cos \theta}{Q} d\varphi \quad (11a)$$

Likewise,

$$x = - \int_{\varphi} \frac{\sin \theta}{\rho Q} d\psi \quad (11b)$$

$$y = \int_{\psi} \frac{\sin \theta}{Q} d\varphi \quad (11c)$$

$$y = \int_{\varphi} \frac{\cos \theta}{\rho Q} d\psi \quad (11d)$$

where the constants of integration are selected to give known values of x or y at one value of φ along each streamline or at one value of ψ along each velocity-potential line. Equations (11a) to (11d) determine the distribution of x and y in the transformed $\varphi\psi$ -plane or, which is the same thing, the shape of the streamlines and velocity-potential lines in the physical xy -plane. In particular, equations (11a) and (11c) when integrated along the boundaries of constant ψ in the $\varphi\psi$ -plane determine the shape of the channel walls.

Turning angle.—In general, if the prescribed velocity distribution along one channel wall differs from the distribution along the other wall, the channel deflects an amount $\Delta\theta$, which is the difference in flow direction far downstream and far upstream of the region in which the prescribed velocity

distribution varies. In part II it is shown that for incompressible flow the turning angle $\Delta\theta$ is given by

$$\begin{aligned} \Delta\theta &= \theta_a - \theta_u \\ &= - \int_{-\infty}^{\infty} \varphi \left[\left(\frac{\partial \log_e Q}{\partial \varphi} \right)_{1.0} - \left(\frac{\partial \log_e Q}{\partial \varphi} \right)_0 \right] d\varphi \end{aligned} \quad (12)$$

where the subscript u refers to conditions upstream at infinity and where the subscripts 0 and 1.0 refer to the channel boundaries along which ψ equals 0 and 1.0, respectively. A similar equation will be given later for the case of linearized compressible flow.

LINEARIZED COMPRESSIBLE FLOW

The nonlinear differential equation (7) for the distribution of velocity in the $\varphi\psi$ -plane with compressible flow becomes linear and is considerably simplified if a linear variation in pressure with specific volume ($1/\rho$) is assumed. This linear relation between pressure and specific volume was first suggested by Chaplygin (ref. 6) in order to linearize the differential equations for two-dimensional compressible flow in the hodograph plane.

Density.—If a linear variation in pressure with specific volume is assumed, the density ρ^* is related to the velocity q^* by (appendix C)

$$\rho^* = (1 + q^{*2})^{-1/2} \quad (13)$$

where

$$\rho^* = k_1 \rho \quad (13a)$$

and

$$q^* = k_2 q \quad (13b)$$

where the constants k_1 and k_2 have been determined so that values of ρ given by equation (13) equal the values of ρ given by equation (8a) for any two selected values of q (designated by q_a and q_b). Thus,

$$k_1 = \frac{1}{\rho_a} \sqrt{\frac{1 - \left(\frac{\rho_a q_a}{\rho_b q_b} \right)^2}{1 - \left(\frac{q_a}{q_b} \right)^2}} \quad (14a)$$

and

$$k_2 = \frac{1}{q_b} \sqrt{\frac{\left(\frac{\rho_a}{\rho_b} \right)^2 - 1}{1 - \left(\frac{\rho_a q_a}{\rho_b q_b} \right)^2}} \quad (14b)$$

where ρ_a and ρ_b are determined by equation (8a) for the selected values of q_a and q_b , respectively. A discussion of the selection of q_a and q_b is given in appendix C. It will be noted that, if γ is equal to -1.0 , equation (8a) has the same form as equation (13).

Stream function and velocity potential.—For the case of linearized compressible flow it is convenient to define the stream function ψ^* and the velocity potential φ^* by

$$d\psi^* = \rho^* q^* d\eta \quad (15)$$

and

$$d\varphi^* = q^* ds \quad (16)$$

Continuity.—The continuity equation expressed in terms of φ^* and ψ^* becomes (appendix D)

$$\frac{\partial \log_e u}{\partial \varphi^*} + \frac{\partial \theta}{\partial \psi^*} = 0 \quad (17)$$

where

$$u = \frac{q^*}{1 + \sqrt{1 + q^{*2}}} \quad (18)$$

or, conversely,

$$q^* = \frac{2u}{1 - u^2} \quad (19)$$

Irrotational fluid motion.—The equation for irrotational fluid motion, expressed in terms of φ^* and ψ^* , becomes (appendix D)

$$\frac{\partial \log_e u}{\partial \psi^*} - \frac{\partial \theta}{\partial \varphi^*} = 0 \quad (20)$$

Differential equation for distribution of $\log_e u$.—The partial differential equation for the distribution of $\log_e u$ in the $\varphi^*\psi^*$ -plane is obtained by differentiating equations (17) and (20) with respect to φ^* and ψ^* , respectively, and combining to eliminate $\frac{\partial^2 \theta}{\partial \varphi^* \partial \psi^*}$. Thus

$$\frac{\partial^2 \log_e u}{\partial \varphi^{*2}} + \frac{\partial^2 \log_e u}{\partial \psi^{*2}} = 0 \quad (21)$$

Equation (21) determines the distribution of $\log_e u$ in the $\varphi^*\psi^*$ -plane for linearized compressible flow with a given value of q_a and for arbitrarily prescribed variations in $\log_e Q$, related to $\log_e u$ by equations (1), (13b), and (18), along the boundaries of constant ψ^* . Equation (21) is linear and is, like equation (9) for the case of incompressible flow, the equation of Laplace. Thus an incompressible flow solution for the distribution of $\log_e Q$ in the $\varphi\psi$ -plane is also a linearized compressible flow solution for the distribution of $\log_e u$ in the $\varphi^*\psi^*$ -plane. The transformation from the $\varphi\psi$ -plane is different, however, from the transformation from the $\varphi^*\psi^*$ -plane so that different channel shapes result in the xy -plane.

Flow direction θ .—The distribution of flow direction θ along a streamline (constant ψ^*) is obtained from equation (20), which integrates to give

$$\theta = \int_{\psi^*} \frac{\partial \log_e u}{\partial \psi^*} d\psi^* \quad (22a)$$

Likewise, the distribution of flow direction θ along a velocity-potential line (constant φ^*) is obtained from equation (17), which integrates to give

$$\theta = - \int_{\varphi^*} \frac{\partial \log_e u}{\partial \varphi^*} d\varphi^* \quad (22b)$$

Equations (22a) and (22b) for linearized compressible flow correspond to, and are used in the same manner as, equations (10a) and (10b) for the usual type of compressible or incompressible flow.

Channel-wall coordinates.—The variation in x along a line of constant ψ^* in the $\varphi^*\psi^*$ -plane is given by

$$\frac{\partial x}{\partial \varphi^*} = \left(\frac{dx}{ds} \frac{ds}{d\varphi^*} \right)_{\psi^*}$$

which combined with equations (2a) and (16) integrates to give

$$x = \int_{\psi^*} \frac{\cos \theta}{q^*} d\varphi^* \quad (23a)$$

Likewise,

$$x = - \int_{\varphi^*} \frac{\sin \theta}{\rho^* q^*} d\psi^* \quad (23b)$$

$$y = \int_{\psi^*} \frac{\sin \theta}{q^*} d\varphi^* \quad (23c)$$

$$y = \int_{\varphi^*} \frac{\cos \theta}{\rho^* q^*} d\psi^* \quad (23d)$$

Equations (23a) to (23d) determine the distribution of x and y in the transformed $\varphi^*\psi^*$ -plane or, which is the same thing, the shape of the streamline and velocity-potential lines in the physical xy -plane. In particular, equations (23a) and (23c), when integrated along the boundaries of constant ψ^* in the $\varphi^*\psi^*$ -plane, determine the shape of the channel walls. Equations (23a) to (23d) for linearized compressible flow correspond to, and are used in the same manner as, equations (11a) to (11d) for the usual type of compressible or incompressible flow.

Turning angle.—In part II it is shown that for linearized compressible flow the turning angle, or difference in flow direction far downstream and far upstream of the region in which the prescribed velocity distribution varies along the channel walls, is given by

$$\Delta \theta = \frac{-1}{\Delta \psi^*} \int_{-\infty}^{\infty} \varphi^* \left[\left(\frac{\partial \log_e u}{\partial \varphi^*} \right)_{\Delta \psi^*} - \left(\frac{\partial \log_e u}{\partial \varphi^*} \right)_0 \right] d\varphi^* \quad (24)$$

where $\Delta \psi^*$ is the value of ψ^* along the left boundary (channel wall) when faced in the direction of flow if the value of ψ^* along the right boundary is zero, and where the subscript $\Delta \psi^*$ refers to the boundary along which ψ^* is equal to $\Delta \psi^*$.

NUMERICAL PROCEDURE

The channel design method in part I of this report was developed for three types of fluid flow: (1) compressible, (2) incompressible, and (3) linearized compressible. Although the numerical procedures of the design method are similar for each type of fluid, the procedures differ in detail and are therefore considered separately in this section.

COMPRESSIBLE FLOW

The numerical procedure for channel design with compressible flow ($\gamma=1.4$, for example) is as follows:

(1) The velocity is specified as a function of arc length along that portion of the channel walls over which the velocity varies

$$q = q(s)$$

or q_a is specified and

$$Q = Q(s) \quad (25)$$

where s is arbitrarily equal to zero at that point along one channel wall where the velocity first begins to vary.

(2) The channel-wall boundaries of the flow field in the transformed $\varphi\psi$ -plane are straight, parallel lines of constant ψ extending indefinitely far upstream and downstream between φ equals $\pm\infty$, where φ is arbitrarily equal to zero at that point on the channel wall at which s is equal to zero. The value of ψ along the right channel wall when faced in the direction of flow (direction of positive φ) is arbitrarily set equal to zero in which case the value of ψ along the left channel wall ($\Delta\psi$) is obtained by integrating equation (3) across the channel at a position far downstream where flow conditions are uniform

$$\Delta\psi = \rho_a \quad (26)$$

(3) The distribution of $\log_e Q$ as a function of φ along the boundaries in the $\varphi\psi$ -plane is obtained by integrating equation (4) between limits so that

$$\varphi = \int_0^s Q ds = \varphi(s) \quad (27)$$

which together with equation (25) gives the distribution of $\log_e Q$ along the boundaries in the $\varphi\psi$ -plane

$$\log_e Q = f(\varphi) \quad (28)$$

The integration indicated by equation (27) is carried out numerically for arbitrary distributions of Q as a function of s .

(4) If the velocities prescribed along one channel wall differ from those along the other wall, the channel will, in general, turn the flow. This turning angle cannot be determined exactly for compressible flow until the channel design is completed. However, it will be shown that this turning angle is only slightly greater than that resulting for linearized compressible flow with the same prescribed velocity and with a suitable selection for q_a and q_b in equations (14a) and (14b). This latter turning angle for linearized compressible flow is given by equation (24), which can be integrated numerically for the arbitrary distribution of $\log_e u = f(\varphi)$ corresponding to equation (28).

(5) In order to solve equation (7) for the distribution of $\log_e Q$ in the $\varphi\psi$ -plane, it is convenient to eliminate the density terms from equation (7) by means of equation (8b). Thus, equation (7) becomes

$$A \frac{\partial^2 \log_e q}{\partial \varphi^2} + B \frac{\partial^2 \log_e q}{\partial \psi^2} + 4C \left(\frac{\partial \log_e q}{\partial \varphi} \right)^2 + 4D \left(\frac{\partial \log_e q}{\partial \psi} \right)^2 = 0 \quad (29)$$

where

$$A = \frac{1 - \frac{\gamma+1}{2} q^2}{\left(1 - \frac{\gamma-1}{2} q^2\right)^{\frac{\gamma+1}{\gamma-1}}}$$

$$B = 1.0$$

$$4C = \frac{-q^2 \left(1 + \frac{\gamma+1}{2} q^2\right)}{\left(1 - \frac{\gamma-1}{2} q^2\right)^{\frac{2\gamma}{\gamma-1}}}$$

and

$$4D = \frac{-q^2}{1 - \frac{\gamma-1}{2} q^2}$$

Equation (29) is nonlinear, and it can be solved by relaxation methods (refs. 4 and 7, for example). A grid of equally spaced points, at each of which the value of $\log_e Q$ is to be determined, is placed in the flow field between the channel-wall boundaries. The grid is extended upstream and downstream sufficiently far so that constant values of $\log_e Q$ are obtained across the channel by the relaxation methods. In the numerical examples to be presented six or eight grid spaces were used across the channel. In example III the number of grid spaces was reduced from eight to four with negligible effect on the resulting channel design. The values of $\log_e Q$ at each grid point were relaxed to five significant figures. If the same velocity distribution is prescribed along both walls, the channel is symmetrical so that the velocity distribution in only one half of the channel need be determined by relaxation methods.

(6) After $\log_e Q$ has been determined at each grid point in the $\varphi\psi$ -plane, the distribution of θ is determined by equations (10a) and (10b), which are integrated numerically. The constants of integration in equations (10a) and (10b) are determined to give a specified value of θ at one point in the channel (far upstream, for example). The integrands in equations (10a) and (10b) are determined by numerical methods (tables I to VII, ref. 4, for example) from the known values of ρ and $\log_e Q$ at each of the grid points. If it is desired to know the flow direction along the channel-walls only, equation (10a) can be solved along the channel-wall boundaries $\psi=0$ and $\psi=\Delta\psi$ only. If it is desired to know θ everywhere in the channel, the recommended procedure is to determine the variation in θ along the mean streamline ($\psi=(\Delta\psi)/2$) by equation (10a) and to determine the variation in θ along each velocity-potential line from the previously determined values on the mean streamline by equation (10b).

(7) After the distributions of $\log_e Q$ and θ are known in the $\varphi\psi$ -plane, the shapes of the streamlines and the velocity-potential lines in the physical xy -plane or, which is the same thing, the distributions of x and y in the transformed $\varphi\psi$ -plane are determined by the numerical integration of equations (11a) to (11d). The constants of integration in these equations are determined so that specified values of x and y occur at one point in the flow field. The recommended procedure is to determine the variation in x and y along the mean streamline by equations (11a) and (11c) and to determine the variation in x and y along each velocity-potential line for the previously determined values on the mean streamline by equations (11b) and (11d). If it is desired to know the x and y coordinates for the channel walls only, equations (11a) and (11c) can be solved along the channel-wall boundaries $\psi=0$ and $\psi=\Delta\psi$ only.

INCOMPRESSIBLE FLOW

The numerical procedure for channel design with incompressible flow ($\rho=1$) is similar to that just outlined for compressible flow, but with the following differences:

- (1) The velocity is specified as a function of arc length by equation (25) alone.
- (2) The value of ψ along the left channel wall ($\Delta\psi$) is equal to 1.0 instead of the value given by equation (26).
- (3) The distribution of $\log_e Q$ as a function of φ along the channel-wall boundaries in the $\varphi\psi$ -plane is the same as that obtained from equations (25) and (27) and given by equation (28).
- (4) The turning angle $\Delta\theta$ of the channel is given by equation (12).
- (5) The distribution of $\log_e Q$ in the $\varphi\psi$ -plane is obtained from the solution of equation (9) by relaxation methods.
- (6) After $\log_e Q$ has been determined at each grid point between the channel-wall boundaries in the $\varphi\psi$ -plane, the distribution of θ is determined by equations (10a) and (10b) as indicated previously for compressible flow, but with ρ equal to unity.
- (7) After the distributions of $\log_e Q$ and θ are known in the $\varphi\psi$ -plane, the shapes of the streamlines and velocity-potential lines in the physical xy -plane are determined by equations (11a) to (11d) as indicated previously for compressible flow, but with ρ equal to unity.

LINEARIZED COMPRESSIBLE FLOW

The numerical procedure for channel design with linearized compressible flow ($\gamma=-1.0$) is similar to that previously outlined for compressible flow, but with the following differences:

- (1) The velocity q is specified as a function of arc length along the channel walls by $q(s)$ or by q_a and equation (25). For each prescribed velocity, there are an infinite number of linearized compressible flow solutions depending on the selected values of q_a and q_b in equations (14a) and (14b). However, for values of q_a and q_b within the range of q prescribed along the channel walls (and therefore everywhere in the channel), the solutions, that is, channel shapes, probably differ only in small detail. The best solution is that most nearly like the nonlinear compressible solution with arbitrary value of γ (1.4, for example). In the numerical examples of this report it is shown that, if q_a and q_b are equal to the maximum and minimum values of q , a good solution results, at least if the ratio of these prescribed velocities is not too large (2:1 in the numerical examples). On the other hand, if continuity is to be satisfied for a gas with the correct value of γ (1.4, for example) upstream and downstream of the region of the channel in which the prescribed velocities vary, then q_a and q_b must equal q_u and q_d .

After q_a and q_b have been selected, the velocity distribution $q(s)$ is expressed as $q^*(s)$ by equation (13b) where k_2 is given by equation (14b) so that

$$q^*=q^*(s) \quad (30)$$

The velocity q^* is then expressed as u by equation (18) so that

$$u=u(s) \quad (31)$$

In the particular case where the selected value of q_a is equal to q_b , the value of k_2 is given by equation (C4b) in appendix C, where the significance of this particular case is also discussed.

- (2) The solution is obtained in the transformed $\varphi^*\psi^*$ -plane where φ^* and ψ^* are defined by equations (16) and (15), respectively. If the value of ψ^* along the right channel wall when faced in the direction of q^* is zero, the value of ψ^* along the left wall ($\Delta\psi^*$) is obtained by integrating equation (15) across the channel at a position far downstream where flow conditions are uniform

$$\Delta\psi^*=\rho_a^*q_a^* \quad (32)$$

- (3) The distribution of $\log_e u$ as a function of φ^* along the channel-wall boundaries in the $\varphi^*\psi^*$ -plane is obtained by integrating equation (16) between limits similar to those discussed previously for compressible flow so that

$$\varphi^*=\int_0^s q^* ds=\varphi^*(s) \quad (33)$$

which together with equation (31) determines the distribution of $\log_e u$ along the channel-wall boundaries in the $\varphi^*\psi^*$ -plane

$$\log_e u=f(\varphi^*) \quad (34)$$

- (4) The turning angle $\Delta\theta$ of the channel is given by equation (24).
- (5) The distribution of $\log_e u$ in the $\varphi^*\psi^*$ -plane is obtained from the solution of equation (21) by relaxation methods.
- (6) After $\log_e u$ has been determined at each grid point between the channel-wall boundaries in the $\varphi^*\psi^*$ -plane, the distribution of θ is determined by equations (22a) and (22b) in a manner similar to that outlined previously for compressible flow.
- (7) After the distributions of $\log_e u$ and θ are known in the $\varphi^*\psi^*$ -plane, the shapes of the streamlines and the velocity-potential lines in the physical xy -plane are determined by equations (23a) to (23d) in a manner similar to that outlined previously for compressible flow. The velocities q^* in equations (23) are obtained from the known values of u , and the densities ρ^* are given by equation (13).

NUMERICAL EXAMPLES

The channel design method has been applied in part I to the five examples listed below:

Examples	Type of channel	Type of flow
I	Reducing section	Incompressible
II	Converging section	Incompressible
III	Elbow	Incompressible
IV	Elbow	Linearized compressible
V	Elbow	Compressible ($\gamma=1.4$)

EXAMPLE I

The first numerical example is the design of a reducing section in a straight channel such that the upstream velocity is half the downstream velocity. The solution is for incompressible flow.

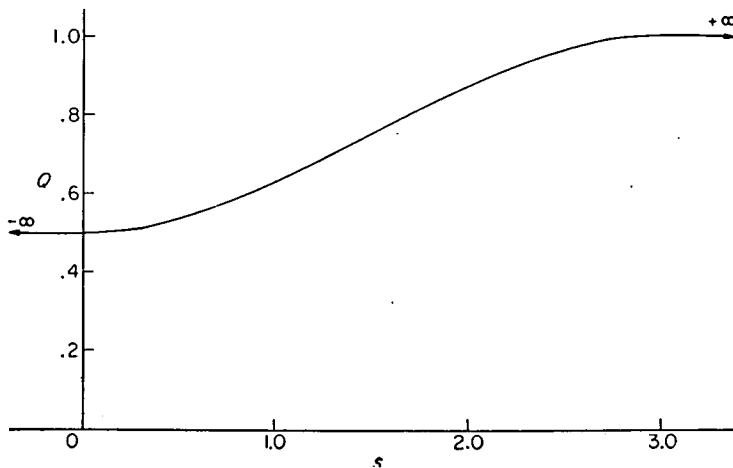


FIGURE 2.—Prescribed velocity distribution as function of arc length along channel wall for examples I, III, IV, and V. Equation (35).

Prescribed velocity distribution.—The prescribed velocity as a function of arc length s along both channel walls is given by

$$\left. \begin{aligned} Q &= 0.5 & (s \leq 0) \\ Q &= \frac{1}{2} + \frac{s^2}{6} - \frac{s^3}{27} & (0 \leq s \leq 3.0) \\ Q &= 1.0 & (s \geq 3.0) \end{aligned} \right\} \quad (35)$$

The prescribed velocity given by equation (35) is plotted in figure 2.

Equation (35) together with equation (27) results in

$$\left. \begin{aligned} \varphi &= 0.5s & (s \leq 0) \\ \varphi &= \frac{s}{2} + \frac{s^3}{18} - \frac{s^4}{108} & (0 \leq s \leq 3.0) \\ \varphi &= -0.75 + s & (s \geq 3.0) \end{aligned} \right\} \quad (36)$$

From equations (35) and (36), $\log_e Q$ is a known function of φ , which function is plotted in figure 3.

Results.—The results of example I are presented in figures 4 to 7.

In figure 4, lines of constant velocity Q and flow direction θ are plotted in the transformed $\varphi\psi$ -plane. The flow direction θ is constant and equal to zero along the mean streamline ($\psi=0.5$), indicating that the center line of the channel is straight. The maximum absolute values of θ occur along the channel walls. The solution is symmetrical about the mean streamline. The lines of constant Q and θ are orthogonal.

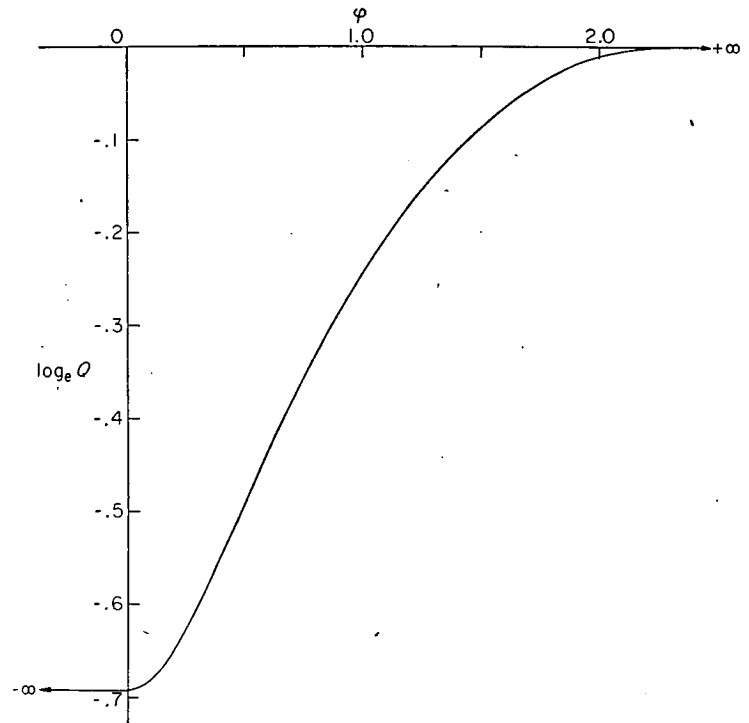


FIGURE 3.—Prescribed distribution of $\log_e Q$ as function of φ along channel walls for example I.

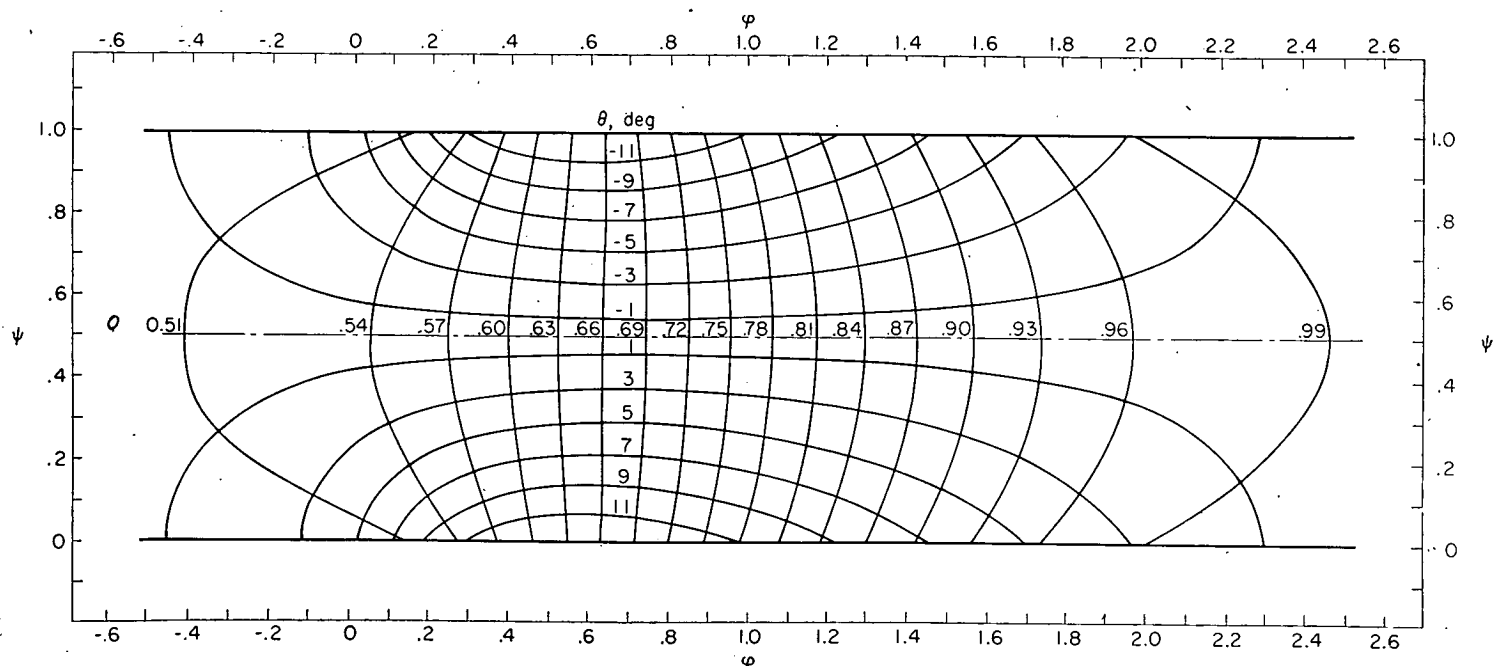
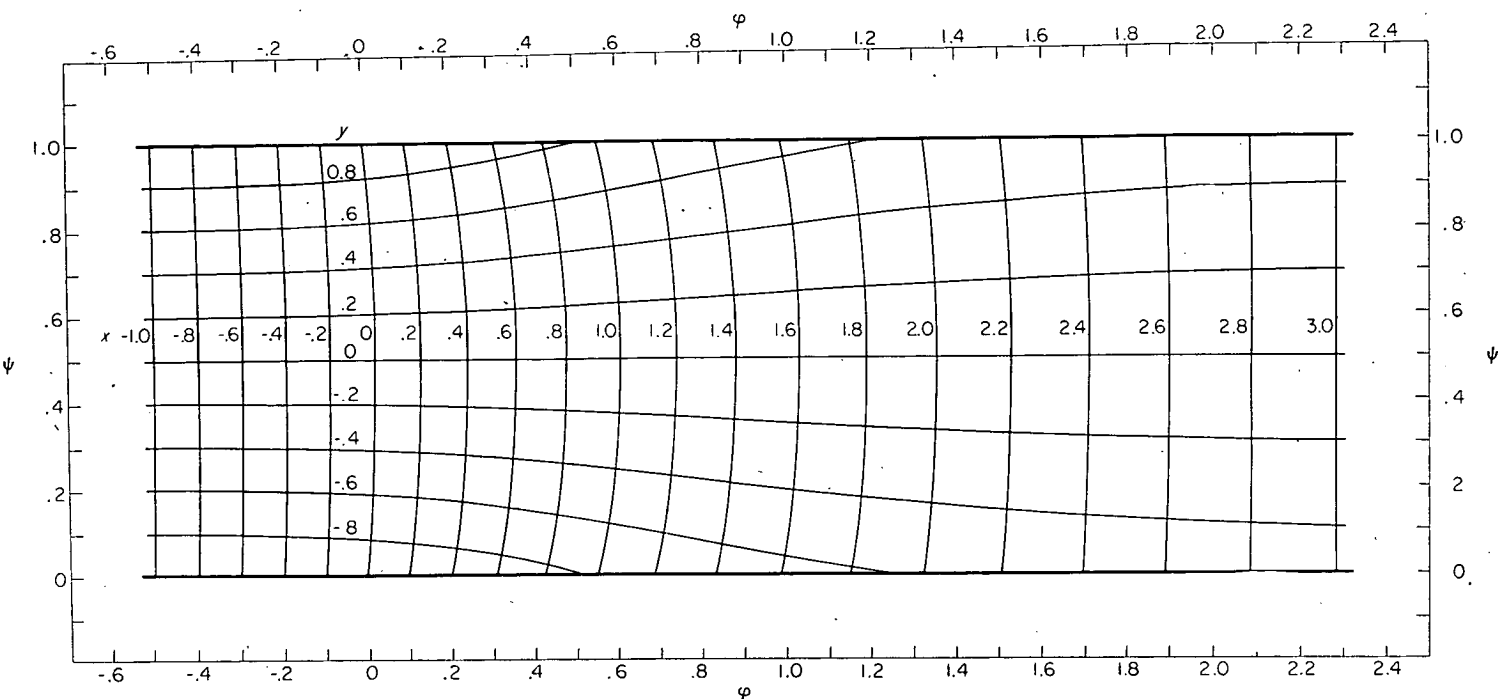
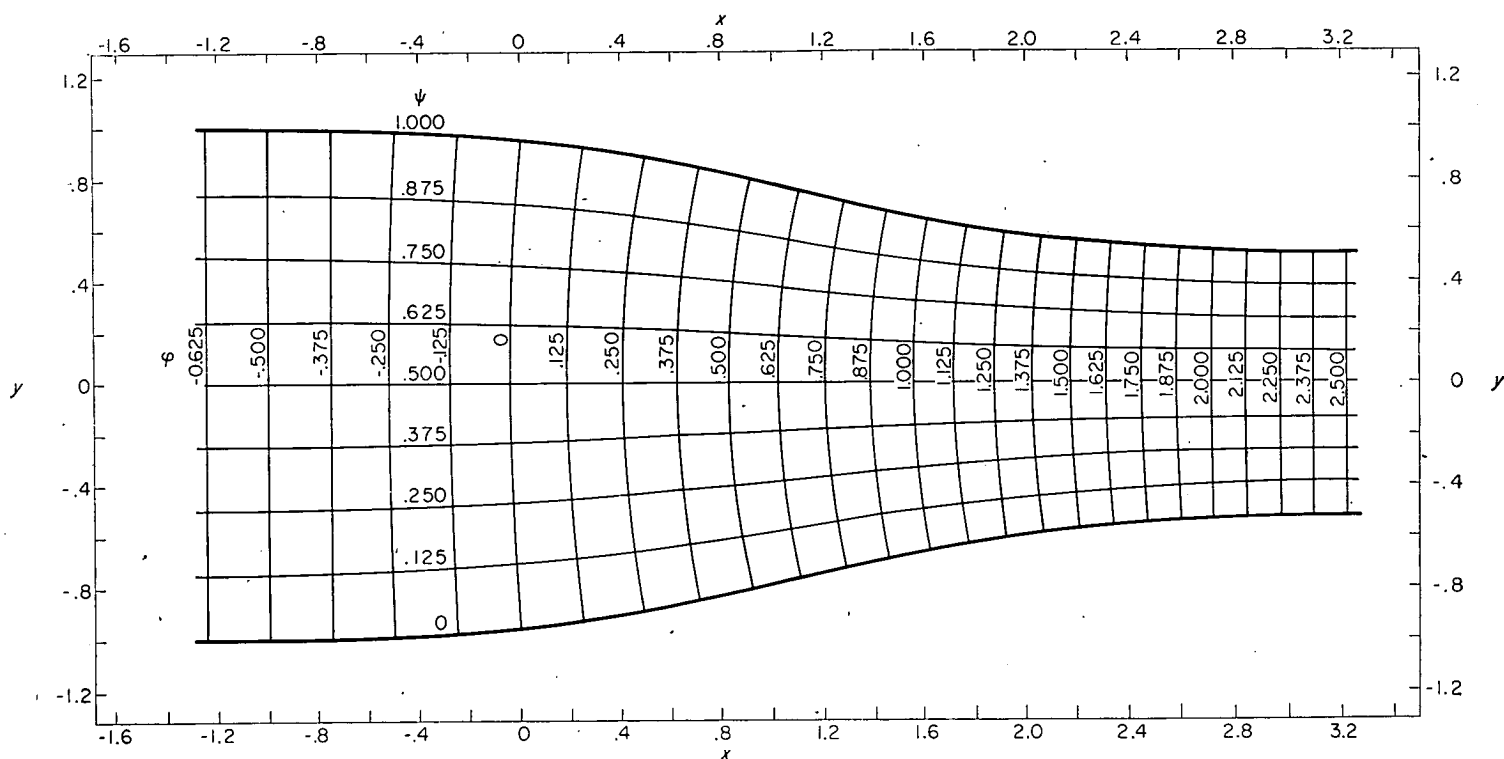


FIGURE 4.—Lines of constant velocity Q and flow direction θ in transformed $\varphi\psi$ -plane for example I. Incompressible flow; prescribed velocity given in figure 2.

FIGURE 5.—Lines of constant x and y coordinates in transformed $\phi\psi$ -plane for example I. Incompressible flow; prescribed velocity given in figure 2.FIGURE 6.—Streamlines and velocity-potential lines on physical xy -plane for example I. Incompressible flow; prescribed velocity given in figure 2.

In figure 5, lines of constant x and y are plotted on the transformed $\phi\psi$ -plane. Along the mean streamline ($\psi=0.5$) the value of y is constant and equal to zero indicating, as before, that the center line of the channel is straight. The lines of constant x and y are orthogonal, and the system of curves forms a square network. The solution is symmetrical.

In figure 6, lines of constant ϕ and ψ (velocity potential and streamlines, respectively) are plotted in the physical xy -plane. The shape of the channel walls is that required to result in the prescribed velocity distribution given by equation (35) and plotted in figure 2. The downstream channel width is 1.0 by definition. The upstream channel width is 2.0 in order that the upstream velocity be half the down-

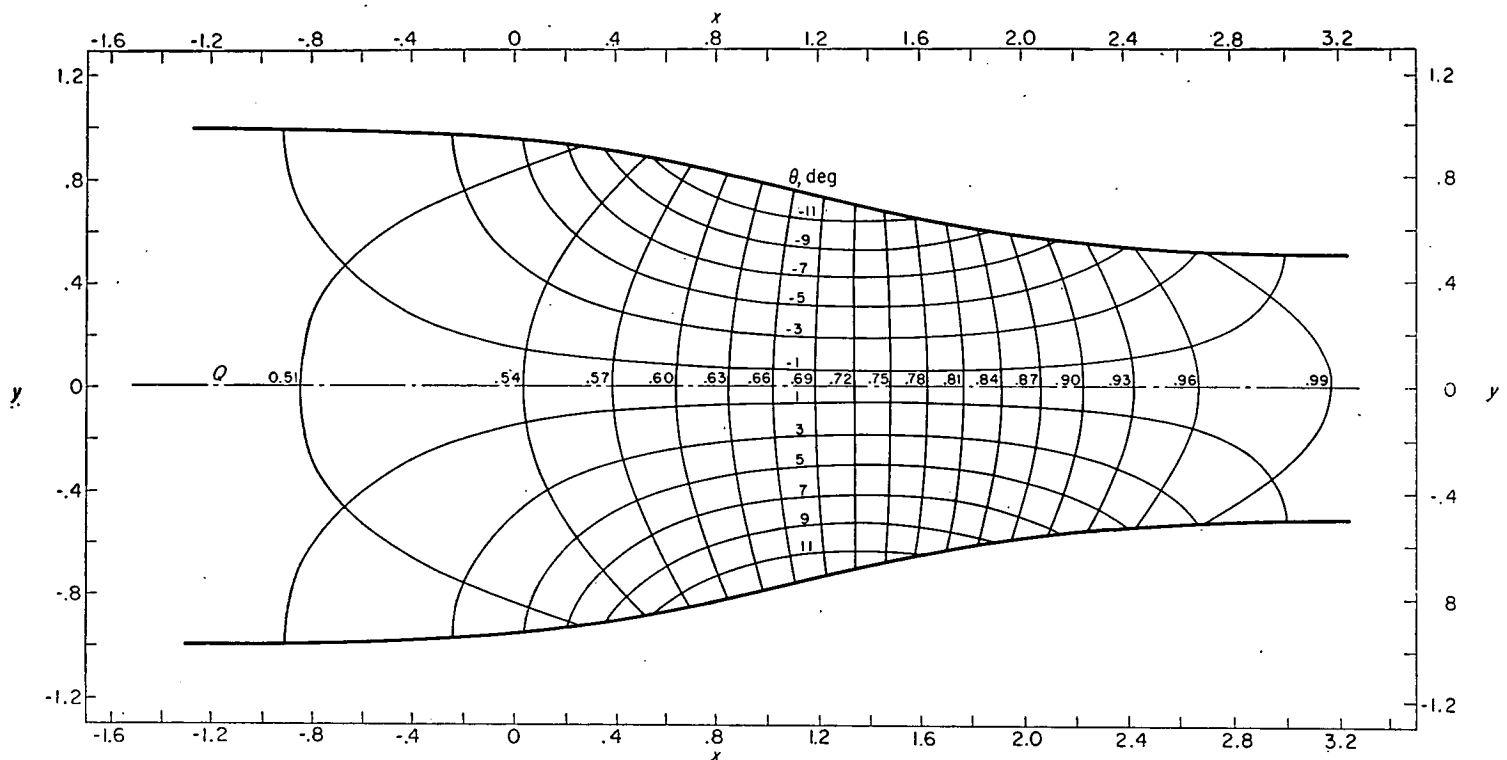


FIGURE 7.—Lines of constant velocity Q and flow direction θ in physical xy -plane for example I. Incompressible flow; prescribed velocity given in figure 2.

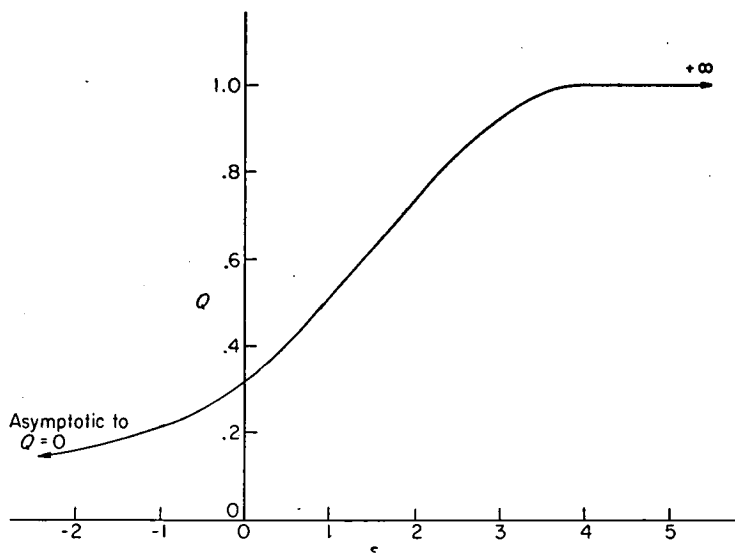


FIGURE 8.—Prescribed velocity distribution as function of arc length along channel wall for example II. Equation (38).

stream velocity. As usual, the streamlines and velocity potential lines are orthogonal and, with equal increments of φ and ψ , form a square network for incompressible flow.

In figure 7, lines of constant Q and θ are plotted in the physical xy -plane. The lines of constant Q and θ are orthogonal.

EXAMPLE II

The second numerical example is the design of a converging section that funnels the fluid from an infinite expanse into a

straight channel of unit width. Far upstream the channel walls are straight and converge at a 90° angle. The solution is for incompressible flow.

Prescribed velocity distribution.—The prescribed velocity as a function of arc length s along both channel walls is given by

$$\left. \begin{aligned} Q &= \frac{-2}{\pi(s-2)} & (s \leq 0) \\ Q &= \frac{1}{\pi} + \frac{1}{2\pi}s - \frac{1}{8}\left(\frac{7}{2\pi} - \frac{3}{2}\right)s^2 + \frac{1}{32}\left(\frac{2}{\pi} - 1\right)s^3 & (0 \leq s \leq 4) \\ Q &= 1.0 & (s \geq 4) \end{aligned} \right\} (37)$$

The prescribed velocity given by equation (37) is plotted in figure 8.

Equation (37) together with equation (27) results in

$$\left. \begin{aligned} \varphi &= \frac{-2}{\pi} \log_e \left(1 - \frac{s}{2}\right) & (s \leq 0) \\ \varphi &= \frac{1}{\pi}s + \frac{1}{2\pi}\frac{s^2}{2} - \frac{1}{8}\left(\frac{7}{2\pi} - \frac{3}{2}\right)\frac{s^3}{3} + \frac{1}{32}\left(\frac{2}{\pi} - 1\right)\frac{s^4}{4} & (0 \leq s \leq 4) \\ \varphi &= \frac{8}{3\pi} - 2 + s & (s \geq 4) \end{aligned} \right\} (38)$$

From equations (37) and (38), $\log_e Q$ is a known function of φ , which function is plotted in figure 9.

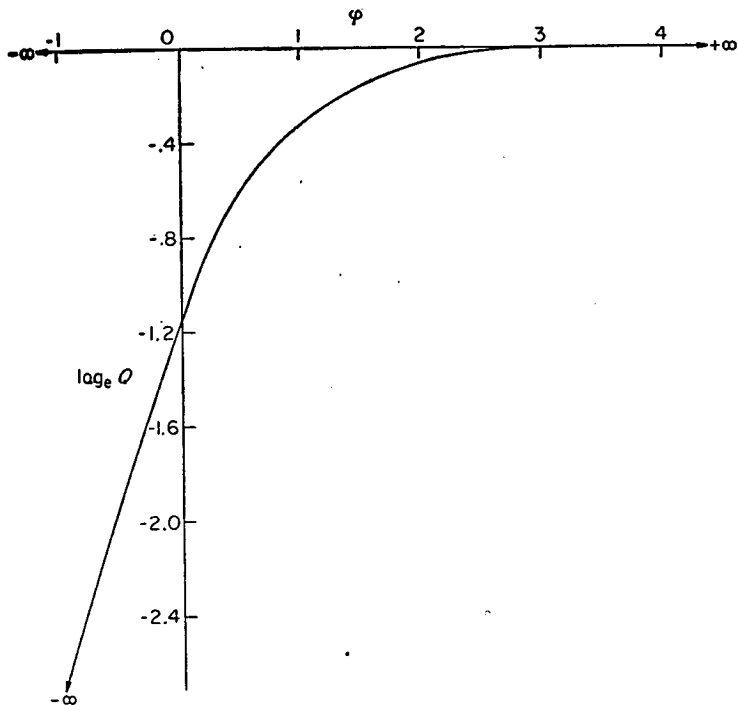


FIGURE 9.—Prescribed distribution of $\log_e Q$ as function of φ along channel walls for example II.

Results.—The results of example II are presented in figures 10 to 12.

In figure 10, lines of constant velocity Q and flow direction θ are plotted in the transformed $\varphi\psi$ -plane. The flow direction θ is constant and equal to zero along the mean streamline ($\psi=0.5$), indicating that the center line of the channel is straight. The solution is symmetrical about the mean streamline. As for example I, the lines of constant Q and θ are orthogonal.

In figure 11, lines of constant φ and ψ are plotted in the physical xy -plane. The shape of the channel walls is that

required to result in the prescribed velocity distribution given by equation (37) and plotted in figure 8. As usual, the streamlines and velocity-potential lines are orthogonal and, for incompressible flow with equal increments of φ and ψ , form a square network.

In figure 12, lines of constant Q and θ are plotted in the physical xy -plane. The lines of constant Q and θ are orthogonal.

EXAMPLE III

The third numerical example is the design of an elbow for which the upstream velocity is half the downstream velocity. The prescribed velocities are such that no deceleration occurs anywhere along the channel walls. The solution is for incompressible flow.

Prescribed velocity distribution.—Along both walls upstream of the elbow the velocity Q is equal to 0.5, and along both walls downstream of the elbow Q is equal to 1.0. The transition from Q equals 0.5 to 1.0 along both walls of the elbow will be the prescribed velocity distribution as a function of arc length given by equation (35) for example I and plotted in figure 2. In terms of $\log_e Q$ as a function of φ , this prescribed velocity distribution is given by equation (36) and is plotted in figure 3. Although this velocity distribution is the same for both walls, the distribution on the outer wall (wall with larger radii of curvature) is shifted in the positive φ direction an amount equal to 2.25 relative to the distribution on the inner wall. Thus, a velocity difference exists on the two walls at equal values of φ , as shown in figure 13. The greater this difference in velocity and the greater the range in φ over which velocity differences exist, the greater is the elbow turning angle. For the prescribed velocity distribution given in figure 13, the elbow turning angle given by equation (12) was 89.37° compared with a value of 89.36° obtained from the relaxation solution.

Results.—The results of example III are presented in figures 14 to 16 and in tables I and II. (The numerical

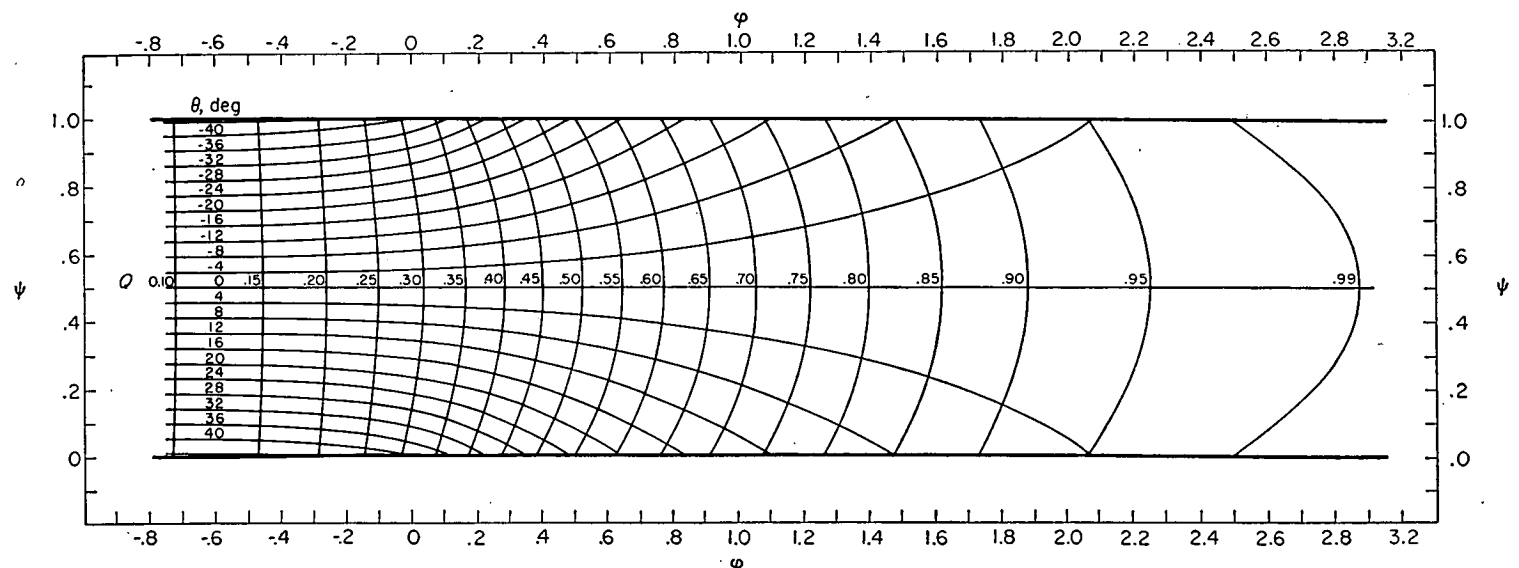


FIGURE 10.—Lines of constant velocity Q and flow direction θ in transformed $\varphi\psi$ -plane for example II. Incompressible flow; prescribed velocity given in figure 8.

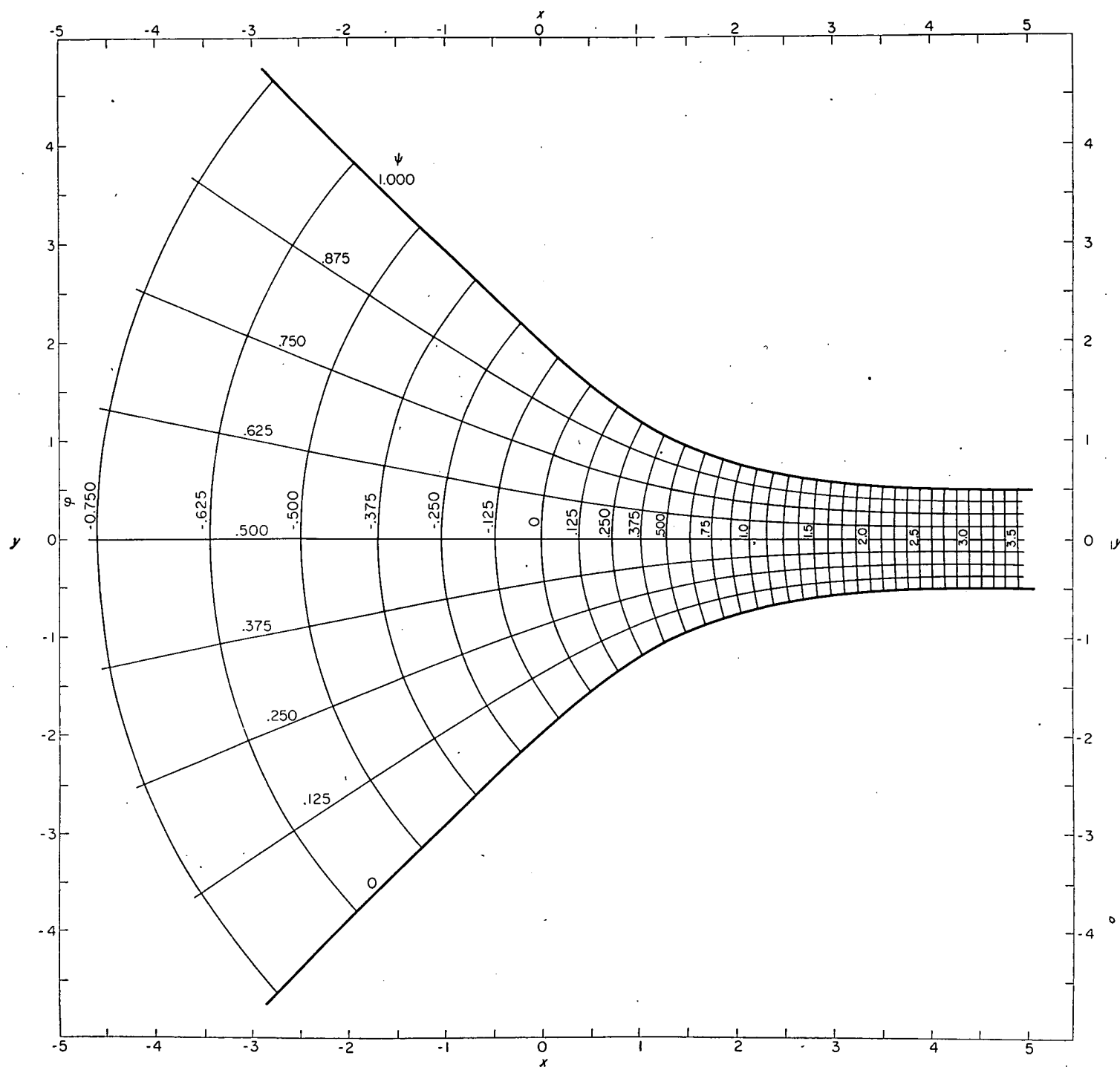


FIGURE 11.—Streamlines and velocity-potential lines in physical xy -plane for example II. Incompressible flow; prescribed velocity given in figure 8.

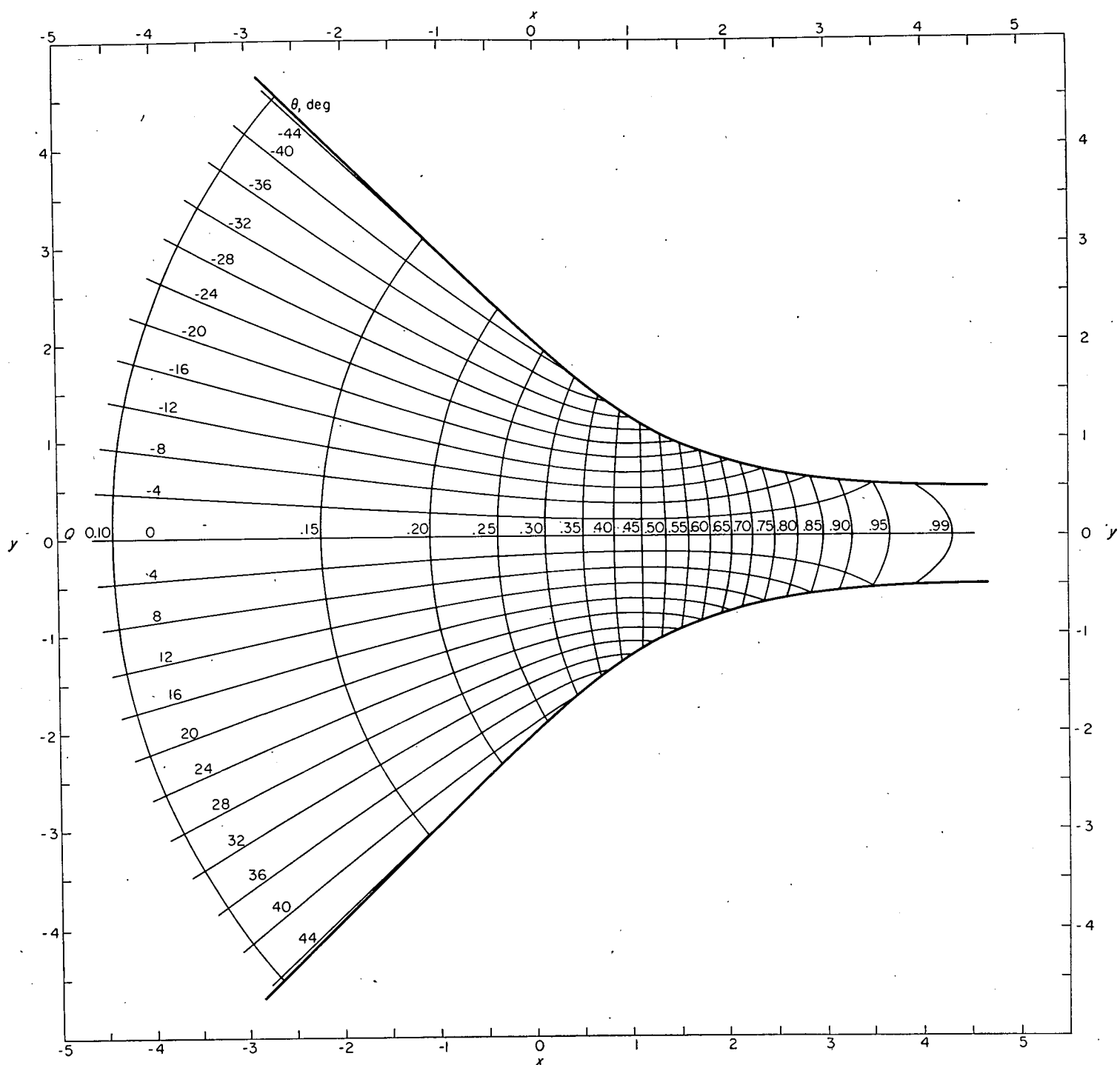


FIGURE 12.—Lines of constant velocity Q and flow direction θ in physical xy -plane for example II. Incompressible flow; prescribed velocity given in figure 8.

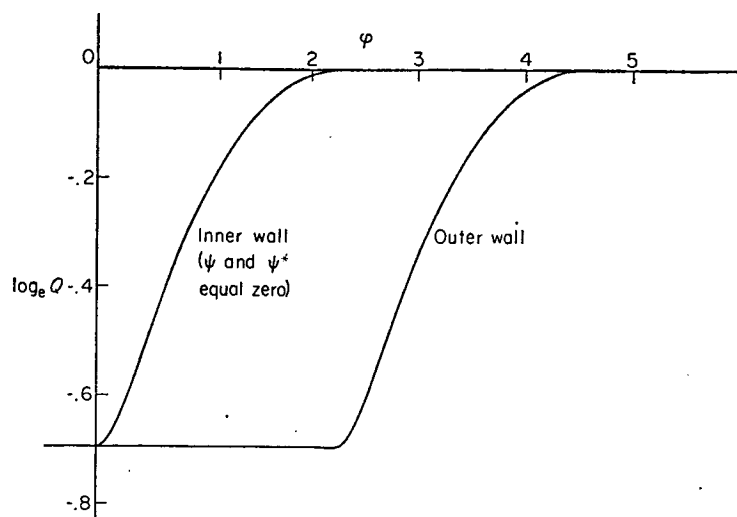


FIGURE 13.—Prescribed distribution of $\log_e Q$ as function of ϕ along channel walls for examples III, IV, and V.

results for examples III, IV, and V are tabulated in tables I to VI to enable a detailed comparison of the three elbow designs with the same prescribed velocity Q distribution as a function of arc length but with incompressible (example III), linearized compressible (example IV), and compressible (example V) flow.)

In figure 14, lines of constant Q and θ are plotted in the $\phi\psi$ -plane. The flow direction θ varies along the mean streamline ($\psi=0.5$), indicating that the channel is curved. The solution is unsymmetrical. As for examples I and II, the lines of constant Q and θ are orthogonal.

In figure 15, lines of constant ϕ and ψ are plotted in the physical xy -plane. The shape of the channel walls is that required to result in the prescribed velocity distribution given by equations (35) and (36) and plotted in figures 2 and 13. The upstream channel width is twice the downstream width in order that the upstream velocity be half the downstream velocity. It is interesting to note that, before curving in the direction of the elbow turning angle, the inner wall first curves in the opposite direction. This behavior of the inner-wall geometry is necessary in order to

maintain the prescribed constant velocity along the outer wall where the velocity would otherwise decelerate because of the necessary curvature in the direction of elbow turning. This feature of the elbow geometry will also be noted in examples IV and V. As usual, the streamlines and velocity-potential lines are orthogonal and, for equal increments of ϕ and ψ , form a square network.

In figure 16, lines of constant Q and θ are plotted in the physical xy -plane. The lines of constant Q and θ are orthogonal.

EXAMPLE IV

The fourth numerical example is the design of an elbow with the same prescribed velocity Q , as a function of arc length, used in example III but for linearized compressible flow ($\gamma=-1.0$).

Prescribed velocity distribution.—The prescribed velocity distribution Q is the same as that for example III and with q_a equal to 0.80176. The variation in Q with s along one channel wall is plotted in figure 2. The values of q_a and q_b in equations (14a) and (14b) are equal to q_a and q_b , or 0.40088 and 0.80176, respectively. For these values of q_a and q_b and for the prescribed velocity distribution with linearized compressible flow, the elbow turning angle given by equation (24) was 104.08° compared with a value of 104.07° obtained from the relaxation solution and a value of 89.36° obtained for incompressible flow (example III).

Results.—The results of example IV are presented in figures 17 to 19 and in tables III and IV.

In figure 17, lines of constant q and θ are plotted in the transformed $\phi^*\psi^*$ -plane. The solution is unsymmetrical and the lines of constant q and θ are orthogonal.

In figure 18, lines of constant $\phi^*/\Delta\psi^*$ and $\psi^*/\Delta\psi^*$ are plotted in the physical xy -plane (where the constant $\Delta\psi^*$ is given by equation (32) and is equal to 0.73782 for q_a equal to 0.80176). The shape of the channel walls is that required to result in the prescribed velocity distribution used in example III but with linearized compressible flow and for q_a equal to 0.80176. From continuity considerations the upstream channel width is 1.5385 times the downstream width. As in example III, the inner wall of the elbow first

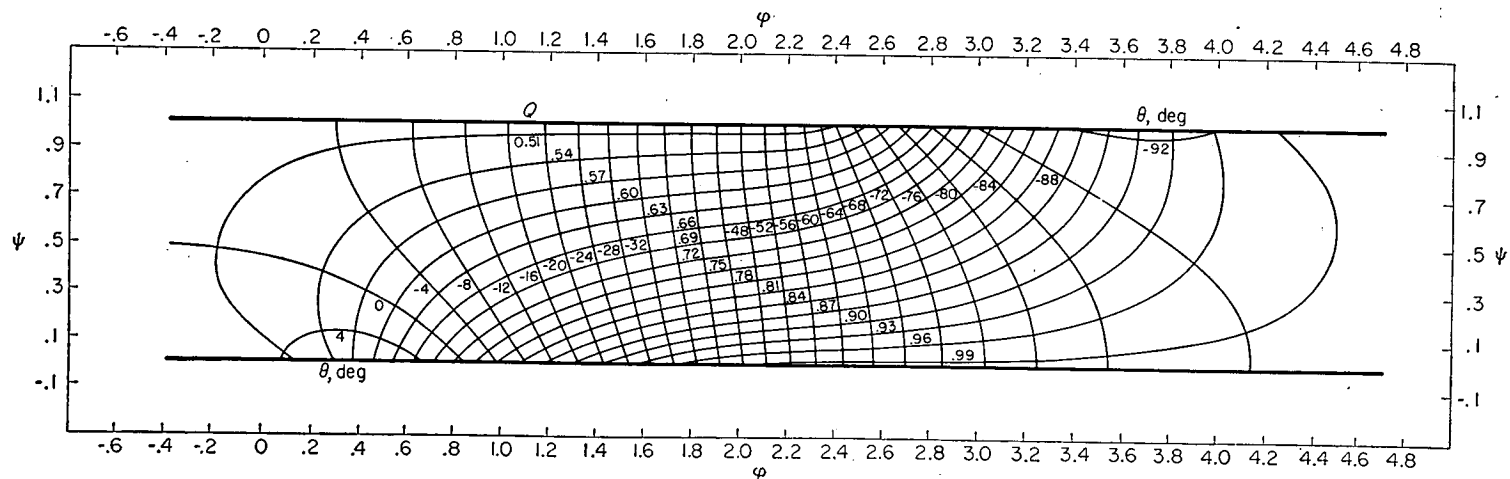


FIGURE 14.—Lines of constant velocity Q and flow direction θ in transformed $\phi\psi$ -plane for example III. Incompressible flow; prescribed velocity given in figures 2 and 13.

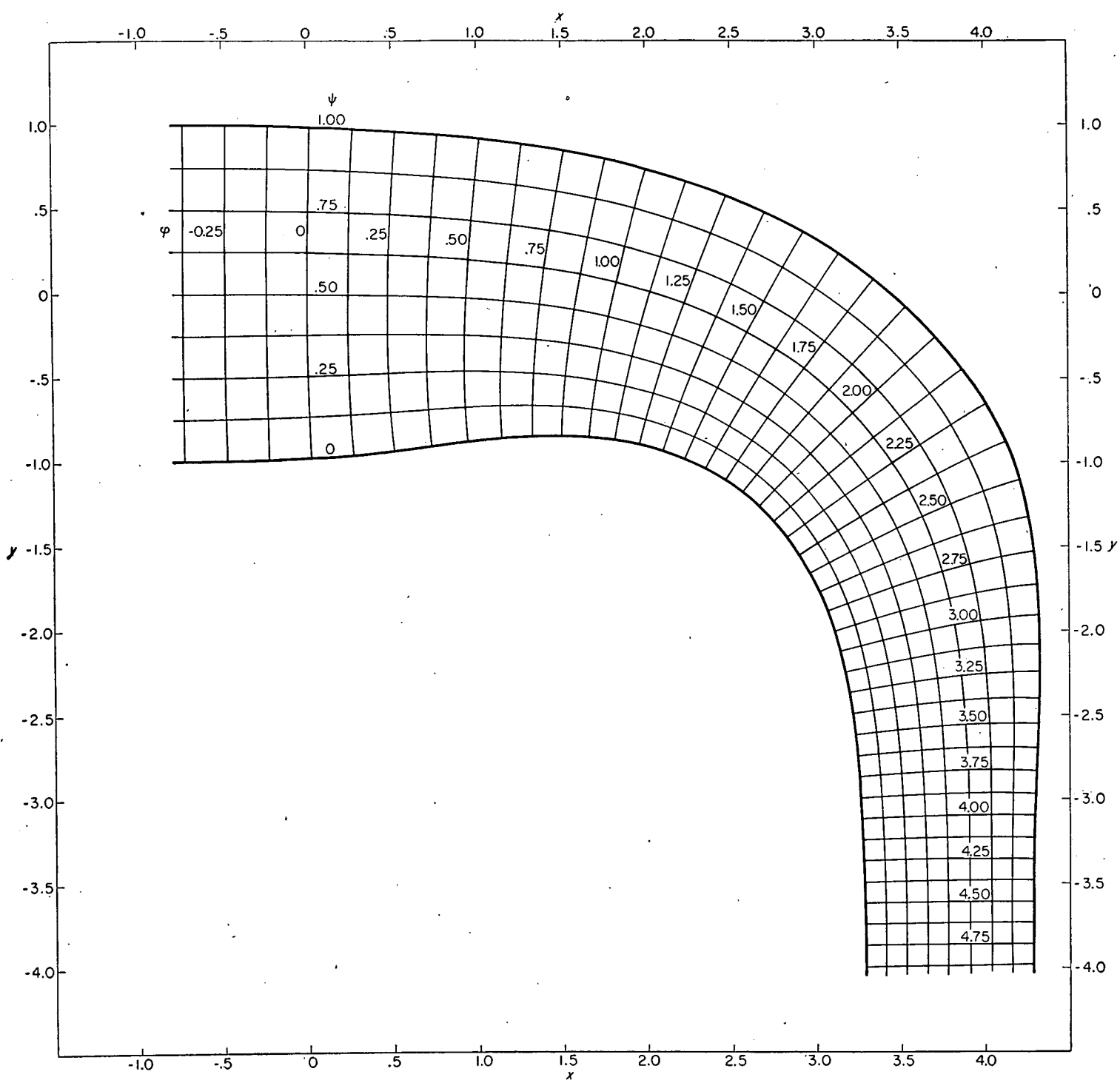


FIGURE 15.—Streamlines and velocity-potential lines in physical xy -plane for example III. Incompressible flow; prescribed velocity given in figures 2 and 13.

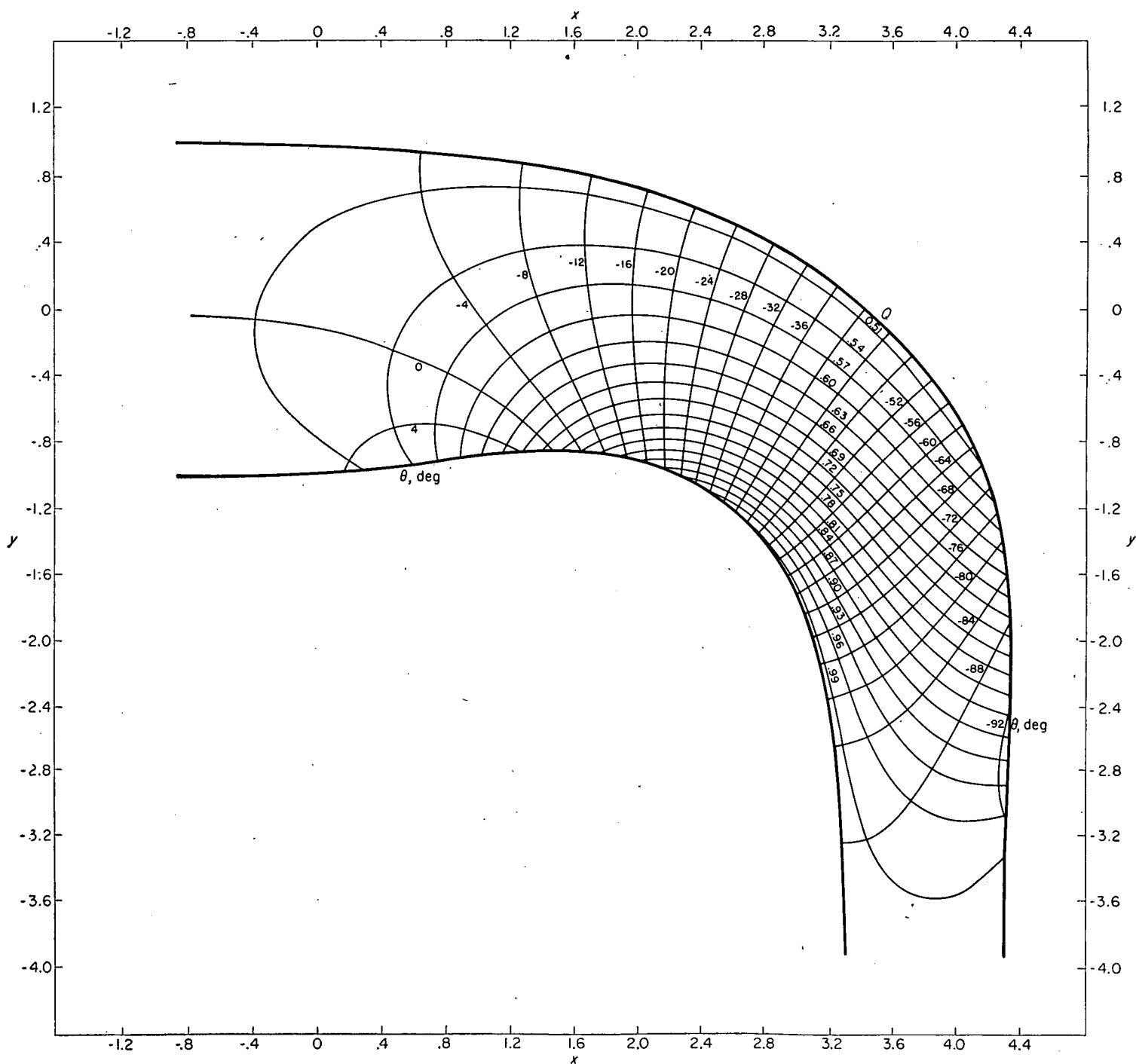


FIGURE 16.—Lines of constant velocity Q and flow direction θ in physical xy -plane for example III. Incompressible flow; prescribed velocity given in figures 2 and 13.

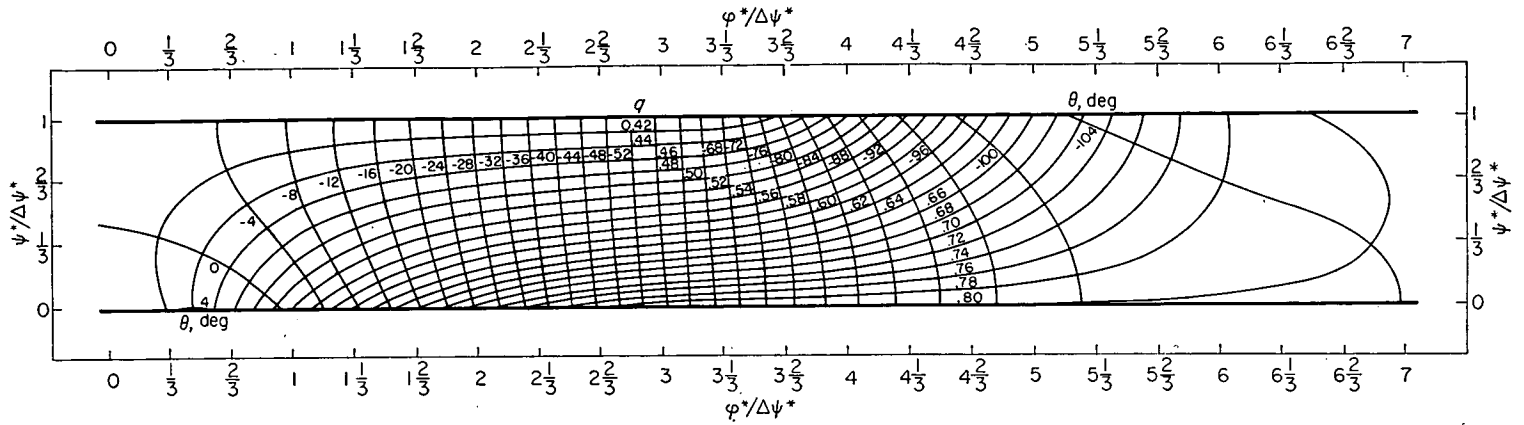


FIGURE 17.—Lines of constant velocity q and flow direction θ in transformed $\phi^*\psi^*$ -plane for example IV. Linearized compressible flow; prescribed velocity as function of arc length along channel walls same as for example III (fig. 2) and with q_a equal to 0.80176.

turns in the opposite direction to the elbow turning angle. As usual, the streamlines and velocity-potential lines are orthogonal.

In figure 19, lines of constant q and θ are plotted in the physical xy -plane. The lines of constant q and θ are not, in general, orthogonal.

EXAMPLE V

The fifth numerical example is the design of an elbow with the same prescribed velocity Q , as a function of arc length, used in examples III and IV but for compressible flow ($\gamma=1.4$).

Prescribed velocity distribution.—The prescribed velocity distribution Q is the same as that for examples III and IV but with q_a equal to 0.79927. The variation in Q with s along one channel wall is plotted in figure 2.

Results.—The results of example V are presented in figures 20 and 21 and in tables V and VI.

In figure 20, lines of constant $\phi/\Delta\psi$ and $\psi/\Delta\psi$ are plotted in the physical xy -plane (where the constant $\Delta\psi$ is given by equation (26) and is equal to 0.71054 for q_a equal to 0.79927). The shape of the channel walls is that required to result in the prescribed velocity distribution used in examples III and IV but with compressible flow ($\gamma=1.4$) and for q_a equal to 0.79927. The upstream channel width is 1.5412 times the downstream width, and the turning angle is 105.31° compared with 104.07° for linearized compressible flow (example IV) and 89.36° for incompressible flow (example III). The streamlines and velocity-potential lines are orthogonal.

The shape of the elbow for compressible flow (example V, fig. 20) is nearly the same as the shape of the elbow for linearized compressible flow (example IV, fig. 18). Therefore, in figure 21 the contours of the walls for both examples are compared. The difference in contours is very small and it is concluded that, if a nonviscous gas with arbitrary γ (1.4, for example) were to flow through a channel designed for linearized compressible flow ($\gamma=-1.0$), the resulting velocity distribution along the channel walls would be nearly the velocity distribution prescribed for the linearized compressible flow, at least if the linearized flow were selected (by the choice of q_a and q_b) so that the densities were equal for both types of flow at the maximum and minimum velocities

and if the ratio of these prescribed velocities is not too large (2:1 in the numerical examples). This conclusion is important because the design method for linearized compressible flow is considerably faster than the design method for compressible flow with γ other than -1.0 .

PART II

SOLUTION BY GREEN'S FUNCTION

In part II a method of solution for the design of two-dimensional channels with prescribed velocity distributions along the walls is developed by means of the appropriate Green's function. The method applies to incompressible and linearized compressible, irrotational flow. One numerical example is presented for an accelerating elbow with linearized compressible flow and with the same prescribed conditions as example IV of part I.

METHOD OF SOLUTION

The method of solution by Green's function is in conjunction with a formula derived from Green's theorem.

PRELIMINARY CONSIDERATIONS

Stream function Ψ .—In part II it is convenient to define the stream function by Ψ , where for incompressible flow

$$d\Psi = \frac{\pi}{2} d\psi \quad (39a)$$

and for linearized compressible flow ($\gamma=-1.0$)

$$d\Psi = \frac{\pi d\psi^*}{2\Delta\psi^*} \quad (39b)$$

For both types of flow Ψ varies from zero along the right side of the channel, when faced in the direction of flow, to $\pi/2$ along the left side.

Velocity potential Φ .—In part II it is convenient to define the velocity potential by Φ , where for incompressible flow

$$d\Phi = \frac{\pi}{2} d\phi \quad (40a)$$

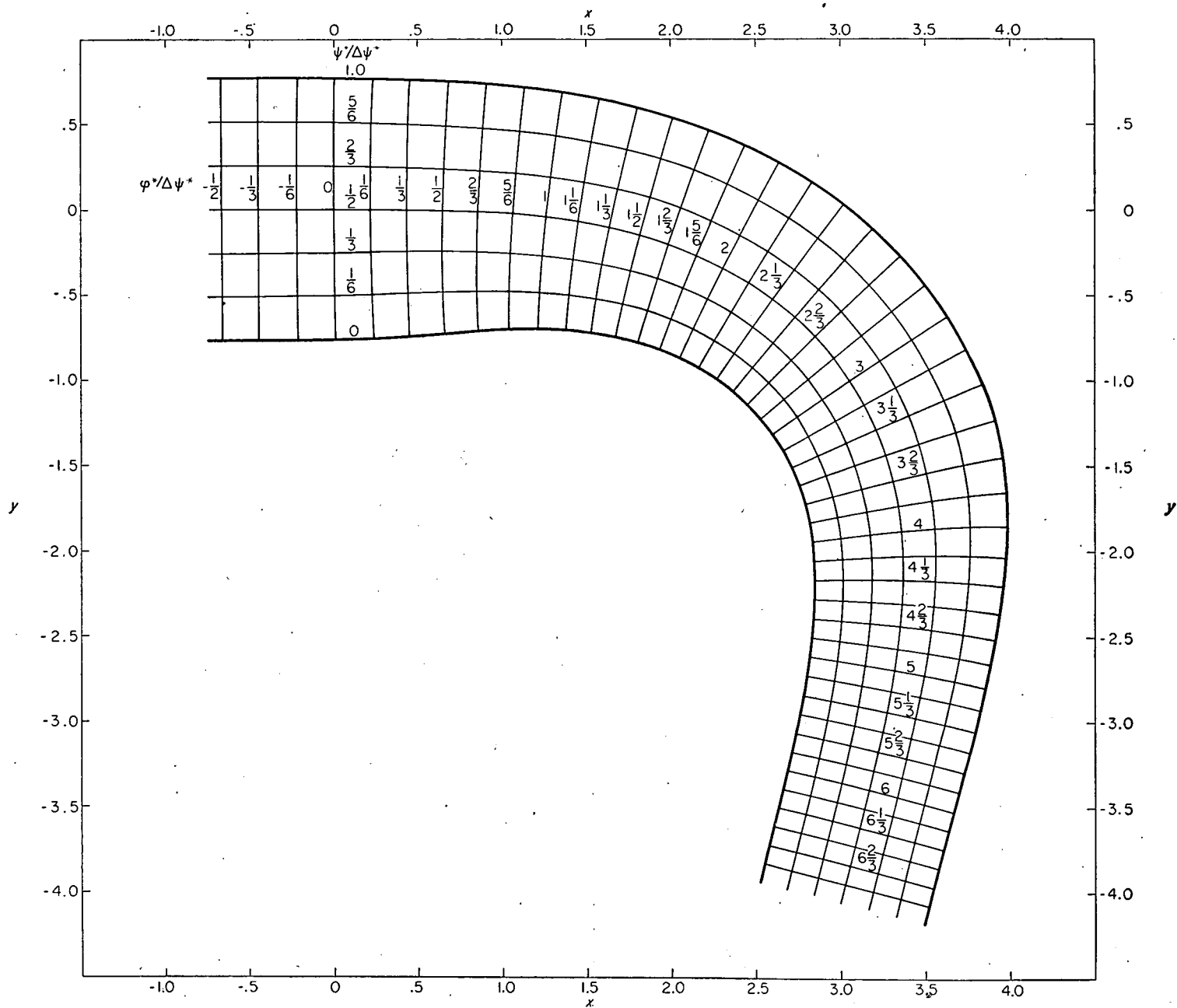


FIGURE 18.—Streamlines and velocity-potential lines in physical xy -plane for example IV. Linearized compressible flow; prescribed velocity as function of arc length along channel walls same as for example III (fig. 2) and with γ equal to 0.80176.

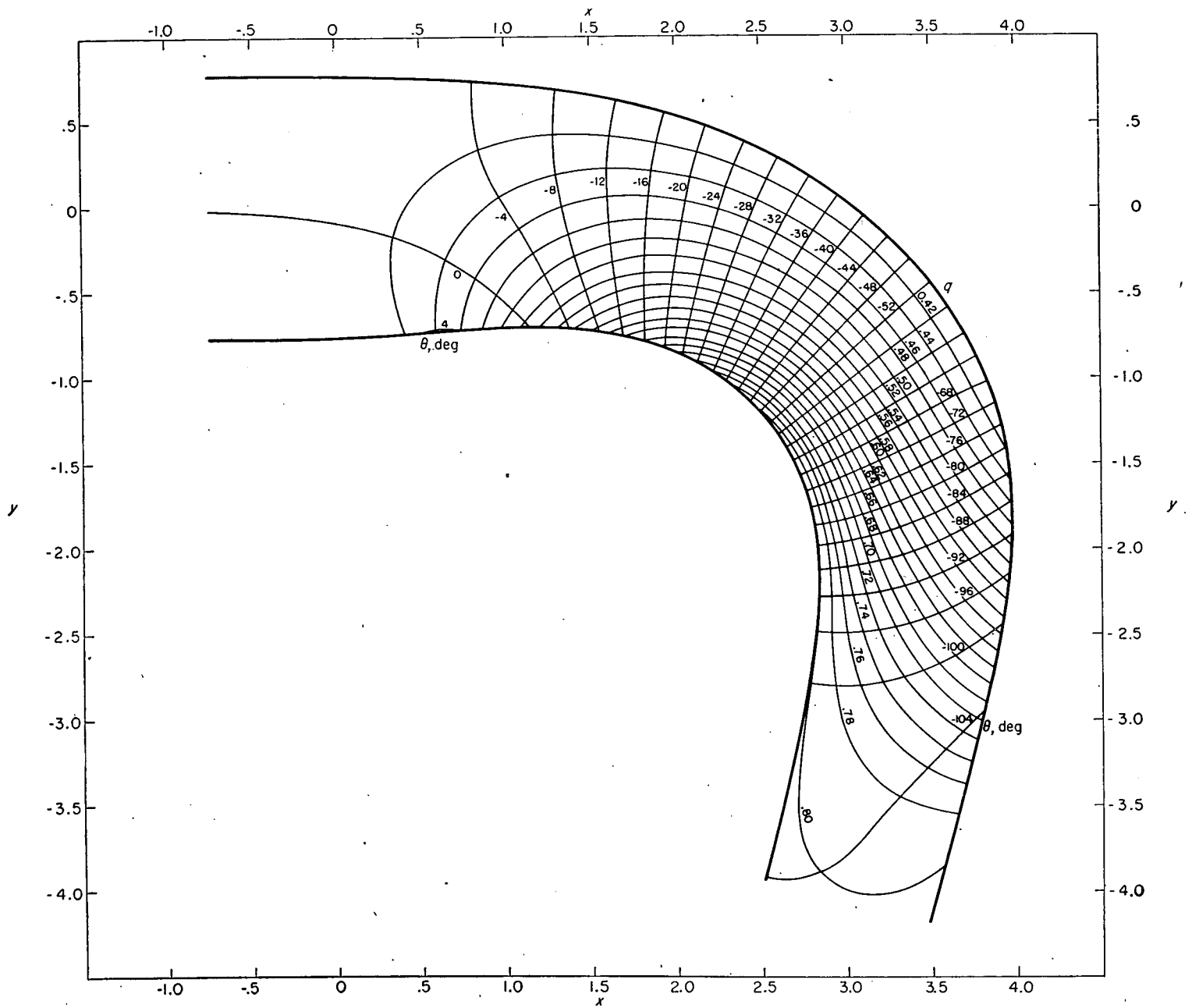


FIGURE 19.—Lines of constant velocity q and flow direction θ in physical xy -plane for example IV. Linearized compressible flow; prescribed velocity as function of arc length along channel walls same as for example III (fig. 2) and with q_d equal to 0.80176.

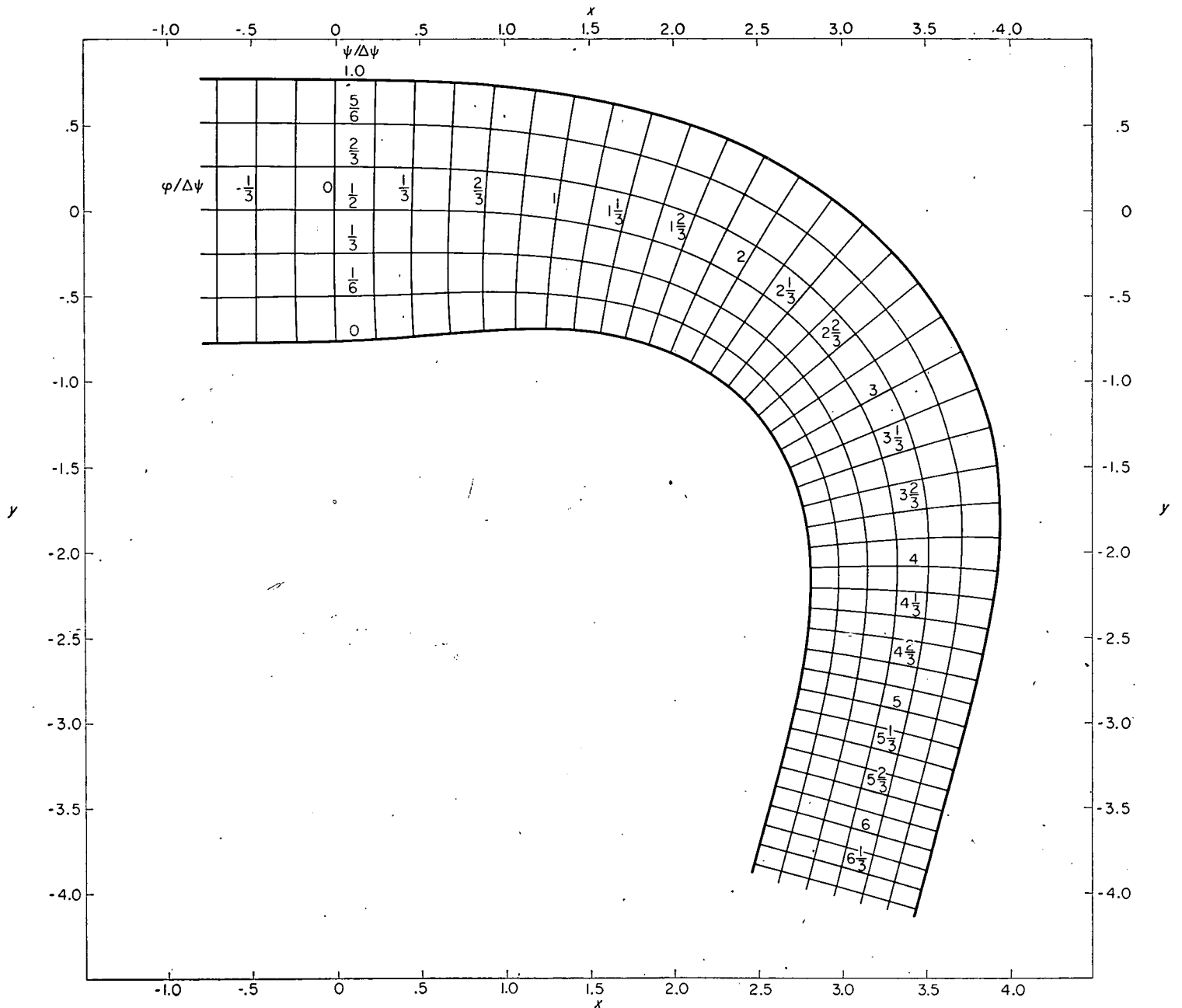


FIGURE 20.—Streamlines and velocity-potential lines in physical xy -plane for example V. Compressible flow ($\gamma=1.4$); prescribed velocity as function of arc length along channel walls same as for examples III and IV (fig. 2) but with q_a equal to 0.79927.

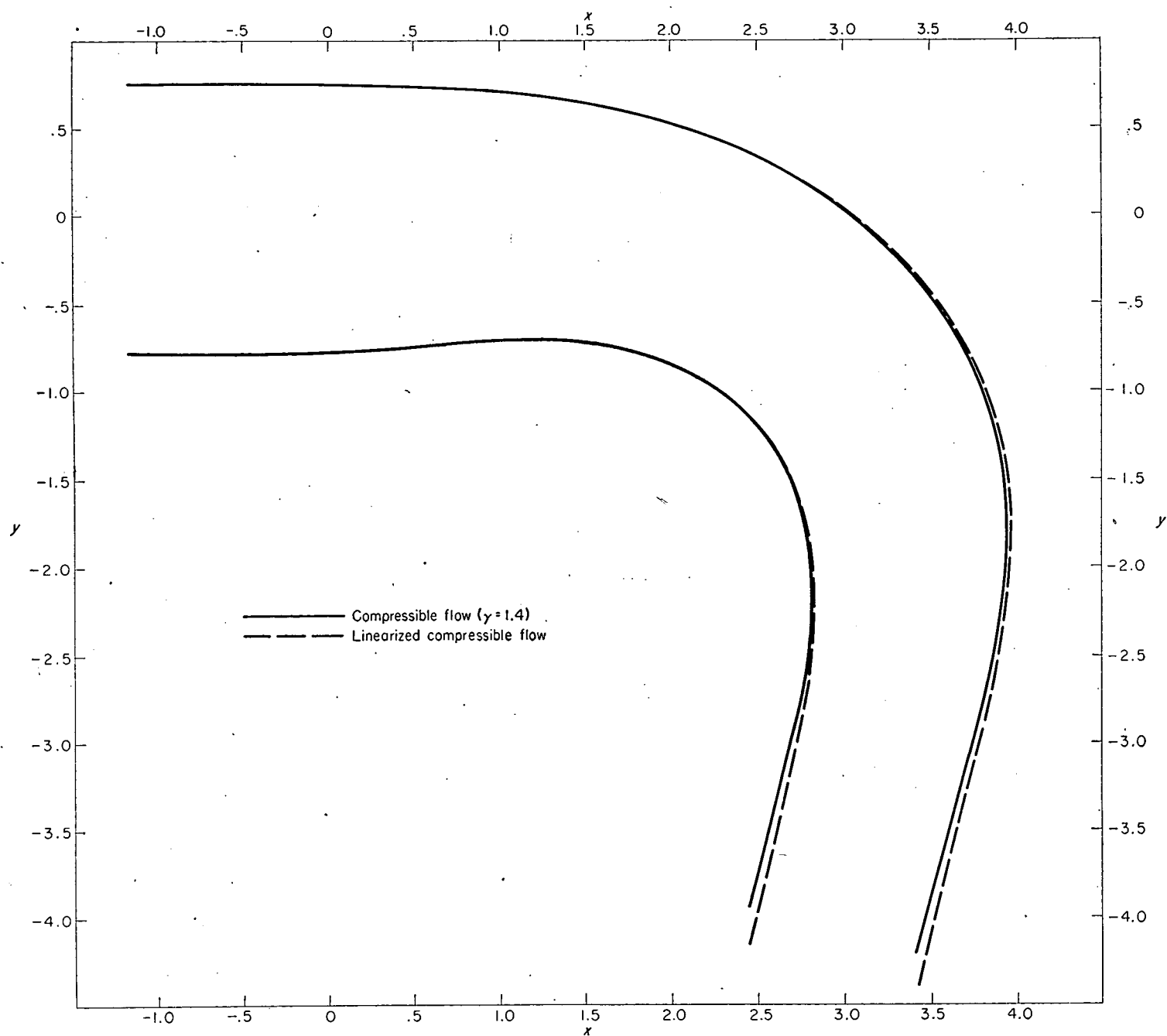


FIGURE 21.—Comparison of channel-wall shapes for compressible flow (example V) with γ equal to 1.4 and for linearized compressible flow (example IV) for same prescribed velocity as function of arc length along channel walls (fig. 2).

and for linearized compressible flow

$$d\Phi = \frac{\pi}{2} \frac{d\varphi^*}{\Delta\psi^*} \quad (40b)$$

Channel-wall coordinates.—From part I the distribution of channel-wall coordinates x and y along the boundaries of constant Ψ equal to 0 and $\pi/2$ in the transformed $\Phi\Psi$ -plane is given by

$$x = \frac{2}{\pi} \Delta\psi^* \int_{\Psi} \frac{\cos \theta}{q^*} d\Phi \quad (41a)$$

and

$$y = \frac{2}{\pi} \Delta\psi^* \int_{\Psi} \frac{\sin \theta}{q^*} d\Phi \quad (41b)$$

for linearized compressible flow, and for incompressible flow is given by

$$x = \frac{2}{\pi} \int_{\Psi} \frac{\cos \theta}{Q} d\Phi \quad (42a)$$

and

$$y = \frac{2}{\pi} \int_{\Psi} \frac{\sin \theta}{Q} d\Phi \quad (42b)$$

where the constants of integration are selected to give known (specified) values of x or y at one value of Φ along each boundary. Because q^* and Q are known functions of Φ from the prescribed velocity as a function of arc length along the channel walls, the shape of the channel walls in the physical xy -plane is given by equation (41) or (42) if θ is determined as a function of Φ along the channel walls. In part II the solution for θ as a function of Φ along the channel walls in the $\Phi\Psi$ -plane is obtained by Green's function.

SOLUTION BY GREEN'S FUNCTION

Continuity.—From part I the continuity equation becomes in the transformed $\Phi\Psi$ -plane

$$\frac{\partial \log_e V}{\partial \Phi} + \frac{\partial \theta}{\partial \Psi} = 0 \quad (43a)$$

where for incompressible flow

$$V = Q \quad (43b)$$

and for linearized compressible flow

$$V = \frac{q^*}{1 + \sqrt{1 + q^{*2}}} \quad (43c)$$

Irrotational motion.—From part I the equation for irrotational motion becomes in the transformed $\Phi\Psi$ -plane

$$\frac{\partial \log_e V}{\partial \Psi} - \frac{\partial \theta}{\partial \Phi} = 0 \quad (44)$$

Integral equation for $\theta(\Phi_o, \Psi_o)$.—From equations (43a) and (44)

$$\frac{\partial^2 \theta}{\partial \Phi^2} + \frac{\partial^2 \theta}{\partial \Psi^2} = 0 \quad (45)$$

so that from appendix E the value of θ at a point (Φ_o, Ψ_o)

within, or on, the channel walls in the transformed $\Phi\Psi$ -plane is given by the integral equation

$$\theta(\Phi_o, \Psi_o) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left[\left(G \frac{\partial \log_e V}{\partial \Phi} \right)_{\frac{\pi}{2}} - \left(G \frac{\partial \log_e V}{\partial \Phi} \right)_0 \right] d\Phi \quad (46)$$

where the subscripts 0 and $\frac{\pi}{2}$ refer to the channel-wall boundaries along which Ψ is 0 and $\frac{\pi}{2}$, respectively, and G is the Green's function of the second kind for the channel, which is an infinite strip of width $\frac{\pi}{2}$ extending in the Φ -direction to $\pm \infty$.

Green's function G .—The Green's function of the second kind G for the infinite channel in the $\Phi\Psi$ -plane is given along the channel-wall boundaries (Ψ equals 0 and $\frac{\pi}{2}$) by (appendix F)

$$G_{0 \text{ or } \frac{\pi}{2}} = -\log_e [\cosh^2(\Phi - \Phi_o) - \cos^2(\Psi - \Psi_o)] \quad (47)$$

where (Φ, Ψ) is any point on the channel-wall boundary and (Φ_o, Ψ_o) is the point in the channel or on the boundary at which θ is to be determined.

Numerical integration for $\theta(\Phi_o, \Psi_o)$.—From equations (46) and (47)

$$\begin{aligned} 2\pi\theta(\Phi_o, \Psi_o) = & \int_{-\infty}^{\infty} \left\{ \frac{\partial \log_e V}{\partial \Phi} \log_e [\cosh^2(\Phi - \Phi_o) - \right. \\ & \left. \sin^2 \Psi_o] \right\}_{\frac{\pi}{2}} d(\Phi - \Phi_o) - \\ & \int_{-\infty}^{\infty} \left\{ \frac{\partial \log_e V}{\partial \Phi} \log_e [\cosh^2(\Phi - \Phi_o) - \right. \\ & \left. \cos^2 \Psi_o] \right\}_0 d(\Phi - \Phi_o) \end{aligned} \quad (48)$$

in which the independent variable of integration has been changed from $d\Phi$ to $d(\Phi - \Phi_o)$ so that the origin, for purposes of integration, lies at Φ_o rather than $\Phi = 0$. If for small changes in $(\Phi - \Phi_o)$, that is, for small $\Delta\Phi$, the term $\frac{\partial \log_e V}{\partial \Phi}$ may be considered constant and equal to its average value over the interval $\Delta\Phi$, then

$$\frac{\partial \log_e V}{\partial \Phi} = \frac{\Delta \log_e V}{\Delta \Phi}$$

and equation (48) becomes

$$\begin{aligned} 2\pi\theta(\Phi_o, \Psi_o) = & \sum_{(\Phi - \Phi_o) = -\infty}^{\infty} \left\{ \frac{\Delta \log_e V}{\Delta \Phi} \int_{(\Phi - \Phi_o)}^{(\Phi - \Phi_o) + \Delta \Phi} \log_e [\cosh^2(\Phi - \Phi_o) - \right. \\ & \left. \sin^2 \Psi_o] d(\Phi - \Phi_o) \right\}_{\frac{\pi}{2}} - \\ & \sum_{(\Phi - \Phi_o) = -\infty}^{\infty} \left\{ \frac{\Delta \log_e V}{\Delta \Phi} \int_{(\Phi - \Phi_o)}^{(\Phi - \Phi_o) + \Delta \Phi} \log_e [\cosh^2(\Phi - \Phi_o) - \right. \\ & \left. \cos^2 \Psi_o] d(\Phi - \Phi_o) \right\}_0 \end{aligned} \quad (49)$$

where the summation sign is understood to mean that the quantity within the braces is summed over the entire range of $(\Phi - \Phi_0)$ between $\pm \infty$.

Equation (49) determines θ at any point in the flow field (channel). For a point (Φ_0, Ψ_0) on the channel walls Ψ_0 is equal to 0 or $\pi/2$ and the integrands in equation (49) become

$$2 \log_e \cosh |(\Phi - \Phi_0)|$$

or

$$2 \log_e \sinh |(\Phi - \Phi_0)|$$

so that equation (49) becomes

$$\pi \theta(\Phi_0, \Psi_0) = \sum_{(\Phi - \Phi_0) = -\infty}^{\infty} \left[\left(\frac{\Delta \log_e V}{\Delta \Phi} \Delta I \right)_{\frac{\pi}{2}} - \left(\frac{\Delta \log_e V}{\Delta \Phi} \Delta I \right)_0 \right] \quad (50a)$$

where

$$\Delta I = I_{(\Phi - \Phi_0) + \Delta \Phi} - I_{(\Phi - \Phi_0)} \quad (50b)$$

$$\left. \begin{aligned} I_{\frac{\pi}{2}} &= \alpha \text{ if } \Psi_0 = 0 \\ I_{\frac{\pi}{2}} &= \beta \text{ if } \Psi_0 = \frac{\pi}{2} \\ I_0 &= \alpha \text{ if } \Psi_0 = \frac{\pi}{2} \\ I_0 &= \beta \text{ if } \Psi_0 = 0 \end{aligned} \right\} \quad (50c)$$

where

$$\alpha = \pm \int_0^{|\Phi - \Phi_0|} \log_e \cosh |(\Phi - \Phi_0)| d|(\Phi - \Phi_0)| \quad (50d)$$

$$\beta = \pm \int_0^{|\Phi - \Phi_0|} \log_e \sinh |(\Phi - \Phi_0)| d|(\Phi - \Phi_0)| \quad (50e)$$

where the + signs apply for positive values of $(\Phi - \Phi_0)$ and the - signs apply for negative values of $(\Phi - \Phi_0)$. Methods of evaluating α and β are given in appendix G, and tabulated values are given for a wide range of $|(\Phi - \Phi_0)|$ in table VII. Equation (50a) determines $\theta(\Phi_0, \Psi_0)$ at any point on the channel-wall boundaries. Thus from equations (41a) and (41b) or (42a) and (42b) the coordinates for the channel-wall shape in the physical xy -plane can be determined.

NUMERICAL PROCEDURE

The numerical procedure for the channel design solution by Green's function is the same, except for minor details, for incompressible and linearized compressible flow. The stepwise procedure is outlined as follows:

(1) For incompressible flow the velocity Q and for linearized compressible flow the velocity q , or which is the same thing the velocity Q and the constant downstream velocity q_a , are specified as functions of arc length along the channel walls

$$Q = Q(s) \quad (51a)$$

or

$$q = q(s) \quad (51b)$$

where s is arbitrarily equal to 0 at that point along one channel wall where the velocity first begins to vary.

(2) Compute V as a function of s from equations (43b) and (51a) for incompressible flow or from equations (13b), (14b), (43c), and (51b) for linearized compressible flow

$$V = V(s) \quad (52)$$

(3) Compute Φ as a function of s from equations (4) and (40a) for incompressible flow or from equations (16), (32), (40b), and (51b) for linearized compressible flow. In equation (32) ρ_a^* is obtained from equations (8a), (13a), and (14a). For arbitrary distributions of Q or q equation (40a) or (40b) is integrated numerically by using, for example, Simpson's one-third rule. Thus

$$\Phi = \Phi(s) \quad (53)$$

(4) From equations (52) and (53) V and Φ are known functions of s so that

$$V = V(\Phi) \quad (54)$$

Thus V is a known function of Φ along the channel-wall boundaries in the transformed $\Phi\Psi$ -plane.

(5) If the prescribed velocity distribution along one wall is different from that along the other, the channel will, in general, turn the flow. This turning angle $\Delta\theta$ is given by equation (H5) in appendix H. If the turning angle is unsatisfactory, a new distribution of velocity as a function of s (eqs. (51a) and (51b)) is prescribed and steps (1) to (5) repeated until the desired value of $\Delta\theta$ is obtained. Equation (H5) is integrated numerically by using Simpson's one-third rule, for example, and equation (54).

(6) The channel-wall boundaries are straight parallel lines of constant Ψ equal to 0 and $\pi/2$, and extending to $\pm \infty$ in the Φ -direction. Along these boundaries of constant Ψ , a series of equally spaced points are located at each of which the flow direction θ and the x, y -coordinates of the channel walls will be determined by numerical integration. In order to use the tables of α and β presented in this report, the point spacing $\Delta\Phi$ must be an even multiple of $\pi/24$. Thus the smallest point spacing $\pi/24$ is equal to $1/2$ of the channel width ($\pi/2$). For a particular prescribed velocity distribution along the channel walls the accuracy of the solution increases, and so does the amount of computing, as the point spacing is reduced. The error for a given point spacing depends on the prescribed velocity distribution, and its order of magnitude is given by the leading term of the error series of the formula used for numerical integration (table VIII, ref. 4, for example). For the numerical example presented in part II of this report the point spacing $\Delta\Phi$ was $\pi/12$. From equation (54)

$$\frac{\Delta \log_e V}{\Delta \Phi} = \frac{(\log_e V)_{\Phi + \Delta \Phi} - (\log_e V)_{\Phi}}{\Delta \Phi} \quad (55)$$

where the subscripts Φ and $\Phi + \Delta\Phi$ refer to adjacent points along the channel boundaries.

(7) The value of θ at each point (Φ_0, Ψ_0) on the channel-wall boundaries is obtained from equation (50a) in which $(\Delta \log_e V)/\Delta \Phi$ is given by equation (55) and ΔI is given by equations (50b), (50c), and table VII. Note that in equation

(50a) the origin has been moved to Φ_0 by changing from Φ to $(\Phi - \Phi_0)$. Thus the value of $(\Delta \log_e V)/\Delta \Phi$ for a given value of $(\Phi - \Phi_0)$ varies with Φ_0 .

(8) The physical x, y -coordinates at each point on the channel-wall boundaries are obtained by the numerical integration of equations (42a) and (42b) for incompressible flow, or equations (41a) and (41b) for linearized compressible flow where $\Delta \psi^*$ is given by equation (32). The constants of integration in equations (41) and (42) are selected to give known values of x and y at upstream or downstream positions where flow conditions can be considered uniform.

NUMERICAL EXAMPLE

The channel design method of part II has been applied to the design of an elbow for the same conditions as example IV of part I. The design is for an accelerating elbow with no local decelerations of the prescribed velocities along the channel walls and with linearized compressible flow.

Prescribed velocity distribution.—The prescribed velocity distribution along the channel walls is the same as that for example IV of part I. The prescribed velocity as a function of Φ is plotted in figure 22.

Results.—As indicated in table VIII, the elbow design resulting from the prescribed velocities given in figure 22 is the same as that obtained by relaxation methods (fig. 21) for the same prescribed conditions (example IV, part I).

The solution obtained by Green's function (part II) required one experienced computer 3 days, whereas the solution by relaxation methods (part I) required about 10 days. The relaxation solutions provide additional information, such as the distribution of velocity across the channel; but for the most part this additional information is of secondary importance, and the design of channels by Green's function is more rapid and therefore to be preferred over the design by relaxation methods.

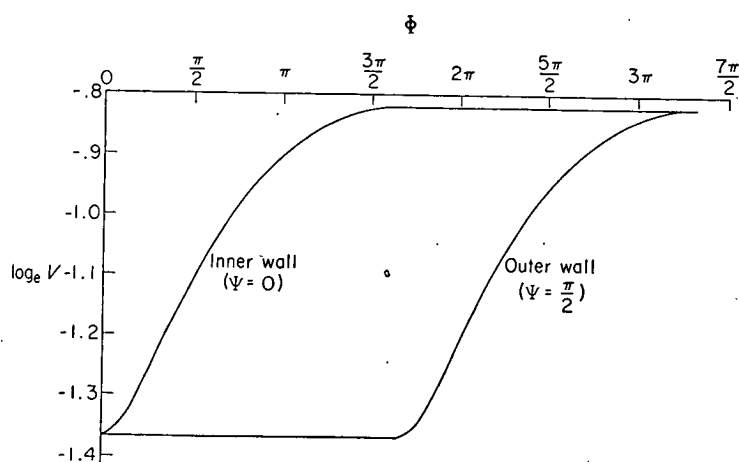


FIGURE 22.—Variation in prescribed values of $\log_e V$ with Φ along channel walls of numerical example in part II.

SUMMARY OF RESULTS AND CONCLUSIONS

A general method of design is developed for two-dimensional unbranched channels with prescribed velocities as a function of arc length along the channel walls. The method is developed for both compressible and incompressible, irrotational, nonviscous flow and applies to the design of elbows, diffusers, nozzles, and so forth. Two types of compressible flow are considered: the general type with arbitrary value for the ratio of specific heats γ (1.4, for example) and the linearized type in which γ is equal to -1.0 . In part I solutions are obtained by relaxation methods on a transformed plane the coordinates of which are the streamlines and velocity-potential lines in the physical plane; in part II solutions are obtained by a Green's function. The method of solution in part I gives complete information concerning the flow throughout the channel, whereas the method of solution in part II gives the channel-wall coordinates only.

Five numerical examples are given in part I and the results are presented by (1) lines of constant velocity and flow direction or lines of constant physical coordinates in the transformed plane and (2) streamlines and velocity-potential lines or lines of constant velocity and flow direction in the physical plane. Among the five examples are three elbow designs for the same prescribed velocity as a function of arc length along the channel walls but with incompressible, linearized compressible, and compressible flow. The numerical results of these three elbow designs are tabulated to enable a detailed comparison of the three designs.

The shapes of the elbows for compressible flow and for linearized compressible flow are very nearly the same; and it is concluded that, if a nonviscous gas with arbitrary γ (1.4, for example) were to flow through a channel designed for linearized compressible flow ($\gamma = -1.0$), the resulting velocity distribution along the channel walls would be nearly the velocity distribution prescribed for the linearized compressible flow. This conclusion is important because the design method for linearized compressible flow is considerably faster than that for compressible flow.

One numerical example is presented in part II for an accelerating elbow with linearized compressible flow. The elbow shape obtained from the solution by Green's function in part II is the same as that obtained from a solution by relaxation methods in part I for the same prescribed conditions. The time required for the calculations was considerably less for the solution by Green's function.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, July 25, 1951

APPENDIX A

SYMBOLS

The following symbols are used in this report:

A, B, C, D	coefficients, equation (29)	θ	flow direction in physical xy -plane (measured in counterclockwise direction from positive x -axis)
A, B	arbitrary constants, equation (C1a)	$\Delta\theta$	channel turning angle, equation (12)
$B_1, B_2, \dots$	Bernoulli's numbers	ρ	density (expressed as ratio of stagnation density)
c	constant, equation (E3)	ρ^*	density in linearized compressible flow and related to ρ by equation (13a)
G	Green's function of the second kind, equations (E2) and (47)	Φ	velocity potential used as Cartesian coordinate in transformed $\Phi\Psi$ -plane and related to φ or φ^* by equation (40a) or (40b), respectively
I	integral (α or β)	φ and φ^*	velocity potential for incompressible and linearized compressible flow, respectively, equations (4) and (16)
k_1	coefficient, equation (14a)	Ψ	stream function used as Cartesian coordinate in transformed $\Phi\Psi$ -plane and related to ψ or ψ^* by equation (39a) or (39b), respectively
k_2	coefficient, equation (14b)	ψ and ψ^*	stream function for incompressible and linearized compressible flow, respectively, equations (3) and (15)
l	length of closed boundary	$\Delta\psi^*$	boundary value of ψ^* , for linearized compressible flow, along left channel wall when faced in the direction of flow, equation (32)
n	distance in xy -plane measured normal to direction of flow (expressed as ratio of characteristic length equal to channel width downstream at infinity)	ω	any harmonic function in $\Phi\Psi$ -plane
p	static pressure (expressed as ratio of stagnation density multiplied by stagnation speed of sound squared)	Subscripts:	
Q	velocity (expressed as ratio of characteristic velocity equal to constant channel velocity downstream at infinity)	a, b	quantities related to two velocities (q_a and q_b , respectively) for which density given by equation (8a) is equal to density ρ given by equations (13), (13a), and (13b)
q	velocity (expressed as ratio of stagnation speed of sound)	d	conditions downstream at infinity
q^*	velocity used in linearized compressible flow and related to q by equation (13b)	o	point in $\Phi\Psi$ -plane at which θ is determined
r	distance from any point in $\Phi\Psi$ -plane to point (Φ_o, Ψ_o) at which logarithmic singularity exists	u	conditions upstream at infinity
s	distance in xy -plane measured along direction of flow (expressed as ratio of characteristic length equal to channel width downstream at infinity)	$\Delta\psi^*$	left channel wall, when faced in direction of flow, along which ψ^* is equal to $\Delta\psi^*$
u	velocity parameter related to q^* by equation (18)	$(\Phi - \Phi_o)$	point at $(\Phi - \Phi_o)$ on either channel-wall boundary
V	velocity parameter defined by equations (43b) and (43c) for incompressible and linearized compressible flow, respectively	$(\Phi - \Phi_o) + \Delta\Phi$	point at $[(\Phi - \Phi_o) + \Delta\Phi]$ on either channel-wall boundary
w, w_1, w_2	complex functions defined by equations (F3), (F1a), and (F2a), respectively	$\varphi, \psi, \varphi^*, \psi^*$	along lines of constant φ, ψ, φ^* , and ψ^* , respectively
x, y	Cartesian coordinates in physical plane (expressed as ratios of characteristic length equal to channel width downstream at infinity)	0	right channel wall, when faced in direction of flow, along which Ψ, ψ , or ψ^* is equal to 0
z	complex coordinate, equation (F1b)	1.0	left channel wall, when faced in direction of flow, along which ψ is equal to 1.0
$\bar{z}$	conjugate of z	$\frac{\pi}{2}$	left channel wall, when faced in direction of flow, along which Ψ is equal to $\frac{\pi}{2}$
α	integral, equation (50d)		
β	integral, equation (50e)		
γ	ratio of specific heats		
Δ	finite increment		
δ	increment of		

APPENDIX B

EQUATIONS OF CONTINUITY AND IRROTATIONAL FLUID MOTION IN TERMS OF TRANSFORMED φ, ψ -COORDINATES

Consider the two-dimensional irrotational motion of a fluid particle in the physical xy -plane. The fluid particle is defined by adjacent streamlines (constant ψ) and velocity-potential lines (constant φ) spaced δn and δs apart as indicated in figure 23. The velocity Q is parallel to the streamlines and normal to the velocity-potential lines.

Continuity.—From continuity considerations of the fluid particle in figure 23

$$\frac{\partial}{\partial s} (\rho Q \delta n) = 0$$

or

$$\frac{\partial \log_e \rho}{\partial s} + \frac{\partial \log_e Q}{\partial s} + \frac{1}{\delta n} \frac{\partial(\delta n)}{\partial s} = 0 \quad (B1)$$

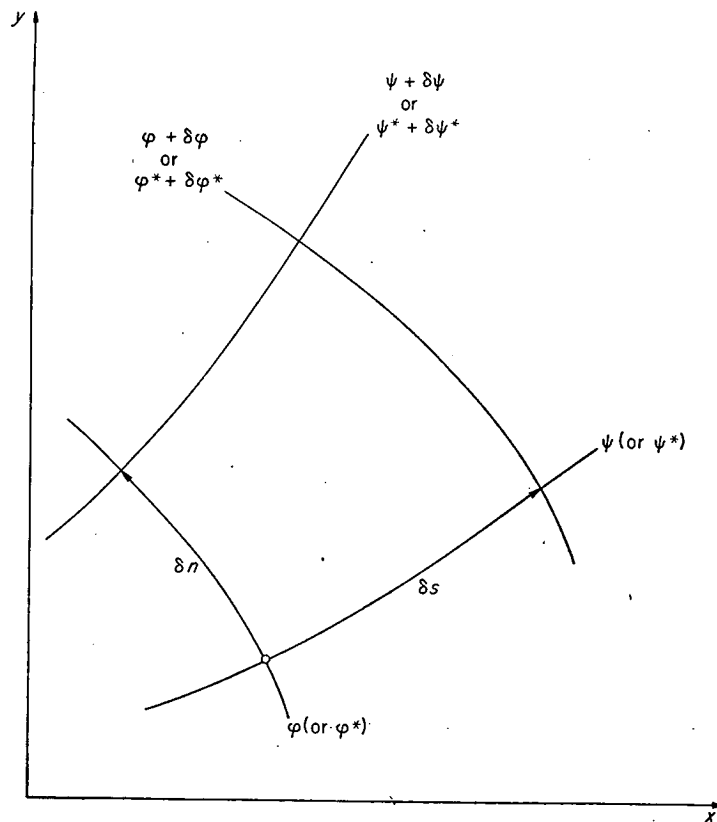


FIGURE 23.—Fluid particle bounded by streamlines and velocity-potential lines in physical xy -plane.

But, from geometrical considerations (ref. 5, p. 167, for example)

$$\frac{1}{\delta n} \frac{\partial(\delta n)}{\partial s} = \frac{\partial \theta}{\partial n} \quad (B2a)$$

and

$$\frac{1}{\delta s} \frac{\partial(\delta s)}{\partial n} = -\frac{\partial \theta}{\partial s} \quad (B2b)$$

so that equation (B1) becomes

$$\frac{\partial \log_e \rho}{\partial s} + \frac{\partial \log_e Q}{\partial s} + \frac{\partial \theta}{\partial n} = 0$$

or

$$\frac{\partial \log_e \rho}{\partial \varphi} \frac{d\varphi}{ds} + \frac{\partial \log_e Q}{\partial \varphi} \frac{d\varphi}{ds} + \frac{\partial \theta}{\partial \psi} \frac{d\psi}{dn} = 0$$

which, combined with equations (3) and (4), becomes

$$\frac{1}{\rho} \left(\frac{\partial \log_e \rho}{\partial \varphi} + \frac{\partial \log_e Q}{\partial \varphi} \right) + \frac{\partial \theta}{\partial \psi} = 0 \quad (5)$$

Equation (5) is the continuity equation expressed in terms of φ, ψ -coordinates.

Irrotational fluid motion.—For irrotational motion of the fluid particle in figure 23

$$\frac{\partial}{\partial n} (Q \delta s) = 0$$

or

$$\frac{\partial \log_e Q}{\partial n} + \frac{1}{\delta s} \frac{\partial(\delta s)}{\partial n} = 0 \quad (B3)$$

But, from equations (B2b) and (B3)

$$\frac{\partial \log_e Q}{\partial n} - \frac{\partial \theta}{\partial s} = 0$$

or

$$\frac{\partial \log_e Q}{\partial \psi} \frac{d\psi}{dn} - \frac{\partial \theta}{\partial \varphi} \frac{d\varphi}{ds} = 0$$

which, combined with equations (3) and (4), becomes

$$\rho \frac{\partial \log_e Q}{\partial \psi} - \frac{\partial \theta}{\partial \varphi} = 0 \quad (6)$$

Equation (6) is the equation for irrotational fluid motion expressed in terms of the φ, ψ -coordinates.

APPENDIX C

RELATION BETWEEN VELOCITY AND DENSITY ASSUMING LINEAR VARIATION IN PRESSURE WITH SPECIFIC VOLUME

The approximate, linear relation between pressure p and specific volume $1/\rho$ first suggested by Chaplygin (ref. 6) is given by

$$p = A - \frac{B}{\rho} \quad (C1a)$$

from which

$$\frac{dp}{d\rho} = \frac{B}{\rho^2} \quad (C1b)$$

where A and B are arbitrary constants.

If p denotes the static pressure expressed as a ratio of the stagnation density multiplied by the stagnation speed of sound squared, Bernoulli's equation is

$$\frac{dp}{\rho} + q dq = 0$$

which combined with equation (C1b) integrates to give the approximate relation between velocity and density

$$\frac{B}{2\rho^2} - \frac{q^2}{2} = \text{constant} \quad (\text{C2})$$

For convenience equation (C2) can be written as

$$\frac{1}{\rho^{*2}} - q^{*2} = 1$$

or

$$\rho^* = (1 + q^{*2})^{-1/2} \quad (\text{13})$$

where

$$\rho^* = k_1 \rho \quad (\text{13a})$$

and

$$q^* = k_2 q \quad (\text{13b})$$

The constants k_1 and k_2 replace the two arbitrary constants in equation (C2), and their values are determined so that for any two arbitrary values of q (designated by q_a and q_b) the values of ρ given by equation (13) equal the values of ρ given by equation (8a). Thus the values of ρ given by equation (13) for q equal to q_a or q_b are correct; for all other values of q the values of ρ are approximate. The constants k_1 and k_2 are determined from the conditions

$$\left. \begin{aligned} \rho_a^* &= k_1 \rho_a \\ q_a^* &= k_2 q_a \\ \rho_b^* &= k_1 \rho_b \\ q_b^* &= k_2 q_b \end{aligned} \right\} \quad (\text{C3})$$

From equation (13) and the conditions given by equation (C3)

$$k_1 = \frac{1}{\rho_a} \sqrt{\frac{1 - \left(\frac{\rho_a q_a}{\rho_b q_b}\right)^2}{1 - \left(\frac{q_a}{q_b}\right)^2}} \quad (\text{14a})$$

and

$$k_2 = \frac{1}{q_b} \sqrt{\frac{\left(\frac{\rho_a}{\rho_b}\right)^2 - 1}{1 - \left(\frac{\rho_a q_a}{\rho_b q_b}\right)^2}} \quad (\text{14b})$$

where ρ_a and ρ_b are determined by equation (8a) for the selected values of q_a and q_b , respectively.

The values of q_a and q_b might, for example, be selected to equal the maximum and minimum values of q (which values of q must occur on the channel walls and are therefore known). Also, the values of q_a and q_b might be selected to equal the upstream and downstream velocities q_u and q_d . In this case the upstream and downstream channel widths would then satisfy continuity for a gas with the correct value of γ (1.4, for example). If the upstream and downstream velocities are equal, their value and the value of some other velocity (the maximum or minimum velocity, for example) can be selected for q_a and q_b ; or, if desired, q_a can be equal to q_b , in which case if

$$q_a = q + \epsilon \text{ where } \epsilon \rightarrow 0$$

$$q_b = q$$

it can be shown from equations (14a) and (14b) that

$$k_1 = \frac{1}{\rho} \sqrt{\frac{1 - \frac{\gamma+1}{2} q^2}{1 - \frac{\gamma-1}{2} q^2}} \quad (\text{C4a})$$

and

$$k_2 = \sqrt{\frac{1}{1 - \frac{\gamma+1}{2} q^2}} \quad (\text{C4b})$$

This latter case, in which $q_a = q_b = q$, corresponds to the method used by Chaplygin (ref. 6) and Kármán-Tsien (ref. 8) in which the correct relation between p and $\frac{1}{\rho}$ is replaced by a straight line (eq. (C1a)) that is tangent to the correct relation at one point (where $q_a = q_b$).

APPENDIX D

EQUATIONS OF CONTINUITY AND IRROTATIONAL FLUID MOTION IN TERMS OF TRANSFORMED φ^* , ψ^* -COORDINATES

Consider the two-dimensional irrotational motion of a fluid particle in the physical xy -plane. The fluid particle is defined by adjacent streamlines (constant ψ^*) and velocity-potential lines (constant φ^*) spaced δn and δs apart as indicated in figure 23. The velocity q^* is parallel to the streamlines and normal to the velocity-potential lines.

Continuity.—From continuity considerations of the fluid particle in figure 23

$$\frac{\partial}{\partial s} (\rho^* q^* \delta n) = 0$$

or

$$\frac{\partial \log_e \rho^*}{\partial s} + \frac{\partial \log_e q^*}{\partial s} + \frac{1}{\delta n} \frac{\partial (\delta n)}{\partial s} = 0$$

which combined with equation (B2a) becomes

$$\frac{\partial \log_e \rho^*}{\partial \varphi^*} \frac{d\varphi^*}{ds} + \frac{\partial \log_e q^*}{\partial \varphi^*} \frac{d\varphi^*}{ds} + \frac{\partial \theta}{\partial \psi^*} \frac{d\psi^*}{dn} = 0$$

or, from equations (15) and (16)

$$\frac{1}{\rho^*} \left(\frac{\partial \log_e \rho^*}{\partial \varphi^*} + \frac{\partial \log_e q^*}{\partial \varphi^*} \right) + \frac{\partial \theta}{\partial \psi^*} = 0 \quad (\text{D1})$$

But, from equation (13)

$$\frac{1}{\rho^*} \frac{\partial \log_e \rho^*}{\partial \varphi^*} = \frac{-q^{*2}}{\sqrt{1+q^{*2}}} \frac{\partial \log_e q^*}{\partial \varphi^*}$$

so that equation (D1) becomes

$$\frac{1}{\sqrt{1+q^{*2}}} \frac{\partial \log_e q^*}{\partial \varphi^*} + \frac{\partial \theta}{\partial \psi^*} = 0 \quad (\text{D2})$$

Finally, if

$$u = \frac{q^*}{1 + \sqrt{1 + q^{*2}}} \quad (18)$$

then

$$\frac{\partial \log_e q^*}{\sqrt{1 + q^{*2}}} = \partial \log_e u \quad (D3)$$

so that equation (D2) becomes

$$\frac{\partial \log_e u}{\partial \varphi^*} + \frac{\partial \theta}{\partial \psi^*} = 0 \quad (17)$$

Equation (17) is the continuity equation expressed in terms of φ^*, ψ^* -coordinates and $\log_e u$.

Irrotational fluid motion.—For irrotational motion of the fluid particle in figure 23

$$\frac{\partial}{\partial n} (q^* \delta s) = 0$$

or

$$\frac{\partial \log_e q^*}{\partial n} + \frac{1}{\delta s} \frac{\partial (\delta s)}{\partial n} = 0$$

which combined with equation (B2b) becomes

$$\frac{\partial \log_e q^*}{\partial \psi^*} \frac{d\psi^*}{dn} - \frac{\partial \theta}{\partial \varphi^*} \frac{d\varphi^*}{ds} = 0$$

or, from equations (13), (15), and (16)

$$\frac{1}{\sqrt{1 + q^{*2}}} \frac{\partial \log_e q^*}{\partial \psi^*} - \frac{\partial \theta}{\partial \varphi^*} = 0 \quad (D4)$$

Finally, from equations (D3) and (D4)

$$\frac{\partial \log_e u}{\partial \psi^*} - \frac{\partial \theta}{\partial \varphi^*} = 0 \quad (20)$$

Equation (20) is the equation for irrotational fluid motion expressed in terms of φ^*, ψ^* -coordinates and $\log_e u$.

APPENDIX E

INTEGRAL EQUATION FOR $\theta(\Phi, \Psi)$

If the distribution of the angle $\theta(\Phi, \Psi)$ in the transformed $\Phi\Psi$ -plane is harmonic, that is, satisfies equation (45) within and on the channel walls (Ψ equals 0 and $\frac{\pi}{2}$), then from Green's theorem and the theorem of mean value it can be shown that the value of θ at a point (Φ_o, Ψ_o) within (or on) the channel walls is given by (ref. 9, p. 204, for example)

$$\theta(\Phi_o, \Psi_o) = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} \left(\theta \frac{\partial G}{\partial \Psi} - G \frac{\partial \theta}{\partial \Psi} \right)_0 d\Phi - \int_{-\infty}^{\infty} \left(-\theta \frac{\partial G}{\partial \Psi} + G \frac{\partial \theta}{\partial \Psi} \right)_{\frac{\pi}{2}} d\Phi \right] \quad (E1)$$

where the two integrals on the right side of equation (E1) represent the line integral around the channel walls in the counterclockwise direction with the signs adjusted so that $\frac{\partial}{\partial \Psi}$ represents the inner normal to the path of integration.

The function $G(\Phi, \Psi)$ in equation (E1) is of the form (ref. 9, p. 204)

$$G(\Phi, \Psi) = \log_e \frac{1}{r} + \omega(\Phi, \Psi) \quad (E2)$$

where r is the distance from any point (Φ, Ψ) to the point (Φ_o, Ψ_o) and where $\omega(\Phi, \Psi)$ is an arbitrary function that is harmonic within and on the channel walls. (Thus from equation (E2), $G(\Phi, \Psi)$ is harmonic within and on the channel walls except at the point (Φ_o, Ψ_o) where a logarithmic singularity exists.) Because the harmonic function $\omega(\Phi, \Psi)$ is arbitrary, the function $G(\Phi, \Psi)$ can be selected so that along

the channel-wall boundaries (Ψ equals 0 and $\frac{\pi}{2}$) $\frac{\partial G}{\partial \Psi}$ is a constant c given by the following equation (obtained from notes presented by Tamarkin and Feller in the 1941 Summer Session for Advanced Instruction and Research in Mechanics at Brown Univ.):

$$c = \frac{2\pi}{l} \quad (E3)$$

where l is the length of the path along which the line integral is taken. For the path under consideration l is infinite and therefore $G(\Phi, \Psi)$ can be selected so that $\frac{\partial G}{\partial \Psi}$ is zero along the channel walls. A function with this property is a Green's function of the second kind. Equation (E1) becomes

$$\theta(\Phi_o, \Psi_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left(G \frac{\partial \theta}{\partial \Psi} \right)_{\frac{\pi}{2}} - \left(G \frac{\partial \theta}{\partial \Psi} \right)_0 \right] d\Phi$$

or, combined with equation (43a)

$$\theta(\Phi_o, \Psi_o) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left[\left(G \frac{\partial \log_e V}{\partial \Phi} \right)_{\frac{\pi}{2}} - \left(G \frac{\partial \log_e V}{\partial \Phi} \right)_0 \right] d\Phi \quad (46)$$

Along the channel walls $\frac{\partial \log_e V}{\partial \Phi}$ is known from the prescribed velocity distribution so that, after the proper Green's function G has been determined (appendix F), equation (46) determines the value of θ at any point (Φ_o, Ψ_o) . The value of $\theta(\Phi_o, \Psi_o)$ given by equation (46) can be adjusted by an arbitrary constant of integration to give a specified value of θ at one point in the flow field.

APPENDIX F

GREEN'S FUNCTION OF SECOND KIND

From appendix E Green's function of the second kind G satisfies the condition

$$\frac{\partial G}{\partial \Psi} = 0$$

along the channel walls, which are straight and parallel boundaries (Ψ equals 0 and $\frac{\pi}{2}$) extending to $\pm \infty$ in the Φ -direction, and satisfies the equation

$$\frac{\partial^2 G}{\partial \Phi^2} + \frac{\partial^2 G}{\partial \Psi^2} = 0$$

everywhere in the channel except at the point (Φ_o, Ψ_o) where G has a logarithmic pole. For these conditions the Green's function G can be obtained by analogy from the velocity potential for incompressible flow into a point sink at (Φ_o, Ψ_o) between straight parallel boundaries at Ψ equal to 0 and $\frac{\pi}{2}$. The logarithmic pole for G at (Φ_o, Ψ_o) corresponds

to the point sink, and the condition $\frac{\partial G}{\partial \Psi} = 0$ at the boundaries corresponds to zero velocity, that is, no flow normal to the boundaries.

The velocity potential for fluid flow with the boundary conditions just described is obtained from two infinite series of point sinks with the sinks of each series spaced π distance apart in the Ψ -direction and the two series arranged by the method of images in such a manner that no flow crosses the boundaries, that is, $\frac{\partial G}{\partial \Psi} = 0$. This arrangement of point sinks is shown in figure 24.

The complex function w_1 for the first infinite series of point sinks is given by (ref. 10, p. 112, for example)

$$w_1 = -\log_e \sinh (z - z_o) \quad (\text{F1a})$$

where

$$z = \Phi + i\Psi \quad (\text{F1b})$$

The complex function w_2 for the second infinite series of point sinks (mirror image of the first series in order to prevent flow across the boundaries Ψ equals 0 and $\frac{\pi}{2}$) is given by

$$w_2 = -\log_e \sinh (z - \bar{z}_o) \quad (\text{F2a})$$

where

$$\bar{z} = \Phi - i\Psi \quad (\text{F2b})$$

The complex function w for the combined flow becomes from equations (F1a) to (F2b)

$$w = w_1 + w_2 = -\log_e \sinh [(\Phi - \Phi_o) + i(\Psi - \Psi_o)] - \log_e \sinh [(\Phi - \Phi_o) + i(\Psi + \Psi_o)] \quad (\text{F3})$$

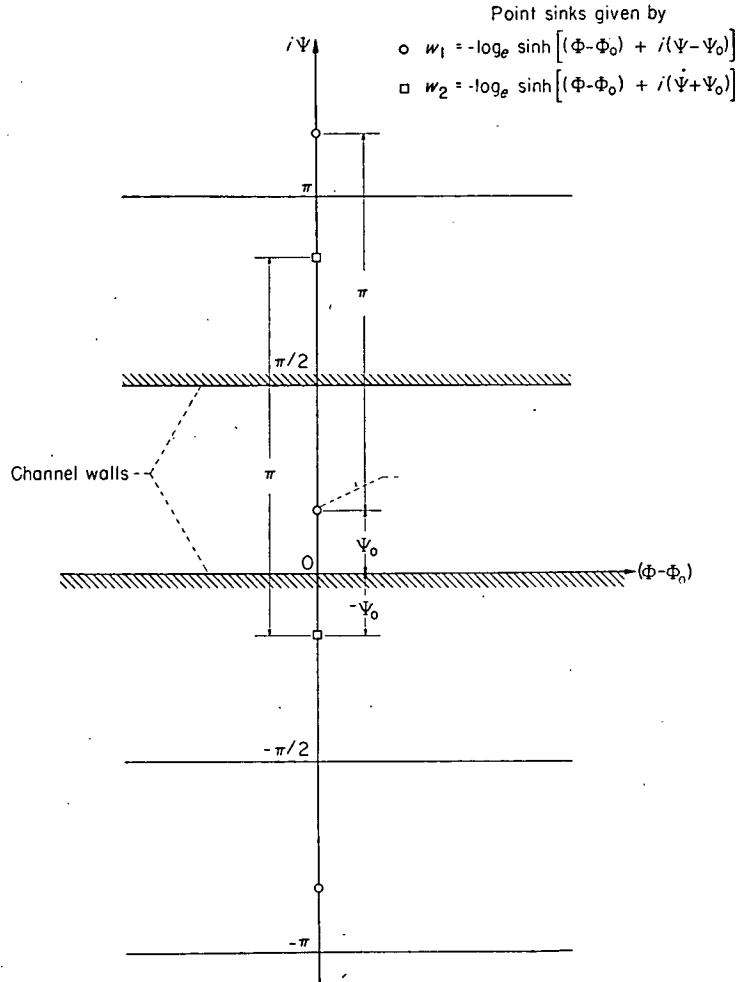


FIGURE 24.—Two infinite series of point sinks required in the development of Green's function of the second kind G .

The Green's function of the second kind G corresponds to the velocity potential for the incompressible flow and is therefore given by the real part of equation (F3)

$$G = -\frac{1}{2} \log_e [\cosh^2 (\Phi - \Phi_o) - \cos^2 (\Psi - \Psi_o)] [\cosh^2 (\Phi - \Phi_o) - \cos^2 (\Psi + \Psi_o)] \quad (\text{F4})$$

But along the channel walls Ψ is equal to 0 or $\frac{\pi}{2}$ so that

$$\cos^2 (\Psi + \Psi_o) = \cos^2 (\Psi - \Psi_o)$$

and equation (F4) becomes

$$G_{\text{on } \frac{\pi}{2}} = -\log_e [\cosh^2 (\Phi - \Phi_o) - \cos^2 (\Psi - \Psi_o)] \quad (\text{F5})$$

Equation (F5) gives the Green's function of the second kind along the channel walls (straight parallel lines of constant Ψ equal to 0 and $\frac{\pi}{2}$ and extending to $\pm \infty$ in the Φ -direction).

APPENDIX G

EVALUATION OF α AND β

Several techniques, depending on the magnitude of the upper limit $|(\Phi - \Phi_0)|$, were used to evaluate the integrals α and β given by equations (50d) and (50e). Each integral is treated separately in this appendix, and the values of $(\Phi - \Phi_0)$ for the upper limit $|(\Phi - \Phi_0)|$ are considered positive. For negative values of $(\Phi - \Phi_0)$ the magnitudes of I (that is, of α or β) are equal for corresponding values of $|(\Phi - \Phi_0)|$ but opposite in sign. As a result the values of ΔI have the same sign.

INTEGRAL α

Small and medium values of $(\Phi - \Phi_0)$.—For small and medium values of the upper limit of integration $(\Phi - \Phi_0)$ in equation (50d), that is, for $0 \leq (\Phi - \Phi_0) \leq 60\pi/24$, the integral α is evaluated by Simpson's one-third rule using increments of $(\Phi - \Phi_0)$ equal to $\pi/48$.

Large values of $(\Phi - \Phi_0)$.—For large values of $(\Phi - \Phi_0)$, that is, for $(\Phi - \Phi_0) > 60\pi/24$, the integrand in equation (50d) becomes

$$\log_e \cosh (\Phi - \Phi_0) \approx (\Phi - \Phi_0) - \log_e 2 \quad (G1)$$

so that equation (50d) becomes

$$\begin{aligned} \alpha &\approx \int_0^{60\pi/24} \log_e \cosh (\Phi - \Phi_0) d(\Phi - \Phi_0) + \\ &\int_{60\pi/24}^{(\Phi - \Phi_0)} [(\Phi - \Phi_0) - \log_e 2] d(\Phi - \Phi_0) \\ &\approx 25.809782 + \left[\frac{(\Phi - \Phi_0)^2}{2} - 0.693147(\Phi - \Phi_0) - 25.398552 \right] \\ &\approx 0.411230 - 0.693147(\Phi - \Phi_0) + \frac{1}{2}(\Phi - \Phi_0)^2 \end{aligned} \quad (G2)$$

Equation (G2) gives values of α for values of $(\Phi - \Phi_0)$ equal to or greater than $60\pi/24$. Values of the integral α are tabulated in table VII for a range of $|(\Phi - \Phi_0)|$ between 0 and $100\pi/24$ in increments of $\pi/24$. For negative values of $(\Phi - \Phi_0)$ the sign of α is negative.

INTEGRAL β

Small values of $(\Phi - \Phi_0)$.—For $(\Phi - \Phi_0)$ equal to zero the integrand of equation (50e) becomes infinite so that Simpson's one-third rule cannot be used to evaluate β in this region of $(\Phi - \Phi_0)$, as was done for α . However, equation (50e) integrates by parts to give

$$\begin{aligned} \int_0^{(\Phi - \Phi_0)} \log_e \sinh (\Phi - \Phi_0) d(\Phi - \Phi_0) &= (\Phi - \Phi_0) \log_e \sinh (\Phi - \Phi_0) - \\ &\int_0^{(\Phi - \Phi_0)} (\Phi - \Phi_0) \operatorname{ctnh} (\Phi - \Phi_0) d(\Phi - \Phi_0) \end{aligned} \quad (G3)$$

where the integrand $(\Phi - \Phi_0) \operatorname{ctnh} (\Phi - \Phi_0)$ on the right side of equation (G3) can be expanded in the following series form:

$$\begin{aligned} (\Phi - \Phi_0) \operatorname{ctnh} (\Phi - \Phi_0) &= 1 + \frac{2^2 B_1 (\Phi - \Phi_0)^2}{2!} - \frac{2^4 B_3 (\Phi - \Phi_0)^4}{4!} + \\ &\frac{2^6 B_5 (\Phi - \Phi_0)^6}{6!} - \frac{2^8 B_7 (\Phi - \Phi_0)^8}{8!} + \frac{2^{10} B_9 (\Phi - \Phi_0)^{10}}{10!} - \\ &\frac{2^{12} B_{11} (\Phi - \Phi_0)^{12}}{12!} + \dots \end{aligned} \quad (G4)$$

where B_1, B_3 , and so forth, are Bernoulli's numbers (ref. 11, p. 90, for example). From equations (G3) and (G4)

$$\begin{aligned} \beta &= (\Phi - \Phi_0) \log_e \sinh (\Phi - \Phi_0) - (\Phi - \Phi_0) - \frac{(\Phi - \Phi_0)^3}{9} + \frac{(\Phi - \Phi_0)^5}{225} - \\ &\frac{2(\Phi - \Phi_0)^7}{6615} + \frac{(\Phi - \Phi_0)^9}{42,525} - \frac{2(\Phi - \Phi_0)^{11}}{1,029,105} + \frac{1382(\Phi - \Phi_0)^{13}}{8,300,667,375} - \dots \end{aligned} \quad (G5)$$

Equation (G5) was used to obtain β as a function of $(\Phi - \Phi_0)$ for $0 \leq (\Phi - \Phi_0) \leq 8\pi/24$.

Medium values of $(\Phi - \Phi_0)$.—For medium values of the upper limit of integration $(\Phi - \Phi_0)$ in equation (50e), that is, for $8\pi/24 < (\Phi - \Phi_0) \leq 60\pi/24$, the integral β is evaluated by Simpson's one-third rule as was done for α .

Large values of $(\Phi - \Phi_0)$.—For large values of $(\Phi - \Phi_0)$, that is, for $(\Phi - \Phi_0) > 60\pi/24$, the integrand in equation (50e) becomes

$$\log_e \sinh (\Phi - \Phi_0) \approx (\Phi - \Phi_0) - \log_e 2 \quad (G6)$$

so that equation (50e) becomes

$$\begin{aligned} \beta &\approx \int_0^{60\pi/24} \log_e \sinh (\Phi - \Phi_0) d(\Phi - \Phi_0) + \\ &\int_{60\pi/24}^{(\Phi - \Phi_0)} [(\Phi - \Phi_0) - \log_e 2] d(\Phi - \Phi_0) \\ &\approx 24.576082 + \left[\frac{(\Phi - \Phi_0)^2}{2} - 0.693147(\Phi - \Phi_0) - 25.398552 \right] \\ &\approx -0.822470 - 0.693147(\Phi - \Phi_0) + \frac{1}{2}(\Phi - \Phi_0)^2 \end{aligned} \quad (G7)$$

Equation (G7) gives values of β for values of $(\Phi - \Phi_0)$ equal to or greater than $60\pi/24$. Values of the integral β are tabulated in table VII for a range of $|(\Phi - \Phi_0)|$ between 0 and $100\pi/24$ in increments of $\pi/24$. For negative values of $(\Phi - \Phi_0)$, the sign of β changes.

APPENDIX H

CHANNEL TURNING ANGLE

If the prescribed velocity distribution along one channel wall differs from the distribution along the other wall, then in general the channel deflects an amount $\Delta\theta$, which is the difference in flow direction far downstream and far upstream

of the region in which the prescribed velocity distribution varies. Thus,

$$\Delta\theta = \theta_d - \theta_u \quad (H1)$$

For large values of $|\Phi - \Phi_o|$ such as occur far upstream and far downstream of the region in which the prescribed velocity varies along the channel walls

$$\cosh^2 (\Phi - \Phi_o) \gg \cos^2 (\Psi - \Psi_o)$$

so that from equation (47)

$$G_o = G_\pi = -2[|\Phi - \Phi_o| - \log_e 2] \quad (\text{H2})$$

Far upstream $\Phi_o < \Phi$ so that

$$|\Phi - \Phi_o| = (\Phi - \Phi_o)$$

and because V is harmonic

$$\int_{-\infty}^{\infty} \left[\left(\frac{\partial \log_e V}{\partial \Phi} \right)_{\frac{\pi}{2}} - \left(\frac{\partial \log_e V}{\partial \Phi} \right)_0 \right] d\Phi = 0$$

so that equation (H2) substituted into equation (46) gives

$$\theta_u = \frac{1}{\pi} \int_{-\infty}^{\infty} \Phi \left[\left(\frac{\partial \log_e V}{\partial \Phi} \right)_{\frac{\pi}{2}} - \left(\frac{\partial \log_e V}{\partial \Phi} \right)_0 \right] d\Phi \quad (\text{H3})$$

Likewise, far downstream $\Phi_o > \Phi$ so that

$$|\Phi - \Phi_o| = -(\Phi - \Phi_o)$$

and equation (H2) substituted into equation (46) gives

$$\theta_d = \frac{-1}{\pi} \int_{-\infty}^{\infty} \Phi \left[\left(\frac{\partial \log_e V}{\partial \Phi} \right)_{\frac{\pi}{2}} - \left(\frac{\partial \log_e V}{\partial \Phi} \right)_0 \right] d\Phi \quad (\text{H4})$$

From equations (H1), (H3), and (H4)

$$\Delta\theta = \frac{-2}{\pi} \int_{-\infty}^{\infty} \Phi \left[\left(\frac{\partial \log_e V}{\partial \Phi} \right)_{\frac{\pi}{2}} - \left(\frac{\partial \log_e V}{\partial \Phi} \right)_0 \right] d\Phi \quad (\text{H5})$$

Equation (H5) determines the channel turning angle $\Delta\theta$.

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[Prescribed variation in Q with arc length s along channel walls plotted in fig. 2: $Q_1=0.5$, $Q_2=1.0$, $\Delta\theta=89.36^\circ$

ψ	0		0.125		0.250		0.375		0.500		0.625		0.750		0.875		1.000	
	Q	θ	Q	θ	Q	θ	Q	θ	Q	θ	Q	θ	Q	θ	Q	θ	Q	θ
-2.000	0.5000	0	0.5000	0	0.5000	0	0.5000	0	0.5000	0	0.5000	0	0.5000	0	0.5000	0	0.5000	0
-1.875	.5000	.01	.5000	.01	.5000	.00	.5000	.00	.5000	.00	.5000	.00	.5000	.00	.5000	-.01	.5000	-.01
-1.750	.5000	.01	.5000	.01	.5000	.01	.5000	.00	.5000	.00	.5000	.00	.5000	-.01	.5000	-.01	.5000	-.01
-1.625	.5000	.01	.5000	.01	.5001	.01	.5001	.01	.5001	.00	.5001	-.01	.5001	-.01	.5000	-.01	.5000	-.01
-1.500	.5000	.02	.5001	.02	.5001	.01	.5001	.01	.5001	.01	.5001	-.01	.5001	-.01	.5001	-.02	.5000	-.02
-1.375	.5000	.03	.5001	.03	.5002	.02	.5002	.01	.5002	.00	.5002	-.01	.5002	-.02	.5001	-.03	.5000	-.03
-1.250	.5000	.04	.5001	.04	.5002	.03	.5003	.02	.5003	.00	.5003	-.02	.5002	-.03	.5001	-.04	.5000	-.04
-1.125	.5000	.06	.5002	.06	.5004	.04	.5005	.03	.5005	.00	.5005	-.03	.5004	-.05	.5002	-.06	.5000	-.06
-1.000	.5000	.09	.5003	.08	.5006	.07	.5007	.04	.5008	.00	.5007	-.04	.5005	-.07	.5003	-.08	.5000	-.09
-.875	.5000	.14	.5005	.12	.5009	.10	.5011	.05	.5012	.00	.5011	-.06	.5008	-.10	.5004	-.12	.5000	-.14
-.750	.5000	.20	.5007	.18	.5013	.14	.5016	.08	.5017	-.01	.5016	-.09	.5012	-.14	.5006	-.18	.5000	-.20
-.625	.5000	.30	.5011	.27	.5019	.21	.5025	.11	.5026	-.01	.5023	-.13	.5018	-.21	.5009	-.27	.5000	-.30
-.500	.5000	.45	.5016	.40	.5029	.30	.5037	.16	.5038	-.02	.5034	-.19	.5026	-.31	.5014	-.39	.5000	-.45
-.375	.5000	.69	.5025	.60	.5044	.45	.5055	.24	.5056	-.04	.5050	-.28	.5037	-.45	.5019	-.56	.5000	-.69
-.250	.5000	1.04	.5039	.89	.5068	.65	.5082	.32	.5083	-.09	.5072	-.43	.5053	-.67	.5028	-.83	.5000	-.99
-.125	.5000	1.63	.5065	1.34	.5107	.94	.5124	.44	.5121	-.17	.5103	-.64	.5074	-.98	.5039	-1.19	.5000	-1.26
0	.5000	2.73	.5115	2.04	.5171	1.30	.5187	.50	.5175	-.36	.5145	-.99	.5103	-1.44	.5053	-1.71	.5000	-1.80
.125	.5097	5.06	.5226	3.21	.5276	1.64	.5278	.34	.5249	-.70	.5200	-1.50	.5139	-2.07	.5071	-2.41	.5000	-2.52
.250	.5354	6.83	.5424	4.02	.5433	1.73	.5402	-.03	.5344	-1.27	.5269	-2.24	.5184	-2.93	.5093	-3.34	.5000	-3.46
.375	.5715	7.36	.5692	4.02	.5637	1.32	.5557	-.75	.5460	-2.18	.5351	-3.29	.5236	-4.07	.5118	-4.54	.5000	-4.68
.500	.6134	6.69	.6006	3.19	.5874	.33	.5736	-1.89	.5592	-3.46	.5444	-4.66	.5295	-5.51	.5146	-6.01	.5000	-6.16
.625	.6576	4.95	.6344	1.52	.6132	-1.31	.5930	-3.55	.5735	-5.19	.5544	-6.43	.5357	-7.31	.5176	-7.83	.5000	-7.99
.750	.7018	2.40	.6690	-.82	.6399	-3.52	.6132	-5.69	.5883	-7.33	.5647	-8.57	.5422	-9.45	.5207	-9.97	.5000	-10.13
.875	.7448	-.81	.7030	-3.76	.6663	-6.26	.6334	-8.31	.6032	-9.89	.5751	-11.09	.5487	-11.95	.5237	-12.46	.5000	-12.62
1.000	.7855	-4.52	.7356	-7.18	.6919	-9.46	.6530	-11.34	.6177	-12.84	.5852	-13.98	.5550	-14.79	.5267	-15.27	.5000	-15.43
1.125	.8235	-8.64	.7662	-11.01	.7162	-13.05	.6717	-14.76	.6316	-16.14	.5949	-17.20	.5611	-17.96	.5296	-18.41	.5000	-18.56
1.250	.8583	-13.03	.7945	-15.16	.7387	-16.97	.6891	-18.51	.6446	-19.76	.6041	-20.74	.5668	-21.44	.5323	-21.86	.5000	-22.00
1.375	.8898	-17.77	.8202	-19.59	.7593	-21.17	.7052	-22.53	.6567	-23.66	.6126	-24.56	.5721	-25.19	.5348	-25.58	.5000	-25.71
1.500	.9177	-22.67	.8432	-24.22	.7778	-25.60	.7199	-26.79	.6678	-27.81	.6204	-28.62	.5771	-29.20	.5371	-29.56	.5000	-29.68
1.625	.9418	-27.72	.8633	-29.03	.7944	-30.20	.7331	-31.25	.6779	-32.16	.6278	-32.90	.5817	-33.44	.5393	-33.90	.5000	-33.90
1.750	.9620	-32.88	.8806	-33.95	.8089	-34.94	.7450	-35.84	.6873	-36.67	.6347	-37.37	.5862	-37.89	.5414	-38.21	.5000	-38.34
1.875	.9782	-38.10	.8949	-38.94	.8215	-39.75	.7558	-40.54	.6962	-41.32	.6415	-42.01	.5908	-42.53	.5437	-42.86	.5000	-43.02
2.000	.9901	-43.30	.9065	-43.91	.8325	-44.58	.7659	-45.28	.7050	-46.04	.6486	-46.78	.5959	-47.35	.5463	-47.73	.5000	-47.94
2.125	.9975	-48.43	.9153	-48.83	.8422	-49.36	.7757	-50.02	.7143	-50.80	.6569	-51.66	.6023	-52.35	.5499	-52.86	.5000	-53.19
2.250	1.0000	-53.34	.9218	-53.57	.8510	-54.00	.7858	-54.65	.7249	-55.51	.6671	-56.56	.6112	-57.49	.5560	-58.32	.5000	-59.04
2.375	1.0000	-57.83	.9271	-58.01	.8596	-58.44	.7968	-59.14	.7374	-60.11	.6805	-61.28	.6248	-62.68	.5686	-64.31	.5097	-66.18
2.500	1.0000	-62.00	.9321	-62.16	.8687	-62.63	.8091	-63.40	.7524	-64.48	.6977	-65.82	.6442	-67.65	.5905	-69.98	.5354	-72.81
2.625	1.0000	-66.84	.9374	-66.01	.8785	-66.51	.8228	-67.35	.7697	-68.52	.7186	-70.02	.6689	-72.14	.6200	-74.88	.5715	-78.23
2.750	1.0000	-69.39	.9429	-69.55	.8890	-70.07	.8378	-70.96	.7891	-72.20	.7424	-73.82	.6975	-76.07	.6544	-78.95	.6134	-82.46
2.875	1.0000	-72.57	.9487	-72.74	.8999	-73.27	.8537	-74.17	.8097	-75.44	.7680	-77.11	.7285	-79.37	.6915	-82.21	.6576	-85.65
3.000	1.0000	-75.43	.9544	-75.59	.9111	-76.12	.8699	-77.02	.8309	-78.27	.7943	-79.92	.7603	-82.10	.7292	-84.80	.7018	-88.03
3.125	1.0000	-77.93	.9601	-78.10	.9221	-78.60	.8860	-79.46	.8521	-80.67	.8205	-82.26	.7918	-84.30	.7603	-86.80	.7448	-89.76
3.250	1.0000	-80.11	.9656	-80.27	.9327	-80.75	.9016	-81.55	.8726	-82.69	.8460	-84.17	.8222	-86.05	.8018	-88.32	.7855	-90.99
3.375	1.0000	-81.97	.9708	-82.11	.9427	-82.56	.9164	-83.30	.8920	-84.35	.8700	-85.71	.8508	-87.41	.8351	-89.44	.8235	-91.81
3.500	1.0000	-83.54	.9755	-83.67	.9521	-84.07	.9301	-84.74	.9100	-85.69	.8923	-86.92	.8773	-88.43	.8658	-90.22	.8583	-92.30
3.625	1.0000	-84.84	.9798	-84.96	.9605	-85.32	.9426	-85.91	.9264	-86.75	.9125	-87.83	.9014	-89.16	.8936	-90.72	.8898	-92.62
3.750	1.0000	-85.90	.9837	-86.01	.9680	-86.32	.9537	-86.84	.9410	-87.57	.9305	-88.51	.9228	-89.65	.9183	-90.99	.9177	-92.52
3.875	1.0000	-86.75	.9870	-86.84	.9746	-87.11	.9634	-87.56	.9538	-88.18	.9462	-88.98	.9414	-89.94	.9397	-91.06	.9418	-92.34
4.000	1.0000	-87.43	.9898	-87.50	.9802	-87.73	.9717	-88.10	.9646	-88.62	.9596	-89.25	.9571	-90.07	.9577	-90.98	.9620	-92.01
4.125	1.0000	-87.95	.9922	-88.01	.9849	-88.20	.9786	-88.50	.9736	-88.92	.9705	-89.48	.9698	-90.77	.9722	-90.98	.9782	-91.57
4.250	1.0000	-88.34	.9942	-88.39	.9887	-88.54	.9842	-88.78	.9808	-89.11	.9792	-89.53	.9798	-89.99	.9831	-90.50	.9901	-91.06
4.375	1.0000	-88.64	.9957	-88.68	.9917	-88.79	.9885	-88.98	.9864	-89.23	.9857	-89.55	.9869	-89.87	.9905	-90.19	.9975	-90.52
4.500	1.0000	-88.85	.9969	-88.88	.9941	-88.97	.9919	-89.10	.9905	-89.29	.9904	-89.53	.9917	-89.73	.9948	-89.90	1.0000	-90.03
4.625	1.0000	-89.01	.9978	-89.03	.9958	-89.09	.9943	-89.19	.9935	-89.33	.9935	-89.49	.9946	-89.62	.9969	-89.72	1.0000	-89.78
4.750	1.0000	-89.11	.9984	-89.13	.9971	-89.18	.9961	-89.25	.9956	-89.34	.9957	-89.45	.9965	-89.54	.9980	-89.60	1.0000	-89.64
4.875	1.0000	-89.19	.9989	-89.21	.9980	-89.24	.9973	-89.29	.9970	-89.35	.9971	-89.43	.9977	-89.48	.9987	-89.52	1.0000	-89.54
5.000	1.0000	-89.24	.9993	-89.25	.9986	-89.28	.9982	-89.31	.9980	-89.36	.9981	-89.41	.9985	-89.44	.9992	-89.47	1.0000	-89.48
5.125	1.0000	-89.28	.9995	-89.29	.9991	-89.30	.9988	-89.33	.9986	-89.36	.9987	-89.39	.9990	-89.42	.9995	-89.43	1.0000	-89.44
5.250	1.0000	-89.31	.9997	-89.31	.9994	-89.32	.9992	-89.34	.9991	-89.36	.9992	-89.38	.9994	-89.40	.9997	-89.41	1.0000	-89.42
5.375	1.0000	-89.32	.9998	-89.33	.9996	-89.33	.9995	-89.35	.9994	-89.36	.9995	-89.38	.9996	-89.39	.9998	-89.39	1.0000	-89.40
5.500	1.0000	-89.34	.9999	-89.34	.9997	-89.34	.9997	-89.35	.9996	-89.36	.9996	-89.37	.9997	-89.38	.9999	-89.38	1.0000	-89.39
5.625	1.0000	-89.34	.9999	-89.35	.9998	-89.35	.9998	-89.35	.9998	-89.36	.9998	-89.37	.9998	-89.37	.9999	-89.38	1.0000	-89.38
5.750	1.0000	-89.35	.9999	-89.35	.9999	-89.35	.9999	-89.36	.9999	-89.36	.9999	-89.37	.9999	-89.37	.9999	-89.37	1.0000	-89.37
5.875	1.0000	-89.35	1.0000	-89.35	1.0000	-89.36	.9999	-89.36	.9999	-89.36	.9999	-89.37	1.0000	-89.37	1.0000	-89.37	1.0000	-89.37
6.000	1.0000	-89.36	1.0000	-89.36	1.0000	-89.36	1.0000	-89.36	1.0000	-89.36	1.0000	-89.36	1.0000	-89.37	1.0000	-89.37	1.0000	-89.37

TABLE II—DISTRIBUTION OF PHYSICAL COORDINATES x AND y IN TRANSFORMED $\phi\psi$ -PLANE FOR EXAMPLE III (ELBOW WITH INCOMPRESSIBLE FLOW)[Prescribed variation in Q with arc length s along channel walls plotted in fig. 2; $Q_a=0.5$, $Q_b=1.0$, $\Delta\theta=89.36^\circ$]

ψ ϕ	0		0.125		0.250		0.375		0.500		0.625		0.750		0.875		1.000	
	x	y	x	y	x	y	x	y	x	y	x	y	x	y	x	y	x	y
-2.000	-3.978	-0.998	-3.978	-0.748	-3.978	-0.498	-3.978	-0.248	-3.978	0.002	-3.978	0.252	-3.978	0.502	-3.978	0.752	-3.978	1.002
-1.875	-3.727	-0.998	-3.728	-0.748	-3.728	-0.498	-3.728	-0.248	-3.728	0.002	-3.728	0.252	-3.728	0.502	-3.728	0.752	-3.727	1.002
-1.750	-3.477	-0.998	-3.478	-0.748	-3.478	-0.498	-3.478	-0.248	-3.478	0.002	-3.478	0.252	-3.478	0.502	-3.478	0.752	-3.477	1.002
-1.625	-3.227	-0.998	-3.228	-0.748	-3.228	-0.498	-3.228	-0.248	-3.228	0.002	-3.228	0.252	-3.228	0.502	-3.228	0.752	-3.227	1.002
-1.500	-2.977	-0.998	-2.977	-0.748	-2.978	-0.498	-2.978	-0.248	-2.978	0.002	-2.978	0.252	-2.978	0.502	-2.978	0.752	-2.977	1.002
-1.375	-2.727	-0.998	-2.728	-0.748	-2.728	-0.498	-2.728	-0.248	-2.728	0.002	-2.728	0.252	-2.728	0.502	-2.728	0.752	-2.727	1.002
-1.250	-2.477	-0.997	-2.478	-0.747	-2.478	-0.498	-2.478	-0.248	-2.478	0.002	-2.478	0.252	-2.478	0.502	-2.478	0.752	-2.477	1.002
-1.125	-2.227	-0.997	-2.228	-0.747	-2.228	-0.497	-2.228	-0.248	-2.228	0.002	-2.228	0.252	-2.228	0.502	-2.228	0.752	-2.227	1.002
-1.000	-1.977	-0.997	-1.978	-0.747	-1.978	-0.497	-1.978	-0.247	-1.978	0.002	-1.978	0.252	-1.978	0.501	-1.978	0.751	-1.977	1.001
-0.875	-1.727	-0.996	-1.728	-0.747	-1.728	-0.497	-1.729	-0.247	-1.729	0.002	-1.729	0.252	-1.729	0.501	-1.729	0.751	-1.727	1.001
-0.750	-1.477	-0.996	-1.478	-0.746	-1.479	-0.496	-1.479	-0.247	-1.480	0.002	-1.479	0.251	-1.479	0.501	-1.478	0.750	-1.477	1.000
-0.625	-1.227	-0.995	-1.229	-0.745	-1.230	-0.496	-1.230	-0.247	-1.231	0.002	-1.230	0.251	-1.230	0.500	-1.229	0.749	-1.227	0.999
-0.500	-0.977	-0.993	-0.979	-0.743	-0.981	-0.495	-0.982	-0.246	-0.982	0.002	-0.982	0.250	-0.931	0.499	-0.979	0.748	-0.977	0.997
-0.375	-0.727	-0.990	-0.730	-0.741	-0.733	-0.493	-0.734	-0.245	-0.735	0.002	-0.734	0.249	-0.732	0.497	-0.730	0.746	-0.727	0.995
-0.250	-0.478	-0.987	-0.482	-0.738	-0.485	-0.491	-0.487	-0.244	-0.488	0.002	-0.487	0.248	-0.484	0.495	-0.481	0.743	-0.477	0.992
-0.125	-0.228	-0.981	-0.235	-0.733	-0.239	-0.487	-0.242	-0.242	-0.243	0.001	-0.241	0.245	-0.238	0.491	-0.233	0.739	-0.227	0.987
0	0.022	-0.972	0.011	-0.726	0.004	-0.482	0.000	-0.240	0.000	0.000	0.003	0.242	0.008	0.486	0.015	0.732	0.023	0.981
0.125	0.270	-0.955	0.254	-0.715	0.243	-0.476	0.239	-0.239	0.240	-0.002	0.245	0.237	0.252	0.479	0.261	0.724	0.273	0.971
0.250	0.510	-0.930	0.489	-0.700	0.477	-0.469	0.472	-0.238	0.476	-0.006	0.484	0.229	0.494	0.468	0.507	0.711	0.522	0.958
0.375	0.734	-0.901	0.713	-0.684	0.702	-0.463	0.700	-0.239	0.707	-0.013	0.719	0.218	0.734	0.454	0.751	0.695	0.772	0.941
0.500	0.942	-0.875	0.925	-0.670	0.918	-0.460	0.921	-0.244	0.933	-0.024	0.950	0.202	0.970	0.434	0.994	0.672	1.021	0.917
0.625	1.139	-0.855	1.129	-0.662	1.128	-0.461	1.136	-0.254	1.153	-0.041	1.177	0.180	1.204	0.408	1.234	0.643	1.269	0.886
0.750	1.322	-0.843	1.321	-0.660	1.327	-0.470	1.343	-0.271	1.367	-0.064	1.398	0.151	1.433	0.375	1.472	0.607	1.516	0.848
0.875	1.495	-0.840	1.502	-0.668	1.518	-0.486	1.542	-0.295	1.575	-0.095	1.614	0.113	1.658	0.332	1.707	0.560	1.761	0.798
1.000	1.658	-0.848	1.675	-0.684	1.701	-0.511	1.734	-0.328	1.776	-0.135	1.824	0.067	1.879	0.279	1.938	0.503	2.003	0.738
1.125	1.812	-0.866	1.840	-0.710	1.875	-0.545	1.918	-0.370	1.969	-0.185	2.028	0.010	2.094	0.216	2.165	0.435	2.242	0.665
1.250	1.958	-0.893	1.995	-0.747	2.041	-0.590	2.094	-0.423	2.156	-0.245	2.225	-0.058	2.302	0.142	2.386	0.354	2.477	0.578
1.375	2.096	-0.931	2.143	-0.793	2.198	-0.644	2.262	-0.485	2.334	-0.316	2.415	-0.137	2.501	0.055	2.601	0.260	2.705	0.477
1.500	2.226	-0.979	2.282	-0.849	2.348	-0.709	2.422	-0.558	2.504	-0.398	2.596	-0.228	2.697	-0.044	2.807	0.152	2.927	0.361
1.625	2.347	-1.036	2.413	-0.914	2.488	-0.783	2.572	-0.642	2.665	-0.491	2.768	-0.330	2.882	-0.156	3.005	0.030	3.139	0.229
1.750	2.461	-1.102	2.535	-0.989	2.621	-0.867	2.713	-0.735	2.816	-0.594	2.930	-0.444	3.055	-0.281	3.192	-0.106	3.341	0.082
1.875	2.566	-1.177	2.649	-1.073	2.741	-0.960	2.844	-0.838	2.957	-0.708	3.081	-0.569	3.218	-0.418	3.368	-0.255	3.531	-0.081
2.000	2.662	-1.260	2.753	-1.164	2.853	-1.061	2.964	-0.950	3.086	-0.831	3.219	-0.705	3.367	-0.567	3.529	-0.418	3.706	-0.259
2.125	2.749	-1.350	2.847	-1.264	2.955	-1.170	3.074	-1.070	3.202	-0.963	3.344	-0.850	3.501	-0.727	3.675	-0.594	3.865	-0.451
2.250	2.828	-1.447	2.932	-1.369	3.047	-1.286	3.172	-1.197	3.307	-1.102	3.455	-1.002	3.620	-0.895	3.803	-0.780	4.004	-0.658
2.375	2.899	-1.550	3.005	-1.481	3.128	-1.407	3.258	-1.329	3.398	-1.247	3.550	-1.160	3.721	-1.070	3.910	-0.977	4.119	-0.880
2.500	2.961	-1.659	3.075	-1.598	3.199	-1.533	3.332	-1.466	3.476	-1.395	3.631	-1.322	3.803	-1.249	3.994	-1.177	4.203	-1.105
2.625	3.016	-1.771	3.134	-1.718	3.260	-1.663	3.396	-1.605	3.541	-1.546	3.697	-1.485	3.869	-1.428	4.056	-1.374	4.259	-1.324
2.750	3.064	-1.886	3.184	-1.841	3.313	-1.794	3.450	-1.746	3.595	-1.697	3.751	-1.648	3.919	-1.604	4.100	-1.565	4.294	-1.532
2.875	3.105	-2.005	3.227	-1.966	3.357	-1.927	3.494	-1.887	3.638	-1.847	3.792	-1.808	3.956	-1.775	4.130	-1.748	4.314	-1.727
3.000	3.139	-2.125	3.263	-2.092	3.393	-2.060	3.530	-2.028	3.673	-1.996	3.824	-1.965	3.983	-1.941	4.150	-1.923	4.324	-1.911
3.125	3.168	-2.246	3.292	-2.219	3.423	-2.193	3.559	-2.167	3.700	-2.142	3.848	-2.119	4.002	-2.101	4.162	-2.089	4.328	-2.084
3.250	3.191	-2.369	3.317	-2.347	3.447	-2.326	3.582	-2.305	3.721	-2.285	3.866	-2.267	4.015	-2.256	4.168	-2.249	4.326	-2.247
3.375	3.211	-2.493	3.337	-2.475	3.466	-2.457	3.600	-2.441	3.737	-2.426	3.879	-2.413	4.024	-2.405	4.171	-2.401	4.323	-2.402
3.500	3.227	-2.617	3.362	-2.602	3.481	-2.588	3.614	-2.576	3.749	-2.564	3.888	-2.555	4.029	-2.549	4.172	-2.548	4.317	-2.551
3.625	3.239	-2.741	3.365	-2.729	3.493	-2.719	3.624	-2.709	3.758	-2.693	3.894	-2.693	4.032	-2.690	4.171	-2.690	4.311	-2.693
3.750	3.249	-2.865	3.375	-2.856	3.503	-2.848	3.633	-2.840	3.764	-2.834	3.898	-2.829	4.033	-2.827	4.169	-2.828	4.305	-2.832
3.875	3.257	-2.990	3.383	-2.983	3.510	-2.976	3.639	-2.971	3.769	-2.965	3.901	-2.962	4.033	-2.961	4.166	-2.962	4.299	-2.966
4.000	3.264	-3.115	3.389	-3.109	3.516	-3.104	3.644	-3.100	3.773	-3.096	3.903	-3.093	4.033	-3.093	4.164	-3.094	4.294	-3.097
4.125	3.269	-3.240	3.394	-3.235	3.520	-3.231	3.648	-3.228	3.776	-3.225	3.905	-3.223	4.033	-3.222	4.162	-3.223	4.290	-3.226
4.250	3.273	-3.365	3.398	-3.361	3.524	-3.358	3.651	-3.355	3.778	-3.352	3.906	-3.351	4.033	-3.350	4.160	-3.351	4.287	-3.353
4.375	3.276	-3.490	3.401	-3.487	3.527	-3.484	3.653	-3.482	3.780	-3.480	3.907	-3.478	4.033	-3.478	4.160	-3.478	4.285	-3.478
4.500	3.279	-3.615	3.404	-3.612	3.529	-3.610	3.655	-3.608	3.781	-3.606	3.908	-3.605	4.034	-3.604	4.159	-3.603	4.285	-3.603
4.625	3.281	-3.740	3.406	-3.738	3.531	-3.736	3.657	-3.734	3.783	-3.732	3.909	-3.731	4.034	-3.730	4.160	-3.729	4.285	-3.729
4.750	3.283	-3.8																

TABLE III—DISTRIBUTION OF VELOCITY q AND FLOW DIRECTION θ IN TRANSFORMED $\phi^*\psi^*$ -PLANE FOR EXAMPLE IV (ELBOW WITH LINEARIZED COMPRESSIBLE FLOW)[Prescribed variation in Q with arc length s along channel walls plotted in fig. 2; $Q_0=0.5$, $Q_d=1.0$, $q_d=0.80176$, $\Delta\psi^*=0.73782$, $\Delta\theta=104.07^\circ$]

$\frac{\psi^*}{\Delta\psi^*}$	0		$\frac{1}{8}$		$\frac{1}{4}$		$\frac{3}{8}$		$\frac{1}{2}$		$\frac{5}{8}$		$\frac{3}{4}$		1.0	
	q	θ	q	θ	q	θ	q	θ	q	θ	q	θ	q	θ	q	θ
-11/6	0.4009	0	0.4009	0	0.4009	0	0.4009	0	0.4009	0	0.4009	0	0.4009	0	0.4009	0
-10/6	.4009	.01	.4009	.01	.4009	.00	.4009	.00	.4009	.00	.4009	.00	.4009	-.01	.4009	-.01
-9/6	.4009	.01	.4009	.01	.4009	.01	.4010	.00	.4010	.00	.4009	-.01	.4009	-.01	.4009	-.01
-8/6	.4009	.02	.4010	.02	.4010	.01	.4010	.00	.4010	.00	.4010	-.01	.4010	-.02	.4009	-.02
-7/6	.4009	.03	.4010	.03	.4011	.01	.4011	.00	.4011	.00	.4010	-.01	.4010	-.02	.4009	-.03
-6/6	.4009	.05	.4011	.04	.4012	.02	.4013	.00	.4012	.00	.4012	-.02	.4011	-.04	.4009	-.05
-5/6	.4009	.08	.4012	.07	.4015	.04	.4015	.00	.4014	.00	.4014	-.04	.4012	-.07	.4009	-.08
-4/6	.4009	.14	.4015	.12	.4019	.07	.4020	.00	.4018	.00	.4014	-.07	.4014	-.11	.4009	-.13
-3/6	.4009	.24	.4019	.20	.4025	.11	.4027	-.01	.4024	-.11	.4017	-.11	.4017	-.18	.4009	-.21
-2/6	.4009	.40	.4026	.33	.4037	.17	.4039	-.03	.4033	-.20	.4022	-.31	.4009	-.35	.4009	-.35
-1/6	.4009	.70	.4041	.57	.4057	.27	.4059	-.05	.4048	-.33	.4030	-.50	.4009	-.56	.4009	-.56
0	.4009	1.31	.4070	.96	.4093	.36	.4090	-.17	.4071	-.58	.4042	-.84	.4009	-.92	.4009	-.92
1/6	.4072	2.82	.4141	1.49	.4155	.43	.4139	-.40	.4104	-.99	.4058	-1.34	.4009	-1.45	.4009	-1.45
2/6	.4243	3.88	.4268	1.77	.4251	.24	.4207	-.87	.4148	-1.63	.4080	-2.07	.4009	-2.22	.4009	-2.22
3/6	.4489	4.00	.4444	1.46	.4377	-.38	.4295	-1.70	.4202	-2.59	.4106	-3.11	.4009	-3.28	.4009	-3.28
4/6	.4780	3.17	.4654	.50	.4526	-1.49	.4396	-2.92	.4265	-3.90	.4135	-4.46	.4009	-4.65	.4009	-4.65
5/6	.5094	1.44	.4882	-1.17	.4689	-3.16	.4506	-4.62	.4333	-5.63	.4167	-6.21	.4009	-6.41	.4009	-6.41
6/6	.5415	-.98	.5118	-3.43	.4857	-5.34	.4621	-6.76	.4403	-7.75	.4200	-8.33	.4009	-8.52	.4009	-8.52
7/6	.5732	-4.00	.5352	-6.23	.5024	-8.01	.4734	-9.35	.4472	-10.29	.4232	-10.85	.4009	-11.04	.4009	-11.04
8/6	.6038	-7.48	.5578	-9.48	.5186	-11.10	.4844	-12.34	.4539	-13.22	.4262	-13.74	.4009	-13.91	.4009	-13.91
9/6	.6329	-11.35	.5793	-13.12	.5340	-14.57	.4947	-15.70	.4601	-16.49	.4291	-16.97	.4009	-17.13	.4009	-17.13
10/6	.6602	-15.52	.5994	-17.09	.5482	-18.37	.5043	-19.38	.4659	-20.09	.4317	-20.52	.4009	-20.67	.4009	-20.67
11/6	.6855	-19.96	.6178	-21.33	.5613	-22.46	.5131	-23.34	.4712	-23.98	.4341	-24.37	.4009	-24.49	.4009	-24.49
12/6	.7086	-24.62	.6346	-25.80	.5732	-26.79	.5210	-27.56	.4759	-28.12	.4362	-28.46	.4009	-28.58	.4009	-28.58
13/6	.7293	-29.46	.6496	-30.47	.5837	-31.32	.5281	-32.00	.4801	-32.49	.4381	-32.79	.4009	-32.89	.4009	-32.89
14/6	.7477	-34.45	.6629	-35.31	.5931	-36.03	.5343	-36.62	.4838	-37.05	.4398	-37.31	.4009	-37.40	.4009	-37.40
15/6	.7636	-39.56	.6744	-40.27	.6013	-40.89	.5398	-41.39	.4872	-41.77	.4413	-42.00	.4009	-42.08	.4009	-42.08
16/6	.7769	-44.76	.6842	-45.34	.6084	-45.86	.5447	-46.30	.4902	-46.64	.4427	-46.85	.4009	-46.93	.4009	-46.93
17/6	.7875	-50.07	.6924	-50.47	.6146	-50.90	.5492	-51.30	.4931	-51.63	.4440	-51.85	.4009	-51.93	.4009	-51.93
18/6	.7953	-55.27	.6991	-55.61	.6202	-55.98	.5537	-56.36	.4961	-56.72	.4456	-56.98	.4009	-57.08	.4009	-57.08
19/6	.8001	-60.49	.7045	-60.72	.6257	-61.06	.5586	-61.47	.4999	-61.91	.4477	-62.28	.4009	-62.44	.4009	-62.44
20/6	.8018	-65.55	.7091	-65.68	.6315	-66.03	.5647	-66.53	.5055	-67.13	.4515	-67.79	.4009	-68.16	.4009	-68.16
21/6	.8018	-70.32	.7135	-70.45	.6385	-70.85	.5730	-71.49	.5142	-72.37	.4598	-73.46	.4009	-74.80	.4009	-74.80
22/6	.8018	-74.81	.7186	-74.96	.6471	-75.43	.5840	-76.22	.5272	-77.36	.4745	-78.91	.4009	-81.03	.4009	-81.03
23/6	.8018	-78.97	.7245	-79.15	.6573	-79.68	.5978	-80.59	.5441	-81.93	.4948	-83.79	.4009	-86.23	.4009	-86.23
24/6	.8018	-82.82	.7311	-83.01	.6690	-83.59	.6138	-84.57	.5641	-86.02	.5190	-88.01	.4009	-90.69	.4009	-90.69
25/6	.8018	-86.28	.7381	-86.48	.6816	-87.07	.6313	-88.08	.5860	-89.55	.5455	-91.55	.4009	-94.16	.4009	-94.16
26/6	.8018	-89.37	.7452	-89.56	.6947	-90.15	.6494	-91.14	.6089	-92.57	.5729	-94.48	.4009	-96.93	.4009	-96.93
27/6	.8018	-92.07	.7523	-92.25	.7077	-92.81	.6676	-93.75	.6318	-95.10	.6002	-96.88	.4009	-99.11	.4009	-99.11
28/6	.8018	-94.40	.7591	-94.57	.7203	-95.09	.6852	-95.97	.6540	-97.21	.6268	-98.83	.4009	-100.83	.4009	-100.83
29/6	.8018	-96.39	.7654	-96.54	.7321	-97.02	.7020	-97.82	.6753	-98.94	.6522	-100.39	.4009	-102.17	.4009	-102.17
30/6	.8018	-98.05	.7713	-98.20	.7432	-98.62	.7177	-99.34	.6952	-100.34	.6759	-101.63	.4009	-103.19	.4009	-103.19
31/6	.8018	-99.44	.7766	-99.56	.7532	-99.94	.7321	-100.58	.7135	-101.46	.6978	-102.59	.4009	-103.95	.4009	-103.95
32/6	.8018	-100.56	.7813	-100.67	.7623	-101.01	.7451	-101.56	.7301	-102.33	.7178	-103.31	.4009	-104.49	.4009	-104.49
33/6	.8018	-101.47	.7855	-101.57	.7702	-101.85	.7566	-102.33	.7449	-102.99	.7357	-103.83	.4009	-104.84	.4009	-104.84
34/6	.8018	-102.18	.7890	-102.26	.7772	-102.51	.7667	-102.91	.7580	-103.47	.7515	-104.18	.4009	-105.03	.4009	-105.03
35/6	.8018	-102.73	.7921	-102.80	.7831	-103.00	.7753	-103.34	.7691	-103.80	.7651	-104.39	.4009	-105.16	.4009	-105.16
36/6	.8018	-103.14	.7946	-103.20	.7880	-103.37	.7825	-103.64	.7785	-104.01	.7764	-104.48	.4009	-105.25	.4009	-105.25
37/6	.8018	-103.45	.7966	-103.49	.7920	-103.62	.7883	-103.83	.7860	-104.12	.7855	-104.49	.4009	-105.31	.4009	-105.31
38/6	.8018	-103.66	.7982	-103.69	.7950	-103.79	.7927	-103.95	.7917	-104.16	.7923	-104.42	.4009	-105.35	.4009	-105.35
39/6	.8018	-103.81	.7994	-103.83	.7973	-103.90	.7960	-104.02	.7957	-104.16	.7968	-104.33	.4009	-105.38	.4009	-105.38
40/6	.8018	-103.90	.8002	-103.92	.7989	-103.97	.7982	-104.05	.7982	-104.14	.7993	-104.23	.4009	-105.40	.4009	-105.40
41/6	.8018	-103.97	.8008	-103.98	.8000	-104.01	.7996	-104.06	.7997	-104.11	.8004	-104.16	.4009	-105.41	.4009	-105.41
42/6	.8018	-104.00	.8011	-104.01	.8006	-104.03	.8004	-104.06	.8005	-104.09	.8010	-104.12	.4009	-105.42	.4009	-105.42
43/6	.8018	-104.03	.8014	-104.03	.8011	-104.05	.8009	-104.06	.8010	-104.08	.8013	-104.10	.4009	-105.43	.4009	-105.43
44/6	.8018	-104.04	.8015	-104.05	.8013	-104.06	.8013	-104.06	.8013	-104.08	.8015	-104.08	.4009	-105.44	.4009	-105.44
45/6	.8018	-104.05	.8016	-104.06	.8015	-104.06	.8015	-104.07	.8015	-104.07	.8016	-104.08	.4009	-105.45	.4009	-105.45
46/6	.8018	-104.06	.8017	-104.06	.8016	-104.06	.8016	-104.07	.8016	-104.07	.8017	-104.07	.4009	-105.46	.4009	-105.46
47/6	.8018	-104.06	.8017	-104.06	.8017	-104.06	.8017	-104.07	.8017	-104.07	.8017	-104.07	.4009	-105.47	.4009	-105.47
48/6	.8018	-104.06	.8017	-104.06	.8017	-104.06	.8017	-104.07	.8017	-104.07	.8017	-104.07	.4009	-105.47	.4009	-105.47
49/6	.8018	-104.06	.8017	-104.06	.8017	-104.06	.8017	-104.07	.8017	-104.07	.8017	-104.07	.4009	-105.47	.4009	-105.47
50/6	.8018	-104.06	.8018	-104.06	.8018	-104.06	.8018	-104.07	.8018	-104.07	.8018	-104.07	.4009	-105.47	.4009	-105.47

TABLE IV—DISTRIBUTION OF PHYSICAL COORDINATES x AND y IN TRANSFORMED $\varphi^*\psi^*$ -PLANE FOR EXAMPLE IV (ELBOW WITH LINERARIZED COMPRESSIBLE FLOW)[Prescribed variation in Q with arc length s along channel walls plotted in fig. 2; $Q_u=0.5$, $Q_d=1.0$, $q_d=0.80176$, $\Delta\psi^*=0.73782$, $\Delta\theta=104.07^\circ$]

$\frac{\psi^*}{\Delta\psi^*}$	0		$\frac{1}{6}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{5}{6}$		1.0	
	x	y	x	y	x	y	x	y	x	y	x	y	x	y
-11/6	-2.466	-0.769	-2.466	-0.512	-2.466	-0.256	-2.466	0.001	-2.466	0.257	-2.466	0.513	-2.466	0.770
-10/6	-2.241	-0.769	-2.241	-0.512	-2.241	-0.256	-2.241	0.001	-2.241	0.257	-2.241	0.513	-2.241	0.770
-9/6	-2.016	-0.769	-2.016	-0.512	-2.016	-0.256	-2.016	0.001	-2.016	0.257	-2.016	0.513	-2.016	0.770
-8/6	-1.791	-0.769	-1.791	-0.512	-1.791	-0.256	-1.791	0.001	-1.791	0.257	-1.791	0.513	-1.791	0.770
-7/6	-1.566	-0.768	-1.566	-0.512	-1.566	-0.256	-1.566	0.001	-1.566	0.257	-1.566	0.513	-1.566	0.770
-6/6	-1.341	-0.768	-1.341	-0.512	-1.341	-0.256	-1.341	0.001	-1.341	0.257	-1.341	0.513	-1.341	0.769
-5/6	-1.116	-0.768	-1.116	-0.512	-1.117	-0.256	-1.117	0.001	-1.117	0.257	-1.116	0.513	-1.116	0.769
-4/6	-0.891	-0.768	-0.892	-0.511	-0.892	-0.255	-0.892	0.001	-0.892	0.256	-0.892	0.513	-0.891	0.769
-3/6	-0.666	-0.767	-0.667	-0.511	-0.668	-0.255	-0.668	0.001	-0.668	0.256	-0.667	0.512	-0.666	0.768
-2/6	-0.441	-0.766	-0.443	-0.510	-0.444	-0.254	-0.444	0.001	-0.444	0.256	-0.443	0.511	-0.441	0.767
-1/6	-0.216	-0.763	-0.219	-0.508	-0.221	-0.254	-0.221	0.000	-0.220	0.255	-0.219	0.510	-0.216	0.765
0	0.008	-0.760	0.003	-0.505	0.000	-0.252	0.000	0.000	0.002	0.253	0.005	0.507	0.009	0.763
1/6	0.233	-0.752	0.223	-0.500	0.219	-0.251	0.219	-0.001	0.222	0.250	0.228	0.503	0.234	0.758
2/6	0.450	-0.739	0.438	-0.494	0.434	-0.249	0.436	-0.003	0.441	0.245	0.449	0.496	0.459	0.751
3/6	0.656	-0.724	0.645	-0.488	0.643	-0.250	0.648	-0.008	0.657	0.237	0.670	0.486	0.684	0.740
4/6	0.851	-0.712	0.844	-0.485	0.846	-0.253	0.855	-0.016	0.870	0.225	0.888	0.472	0.908	0.725
5/6	1.033	-0.704	1.033	-0.485	1.042	-0.261	1.057	-0.029	1.079	0.208	1.104	0.452	1.132	0.703
6/6	1.205	-0.703	1.213	-0.492	1.230	-0.274	1.254	-0.049	1.284	0.184	1.318	0.425	1.355	0.674
7/6	1.366	-0.710	1.385	-0.507	1.411	-0.295	1.445	-0.076	1.485	0.152	1.529	0.389	1.577	0.636
8/6	1.519	-0.725	1.548	-0.529	1.586	-0.325	1.630	-0.111	1.681	0.112	1.737	0.344	1.797	0.587
9/6	1.663	-0.749	1.704	-0.560	1.753	-0.363	1.809	-0.156	1.871	0.061	1.940	0.288	2.013	0.527
10/6	1.798	-0.781	1.852	-0.600	1.913	-0.410	1.981	-0.210	2.056	0.000	2.138	0.221	2.226	0.455
11/6	1.926	-0.822	1.992	-0.649	2.065	-0.466	2.146	-0.274	2.235	-0.072	2.331	0.142	2.434	0.368
12/6	2.046	-0.871	2.124	-0.706	2.209	-0.532	2.304	-0.349	2.406	-0.156	2.517	0.050	2.635	0.268
13/6	2.157	-0.928	2.247	-0.772	2.346	-0.608	2.453	-0.435	2.569	-0.251	2.694	-0.055	2.829	0.153
14/6	2.261	-0.993	2.363	-0.847	2.473	-0.693	2.593	-0.530	2.723	-0.357	2.862	-0.173	3.013	0.024
15/6	2.356	-1.064	2.469	-0.930	2.591	-0.787	2.723	-0.636	2.866	-0.475	3.020	-0.304	3.186	-0.120
16/6	2.443	-1.143	2.567	-1.020	2.700	-0.889	2.843	-0.751	2.999	-0.604	3.166	-0.447	3.346	-0.278
17/6	2.521	-1.228	2.655	-1.117	2.798	-1.000	2.952	-0.875	3.118	-0.743	3.298	-0.601	3.492	-0.449
18/6	2.590	-1.318	2.732	-1.220	2.885	-1.117	3.049	-1.007	3.225	-0.890	3.416	-0.766	3.623	-0.632
19/6	2.650	-1.414	2.800	-1.330	2.960	-1.240	3.132	-1.146	3.317	-1.046	3.518	-0.940	3.736	-0.826
20/6	2.701	-1.514	2.858	-1.443	3.024	-1.369	3.203	-1.290	3.395	-1.208	3.603	-1.122	3.830	-1.030
21/6	2.743	-1.619	2.905	-1.561	3.076	-1.501	3.259	-1.438	3.456	-1.374	3.669	-1.309	3.901	-1.242
22/6	2.777	-1.726	2.942	-1.681	3.117	-1.635	3.303	-1.588	3.501	-1.541	3.715	-1.496	3.947	-1.455
23/6	2.803	-1.836	2.970	-1.803	3.147	-1.770	3.333	-1.737	3.531	-1.707	3.743	-1.680	3.969	-1.661
24/6	2.820	-1.947	2.990	-1.926	3.167	-1.905	3.353	-1.885	3.548	-1.869	3.755	-1.858	3.974	-1.856
25/6	2.831	-2.059	3.001	-2.048	3.177	-2.038	3.362	-2.030	3.554	-2.026	3.756	-2.027	3.966	-2.038
26/6	2.835	-2.171	3.005	-2.169	3.181	-2.169	3.363	-2.171	3.552	-2.177	3.747	-2.189	3.950	-2.209
27/6	2.834	-2.283	3.003	-2.290	3.177	-2.297	3.357	-2.308	3.542	-2.322	3.732	-2.342	3.927	-2.369
28/6	2.827	-2.396	2.996	-2.409	3.168	-2.423	3.345	-2.440	3.527	-2.461	3.712	-2.487	3.900	-2.530
29/6	2.817	-2.508	2.984	-2.527	3.155	-2.547	3.330	-2.570	3.508	-2.596	3.688	-2.626	3.871	-2.683
30/6	2.803	-2.619	2.969	-2.643	3.139	-2.668	3.311	-2.695	3.485	-2.725	3.662	-2.760	3.841	-2.799
31/6	2.786	-2.731	2.951	-2.758	3.119	-2.787	3.289	-2.818	3.461	-2.851	3.635	-2.888	3.809	-2.929
32/6	2.766	-2.841	2.931	-2.872	3.097	-2.904	3.266	-2.938	3.435	-2.973	3.606	-3.012	3.777	-3.055
33/6	2.744	-2.952	2.908	-2.985	3.074	-3.010	3.241	-3.055	3.409	-3.093	3.577	-3.133	3.746	-3.176
34/6	2.721	-3.062	2.885	-3.097	3.049	-3.133	3.215	-3.171	3.381	-3.209	3.548	-3.250	3.714	-3.294
35/6	2.697	-3.172	2.860	-3.209	3.024	-3.246	3.188	-3.284	3.353	-3.324	3.518	-3.366	3.683	-3.409
36/6	2.672	-3.281	2.834	-3.319	2.998	-3.358	3.161	-3.397	3.325	-3.437	3.489	-3.479	3.653	-3.522
37/6	2.646	-3.391	2.808	-3.430	2.971	-3.469	3.134	-3.509	3.297	-3.549	3.460	-3.591	3.623	-3.633
38/6	2.620	-3.500	2.782	-3.540	2.944	-3.579	3.107	-3.619	3.269	-3.660	3.432	-3.701	3.594	-3.744
39/6	2.593	-3.609	2.755	-3.649	2.917	-3.689	3.079	-3.730	3.241	-3.770	3.403	-3.811	3.565	-3.853
40/6	2.566	-3.719	2.728	-3.759	2.890	-3.799	3.052	-3.839	3.214	-3.880	3.376	-3.921	3.537	-3.962
41/6	2.539	-3.828	2.701	-3.868	2.862	-3.908	3.024	-3.949	3.186	-3.990	3.348	-4.030	3.510	-4.071
42/6	2.512	-3.937	2.673	-3.977	2.835	-4.018	2.997	-4.058	3.159	-4.099	3.320	-4.139	3.482	-4.180
43/6	2.484	-4.046	2.646	-4.087	2.808	-4.127	2.970	-4.168	3.131	-4.208	3.293	-4.249	3.455	-4.289
44/6	2.457	-4.155	2.619	-4.196	2.780	-4.236	2.942	-4.277	3.104	-4.317	3.266	-4.358	3.427	-4.398
45/6	2.430	-4.264	2.591	-4.305	2.753	-4.345	2.915	-4.386	3.077	-4.426	3.238	-4.467	3.400	-4.507
46/6	2.402	-4.374	2.564	-4.414	2.726	-4.455	2.887	-4.495	3.049	-4.536	3.211	-4.576	3.372	-4.617
47/6	2.375	-4.483	2.537	-4.523	2.698	-4.564	2.860	-4.604	3.022	-4.645	3.183	-4.685	3.345	-4.726
48/6	2.348	-4.592	2.509	-4.632	2.671	-4.673	2.833	-4.713	2.994	-4.754	3.156	-4.794	3.318	-4.835
49/6	2.320	-4.701	2.482	-4.741	2.644	-4.782	2.805	-4.822	2.967	-4.863	3.129	-4.903	3.290	-4.944
50/6	2.293	-4.810	2.455	-4.851	2.616	-4.891	2.778	-4.932	2.940	-4.972	3.101	-5.013	3.263	-5.053

TABLE VI—DISTRIBUTION OF PHYSICAL COORDINATES x AND y IN TRANSFORMED $\varphi\psi$ -PLANE FOR EXAMPLE V (ELBOW WITH COMPRESSIBLE FLOW ($\gamma=1.4$))[Prescribed variation in Q with arc length s along channel walls plotted in fig. 2; $O_a=0.5$, $O_d=1.0$, $q_d=0.79927$, $\Delta\psi=0.71054$, $\Delta\theta=105.31^\circ$]

$\frac{\varphi}{\Delta\psi}$	0		$\frac{1}{6}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{5}{6}$		1.0	
	x	y	x	y	x	y	x	y	x	y	x	y	x	y
-12/6	-2.832	-0.770	-2.832	-0.513	-2.832	-0.256	-2.832	0.001	-2.832	0.258	-2.832	0.514	-2.832	0.771
-11/6	-2.595	-0.770	-2.595	-0.513	-2.595	-0.256	-2.595	0.001	-2.595	0.258	-2.595	0.514	-2.595	0.771
-10/6	-2.358	-0.770	-2.358	-0.513	-2.358	-0.256	-2.358	0.001	-2.358	0.258	-2.358	0.514	-2.358	0.771
-9/6	-2.122	-0.770	-2.122	-0.513	-2.122	-0.256	-2.122	0.001	-2.122	0.258	-2.122	0.514	-2.122	0.771
-8/6	-1.885	-0.770	-1.885	-0.513	-1.885	-0.256	-1.885	0.001	-1.885	0.258	-1.885	0.514	-1.885	0.771
-7/6	-1.648	-0.770	-1.648	-0.513	-1.648	-0.256	-1.648	0.001	-1.648	0.257	-1.648	0.514	-1.648	0.771
-6/6	-1.411	-0.769	-1.411	-0.513	-1.411	-0.256	-1.412	0.001	-1.411	0.257	-1.411	0.514	-1.411	0.771
-5/6	-1.174	-0.769	-1.175	-0.512	-1.175	-0.256	-1.175	0.001	-1.175	0.257	-1.175	0.514	-1.174	0.771
-4/6	-0.937	-0.769	-0.938	-0.512	-0.939	-0.256	-0.939	0.001	-0.938	0.257	-0.938	0.513	-0.937	0.770
-3/6	-0.781	-0.768	-0.702	-0.511	-0.702	-0.255	-0.703	0.001	-0.702	0.257	-0.702	0.513	-0.701	0.769
-2/6	-0.464	-0.767	-0.466	-0.510	-0.467	-0.255	-0.467	0.001	-0.467	0.256	-0.465	0.512	-0.464	0.768
-1/6	-0.227	-0.764	-0.230	-0.508	-0.232	-0.254	-0.233	0.001	-0.232	0.255	-0.230	0.510	-0.227	0.766
0	0.009	-0.760	0.004	-0.505	0.000	-0.252	0.000	0.000	0.002	0.253	0.005	0.507	0.010	0.763
1/6	0.245	-0.750	0.235	-0.499	0.230	-0.250	0.230	-0.001	0.234	0.250	0.240	0.602	0.247	0.758
2/6	0.473	-0.735	0.460	-0.492	0.456	-0.249	0.458	-0.004	0.463	0.244	0.473	0.495	0.484	0.749
3/6	0.688	-0.719	0.677	-0.485	0.675	-0.249	0.680	-0.009	0.690	0.235	0.704	0.483	0.720	0.737
4/6	0.891	-0.705	0.884	-0.482	0.887	-0.253	0.897	-0.019	0.913	0.220	0.934	0.466	0.956	0.719
5/6	1.080	-0.697	1.081	-0.483	1.091	-0.263	1.109	-0.035	1.132	0.200	1.161	0.443	1.192	0.693
6/6	1.257	-0.698	1.268	-0.492	1.287	-0.279	1.314	-0.058	1.347	0.172	1.385	0.411	1.426	0.659
7/6	1.424	-0.707	1.446	-0.510	1.475	-0.304	1.513	-0.089	1.556	0.135	1.605	0.369	1.659	0.615
8/6	1.581	-0.726	1.615	-0.537	1.656	-0.338	1.705	-0.130	1.760	0.088	1.822	0.317	1.888	0.558
9/6	1.729	-0.755	1.775	-0.574	1.828	-0.383	1.890	-0.182	1.958	0.029	2.033	0.252	2.115	0.488
10/6	1.867	-0.794	1.926	-0.620	1.992	-0.438	2.067	-0.245	2.149	-0.042	2.239	0.174	2.336	0.403
11/6	1.997	-0.842	2.069	-0.677	2.148	-0.503	2.236	-0.320	2.332	-0.125	2.437	0.082	2.550	0.303
12/6	2.117	-0.899	2.202	-0.743	2.295	-0.579	2.396	-0.406	2.507	-0.221	2.627	-0.024	2.757	0.187
13/6	2.229	-0.966	2.326	-0.820	2.432	-0.666	2.546	-0.503	2.671	-0.330	2.806	-0.145	2.953	0.055
14/6	2.331	-1.040	2.441	-0.905	2.558	-0.763	2.686	-0.612	2.824	-0.452	2.974	-0.279	3.137	-0.094
15/6	2.424	-1.122	2.545	-0.999	2.674	-0.869	2.814	-0.732	2.965	-0.585	3.129	-0.428	3.308	-0.258
16/6	2.506	-1.211	2.638	-1.100	2.778	-0.984	2.929	-0.862	3.092	-0.730	3.270	-0.589	3.463	-0.436
17/6	2.579	-1.306	2.720	-1.209	2.870	-1.108	3.031	-1.000	3.205	-0.886	3.394	-0.762	3.601	-0.629
18/6	2.642	-1.407	2.791	-1.325	2.949	-1.238	3.118	-1.147	3.301	-1.050	3.501	-0.946	3.719	-0.834
19/6	2.694	-1.513	2.850	-1.445	3.015	-1.374	3.191	-1.299	3.381	-1.221	3.588	-1.138	3.816	-1.050
20/6	2.737	-1.624	2.898	-1.570	3.068	-1.514	3.248	-1.456	3.443	-1.396	3.654	-1.335	3.887	-1.274
21/6	2.770	-1.738	2.935	-1.697	3.107	-1.656	3.290	-1.614	3.486	-1.572	3.698	-1.533	3.928	-1.499
22/6	2.794	-1.854	2.961	-1.826	3.135	-1.799	3.319	-1.772	3.513	-1.747	3.722	-1.727	3.945	-1.714
23/6	2.809	-1.971	2.977	-1.956	3.152	-1.940	3.334	-1.927	3.527	-1.917	3.729	-1.912	3.944	-1.916
24/6	2.816	-2.089	2.985	-2.085	3.159	-2.081	3.339	-2.079	3.528	-2.081	3.724	-2.089	3.929	-2.106
25/6	2.816	-2.208	2.985	-2.212	3.157	-2.218	3.336	-2.226	3.520	-2.238	3.710	-2.256	3.906	-2.281
26/6	2.810	-2.326	2.978	-2.339	3.149	-2.353	3.325	-2.369	3.505	-2.389	3.689	-2.414	3.878	-2.446
27/6	2.799	-2.444	2.965	-2.463	3.135	-2.484	3.308	-2.507	3.484	-2.533	3.664	-2.564	3.846	-2.601
28/6	2.783	-2.561	2.948	-2.586	3.116	-2.613	3.287	-2.641	3.460	-2.672	3.635	-2.708	3.812	-2.748
29/6	2.763	-2.678	2.928	-2.708	3.094	-2.739	3.263	-2.771	3.433	-2.807	3.604	-2.845	3.777	-2.887
30/6	2.740	-2.794	2.904	-2.828	3.069	-2.862	3.236	-2.898	3.404	-2.936	3.573	-2.977	3.742	-3.021
31/6	2.715	-2.910	2.879	-2.947	3.043	-3.004	3.208	-3.022	3.374	-3.063	3.540	-3.105	3.707	-3.150
32/6	2.689	-3.025	2.851	-3.064	3.014	-3.104	3.178	-3.144	3.342	-3.186	3.507	-3.229	3.672	-3.274
33/6	2.660	-3.140	2.822	-3.181	2.985	-3.222	3.148	-3.264	3.311	-3.307	3.474	-3.351	3.637	-3.396
34/6	2.631	-3.255	2.793	-3.297	2.954	-3.339	3.117	-3.382	3.279	-3.425	3.441	-3.470	3.603	-3.515
35/6	2.601	-3.369	2.762	-3.412	2.924	-3.455	3.085	-3.498	3.247	-3.542	3.409	-3.587	3.570	-3.632
36/6	2.570	-3.484	2.731	-3.527	2.893	-3.571	3.054	-3.614	3.215	-3.658	3.376	-3.703	3.537	-3.748
37/6	2.540	-3.598	2.701	-3.642	2.862	-3.685	3.023	-3.729	3.184	-3.773	3.344	-3.818	3.505	-3.862
38/6	2.509	-3.712	2.669	-3.756	2.830	-3.800	2.991	-3.844	3.152	-3.888	3.313	-3.932	3.474	-3.977
39/6	2.477	-3.827	2.638	-3.871	2.799	-3.915	2.960	-3.959	3.121	-4.003	3.281	-4.047	3.442	-4.091
40/6	2.446	-3.941	2.607	-3.985	2.768	-4.029	2.929	-4.073	3.089	-4.117	3.250	-4.161	3.411	-4.205
41/6	2.415	-4.055	2.576	-4.099	2.736	-4.143	2.897	-4.187	3.058	-4.232	3.219	-4.275	3.380	-4.319
42/6	2.384	-4.169	2.544	-4.213	2.705	-4.257	2.866	-4.301	3.027	-4.345	3.187	-4.389	3.348	-4.433
43/6	2.352	-4.284	2.513	-4.328	2.674	-4.372	2.835	-4.416	2.995	-4.460	3.156	-4.504	3.317	-4.548
44/6	2.321	-4.398	2.482	-4.442	2.643	-4.486	2.803	-4.530	2.964	-4.574	3.125	-4.618	3.286	-4.662

TABLE VII—TABULATED VALUES OF THE INTEGRALS α AND β FOR A RANGE OF $|(\Phi - \Phi_0)|$

[Computational methods given in appendix G]

$ (\Phi - \Phi_0) $	$\alpha^{(1)}$	$\Delta I = \Delta \alpha$		$\beta^{(1)}$	$\Delta I = \Delta \beta$	
		$\Delta_{1\alpha}$ ($\Delta\Phi = \pi/24$)	$\Delta_{2\alpha}$ ($\Delta\Phi = 2\pi/24$)		$\Delta_{1\beta}$ ($\Delta\Phi = \pi/24$)	$\Delta_{2\beta}$ ($\Delta\Phi = 2\pi/24$)
0	0	0.000373	0.002970	0	-0.396937	-0.611660
1($\pi/24$)	.000373	.002597		-.396937	-.214723	
2($\pi/24$)	.002970	.006972	.020331	-.611660	-.144746	-.242791
3($\pi/24$)	.009942	.013359		-.756406	-.098045	
4($\pi/24$)	.023301	.021574	.052976	-.854451	-.062035	-.094079
5($\pi/24$)	.044875	.031402		-.916486	-.032044	
6($\pi/24$)	.076277	.042620	.097632	-.948530	-.005816	.012092
7($\pi/24$)	.118897	.055012		-.954346	.017908	
8($\pi/24$)	.173909	.068374	.150903	-.936438	.039896	.100542
9($\pi/24$)	.242283	.082529		-.896542	.060646	
10($\pi/24$)	.324812	.097324	.209951	-.835896	.080497	.180181
11($\pi/24$)	.422136	.112627		-.755399	.099684	
12($\pi/24$)	.534763	.128335	.272697	-.655715	.118377	.255075
13($\pi/24$)	.663098	.144362		-.537338	.136698	
14($\pi/24$)	.807460	.160636	.337741	-.400640	.154739	.327305
15($\pi/24$)	.968096	.177105		-.245901	.172566	
16($\pi/24$)	1.145201	.193725	.404188	-.073335	.190232	.398006
17($\pi/24$)	1.338926	.210463		.116897	.207774	
18($\pi/24$)	1.549389	.227290	.471478	.324671	.225221	.467817
19($\pi/24$)	1.776679	.244188		.549892	.242596	
20($\pi/24$)	2.020867	.261141	.539276	.792488	.259915	.537106
21($\pi/24$)	2.282008	.278135		1.052403	.277191	
22($\pi/24$)	2.560143	.295161	.607373	1.329594	.294435	.606089
23($\pi/24$)	2.855304	.312212		1.624029	.311654	
24($\pi/24$)	3.167516	.329283	.675652	1.935683	.328853	.674890
25($\pi/24$)	3.496799	.346369		2.264536	.346037	
26($\pi/24$)	3.843168	.363465	.744035	2.610573	.363210	.743585
27($\pi/24$)	4.206633	.380570		2.973783	.380375	
28($\pi/24$)	4.587203	.397682	.812482	3.354158	.397531	.812215
29($\pi/24$)	4.984885	.414800		3.751689	.414684	
30($\pi/24$)	5.399685	.431922	.880967	4.166373	.431832	.880808
31($\pi/24$)	5.831607	.449045		4.598205	.448976	
32($\pi/24$)	6.280652	.466173	.949474	5.047181	.466120	.949380
33($\pi/24$)	6.746825	.483301		5.513301	.483260	
34($\pi/24$)	7.230126	.500431	1.017993	5.996561	.500400	1.017938
35($\pi/24$)	7.730557	.517562		6.496961	.517538	
36($\pi/24$)	8.248119	.534694	1.086521	7.014499	.534676	1.086488
37($\pi/24$)	8.782813	.551827		7.549175	.551812	
38($\pi/24$)	9.334640	.568960	1.155053	8.100987	.568949	1.155034
39($\pi/24$)	9.903600	.586093		8.669936	.586085	
40($\pi/24$)	10.489693	.603227	1.223588	9.256021	.603220	1.223576
41($\pi/24$)	11.092920	.620361		9.859241	.620356	
42($\pi/24$)	11.713281	.637496	1.292125	10.479597	.637492	1.292118
43($\pi/24$)	12.350777	.654629		11.117089	.654626	
44($\pi/24$)	13.005406	.671764	1.360662	11.771715	.671762	1.360658
45($\pi/24$)	13.677170	.688898		12.443477	.688896	
46($\pi/24$)	14.366068	.706033	1.429200	13.132373	.706032	1.429198
47($\pi/24$)	15.072101	.723167		13.838405	.723166	
48($\pi/24$)	15.795268	.740303	1.497739	14.561571	.740302	1.497738
49($\pi/24$)	16.535571	.757436		15.301873	.757436	

(1) For negative values of $(\Phi - \Phi_0)$ the signs of α and β change, but the signs of $\Delta\alpha$ and $\Delta\beta$ remain unchanged.

TABLE VII—TABULATED VALUES OF THE INTEGRALS α AND β FOR A RANGE OF $|(\Phi - \Phi_0)|$ —Concluded.

[Computation methods given in appendix G.]

$ (\Phi - \Phi_0) $	$\alpha^{(1)}$	$\Delta I = \Delta \alpha$		$\beta^{(1)}$	$\Delta I = \Delta \beta$	
		$\Delta_1 \alpha$ ($\Delta \Phi = \pi/24$)	$\Delta_2 \alpha$ ($\Delta \Phi = 2\pi/24$)		$\Delta_1 \beta$ ($\Delta \Phi = \pi/24$)	$\Delta_2 \beta$ ($\Delta \Phi = 2\pi/24$)
50($\pi/24$)	17.293007			16.059309		
51($\pi/24$)	18.067579	0.774572	1.566278	16.833880	0.774571	1.566276
52($\pi/24$)	18.859285	.791706		17.625585	.791705	
53($\pi/24$)	19.668125	.808840	1.634816	18.434426	.808841	1.634816
54($\pi/24$)	20.494101	.825976		19.260401	.825975	
55($\pi/24$)	21.337211	.843110	1.703355	20.103511	.843110	1.703354
56($\pi/24$)	22.197456	.860245		20.963755	.860244	
57($\pi/24$)	23.074835	.877379	1.771893	21.841135	.877380	1.771894
58($\pi/24$)	23.969349	.894514		22.735649	.894514	
59($\pi/24$)	24.880998	.911649	1.840433	23.647298	.911649	1.840433
60($\pi/24$)	25.809782	.928784		24.576082	.928784	
61($\pi/24$)	26.755694	.945912	1.908968	25.521994	.945912	1.908968
62($\pi/24$)	27.718750	.963056		26.485050	.963056	
63($\pi/24$)	28.698940	.980190	1.977507	27.465240	.980190	1.977507
64($\pi/24$)	29.696257	.997317		28.462557	.997317	
65($\pi/24$)	30.710717	1.014460	2.046044	29.477017	1.014460	2.046044
66($\pi/24$)	31.742311	1.031594		30.508611	1.031594	
67($\pi/24$)	32.791033	1.048722	2.114585	31.557333	1.048722	2.114585
68($\pi/24$)	33.856896	1.065863		32.623196	1.065863	
69($\pi/24$)	34.939895	1.082999	2.183123	33.706195	1.082999	2.183123
70($\pi/24$)	36.040029	1.100134		34.806329	1.100134	
71($\pi/24$)	37.157289	1.117260	2.251663	35.923589	1.117260	2.251663
72($\pi/24$)	38.291692	1.134403		37.057992	1.134403	
73($\pi/24$)	39.443230	1.151538	2.320201	38.209530	1.151538	2.320201
74($\pi/24$)	40.611893	1.168663		39.378193	1.168663	
75($\pi/24$)	41.797701	1.185808	2.388740	40.564001	1.185808	2.388740
76($\pi/24$)	43.000643	1.202942		41.766943	1.202942	
77($\pi/24$)	44.220710	1.220067	2.457279	42.987010	1.220067	2.457279
78($\pi/24$)	45.457922	1.237212		44.224222	1.237212	
79($\pi/24$)	46.712269	1.254347	2.525818	45.478569	1.254347	2.525818
80($\pi/24$)	47.983750	1.271481		46.750050	1.271481	
81($\pi/24$)	49.272356	1.288606	2.594357	48.038656	1.288606	2.594357
82($\pi/24$)	50.578107	1.305751		49.344407	1.305751	
83($\pi/24$)	51.900992	1.322885	2.662895	50.667292	1.322885	2.662895
84($\pi/24$)	53.241002	1.340010		52.007302	1.340010	
85($\pi/24$)	54.598158	1.357156	2.731435	53.364458	1.357156	2.731435
86($\pi/24$)	55.972447	1.374289		54.738747	1.374289	
87($\pi/24$)	57.363861	1.391414	2.799974	56.130161	1.391414	2.799973
88($\pi/24$)	58.772421	1.408560		57.538720	1.408560	
89($\pi/24$)	60.198115	1.425694	2.868512	58.964415	1.425694	2.868513
90($\pi/24$)	61.640933	1.442818		60.407233	1.442818	
91($\pi/24$)	63.100897	1.459964	2.937052	61.867197	1.459964	2.937052
92($\pi/24$)	64.577995	1.477098		63.344295	1.477098	
93($\pi/24$)	66.072228	1.494233	3.005590	64.838528	1.494233	3.005590
94($\pi/24$)	67.583585	1.511357		66.349885	1.511357	
95($\pi/24$)	69.112088	1.528503	3.074130	67.878388	1.528503	3.074130
96($\pi/24$)	70.657725	1.545637		69.424025	1.545637	
97($\pi/24$)	72.220486	1.562761	3.142668	70.986786	1.562761	3.142668
98($\pi/24$)	73.800393	1.579907		72.566693	1.579907	
99($\pi/24$)	75.397435	1.597042	3.211206	74.163735	1.597042	3.211206
100($\pi/24$) ⁽²⁾	77.011599	1.614164		75.777899	1.614164	

⁽¹⁾ For negative values of $(\Phi - \Phi_0)$ the signs of α and β change, but the signs of $\Delta \alpha$ and $\Delta \beta$ remain unchanged.⁽²⁾ For values of $|(\Phi - \Phi_0)| > 100(\pi/24)$ use equation (G2) for α and equation (G7) for β .

TABLE VIII—COMPARISON OF ELBOW DESIGNS OBTAINED FROM SOLUTIONS BY RELAXATION METHODS AND BY GREEN'S FUNCTION

[Linearized compressible flow; prescribed velocity distribution given in figs. 2 and 22.]

Φ	$\Psi=0$ (Inner wall)									$\Psi=\pi/2$ (Outer wall)								
	Q	q	Solution by relaxation methods (Part I)			Solution by Green's function (Part II)				Q	q	Solution by relaxation methods (Part I)			Solution by Green's function (Part II)			
			x	y	θ (deg)	x	y	θ (deg)				x	y	θ (deg)	x	y	θ (deg)	
-22($\pi/24$)	0.5000	0.4009	-2.466	-0.769	0	-2.466	-0.769	0		0.5000	0.4009	-2.466	0.770	0	-2.466	0.770	0	
-20($\pi/24$)	.5000	.4009	-2.241	-.769	.01	-2.241	-.769	.01		.5000	.4009	-2.241	.770	-.01	-2.241	.770	-.01	
-18($\pi/24$)	.5000	.4009	-2.016	-.769	.01	-2.016	-.769	.01		.5000	.4009	-2.016	.770	-.01	-2.016	.770	-.01	
-16($\pi/24$)	.5000	.4009	-1.791	-.769	.02	-1.791	-.769	.02		.5000	.4009	-1.791	.770	-.02	-1.791	.770	-.02	
-14($\pi/24$)	.5000	.4009	-1.566	-.768	.03	-1.566	-.768	.03		.5000	.4009	-1.566	.770	-.03	-1.566	.770	-.03	
-12($\pi/24$)	.5000	.4009	-1.341	-.768	.05	-1.341	-.768	.05		.5000	.4009	-1.341	.769	-.05	-1.341	.770	-.05	
-10($\pi/24$)	.5000	.4009	-1.116	-.768	.08	-1.116	-.768	.08		.5000	.4009	-1.116	.769	-.08	-1.116	.769	-.08	
-8($\pi/24$)	.5000	.4009	-.891	-.768	.14	-.891	-.768	.14		.5000	.4009	-.891	.769	-.13	-.891	.769	-.13	
-6($\pi/24$)	.5000	.4009	-.666	-.767	.24	-.666	-.767	.24		.5000	.4009	-.666	.768	-.21	-.666	.768	-.22	
-4($\pi/24$)	.5000	.4009	-.441	-.766	.40	-.441	-.766	.41		.5000	.4009	-.441	.767	-.35	-.441	.767	-.36	
-2($\pi/24$)	.5000	.4009	-.216	-.763	.70	-.216	-.763	.74		.5000	.4009	-.216	.765	-.56	-.216	.765	-.59	
0	.5000	.4009	.008	-.760	1.31	.010	-.759	1.52		.5000	.4009	.009	.763	-.92	.009	.762	-.94	
2($\pi/24$)	.5079	.4072	.233	-.752	2.82	.233	-.751	2.76		.5000	.4009	.234	.758	-1.45	.234	.758	-1.48	
4($\pi/24$)	.5293	.4243	.450	-.739	3.88	.449	-.738	3.76		.5000	.4009	.459	.751	-2.22	.459	.750	-2.24	
6($\pi/24$)	.5599	.4489	.656	-.724	4.00	.656	-.724	3.89		.5000	.4009	.684	.740	-3.28	.684	.740	-3.29	
8($\pi/24$)	.5962	.4780	.851	-.712	3.17	.850	-.713	3.07		.5000	.4009	.908	.725	-4.65	.908	.724	-4.68	
10($\pi/24$)	.6354	.5094	1.033	-.704	1.44	1.033	-.705	1.38		.5000	.4009	1.132	.703	-6.41	1.132	.702	-6.43	
12($\pi/24$)	.6754	.5415	1.205	-.703	-.98	1.205	-.704	-1.04		.5000	.4009	1.355	.674	-8.52	1.355	.673	-8.56	
14($\pi/24$)	.7149	.5732	1.366	-.710	-4.00	1.367	-.711	-4.04		.5000	.4009	1.577	.636	-11.04	1.577	.635	-11.06	
16($\pi/24$)	.7531	.6038	1.519	-.725	-7.48	1.519	-.727	-7.51		.5000	.4009	1.797	.587	-13.91	1.797	.586	-13.94	
18($\pi/24$)	.7894	.6329	1.663	-.749	-11.35	1.663	-.750	-11.37		.5000	.4009	2.013	.527	-17.13	2.013	.526	-17.16	
20($\pi/24$)	.8235	.6602	1.798	-.781	-15.52	1.799	-.783	-15.54		.5000	.4009	2.226	.455	-20.67	2.226	.453	-20.69	
22($\pi/24$)	.8550	.6855	1.926	-.822	-19.96	1.926	-.823	-19.98		.5000	.4009	2.434	.368	-24.49	2.434	.367	-24.52	
24($\pi/24$)	.8838	.7086	2.046	-.871	-24.62	2.046	-.871	-24.63		.5000	.4009	2.635	.268	-28.58	2.635	.266	-28.59	
26($\pi/24$)	.9097	.7293	2.157	-.928	-29.46	2.158	-.929	-29.47		.5000	.4009	2.829	.153	-32.89	2.828	.152	-32.90	
28($\pi/24$)	.9326	.7477	2.261	-.993	-34.45	2.261	-.994	-34.46		.5000	.4009	3.013	.024	-37.40	3.012	.022	-37.41	
30($\pi/24$)	.9524	.7636	2.356	-1.064	-39.56	2.356	-1.066	-39.57		.5000	.4009	3.186	-.120	-42.08	3.185	-.122	-42.09	
32($\pi/24$)	.9690	.7769	2.443	-1.143	-44.76	2.443	-1.144	-44.77		.5000	.4009	3.346	-.278	-46.93	3.346	-.279	-46.94	
34($\pi/24$)	.9822	.7875	2.521	-1.228	-50.02	2.521	-1.229	-50.01		.5000	.4009	3.492	-.449	-51.93	3.492	-.450	-51.94	
36($\pi/24$)	.9919	.7953	2.590	-1.318	-55.27	2.590	-1.320	-55.27		.5000	.4009	3.623	-.632	-57.08	3.623	-.633	-57.10	
38($\pi/24$)	.9979	.8001	2.650	-1.414	-60.49	2.650	-1.415	-60.47		.5000	.4009	3.736	-.826	-62.44	3.736	-.828	-62.49	
40($\pi/24$)	1.0000	.8018	2.701	-1.514	-65.55	2.701	-1.516	-65.51		.5000	.4009	3.830	-1.030	-68.16	3.829	-1.034	-68.37	
42($\pi/24$)	1.0000	.8018	2.743	-1.619	-70.32	2.744	-1.620	-70.29		.5079	.4072	3.901	-1.242	-74.80	3.901	-1.244	-74.75	
44($\pi/24$)	1.0000	.8018	2.777	-1.726	-74.81	2.777	-1.727	-74.78		.5293	.4243	3.947	-1.455	-81.03	3.945	-1.456	-80.91	
46($\pi/24$)	1.0000	.8018	2.803	-1.836	-78.97	2.803	-1.837	-78.96		.5599	.4489	3.969	-1.661	-86.33	3.969	-1.662	-86.23	
48($\pi/24$)	1.0000	.8018	2.820	-1.947	-82.82	2.821	-1.948	-82.80		.5962	.4780	3.974	-1.856	-90.69	3.975	-1.856	-90.59	
50($\pi/24$)	1.0000	.8018	2.831	-2.059	-86.28	2.831	-2.059	-86.26		.6354	.5094	3.966	-2.038	-94.16	3.967	-2.040	-94.09	
52($\pi/24$)	1.0000	.8018	2.835	-2.171	-89.37	2.836	-2.172	-89.34		.6754	.5415	3.950	-2.209	-96.93	3.951	-2.210	-96.88	
54($\pi/24$)	1.0000	.8018	2.834	-2.283	-92.07	2.834	-2.285	-92.04		.7149	.5732	3.927	-2.369	-99.11	3.928	-2.371	-99.07	
56($\pi/24$)	1.0000	.8018	2.827	-2.396	-94.40	2.828	-2.397	-94.37		.7531	.6038	3.900	-2.520	-100.83	3.901	-2.522	-100.80	
58($\pi/24$)	1.0000	.8018	2.817	-2.508	-96.39	2.817	-2.509	-96.36		.7894	.6329	3.871	-2.663	-102.17	3.872	-2.665	-102.14	
60($\pi/24$)	1.0000	.8018	2.803	-2.619	-98.05	2.803	-2.620	-98.03		.8235	.6602	3.841	-2.799	-103.19	3.842	-2.801	-103.17	
62($\pi/24$)	1.0000	.8018	2.786	-2.731	-99.44	2.786	-2.732	-99.42		.8550	.6855	3.809	-2.929	-103.95	3.810	-2.931	-103.94	
64($\pi/24$)	1.0000	.8018	2.766	-2.841	-100.56	2.767	-2.842	-100.55		.8838	.7086	3.777	-3.055	-104.49	3.778	-3.056	-104.48	
66($\pi/24$)	1.0000	.8018	2.744	-2.952	-101.47	2.745	-2.953	-101.45		.9097	.7293	3.746	-3.176	-104.84	3.746	-3.178	-104.83	
68($\pi/24$)	1.0000	.8018	2.721	-3.062	-102.18	2.722	-3.063	-102.17		.9326	.7477	3.714	-3.294	-105.03	3.715	-3.296	-105.03	
70($\pi/24$)	1.0000	.8018	2.697	-3.172	-102.73	2.698	-3.173	-102.72		.9524	.7636	3.683	-3.409	-105.10	3.684	-3.411	-105.09	
72($\pi/24$)	1.0000	.8018	2.672	-3.281	-103.14	2.672	-3.283	-103.14		.9690	.7769	3.653	-3.522	-105.05	3.654	-3.524	-105.01	
74($\pi/24$)	1.0000	.8018	2.646	-3.391	-103.45	2.647	-3.392	-103.44		.9822	.7875	3.623	-3.633	-104.91	3.624	-3.635	-104.91	
76($\pi/24$)	1.0000	.8018	2.620	-3.500	-103.66	2.620	-3.501	-103.66		.9919	.7953	3.591	-3.744	-104.71	3.595	-3.746	-104.72	
78($\pi/24$)	1.0000	.8018	2.593	-3.609	-103.81	2.594	-3.611	-103.79		.9979	.8001	3.565	-3.853	-104.49	3.566	-3.855	-104.49	
80($\pi/24$)	1.0000	.8018	2.566	-3.719	-103.90	2.567	-3.720	-103.90		1.0000	.8018	3.537	-3.962	-104.28	3.538	-3.964	-104.30	
82($\pi/24$)	1.0000	.8018	2.539	-3.828	-103.97	2.540	-3.829	-103.97		1.0000	.8018	3.510	-4.071	-104.18	3.510	-4.073	-104.19	

REPORT 1116

APPLICATION OF A CHANNEL DESIGN METHOD TO HIGH-SOLIDITY CASCADES AND TESTS OF AN IMPULSE CASCADE WITH 90° OF TURNING¹

By JOHN D. STANITZ and LEONARD J. SHELDRAKE

SUMMARY

A technique is developed for the application of a channel design method to the design of high-solidity cascades with prescribed velocity distributions as a function of arc length along the blade-element profile. The technique applies to both incompressible and subsonic compressible, nonviscous, irrotational fluid motion. For compressible flow, the ratio of specific heats is assumed equal to $\gamma = 1.0$. An impulse cascade with 90° turning was designed for incompressible flow and was tested at the design angle of attack over a range of downstream Mach number from 0.2 to choke flow. To achieve good efficiency, the cascade was designed for prescribed velocities and maximum blade loading according to limitations imposed by considerations of boundary-layer separation.

INTRODUCTION

In order to obtain large pressure ratios per stage in axial-flow compressors and turbines, cascades of blade elements with large fluid-turning angles are required. High-solidity blading is required to achieve these large turning angles without serious shock losses due to supersonic peak velocities and without boundary-layer separation due to excessive blade loading. However, because friction losses increase with the ratio of wetted surface to flow area, it is desirable that the blades be as highly loaded as possible so that the solidity be no greater than necessary. Thus it is desirable to have design methods for high-solidity cascades with maximum prescribed velocities that do not result in shock losses and with maximum prescribed deceleration rates that do not result in boundary-layer separation. Such blade elements should have optimum efficiency for the prescribed turning angle of the fluid.

Two types of method are used for the design of blade elements in cascade: (1) airfoil methods, and (2) channel-flow methods. Airfoil methods have been developed for incompressible and linearized compressible flow (refs. 1 to 5, for example). These methods are exact for irrotational, nonviscous fluid motion but generally become difficult to apply if the blade-element solidity is large (1.5 or larger).

For high blade-element solidities, channel-flow methods of design are used, in which the channel between blades is designed and the "islands" between adjacent channels (fig. 1) constitute the blade elements, with the nose and tail of the islands rounded off. Geometric methods for channel design based on the combination of several circular arcs have been used extensively (ref. 6, for example), but these methods have no direct control over the velocity distribution along the blade-element profile.

In reference 7 the shape of the mean streamline between

blade-element profiles and the velocity distribution along this streamline are prescribed together with the blade spacing, and the resulting blade-element profile and velocity distribution along it are determined. The method involves approximations that are accurate for high-solidity cascades, but has no direct control over the velocity distribution along the profile surface.

Design methods for blade elements of high-solidity cascades with prescribed velocities along the blade-element profile are given in references 8 to 10. All these methods involve approximations. In reference 8 the desired velocity distribution is obtained by trial-and-error methods. In reference 9 the shape of one channel wall and the velocity distribution along it are prescribed so that, as in reference 7, the problem is overdetermined and therefore approximate. In reference 10 the manner in which various flow conditions vary across the channel of high-solidity cascades is assumed.

A technique for application of the channel design methods of reference 11 to the design of high-solidity cascades with prescribed velocity distributions as a function of arc length along the blade-element profiles is presented herein. The technique applies to both incompressible and subsonic compressible, nonviscous, irrotational fluid motion. For compressible flow the ratio of specific heats is assumed equal to

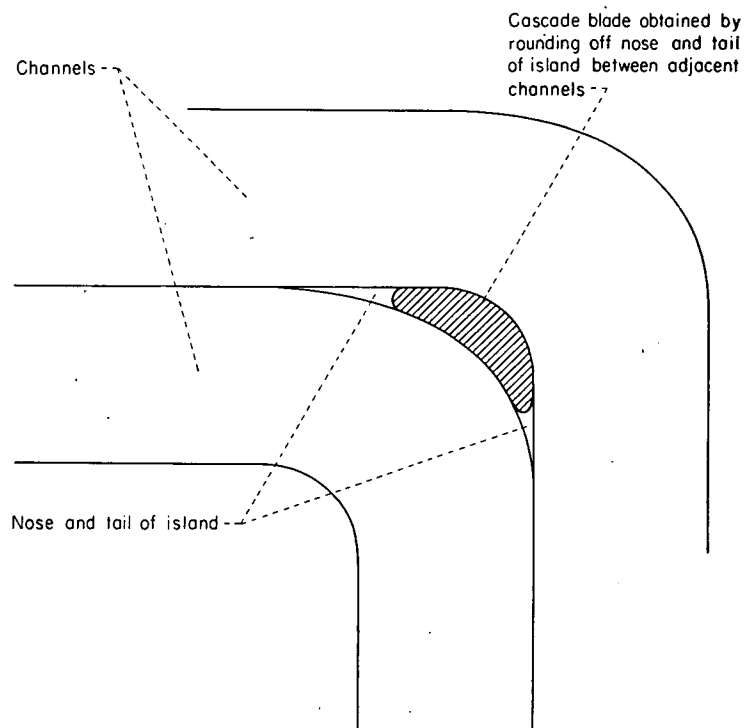


FIGURE 1.—Cascade blade obtained in xy -plane by rounding off nose and tail of island between adjacent channels designed for prescribed velocity distribution along walls.

¹ Supersedes NACA TN 2652, "Application of a Channel Design Method to High-Solidity Cascades and Tests of an Impulse Cascade With 90° of Turning" by John D. Stanitz and Leonard J. Sheldrake, 1952

—1.0. The design method gives exact results except for the approximation resulting from rounding off the nose and tail of the blade element. In order to investigate the validity of rounding off the nose and tail and to investigate effects of compressibility and viscosity, a high-solidity, 90° impulse cascade was designed and tested. In order to achieve good efficiency, the cascade was designed for prescribed velocities and maximum blade loading according to limitations imposed by boundary-layer separation (ref. 12). The cascade was developed for incompressible flow and was tested at the design angle of attack over a range of downstream Mach number from 0.2 to choke flow. The data were analyzed and correlated by methods developed in the report.

The application of a channel design method to high-solidity cascades reported herein was developed at the NACA Lewis laboratory during 1951 and is part of a doctoral thesis conducted by the senior author with the advice of Professor Ascher H. Shapiro of the Massachusetts Institute of Technology.

CASCADE DESIGN METHOD

A cascade design method based on the channel design methods of reference 11 is developed for nonviscous, irrotational, incompressible or compressible fluid motion.

THEORY OF METHOD

Outline.—Consider the flow of fluid past the high-solidity cascade in figure 2. Any two blade elements and their respec-

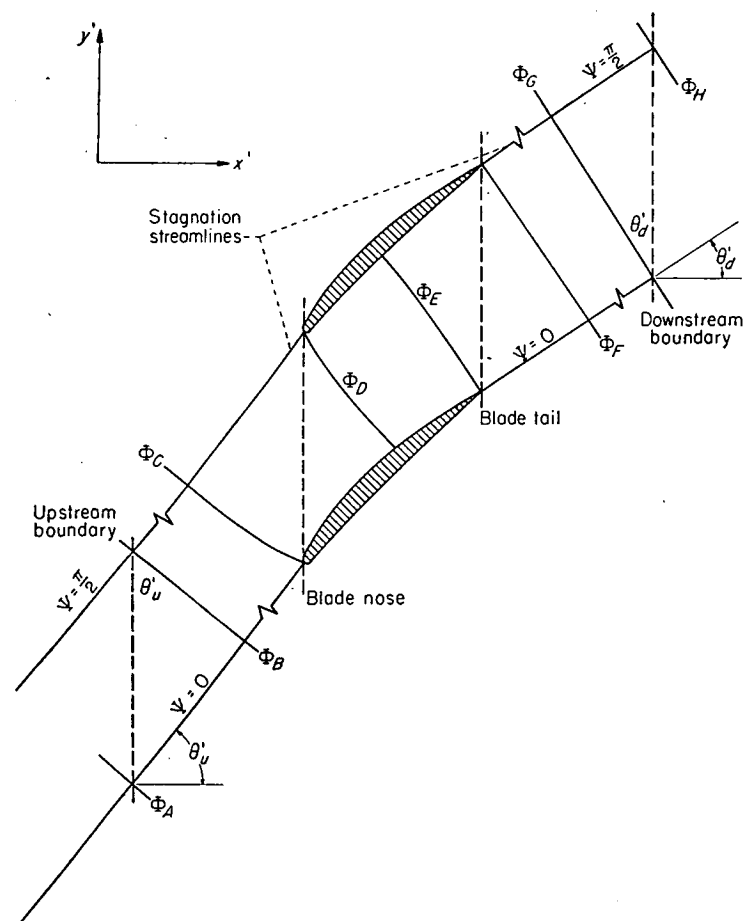


FIGURE 2.—Channel between adjacent stagnation streamlines with velocity potential lines and subscript conventions for flow through cascade.

tive stagnation streamlines upstream and downstream of the cascade constitute a flow channel. In the proposed high-solidity cascade design method the shape of this channel will be determined, except for regions in the vicinity of stagnation points, for prescribed variations in velocity as a function of arc length s along the channel walls between points corresponding to the nose and tail of the blade elements. The channel design methods of reference 11 will be used to solve for the shape of this channel between high-solidity blade elements.

The flow field of the two-dimensional channel between blade elements is considered to lie in the physical xy -plane where x and y are Cartesian coordinates for which the units are so chosen that the channel width downstream at infinity is unity. (All symbols are defined in appendix A.)

At each point in the channel between blade elements the velocity vector (fig. 3) has a magnitude Q and a direction θ , where Q is the fluid velocity for which the units are so chosen that the channel velocity downstream at infinity is unity. For compressible flow, the velocity q is related to the velocity ratio Q by

$$q = Qq_d$$

where q is the velocity for which the units are so chosen that the stagnation speed of sound is unity and where the subscript d refers to conditions downstream at infinity.

Solutions for two-dimensional flow are boundary-value problems. That is, the solutions depend on known conditions imposed along the boundaries of the problem. In the inverse problem of channel design the geometry of the channel walls in the physical plane is unknown. This unknown geometry apparently precludes the possibility of solving the problem in the physical plane and necessitates the use of some

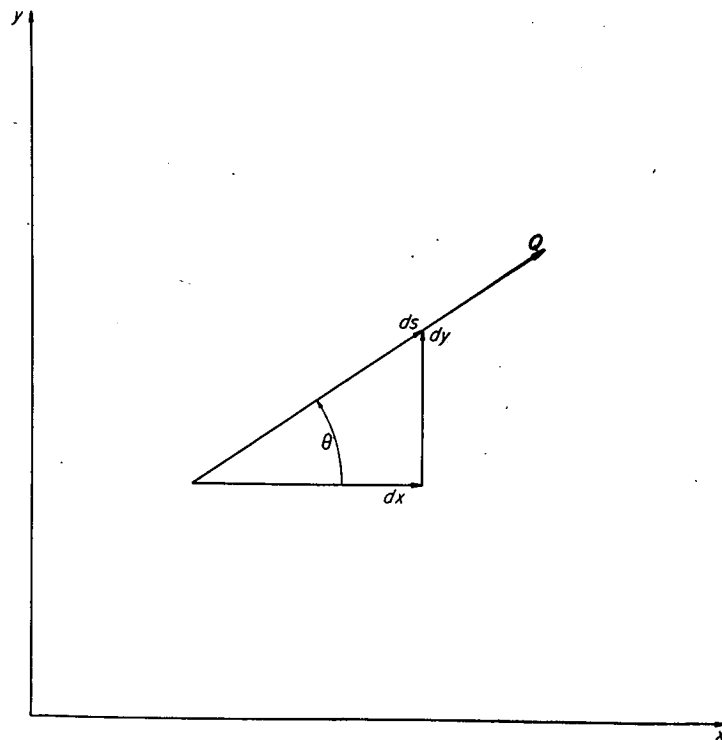


FIGURE 3.—Magnitude and direction of velocity at point in xy -plane.

new plane. This new plane must be such that the shape of the boundaries along which the velocities are prescribed is known. It is also desirable that the coordinate system of the new plane be orthogonal in the physical plane. A set of coordinates that satisfies these requirements is provided by the velocity potential lines of constant Φ and the streamlines of constant Ψ (where Φ and Ψ are defined in ref. 11), which are orthogonal in the xy -plane and for which the geometric boundaries are known constant values of Ψ (equal to 0 and $\pi/2$) in the $\Phi\Psi$ -plane. The distribution of velocity as a function of Φ along these boundaries of constant Ψ is known because, if

$$Q=Q(s)$$

or

$$q=q(s)$$

is prescribed, the definition of Φ (ref. 11) gives

$$\Phi=\Phi(s)$$

from which

$$Q=Q(\Phi)$$

or

$$q=q(\Phi)$$

The technique of the channel design methods developed in reference 11 is therefore to solve for the physical xy -coordinates of the channel walls in the $\Phi\Psi$ -plane where the prescribed boundary conditions for the two-dimensional flow problem are known.

The channel design methods of reference 11 are applied to the design of high-solidity cascades as follows: Along the upstream and downstream stagnation streamlines (fig. 2) the velocity will be assumed constant and equal to the upstream and downstream velocities, respectively. The stagnation points on the blade surface are ignored (which practice results in a cusped nose and tail that are rounded off, as indicated previously), and the velocity at the nose and tail are assumed equal to the upstream and downstream velocities, respectively. Along the channel wall corresponding to the suction surface of the blade element, the velocity accelerates from its upstream value at the nose to some maximum value, after which it decelerates to its downstream value at the tail. Along the channel wall corresponding to the pressure surface of the blade element, the velocity decelerates from its upstream value at the blade nose to some minimum value, after which it accelerates to the downstream value at the tail.

Because of the velocity difference over that portion of the channel walls corresponding to the blade-element surfaces, the channel turns the fluid an amount $\Delta\theta$ that can be computed by equation (H5) of reference 11. If a specified value of $\Delta\theta$ is desired, the prescribed velocity distribution must be adjusted, by methods to be considered later, to obtain this turning.

The physical coordinates of the channel are determined in the $\Phi\Psi$ -plane for the prescribed velocity distribution according to the design methods of reference 11. The islands between adjacent channels (fig. 1) in the physical xy -plane constitute the blade-element profiles. The cusped nose and tail of these islands are rounded off at the previously selected positions for the nose and tail of the blade-element profiles.

Nose and tail positions in $\Phi\Psi$ -plane.—Because the channel

design is carried out in the $\Phi\Psi$ -plane, it is necessary to determine the positions of the nose and tail on the pressure and suction surfaces of the channel walls in the $\Phi\Psi$ -plane. Consider the flow of fluid corresponding to the channel between blades (fig. 2). The change in velocity potential Φ from the upstream boundary, at which conditions are considered uniform, to the nose must be equal along both upstream stagnation streamlines so that

$$\Phi_C - \Phi_A = \Phi_D - \Phi_B$$

from which

$$\Phi_D - \Phi_C = \Phi_B - \Phi_A \quad (1)$$

where the subscripts A, B, C, D , and so forth, refer to positions defined by the velocity potential lines in figure 2. But, because conditions are uniform along the upstream boundary,

$$\frac{\Phi_B - \Phi_A}{\pi/2} = \tan \theta'_u \quad (2)$$

where $\pi/2$ is the change in Ψ across the channel and where the angle θ' is measured counterclockwise from the positive x' -axis of the x', y' -coordinate system in which the cascade lies along the y' -axis. From equations (1) and (2)

$$\Phi_D - \Phi_C = \frac{\pi}{2} \tan \theta'_u \quad (3)$$

Equation (3) determines the difference in Φ on the two walls of the channel at the points in the $\Phi\Psi$ -plane corresponding to the nose of the cascade blade. Likewise, the difference in Φ on the two walls of the channel at the points in the $\Phi\Psi$ -plane corresponding to the tail of the cascade blade is given by

$$\Phi_F - \Phi_E = \frac{\pi}{2} \tan \theta'_d \quad (4)$$

Equations (3) and (4) determine the relative positions on the channel walls in the $\Phi\Psi$ -plane of points corresponding to the nose and tail of the blade profile, respectively.

Prescribed velocity distribution.—In general, the prescribed distribution of velocity as a function of arc length along the channel walls between blade-element profiles can be arbitrary for the proposed blade-element design method except that the velocity is higher on the suction surface than on the pressure surface, the resulting blade-element profile must be practical, and the difference in velocity distribution on the two walls must satisfy equations (3) and (4). In addition, the prescribed velocity distribution must result in the prescribed turning angle. This last condition can be determined by computing the turning angle from equation (H5) of reference 11 and, in general, the original velocity distribution must be adjusted by trial-and-error methods to achieve the correct (prescribed) turning angle. Along the channel walls upstream and downstream of the points (and at the points) corresponding to the nose and tail of the blade-element profile (fig. 2), the velocity is assumed constant and equal to the upstream and downstream velocities, respectively.

In the remainder of this report it is assumed that the velocity distribution is prescribed, for convenience, by $\log_e Q$ (for incompressible flow) as a linear function of Φ

along the channel walls in the $\Phi\Psi$ -plane. A typical example of such a velocity distribution is given in figure 4. The velocity accelerates along the suction surface ($\Psi=0$) from the upstream value Q_u at the nose Φ_C to a maximum value Q_{max} at which it remains constant until it decelerates to the downstream value Q_d at the tail Φ_E . The velocity decelerates along the pressure surface ($\Psi=\pi/2$) from the upstream value Q_u at the nose Φ_D to a minimum value Q_{min} at which it remains constant until it accelerates to the downstream value Q_d at the tail Φ_F . The accelerating flow along the suction surface and the decelerating flow along the pressure surface near the blade nose and vice versa near the tail will, in general, result in a physically practical blade-element profile. An equation for the turning angle $\Delta\theta$ that results from this linear distribution of $\log_e Q$ and which must satisfy the prescribed turning angle is developed next.

Turning angle $\Delta\theta$.—From appendix H of reference 11 the channel or cascade turning angle $\Delta\theta$ is given by (for incompressible flow)

$$\Delta\theta = \frac{-2}{\pi} \int_{-\infty}^{\infty} \Phi \left[\left(\frac{\partial \log_e Q}{\partial \Phi} \right)_{\frac{\pi}{2}} - \left(\frac{\partial \log_e Q}{\partial \Phi} \right)_0 \right] d\Phi \quad (5)$$

For linear variations in $\log_e Q$ with Φ

$$\frac{\partial \log_e Q}{\partial \Phi} = \frac{\Delta \log_e Q}{\Delta \Phi} = \text{constant}$$

and for the type of linear velocity distribution given in figure 4, equation (5) integrates to give

$$\Delta\theta = \frac{1}{\pi} \left[\left(\log_e \frac{Q_{max}}{Q_u} \right) (\Phi_C + \Phi_{CC}) + \left(\log_e \frac{Q_d}{Q_{max}} \right) (\Phi_{EE} + \Phi_E) - \right. \\ \left. \left(\log_e \frac{Q_{min}}{Q_u} \right) (\Phi_D + \Phi_{DD}) - \left(\log_e \frac{Q_d}{Q_{min}} \right) (\Phi_{FF} + \Phi_F) \right] \quad (6)$$

which is the cascade turning angle. For a prescribed value of $\Delta\theta$, equation (6) establishes a relation among Q_{max} , Q_{min} , Q_u , Q_d , Φ_C , Φ_{CC} , Φ_D , Φ_{DD} , Φ_E , Φ_{EE} , Φ_F , and Φ_{FF} . For linearized compressible flow, linear distributions of $\log_e V$ (defined in ref. 11) with Φ of the type shown in figure 4 could be prescribed, in which case the turning angle $\Delta\theta$ would be given by equation (6) with Q replaced by V .

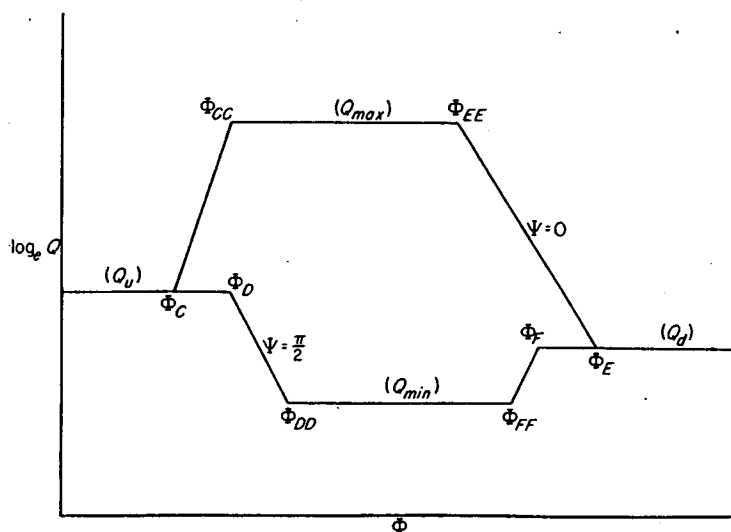


FIGURE 4.—Typical example of linear, prescribed variation in $\log_e Q$ with Φ , including subscript convention for Φ and Q .

Allowable deceleration on suction surface.—In order to achieve the desired turning $\Delta\theta$ with the minimum number of blades, the difference ($Q_{max} - Q_{min}$) for incompressible flow or ($V_{max} - V_{min}$) for compressible flow must be large; and the arc length over which Q_{max} and Q_{min} are prescribed should be extensive. However, the magnitude of Q_{max} is limited by shock losses (compressible flow) and cavitation (incompressible flow). Also, the arc length over which Q_{max} is prescribed is limited (in percentage of total suction surface length) by the allowable rate of deceleration from Q_{max} to Q_d along the suction surface near the blade tail. This limitation will be considered next. The deceleration from Q_u to Q_{min} along the pressure surface near the blade nose is not so critical because the boundary layer is thin in this region.

The allowable deceleration without boundary-layer separation on the suction surface is determined by the ratio Q_d/Q_{max} and the blade-element Reynolds number based on blade chord (ref. 12). This allowable deceleration can be expressed as a ratio λ of arc lengths, where λ is defined by

$$\lambda = \frac{s_{EE} - s_C}{s_E - s_C} \quad (7)$$

where the reference point for s is arbitrary. The maximum allowable value of λ for given values of Q_d/Q_{max} and Reynolds number is given in reference 12.

From equations (4), (16), and (40) of reference 11 the arc length ($s_{EE} - s_C$) is given by

$$\frac{\pi}{2} (s_{EE} - s_C) = \int_{\Phi_C}^{\Phi_{CC}} \frac{d\Phi}{Q} + \int_{\Phi_{CC}}^{\Phi_{EE}} \frac{d\Phi}{Q} \quad (8)$$

where for the type of velocity distribution in figure 4

$$\left. \begin{aligned} \log_e Q &= \log_e Q_u + (\log_e Q_{max} - \log_e Q_u) \left(\frac{\Phi - \Phi_C}{\Phi_{CC} - \Phi_C} \right) & (\Phi_C \leq \Phi \leq \Phi_{CC}) \\ \text{and} \quad \log_e Q &= \log_e Q_{max} & (\Phi_{CC} \leq \Phi \leq \Phi_{EE}) \end{aligned} \right\} \quad (9)$$

From equations (8) and (9)

$$\frac{\pi}{2} (s_{EE} - s_C) = \frac{\Phi_{CC} - \Phi_C}{\log_e \frac{Q_{max}}{Q_u}} \left(\frac{Q_{max} - Q_u}{Q_{max} Q_u} \right) + \frac{\Phi_{EE} - \Phi_{CC}}{Q_{max}}$$

and, likewise,

$$\frac{\pi}{2} (s_E - s_C) = \frac{\pi}{2} (s_{EE} - s_C) + \frac{\Phi_E - \Phi_{EE}}{\log_e \frac{Q_d}{Q_{max}}} \left(\frac{Q_d - Q_{max}}{Q_d Q_{max}} \right)$$

so that equation (7) becomes

$$\Phi_E = \Phi_{EE} + \frac{1 - \lambda}{\lambda} \left[\frac{\Phi_{CC} - \Phi_C}{\log_e \frac{Q_{max}}{Q_u}} \left(\frac{Q_{max} - Q_u}{Q_u} \right) + \right. \\ \left. \Phi_{EE} - \Phi_{CC} \right] \frac{Q_d \log_e \left(\frac{Q_d}{Q_{max}} \right)}{Q_d - Q_{max}} \quad (10)$$

Equation (10) determines the minimum value for Φ_E if the maximum allowable rate of deceleration between Φ_{EE} and Φ_E is not to be exceeded.

The values for λ in reference 12 are based on maximum allowable safe rates of deceleration which decrease in the direction of flow as the boundary-layer thickness increases. Thus the required distribution of $\log_e Q$ along Φ is not necessarily linear as assumed in this report. However, as will be shown in the section **Prescribed velocity distribution**, the assumed distribution of $\log_e Q$ with Φ has similar characteristics to those required in reference 12 and is considered accurate enough for engineering purposes.

Rounding off nose and tail of blade-element profile.—After the prescribed velocity distribution has been selected to meet the conditions discussed previously, the channel shape is determined by the numerical methods developed in reference 11. The islands between adjacent channels (fig. 1) in the physical xy -plane constitute the blade-element profiles. Because the stagnation points are not considered in this design method, the nose and tail of the blade element are cusped. These cusps are eliminated by faired curves (circular arcs, for example) that are tangent to the channel walls at the previously selected positions for the nose and tail of the blade-element profiles.

DESIGN PROCEDURE

The various conditions to be satisfied in the application of the channel design method to the design of high-solidity cascades of blades with prescribed velocity distributions along the blade contours have been discussed, and the details of the numerical procedure for the channel design itself are the same as those outlined in reference 11. A brief step-by-step outline of the conditions to be satisfied and the numerical procedure follows:

(1) The cascade, or channel, turning angle $\Delta\theta$ and the upstream and downstream velocities q_u and q_d are prescribed. (For incompressible flow the upstream velocity Q_u is sufficient because Q_d equals 1.0.) The cascade stagger angle is fixed by these prescribed conditions and the equation of continuity. The flow may be incompressible or compressible (γ equal to -1.0).

(2) The solution for the equivalent channel wall coordinates will be carried out in the $\Phi\Psi$ -plane. The relative positions of points on the channel walls in the $\Phi\Psi$ -plane, $(\Phi_D - \Phi_C)$ and $(\Phi_F - \Phi_E)$ in figure 2, corresponding to the nose and tail of the blade profile are determined by equations (3) and (4).

(3) The prescribed velocity along the channel walls upstream of Φ_C and Φ_D is equal to the upstream velocity.

(4) The prescribed velocity along the channel walls downstream of Φ_E and Φ_F is equal to the downstream velocity.

(5) The velocity distribution along the suction surface of the blade-element profile, between the points Φ_C and Φ_E (fig. 2) on the channel wall in the transformed $\Phi\Psi$ -plane, is prescribed as an arbitrary function of arc length. Usually a practical blade shape results if the prescribed velocity increases from the upstream value at the nose (Φ_C) and then decreases to the downstream value at the tail (Φ_E).

(6) The prescribed velocity along the pressure surface of the blade-element profile, between the points Φ_D and Φ_F (fig. 2) on the channel wall in the transformed $\Phi\Psi$ -plane, is

prescribed as an arbitrary function of arc length along the blade profile. Usually a practical blade shape results if the prescribed velocity decreases from the upstream value at the nose (Φ_D) and then increases to the downstream value at the tail (Φ_F).

(7) In order to obtain an efficient high-solidity cascade, the difference in prescribed velocities on the channel walls corresponding to the suction and pressure surfaces of the blades should be large so that the blade spacing is large enough to prevent serious friction losses. But the maximum prescribed velocity on the suction surface should not be so large that losses result from shock or that serious boundary-layer separation losses result from rapid deceleration to the downstream velocity at the blade tail (Φ_E in fig. 2).

(8) The prescribed velocity distribution on the channel walls must satisfy the prescribed cascade, or channel, turning angle $\Delta\theta$. This angle is determined by equation (H5) of reference 11. If the prescribed velocity distribution does not satisfy the prescribed turning angle, the velocity distribution is adjusted by trial-and-error methods; or, for the type of linear velocity distributions given in figure 4, the proper adjustment in velocity can be determined directly from equation (6).

(9) After the prescribed velocity distribution that satisfies the conditions just outlined has been selected, the channel design is determined by methods outlined in reference 11.

(10) The cusped nose and tail of the islands that result between adjacent channels in the physical plane (fig. 1) are rounded off by faired curves (circular arcs, for example) that are tangent to the channel walls at points corresponding to the nose and tail of the blade element (Φ_C through Φ_F in fig. 2). Finally, if desired, the displacement thickness of the boundary layer can be estimated by boundary-layer theory and subtracted from the preceding contours to obtain the final blade profile. Thus the high-solidity cascade design is complete.

NUMERICAL EXAMPLE

In general, as the percentage reaction decreases and as the blade camber (or turning angle) increases, the cascade efficiency decreases (ref. 13, p. 232). The problem selected is therefore to design an efficient impulse cascade (zero percent reaction) with large turning angle.

Prescribed conditions.—An impulse cascade with 90° of turning was designed for incompressible flow with the following prescribed conditions:

- $$\left. \begin{aligned} (1) \theta'_u &= \pi/4 \\ (2) \theta'_d &= -\pi/4 \end{aligned} \right\} \Delta\theta = \frac{\pi}{2} = 90^\circ$$

$$(3) Q_u = Q_d = 1.0$$

Prescribed velocity distribution.—For convenience the velocity distribution has been specified by linear variations in $\log_e Q$ with Φ . The following conditions were arbitrarily selected:

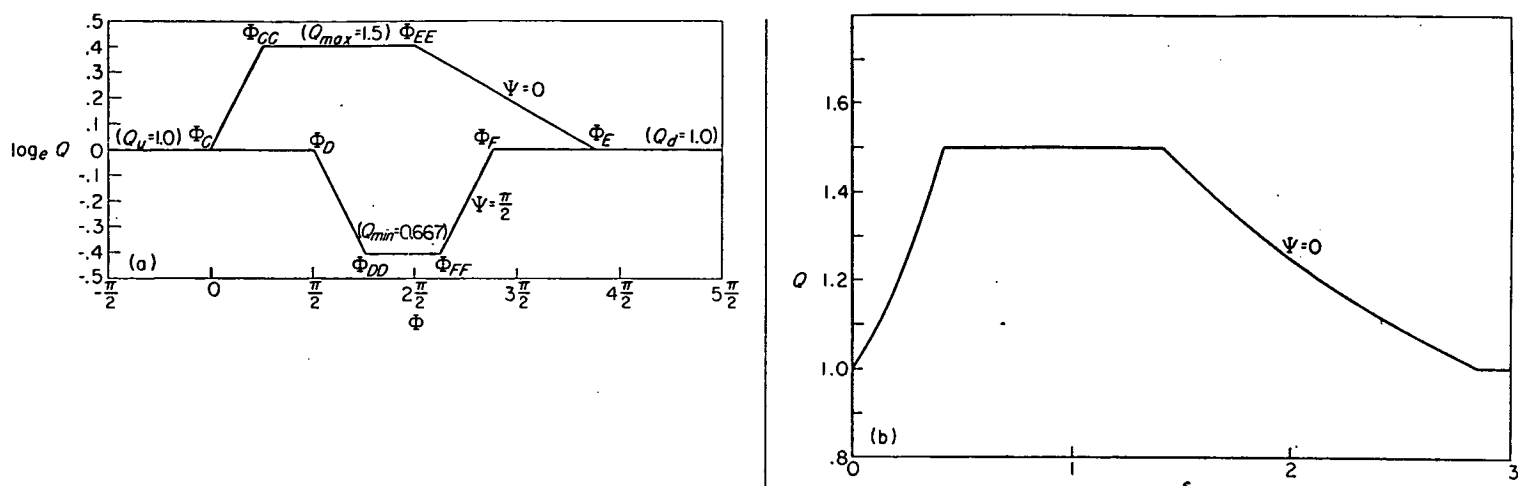
$$(1) Q_{max} = 1.5$$

$$(2) \Phi_C = 0$$

$$(3) \Phi_{CC} - \Phi_C = \Phi_{DD} - \Phi_D = \Phi_F - \Phi_{FF} = \frac{\pi}{4}$$

$$(4) \Phi_{EE} = \pi$$

The quantities $(\Phi_D - \Phi_C)$ and $(\Phi_F - \Phi_E)$ are obtained from equations (3) and (4) and are equal to $\pi/2$ and $-\pi/2$, re-

(a) Variation in $\log_e Q$ with Φ .

(b) Velocity as function of arc length along suction surface.

Figure 5.—Prescribed velocity distribution for 90° impulse cascade blade.

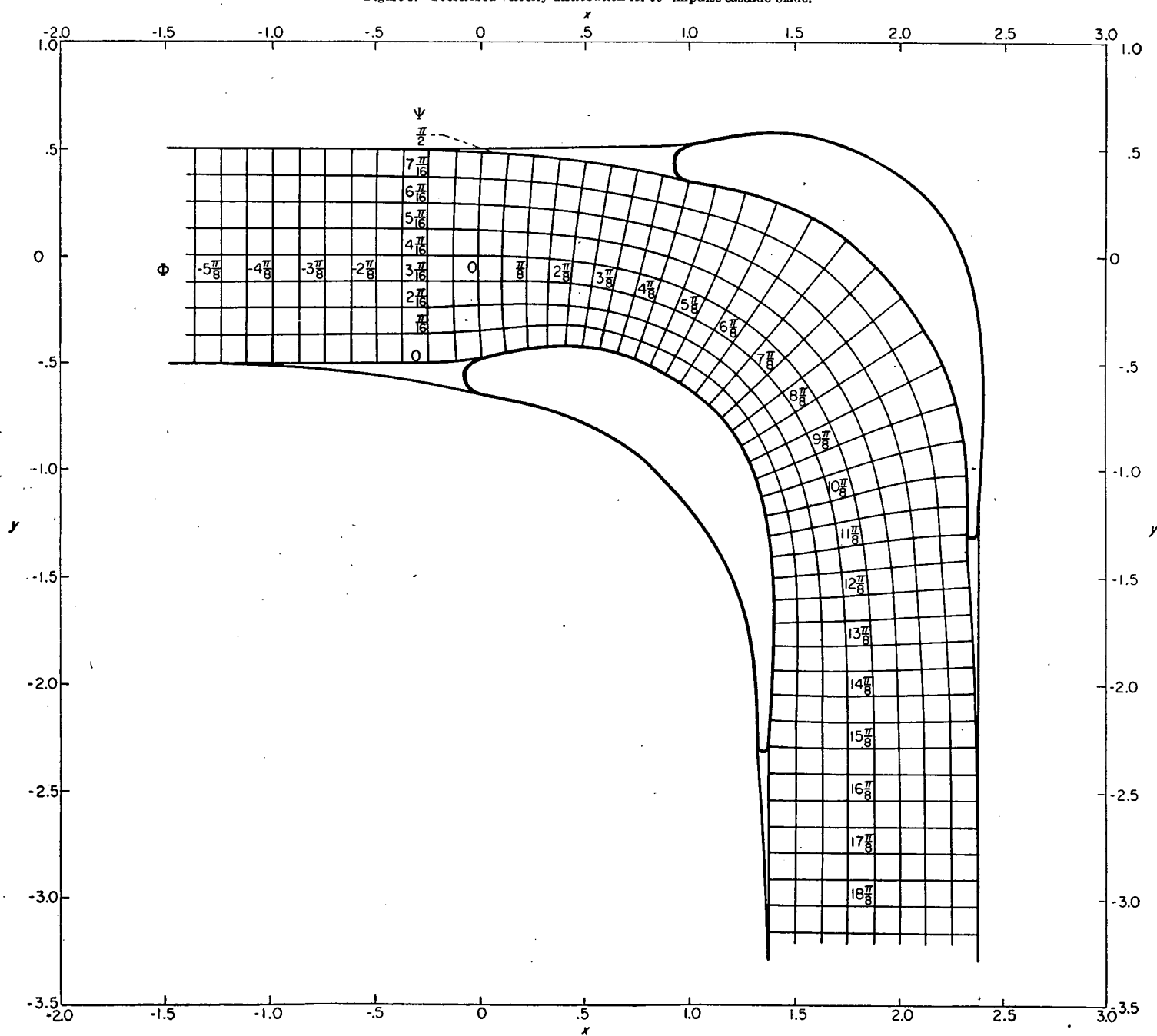


FIGURE 6.—Streamlines and velocity-potential lines for channel between blade elements of 90° impulse cascade. Designed for prescribed velocity distribution given in figure 5 (a).

spectively. The quantity Φ_E (fig. 2) is given by equation (10) with λ equal to 0.5, and is equal to $3.75 \pi/2$. (The value of 0.5 for λ was obtained from fig. 5 (a) of ref. 12 for Q_a/Q_{max} equal to 0.667 and for a blade Reynolds number equal to ∞ , which Reynolds number gives the minimum, and therefore safest, value of λ .) The value of Q_{min} was obtained from equation (6) with $\Delta\theta$ equal to $\pi/2$:

$$Q_{min} = 0.66687$$

The resulting prescribed distribution of $\log_e Q$ with Φ is given in figure 5 (a). The corresponding distribution of Q with arc length s along the suction surface can be obtained from the definition of Φ in reference 11 and is given in figure 5 (b). The velocity distribution between Q_{max} (1.5) and Q_a (1.0) is similar in shape to that resulting from the maximum allowable safe rates of deceleration in reference 12, so that the assumed linear variation in $\log_e Q$ with Φ satisfies approximately the conditions on which the results of reference 12 are based.

Cascade design.—The channel shape corresponding to the prescribed velocity distribution in figure 5 (a) was determined by the relaxation methods used in reference 11 and is plotted in figure 6 together with the resulting high-solidity cascade of blades formed by rounding off the cusped nose and tail of the islands formed between adjacent channels. For the experimental investigation, a cascade of these blades with a chord of 5.5 inches was constructed, and the coordinates for this blade profile are given in table I. The blade profile was not adjusted to provide for the displacement thickness of the boundary layer. The characteristics of the resulting cascade are given in figure 7 on the $x'y'$ -plane in which all linear distances are dimensionless, being divided by the blade chord c . The reciprocal s'/c of the cascade solidity (where s' is the blade spacing) is 0.6130. The maximum blade thickness is approximately 18 percent of the chord, the trailing-edge thickness is approximately 2.8 percent of the chord, and the

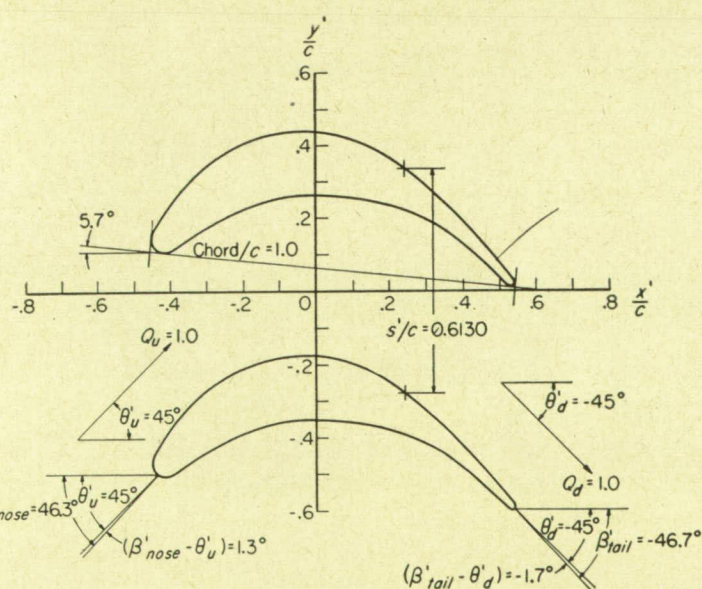


FIGURE 7.—Characteristics of 90° impulse cascade in $x'y'$ -plane. Coordinates of blade-element profile are given in table I.

radius of the circular arc at the blade nose is 3.7 percent of the blade chord. The average angle β'_{nose} of the blade surfaces tangent to the circular arc at the blade nose is 46.3° so that the blade is overturned 1.3° at the nose (fig. 7). The average angle β'_{tail} of the blade surfaces tangent to the circular arc at the blade tail is -46.7° so that the blade is overturned 1.7° at the tail. The blade profile is similar in appearance to the best shape developed in reference 6 by combination of circular arcs.

CASCADE TESTS

An experimental investigation was made on the blade profile just designed in order to determine if rounding off the cusped nose and tail has a serious effect on the resulting agreement between the prescribed velocity distribution on the blade surface and the distribution measured by test and to determine if the design procedure taking into account present knowledge of boundary-layer separation results in efficient blade shapes. In addition, the blade profile, which was designed for incompressible flow, was tested over a range of downstream Mach number from 0.2 to choke flow in order to determine effects of compressibility.

DESCRIPTION OF APPARATUS

Flow tank.—As indicated in the line drawing of figure 8, the cascade of 90° impulse blades was attached to a short tunnel of straight parallel walls that was mounted on a rounded approach at the flow test tank. Dimensions of the tank and piping are given in figure 8. The tank contained a honeycomb of square cells (2 by 2 in.) 8 inches deep. Immediately upstream of the honeycomb were three screens; one 28×30 mesh and two 40×60 mesh with the mesh oriented 90° apart. The tank pressure, and therefore the flow rate, was controlled by a valve upstream of the tank. The maximum flow rate through the tank during the tests was 88

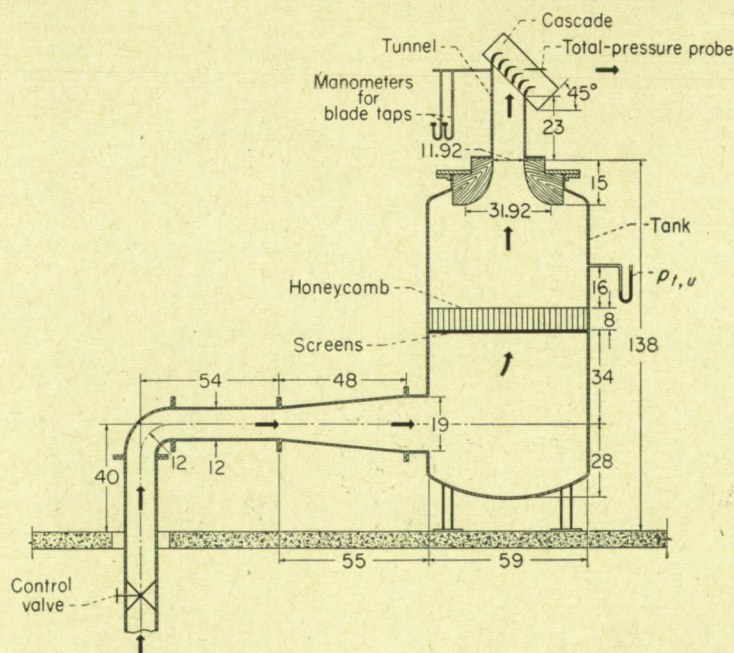


FIGURE 8.—Line drawing of test setup. All linear dimensions in inches.

pounds per second. The rounded approach to the tunnel had an elliptical profile. A photograph of the test setup is shown in figure 9.

Tunnel.—The tunnel consisted of straight parallel walls that could not be adjusted to account for boundary-layer growth or to simulate the shape of the stagnation streamlines upstream of the cascade. (For high-solidity cascades the position and shape of the upstream tunnel walls have little effect on flow conditions in the channel between blades where the character of the flow is almost exclusively influenced by the shape of the blades.) The tunnel length was short to prevent large boundary-layer growth on the tunnel walls. The cross section of the tunnel normal to the direction of flow was 11.91 by 16.50 inches.

Cascade.—The blade-element profile is described in table I, and the cascade characteristics are given in figure 7. Six blades with a chord of 5.5 inches and an aspect ratio of 3.0 were used (see ref. 14, p. 3). The blade span was therefore 16.5 inches. Based on a blade chord of 5.5 inches, the Reynolds number Re was approximately related to the downstream Mach number M_a by

$$Re \approx 3 \times 10^6 M_a$$

Thus for the minimum test value of M_a equal to 0.2 the Reynolds number was 600,000, which is well above the critical values indicated in reference 12. A photograph of the assembled cascade is shown in figure 10.

INSTRUMENTATION

Tank and tunnel.—The total pressure upstream of the cascade was measured by a static tap downstream of the honeycomb in the tank (fig. 8). The total temperature of the air was measured by a thermocouple in the tank. Static pressure was measured on the center line of the tunnel walls, 12 inches upstream of the cascade.

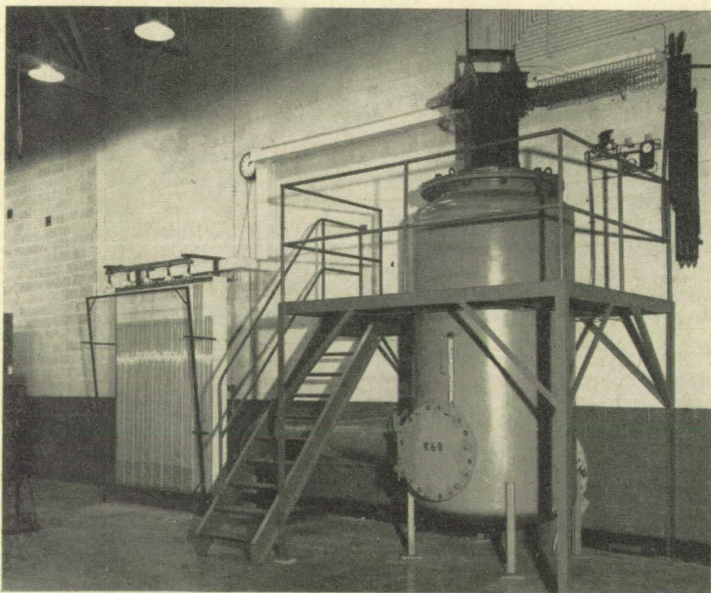


FIGURE 9.—Photograph of test setup.

Cascade.—Static pressures at midspan on the blade surfaces of the center channel in the cascade were measured at 48 locations indicated in table II. In addition total-pressure and flow-direction surveys were made at midspan across the center channel between blades in a plane $1\frac{1}{2}$ inches downstream of the exit plane of the cascade. (The flow direction was essentially constant in the survey plane.) The total-pressure probe was unshielded and the yaw probe was of the wedge type. The static pressure p_a , used to determine flow conditions downstream of the cascade, was measured at a wall tap located approximately $1\frac{3}{4}$ inches downstream of the exit plane of the cascade and was for all values of M_a approximately equal to atmospheric room pressure.

TEST RESULTS

Static pressures on the blade surfaces of the center channel in the cascade were obtained for eight values of the downstream Mach number between 0.2 and 0.79. (The cascade choked at a downstream Mach number between 0.75 and 0.79.) In addition, total-pressure surveys were made and the flow direction was measured downstream of the cascade for five values of the downstream Mach number between 0.3 and 0.7. The results are plotted in figures 11 to 14.

Pressure coefficient P .—The pressure coefficient P is plotted in figure 11 as a function of the coordinate x'/c along the blade surface (fig. 7). The pressure coefficient P is defined by

$$P = \frac{p - p_a}{p_{t,a} - p_a} \quad (11)$$

where p is the static pressure and $p_{t,a}$ is the total pressure downstream of the cascade exclusive of the wake and is therefore equal to the upstream total pressure. For incompressible flow the pressure coefficient P , defined by equation (11),

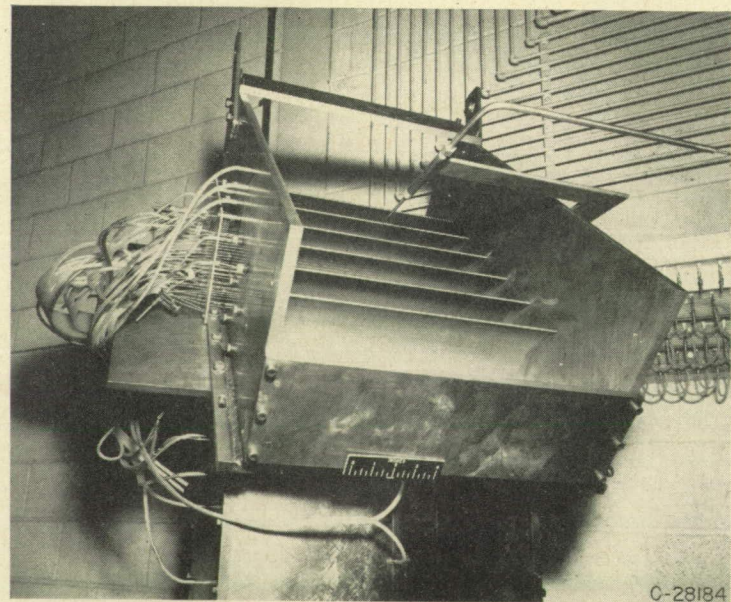


FIGURE 10.—Photograph of cascade with downstream total-pressure probe in position.

reduces to the usual definition for pressure coefficient, that is, pressure difference divided by downstream, or upstream, velocity head. The pressure tap corresponding to a given data point can be determined from the value of x'/c and table II. For incompressible flow P is related to Q by

$$P = 1 - Q^2$$

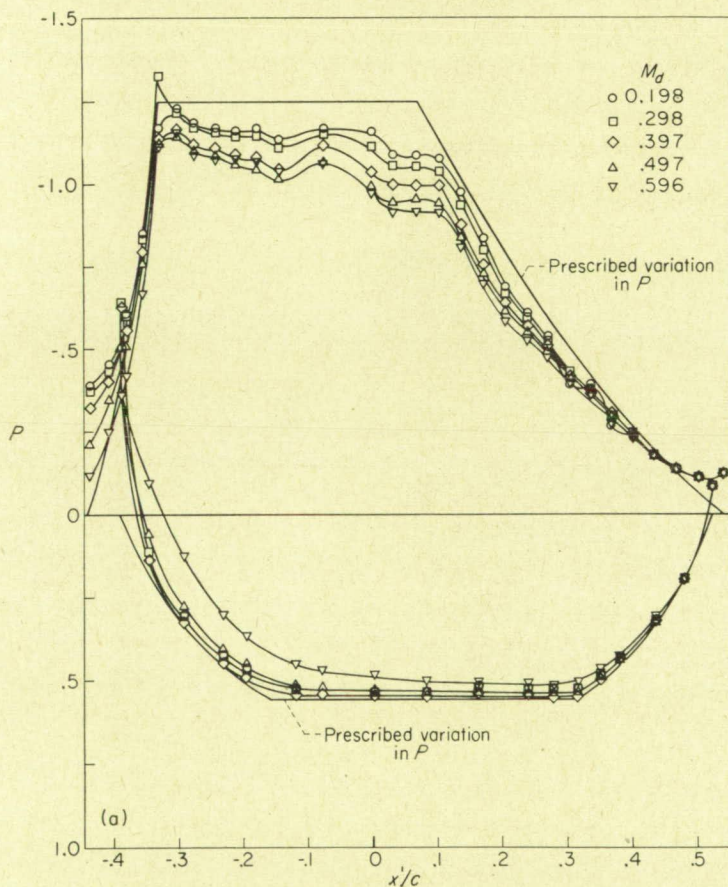
The variation in P with x'/c for the design variation in Q is given by the regular solid lines in figure 11.

In figure 11 (a) the results are plotted for the downstream Mach numbers between 0.2 and 0.6. For these Mach numbers the flow is everywhere subsonic. The agreement between the measured and prescribed (design) values of P is good for M_d equal to 0.198, but becomes progressively worse because of compressibility effects as M_d increases. In general, the discrepancy between the measured and prescribed values of P at M_d equal to 0.198 can be attributed in part to the lower-than-design flow rate that results from the reduced effective flow area due to the wake displacement downstream of the cascade. That is, the channel between blades turned a slightly smaller quantity of fluid than designed for, and therefore required slightly less pressure difference on the blade surfaces. The important discrepancy between the

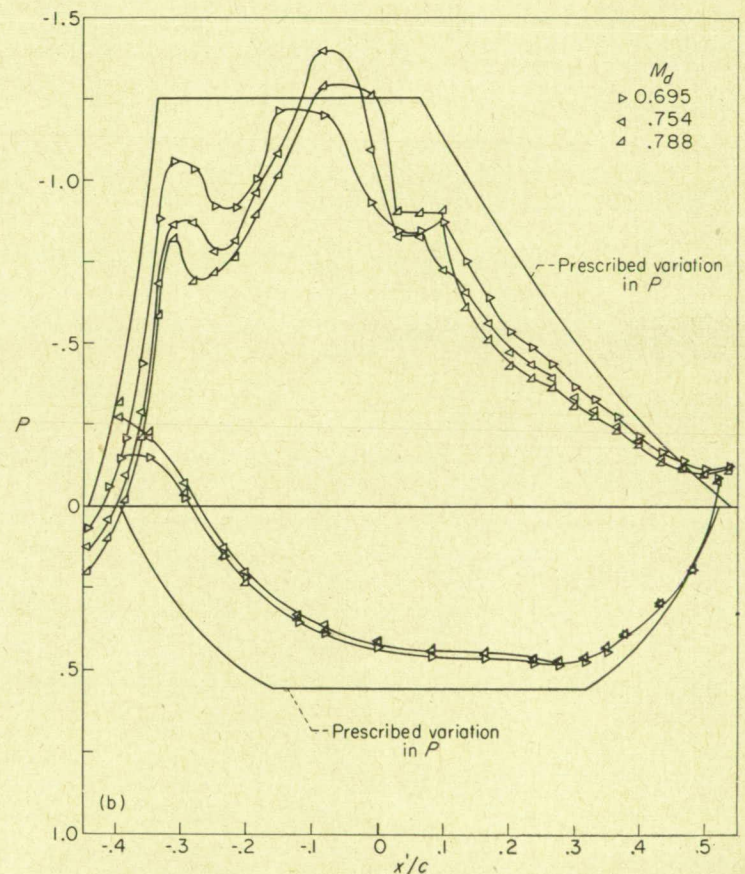
design and experimental value of P on the pressure surface at the nose (x'/c equal to -0.393) results from rounding off the blade nose and will be discussed later.

In figure 11 (b) the results (P against x'/c) are plotted for downstream Mach numbers of 0.70, 0.75, and 0.79. For these Mach numbers it will be shown that local regions of supersonic flow exist on the suction surface of the blade and shock phenomena result as indicated by the rapid fluctuation in pressure. For all three of these values of downstream Mach number two regions of shock appear on the suction surface; one at x'/c approximately equal to -0.275 and the other centrally located around x'/c equal to zero.

Velocity Q .—The velocity Q , which is dimensionless, having been divided by the downstream velocity, is plotted in figure 12 as a function of the ideal (design) velocity potential Φ along the blade surface. The pressure tap corresponding to a given data point can be determined from the value of Φ and table II. The prescribed variation in Q with Φ is given by the regular solid lines in figure 12. The experimental values of Q were obtained from the total pressure and the measured static pressures. The total pressure was assumed equal to the tank pressure, so that in the presence of shock losses the computed velocities are indicative only.



(a) Downstream Mach numbers between 0.2 and 0.6.



(b) Downstream Mach numbers of 0.70, 0.75, and 0.79.

FIGURE 11 — Variation in test values of pressure coefficient P with position x'/c along surface of blade element. (The pressure tap corresponding to a given data point can be determined from the value of x'/c and table II.)

In figure 12 (a) the variation of velocity Q with Φ is plotted for the downstream Mach numbers between 0.2 and 0.6. For these Mach numbers the flow is everywhere subsonic, and the agreement with the prescribed velocity distribution is considered quite good and appears to be independent of the downstream Mach number M_d . (This independence of downstream Mach number will be discussed in the section **Compressibility effects**.) It is concluded that blades for high-solidity cascades can be designed for prescribed velocities by the channel flow methods of this report and that rounding off the nose and tail of the blade-element profile has negligible effect on the velocity distribution along the blade surface except in the vicinity of the blade nose. The discrepancy at the blade nose will be discussed in the section **Continuity**.

In figure 12 (b) the variation in velocity Q with Φ is plotted for downstream Mach numbers of 0.70, 0.75, and 0.79. For these Mach numbers local regions of supersonic flow exist on the suction surface of the blade, and shock phenomena result at the points indicated previously by the pressure distribution in figure 11 (b).

Loss coefficient $\frac{\bar{\omega}}{(p_t - p)_d}$ —The loss coefficient $\frac{\omega}{(p_t - p)_d}$ has been computed from the total-pressure survey data taken downstream of the cascade for five values of M_d between 0.3 and 0.7. This loss coefficient is plotted in figure 13. The average total-pressure loss $\bar{\omega}$ was calculated by methods given

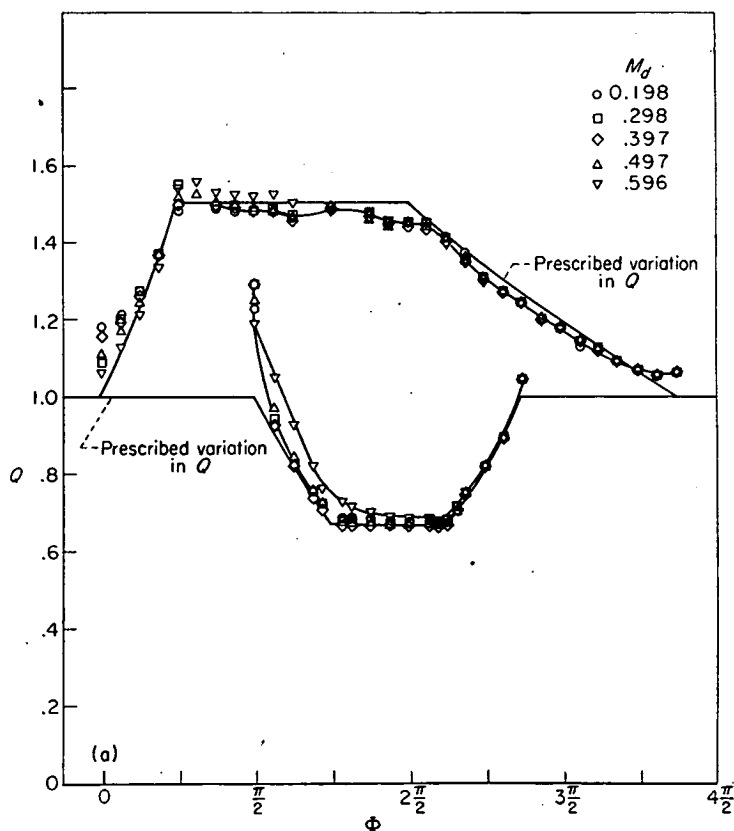
in appendix B. The loss coefficient at first decreases with increasing M_d , probably as a result of the increasing Reynolds number, and then increases rapidly, as a result of shock losses, after M_d equal to 0.6.

Turning angle $\Delta\theta$.—The measured value of the cascade turning angle $\Delta\theta$ is plotted as a function of M_d in figure 14. The sudden increase in $\Delta\theta$ as M_d approaches 0.6 is not reasonable in view of the small increase in loss coefficient (fig. 13) at this Mach number. The measured turning angle may therefore be inaccurate because of the adverse test conditions under which the data were recorded at high Mach numbers. In any event, for values of M_d less than 0.5 the turning angle is insensitive to Mach number, and the measured turning angle agreed within 0.5° with the design turning angle.

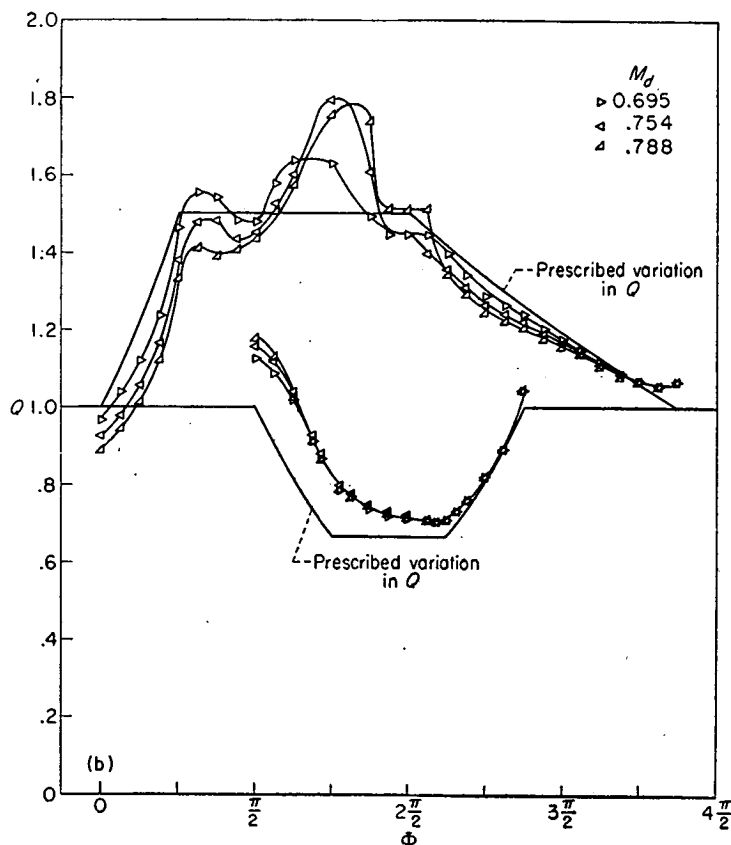
ANALYSIS OF RESULTS

The test results are analyzed for the continuity condition, compressibility effects, and momentum considerations. The cascade performance is then compared with that of similar cascades reported in the literature.

Continuity.—In figure 15 it is shown that, as the downstream Mach number M_d increases, the experimentally determined upstream Mach number M_u becomes progressively less than M_d . For an impulse cascade M_u should equal M_d , and the measured difference between M_d and M_u was sufficiently great to require an investigation of the continuity condition upstream and downstream of the cascade.



(a) Downstream Mach numbers between 0.2 and 0.6.



(b) Downstream Mach numbers of 0.70, 0.75, and 0.79.

FIGURE 12.—Variation in test values of velocity Q with coordinate Φ along surface of blade element. (The tap position corresponding to a given data point can be determined from the value of Φ and table II.)

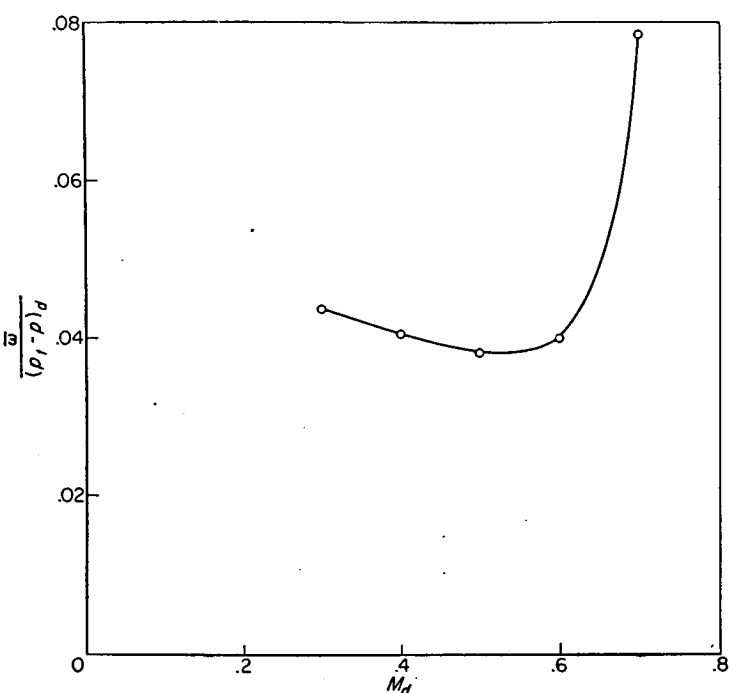


FIGURE 13.—Variation in pressure-loss coefficient with downstream Mach number.

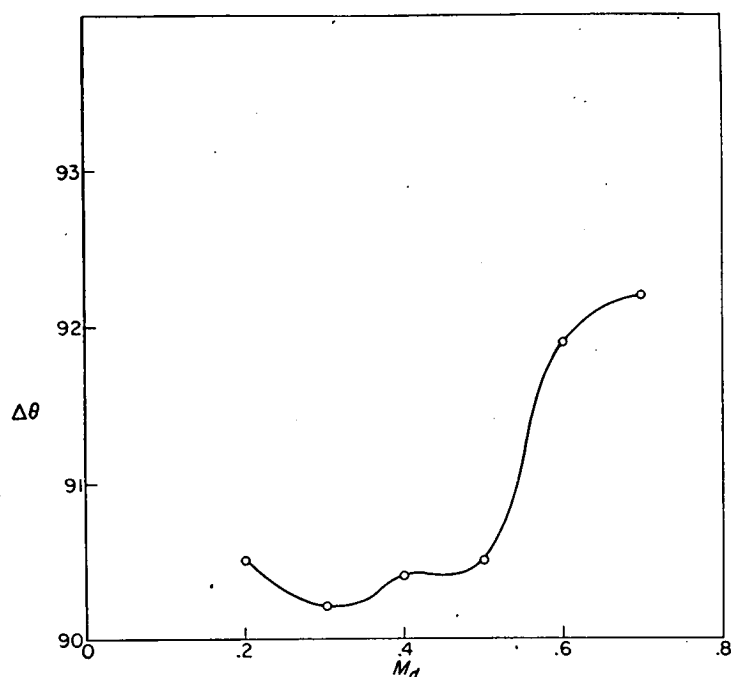


FIGURE 14.—Variation in air-turning angle with downstream Mach number.

From the total-pressure surveys downstream of the cascade, the velocity distribution is obtained as a function of y'/s' , where y' is measured from the center of a wake, and a typical example is given in figure 16 for a downstream Mach number of 0.3. (From the relatively small momentum losses indicated by this velocity distribution, it is concluded that boundary-layer separation on the blade surfaces was negligible.) From this velocity distribution the flow rate W through the channel between blades can be determined from the continuity equation

$$W_a = s' \int_0^{1.0} \rho'' q'' \cos \theta'_a d\left(\frac{y'}{s'}\right)$$

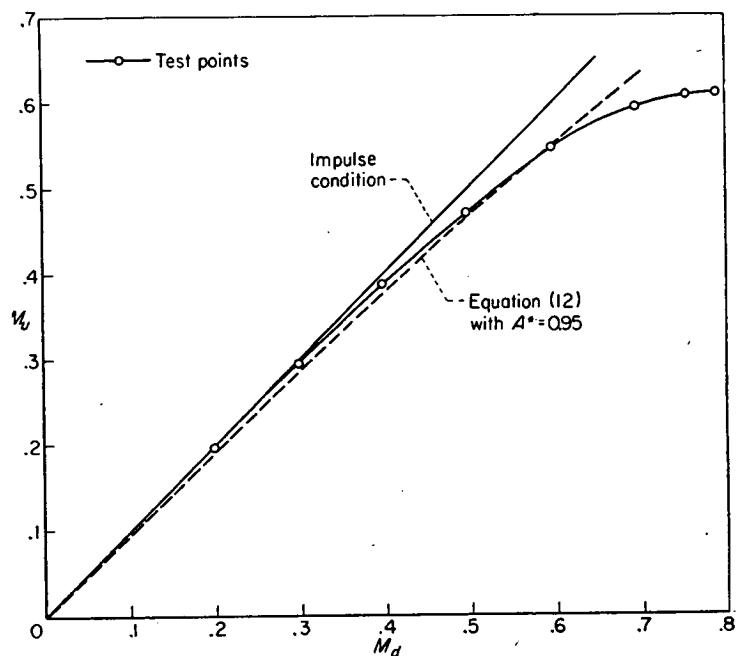
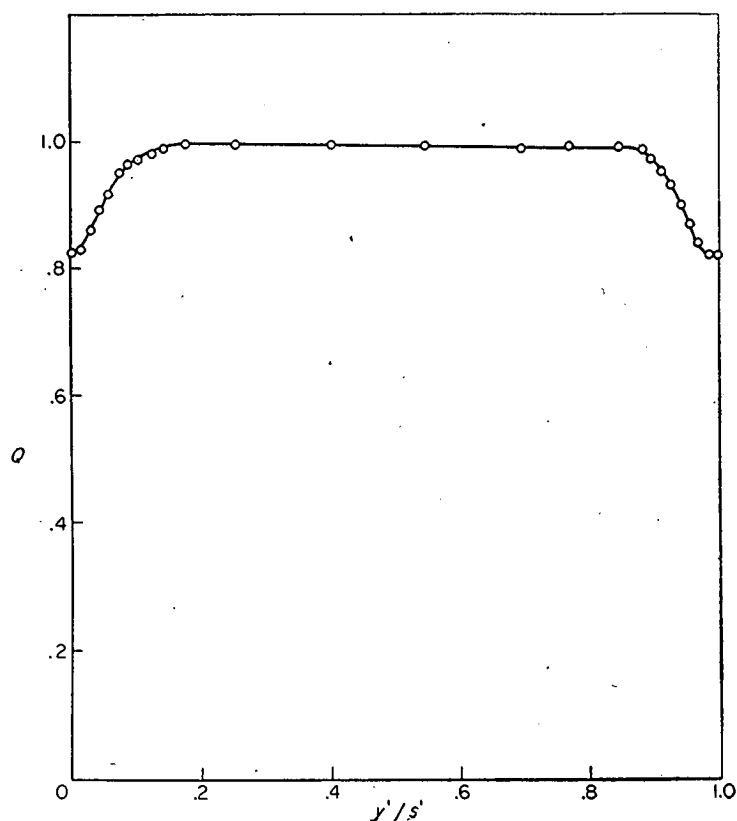


FIGURE 15.—Variation in upstream Mach number with downstream Mach number.

FIGURE 16.—Variation in Q with y'/s' between wakes downstream of cascade. Downstream Mach number M_d , 0.30.

where ρ'' and q'' are the density and velocity, respectively, in dimensional form and θ'_a is the measured flow direction downstream of the cascade in the $x'y'$ -plane. Upstream of the cascade, flow conditions are uniform and the continuity equation becomes

$$W_u = s'(\rho'' q'' \cos \theta')_u$$

The flow rates W_u and W_d are plotted in figure 17, and it is seen that the continuity condition ($W_u = W_d$) is satisfied. The increasing magnitude of $(M_d - M_u)$ with increasing M_d (fig. 15) must therefore be caused by the displacement of the wake downstream of the cascade.

If A^* is the ratio of the effective flow area (geometric area minus the effective displacement area of the wake) downstream of the cascade to the geometric area upstream of the cascade, M_d and M_u are related by (appendix C)

$$\frac{M_u}{M_d} = A^* \left(\frac{1 + \frac{\gamma-1}{2} M_u^2}{1 + \frac{\gamma-1}{2} M_d^2} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad (12)$$

For an impulse cascade with no boundary layer, A^* equals 1.0 and from equation (12) M_u is equal to M_d . If, however, the effective displacement of the boundary layer is 5 percent, A^* is equal to 0.95 and the relation between M_u and M_d is given by the dashed curve in figure 15. Thus, for a given value of A^* a cascade designed for impulse operation without a wake exhibits progressively more reaction as the downstream Mach number increases.

In figure 15 for values of M_d greater than 0.75, the value of M_u remains essentially constant and equal to 0.6, or a little higher, so that the cascade is choked. In figure 12 the high value for Q on the pressure surface at Φ equal to $\pi/2$ indicates that the choke condition occurs along this value of Φ . Figure 6 shows that for this value of Φ the flow area of the channel between blade elements is a minimum. Thus the upstream Mach number for choke flow could probably be increased by a slight modification in the blade-element design in the regions of Φ equal to $\pi/2$. (For example, the manner in which the blade nose is rounded off might be modified to increase the minimum flow area. Also, the blade nose might be extended farther upstream along the channel boundaries (shown in fig. 6) to guide the fluid into the minimum area in the proper direction. Or perhaps near the nose a less rapid velocity deceleration might be prescribed on the pressure surface, or a less rapid acceleration on the suction surface, so that the rate of area convergence and divergence in the vicinity of the minimum area would be reduced.) In addition this design modification would eliminate the large deceleration of the velocity along the pressure surface following the peak velocity at Φ equal to $\pi/2$ and might thus improve the efficiency of the cascade by eliminating a possible region of separated boundary layer.

Compressibility effects.—The effects of compressibility on P and Q are shown by the effects of M_d in figures 11 and 12. Consider the region of constant prescribed velocity along the suction surface. Provided the local velocities are subsonic, the absolute magnitude of the pressure coefficient P decreases with increasing M_d (fig. 11(a)), but the velocity Q remains essentially unchanged (fig. 12(a)). This behavior of P and Q is unlike that for isolated blades (airfoils), but compares favorably with that for the known compressible flow between a curved channel consisting of streamlines from a free compressible vortex. (The regions of constant velocity along the pressure and suction surfaces of the blades

suggest that the channel between these regions can be approximated by the flow between selected streamlines of a free vortex.) In appendix D equations are derived for computing the variation in P and Q with the equivalent M_d for those radii of a free compressible vortex for which the values of P and Q at M_d equal to zero (incompressible flow) are the prescribed values for the cascade design ($Q_{max}=1.5$, $Q_{min}=0.66687$, and so forth). The resulting distributions in Q and P with M_d for the compressible vortex are shown in figures 18 and 19, respectively, and are compared with the test values of Q and P at taps 44 and 20 (see table II) on the pressure and suction surfaces of the blades, respectively. The agreement in trends is good and indicates that the observed variations in P and Q with M_d in the tests are reasonable. Thus, for the high-solidity impulse cascade of this report the distribution of Q is essentially independent of downstream Mach number M_d .

The appearance of supersonic velocities on the suction surface of the blade is indicated (in figs. 11(b) and 12(b)) by sizeable variations in P and Q (with x'/c and Φ , respectively) due to shock phenomena. For a given value of M_d there is a critical value of Q (Q_{cr}) for which the velocity corresponding to Q_{cr} is sonic. This relation is given by (appendix E)

$$Q_{cr} = \sqrt{\frac{1 + \frac{\gamma-1}{2} M_d^2}{\frac{\gamma+1}{2} M_d^2}} \quad (13)$$

which is plotted in figure 20. For test values of Q about equal to 1.5, such as exist on the suction surface at Φ equal to $\pi/4$, the value of M_d in figure 20 is about 0.63. Thus in figures 11(b) and 12(b) shock phenomena are observed for values of M_d equal to 0.70, 0.75, and 0.79.

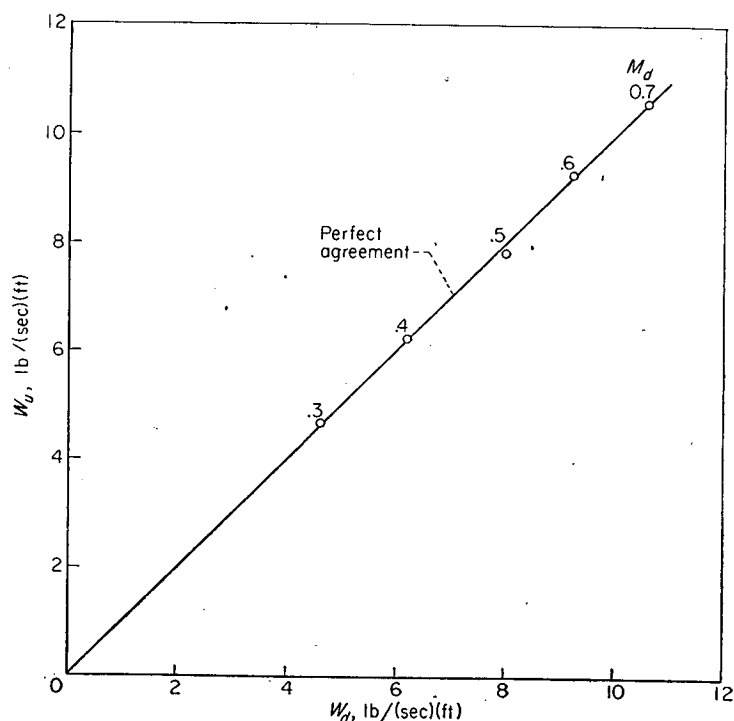


FIGURE 17.—Comparison of measured flow rates upstream and downstream of cascade for five values of downstream Mach number.

Momentum.—From momentum considerations it can be shown (appendix F) that the blade force in the direction of y' (per unit length of blade span) is equal to

$$F_y = \frac{s' \sin \theta'_a \cos \theta'_a}{g} \int_0^{1.0} \rho''_a (q''_a)^2 d\left(\frac{y'}{s'}\right) - \frac{s' \sin \theta'_u \cos \theta'_u}{g} \rho''_u (q''_u)^2 \quad (14)$$

where ρ'' and q'' are evaluated from the test data by equations (B2), (B3), and (B5) of appendix B, and the known value of the stagnation speed of sound a_o .

From the measured pressures on the blade surface, the blade force F_y is also equal to

$$F_y = c(p_{t,a} - p_d) \int_{nose}^{tail} \left(P_{\frac{\pi}{2}} - P_0\right) d\left(\frac{x'}{c}\right) \quad (15)$$

where the subscripts $\pi/2$ and 0 refer to the pressure and suction surfaces of the blade, respectively, and where for a given M_d the integral is equal to the area under the curve in figure 11. The blade force F_y has been computed from the test data by equations (14) and (15), and the values are compared in figure 21. The agreement is considered satisfactory and serves as a check on the accuracy of the experimental data.

Comparison with other impulse cascades.—The test performance [$\Delta\theta$ and minimum $\bar{\omega}/(p_t - p_d)$] of the 90° impulse cascade given in this report is compared in the following table with that of other impulse cascades reported in the literature:

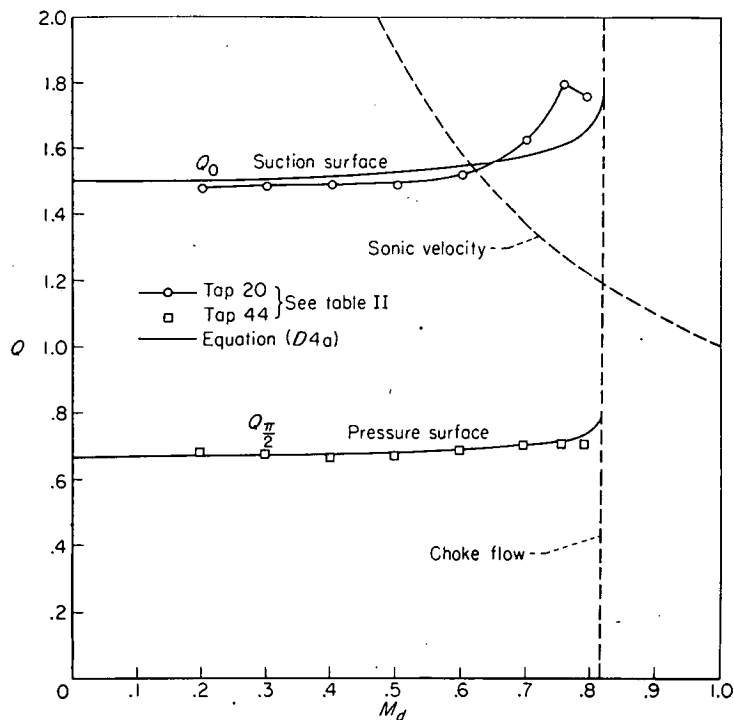


FIGURE 18.—Variation in velocity Q with downstream Mach number M_d at static taps on pressure and suction surfaces of blade and at equivalent radii of a free compressible vortex.

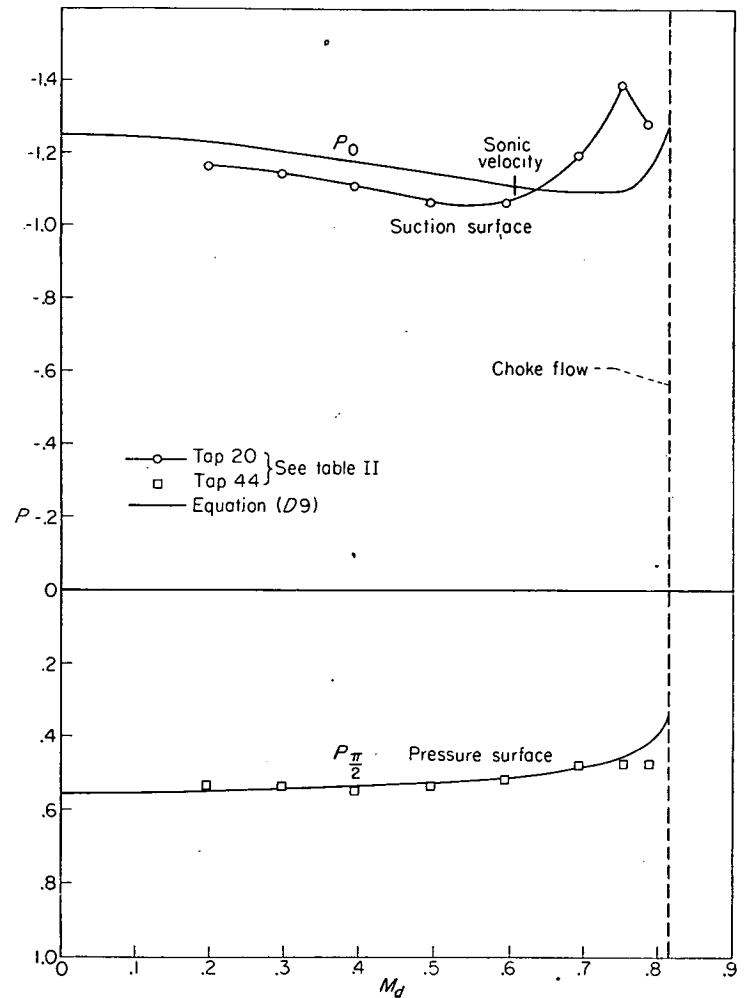


FIGURE 19.—Variation in pressure coefficient P with downstream Mach number M_d at static taps on pressure and suction surface of blade and at equivalent radii of a free compressible vortex.

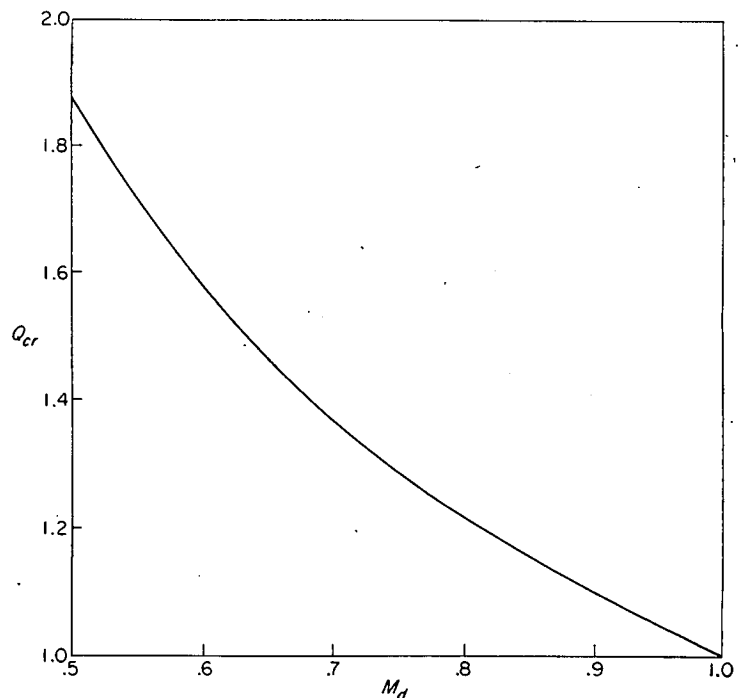


FIGURE 20.—Variation in Q_{cr} with downstream Mach number. (For a given value of M_d sonic velocity occurs for Q equal to Q_{cr} .)

Blade	Inverse of solidity, s/c	Loss coefficient, $\frac{\bar{\omega}}{(p_t - p)_a}$	Turning angle, $\Delta\theta$, deg	Reynolds number, Re	Comments	Reference
A.....	0.750	0.035 (min.).....	90	2×10^5	Airfoil blading $t/c=0.10$	13, fig. 62
B.....	.750	0.035.....	86	Low speed.....	Blade shape similar to blade C.....	15, fig. 178
C.....	.613	0.038 (min.).....	90.5	1.5×10^5	Designed for prescribed velocity gradients that avoid separation.....	This report
D.....	.500	0.038.....	87	Low speed.....	Airfoil blading.....	15, fig. 177
E.....	.500	0.039 (min.).....	90	2×10^5	Airfoil blading, $t/c=0.10$	13, fig. 62
F.....	.574	0.05.....	88.6	1.8×10^5	Two circular arcs plus flat section, similar to blade C.....	6, p. 4
G.....	.625	0.072 (min.).....	90	2×10^5	"Conventional" impulse blade, $t/c=0.22$	13, fig. 62
H.....	.500	0.09.....	88.7	1.8×10^5	Two circular arcs, sharp nose and tail, symmetrical.....	6, p. 4

The blades A to H are arranged in order of increasing minimum loss coefficient $\frac{\bar{\omega}}{(p_t - p)_a}$. All cascades have approximately a 90° turning angle. Blade C, from this report, is seen to have about as low losses as any reported in the literature. The low loss coefficients of blades A and E are questioned in reference 13 (p. 233) because of the experimental technique. Also, the thin profiles (small values of t/c , where t is the maximum blade thickness) of blades A, D, and E prohibit their use in turbines near the blade root (where impulse conditions are usually approached) because the blade taper requires thicker profiles at the root. Blade B has a thicker profile and gives excellent performance. It is similar in shape and performance to blade C, developed in this report.

It is concluded that, if properly applied, the high-solidity blade-element design method developed in this report can result in efficient blade profiles for incompressible flow or for compressible flow with local subsonic velocities. These profiles can be designed directly without extensive experimental trial-and-error development.

SUMMARY OF RESULTS AND CONCLUSIONS

A technique is developed for application of a channel design method to the design of high-solidity cascades with prescribed velocity distributions as a function of arc length along the blade-element profile and for prescribed turning angles of the fluid. The technique applies to both incompressible and subsonic compressible (ratio of specific heats equal to $\gamma=1.0$) fluid motion, and the results are exact except for the usual approximation resulting from rounding off the nose and tail of the blade element. In order to investigate the effect on the velocity distribution of rounding off the nose and tail, a high-solidity 90° impulse cascade was designed and tested. To achieve good efficiency, the cascade was designed for prescribed velocities with maximum allowable blade loading according to limitations imposed by considerations of boundary-layer separation. The cascade was developed for incompressible flow and was tested at the design angle of attack over a range of downstream Mach

numbers from 0.2 to choke flow. From the results of the tests it is concluded that:

1. Blades for high-solidity cascades can be designed for prescribed velocities by the channel flow methods of this report.
2. Rounding off the nose and tail of the blade-element profile has negligible effect on the velocity distribution along the blade surface except in the vicinity of the blade nose.
3. The distribution of the velocity (expressed as a ratio of the downstream velocity) is essentially independent of downstream Mach number, provided the maximum velocity on the blade surface is subsonic.
4. For the velocity distribution that was prescribed (and measured), the boundary-layer separation on the blade surfaces was negligible.
5. For downstream Mach numbers of 0.5 or less the measured turning angle was less than 0.5° greater than the design turning angle (90°).

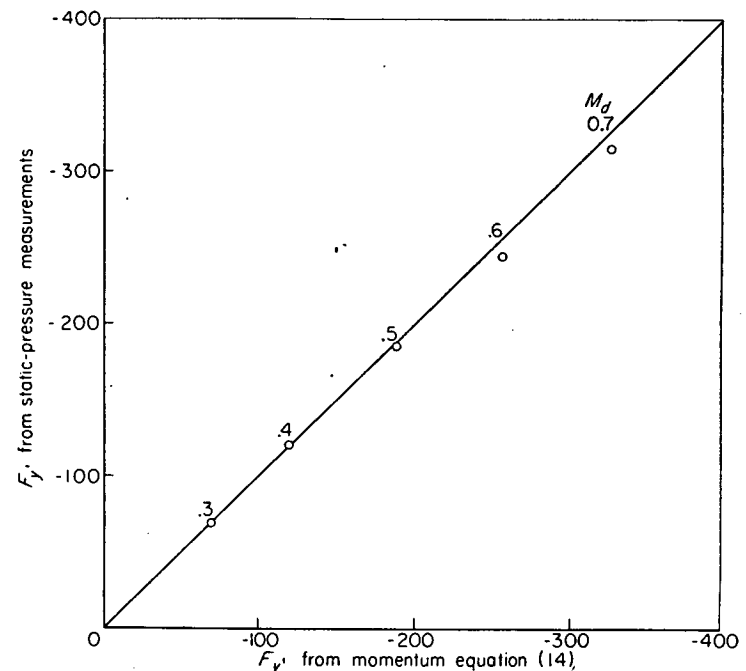


FIGURE 21.—Comparison of blade force in direction of y' as determined from measured static pressures on blade surface and from momentum equation (14).

6. The cascade choked near the inlet at an upstream Mach number slightly greater than 0.6. This Mach number for choke could probably be increased by a modification in the profile design near the blade nose.

7. Sonic velocity first appears on the suction surface of the blade at a downstream Mach number of about 0.63 and for downstream Mach numbers of 0.70, 0.75, and 0.79 shock phenomena were observed on the blade surfaces.

8. A cascade designed for impulse operation without a boundary layer exhibits progressively more reaction in the presence of a constant wake displacement as the downstream Mach number increases.

9. If properly applied, the high-solidity blade-element design method developed in this report can result in efficient blade profiles for incompressible flow or for compressible flow with local subsonic velocities. These profiles can be designed directly without extensive experimental trial-and-error development.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, November 30, 1951.

APPENDIX A

SYMBOLS

The following symbols are used in this report:

A^* ratio of flow area downstream of cascade to flow area upstream

a local speed of sound

a_o stagnation speed of sound

c blade chord (fig. 7)

F_y blade force in direction of y'

g gravitational acceleration

K_1, K_2 constants, eqs. (D1a) and (D4a), respectively, of appendix D

M Mach number, q''/a

P pressure coefficient, eq. (11)

p static pressure (dimensional form)

Δp_i loss in total pressure at point downstream of cascade

Q velocity (for which units are so chosen that channel velocity downstream at infinity is unity)

Q_{cr} critical value of Q for which velocity corresponding to Q_{cr} is sonic; related to M_a by eq. (13)

q velocity (for which units are so chosen that stagnation speed of sound is unity)

q'' velocity (dimensional form)

R perfect gas constant

Re Reynolds number based on blade chord

r radius from center of free vortex (for which radius the units are so chosen that width, downstream at infinity, of channel between cascade blades that is being simulated by free vortex is unity)

s distance in xy -plane measured along direction of flow from arbitrary reference point (for which distance the units are so chosen that channel width downstream at infinity is unity)

s' blade spacing (fig. 7)

T temperature of gas

t maximum thickness of blade-element profile

V velocity parameter defined in ref. 11

W flow rate (per unit length of blade span) through channel between two blade elements

x, y Cartesian coordinates in physical plane (for which coordinates the units are so chosen that channel width downstream at infinity is unity)

x', y' x, y -coordinate system rotated and translated so that cascade lies along y' -axis (fig. 7)

$\beta'_{nose}, \beta'_{tail}$ average angle of blade surfaces tangent to circular arcs at nose and tail, respectively, in $x'y'$ -plane (fig. 7)

γ ratio of specific heats

Δ finite increment

θ flow direction in physical xy -plane (measured in counterclockwise direction from positive x -axis)

$\Delta\theta$ channel, or cascade, turning angle

θ' flow direction in physical $x'y'$ -plane (measured counterclockwise from positive x' -axis)

λ ratio of arc lengths, eq. (7)

ρ density, (expressed as ratio of stagnation density)

ρ'' density (dimensional form)

Φ, Ψ velocity potential and stream function, respectively, used as Cartesian coordinates in transformed $\Phi\Psi$ -plane and defined in ref. 11

$\bar{w}$ average loss in total pressure, eq. (B1) of appendix B

Subscripts:

$A, B, \dots, H$ positions defined by velocity potential lines in fig. 2

CC, DD, EE, FF values of Φ defined in fig. 4

d conditions downstream at infinity

max maximum

min minimum

t total, or stagnation, condition

u conditions upstream at infinity

0 right boundary of channel, when faced in direction of flow, along which Ψ is equal to zero

$\frac{\pi}{2}$ left boundary of channel, when faced in direction of flow, along which Ψ is equal to $\pi/2$

APPENDIX B

CALCULATION OF AVERAGE LOSS IN TOTAL PRESSURE $\bar{\omega}$ FROM TOTAL-PRESSURE SURVEY DOWNSTREAM OF CASCADE

By definition the mass-weighted average value of the loss in total pressure ($\Delta p_{t,d}$) downstream of the cascade is

$$\bar{\omega} = \frac{\int \rho'' q (\Delta p_{t,d}) dy'}{\int \rho'' q dy'} \quad (B1)$$

where the integration is taken across an entire channel equal in width to the blade spacing and therefore including the wake region. The density ρ'' is related to the pressure p by

$$\frac{\rho''}{\rho''_t} = \left(\frac{p}{p_t} \right)^{\frac{1}{\gamma}} \quad (B2)$$

where, from the perfect gas law,

$$\rho''_t = \frac{p_t}{RT_t} \quad (B3)$$

Also, from the general energy equation,

$$\frac{T}{T_t} = 1 - \frac{\gamma-1}{2} q^2 \quad (B4)$$

where T is the temperature of the gas and γ is the ratio of specific heats, so that

$$\frac{p}{p_t} = \left(\frac{T}{T_t} \right)^{\frac{\gamma}{\gamma-1}} = \left(1 - \frac{\gamma-1}{2} q^2 \right)^{\frac{\gamma}{\gamma-1}}$$

from which

$$q = \sqrt{\frac{2}{\gamma-1} \left[1 - \left(\frac{p}{p_t} \right)^{\frac{\gamma-1}{\gamma}} \right]} \quad (B5)$$

Thus, from equations (B1), (B2), (B3), and (B5)

$$\bar{\omega} = \frac{\int p_t (\Delta p_t) \left(\frac{p_d}{p_t} \right)^{\frac{1}{\gamma}} \sqrt{1 - \left(\frac{p_d}{p_t} \right)^{\frac{\gamma-1}{\gamma}}} dy'}{\int p_t \left(\frac{p_d}{p_t} \right)^{\frac{1}{\gamma}} \sqrt{1 - \left(\frac{p_d}{p_t} \right)^{\frac{\gamma-1}{\gamma}}} dy'} \quad (B6)$$

where p_t is the total pressure measured by the survey along y' .

APPENDIX C

RELATION BETWEEN M_u AND M_d FOR VARIOUS AREA RATIOS A^*

From continuity considerations

$$\rho_u (Ma)_u = \rho_d (Ma)_d A^* \quad (C1)$$

where ρ is the fluid density expressed as a ratio of the stagnation density, M is the local Mach number, a is the local speed of sound so that the product (Ma) is equal to the velocity q'' , and A^* is the ratio of the effective downstream flow area to the upstream area. From the general energy equation,

$$\frac{T_t}{T} = 1 + \frac{\gamma-1}{2} M^2 \quad (C2)$$

so that

$$\frac{a_d}{a_u} = \sqrt{\frac{T_d}{T_u}} = \sqrt{\frac{1 + \frac{\gamma-1}{2} M_u^2}{1 + \frac{\gamma-1}{2} M_d^2}}$$

and

$$\frac{\rho_d}{\rho_u} = \left(\frac{T_d}{T_u} \right)^{\frac{1}{\gamma-1}} = \left(\frac{1 + \frac{\gamma-1}{2} M_u^2}{1 + \frac{\gamma-1}{2} M_d^2} \right)^{\frac{1}{\gamma-1}}$$

from which equation (C1) becomes

$$M_u = M_d A^* \left(\frac{1 + \frac{\gamma-1}{2} M_u^2}{1 + \frac{\gamma-1}{2} M_d^2} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad (12)$$

APPENDIX D

EQUIVALENT COMPRESSIBLE FREE VORTEX

The regions of constant, prescribed velocity along the pressure and suction surfaces of the 90° impulse blade suggest that, for purposes of investigating the variation in these velocities with the equivalent downstream Mach number, the channel between these regions of constant velocity can be approximated by the flow between selected streamlines of a compressible free vortex. First, the radii for these selected streamlines are determined for an incompressible free vortex to give the prescribed values of Q_0 and $Q_{\frac{\pi}{2}}$ for incompressible

flow along the suction and pressure surfaces of the blade, respectively.

Incompressible free vortex.—For an incompressible free vortex

$$Qr = K_1 \quad (D1a)$$

so that

$$Q_0 r_0 = Q_{\frac{\pi}{2}} r_{\frac{\pi}{2}} \quad (D1b)$$

where r , like the Cartesian coordinates x and y , is expressed

in units of the channel width downstream at infinity of the channel between cascade blades that is being simulated by the free vortex. From continuity

$$\int_{r_0}^{r_{\frac{\pi}{2}}} Q dr = 1.0 \quad (D2)$$

so that, from equations (D1a) and (D2),

$$K_1 \log_e \frac{r_{\frac{\pi}{2}}}{r_0} = 1.0$$

and therefore, from equations (D1a) and (D1b),

$$r_0 = \frac{1}{Q_0 \log_e \frac{Q_0}{Q_{\frac{\pi}{2}}}} \quad (D3a)$$

and

$$r_{\frac{\pi}{2}} = \frac{1}{Q_{\frac{\pi}{2}} \log_e \frac{Q_0}{Q_{\frac{\pi}{2}}}} \quad (D3b)$$

Equations (D3a) and (D3b) give the radii that determine the channel in the free vortex (compressible or incompressible) which is equivalent to the channel between the cascade blades in the region of constant prescribed velocity along the pressure and suction surfaces. For the prescribed values of Q_0 and $Q_{\frac{\pi}{2}}$ (1.5 and 0.66687, respectively) the values of r_0 and $r_{\frac{\pi}{2}}$ given by equations (D3a) and (D3b) are 0.82260 and 1.84992, respectively.

Compressible free vortex.—For a compressible free vortex

$$Qr = K_2 \quad (D4a)$$

so that

$$Q_0 r_0 = Q_{\frac{\pi}{2}} r_{\frac{\pi}{2}} \quad (D4b)$$

and, from continuity,

$$\int_{r_0}^{r_{\frac{\pi}{2}}} \rho Q dr = \rho_d \quad (D5)$$

where, from equation (B4),

$$\rho = \left(\frac{T}{T_i} \right)^{\frac{1}{\gamma-1}} = \left(1 - \frac{\gamma-1}{2} Q^2 q_d^2 \right)^{\frac{1}{\gamma-1}} \quad (D6a)$$

so that

$$\rho_d = \left(1 - \frac{\gamma-1}{2} q_d^2 \right)^{\frac{1}{\gamma-1}} \quad (D6b)$$

Equation (D6a) expands in series form to give

$$\rho = 1 - \frac{1}{2} Q^2 q_d^2 + \frac{2-\gamma}{8} Q^4 q_d^4 - \dots \quad (D6c)$$

so that equation (D5), together with equations (D4a), (D6b), and (D6c), integrates to give

$$\rho_d = \left(1 - \frac{\gamma-1}{2} q_d^2 \right)^{\frac{1}{\gamma-1}} = K_2 \log_e \frac{r_{\frac{\pi}{2}}}{r_0} + \frac{q_d^2 K_2^3}{4} \left(\frac{1}{r_{\frac{\pi}{2}}^2} - \frac{1}{r_0^2} \right) - \frac{(2-\gamma)}{32} q_d^4 K_2^5 \left(\frac{1}{r_{\frac{\pi}{2}}^4} - \frac{1}{r_0^4} \right) + \dots \quad (D7)$$

where, from equation (C2),

$$q_d = M_d \sqrt{\frac{T_d}{T_i}} = \frac{M_d}{\sqrt{1 + \frac{\gamma-1}{2} M_d^2}} \quad (D8)$$

From equations (D7) and (D8) K_2 is a known function of M_d and therefore Q_0 and $Q_{\frac{\pi}{2}}$ are given by equation (D4a). The variation in Q_0 and $Q_{\frac{\pi}{2}}$ with M_d is plotted in figure 18.

The pressure coefficient P is defined by

$$P = \frac{p - p_d}{p_{t,d} - p_d} = \frac{\frac{p}{p_{t,d}} - \frac{p_d}{p_{t,d}}}{1 - \frac{p_d}{p_{t,d}}}$$

where, from equation (B4),

$$\frac{p}{p_{t,d}} = \left(\frac{T}{T_i} \right)^{\frac{\gamma}{\gamma-1}} = \left(1 - \frac{\gamma-1}{2} Q^2 q_d^2 \right)^{\frac{\gamma}{\gamma-1}}$$

so that, with $p_i = p_{t,d}$,

$$P = \frac{\left(1 - \frac{\gamma-1}{2} Q^2 q_d^2 \right)^{\frac{\gamma}{\gamma-1}} - \left(1 - \frac{\gamma-1}{2} q_d^2 \right)^{\frac{\gamma}{\gamma-1}}}{1 - \left(1 - \frac{\gamma-1}{2} q_d^2 \right)^{\frac{\gamma}{\gamma-1}}} \quad (D9)$$

Equation (D9) determines P for given values of Q and q_d . The variation in P_0 and $P_{\frac{\pi}{2}}$ with M_d is plotted in figure 19.

Choke flow.—Choke, or maximum, flow occurs when the derivative of the flow rate with respect to a characteristic velocity is zero. From equation (D5) the flow rate is proportional to ρ_d , and from equation (D4a) the velocity at each radius is proportional to K_2 so that choke flow occurs when

$$\frac{d\rho_d}{dK_2} = 0$$

Therefore, from equation (D7)

$$0 = \log_e \frac{r_{\frac{\pi}{2}}}{r_0} + \frac{3q_d^2 K_2^3}{4} \left(\frac{1}{r_{\frac{\pi}{2}}^2} - \frac{1}{r_0^2} \right) - \frac{5(2-\gamma)}{32} q_d^4 K_2^5 \left(\frac{1}{r_{\frac{\pi}{2}}^4} - \frac{1}{r_0^4} \right) + \dots$$

or

$$q_d^2 K_2^2 = \frac{\frac{3}{4} \left(\frac{1}{r_{\frac{\pi}{2}}^2} - \frac{1}{r_0^2} \right) - \sqrt{\frac{9}{16} \left(\frac{1}{r_{\frac{\pi}{2}}^2} - \frac{1}{r_0^2} \right)^2 + \frac{5(2-\gamma)}{8} \left(\frac{1}{r_{\frac{\pi}{2}}^4} - \frac{1}{r_0^4} \right) \log_e \frac{r_{\frac{\pi}{2}}}{r_0}}}{\frac{5(2-\gamma)}{16} \left(\frac{1}{r_{\frac{\pi}{2}}^4} - \frac{1}{r_0^4} \right)} \quad (D10)$$

The value of M_d for choke flow is then obtained from equations (D7), (D8), and (D10). For the values of r_0 and $r_{\frac{\pi}{2}}$ given by equations (D3a) and (D3b), the value of M_d thus obtained is 0.815.

APPENDIX E

CRITICAL VALUE OF Q FOR WHICH VELOCITY IS SONIC

For a given value of M_d there is a critical value of Q (Q_{cr}) for which the velocity corresponding to Q_{cr} is sonic. By definition

$$Q = \frac{q}{q_d} = \frac{M}{M_d} \sqrt{\frac{T}{T_d}}$$

which, from equation (C2), becomes

$$Q = \frac{M}{M_d} \sqrt{\frac{1 + \frac{\gamma-1}{2} M_d^2}{1 + \frac{\gamma-1}{2} M^2}} \quad (\text{E1})$$

By definition Q is equal to Q_{cr} if M is equal to 1.0 so that equation (E1) becomes

$$Q_{cr} = \sqrt{\frac{1 + \frac{\gamma-1}{2} M_d^2}{\frac{\gamma+1}{2} M_d^2}} \quad (13)$$

Equation (13) gives the relation between Q_{cr} and M_d , which relation is plotted in figure 20.

APPENDIX F

BLADE FORCE COMPUTED FROM MOMENTUM CONSIDERATIONS

If the viscous shear forces, which are relatively small, are ignored, the blade force F_v , acting on the fluid in the positive y' -direction (fig. 7) must, from momentum considerations, equal the change in the rate of momentum, in the positive y' -direction, of the fluid flowing through the cascade. The rate of momentum flow into the cascade in the positive y' -direction is

$$\frac{s' \sin \theta'_u \cos \theta'_u}{g} = \rho''_u (q''_u)^2$$

where flow conditions are considered uniform upstream of the cascade, and the rate of momentum flow out of the cascade in the positive y' -direction is

$$\frac{s' \sin \theta'_d \cos \theta'_d}{g} \int_0^{1.0} \rho''_d (q''_d)^2 d\left(\frac{y'}{s'}\right)$$

where the flow direction is uniform at a station far enough downstream. Therefore, F_v becomes

$$F_v = \frac{s' \sin \theta'_d \cos \theta'_d}{g} \int_0^{1.0} \rho''_d (q''_d)^2 d\left(\frac{y'}{s'}\right) - \frac{s' \sin \theta'_u \cos \theta'_u}{g} \rho''_u (q''_u)^2 \quad (14)$$

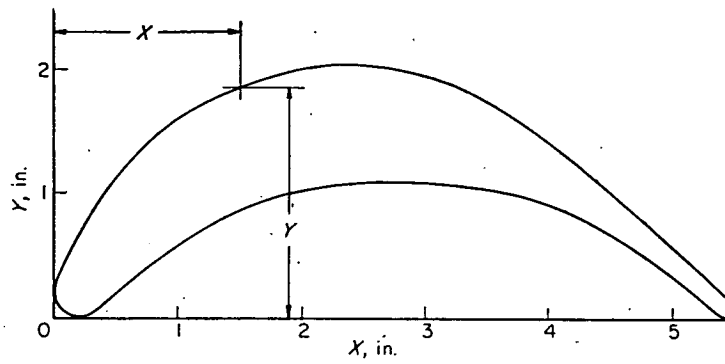
Equation (14) gives the component of the blade force in the positive direction of y' .

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4. Costello, George R.: Method of Designing Cascade Blades with Prescribed Velocity Distributions in Compressible Potential Flows. NACA Rep. 978, 1950. (Supersedes NACA TN's 1913 and 1970.)
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9. Alpert, Sumner: Design Method for Two-Dimensional Channels for Compressible Flow with Application to High-Solidity Cascades. NACA TN 1931, 1949.
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11. Stanitz, John D.: Design of Two-Dimensional Channels with Prescribed Velocity Distributions Along the Walls. NACA Rep. 1115, 1953. (Supersedes NACA TN's 2593 and 2595.)
12. Goldstein, Arthur W., and Mager, Artur: Attainable Circulation about Airfoils in Cascade. NACA Rep. 953, 1950. (Supersedes NACA TN 1941.)
13. Ainley, D. G.: Performance of Axial-Flow Turbines. War Emergency Issue No. 41 pub. by Inst. Mech. Eng. (London). (Reprinted in U. S. by A. S. M. E., April 1949, pp. 230-244.)
14. Howell, A. R.: The Present Basis of Axial Flow Compressor Design. Part I. Cascade Theory and Performance. R. & M. No. 2095, British A. R. C., June 1942.
15. Reeman, J.: The Turbine for the Simple Jet Propulsion Engine. War Emergency Issue No. 12 pub. by Inst. Mech. Eng. (London), 1945. (Reprinted in U. S. by A. S. M. E., Jan. 1947, pp. 495-504.)

TABLE I.—BLADE PROFILE COORDINATES

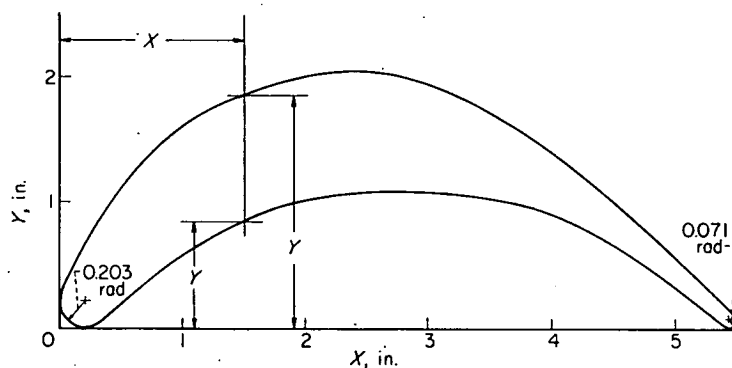
[90° impulse cascade; prescribed velocity distribution, fig. 5; incompressible flow]



X	Y_1	Y_2
0	0.203	0.203
.05	.069	.354
.10	.028	.456
.15	.009	.554
.20	.000	.647
.25	.006	.736
.30	.024	.821
.35	.061	.905
.40	.104	.985
.45	.147	1.063
.50	.189	1.135
.60	.272	1.253
.70	.353	1.352
.80	.430	1.440
.90	.504	1.520
1.0	.573	1.593
1.2	.701	1.718
1.4	.812	1.821
1.6	.899	1.904
1.8	.965	1.967
2.0	1.014	2.012
2.2	1.052	2.036
2.4	1.078	2.044
2.6	1.094	2.032
2.8	1.100	1.996
3.0	1.094	1.938
3.2	1.079	1.863
3.4	1.052	1.770
3.6	1.016	1.660
3.8	.969	1.536
4.0	.909	1.397
4.2	.831	1.246
4.4	.729	1.085
4.6	.608	.914
4.8	.473	.736
5.0	.325	.558
5.1	.246	.468
5.2	.167	.376
5.3	.085	.284
5.35	.044	.239
5.40	.007	.194
5.45	.003	.149
5.50	.071	.071

TABLE II.—STATIC TAP LOCATIONS ON BLADE SURFACE AT MIDSPAN

[90° impulse cascade]



Tap	X	Y	$\frac{x'}{c}$	$\frac{y'}{c}$	Φ	Ψ	Tap	X	Y	$\frac{x'}{c}$	$\frac{y'}{c}$	Φ	Ψ
9	0.015	0.280	-0.443	0.172	0	0	33	4.631	0.888	0.400	0.187	$\frac{26\pi}{16}$	0
10	.171	.595	-.412	.214	$\frac{\pi}{16}$		34	4.832	.709	.433	.151	$\frac{27\pi}{16}$	
11	.297	.816	-.385	.252	$\frac{2\pi}{16}$		35	5.038	.524	.467	.113	$\frac{28\pi}{16}$	
12	.418	1.014	-.360	.286	$\frac{3\pi}{16}$		36	5.248	.332	.502	.075	$\frac{29\pi}{16}$	
13	.535	1.180	-.336	.314	$\frac{4\pi}{16}$		37	5.468	.133	.538	.035	$\frac{30\pi}{16}$	
14	.668	1.321	-.309	.337	$\frac{5\pi}{16}$		38	5.384	.017	.521	.015	$\frac{22\pi}{16}$	$\frac{\pi}{2}$
15	.817	1.454	-.280	.358	$\frac{6\pi}{16}$		39	5.147	.209	.481	.054	$\frac{21\pi}{16}$	
16	.974	1.575	-.249	.377	$\frac{7\pi}{16}$		40	4.871	.421	.435	.098	$\frac{20\pi}{16}$	
17	1.141	1.684	-.217	.394	$\frac{8\pi}{16}$		41	4.556	.635	.382	.142	$\frac{19\pi}{16}$	
18	1.314	1.780	-.184	.408	$\frac{9\pi}{16}$		42	4.369	.746	.350	.166	$\frac{18.5\pi}{16}$	
19	1.494	1.863	-.150	.420	$\frac{10\pi}{16}$		43	4.174	.842	.317	.187	$\frac{18\pi}{16}$	
20	1.873	1.985	-.079	.435	$\frac{12\pi}{16}$		44	3.960	.922	.279	.205	$\frac{17.5\pi}{16}$	
21	2.266	2.041	-.007	.438	$\frac{14\pi}{16}$		45	3.740	.984	.241	.220	$\frac{17\pi}{16}$	
22	2.464	2.042	.029	.435	$\frac{15\pi}{16}$		46	3.304	1.066	.163	.243	$\frac{16\pi}{16}$	
23	2.661	2.023	.064	.428	$\frac{16\pi}{16}$		47	2.859	1.099	.083	.257	$\frac{15\pi}{16}$	
24	2.857	1.981	.099	.417	$\frac{17\pi}{16}$		48	2.414	1.079	.002	.261	$\frac{14\pi}{16}$	
25	3.055	1.919	.134	.402	$\frac{18\pi}{16}$		49	1.975	1.008	-.078	.256	$\frac{13\pi}{16}$	
26	3.253	1.840	.168	.384	$\frac{19\pi}{16}$		50	1.754	.951	-.119	.250	$\frac{12.5\pi}{16}$	
27	3.451	1.744	.202	.363	$\frac{20\pi}{16}$		51	1.538	.875	-.160	.240	$\frac{12\pi}{16}$	
28	3.648	1.632	.236	.339	$\frac{21\pi}{16}$		52	1.342	.782	-.197	.227	$\frac{11.5\pi}{16}$	
29	3.844	1.507	.269	.313	$\frac{22\pi}{16}$		53	1.154	.673	-.233	.211	$\frac{11\pi}{16}$	
30	4.039	1.368	.302	.284	$\frac{23\pi}{16}$		54	.838	.459	-.294	.178	$\frac{10\pi}{16}$	
31	4.235	1.218	.334	.254	$\frac{24\pi}{16}$		55	.565	.244	-.347	.144	$\frac{9\pi}{16}$	
32	4.432	1.058	.367	.221	$\frac{25\pi}{16}$		56	.333	.046	-.393	.112	$\frac{8\pi}{16}$	

REPORT 1117

A STUDY OF ELASTIC AND PLASTIC STRESS CONCENTRATION FACTORS DUE TO NOTCHES AND FILLETS IN FLAT PLATES¹

By HERBERT F. HARDRATH and LACHLAN OHMAN

SUMMARY

Six large 24S-T3 aluminum-alloy-sheet specimens containing various notches or fillets were tested in tension to determine their stress concentration factors in both the elastic and plastic ranges.

The elastic stress concentration factors were found to be slightly higher than those calculated by Neuber's method and those obtained photoelastically by Frocht. The results showed further that the stress concentration factor decreases as strains at the discontinuity enter the plastic range.

A generalization of Stowell's relation for the plastic stress concentration factor at a circular hole in an infinite plate was applied to the specimen shapes tested and gave good agreement with test results.

INTRODUCTION

Theoretical studies of stress concentrations around discontinuities in flat plates have been limited because of analytical difficulties to certain geometric shapes (refs. 1 to 6) and, until recently, to the realm of stresses in the elastic range. Experimental studies on the subject are scarce, such investigations being limited by available loading and strain measuring equipment (refs. 7 to 11). The present investigation was undertaken to obtain experimentally stress concentration data in both the elastic and plastic ranges for sheet specimens containing a variety of notches and fillets.

A formula has been presented by Stowell (ref. 12) for the stress concentration factor in the plastic range for a circular hole in an infinite plate. When the formula is written in a generalized form it appears to be applicable to the specimen shapes of this investigation and to other configurations.

EXPERIMENTAL PROCEDURE

Since the maximum stress in a panel containing a stress concentration occurs at a point and the stress distribution around the point follows a gradient, it is essential for reliable measurements that the ratio of notch dimensions to gage length of the strain gages be as large as possible. Therefore, in order to obtain a high ratio and to be able to use strain gages of practical dimensions, the specimens were made 48 by 142 inches.

The specimens consisted of six 24S-T3 aluminum-alloy

panels $\frac{1}{8}$ inch thick. Three panels, designated herein by N, contained notches; three, designated by F, contained fillets. In each group of three there was a panel designed to have a nominal stress concentration factor of 2, one of 4, and one of 6; hence the specimens are designated N2, N4, N6 and F2, F4, and F6. (Specimen dimensions are shown in detail in figs. 1 and 2.)

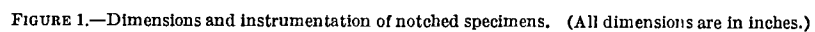
The panels were cut from sheets 54 inches wide. Excess material from the region near the notches or fillets was used to make standard tensile coupons, four for each panel. Standard tensile tests run on these coupons yielded stress-strain curves. Each of the stress-strain curves as presented in figure 3 is an average of the four curves obtained from the four coupons for each panel.

Four types of strain gages were used in the investigation. On panels N2 and F2 Tuckerman optical strain gages with a $\frac{1}{4}$ -inch gage length were mounted on the face at the edge of the panel at the critical points which are defined as the points on each panel where the maximum stress concentrations occur. In a notched specimen the maximum stress occurs at the center of the base of the notch. Photoelastic studies of filleted specimens (ref. 6) indicate that the maximum stress occurs at a point about 10° around the fillet from the tangent between the fillet and the straight side of the reduced part of the panel. Strains at the critical points of panels N4, N6, F4, and F6 were measured with $\frac{1}{8}$ -inch-gage-length electromagnetic extensometers mounted on the edges of the panels at the critical points. (See figs. 1 and 2.)

Baldwin SR-4 type A-5 strain gages were applied at intervals across the width of each panel at the net section to determine the strain distribution at that section (figs. 1 and 2). In a notched panel the net section is defined as the cross section through the panel between the two critical points (fig. 1). In a filleted panel the net section is defined as the section across the panel containing the points where the fillets and the straight sides of the reduced part of the panel are tangent (fig. 2). Baldwin SR-4 type A-1 strain gages were applied at other cross sections (figs. 1 and 2) to ascertain the uniformity of load application.

Load and strain readings were taken throughout each test at successive increments from zero load to a load which produced approximately 2 percent strain at the critical point.

¹ Supersedes NACA TN 2566, "A Study of Elastic and Plastic Stress Concentration Factors Due to Notches and Fillets in Flat Plates" by Herbert F. Hardrath and Lachlan Ohman, 1951.



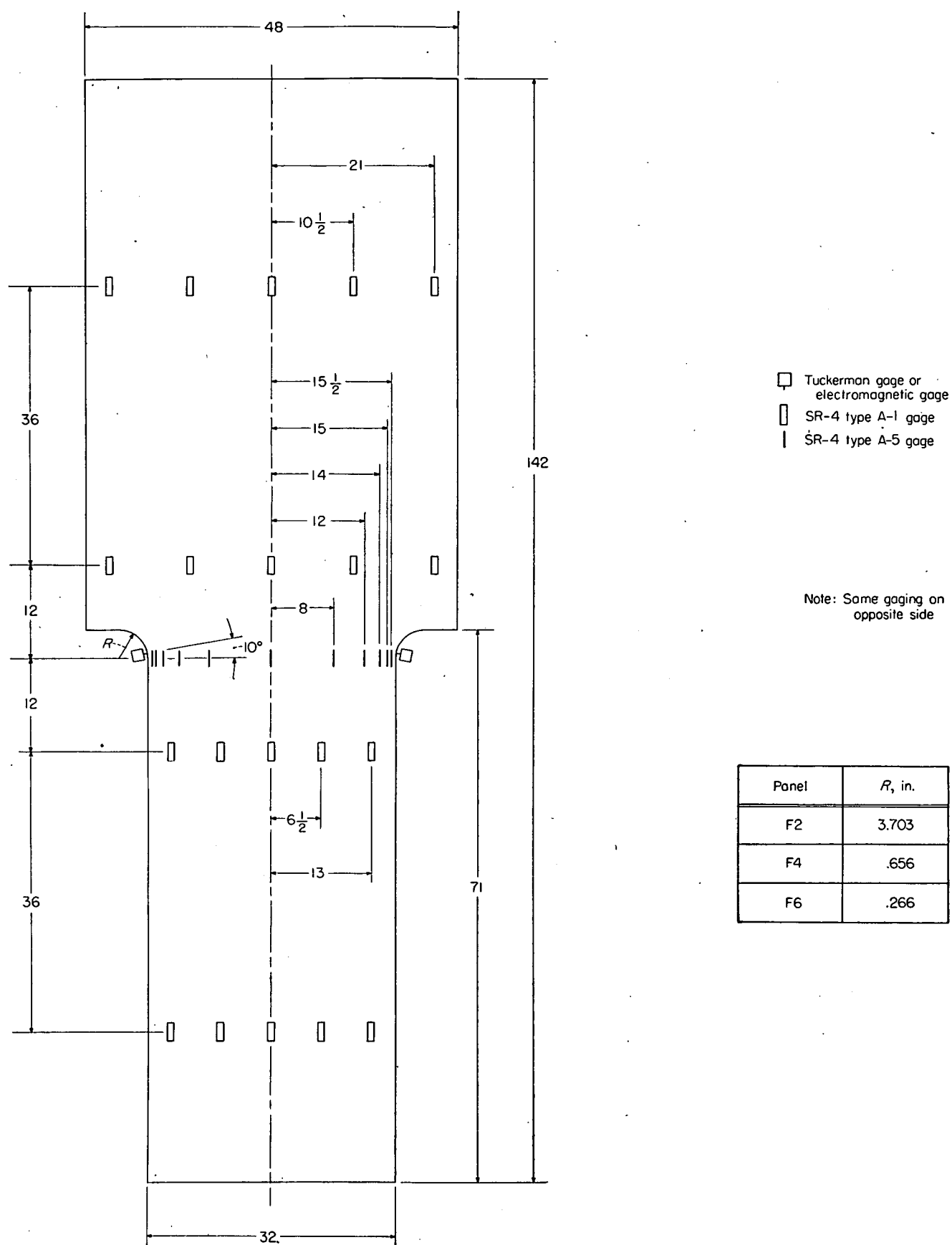


FIGURE 2.—Dimensions and instrumentation of filleted specimens. (All dimensions are in inches.)

RESULTS AND DISCUSSION

ELASTIC STRESS CONCENTRATION FACTORS

The stress concentration factor is defined in this report as the ratio of the stress at the critical point σ_{cp} to the average net section stress S . The elastic stress concentration factor $K_{elastic}$ is computed from measurements taken before plastic yielding occurred in the specimen. The value of $K_{elastic}$ thus obtained for each specimen is presented in the second column of table I.

TABLE I.—ELASTIC STRESS CONCENTRATION FACTORS

Panel	$K_{elastic}$ (experimental)	$K_{elastic}$ (calculated by Neuber's method)	$K_{elastic}$ (determined photoelastically by Frocht)
N2	2.08	2.00	2.02
N4	4.26	4.00	4.02
N6	6.41	6.00	6.00
F2	1.88	----	----
F4	3.87	----	3.49
F6	5.44	----	5.00

The third column of table I lists the elastic stress concentration factors for the notched specimens as calculated by an analytical method presented by Neuber (ref. 1) who, for reasons of mathematical convenience, treated notches with a hyperbolic contour. The computed value of the stress concentration factor in each case is based on a hyperbolic notch that has its depth and minimum radius of curvature equal to the depth and radius of the corresponding experimental notch. The experimental notch contour rather than being hyperbolic, however, is composed of a semicircular base with straight, parallel tangents, a shape which is somewhat sharper than the corresponding hyperbolic notch. The experimental stress concentration factors, therefore, would be expected to be somewhat higher than those computed by Neuber's method. A comparison of values shown in the preceding table indicates the difference to be from 4 to 7 percent.

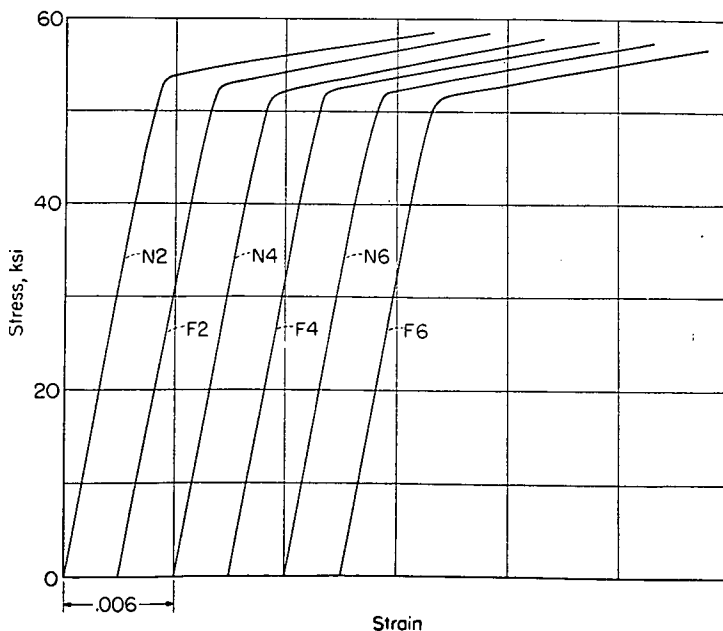


FIGURE 3.—Stress-strain curves of 24S-T3 aluminum alloy for each specimen tested.

A photoelastic investigation of elastic stress concentration factors for shapes geometrically similar to those used in this investigation was performed by Frocht under the sponsorship of the National Advisory Committee for Aeronautics (ref. 10). The results of this investigation are given in the fourth column of table I. The results of the present investigation were found to be 4 to 7 percent higher for the notched specimens and 3 to 10 percent higher for the filleted specimens than the factors obtained photoelastically by Frocht. Although Frocht's values are nearly equal to those determined by Neuber's method, Frocht states that his values can be in error by as much as 10 percent for high stress concentration factors because of difficulties in determining the maximum fringe value near the boundary of sharp notches.

A comparison of elastic stress concentration factors determined by the three methods is given in figures 4 and 5.

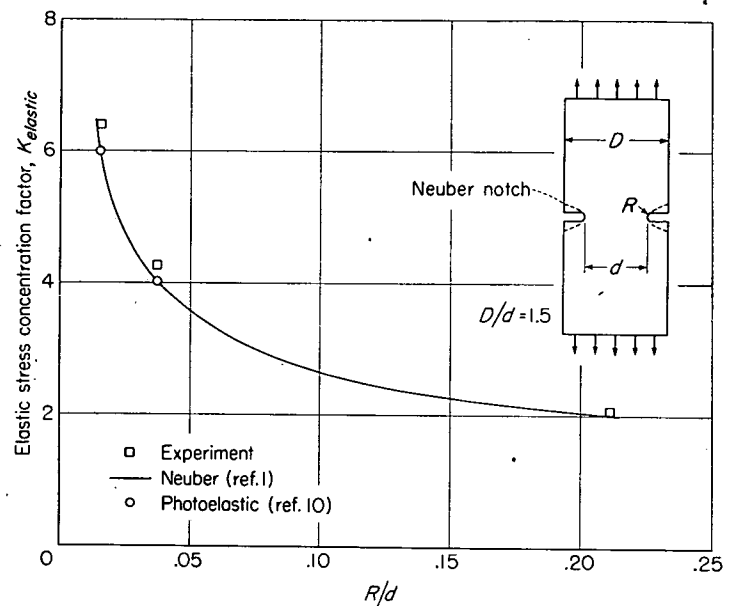


FIGURE 4.—Elastic stress concentration factors for notched specimens.

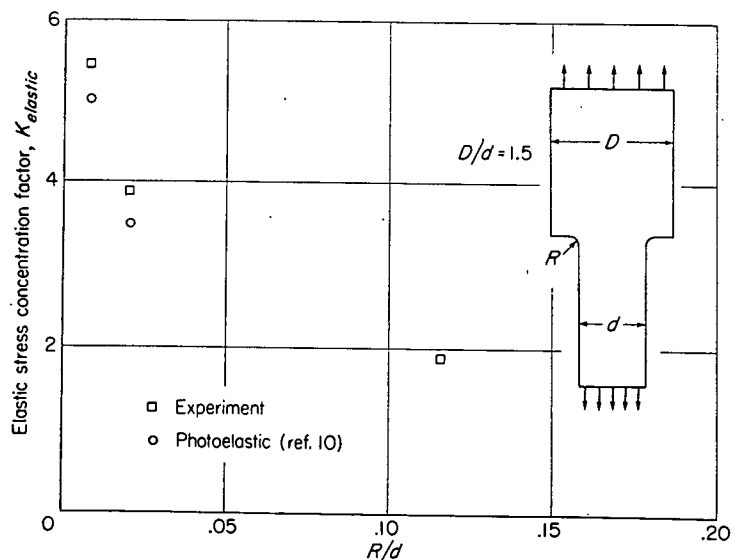


FIGURE 5.—Elastic stress concentration factors for filleted specimens.

PLASTIC STRESS CONCENTRATION FACTORS AND A SUGGESTED FORMULA

Figures 6 and 7 show the experimentally determined stress concentration factors plotted against average net-section stress. The stress concentration factors start to decrease as the load on a panel exceeds a value sufficient to cause plastic yielding at the critical point.

Stowell (ref. 12) presented a relation to be used in calculating the stress concentration factor in the plastic range for a circular hole in an infinite plate subjected to tension. This relation states that at any one instant in the loading

$$\frac{\sigma_{cp}}{S_{\infty}} = 1 + 2 \frac{E_s}{E_{\infty}} \quad (1)$$

where

σ_{cp} stress occurring at the point of maximum stress

S_{∞} average stress at points far removed from hole

E_s secant modulus of material at point of maximum stress

E_{∞} secant modulus of material at points far removed from hole

For the case when all stresses are elastic, the formula yields the value of 3, which is the theoretical elastic stress concentration factor for a circular hole in an infinite plate (ref. 6). Predictions by formula (1) agreed well (ref. 12) with the results of an experimental investigation reported by Griffith (ref. 8) concerning the stress concentration factor for a circular hole in a flat sheet subjected to tension.

By rewriting formula (1) in a generalized form a relation between $K_{elastic}$ and $K_{plastic}$ may be obtained:

$$K_{plastic} = 1 + (K_{elastic} - 1) \frac{E_s}{E_{\infty}} \quad (2)$$

where $K_{plastic}$ is the stress concentration factor in the plastic

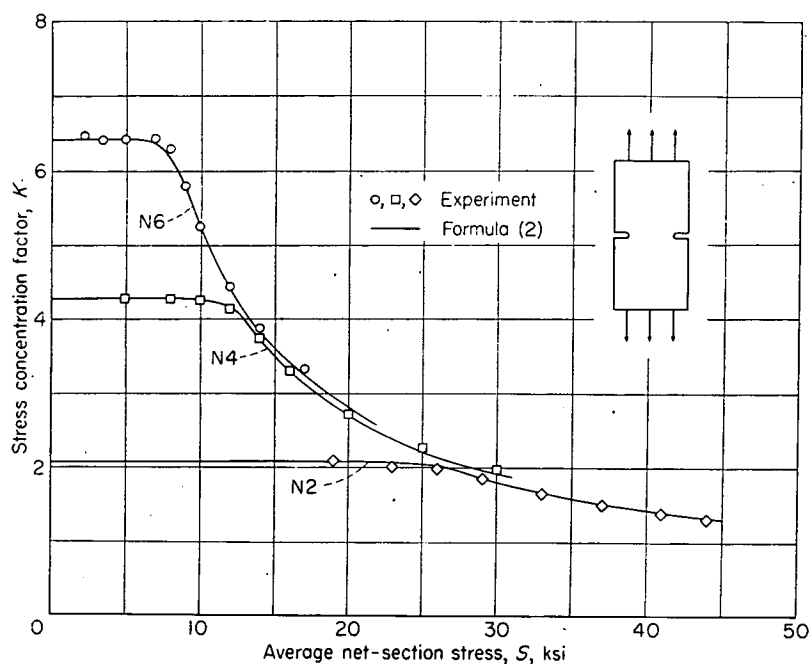


FIGURE 6.—Stress concentration factors for notched specimens.

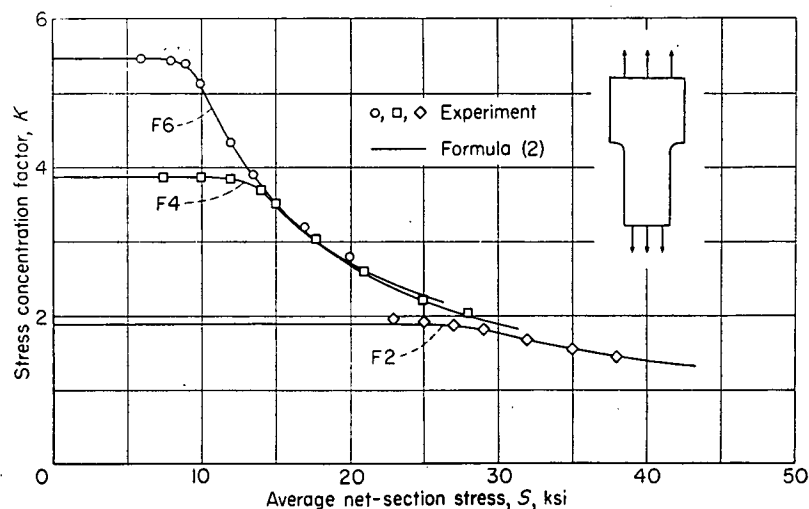


FIGURE 7.—Stress concentration factors for filleted specimens.

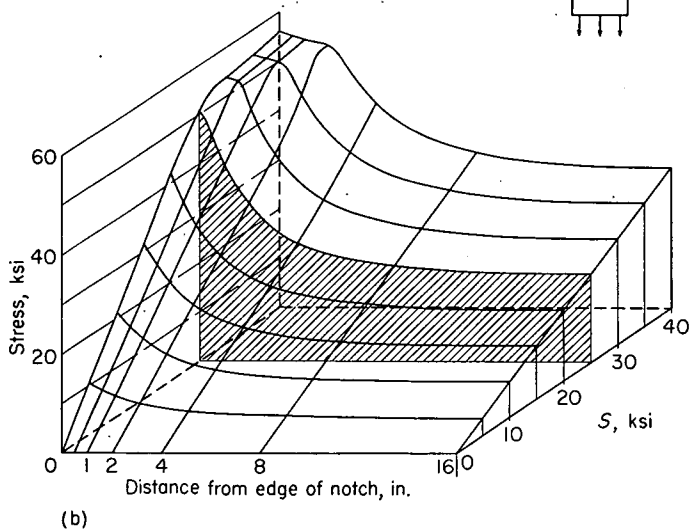
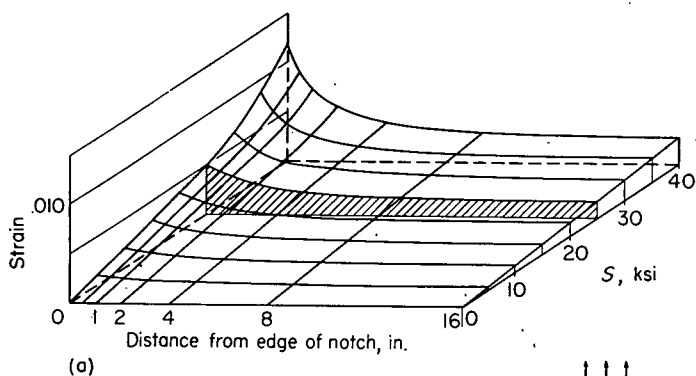
range. This form suggests that the formula might also be made applicable to shapes other than round holes by inserting a value for $K_{elastic}$ to correspond to the particular shape being considered. The plastic stress concentration factor for each of the specimens tested was computed by formula (2) with the experimental $K_{elastic}$ given in table I, and the curves in figures 6 and 7 show that the generalized formula produced excellent agreement with the test results. A comparison between predictions by formula (2) and experimental results obtained by Box (ref. 9) for other configurations is given in a subsequent section.

STRAIN AND STRESS DISTRIBUTIONS

Strains measured by strain gages at four symmetrical locations on the net section of a panel were averaged to obtain the strain at a single station. These values are plotted for each of the six panels as the longitudinal strain

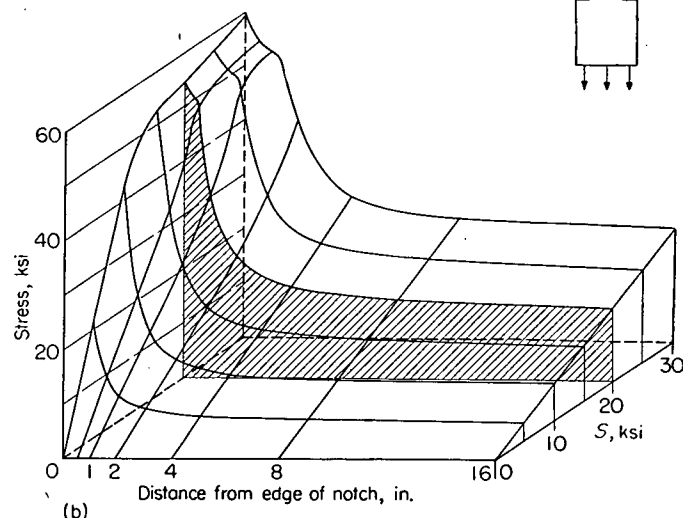
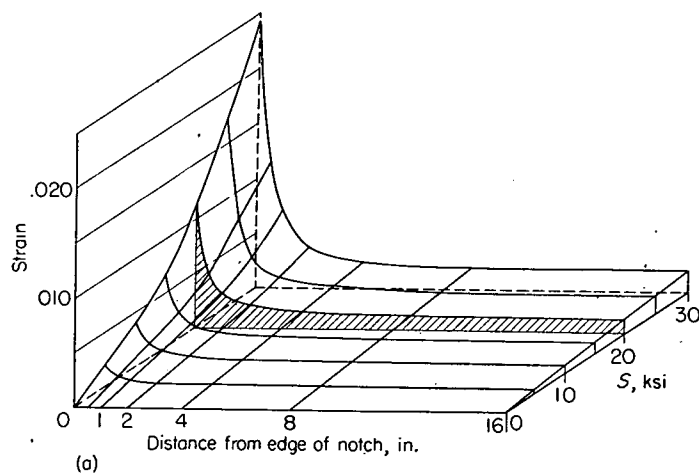
distribution across half of the net section (pts. "a", figs. 8 to 13).

The apparent stresses corresponding to strains at individual stations were obtained by using the stress-strain curves for the material (fig. 3) and are plotted for each panel as the longitudinal stress distribution across half the net section (pts. "b", figs. 8 to 13). The apparent stresses so determined are not the true stresses at any stations other than those at the edges because stress conditions in the sheet are not uniaxial. The illustrated trends of the apparent stress distributions, however, are representative of the actual stress distributions.



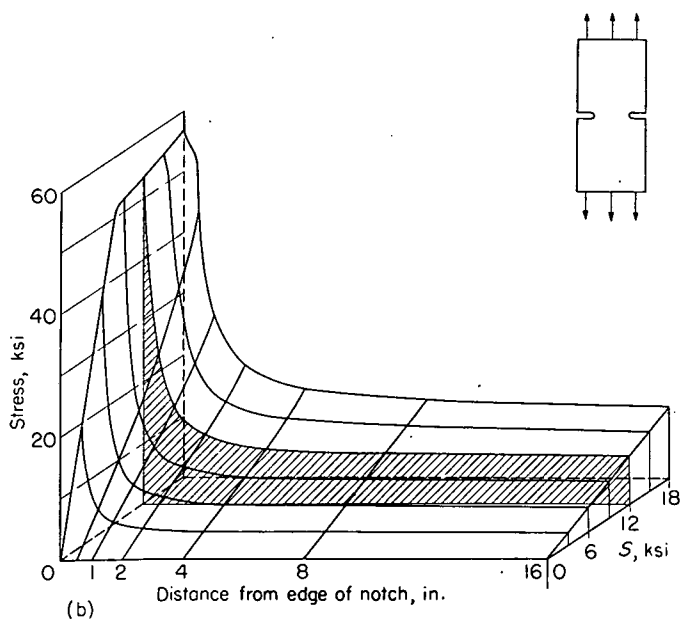
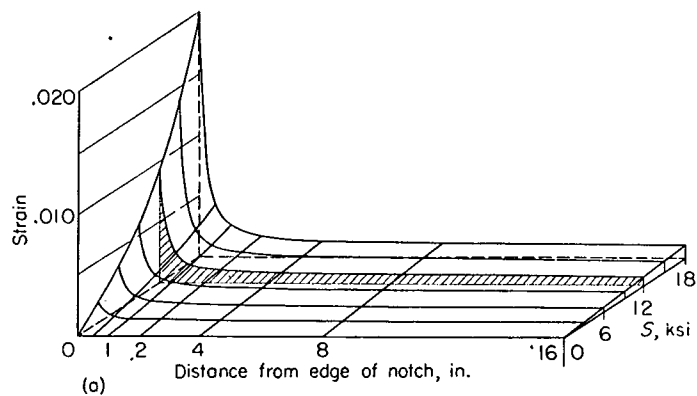
(a) Strain distribution.
(b) Apparent stress distribution.

FIGURE 8.—Strain distribution and apparent stress distribution at the net section of specimen N2.



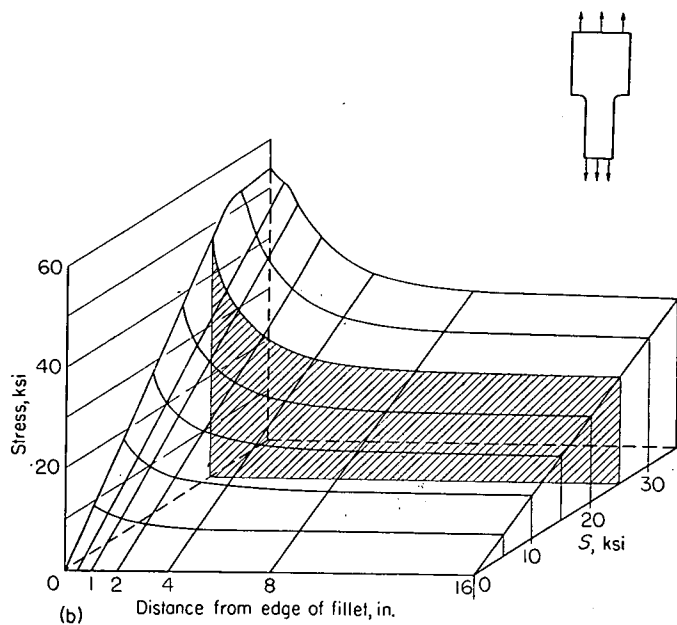
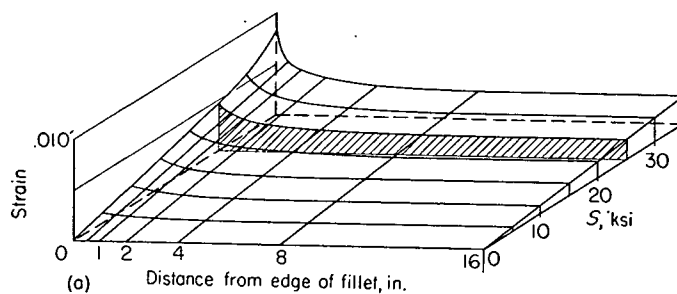
(a) Strain distribution.
(b) Apparent stress distribution.

FIGURE 9.—Strain distribution and apparent stress distribution at the net section of specimen N4.



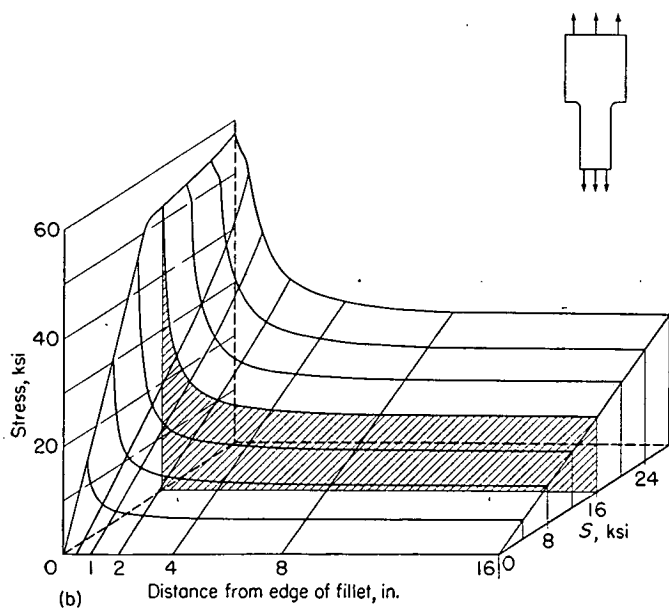
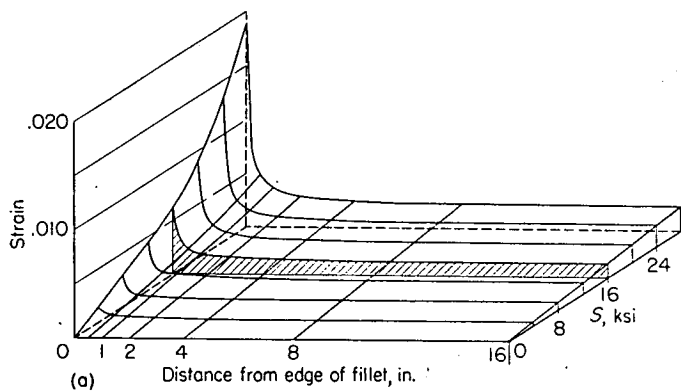
(a) Strain distribution.
(b) Apparent stress distribution.

FIGURE 10.—Strain distribution and apparent stress distribution at the net section of specimen N6.

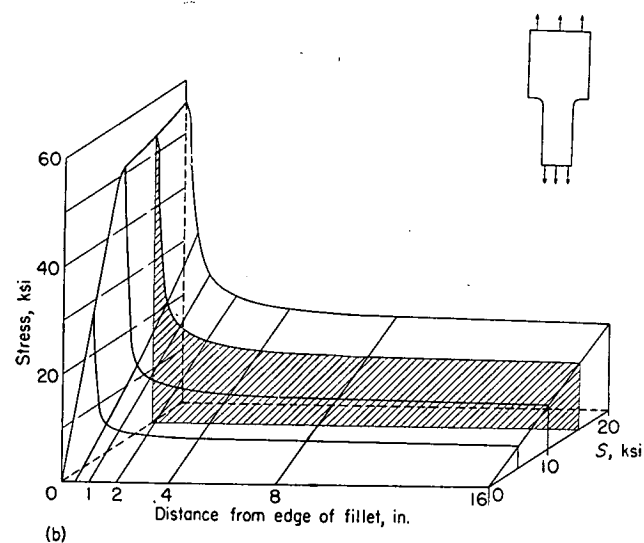
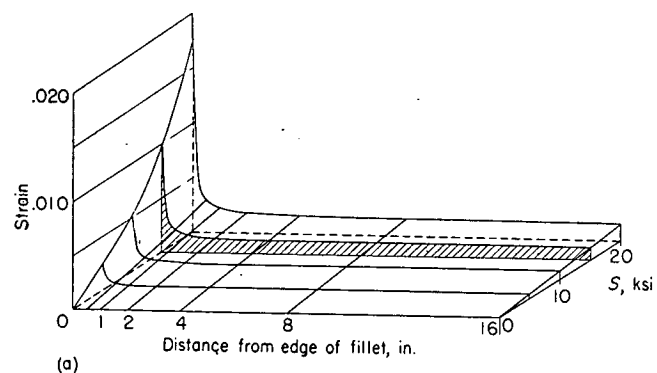


(a) Strain distribution.
(b) Apparent stress distribution.

FIGURE 11.—Strain distribution and apparent stress distribution at the net section of specimen F2.



(a) Strain distribution.
(b) Apparent stress distribution.
FIGURE 12.—Strain distribution and apparent stress distribution at the net section of specimen F4.



(a) Strain distribution.
(b) Apparent stress distribution.
FIGURE 13.—Strain distribution and apparent stress distribution at the net section of specimen F6.

COMPARISON OF SUGGESTED FORMULA WITH ADDITIONAL DATA

Additional experimental data on the effect of plasticity on the stress concentration factors for various configurations have been reported by Box (ref. 9). Six flat sheet specimens were tested in tension: one contained semicircular notches, one had two holes symmetrically spaced in one cross section, one had a single hole elongated in the transverse direction, and three had single holes at various positions across the width of the specimens. The specimens, made of 24S-T3 aluminum alloy, were $4\frac{1}{2}$ inches wide, and the radius of all holes was $\frac{1}{2}$ inch. The stress-strain curve for the material is reproduced as figure 14.

Comparisons of these data and predictions by formula (2) are presented in figures 15 to 20. Since the strain measuring technique employed by Box gave inaccurate results for small strains but improved in accuracy for higher strains, the calculations were based on an experimental value of $K_{elastic}$ obtained near the high end of the elastic range. The resulting predictions of $K_{plastic}$ by formula (2) agree well with the experimental results obtained by Box.

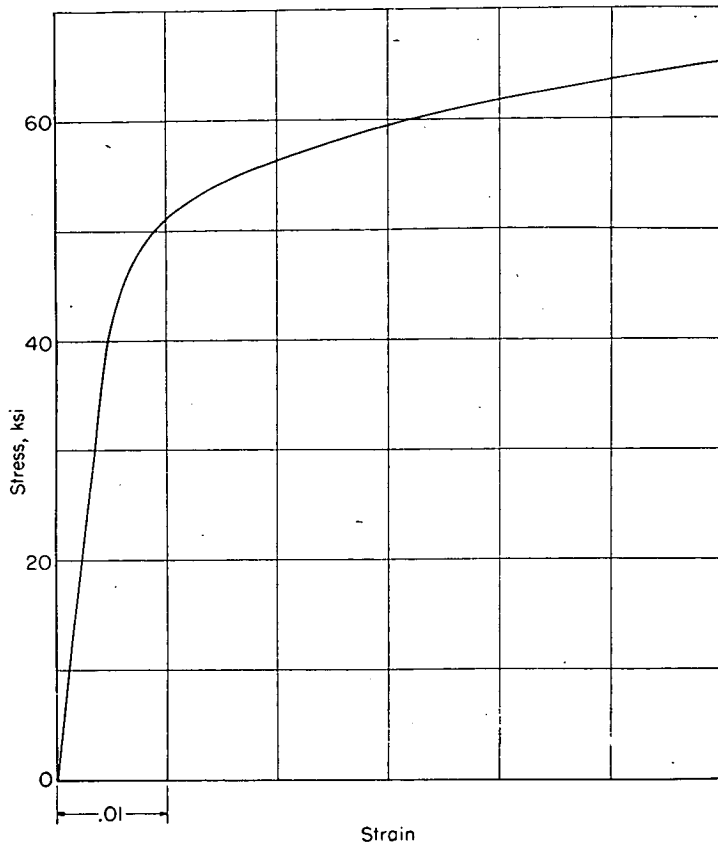


FIGURE 14.—Stress-strain curve for 24S-T3 aluminum alloy used in reference 9.

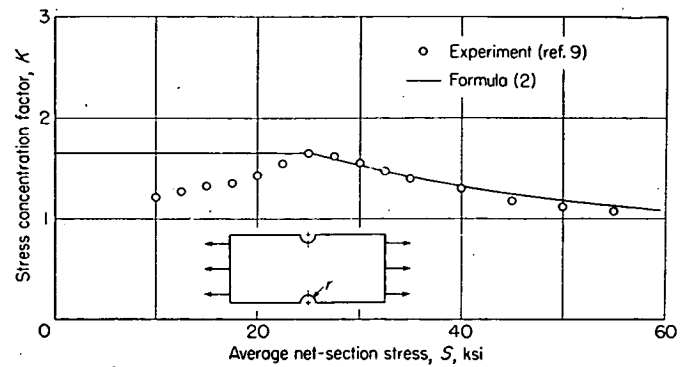


FIGURE 15.—Stress concentration factor for a flat plate with semicircular edge notches.

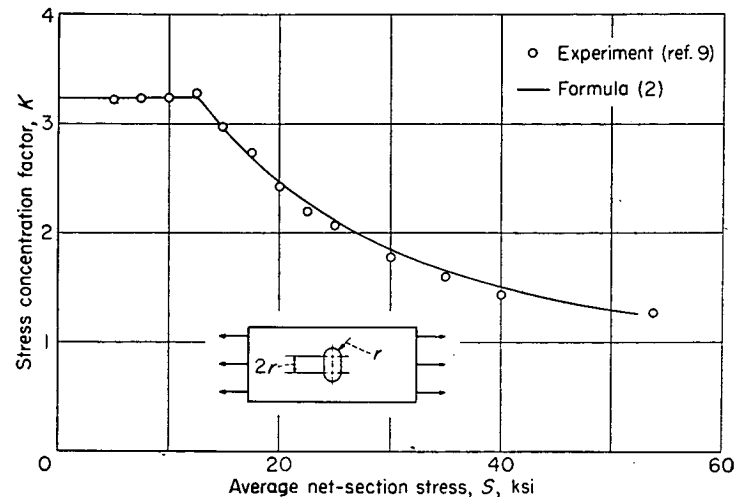


FIGURE 16.—Stress concentration factor for a flat plate with a single central hole elongated in the transverse direction.

CONCLUDING REMARKS

The elastic stress concentration factors were determined experimentally for six different specimens and were found to be 4 to 7 percent higher for the notched specimens and 3 to 10 percent higher for the filleted specimens than the factors obtained photoelastically by Frocht for shapes geometrically similar to those tested. The experimentally determined factors are 4 to 7 percent higher than those predicted for hyperbolic notches by Neuber's analytical method.

The stress concentration factors decreased as the strains at the critical points entered the plastic range and were found to agree well with calculated results obtained from the following equation which is a generalization of a relation originally presented by Stowell for the plastic stress concentration factor at a circular hole in an infinite plate:

$$K_{plastic} = 1 + (K_{elastic} - 1) \frac{E_s}{E_\infty}$$

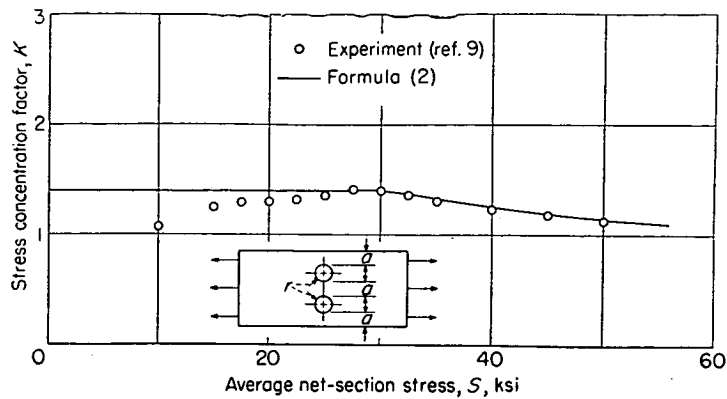


FIGURE 17.—Stress concentration factor for a flat plate with two circular holes equally spaced in one cross section.

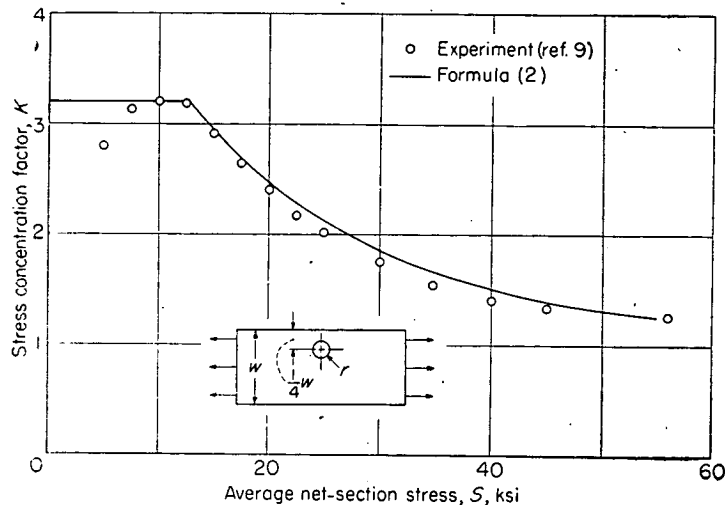


FIGURE 18.—Stress concentration factor for a flat plate with a circular hole located $\frac{1}{4}w$ from the edge.

where

$K_{plastic}$ stress concentration factor in plastic range

$K_{elastic}$ stress concentration factor in elastic range

E_S secant modulus of material at point of maximum stress

E_∞ secant modulus of material at points removed from hole

The generalized equation also checks well with experimental work done by Box on stress concentration factors in the plastic range for several additional configurations.

Langley Aeronautical Laboratory,

National Advisory Committee for Aeronautics,
Langley Field, Va., September 7, 1951.

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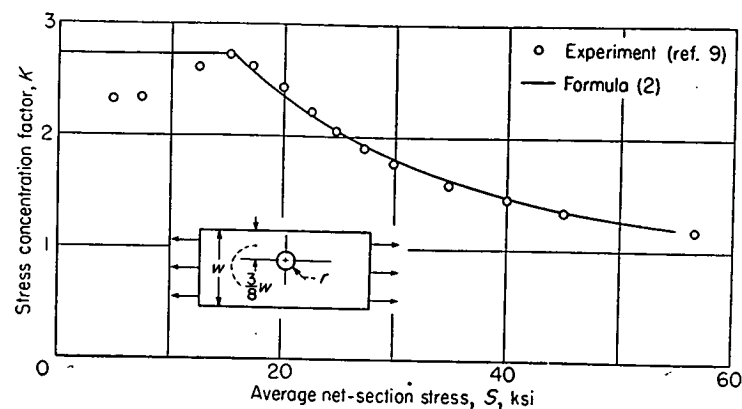


FIGURE 19.—Stress concentration factor for a flat plate with a circular hole located $\frac{3}{8}w$ from the edge.

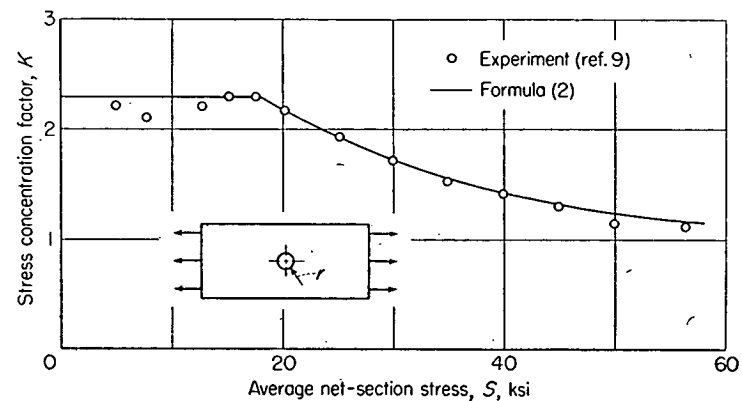


FIGURE 20.—Stress concentration factor for a flat plate with a single, central, circular hole.

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REPORT 1118

METHOD FOR CALCULATION OF LAMINAR HEAT TRANSFER IN AIR FLOW AROUND CYLINDERS OF ARBITRARY CROSS SECTION (INCLUDING LARGE TEMPERATURE DIFFERENCES AND TRANSPIRATION COOLING)¹

By E. R. G. ECKERT and JOHN N. B. LIVINGOOD

SUMMARY

The solution of heat-transfer problems has become vital for many aeronautical applications. The shapes of objects to be cooled can often be approximated by cylinders of various cross sections with flow normal to the axis as, for instance, heat transfer on gas-turbine blades and on air foils heated for deicing purposes. A laminar region always exists near the stagnation point of such objects.

A method previously presented by E. R. G. Eckert permits the calculation of local heat transfer around the periphery of cylinders of arbitrary cross section in the laminar region for flow of a fluid with constant property values with an accuracy sufficient for engineering purposes. The method is based on exact solutions of the boundary-layer equations for incompressible wedge-type flow and on the postulate that at any point on the cylinder the boundary-layer growth is the same as that on a wedge with comparable flow conditions. This method is extended herein to take into account the influence of large temperature differences between the cylinder wall and the flow as well as the influence of transpiration cooling when the same medium as the outside flow is used as coolant. Prepared charts make the calculation procedure very rapid. For cylinders with solid walls and elliptic cross sections, a comparison is made of the results of calculations based on the presented method, the results of calculations by other known methods, and results obtained in experimental investigations.

INTRODUCTION

Calculation of the heat transferred to cylinders with arbitrary cross sections from air flowing normal to the axis by a solution of the boundary-layer equations is a difficult problem, even when the laminar region is considered. The problem is especially complicated by the large number of parameters influencing heat transfer. Such parameters are: the shape of the cross section of the cylinder, the Mach number which determines the flow outside the boundary layer, the temperatures on the surface of the cylinder as well as in the stream, the stream velocity determining the internal heat generation, and the temperature distribution around the circumference of the cylinder. If the cylinder is cooled by the transpiration-cooling method in which a coolant is ejected through a porous surface into the outside stream, the amount of coolant and its distribution around the circumference of the cross section of the cylinder are additional parameters. Even if a solution is obtained for

such a problem, for instance by use of an electronic computer, the solution is very restricted because of the many parameters. Up to the present time, therefore, the problem has been attacked only under simplifying restrictions.

The restrictions most commonly used are: (1) low velocities, (2) constant property values, (3) constant wall temperatures, and (4) impermeable surfaces (no transpiration cooling). Under restriction (2), the development of the boundary layer along the cylindrical surface is independent of the heat transfer; available knowledge on the flow boundary layer can therefore be used as a basis for a heat-transfer calculation. Under the simplifying assumptions, which are necessary in order to transform the general viscous-flow equations into the boundary-layer equations, the development of the flow boundary layer does not depend immediately on the shape of the cross section of the cylinder but only on the velocity distribution in the stream outside the boundary layer and along its surface.

One method which was applied successfully to obtain a solution of the flow boundary-layer equation developed the stream velocity along the surface of the cylinder in a power series of the distance from the stagnation point measured along the circumference of the cylinder. In reference 1, this method is used to solve the heat-transfer problem. It is also shown that the temperature field within the boundary layer can be presented in a power series of the distance from the stagnation point in which the single terms contain only universal functions of a dimensionless wall distance and of the Prandtl number of the fluid. The heat transfer to the surface is given by an analogous series with terms depending on the Prandtl number. The calculation of the universal functions, however, is a tedious process, and accordingly these functions are known only for a limited number of terms. For air with a Prandtl number of 0.7, they are presented in reference 1. For a gas with a Prandtl number of 1, they are contained in reference 2, which is based on reference 3, in which the boundary-layer flow on a yawed cylinder is calculated. The fact that the boundary-layer equation for the velocity component parallel to the axis of a yawed cylinder is identical in form to the boundary-layer equation describing the temperature field for a fluid with a Prandtl number of 1, flowing normal to the axis of the cylinder, was used in reference 2 to determine heat transfer to such cylinders. The presentation of more terms of the series is announced in reference 4. It was found, however, that the velocity distribution for only a limited

¹ Supersedes NACA TN 2733, "Method for Calculation of Heat Transfer in Laminar Region of Air Flow Around Cylinders of Arbitrary Cross Section (Including Large Temperature Differences and Transpiration Cooling)" by E. R. G. Eckert and John N. B. Livingood, 1952.

range of cross sections of cylinders can be represented by a power series converging rapidly enough that the number of the known universal functions is sufficient to calculate the heat transfer.

The difficulties connected with a solution of the boundary-layer equations point out the need for an approximate approach by which, with a small expenditure of time, heat-transfer coefficients can be determined with an accuracy sufficient for engineering purposes. A considerable number of such approaches have been tried in the past; the results differ greatly as shown in figure 1, taken from reference 2.

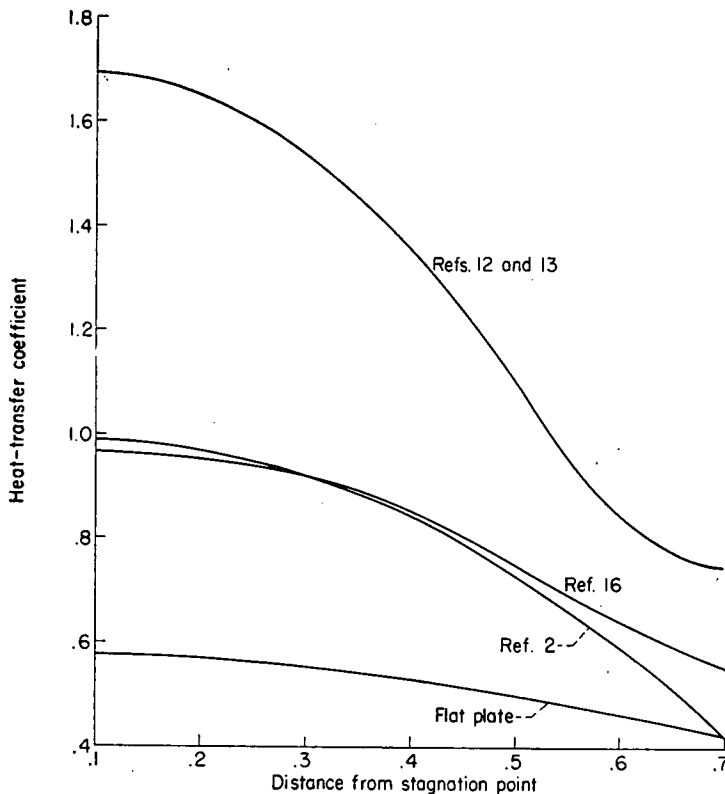


FIGURE 1.—Heat-transfer coefficients for cylinder (ref. 2).

The simplest procedure is probably that in which the heat-transfer coefficients as calculated in reference 5 are used for flow of constant velocity along a flat plate. The fact that in reality the stream velocity varies along the cross section of the cylinder is taken into account by calculating the local heat-transfer coefficients by use of the velocity found in the stream at the considered distance from the stagnation point. This method is contained in a summary presented in reference 6. Unfortunately, such an approach gives heat-transfer coefficients which are considerably low in many cases (see fig. 1).

Better agreement was obtained by another approach (ref. 7) which uses, instead of the flat-plate solution, a family of solutions of the boundary-layer equations which can be obtained in a general form, namely, for the case where the stream velocity varies along the surface as a certain power of the distance from the stagnation point. Such a velocity variation is obtained in incompressible flow around wedges. The solutions for such a type of flow were used to obtain approximate heat-transfer coefficients for a cylinder with arbitrary cross section by stipulating that the local heat-

transfer coefficient on any location along the cylinder is identical with the local heat-transfer coefficient on a wedge for which, at the same distance from the stagnation point, the stream velocity and its gradient are the same as those on the investigated cylinder. This approach was subsequently used by different authors, and is described, for instance, in references 8 and 9. It takes into account the stream conditions which influence the boundary-layer growth at the location at which the heat transfer is going to be determined; however, it does not properly account for the development of the boundary layer in the range upstream of the point considered. This development may be different on the cylinder and on the equivalent wedge.

Another group uses an integrated momentum equation for the boundary-layer flow as proposed by von Kármán and K. Pohlhausen (refs. 10 and 11, respectively) to calculate the velocity boundary layer. Different procedures were proposed for determining local heat-transfer coefficients from the known velocity boundary layer. Some investigators use Reynolds analogy directly (ref. 12) or with a correction for Prandtl numbers different from 1 (ref. 13). Such approaches give heat-transfer coefficients which are considerably high in many cases, as shown, for instance, in figure 1. More accurate results were obtained when the heat transfer was determined by solving an integrated heat-flow equation for the boundary layer. The velocity field within the boundary layer has to be known in this approach, since the flow velocities within the boundary layer occur in the mentioned heat-flow equation. This method was originated by Kroujiline (ref. 14). Extensions and simplifications are contained in references 15 to 18, and an extension to compressible flow of a fluid having a Prandtl number equal to 1 is found in references 19 and 20. Useful information is also contained in a summarizing report (ref. 21).

Another approach is based on the fact that the use of the heat-transfer coefficients for wedge-type profiles as described previously was found to give fairly accurate heat-transfer coefficients. It should be expected that these heat-transfer coefficients can be improved to a degree which is sufficient for all engineering purposes by a method which takes into account in some approximate way the previous history of the boundary layer. Such a method, called the equivalent wedge-type flow method, is proposed in reference 22, extended to heat transfer at high flow velocities and variable wall temperature in reference 23, and extended to transpiration cooling with small temperature differences in reference 24. The advantages of this method are that no knowledge of the velocity boundary layer is required and that it can be readily extended to take into account the effects of large temperature differences, of transpiration cooling, and of variable wall temperature as soon as the corresponding solutions for the wedge-type flow are available.

Such an extension was made at the NACA Lewis laboratory during 1950-51 and is described herein. It is based on exact boundary-layer solutions for wedge-type flow with large temperature differences and with transpiration cooling (refs. 25 and 26). Charts were prepared which make the calculation of heat transfer around cylinders of any arbitrary cross section more rapid.

SOLUTION OF BOUNDARY-LAYER EQUATIONS FOR WEDGE-TYPE FLOW

BOUNDARY-LAYER EQUATIONS

The following boundary-layer equations describe the velocity and temperature fields in a laminar steady two-dimensional gas flow: the momentum equation, the continuity equation, and the energy equation. The momentum equation is

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) - \frac{\partial p}{\partial x} \quad (1)$$

when body forces are neglected. (All symbols are defined in appendix A; consistent units are used throughout the report.) Since the pressure variation normal to the surface throughout the boundary layer may be neglected, it follows that the pressure is prescribed by the conditions in the stream outside the boundary layer and can be connected with the velocity u_s in the stream and just outside the boundary layer by the Bernoulli equation

$$-\frac{\partial p}{\partial x} = \rho_s u_s \frac{\partial u_s}{\partial x}$$

The introduction of this expression changes the momentum equation to the form

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + \rho_s u_s \frac{\partial u_s}{\partial x} \quad (2)$$

The continuity equation is

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \quad (3)$$

and the energy equation is

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 + u \frac{\partial p}{\partial x} \quad (4)$$

The heat generated by internal friction, described by the second term on the right side of equation (4), and the temperature variation connected with expansion, described by the third term, can be neglected as long as the difference between the total and the static temperature in the gas stream is small compared with the difference between the wall temperature and the temperature in the gas stream. For this condition, then, only the first term on the right side of equation (4) is retained, and the energy equation assumes the form

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \quad (5)$$

Equations (2), (3), and (5) include the case of transpiration cooling when the same medium as that in the outside flow is used as coolant and the boundary conditions are properly defined.

$$\begin{aligned} u=0, v=v_w, \text{ and } T=T_w \text{ when } y=0 \\ u \rightarrow u_s \text{ and } T \rightarrow T_s \text{ when } y \rightarrow \infty \end{aligned} \quad (6)$$

The property values μ , k , c_p , and ρ appearing in the equation depend on temperature and pressure. The variation with pressure can be neglected at the low velocities to which the energy equation was already restricted by disregarding the internal friction and the expansion terms. The influence of the temperature dependency, however, may be appreciable in applications with large temperature differences within the boundary layer. Solutions of the boundary-layer equations which take into account the temperature variation of the property values were obtained in references 9, 25, and 26, in which the partial differential equations were transformed into total differential equations.

CHANGE OF VARIABLES

The transformation of the partial differential equations into total differential equations is possible under the following specialized conditions: The stream velocity is assumed to vary as a power function of the distance from the stagnation point measured along the surface of the cylinder.

$$u_s = Cx^m \quad (7)$$

It has recently become customary to refer to the exponent m in this equation as "Euler number." The Euler number can be expressed by the Bernoulli equation in the following way:

$$m = \frac{-\partial p / \partial x}{\rho_s u_s^2 / x} \quad (8)$$

In addition, the temperature of the wall is assumed to be constant and the property values are assumed to vary proportionally to a power of the absolute temperature T . The numerical calculations were made for air. The exponents used were 0.7 for the viscosity, 0.85 for the heat conductivity, 0.19 for the specific heat, and -1.0 for the density.

The variables

$$\left. \begin{aligned} \eta &= y \sqrt{\frac{\rho_w u_s}{\mu_w x}} \\ f &= \frac{\rho_w \psi}{\sqrt{\mu_w \rho_w x u_s}} \\ \theta &= \frac{T - T_w}{T_s - T_w} \end{aligned} \right\} \quad (9)$$

are used to transform equations (2), (3), and (5) into total differential equations presenting f and θ as functions of η only. The stream function ψ appearing in equations (9) is defined in such a way as to eliminate the continuity equation (3).

$$\left. \begin{aligned} \rho u &= \frac{\partial(\rho_w \psi)}{\partial y} \\ \rho v &= -\frac{\partial(\rho_w \psi)}{\partial x} \end{aligned} \right\} \quad (10)$$

Introducing the new variables into the second of equations (10) leads to the following expression for the velocity component normal to the surface:

$$-\rho v = \rho_w \left[\frac{1+m}{2} f \sqrt{\frac{\mu_w u_s}{\rho_w x}} + \frac{y}{2} f'(m-1) \frac{u_s}{x} \right] \quad (11)$$

The velocity at the surface itself follows:

$$v_w = -\frac{1+m}{2} f_w \sqrt{\frac{\mu_w u_s}{\rho_w x}} \quad (12)$$

The transformation therefore prescribes a certain variation of the coolant velocity v_w along the surface, since the function f_w has to be constant (independent of x). The stream velocity is described by equation (7); thus, the coolant velocity v_w is also proportional to some power of x . Such a variation of the coolant velocity leads to a constant wall temperature and is therefore consistent with the assumed constant wall temperature when heat transfer by radiation may be neglected (ref. 27). The transformed equations are presented in references 9, 25, and 26, together with the solutions for a Prandtl number Pr of 0.7, and for a range of Euler number m , temperature ratio T_s/T_w , and the parameter f_w describing the cooling-air flow through a porous surface. The results contain expressions for the thickness of the flow boundary layer which are defined in two ways: the displacement thickness

$$\delta_d = \int_0^\infty \left(1 - \frac{\rho u}{\rho_s u_s}\right) dy \quad (13)$$

and the momentum thickness

$$\delta_i = \int_0^\infty \frac{\rho u}{\rho_s u_s} \left(1 - \frac{u}{u_s}\right) dy \quad (14)$$

The thermal boundary layer is characterized in this report by the convection thickness

$$\delta_c = \int_0^\infty \frac{\rho u}{\rho_s u_s} \frac{T - T_s}{T_w - T_s} dy \quad (15)$$

In addition, a thermal boundary-layer thickness, which is defined as follows, will be used herein:

$$\delta_t = \int_0^\infty \left(\frac{T - T_s}{T_w - T_s}\right) dy \quad (16)$$

Values for this boundary-layer thickness can be easily calculated from results presented in references 25 and 26.

APPLICATION TO HIGH VELOCITIES

The solutions described apply exactly only to flow with low velocities. Practically, the limiting velocity up to which it is possible to neglect the frictional and the expansion terms can be set quite high for a gas; this fact can be understood from the following transformation of the energy equation, in which only the specific heat is regarded constant. If the momentum equation (1) is multiplied by the velocity u and added to the energy equation (4) and if, in addition, the total temperature $T_T = T + u^2/2c_p$ is introduced, the following expression is obtained:

$$\rho c_p \left(u \frac{\partial T_T}{\partial x} + v \frac{\partial T_T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k \frac{\partial T_T}{\partial y} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left[\mu \left(1 - \frac{1}{Pr} \right) \frac{\partial(u^2)}{\partial y} \right] \quad (17)$$

The last term on the right side of the equation vanishes for a Prandtl number equal to 1. In this case, the energy equation has the same form as the one for low velocities in which the friction and the expansion terms were neglected. The only difference lies in the fact that the total temperature appears in the energy equation. When the Prandtl number is approximately 1, the last term in equation (17) will be comparatively small up to considerable velocities, and the energy equation (5) used in the following considerations applies to this condition when the temperature T is interpreted as total temperature. It will be shown later that as far as heat transfer is concerned, the range in which the results of a calculation with equation (5) may be used can be extended even further by using a properly defined adiabatic wall temperature instead of the total gas temperature.

The property values μ , k , c_p , and ρ depend for gases on the temperature. This dependency was taken into account in the described calculations. The density depends, in addition, on the pressure, and the pressure variation may become considerable at high Mach numbers. There are indications, however, that calculations which neglect this pressure variation can be used with sufficient accuracy over the entire subsonic range, as is pointed out in reference 28, in which an investigation of results obtained by L. Howarth (ref. 29) is reported.

EXTENSION OF THEORY TO ARBITRARY BODIES

DETERMINATION OF EQUIVALENT WEDGE

The solutions discussed in the previous paragraph are in an exact sense restricted to a certain type of velocity variation along a cylindrical surface, namely, a stream velocity which just outside the boundary layer is proportional to some power of the distance from the stagnation point. Such a velocity distribution is realized, for instance, in incompressible flow around wedges. The wedge-type solutions may be used, however, to obtain approximate heat-transfer coefficients on cylinders of arbitrary cross section. In one approach in this direction, it is assumed that the heat-transfer coefficient on any point along the circumference of a profile with arbitrary cross section is the same as that on a wedge at the same distance from the leading edge, provided the stream velocity and its gradient on the wedge and on the arbitrary profile have the same value at the location considered and that the temperature ratio T_s/T_w is the same. It will be shown that such an approach takes into account the right stream conditions at the local spot for which the heat-transfer coefficient is to be determined. However, the previous history within the boundary layer is not properly considered. Numerical calculations presented herein show that heat-transfer coefficients obtained in such a manner are in most cases within about 15-percent agreement with experimental data. It is to be expected that a modification which accounts in some approximate manner for the conditions in the boundary layer

upstream of the point under consideration should improve this approximation to the desired degree. This modification is made in reference 22 by the stipulation that the rate of increase of the boundary-layer thickness is the same on the considered point of an arbitrary profile and on the point of a wedge which has the same boundary-layer thickness, the same stream velocity, and the same stream velocity gradient. This same stipulation will be used in the present report. For a given temperature ratio T_s/T_w , the heat-transfer coefficients on a wedge depend on the Euler number m and the value f_w characterizing the coolant flow through a porous surface. These parameters which define the equivalent wedge profile will now be expressed by the boundary-layer thickness and the local stream velocity gradient.

For the wedge-type profile, the stream velocity is expressed by the power law

$$u_s = C\xi^m \quad (18)$$

in which the value ξ expresses the distance from the leading edge measured along the wedge surface in order to distinguish it from the distance of the point under consideration from the stagnation point on the arbitrary profile, which is denoted by x . The variables used for the transformation of the original boundary-layer equations in the previous section may now be written

$$\eta = y \sqrt{\frac{\rho_w u_s}{\mu_w \xi}} \quad (19)$$

and

$$f_w = -\frac{2}{m+1} v_w \sqrt{\frac{\rho_w \xi}{\mu_w u_s}} \quad (20)$$

Corresponding to a certain value y , which indicates the boundary-layer thickness δ , there is a value η_b of the coordinate η defined by the equation

$$\eta_b = \delta \sqrt{\frac{\rho_w u_s}{\mu_w \xi}} \quad (21)$$

In order to eliminate the distance ξ from this equation, equation (18) is differentiated to obtain

$$\frac{\partial u_s}{\partial \xi} = mC\xi^{m-1} = m \frac{u_s}{\xi} \quad (22)$$

Since the velocity gradient on the wedge profile is assumed the same as that on the profile under consideration, it follows that $\partial u_s / \partial \xi = \partial u_s / \partial x$. This equality gives for the coordinate ξ the expression

$$\xi = \frac{m u_s}{\partial u_s / \partial x} \quad (23)$$

When this expression is introduced into equation (21), there is obtained

$$\eta_b = \delta \sqrt{\frac{\rho_w}{\mu_w m} \left(\frac{\partial u_s}{\partial x} \right)}$$

In this expression, η_b (denoted as $(\delta/x)\sqrt{Re}$ in refs. 25 and 26) is a function of the Euler number m and of the coolant-flow parameter f_w . Therefore, if this equation is written in the form

$$\eta_b^2 m = \frac{\rho_w \delta^2}{\mu_w} \frac{\partial u_s}{\partial x} \quad (24)$$

the left side is a function of m and f_w , and equation (24) relates both values to the boundary-layer thickness δ and the velocity gradient $\partial u_s / \partial x$. In order to obtain a second relation for m and f_w , the coordinate ξ is replaced in equation (20). The result is

$$-\frac{m+1}{2} f_w \eta_b = \frac{\rho_w v_w \delta}{\mu_w} \quad (25)$$

which is written again in such a way that the left side is a function of the Euler number m and the flow parameter f_w , which can be calculated from the results in references 25 and 26. Both equations (24) and (25) are therefore sufficient to determine the equivalent wedge profile.

EQUATIONS FOR BOUNDARY-LAYER THICKNESS AND HEAT TRANSFER

The next step is to develop a differential equation for the boundary-layer thickness from the postulate that the boundary-layer gradient $d\delta/dx$ is the same for the real profile as for the equivalent wedge profile. For the wedge profile, the boundary-layer thickness is given by the expression

$$\delta = \eta_b \sqrt{\frac{\mu_w}{C \rho_w}} \xi^{\frac{1-m}{2}}$$

which is obtained from equation (21) by replacing the stream velocity with equation (18) and solving for the boundary-layer thickness. A differentiation of this equation and the use of equations (23) and (24) result in

$$\frac{d\delta}{dx} = \frac{1-m}{2m} \delta \frac{\partial u_s / \partial x}{u_s} = \frac{1-m}{2} \eta_b^2 \frac{\mu_w}{\rho_w \delta u_s} \quad (26)$$

This is a differential equation for the boundary-layer thickness which contains only values which are known for the profile under consideration or which are determined from equations (24) and (25) for the equivalent wedge-type flow. An integration of the differential equation gives the boundary layer along the circumference of the profile under consideration.

The local heat-transfer coefficient is defined by the following equation:

$$h(T_s - T_w) = k_w \left(\frac{\partial T}{\partial y} \right)_w$$

Introducing the dimensionless temperature ratio given in equation (9) and the coordinate ξ results in

$$h = k_w \theta'_w \frac{\partial \eta}{\partial y} = k_w \theta'_w \frac{\mu_s}{\delta} \quad (27)$$

The heat-transfer coefficient may be calculated from this expression as soon as the boundary-layer thickness δ is known, since θ'_w and η_b are functions of m and f_w contained in references 25 and 26.

Up to the present time no recommendation has been made as to which boundary-layer thickness should be used in the prescribed procedure. When the momentum thickness is used in the foregoing equations, it is easily understandable that the integrated momentum equation is satisfied and that the method of calculation becomes the same as the one proposed by von Kármán in reference 10. This fact can be proved mathematically by a procedure completely analogous to the one used in appendix B. On the other hand, the use of the convection thickness as defined in equation (15) satisfies the integrated heat-flow equation within the boundary layer, as shown in appendix B. The use of both boundary-layer thicknesses leads to somewhat different results for the local heat-transfer coefficient, and a question arises as to which is preferable. It is pointed out by Schuh in reference 23 that, for the purpose of determining heat-transfer coefficients, it is more important to satisfy the heat-flow balance; the use of the convection thickness was therefore recommended. In reference 22, the use of the thermal boundary-layer thickness as defined in equation (16) is investigated, and the results of the calculation with this boundary-layer thickness are found to agree even better with measured values and with other calculations. The convection thickness δ_c and the thermal thickness δ_t for the boundary layer will therefore be used in parallel in the following numerical evaluations.

CALCULATION PROCEDURE USE OF DIMENSIONLESS VARIABLES

The procedure which may be followed in determining local heat-transfer coefficients with the relations developed in the preceding section is now explained. Figure 2 shows a

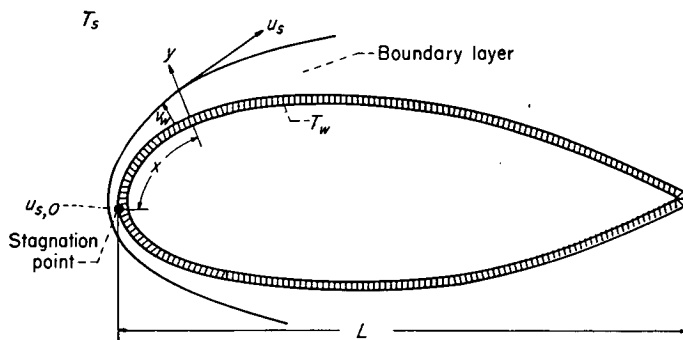


FIGURE 2.—Sketch of cylinder indicating notation used.

sketch of a cylinder with arbitrary cross section and the notation used in the analysis. Before numerical calculations are made, however, it is advisable to change to dimensionless quantities. In order to make this change, the distance x is divided by the major axis L of the cylinder and the mass velocity in the direction of x is divided by an upstream mass velocity. All lengths and mass velocities parallel to y are, in addition, multiplied by the square root of the Reynolds number Re_0 based on the major axis and the upstream mass velocity. The dimensionless variables which are subsequently needed are

$$x^* = \frac{x}{L} \quad (28)$$

$$\delta^* = \frac{\delta}{L} \sqrt{Re_0} \quad (29)$$

$$u_s^* = \frac{\rho_w u_s}{\rho_0 u_{s,0}} \quad (30)$$

$$v_w^* = \frac{\rho_w v_w}{\rho_0 u_{s,0}} \sqrt{Re_0} \quad (31)$$

where

$$Re_0 = \frac{u_{s,0} L \rho_0}{\mu_w} \quad (32)$$

By use of these dimensionless quantities, equation (26) is transformed into

$$\frac{d\delta^*}{dx^*} = \frac{M}{\delta^* u_s^*} \quad (33)$$

where

$$M = \frac{1-m}{2} \eta_b^2 = M \left(\frac{du_s^*}{dx^*} \delta^{*2}, v_w^* \delta^* \right) \quad (34)$$

according to equations (24) and (25), which, in dimensionless values, are

$$\eta_b^2 m = \frac{du_s^*}{dx^*} \delta^{*2}$$

and

$$-\frac{m+1}{2} f_w \eta_b = v_w^* \delta^*$$

Introduction of the dimensionless quantities into equation (27) leads to

$$\frac{Nu}{\sqrt{Re_0}} = \frac{N}{\delta^*} \quad (35)$$

where

$$Nu = \frac{hL}{k_w} \quad (36)$$

and

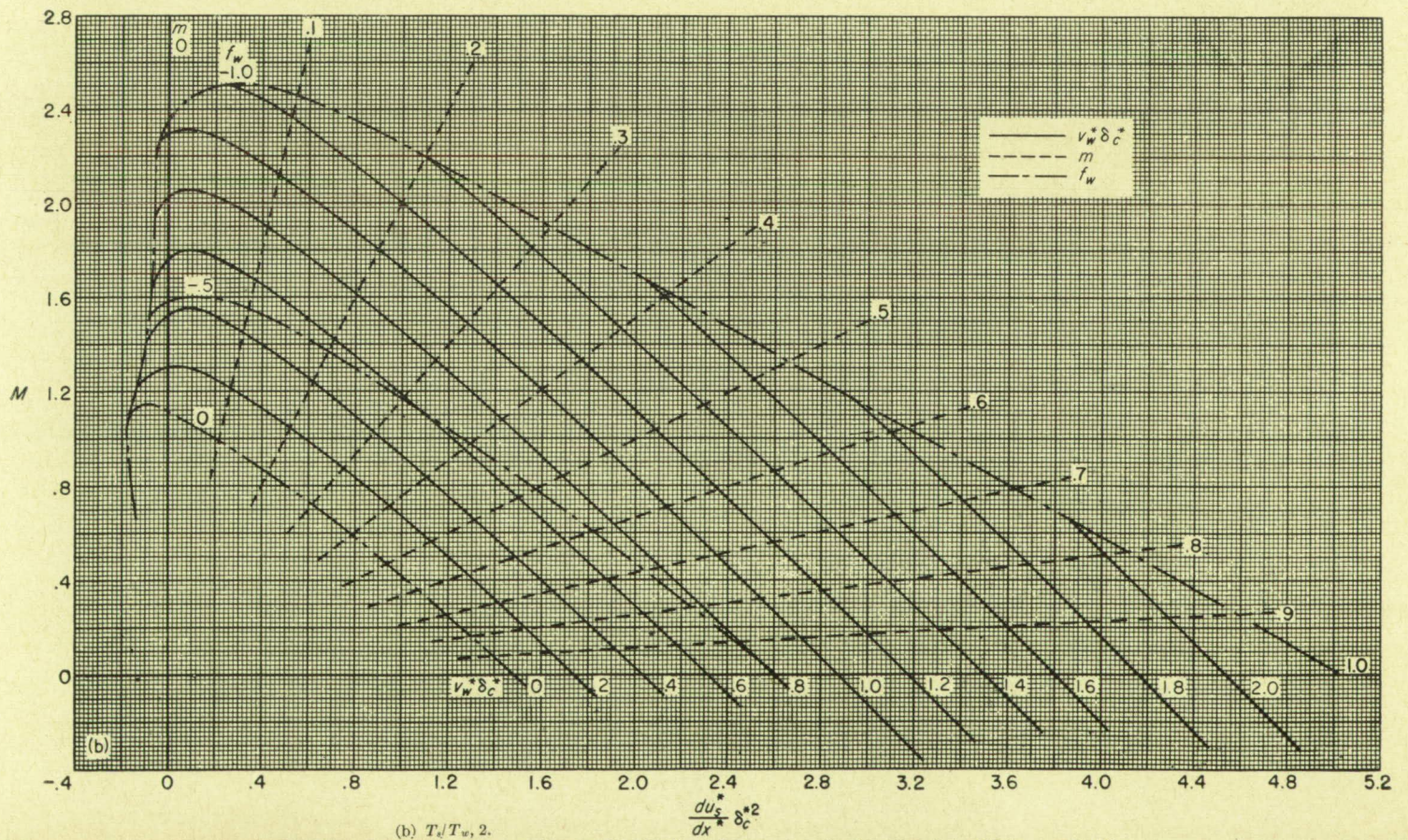
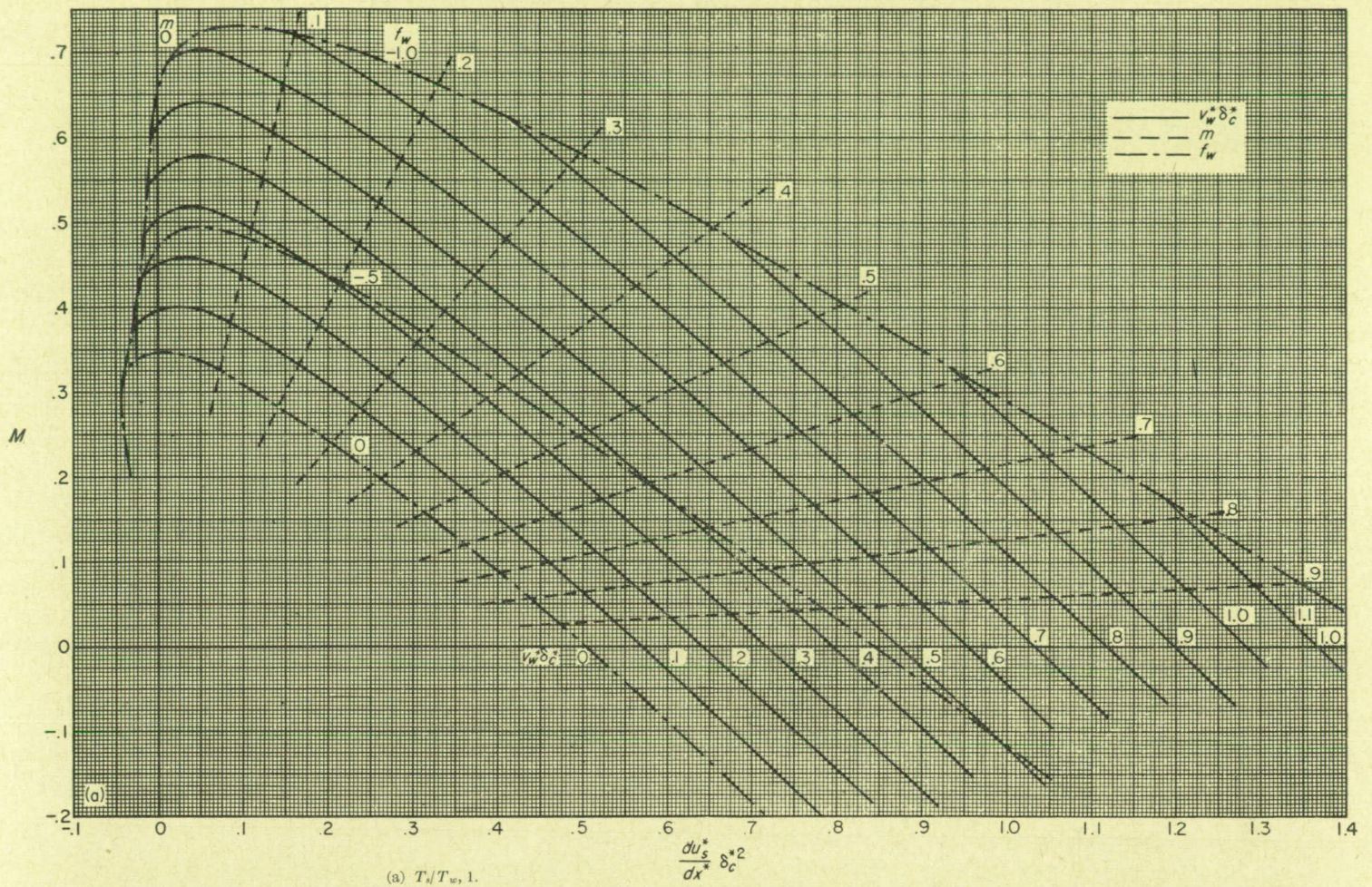
$$N = \theta'_w \eta_b = N \left(\frac{du_s^*}{dx^*} \delta^{*2}, v_w^* \delta^* \right) \quad (37)$$

CHARTS AND CALCULATION PROCEDURE FOR PRESCRIBED COOLANT FLOW

Charts have been prepared which present the functions M and N as expressed by equations (34) and (37) in dependence on $(du_s^*/dx^*)\delta^{*2}$ and $v_w^*\delta^*$. The charts presented herein were constructed from results presented in references 25 and 26. In figures 3 and 4, the dimensionless convection thickness of the boundary layer is used; in figures 5 and 6, the dimensionless thermal boundary-layer thickness is used.

At the stagnation point of any blunt nosed cylindrical body, conditions are the same as those at the stagnation point of a plate normal to the flow. Therefore $m=1$, but the value of δ^* is unknown. However, there exists at the stagnation point a unique relation $v_w^*\delta^* = F[(du_s^*/dx^*)\delta^{*2}]$ which may, for instance, be read along the abscissa in figure 3 or in figure 5. Squaring this equation and dividing both sides by $(du_s^*/dx^*)\delta^{*2}$ result in

$$\frac{v_w^{*2}}{du_s^*/dx^*} = \frac{1}{(du_s^*/dx^*)\delta^{*2}} F^2 \left(\frac{du_s^*}{dx^*} \delta^{*2} \right)$$


 FIGURE 3.—Chart for use in determination of M for dimensionless convection boundary-layer thickness. $Pr, 0.7$.

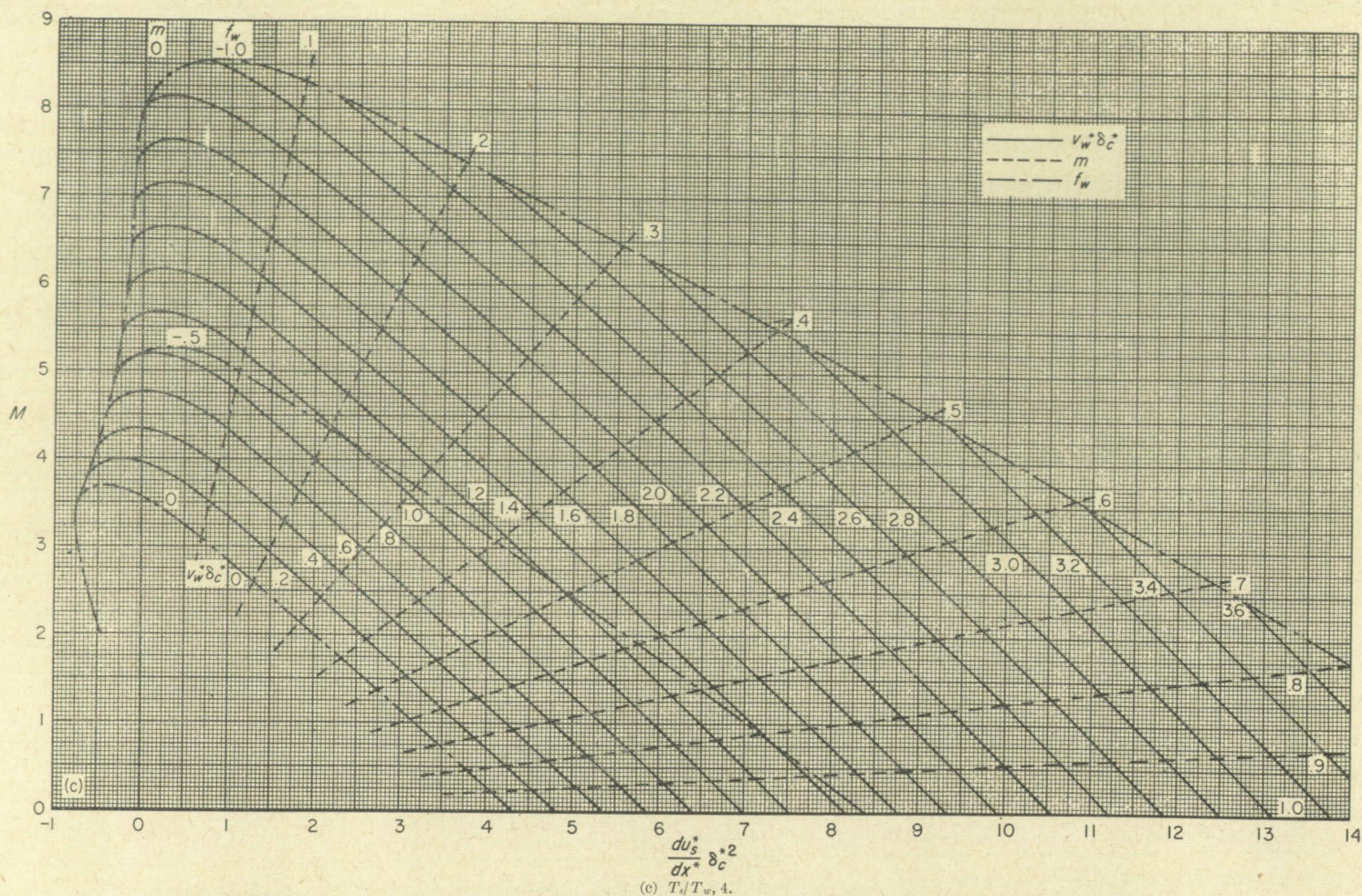


FIGURE 3.—Concluded. Chart for use in determination of M for dimensionless convection boundary-layer thickness. $Pr, 0.7$.

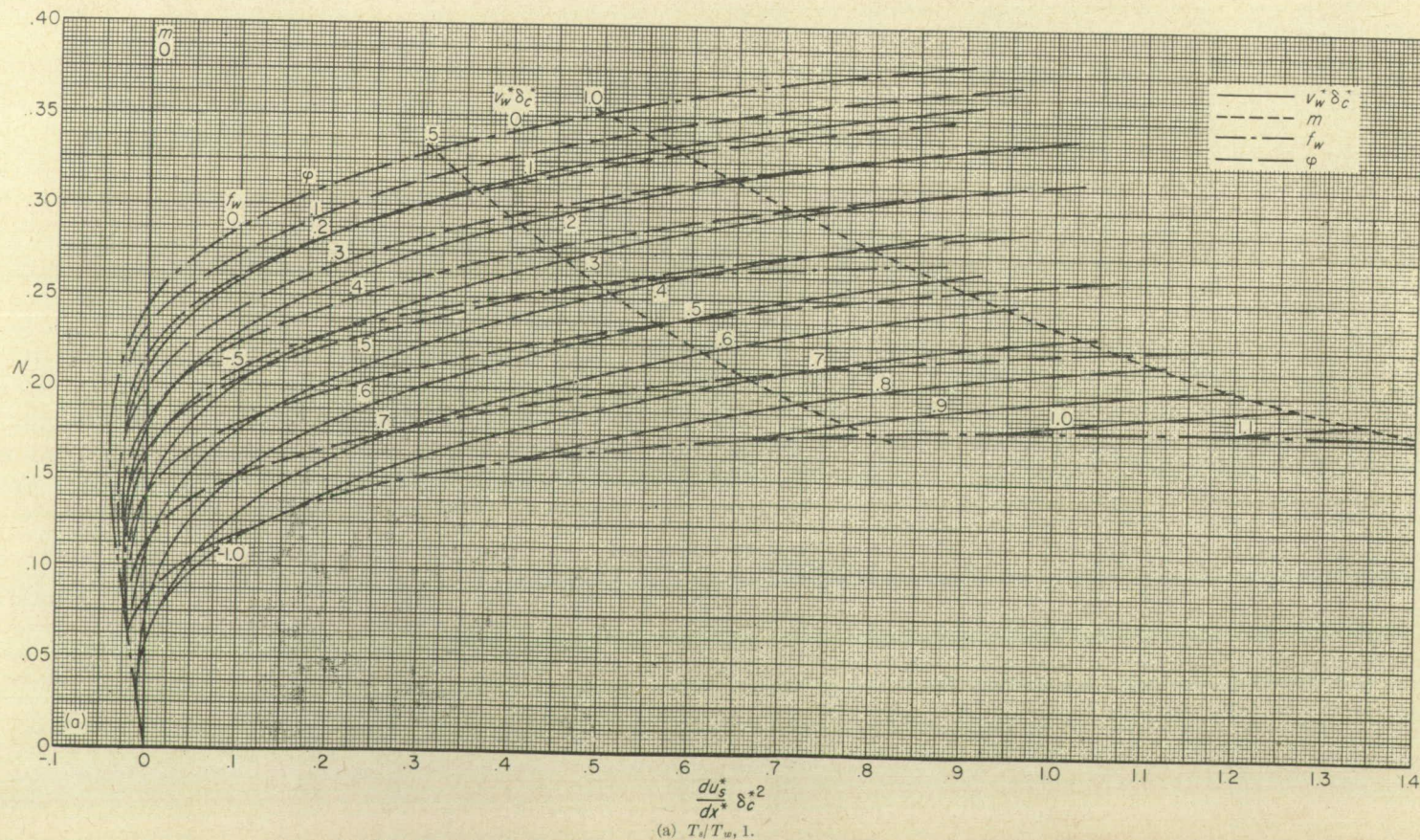
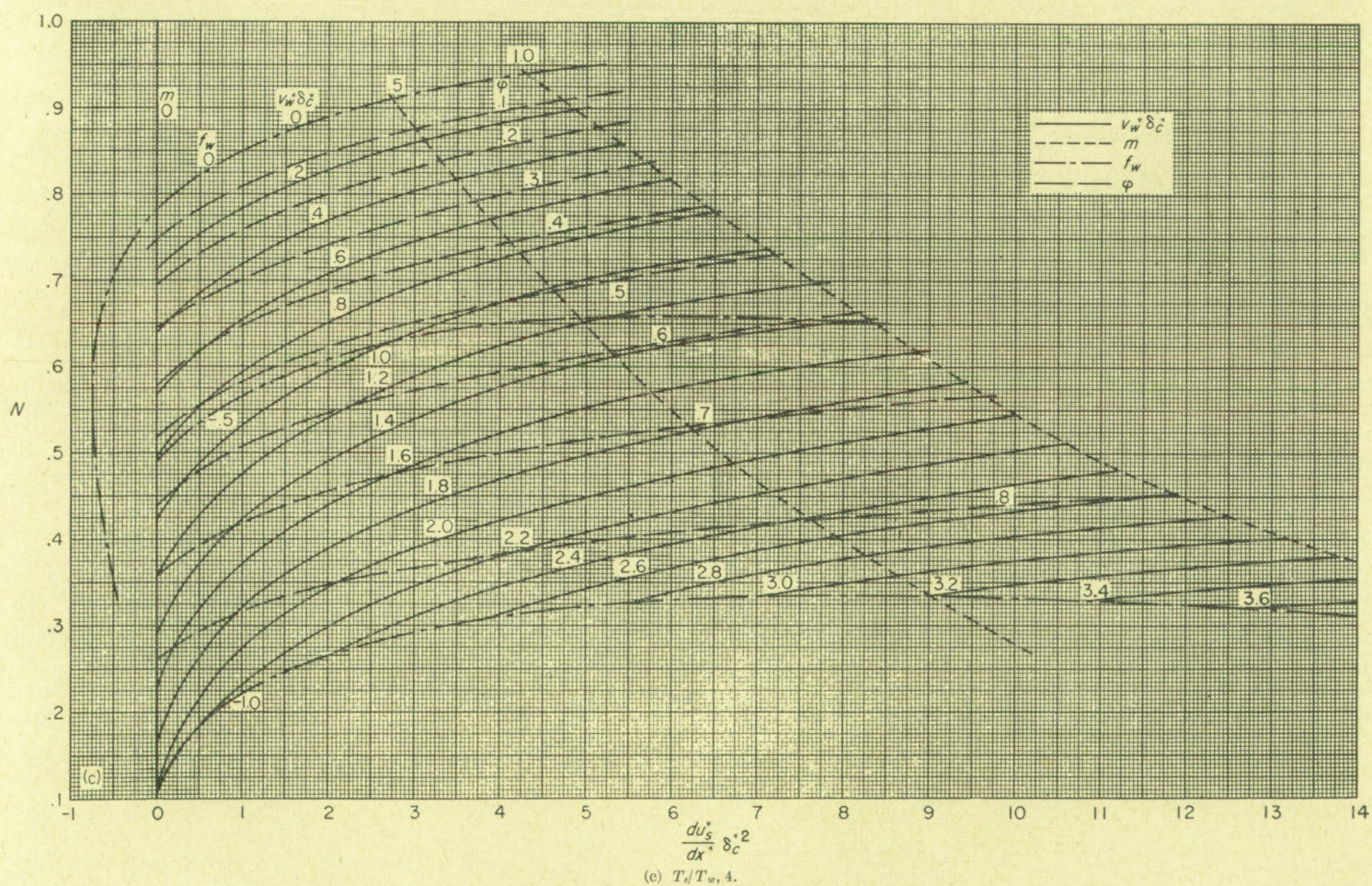
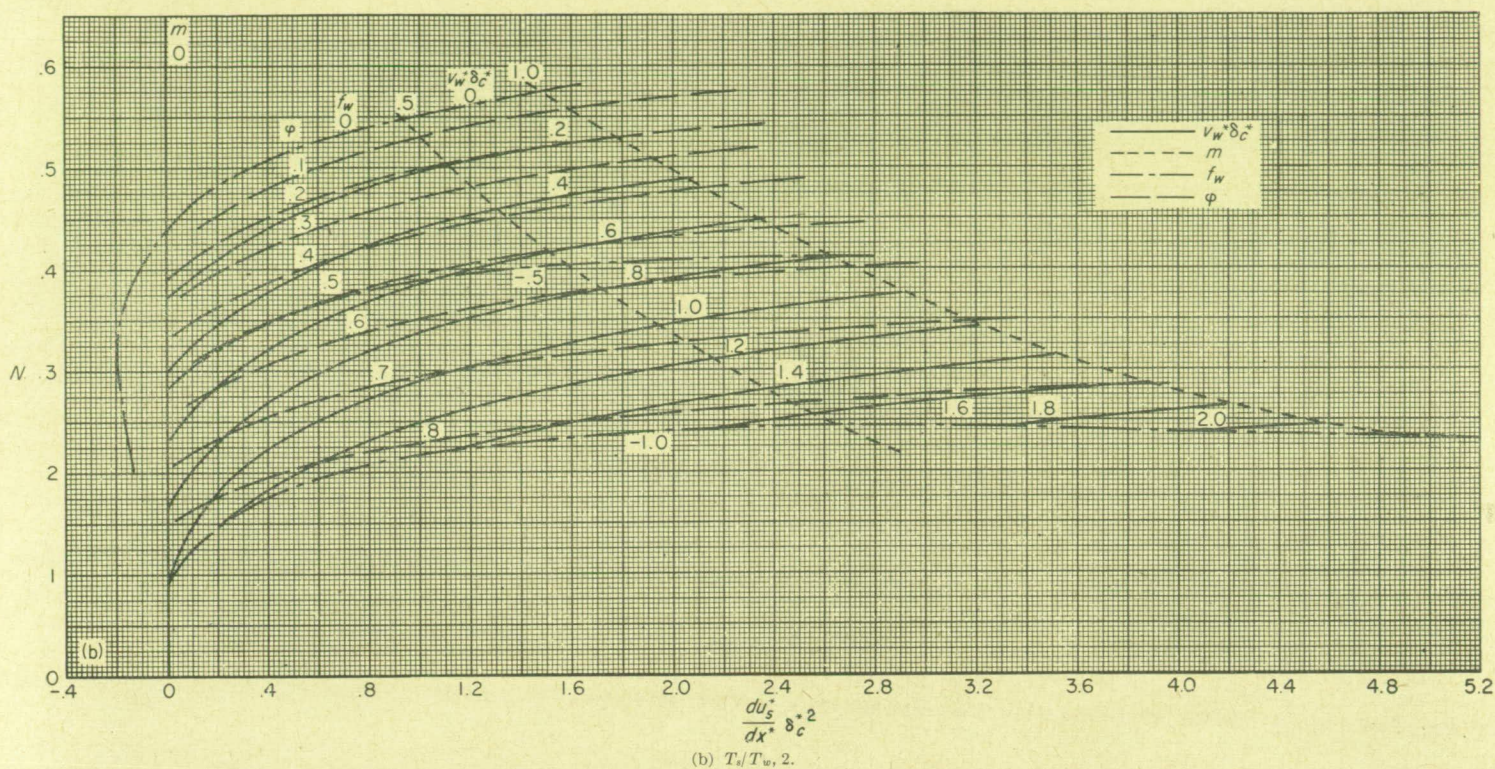


FIGURE 4.—Chart for use in determination of N for dimensionless convection boundary-layer thickness. $Pr, 0.7$.

FIGURE 4.—Concluded. Chart for use in determination of N for dimensionless convection boundary-layer thickness. $Pr, 0.7$.

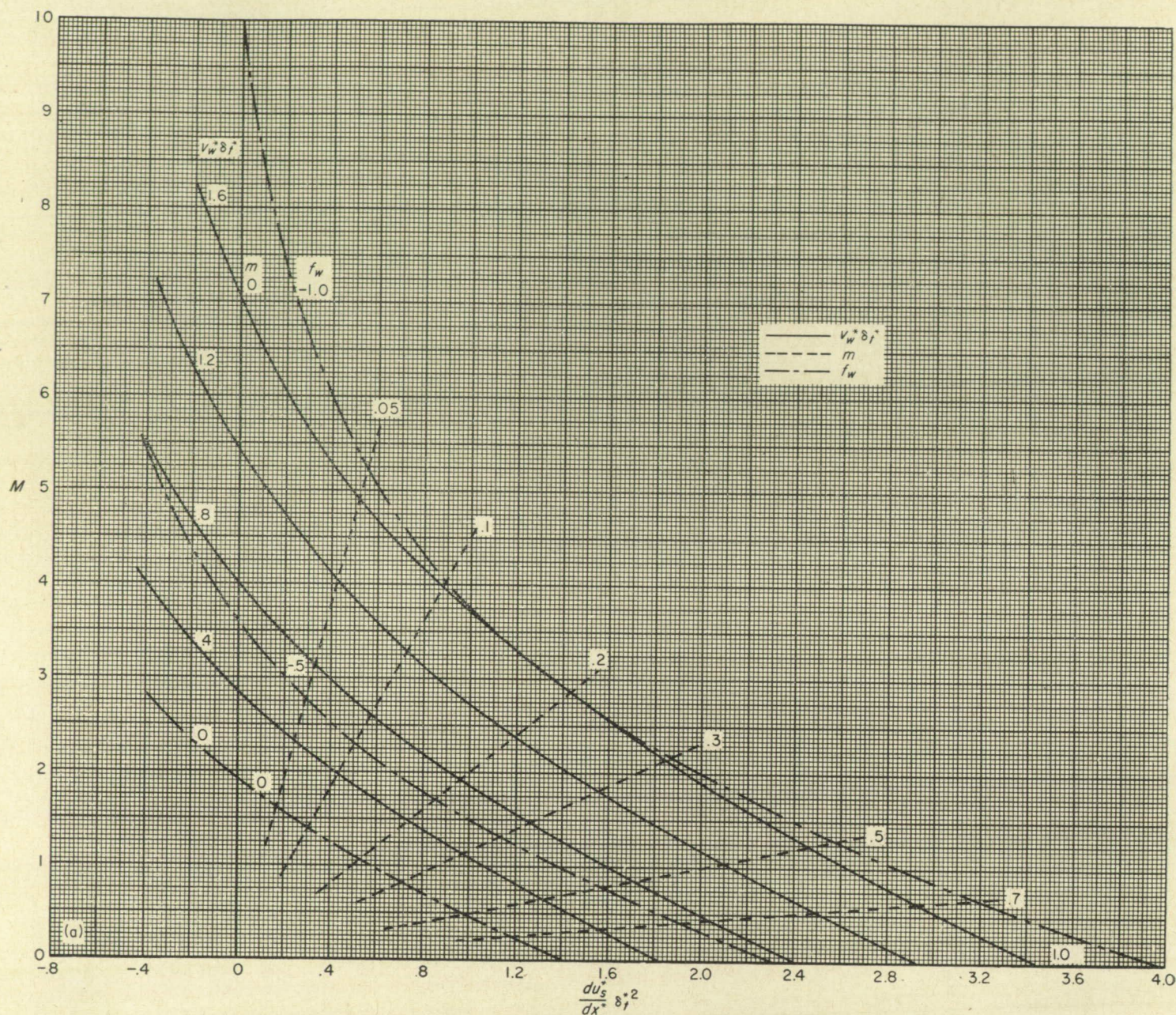


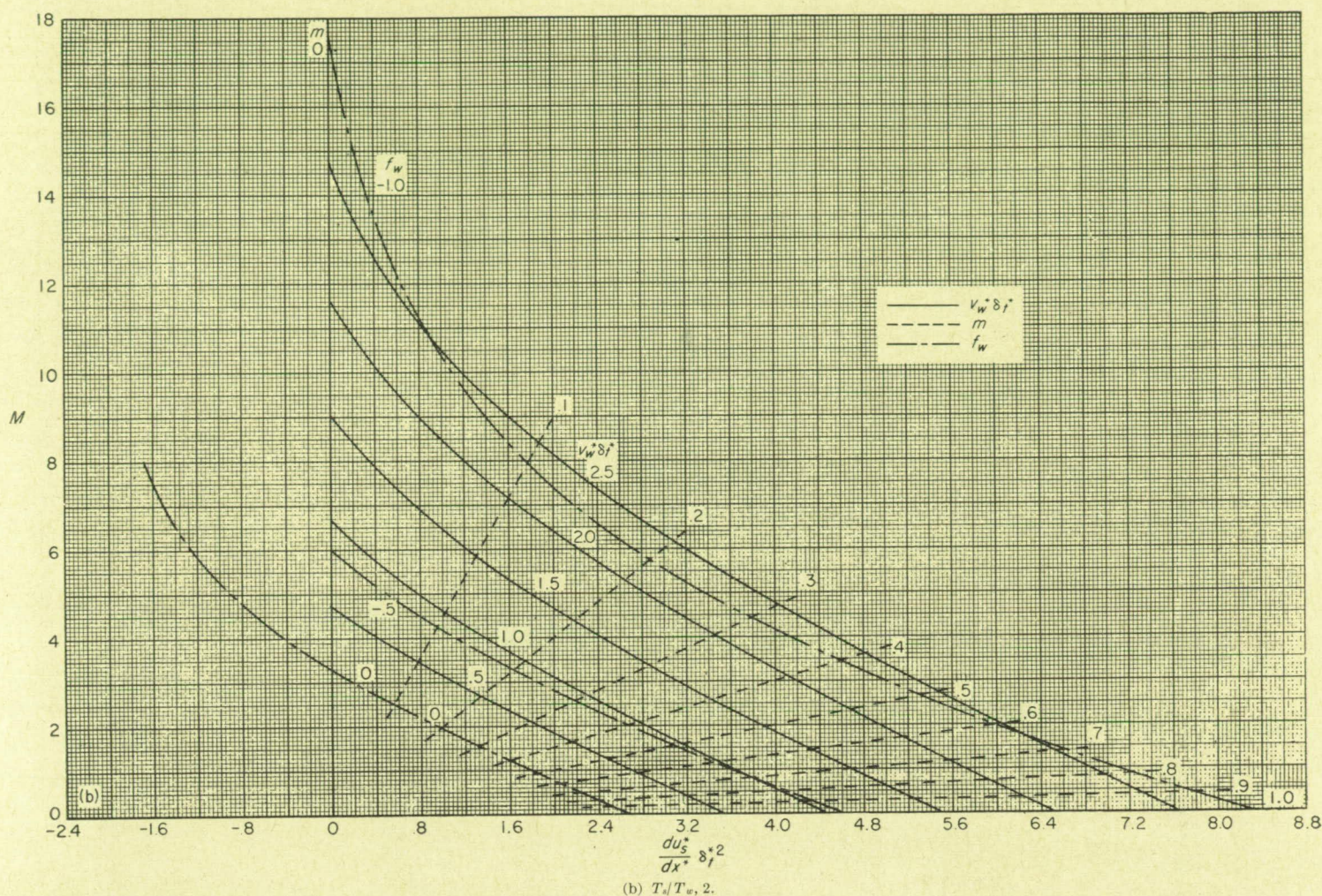
FIGURE 5.—Chart for use in determination of M for dimensionless thermal boundary-layer thickness. $Pr, 0.7$.

These relations are presented in figure 7 for the dimensionless convection boundary-layer thickness and in figure 8 for the dimensionless thermal boundary-layer thickness.

By use of these charts, the calculations for any profile can be made in a very simple manner for either the dimensionless convection or the dimensionless thermal boundary-layer thickness. The method of solution for the convection thicknesses is described subsequently. For the thermal thickness, the procedure is the same.

The values of u_s and du_s/dx must be found for the cylinder

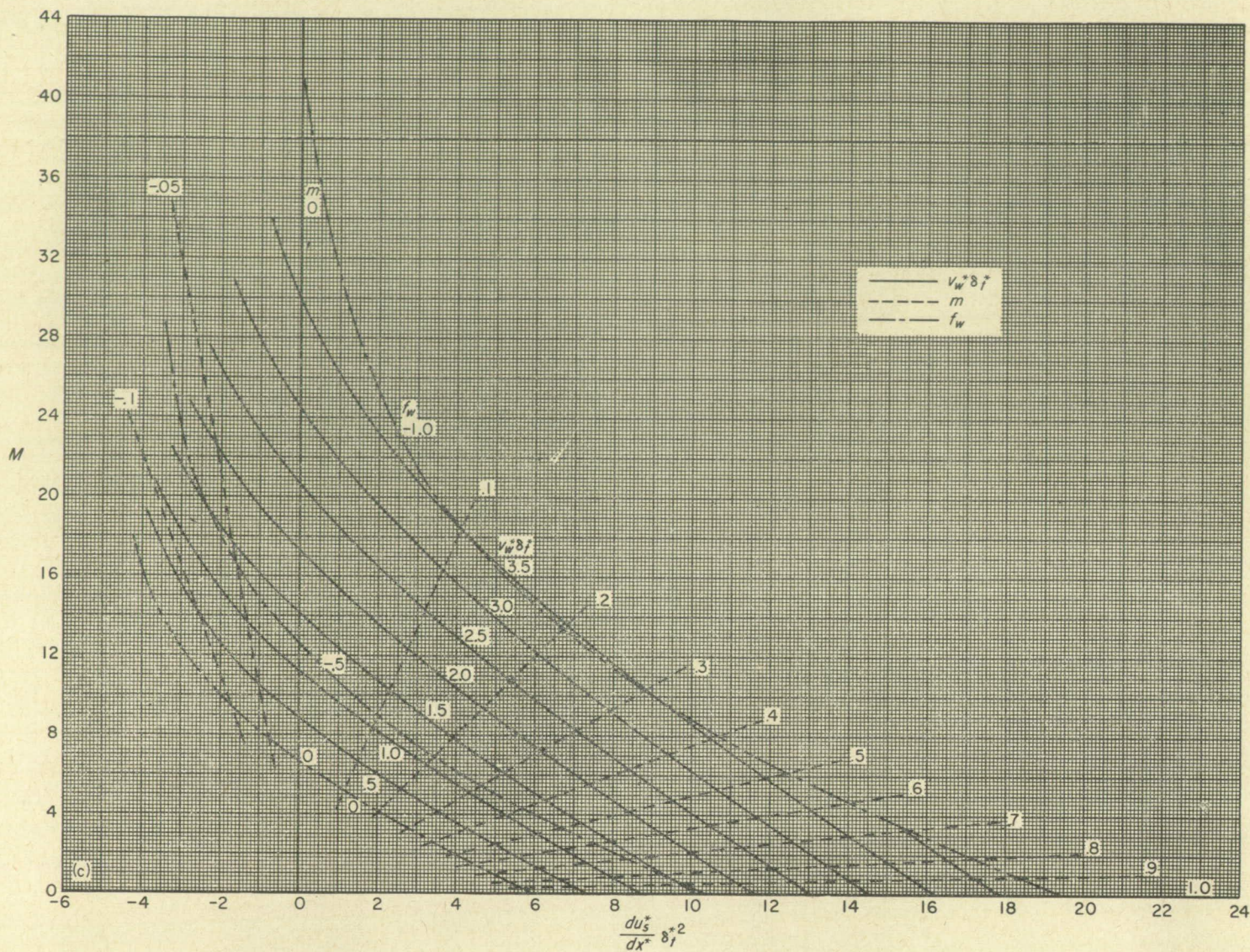
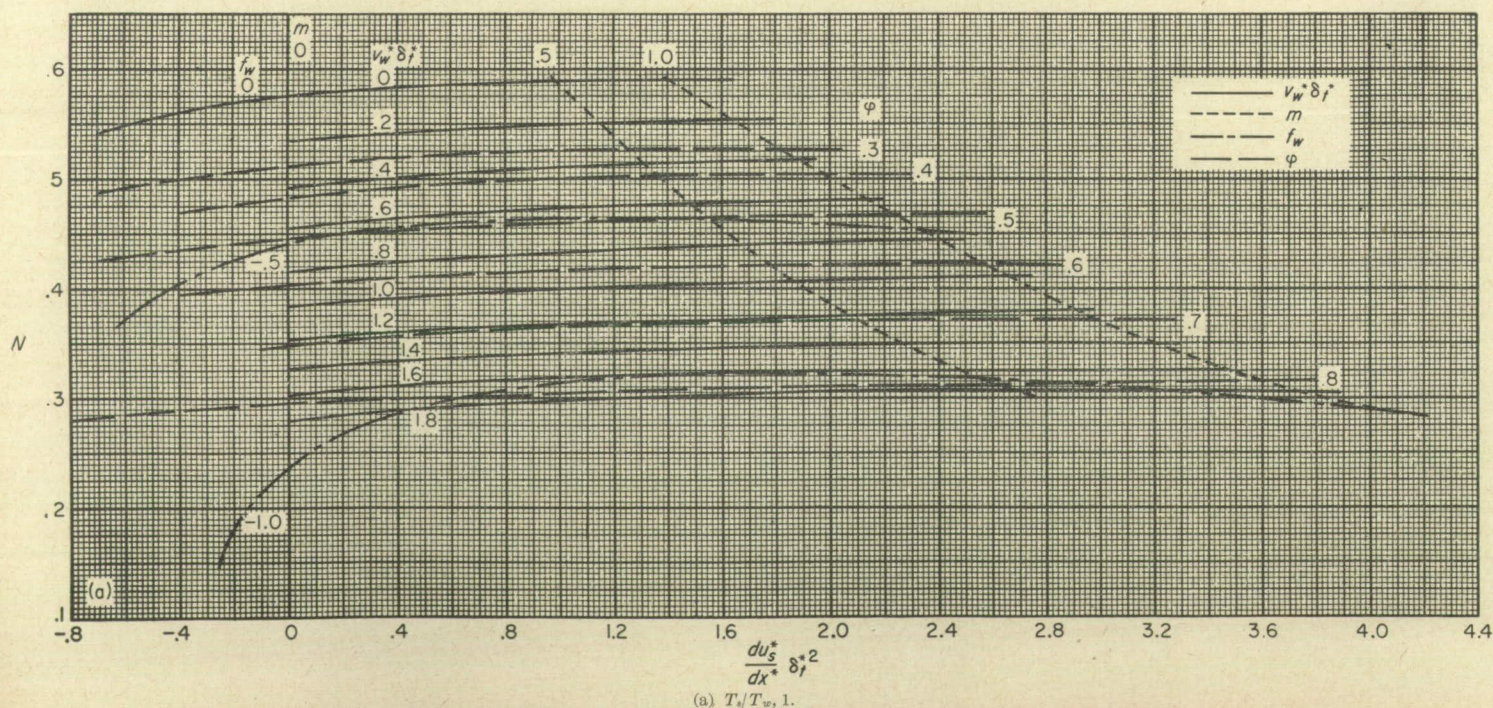
profile under consideration either by measurement or by a solution of the inviscid-flow equations. The coolant velocity v_w is prescribed by the porosity of the wall and by the pressure distribution around the profile. From these terms, the values of u_s^* , du_s^*/dx^* , and v_w^* can be calculated. The value of δ_c^* at the stagnation point can be determined from figure 7 in the following way: The value of $v_w^{*2}/(du_s^*/dx^*)$ is computed, and the corresponding value of $(du_s^*/dx^*)\delta_c^{*2}$ is read from figure 7. A simple algebraic operation then yields the desired value of δ_c^* at the stagnation point.

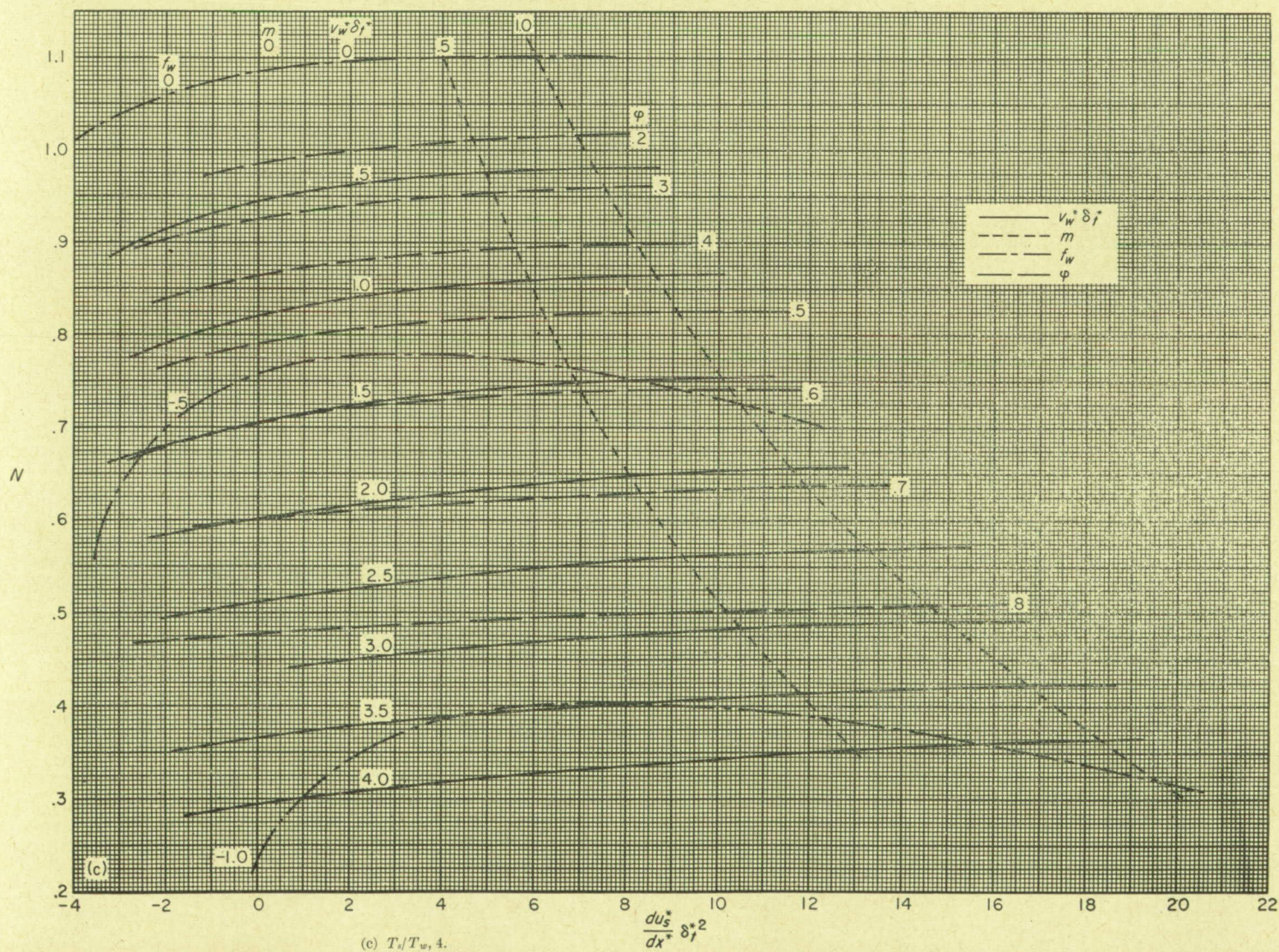
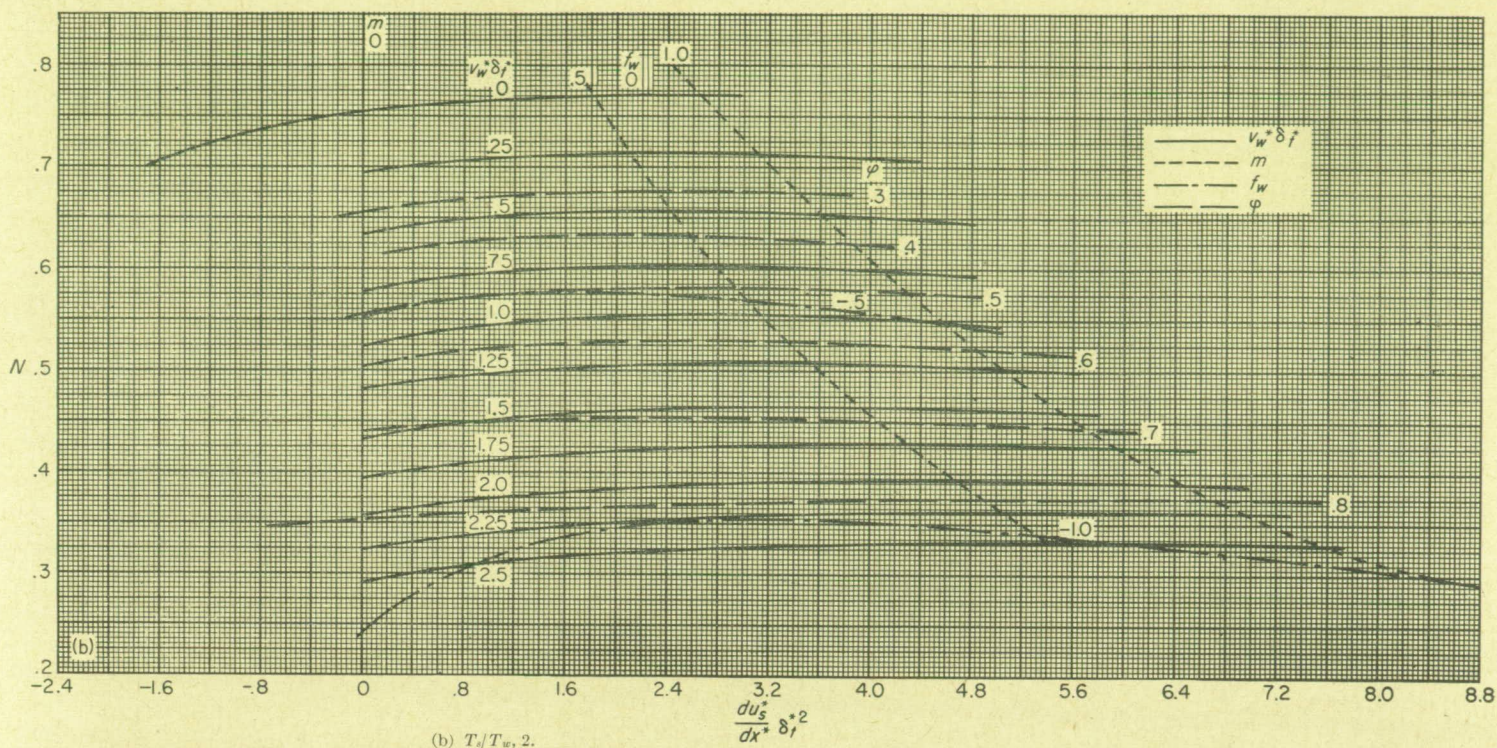
FIGURE 5.—Continued. Chart for use in determination of M for dimensionless thermal boundary-layer thickness. $Pr, 0.7$.

The dimensionless convection boundary-layer thickness δ_c^* along the cylindrical surface is determined from equation (33); for the numerical evaluation presented herein, this equation was solved by the method of isoclines with the aid of figure 3, depending upon which ratio of stream to wall temperature is applied. Equation (33) determines the direction of the tangents to the different δ_c^* -curves which satisfy the equation. The task is to find that curve which contains the δ_c^* -value previously calculated for the stagnation point. For chosen values of x^* and δ_c^* , values of $(du_s^*/dx^*)\delta_c^{*2}$ and $v_w^*\delta_c^*$ are computed and the value of M is read from the appropriate part of figure 3. Equation (33) then gives the slope of the tangent at this selected value of x^* for the assumed δ_c^* . Several values of δ_c^* are used for this x^* . The same calculations are repeated for other values of x^* . If the chosen distance between these x^* -values is small enough, an accurate curve of δ_c^* against x^* can be drawn which starts at the desired previously calculated value of δ_c^* at the stag-

nation point and which will have the correct slope at each value of x^* considered. Figure 9 illustrates this method of solution. Values of N can then be obtained for each of the correct δ_c^* -values and the considered v_w^* -value for each x^* from figure 4 after $(du_s^*/dx^*)\delta_c^{*2}$ and $v_w^*\delta_c^*$ are computed (the ratio of stream to wall temperature under consideration determines which part of figure 4 should be used). The value of $Nu/\sqrt{Re_0}$ can finally be obtained from equation (35).

The same calculation procedure can be used when the dimensionless thermal boundary-layer thickness is considered. Figure 8 is used for the determination of the value of δ_t^* at the stagnation point; figure 5 is used to determine M ; and figure 6 is used to determine N . The particular ratio of stream to wall temperature under consideration determines which parts of these figures apply for the calculation of the values of M and N . Finally, equation (35) gives the desired value of $Nu/\sqrt{Re_0}$.

FIGURE 5.—Concluded. Chart for use in determination of M for dimensionless thermal boundary-layer thickness. $Pr, 0.7.$ FIGURE 6.—Chart for use in determination of N for dimensionless thermal boundary-layer thickness. $Pr, 0.7.$


 FIGURE 6.—Concluded. Chart for use in determination of N for dimensionless thermal boundary-layer thickness. $Pr, 0.7.$

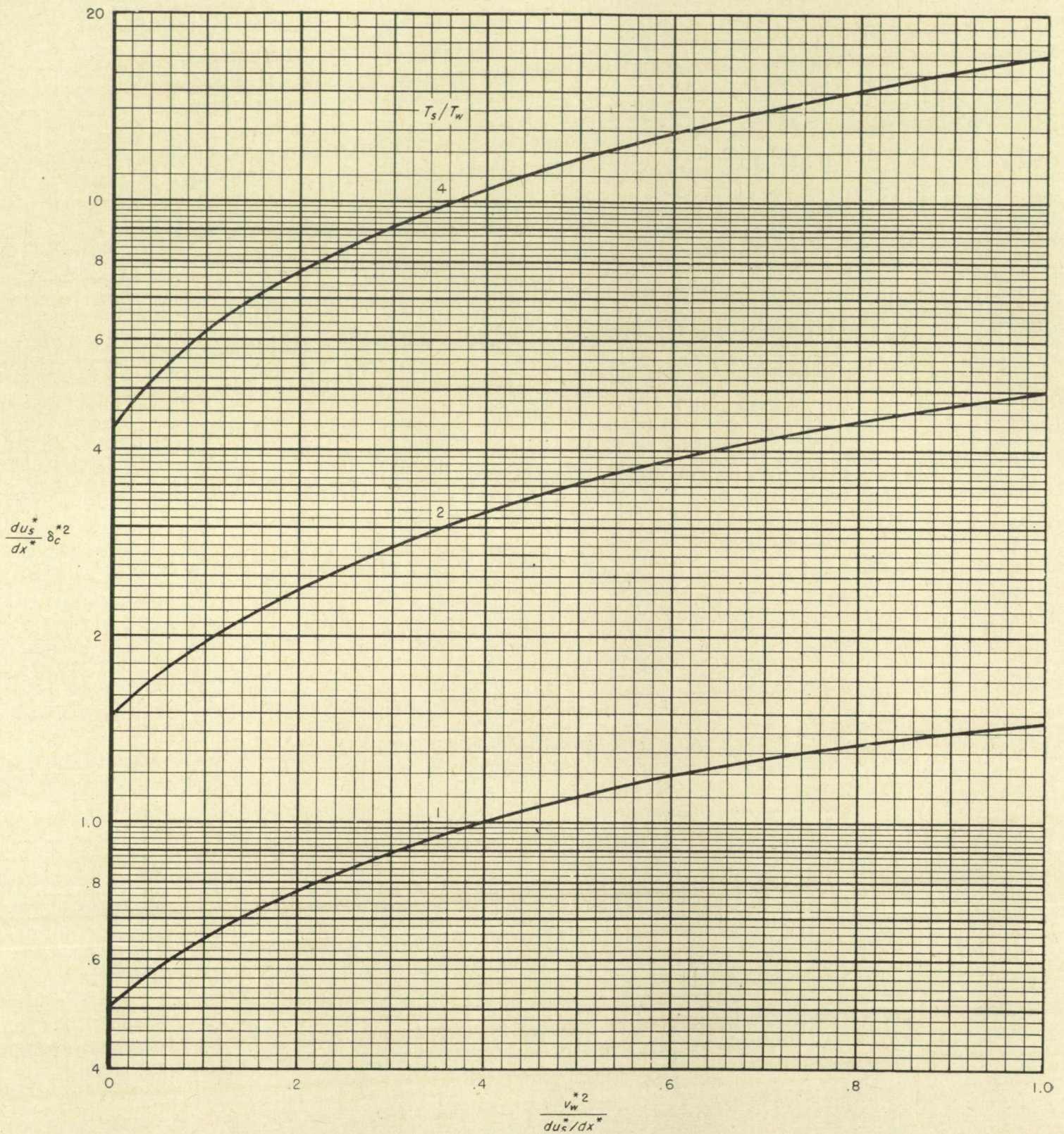


FIGURE 7.—Chart for use in determination of dimensionless convection boundary-layer thickness at stagnation point.

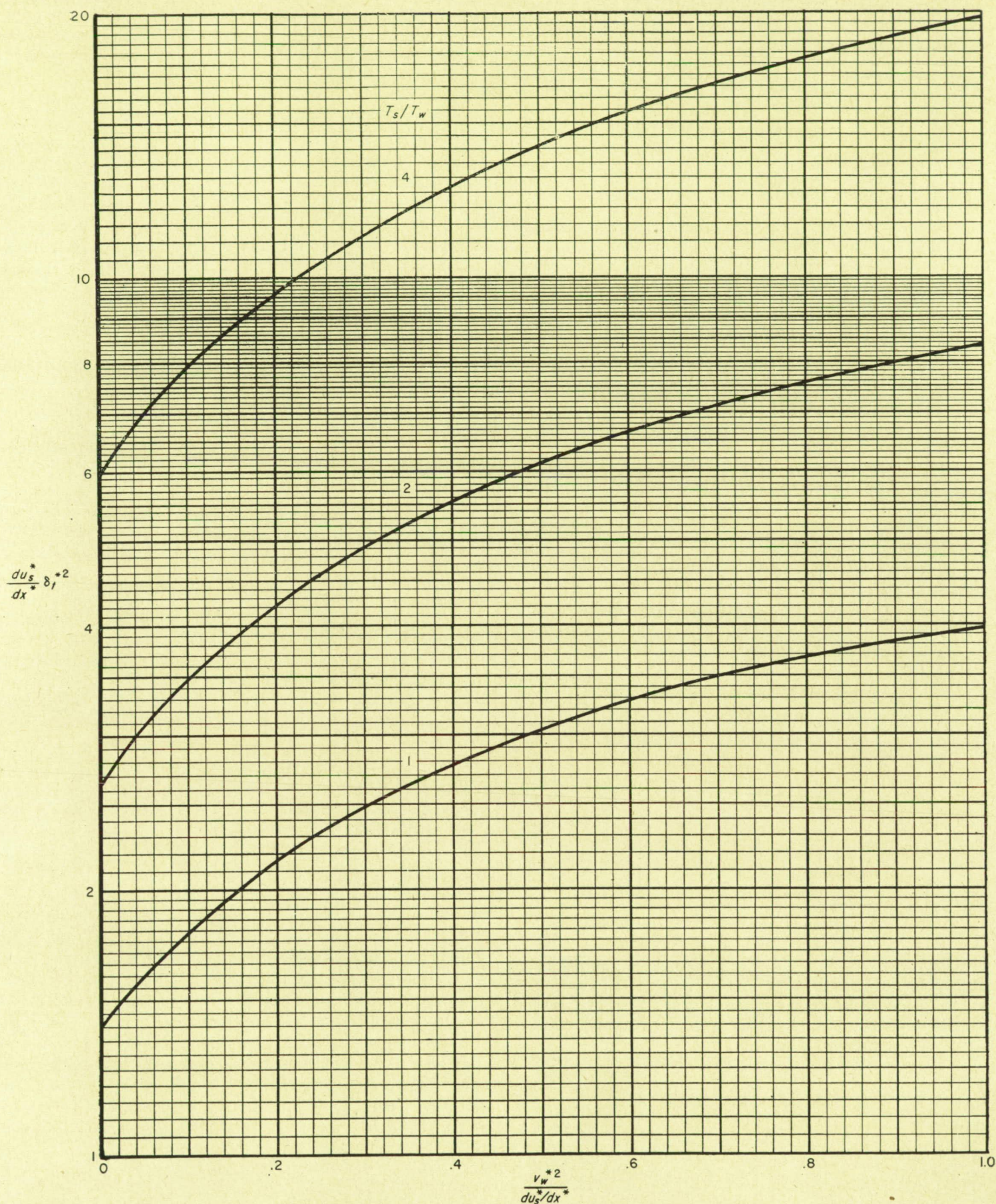


FIGURE 8.—Chart for use in determination of dimensionless thermal boundary-layer thickness at stagnation point.

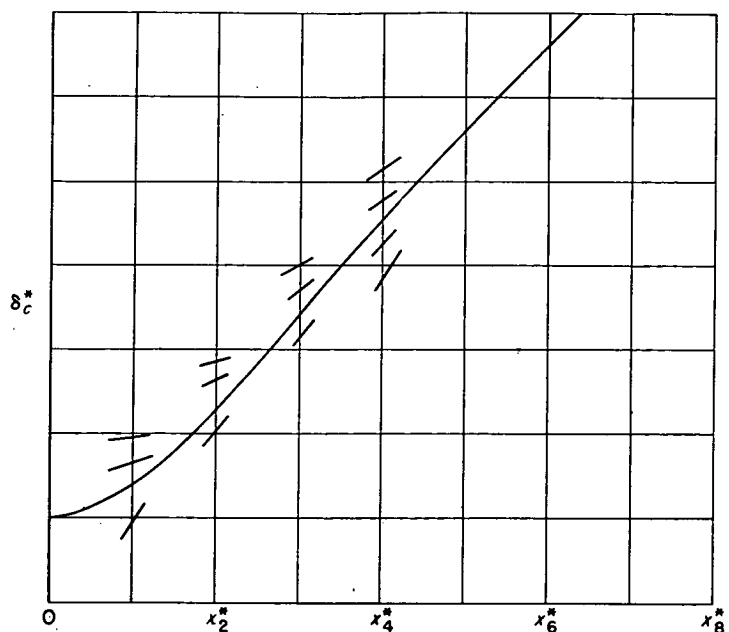


FIGURE 9.—Isocline solution of boundary-layer equation.

CHARTS AND CALCULATION PROCEDURE FOR PRESCRIBED WALL TEMPERATURE

The heat-transfer coefficients determined by the values of $Nu/\sqrt{Re_0}$ can now be used to calculate the surface temperature of the cylinder when the outside stream temperature and the temperature at which the coolant is supplied to the interior of the cylinder are known. For this purpose, a heat balance for an element of the wall as shown in figure 10 is set up. The cylindrical volume element considered may have two plane surfaces, one surface (1) coinciding with the outside surface of the cylinder wall and the other (2) apart from the inside surface of the wall by such a distance that it is situated outside the boundary layer present on this side. (The inside surface has to be considered as a surface of a

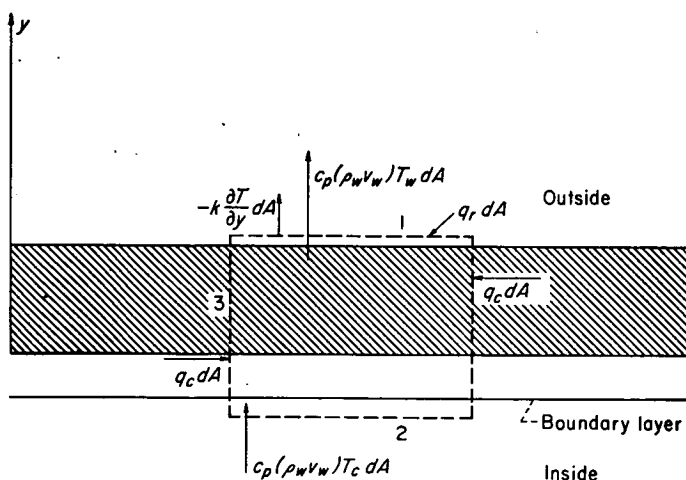


FIGURE 10.—Cross section through part of cylinder wall used in setting up heat balance.

wall to which suction is applied and on which a boundary layer builds up as shown in ref. 30.) The mantle surface (3) of the cylinder may be normal to the wall surfaces. Heat is carried by convection with the cooling air through surfaces 1 and 2. The amount per unit time is indicated in figure 10. It is assumed that the coolant is heated up to the wall surface temperature T_w when it leaves the wall. This assumption is usually well fulfilled. Heat will also be transferred by conduction through the fluid layers immediately adjacent to the outside wall surfaces, the amount being $-k_w (\partial T / \partial y)_w dA$. In addition, heat may be transferred to the outside wall by radiation; it may be $q_r dA$. Heat may also flow into the volume element by conduction in the solid material or by transverse flow of the cooling air. The sum of all these individual flows may be $q_c dA$. Then the heat balance is

$$q_r + q_c + c_p \rho_w v_w T_c = c_p \rho_w v_w T_w - k_w \left(\frac{\partial T}{\partial y} \right)_w$$

The heat $-k_w (\partial T / \partial y)_w$ transferred per unit area from the gas to the wall is expressed in this report by a heat-transfer coefficient

$$h(T_w - T_s) = -k_w \left(\frac{\partial T}{\partial y} \right)_w$$

Combining these two equations results in

$$q_r + q_c + h(T_s - T_w) = c_p \rho_w v_w (T_w - T_c) \quad (38)$$

This equation permits a calculation of the wall temperature for any place on the cylindrical surface when the coolant velocity v_w is prescribed, when the local radiative heat flow q_r and the conductive heat flow q_c are known, and when the heat-transfer coefficient h has been obtained. The conductive heat flow q_c is usually small and can be neglected. Such a calculation results in a wall surface temperature which generally will vary along the circumference of the cylinder. When the variations are large, the temperature distribution obtained can be regarded only as an approximation, since the wedge solutions (refs. 22, 25, and 26) on which the method in this paper is based were obtained for the case of a constant wall temperature.

Usually, however, the problem which faces the designer in an application is somewhat different from the one treated. The purpose of transpiration cooling is mostly to keep the wall temperature of some structural element below the limits which the material can withstand. On the other hand, the amount of coolant almost always must be kept small, which means that local overcooling should be avoided. For the wall surface under these conditions, a temperature is prescribed which should be uniform about the circumference of the cylinder and the problem is to find that distribution of the coolant velocity v_w which results in the desired wall temperature. Generally, such an investigation requires a trial-and-

error procedure which is very involved. The procedure becomes simple and straightforward, however, when the radiative heat flow q_r and the conductive heat flow q_c can be neglected. Such a solution, then, is also useful as a starting point for the trial-and-error procedure when radiation is present.

The heat balance (eq. (38)) can be transformed to

$$\frac{h}{c_p \rho_s u_{s,0}} = \frac{\rho_w v_w}{\rho_s u_{s,0}} \frac{T_w - T_c}{T_s - T_w} \quad (39)$$

when $q_r = q_c = 0$. The ratio of temperature differences in this equation is now a prescribed value. A similar ratio $(T_s - T_w)/(T_s - T_c)$ often appears in turbine-cooling work and is denoted by φ . Introduction of this value and conversion to dimensionless values result in

$$\frac{Nu}{\sqrt{Re_0}} = v_w^* Pr \frac{1 - \varphi}{\varphi} \quad (40)$$

Another expression for $Nu/\sqrt{Re_0}$ is given by equation (35). Combining both equations gives

$$N = v_w^* \delta_c^* Pr \frac{1 - \varphi}{\varphi} \quad (41)$$

This equation expresses a relation between the parameters N and $v_w^* \delta_c^*$ in figures 4 and 6 which may be used to insert lines of constant φ into these figures. With the use of these lines, the calculation procedure for any specific problem becomes quite simple. The procedure will be described for $T_s/T_w = 1$ (or near 1) and with the use of the convection boundary-layer thickness δ_c^* . The prescribed temperatures fix the value of φ .

At the stagnation point, $m=1$ and du_s^*/dx^* is known. In figure 4 (a) the intersection between the line $m=1$ and the line for the prescribed φ determines $v_w^* \delta_c^*$ and $(du_s^*/dx^*) \delta_c^{*2}$, and, from both values, δ_c^* and v_w^* may be calculated.

The method of isoclines may again be used to determine the development of the boundary layer along the cylindrical surface. The use of this method implies that the gradient $d\delta_c^*/dx^*$ has to be determined for any pair of values x^* and δ_c^* . For an assumed δ_c^* , the value $v_w^* \delta_c^*$ can be found in figure 4 (a) as the value on the prescribed φ -curve above the known abscissa value $(du_s^*/dx^*) \delta_c^{*2}$. Figure 3 (a) then gives M and equation (33), the gradient $d\delta_c^*/dx^*$. A plot similar to figure 9 determines the boundary-layer thickness, and the values v_w^* belonging to these boundary-layer thicknesses represent the coolant-flow distribution for the particular temperature-difference ratio φ .

NUMERICAL EVALUATIONS AND COMPARISONS WITH KNOWN RESULTS SOLID SURFACES

The results of the outlined procedure for calculating local heat-transfer coefficients have to be compared with experi-

mental results or calculations by some other method in order to check the accuracy. The only cylindrical shape for which experimental data or solutions of the boundary-layer equations suitable for such a comparison are available seems to be the cylinder with a circular cross section. Accordingly, local heat-transfer coefficients were calculated according to the method proposed in this report by use of the dimensionless thermal boundary-layer thickness as well as of the dimensionless convection boundary-layer thickness. The results of these calculations are plotted in figure 11 over the dimensionless distance from the stagnation point. Also inserted in the figure is a curve representing the average curve through the experimentally determined local heat-transfer coefficients

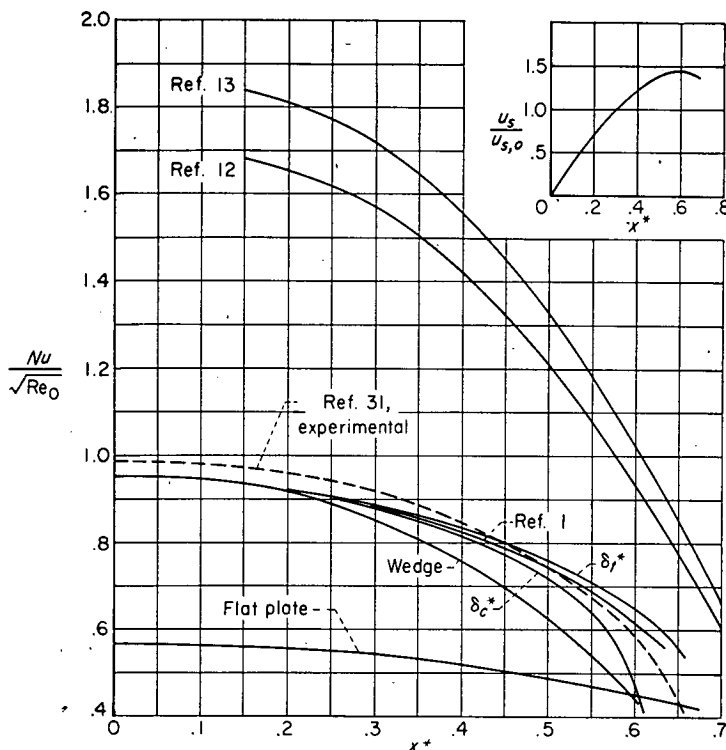


FIGURE 11.—Comparison of present method with formerly used methods for calculation of local heat-transfer coefficients around circular cylinder with impermeable wall. $\varphi_w^*, 0$; $Pr, 0.7$; $T_s/T_w, 1$.

mentioned in reference 31. It is shown in reference 22 that the measurements correlated well into a single curve when the experiments with Reynolds numbers near the critical value for transition to turbulence within the boundary layer were excluded. The tests with high Reynolds numbers gave values of $Nu/\sqrt{Re_0}$ which over the whole upstream side of the cylinder were about 10 percent higher than the ones for the lower Reynolds numbers. The same behavior is reported in references 32 and 33 in which it is shown that an increase up to 50 percent in the heat-transfer coefficients over the expected laminar values was caused by the turbulence level in the wind tunnels used. The result of a solution of the boundary-layer equation as presented in reference 1 is also included in figure 11.

This method solves the boundary-layer equations and obtains results as a series in the distance along the surface. Also inserted are values obtained by use of the Pohlhausen flat-plate solution when the free-stream velocity is based on the local values and results obtained by the methods of references 12 and 13. Heat-transfer coefficients on wedges with the same local stream velocity and velocity gradient at the same distance from the stagnation point are also included (ref. 8). Appendix C explains how these wedge solutions were obtained.

On a cylinder with a circular cross section, separation occurs in the subcritical range near the value $x^*=0.7$. The stream velocity distribution around the surface of the cylinder which was needed for the calculations was obtained from pressure distributions given in reference 31 and is contained in reference 22.

It may be seen from figure 11 that the use of flat-plate values results in heat-transfer coefficients which are considerably lower than experimental values, whereas the methods in references 12 and 13 result in values which are too high. Much better agreement is found between the wedge heat-transfer coefficients and the experimental results, especially near the stagnation point. Farther downstream, the accuracy is improved by the method of this report. For the largest distance from the stagnation point, the use of the dimensionless thermal boundary-layer thickness results in values which are higher and the use of the dimensionless convection thickness, in values which are lower than the experimental ones. The values calculated by Frössling's solution of the boundary-layer equations are also higher than the experimental ones. Frössling's method has to be considered as an exact solution of the boundary-layer equations. In reference 22 it is recommended, on the basis of the good agreement between Frössling's curve and the values obtained by the use of the thermal boundary-layer thickness, that the method of the equivalent wedge flow be based on the thermal boundary-layer thickness. The values of the heat-transfer coefficients depend primarily on the velocity distribution in the stream around the cross section of the cylinder. The velocity distribution used for the calculation on the circular cylinder is also shown on figure 11. The calculations are made for a Prandtl number of 0.7, for a solid surface ($v_w=0$) and a temperature ratio T_s/T_w of 1, equivalent to the assumption of constant property values. These calculations agree within 5 percent with the exact calculation and within 8 percent with experiment when the immediate neighborhood of the separation point is excluded. Similar comparisons have already been made in reference 2 for a gas with a Prandtl number of 1 and a different velocity distribution (see fig. 1). This comparison shows that the method proposed by Squire (ref. 16) gives heat-transfer coefficients which agree with the exact boundary-layer solution to about the same degree as those of the method of

the equivalent wedge flow. The same fact holds for the method indicated in references 15 and 17 especially with the improvement given in reference 4. It can be stated in summary, therefore, that a number of methods exist today which, at least for the circular cylinder, permit the determination of heat-transfer coefficients on solid surfaces in the laminar region of a gas having constant property values with a very good accuracy. The advantage of the equivalent wedge flow method over those methods just discussed is that it gives solutions in a very short time and that it can be readily extended to include variable property values and transpiration cooling, as was done in this report. The wedge solution, according to reference 7, is still more rapid; however, the results differ from the experimental values up to 15 percent.

Figure 12 gives the analogous results for an elliptic cylinder with the axis ratio 1:2. It may be observed that heat-transfer coefficients on wedges differ only slightly from those obtained for equivalent wedge-type flow, whereas the flat-plate values and the ones calculated with references 12 and 13 are considerably different. No experimental results or solutions of the boundary-layer equations for a cylinder with such a cross section which could be compared with the approximate solutions are known to the authors. The calcu-

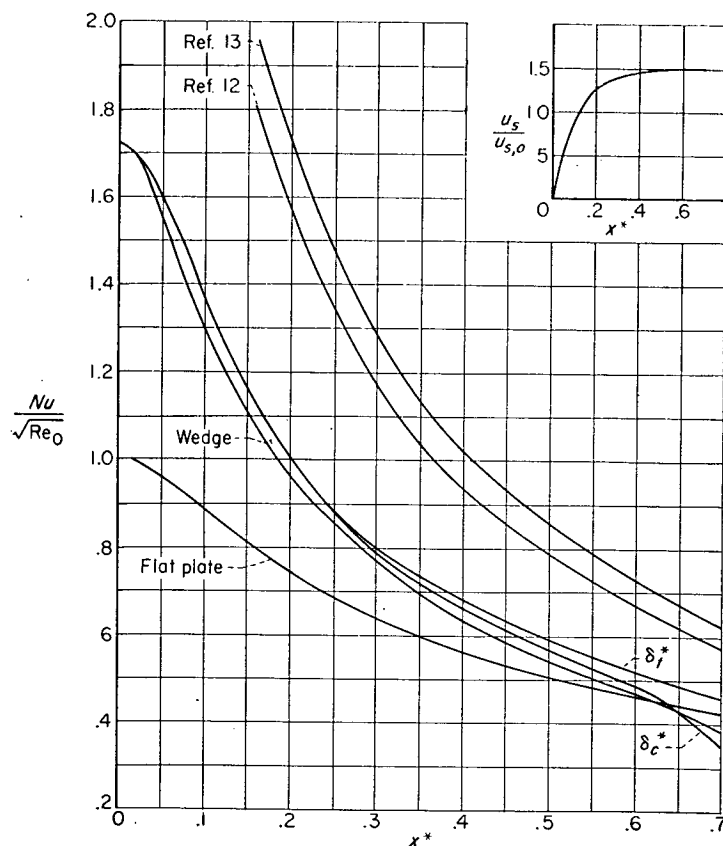


FIGURE 12.—Comparison of present method with formerly used methods for calculation of local heat-transfer coefficients around elliptic cylinder with axis ratio of 1:2. $v_w=0$; $Pr, 0.7$; $T_s/T_w, 1$.

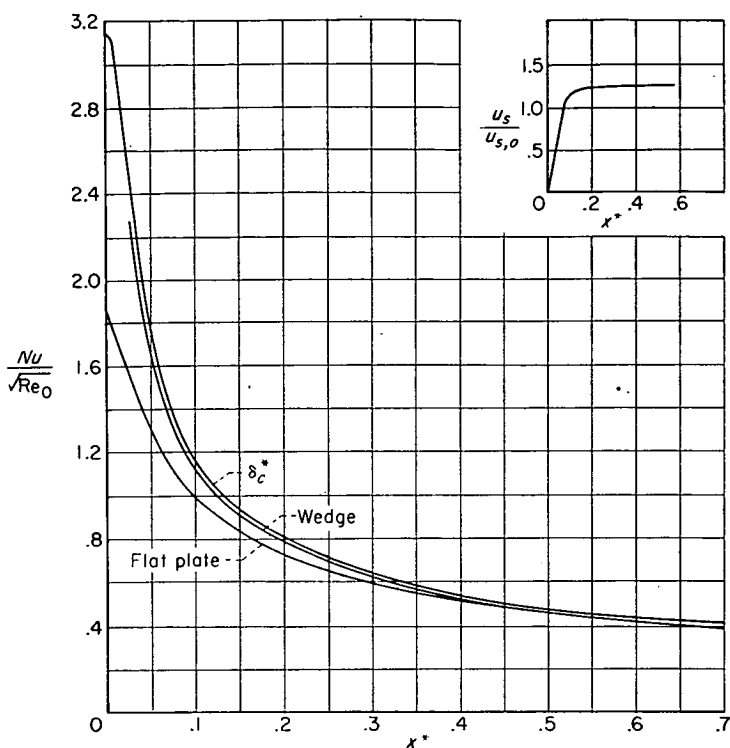


FIGURE 13.—Comparison of present method with formerly used methods for calculation of local heat-transfer coefficients around elliptic cylinder with axis ratio of 1:4. $v_w = 0$; $Pr = 0.7$; $T_s/T_w = 1$.

lation with Kroujiline's method, presented in reference 22, agrees well with the solutions obtained with the equivalent wedge-type flow method. Separation of the flow occurs on such a profile near $x^* = 0.8$. The stream velocities used are calculated values contained in reference 22.

The agreement between the wedge solutions and the results obtained by the method herein is still closer for the elliptic cylinder with axis ratio 1:4 (fig. 13). The reason for this fact is the type of stream velocity variation occurring on elliptic cylinders. Flow separation occurs on this cylinder near $x^* = 0.85$. The curves in figures 12 and 13 show that the stream velocity is comparatively constant over a considerable part of its circumference after a steep increase near the stagnation point. This behavior is more pronounced for an axis ratio of 1:4 than for one of 1:2. An inspection of figure 13 shows that, apart from the region near the stagnation point, even the flat-plate values give a reasonably good approximation. Calculations obtained by use of the dimensionless thermal boundary-layer thickness extended to the flow separation point, whereas those for the dimensionless convection boundary-layer thickness did not. It therefore appears advisable to use the dimensionless thermal boundary-layer thickness.

Experimental heat-transfer coefficients found at the University of California for an elliptic cylinder with an axis ratio of 1:4 (ref. 34) are about 50 percent higher than the

theoretical values shown in figure 13. There are several reasons for this discrepancy. The measured stream-velocity distribution was different from the one on which the present calculations are based, probably because of a limited width of the wind tunnel. The cylinder in the experimental investigation was heated by an electric resistance which produced a constant heat flow through the surface per unit area. Accordingly, the surface temperature varied along the circumference of the cylinder, being lowest at the forward stagnation point and increasing in the downstream direction. Calculations in reference 34 indicate that the higher values found in the tests are mostly due to this fact. Another increase of the experimental heat-transfer coefficients may again be connected with the turbulence level in the wind tunnel used, as discussed in connection with the experimental results for circular cylinders.

From figures 11 to 13, it may be concluded that, for cylinders with a stream velocity which is fairly constant over the greater part of the circumference, local heat-transfer coefficients may be obtained with good accuracy from wedge solutions. In the region in which the stream velocity variation is considerable, the method of the equivalent wedge flow gives heat-transfer coefficients with an accuracy sufficient for engineering purposes.

POROUS SURFACES

Heat-transfer coefficients were calculated by the method of the equivalent wedge flow for cylinders with circular and elliptic cross sections for transpiration-cooled surfaces and different temperature ratios T_s/T_w by using either the thermal or the convection boundary-layer thickness (figs. 14 to 18). In these figures it was more expedient to base the Reynolds numbers appearing on the ordinate and in the coolant flow parameter on the density at wall temperature (Re_w) rather than on the upstream density (Re_0). In figures 11 to 13 and 19, both Reynolds numbers are identical since they are calculated for a temperature ratio $T_s/T_w = 1$. The use of both boundary-layer thicknesses gives different results only for large distances from the stagnation point. The variation of the heat-transfer coefficients with the ratio of stream to wall temperature is comparatively small for solid surfaces. This result is in agreement with previous findings. For transpiration-cooled surfaces, however, the effect of the temperature ratio on the heat-transfer coefficients becomes more pronounced, especially on cylinders with nearly circular cross sections. In reference 24, the case of transpiration cooling with small temperature differences is calculated; this reference includes the effect of the temperature ratio by a correction factor which is based on the assumption that this effect is the same as that determined experimentally for impermeable surfaces. A comparison of results shows that the procedure in reference 24 underestimates the effect of temperature ratio for transpiration-cooled surfaces. In

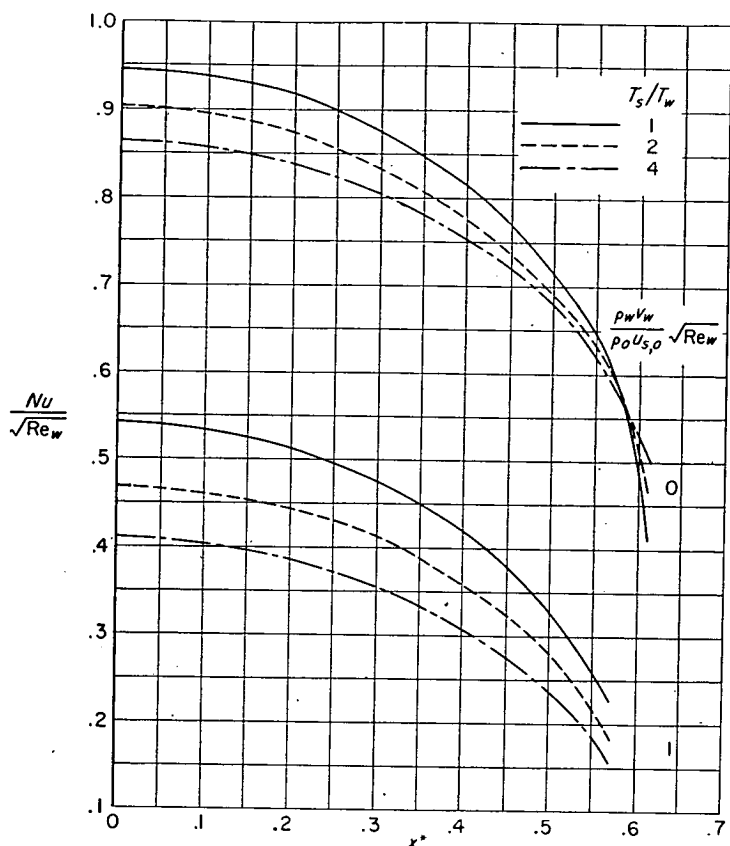


FIGURE 14.—Local heat-transfer coefficients around transpiration-cooled circular cylinder determined by use of dimensionless convection boundary-layer thickness for several cooling-air-flow rates and ratios of stream to wall temperature.

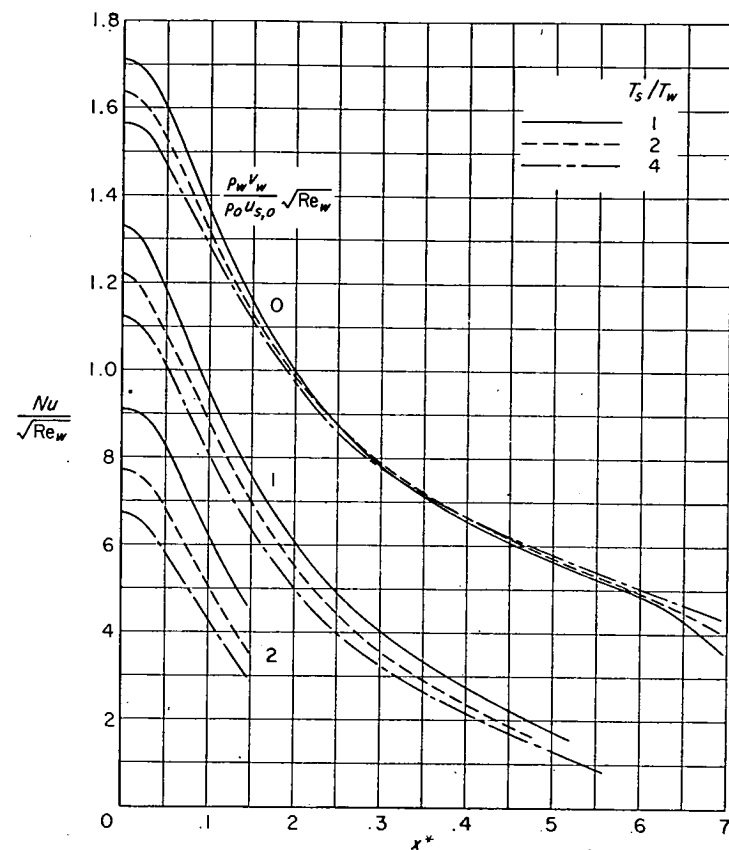


FIGURE 16.—Local heat-transfer coefficients around transpiration-cooled elliptic cylinder with axis ratio of 1:2 determined by use of dimensionless convection boundary-layer thickness for several cooling-air-flow rates and ratios of stream to wall temperature.

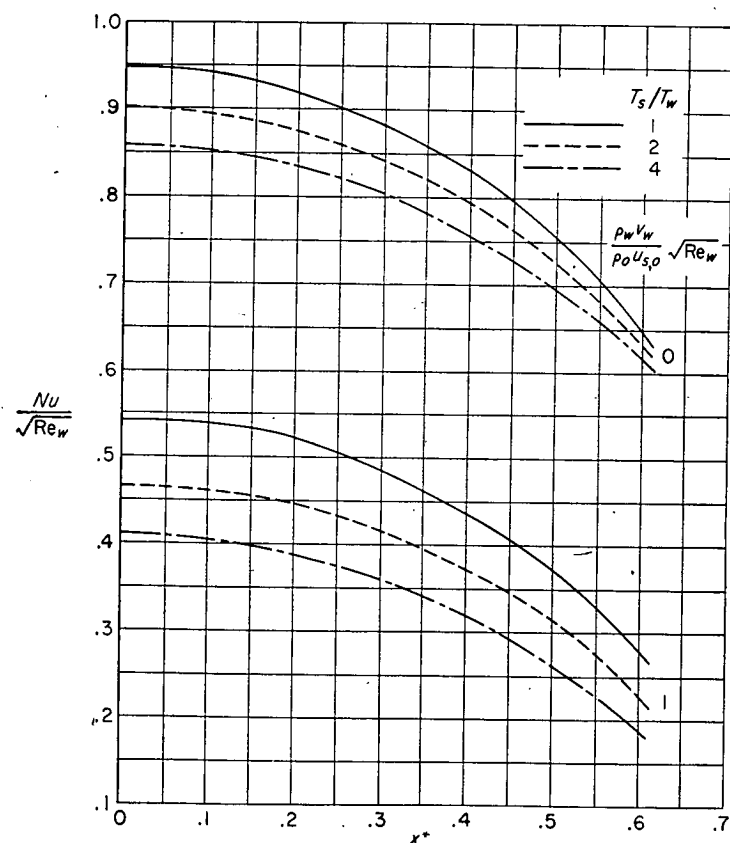


FIGURE 15.—Local heat-transfer coefficients around transpiration-cooled circular cylinder determined by use of dimensionless thermal boundary-layer thickness for several cooling-air-flow rates and ratios of stream to wall temperature.

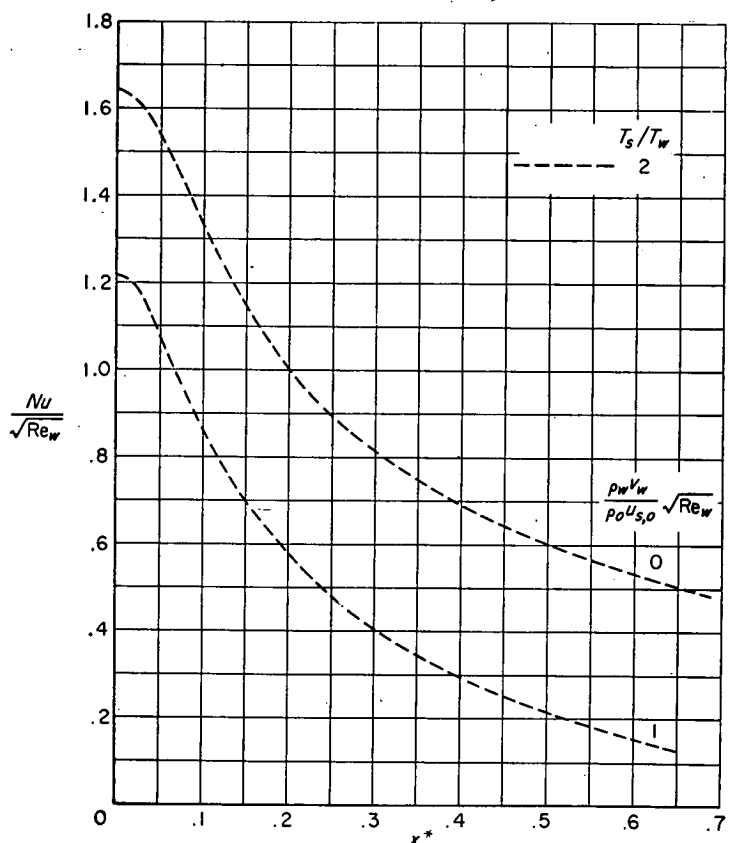


FIGURE 17.—Local heat-transfer coefficients around transpiration-cooled elliptic cylinder with axis ratio of 1:2 determined by use of dimensionless thermal boundary-layer thickness for several cooling-air-flow rates and ratio of stream to wall temperature of 2.

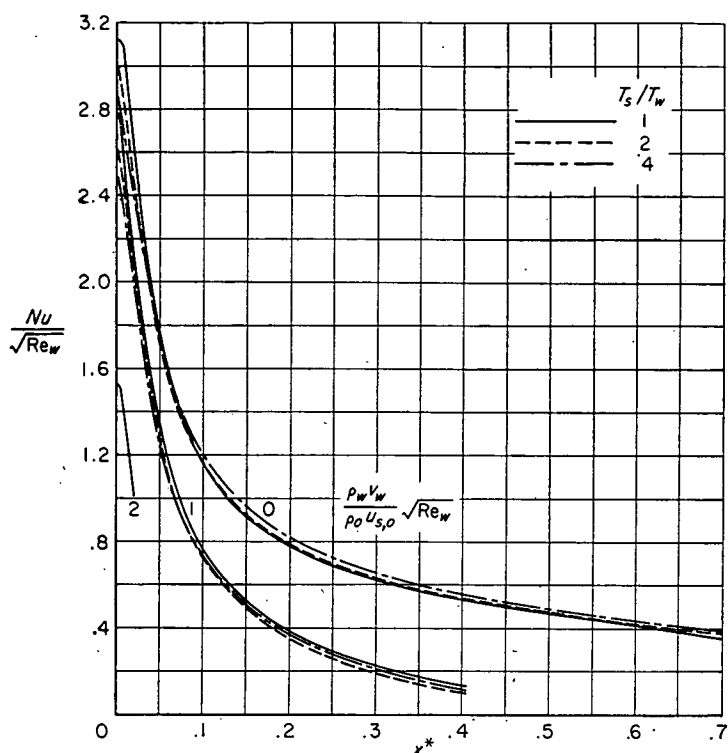


FIGURE 18.—Local heat-transfer coefficients around transpiration-cooled elliptic cylinder with axis ratio of 1:4 determined by use of dimensionless convection boundary-layer thickness for several cooling-air-flow rates and ratios of stream to wall temperature.

addition, it can be observed that transpiration cooling results in a considerable decrease of the heat-transfer coefficients. A larger amount of coolant flow is necessary to reduce the heat-transfer coefficients by the same amounts in regions in which the heat-transfer coefficients are large. Such a region exists at the stagnation point on a cylinder with an axis ratio of 1:2, and especially on a cylinder with an axis ratio of 1:4.

The variation in coolant flow required to maintain constant wall temperature for transpiration-cooled cylinders with circular and elliptic cross sections is shown in figure 19. The calculations were made for a temperature ratio T_s/T_w of 1, a value of ϕ of 0.5, and a Prandtl number Pr of 0.7. Figure 19 shows that the highest local coolant-flow rates are necessary near the stagnation point in order to keep the wall temperature down at that place. The magnitude of the coolant-flow rate at the stagnation point is proportional to the square root of the velocity gradient du_s^*/dx^* ; this in turn is determined mainly by the value of the radius of curvature at this point. As this radius of curvature decreases, the required coolant flow increases. This is in agreement with figure 19, which shows that the maximum coolant flow is required at the stagnation point of the elliptic cylinder with the 1:4 axis ratio. Downstream of the stagnation point, the flow rates decrease for each cylinder. Figure 19 also shows that the use of the thermal rather than the convection boundary-layer thickness results in only a very minor increase in coolant flow required to maintain the circular cylinder wall at a constant temperature.

EXTENSION OF CALCULATION TO HIGH-VELOCITY FLOW

The heat generated by internal friction was neglected in equation (5) according to the assumption of small velocities.

The equation

$$q = h(T_s - T_w) \quad (42)$$

gives the heat-transfer coefficient for this case. It was already explained that the inclusion of the internal friction for a gas with a Prandtl number of 1 results only in the change that the temperature T in equation (5) and the temperature T_s in equation (42) are now total temperatures, as long as the property values may be regarded constant. The heat-transfer coefficients determined in this report may be used in this case. It is shown in reference 35 by use of results obtained in reference 36 that the heat-transfer coefficients determined for low-velocity flow apply to high-velocity flow up to a Mach number of about 4 for a gas with a Prandtl number different from 1, when the stream velocity is constant (flat plate) and the heat flow is not too large. The heat-transfer coefficient, however, has now to be defined by the equation

$$q = h(T_{ad} - T_w) \quad (43)$$

in which the temperature T_{ad} denotes the value which an unheated plate assumes in the high-velocity flow. The adiabatic wall temperature may be determined from the recovery factor

$$r_0 = \frac{T_{ad} - T_s}{T_{T,s} - T_s} \quad (44)$$

which was found to be equal to $\sqrt{Pr}$ for laminar flow and for Prandtl numbers of approximately 1. The difference between the total and the static temperatures in the stream is connected with the stream velocity by the equation

$$T_{T,s} - T_s = \frac{u_s^2}{2c_p} \quad (45)$$

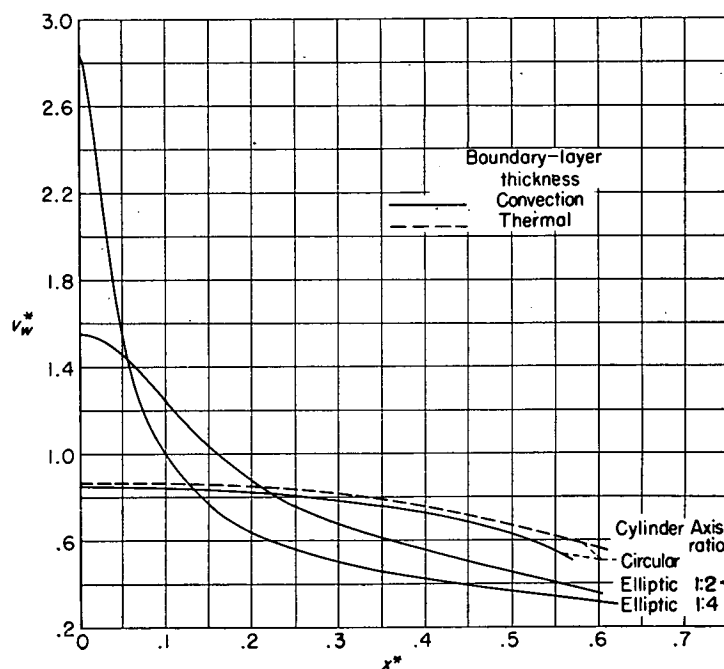


FIGURE 19.—Coolant flow required to maintain constant cylinder wall temperature. ϕ , 0.5; Pr , 0.7; T_s/T_w , 1.

For the flat plate with a constant stream velocity, the adiabatic wall temperature is therefore constant.

Conditions are more involved on a cylinder with a stream velocity which varies along its circumference. Even when the recovery factor is assumed to be constant, equations (44) and (45) give an adiabatic wall temperature which varies along the circumference of the cylinder. The fact that the low-velocity heat-transfer coefficients also represented the high-velocity values on a flat plate, however, followed from the fact that the energy equation for constant property values is linear in T , and that a general solution of the nonhomogeneous equation describing the heat transfer including the internal friction could therefore be obtained by superposition of the solution of the homogeneous equation valid for small velocities and a particular solution of the nonhomogeneous equation. Such a superposition results in a constant wall temperature on the flat plate when the solution of the homogeneous equation for constant wall temperature and the one describing the adiabatic wall temperature is used, since the adiabatic wall temperature is also constant. For a cylinder with an arbitrary cross section, however, the adiabatic wall temperature which represents a particular solution of the nonhomogeneous equation varies along the circumference. Therefore, a superposition of this particular solution with the low-velocity solutions for constant wall temperature does not give a constant wall temperature, which was specified for the problems investigated in this report. Accordingly, the heat transfer has now to be calculated with the equation

$$q = h(T_{eff} - T_w) \quad (46)$$

in which T_{eff} has to be determined for constant wall temperature conditions; that is, T_{eff} is the temperature which a particular spot along the surface, for which the heat-transfer coefficient is to be determined, assumes when the heat flow through the wall at this particular spot is zero and the wall temperature along the circumference of the cylinder is constant.

For flow around wedges, this temperature, which may be referred to as the "effective temperature," can be found from the results in reference 23. It is also determined for several cases in reference 37. The calculation procedure which determines this effective wall temperature from reference 23 is described in appendix D. The calculation shows that this temperature may be again expressed by a recovery factor

$$r_\infty = \frac{T_{eff} - T_s}{T_{t,s} - T_s} \quad (47)$$

The index ∞ is used to indicate that such a recovery factor could be determined experimentally by a model made of a material with a very large heat conductivity so that the internal heat conduction would eliminate all temperature differences along the surface. On the other hand, the recovery factor describing the adiabatic wall temperature in equation (44) has to be determined experimentally by a model made

of a material with an infinitely small heat conductivity so as to eliminate internal heat flow. Values for the recovery factor r_∞ determining the effective temperature of a wedge are presented in figure 20. The recovery factors r_0 describing the adiabatic wall temperature according to equation (44) have been calculated for wedges in reference 7. This calculation had resulted in values which decreased slightly with increasing Euler number m . Repetition of these calculations on an electric computing machine, however, according to a communication from Arthur N. Tifford of Ohio State University, showed that the recovery factors for the adiabatic wall temperature are practically independent of the

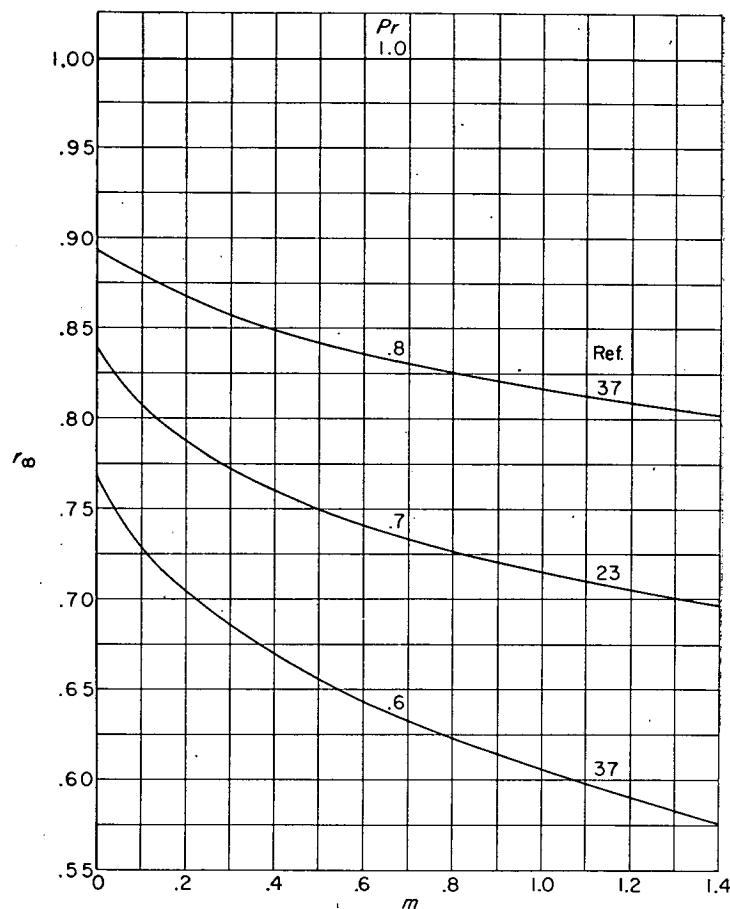


FIGURE 20.—Recovery factor values determining effective temperature of wedge.

Euler number and have the same values as the recovery factor r_∞ shown in figure 20 for an Euler number m equal to zero.

The consideration up to now dealt with solid surfaces. No information was found in the literature on recovery factors for transpiration-cooled surfaces. Some recovery factors were therefore determined for a transpiration-cooled flat plate and a flow with constant property values (the same for outside and coolant flow) by an integration of the boundary-layer equation (4). The integration was carried out in the same way as in reference 5. The dimension-

less stream function f and its second derivative were taken from reference 30. The results of this calculation are presented in figure 21 and the following table where $T_s/T_w=1$ and $Pr=0.7$:

f_w	Recovery factor
-1	0.713
-.75	.750
-.50	.786
0	.838
.50	.874
1	.900

The figure shows that the recovery factors decrease considerably with increasing coolant flow. The calculations were extended to positive values of f_w which apply to a surface with suction.

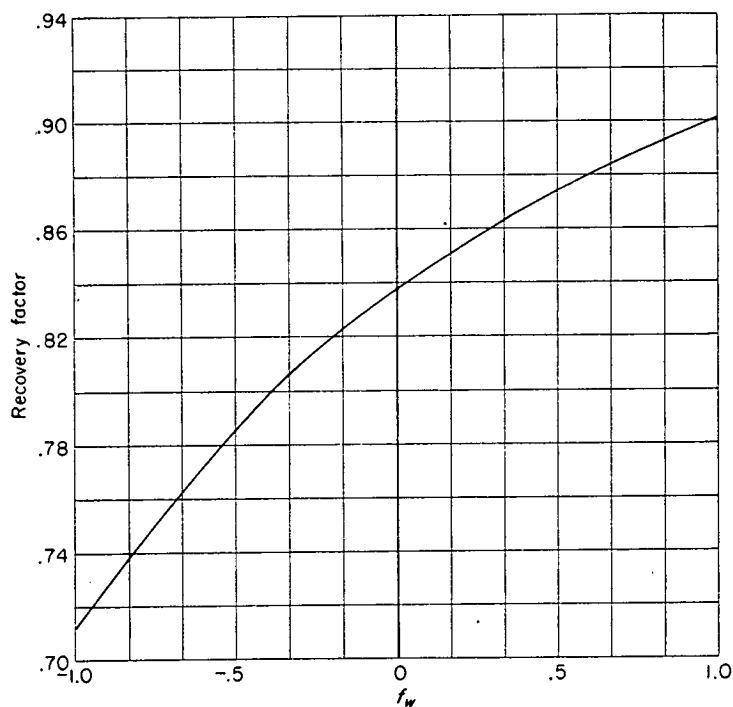


FIGURE 21.—Recovery factors for transpiration-cooled flat plate and constant fluid properties. $T_s/T_w=1$; $Pr, 0.7$.

It might be worthwhile to mention that the accurate determination of the adiabatic or effective wall temperature appreciably influences the heat flow as calculated by equation (43) only when the difference $T_{ad}-T_w$ is of the same order of magnitude as or of a smaller order of magnitude than the difference $T_{r,s}-T_s$ (see also appendix D).

RESULTS AND CONCLUSIONS

An approximate method for the calculation of heat transfer in the laminar region around cylinders of arbitrary cross section was presented. The method, called the equivalent wedge-type-flow method, is based on exact solutions of the laminar boundary-layer equations for wedge-type flow and

takes into account the influence of large temperature differences between the flow and the cylinder wall and the influence of transpiration cooling. The use of prepared charts reduces calculations to a graphical solution of an ordinary first-order differential equation. The method can be based either on the convection thickness or on the thermal thickness of the boundary layer. The results of calculations based on one thickness differ slightly from those based on the other thickness. There are not enough experimental data available to decide which boundary-layer thickness should be used. Near the separation point, however, the results obtained with the thermal boundary-layer thickness seem somewhat more plausible.

The method was applied to circular and elliptic cylinders, and the following results and conclusions are given:

1. Results of experiments and exact calculations were available only for circular cylinders with solid surfaces. Calculations based on the present method and on the thermal boundary-layer thickness agreed within 5 percent with the exact calculation and within 8 percent with experiment when the immediate neighborhood of the separation point was excluded.

2. With the present method, heat-transfer coefficients may be obtained without a knowledge of the flow boundary layer. Consequently, such calculations are more rapid than those based on the momentum and heat-flow equations.

3. Heat-transfer coefficients determined from wedge solutions agreed on the circular cylinder within 15 percent with the results of experiments. The calculation procedure is still more rapid.

4. For elliptic cylinders, the differences between the results of calculations with the various methods decreased as the axis ratio increased from 1:2 to 1:4.

5. The development of the boundary layer is determined by the velocity distribution around the cylinder. The accuracy which has to be expected for the results of calculations with the different methods will therefore depend on the character of the velocity distribution.

6. For cylinders with solid walls, the variation of the heat-transfer coefficients with ratio of stream to wall temperature was comparatively small.

7. For transpiration-cooled surfaces, the effect of temperature ratio on heat-transfer coefficients became pronounced, especially on cylinders with nearly circular cross sections.

8. A considerable decrease in heat-transfer coefficients accompanied transpiration cooling.

9. The influence of transpiration cooling on the recovery factor was investigated for a flat plate and constant property values. It was found that the recovery factor decreased considerably with increasing coolant flow.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, March 19, 1952

APPENDIX A

SYMBOLS

The following symbols are used in this report:	
A	dimensionless wall temperature gradient taken from refs. 22 and 23, $\sqrt{\frac{2}{m+1}} \theta'_w$
C	constant
c_p	specific heat at constant pressure
F	function
f	dimensionless stream function, $(\rho_w \psi)/\sqrt{\mu_w \rho_w x u_s}$
h	heat-transfer coefficient
k	thermal conductivity
L	characteristic dimension (major axis of cylinder)
$M = \frac{1-m}{2} \eta_b^2 = M \left(\frac{du_s^*}{dx^*} \delta^{*2}, v_w^* \delta^* \right)$	(see eq. (34))
m	Euler number, $-\frac{\partial p / \partial x}{\rho_s u_s^2 / x}; u_s = Cx^m$
$N = \theta'_w \eta_b = N \left(\frac{du_s^*}{dx^*} \delta^{*2}, v_w^* \delta^* \right)$	(see eq. (37))
Nu	Nusselt number, hL/k_w
Pr	Prandtl number, $c_p \mu / k$
p	pressure
q	heat flow
$\bar{q}$	approximated heat flow
q_c	heat flow by conduction
q_r	heat flow by radiation
Re_0	Reynolds number, $u_{s,0} L \rho_0 / \mu_w$
Re_w	Reynolds number, $u_{s,0} L \rho_w / \mu_w$
r_0	recovery factor defined by $(T_{ad} - T_s) / (T_{T,s} - T_s)$ (eq. (44))
r_∞	recovery factor defined by $(T_{eff} - T_s) / (T_{T,s} - T_s)$ (eq. (47))
T	temperature in boundary layer
T_s	temperature in stream
u	velocity component along surface
u_s	free-stream velocity
u_s^*	dimensionless mass velocity in free stream, $\rho_w u_s / \rho_0 u_{s,0}$
v	velocity component normal to surface
v^*	dimensionless velocity normal to surface, $\frac{\rho_w v}{\rho_0 u_{s,0}} \sqrt{Re_0}$
x	distance from stagnation point along surface
x^*	dimensionless distance from stagnation point along surface, x/L
y	distance normal to surface
z	dimensionless boundary-layer coordinate taken from refs. 22 and 23, $\sqrt{\frac{m+1}{2}} \eta$
z_b	$\frac{1}{\sqrt{2-\beta}} \frac{\delta_c}{x} \sqrt{\frac{u_s x}{\nu}}$
β	pressure gradient parameter, $2m/(m+1)$
δ	boundary-layer thickness
δ^*	dimensionless boundary-layer thickness, $(\delta/L) \sqrt{Re_0}$
δ_c	convection boundary-layer thickness (eq. (15))
δ_c^*	dimensionless convection boundary-layer thickness, $(\delta_c/L) \sqrt{Re_0}$
δ_d	displacement boundary-layer thickness (eq. (13))
δ_i	momentum boundary-layer thickness (eq. (14))
δ_t	thermal boundary-layer thickness (eq. (16))
δ_t^*	dimensionless thermal boundary-layer thickness, $(\delta_t/L) \sqrt{Re_0}$
η	dimensionless boundary-layer coordinate, $y \sqrt{\frac{\rho_w u_s}{\mu_w x}}$
η_b	$\delta \sqrt{\frac{\rho_w u_s}{\mu_w x}}$
θ	dimensionless temperature-difference ratio, $\frac{T - T_w}{T_s - T_w}$
ϑ	dimensionless temperature-difference ratio, $\frac{T - T_s}{T_{T,s} - T_s}$
λ	dimensionless stream function taken from ref. 27, $-\frac{m+1}{2} f_w$
μ	absolute viscosity
ν	kinematic viscosity, μ/ρ
ξ	distance along wedge, taken from refs. 22 and 23
ρ	density
φ	dimensionless temperature-difference ratio, $\frac{T_s - T_w}{T_s - T_c}$
ψ	stream function
Subscripts:	
ad	adiabatic
c	coolant, when used with T
eff	effective
s	stream
T	total
w	wall
0	except when used with r , refers to a fixed point in the stream
Superscripts:	
m	exponent of distance along surface from stagnation point for stream velocity, $u_s = Cx^m$
$\cdot$	denotes differentiation with respect to η

APPENDIX B

EVALUATION OF HEAT-FLOW EQUATION

The energy equation (5) will be integrated along y throughout the boundary layer under the conditions of small Mach number, constant wall temperature, and constant specific heat

$$c_p \int_0^\infty \left(\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} \right) dy = \int_0^\infty \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) dy$$

The first term on the left side can be transformed by partial differentiation to

$$\rho u \frac{\partial T}{\partial x} = \frac{\partial}{\partial x} (\rho u T) - T \frac{\partial (\rho u)}{\partial x}$$

An analogous transformation of the second term and consideration that the temperature gradient $\partial T / \partial y$ is zero outside the boundary layer (for $y = \infty$) result in

$$\int_0^\infty \frac{\partial}{\partial x} (\rho u T) dy - \int_0^\infty T \frac{\partial (\rho u)}{\partial x} dy + \rho v T \Big|_0^\infty - \int_0^\infty T \frac{\partial (\rho v)}{\partial y} dy = - \frac{k_w}{c_p} \left(\frac{\partial T}{\partial y} \right)_w$$

The second and fourth terms cancel because of the continuity equation (3). In the first term, the sequence of differentiation and integration can be reversed. Introduction of the convection thickness of the boundary layer leads finally to the integrated heat-flow equation

$$\rho_s \frac{d}{dx} (u_s \delta_c) - \rho_w v_w = \frac{k_w}{c_p} \left(\frac{\partial \theta}{\partial y} \right)_w \quad (\text{B1})$$

It will now be proved that equation (26), used for the method of the equivalent wedge-type flow, is the same as this integrated heat-flow equation when the convection thickness for the boundary layer is used. Equation (B1) may be transformed by partial differentiation of the first term into

$$\rho_s u_s \frac{d \delta_c}{dx} + \rho_s \delta_c \frac{du_s}{dx} - \rho_w v_w = \frac{k_w}{c_p} \left(\frac{\partial \theta}{\partial y} \right)_w \quad (\text{B2})$$

For wedge-type flow, the convection thickness is given by the expression

$$\delta_c = \eta_c \sqrt{\frac{\mu_w x}{\rho_w u_s}} = \eta_c \sqrt{\frac{\mu_w}{\rho_w C}} x^{\frac{1-m}{2}} \quad (\text{B3})$$

Differentiation of this equation gives

$$\frac{d \delta_c}{dx} = \frac{1-m}{2} \eta_c \sqrt{\frac{\mu_w}{\rho_w C}} x^{\frac{1+m}{2}} = \frac{1-m}{2} \eta_c \sqrt{\frac{\mu_w}{\rho_w x u_s}} \quad (\text{B4})$$

Introducing this expression as well as equations (9) and (12) into equation (B2) gives the equation

$$\frac{1+m}{2} (\rho_s \eta_c + \rho_w f_w) = \frac{\rho_w}{Pr} \theta'_w \quad (\text{B5})$$

which interconnects the convection thickness with the dimensionless temperature gradient at the wall. The gradient of the convection thickness may now be determined from the integrated energy equation (B2) when the expressions in this equation are transformed to the new variables

$$\frac{d \delta_c}{dx} = -m \eta_c \sqrt{\frac{\mu_w}{\rho_w x u_s}} - \frac{\rho_w}{\rho_s} \frac{1+m}{2} f_w \sqrt{\frac{\mu_w}{\rho_w x u_s}} + \frac{\rho_w}{\rho_s} \frac{1}{Pr} \theta'_w \sqrt{\frac{\mu_w}{\rho_w x u_s}}$$

Replacing the nondimensional temperature gradient in this equation by equation (B5) results in

$$\frac{d \delta_c}{dx} = \frac{1-m}{2} \eta_c \sqrt{\frac{\mu_w}{\rho_w x u_s}} = \frac{1-m}{2} \eta_c^2 \frac{\mu_w}{\rho_w \delta_c u_s}$$

which is the same as equation (26).

It can also be proved by a completely analogous calculation that the method of the equivalent wedge-type flow, when it is used to calculate the momentum thickness of the flow boundary layer, satisfies the integrated momentum equation which is obtained from equation (2) by an integration over y in a manner similar to the derivation of equation (B1)

$$\frac{d}{dx} (\rho_s u_s^2 \delta_i) + \rho_s u_s \frac{du_s}{dx} \delta_d - \rho_w v_w u_s = \mu_w \left(\frac{\partial u}{\partial y} \right)_w$$

APPENDIX C

DETERMINATION OF WEDGE SOLUTIONS

The wedge solutions which were used as a first approximation in figures 11 to 13 can be obtained very easily with the use of figure 22 reproduced from reference 9. The heat-transfer coefficient has to be determined on a wedge which has the same stream velocity and its gradient at the same distance from the stagnation point as the real profile. The Euler number for this wedge can be found from equation (23). In the dimensionless coordinates it is

$$m = \frac{x \frac{du_s}{dx}}{u_s} = \frac{x^* \frac{du_s^*}{dx^*}}{u_s^*} \quad (C1)$$

The parameter f_w which determines the coolant flow through the porous wall is found from equation (20), which reads, when converted to dimensionless quantities,

$$f_w = -\frac{2}{m+1} v_w^* \sqrt{\frac{x^*}{u_s^*}} \quad (C2)$$

The value $(Nu/\sqrt{Re_0})\sqrt{x^*/u_s^*}$ can be determined from figure

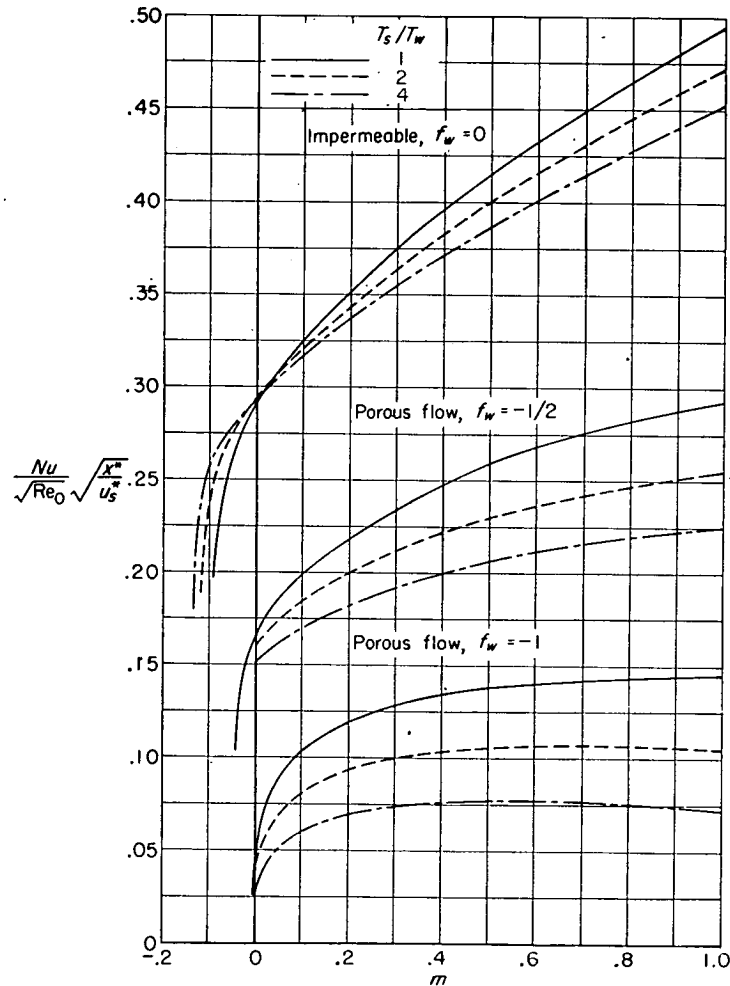


FIGURE 22.—Heat transfer through laminar boundary layer with and without porous flow (ref. 9).

22, and $Nu/\sqrt{Re_0}$ is finally obtained by multiplication by $\sqrt{u_s^*/x^*}$.

When the temperature ratio φ is prescribed, figure 23 reproduced from reference 9 can be used to obtain the parameter f_w for any Euler number m . Equation (C2) then determines the value v_w^* and the distribution of the required coolant flow along the profile.

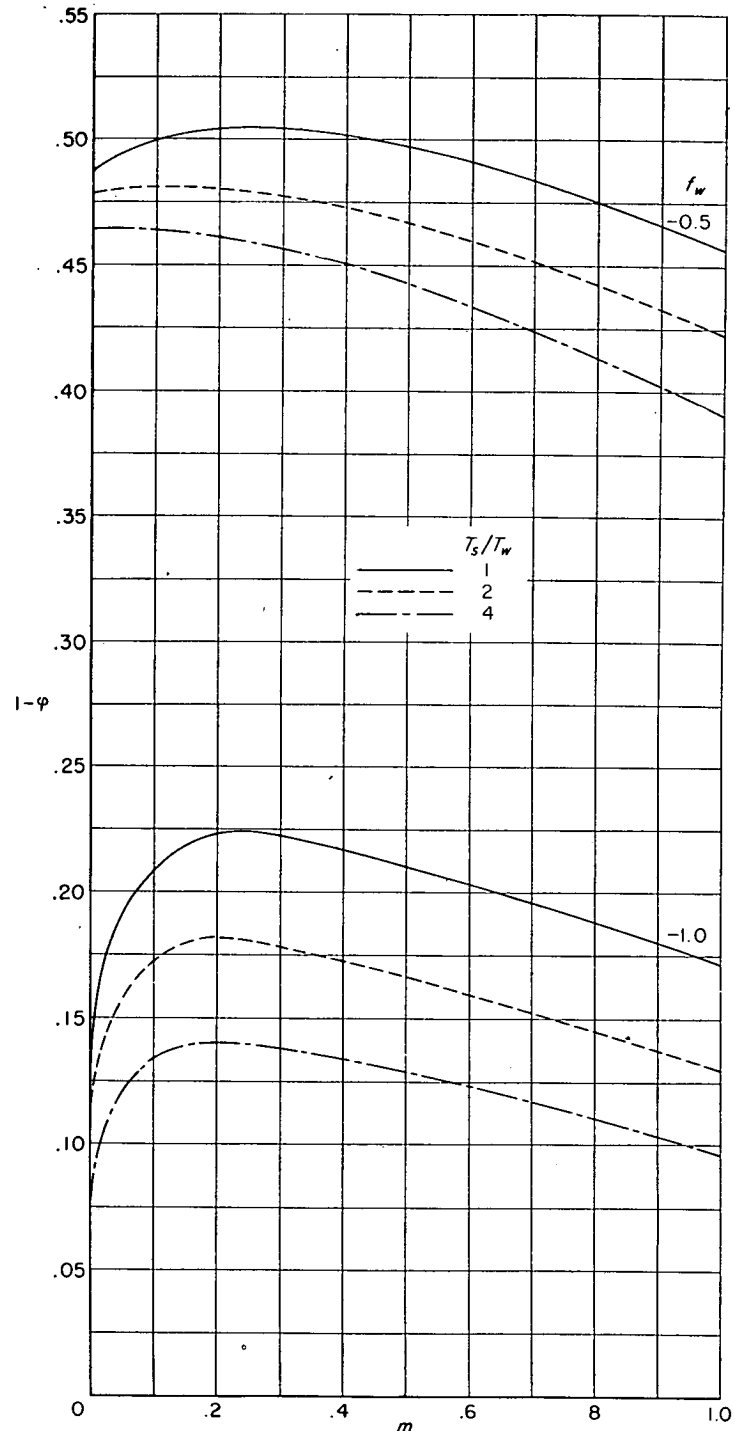


FIGURE 23.—Temperature of porous wall (ref. 9).

APPENDIX D

DETERMINATION OF EFFECTIVE WALL TEMPERATURE

It is shown in reference 23 that for high-velocity flow of a fluid with constant property values around a wedge with constant wall temperature, the temperature field can be expressed by the equation

$$T = (T_w - T_{T,s})(1 - \theta) + (T_{T,s} - T_s)\vartheta + T_s \quad (\text{D1})$$

in which θ represents the nondimensional temperature field for low-velocity flow and ϑ , the nondimensional temperature field for high-velocity flow and a wall temperature equal to the total stream temperature. The heat flow from the wall, obtained by differentiating equation (D1), is

$$q_w = -k \left(\frac{\partial T}{\partial y} \right)_w = k \left[(T_w - T_{T,s}) \left(\frac{\partial \theta}{\partial y} \right)_w - (T_{T,s} - T_s) \left(\frac{\partial \vartheta}{\partial y} \right)_w \right] \quad (\text{D2})$$

With the transformations used in reference 23 (see also appendix E)

$$\left. \begin{aligned} z &= \frac{y}{\sqrt{2-\beta}} \sqrt{\frac{u_s}{\nu x}} \\ u_s &= Cx^m \\ \beta &= \frac{2m}{m+1} \end{aligned} \right\} \quad (\text{D3})$$

equation (D2) can be transformed into

$$q_w = \frac{k}{\sqrt{2-\beta}} \sqrt{\frac{u_s}{\nu x}} \left[(T_w - T_{T,s}) \left(\frac{d\theta}{dz} \right)_w - (T_{T,s} - T_s) \left(\frac{d\vartheta}{dz} \right)_w \right] \quad (\text{D4})$$

This equation is to be brought into the form

$$q_w = \frac{k}{\sqrt{2-\beta}} \sqrt{\frac{u_s}{\nu x}} (T_w - T_{eff}) \left(\frac{d\theta}{dz} \right)_w \quad (\text{D5})$$

A comparison of equations (D4) and (D5) gives

$$T_w - T_{eff} = T_w - T_{T,s} - (T_{T,s} - T_s) \frac{\left(\frac{d\vartheta}{dz} \right)_w}{\left(\frac{d\theta}{dz} \right)_w} \quad (\text{D6})$$

from which the difference between the total and effective gas temperatures can be found. The expression

$$\frac{T_{T,s} - T_{eff}}{T_{T,s} - T_s} = 1 - r_\infty = - \frac{\left(\frac{d\vartheta}{dz} \right)_w}{\left(\frac{d\theta}{dz} \right)_w} \quad (\text{D7})$$

defines this temperature difference and the recovery factor for the effective wall temperature. The nondimensional temperature gradients appearing on the right side of this equation are presented in references 23 and 37. In this way, the values in figure 20 have been determined.

To obtain an estimate of the conditions under which the difference between the adiabatic wall temperature and the effective wall temperature may be neglected, the heat flow into the wall will be approximated by the equation

$$\tilde{q}_w = k(T_w - T_{ad}) \left(\frac{d\theta}{dy} \right)_w \quad (\text{D8})$$

and the error of such an approximation will be determined. The ratio of the exact heat-flow equation (D2) to the one approximated by equation (D8) is

$$\frac{q_w}{\tilde{q}_w} = \frac{T_w - T_{T,s}}{T_w - T_{ad}} + \frac{T_{T,s} - T_s}{T_w - T_{ad}} (1 - r_\infty)$$

Introducing the recovery factor for the adiabatic wall temperature

$$T_{T,s} - T_{ad} = (1 - r_0)(T_{T,s} - T_s)$$

gives

$$\frac{q_w}{\tilde{q}_w} = 1 - \frac{T_{T,s} - T_s}{T_w - T_{ad}} (r_\infty - r_0)$$

For an Euler number equal to 1, which characterizes flow near a stagnation point and which, according to figure 20, shows a large difference between the recovery factors r_0 and r_∞ , the error is smaller than 5 percent when

$$\frac{T_{ad} - T_w}{T_{T,s} - T_s}$$

is larger than 2.5.

APPENDIX E

COMPARISON OF VARIABLES

This appendix gives a comparison of the variables used in references 9, 25, and 26 with the ones used in references 22, 23, and 27. All of these references deal with wedge-type flow.

The values used in this report are related to the ones in the aforementioned references by the following equations:

$$\eta_b = \frac{\delta}{x} \sqrt{\frac{u_s x}{\nu}}$$

$$z_b = \sqrt{\frac{m+1}{2}} \frac{\delta_c}{x} \sqrt{\frac{u_s x}{\nu}} = \frac{1}{\sqrt{2-\beta}} \frac{\delta_c}{x} \sqrt{\frac{u_s x}{\nu}}$$

Symbols from references 9, 25, and 26	Symbols from references 22, 23, and 27	Relation among symbols
$m (Eu)$	β	$\beta = \frac{2m}{m+1}$
f_w	λ	$\lambda = -\frac{m+1}{2} f_w$
η	z	$z = \sqrt{\frac{m+1}{2}} \eta$
θ'_w	A	$A = \sqrt{\frac{2}{m+1}} \theta'_w$

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REPORT 1119

RECIPROCITY RELATIONS IN AERODYNAMICS¹

By MAX. A. HEASLET and JOHN R. SPREITER

SUMMARY

Reverse-flow theorems in aerodynamics are shown to be based on the same general concepts involved in many reciprocity theorems in the physical sciences. Reciprocal theorems for both steady and unsteady motion are found as a logical consequence of this approach. No restrictions on wing plan form or flight Mach number are made beyond those required in linearized compressible-flow analysis. A number of examples are listed, including general integral theorems for lifting, rolling, and pitching wings and for wings in nonuniform downwash fields. Correspondence is also established between the buildup of circulation with time of a wing starting impulsively from rest and the buildup of lift of the same wing moving in the reverse direction into a sharp-edged gust.

INTRODUCTION

Some of the most important results in the recent study of wing theory have been achieved through the development of reverse-flow relations. The theorems already obtained are of outstanding practical utility and it appears obvious that the fullest exploitation of the methods has yet to be accomplished, either from a purely theoretical standpoint or in the routine calculation of wing characteristics. Attention to such problems in aerodynamics was initiated by von Kármán (ref. 1) who first announced the invariance of drag with forward and reversed directions of flight for a nonlifting symmetrical wing at supersonic speed. Subsequently, advances in the theory were made by Munk, Hayes, Brown, Harmon, and Flax (refs. 2 through 7). Up to the present time, the most general results have been expressed by Ursell and Ward (ref. 8) and by R. T. Jones (ref. 9) in his attack on the study of wing shapes of minimum drag. In the forms given in the two latter papers, the derived equivalences could be termed reciprocal or reciprocity relations rather than reverse-flow relations; in fact, this change in terminology divorces attention, momentarily, from the purely aerodynamic aspects of the results and, in this way, suggests a reorientation in terms of the various similar relations appearing in other engineering fields. In the theory of elasticity, for example, a reciprocity theorem for small displacements of an elastic medium is so expressed as to appear in formal agreement with the statement of the result given by Ursell and Ward (see, e. g., ref. 10). This theorem is attributed to E. Betti and was published in 1872. A generalization was given by Lord Rayleigh in 1873, and

in various sections of his two volumes on the theory of sound (ref. 11) discussions of reciprocal relations in an elastic medium and for acoustic sources are given. In 1886, von Helmholtz (ref. 12) obtained, by means of variational methods applied to Hamilton's characteristic function, a reciprocal theorem for small changes in the momenta and coordinates of a general dynamical system in forward and reversed motion. This result was commented on, in turn, by Lamb (ref. 13) and an independent proof based upon Lagrange's generalized equations was given. The paper by Lamb is of particular interest since it contains the essential idea underlying the development of reverse-flow theorems in wing theory. Thus, Lamb remarks, as had Lord Rayleigh previously, that reciprocity relations between sound sources do not apply directly in a moving atmosphere. He points out, however, that the reciprocity can be restored if the direction of the wind is also reversed. Further examples of reciprocal theorems appear in the theories of electricity and magnetism (in particular, reference should be made to Maxwell's discussion of the subject in ref. 14) and of optics.

The generality in the statement of reciprocity relations appears, almost universally, to have held back their application to problems for which they are obviously, in retrospect, particularly fitted. This generality is even more apparent in some of the conclusions of Lord Rayleigh and von Helmholtz which apply to nonconservative systems.

The purpose of the present paper is twofold. First, a close connection will be established between reverse-flow theorems in subsonic and supersonic, steady-state wing theory and known reciprocity relations between two solutions of the equation governing the flow field. In this way, machinery will be provided whereby extensions of existing results to the case of unsteady motion follow directly. Second, a number of particular problems in wing theory in steady and unsteady flow will be considered. It will be shown that, provided attention is limited to force and moment characteristics, the complexity of many solutions involving nonuniform flow fields, control-surface deflections, and unsteady motion can be reduced considerably. In some cases, previously obtained solutions will be calculated. Comparison with the original calculations will almost invariably highlight the economy of effort in obtaining the final result. The utility of reverse-flow theorems is based on the fact that they build from known solutions and thus avoid the necessity of starting each problem anew.

¹ Supersedes NACA TN 2700, "Reciprocity Relations in Aerodynamics" by Max. A. Heaslet and John R. Spreiter.

GENERAL ANALYSIS

RECIPROCITY RELATIONS FOR A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

In this section, integral relations associated with linear partial differential equations will be reviewed from the standpoint of relating independent solutions. The subject matter is precisely Green's theorem and, in common with the usual expression of the theorem, it is preferable to treat the variables initially as abstract quantities. Consider, therefore, a class of linear partial differential equations of second order with independent variables $X_1, X_2, \dots, X_m$ that may be thought of as rectangular coordinates in a space of m dimensions. Denote differentiation of the function $\psi(X_1, X_2, \dots, X_m)$ with respect to the variables X_i and X_j by the subscript notation

$$(\psi)_{ij} \equiv \frac{\partial^2 \psi}{\partial X_i \partial X_j}$$

and consider the differential equation

$$L(\psi) \equiv \sum_{i=1}^m \sum_{j=1}^m A_{ij}(\psi)_{ij} + B\psi = 0 \quad (1)$$

where, for the purposes at hand, $A_{ij} = A_{ji}$ are made independent of X_i and X_j , and B is a constant. Such equations fall within the class of self-adjoint equations.

By Green's theorem (see, e. g., ref. 15), it is possible to relate two arbitrary functions ψ and Ω by the integral expression

$$\iiint [\Omega L(\psi) - \psi L(\Omega)] dV = - \iint [\Omega D_n \psi - \psi D_n \Omega] dS \quad (2)$$

where the left member is a volume integral over a prescribed region in m -dimensional space and the right member is a surface integral over the hypersurface S enclosing the given region. Equation (2) certainly holds for any region in which ψ and Ω and their first and second derivatives are continuous. The directional derivatives D_n are defined in terms of the direction cosines $n_1, n_2, \dots, n_m$ of the normal to the surface S with the stipulation that the normal is directed into the given region. Thus, setting

$$\sum_{j=1}^m n_j A_{ij} = N \nu_i$$

where $\nu_1, \nu_2, \dots, \nu_m$ are the direction cosines of a line termed the "conormal," the directional derivative is defined by the expression

$$D_n \psi = N \sum_{i=1}^m \nu_i (\psi)_{,i} = N(\psi)_{,n} = N \frac{\partial \psi}{\partial \nu} \quad (3)$$

If, finally, ψ and Ω are assumed to satisfy equation (1), the left side of equation (2) vanishes and the resulting expression

$$\iint \Omega D_n \psi dS = \iint \psi D_n \Omega dS \quad (4)$$

is a general reciprocity relation expressing the functional dependence between two arbitrary solutions of equation (1).

An interesting interpretation of equation (4) has been given by several writers (see, e. g., ref. 16, p. 46) and applies

to the particular case when ψ and Ω are identified with the perturbation velocity potential φ in the theory of incompressible-fluid flow. The governing equation is Laplace's equation in three dimensions

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \quad (5)$$

and the reciprocity relation takes the form, for two possible solutions φ and φ' ,

$$\iint \varphi \frac{\partial \varphi'}{\partial n} dS = \iint \varphi' \frac{\partial \varphi}{\partial n} dS \quad (6)$$

where the directional derivatives are now along the true normals to the surface S enclosing a three-dimensional volume. It is known that any actual state of motion of a liquid for which a single-valued velocity potential exists can be produced instantaneously by the application of a properly chosen system of impulsive pressures. These impulsive pressures are directly proportional to the velocity potential plus an arbitrary constant which may, in the present case, be associated with the pressure of the uniform stream. Equation (6) is thus seen to represent summations of the cross products of impulse and normal velocity in two possible motions of a conservative system and is a special case of the dynamical theorem (ref. 11, p. 98)

$$\sum_{r=1}^n p_r \dot{q}_r' = \sum_{r=1}^n p_r' \dot{q}_r \quad (7)$$

where p_r , $\dot{q}_r$ and p_r' , $\dot{q}_r'$ are generalized components of impulse and velocity in any two possible motions of a system which starts from rest.

The interpretation of equation (6) that leads to equation (7) provides an indication of the close connection between reciprocal theorems based upon the principles of least action and the symmetric character of Green's theorem for certain second-order differential equations. In the subsequent applications it will be convenient to proceed directly from equation (4) and seek to establish reciprocal relations between flow fields in wing theory. Such a process is well known when ψ or Ω is replaced by the elementary solution associated with a unit source and, in this case, establishes a general solution in terms of source and doublet distributions determined by arbitrary boundary conditions. The present objective is, however, different in that one wishes to get a symmetrical dependence between two general solutions. Moreover, the apparent symmetry of equation (4) must be consistent with physical considerations so that, for example, the velocity at point A induced by a single source at point B is equal to the velocity at point B induced by a source at point A . If, in the two systems, the effective free streams or flight directions are opposed, a fore and aft symmetry occurs and the possibility of maintaining symmetry in the reciprocal relations becomes feasible.

RECIPROCAL THEOREMS IN WING THEORY

REVERSE FLOW FOR SUBSONIC WINGS

Consider a thin wing at possibly a small angle of attack and situated in the immediate vicinity of a flat surface which is designated the xy plane. For sufficiently small thickness and angle of attack, the perturbation velocity potential or any of

the perturbation velocity components of the wing satisfy the linearized partial differential equation of compressible flow.

$$\beta^2 \psi_{xx} + \psi_{yy} + \psi_{zz} = 0 \quad (8)$$

where $\beta^2 = 1 - M_0^2 = 1 - (U_0/a_0)^2$, and U_0, a_0 are, respectively, the flow velocity and speed of sound in the free stream. Equation (8) applies in forward or reverse flow, provided the corresponding free-stream Mach numbers M_0 and M_0' are equal. In figure (1) a lifting wing with plan form P is indicated along with the vortex wakes as the configuration would appear if the two flow fields were superimposed. It will be assumed that the wing chord is finite and that the profiles are closed.

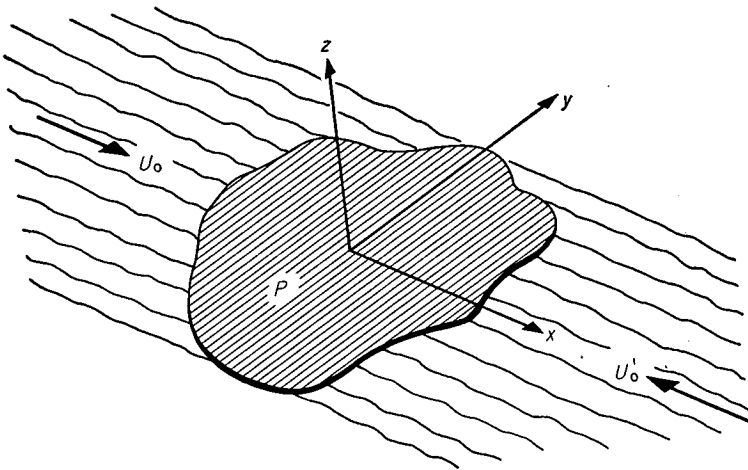


FIGURE 1.—View of wing in combined flow field.

It is proposed to apply the general reciprocity relation (4) to these flow fields in a manner similar to the approach used in developing the basic solutions of the differential equation (see, e. g., refs. 17 and 18). Thus, for subsonic flow, hemispherical regions of large radius and lying first above and then below the plane of the wing will be chosen as the volumes of integration. The surface integrals will therefore extend over a hemisphere with center at the wing and a flat surface that lies immediately adjacent to the $z=0$ plane. This latter surface is subsequently to be brought into coincidence with the plane of the wing but must be considered first in its displaced position since only then can the flow be assumed free of possible singularities in perturbation velocities and their gradients. As in wing theory, in general, the attenuation of the perturbation potential and its gradients may be assumed of such a nature that the integrated contributions of the wing and its wake over the hemispherical surfaces vanish in the limit as the radius becomes infinitely large. It remains, therefore, to consider the integrals over the surfaces at $z=0+\epsilon$ and $z=0-\epsilon$. Denote these surfaces as σ_u and σ_l , respectively, where the subscripts u and l specify values above and below the $z=0$ plane. Equation (4) can then be written as

$$\iint_{\sigma_u} \psi \frac{\partial \psi'}{\partial n} dS = \iint_{\sigma_l} \psi' \frac{\partial \psi}{\partial n} dS; \quad n \text{ directed upward} \quad (9a)$$

and

$$\iint_{\sigma_l} \psi \frac{\partial \psi'}{\partial n} dS = \iint_{\sigma_u} \psi' \frac{\partial \psi}{\partial n} dS; \quad n \text{ directed downward} \quad (9b)$$

where the primes denote conditions for the reversed flow.

In equations (9), ψ is now replaced by the perturbation velocity potential $\varphi(x, y, z)$ and ψ' by the x or streamwise component of perturbation velocity $u(x, y, z)$. By virtue of the irrotationality of the flow, the gradients of u and w are related by the expression

$$\frac{\partial u}{\partial n} = \pm \frac{\partial w}{\partial x}$$

and equations (9) can both be written in the form

$$\iint \varphi \frac{\partial w'}{\partial x} dS = \iint u' w dS$$

Integration by parts over either of the surfaces σ_u or σ_l gives

$$\int_{-\infty}^{\infty} \varphi w' \Big|_{-\infty}^{\infty} dy - \iint u w' dS = \iint u' w dS \quad (10)$$

At $x=-\infty$ the potential φ for the forward flow vanishes and at $x=+\infty$ the upwash w' in the reversed flow vanishes so that the first term on the left is zero. The remaining double integrals have for their surfaces of integration the displaced planes σ_u and σ_l . In order to obtain a concise form of the reverse-flow theorem, it suffices to subtract the integrals extended over σ_l from the integrals over σ_u and let the planes approach coincidence with the plane of the wing. Since w, w' and u, u' are continuous everywhere except possibly in the immediate vicinity of the wing, the integration region can be restricted to planes slightly above and below the wing but extending beyond the wing edges. Provided the singularities at the edges can be disregarded, the analytical expression of the steady-state reverse-flow theorem of Ursell and Ward (ref. 8) becomes

$$-\iint_P (u_u w_u' - u_l w_l') dS = \iint_P (u_u' w_u - u_l' w_l) dS \quad (11)$$

for either lifting surfaces or symmetrical wings where P is the plan form of the wing in the $z=0$ plane.

It remains to discuss the effect of edge singularities. In the case of a lifting surface, square-root singularities in both u and w can occur at the leading and trailing edges just on and off the wing, respectively. In the combined flow fields, the limiting process would then yield residue terms analogous to the leading-edge thrust of a lifting plate. If, however, the Kutta condition is imposed at the trailing edge for both the flow in forward and reverse direction, a combination of singularities does not occur and equation (11) is valid. If the leading edge of a symmetrical wing is rounded so as to produce a square-root singularity in w on the wing, a square-root singularity in u occurs just off the wing and a term corresponding to the leading-edge drag (ref. 19) appears. If the geometry of the wing is fixed in the forward and reverse flow, however, the effect of these terms is canceled and equa-

tion (11) still applies. It is important to point out that if the Kutta condition is not satisfied, then the area of integration of equation (11) cannot be restricted to the plan form.

The two sides of equation (11) are expressed in terms of the same coordinate system but it is usually preferable to associate with each of the two streams an x axis extending in the stream direction. To this end introduce now the subscripts 1 and 2 to denote the forward and reverse flow and the two coordinate systems. Thus, in general,

$$x_1 = -x_2 + \xi, \quad y_1 = -y_2 + \eta, \quad z_1 = z_2 \quad (12)$$

where ξ and η are arbitrary constants, and equation (11) then becomes

$$\iint_{P_1} [u_{u_1}(x_1, y_1) w_{u_2}(x_1, y_1) - u_{l_1}(x_1, y_1) w_{l_2}(x_1, y_1)] dx_1 dy_1 = \iint_{P_2} [u_{u_2}(x_2, y_2) w_{u_1}(x_2, y_2) - u_{l_2}(x_2, y_2) w_{l_1}(x_2, y_2)] dx_2 dy_2 \quad (13)$$

In the case of a symmetrical nonlifting wing, the relations

$$u_u = u_l, \quad w_u = -w_l$$

must apply and, in the case of a lifting surface, linearized theory yields

$$u_u = -u_l, \quad w_u = w_l$$

It follows that in either case, equation (13) reduces to the form

$$\iint_{P_1} u_1(x_1, y_1) w_2(x_1, y_1) dx_1 dy_1 = \iint_{P_2} u_2(x_2, y_2) w_1(x_2, y_2) dx_2 dy_2 \quad (14)$$

where the velocity components can be evaluated on either the upper or lower surface of the plan form. If, furthermore, the linearized pressure relation

$$p - p_0 = -\rho_0 U_0 u \quad (15)$$

is used where p is local static pressure, p_0 is static pressure in the free stream, and the wing profiles are assumed closed, equation (14) becomes

$$\iint_{P_1} p_1(x_1, y_1) w_2(x_1, y_1) dx_1 dy_1 = \iint_{P_2} p_2(x_2, y_2) w_1(x_2, y_2) dx_2 dy_2 \quad (16)$$

If, instead of specifying boundary conditions in a single plane, it is necessary to treat boundary conditions for a system of planes, the expression of the reverse-flow theorem is of the same general form as equation (11). Provided the Kutta condition is imposed at the trailing edges of all lifting surfaces, the relation becomes

$$\iint_{\Sigma} (V_n)_1 u_2 dS = \iint_{\Sigma} (V_n)_2 u_1 dS \quad (17)$$

where the area of integration Σ extends over both sides of all the wing surfaces, V_n is the component of perturbation velocity normal to and directed away from each wing, and

the subscripts 1 and 2 refer to forward and reverse flow in the two axial systems.

REVERSE FLOW FOR SUPERSONIC WINGS

The development of a reverse-flow theorem for supersonic wings parallels closely the analysis for the subsonic case. For either planar or multiplanar systems, the conormal in equation (4) is, in fact, the normal so long as the surface of integration is a plane parallel to the x axis. In the case of the single wing, for example, equations (9) apply where the surfaces σ are slightly removed from the plane of the wing and where ψ satisfies the differential equation

$$(M_0^2 - 1) \psi_{xx} - \psi_{yy} - \psi_{zz} = 0$$

In the limit as σ approaches the $z=0$ plane, the reversibility theorem takes the form of equations (11) and (16), provided the integration extends beyond the edges of the wing. It is necessary to include these edges for wings with subsonic leading and trailing edges since singularities occur in the perturbation components and the solutions are not necessarily unique. For supersonic-type edges, the area of integration can be confined to the plan form of the wing and this is also true for subsonic edges, provided the Kutta condition holds for all subsonic trailing edges in both the forward and reverse flow. Equation (17) relates the two possible flows in the case of multiplanar systems.

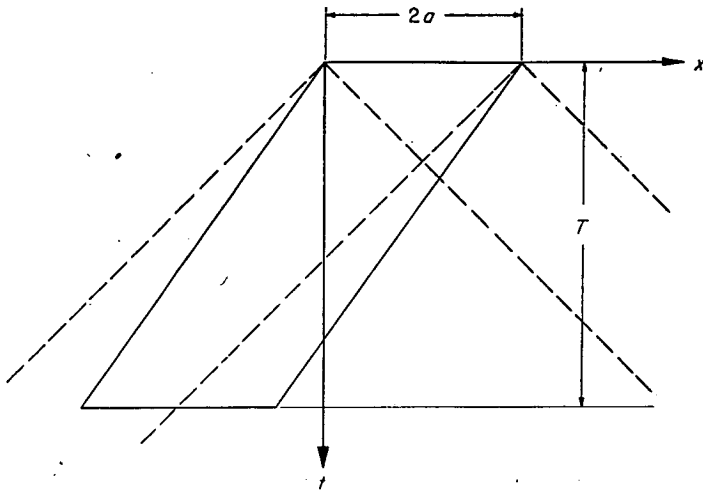
REVERSE UNSTEADY MOTION

In the case of unsteady motion at either subsonic or supersonic flight speed, the basic equation may be taken in the form

$$\psi_{tt} - \psi_{xx} - \psi_{yy} - \psi_{zz} = 0 \quad (18)$$

where $t = a_0 t'$, a_0 is the speed of sound in the undisturbed region of the field, t' is time, and ψ is the perturbation velocity potential or any of the perturbation velocity components. Equation (18) is the acoustic equation for small disturbances in three space dimensions and holds for a system of Cartesian coordinates fixed relative to the undisturbed air. In applications to wing theory, therefore, the wings move relative to fixed axes.

In the derivation of a useful theorem it is convenient to treat thin wings at small angles of attack and to assume that the motion takes place in the xy plane. The visualization of the time and geometry relations is relatively easy for two-dimensional wings moving at a uniform speed, as indicated in figure 2. The airfoil starts at time $t'=0$ and moves to the left at a constant velocity U_0 so that the trace of the leading edge in the xt plane is $x = -U_0 t' = -M_0 t$, and the trailing-edge trace is $x = 2a - M_0 t$. The lines $x = \pm t$ and $x = 2a \pm t$ are the traces of the extremities of the regions affected by the acoustic waves set in motion at $t'=0$ by the leading and trailing edges. In figure 2, the wing has traveled a time $T' = T/a_0$ and the boundary condition determining the wing shape during the motion will be fixed by prescribing the value of vertical induced velocity w over the region "swept out" of the xt plane by the wing. In order to determine a reciprocal theorem, a second wing is assumed to start at the final position of the first wing and to move with negative velocity until it has reached the initial position of the first wing. With these


 FIGURE 2.—Two-dimensional wing in xz plane.

concepts in mind, it follows from equation (4) that the relation

$$\iint \varphi \frac{\partial^2 \varphi'}{\partial z \partial t} dx dt = \iint \frac{\partial \varphi'}{\partial t} w dx dt \quad (19)$$

holds where φ and φ' are, respectively, the perturbation potentials for the forward and reverse motions. The region of integration is determined by the area occupied in the xz plane by the wing, and the Kutta condition is assumed to apply to the trailing edge when in subsonic flight.

If the left side of equation (19) is integrated by parts, the general relation becomes

$$-\iint \frac{\partial \varphi}{\partial t} w' dx dt = \iint \frac{\partial \varphi'}{\partial t} w dx dt \quad (20)$$

If the motion of a three-dimensional wing is to be studied, equations (19) and (20) must be modified to include an integration with respect to y .

Two further changes in equation (20) serve to simplify applications. In the first place, asymmetry is restored to the expression if two distinct systems of axes are used as in equation (14); in the second place, the pressure relation

$$p - p_0 = -\rho_0 \frac{\partial \varphi}{\partial t} = -\rho_0 a_0 \frac{\partial \varphi}{\partial t} \quad (21)$$

where p_0 denotes undisturbed pressure, permits the introduction of pressure p in the integrands. The final expression for the three-dimensional case is, therefore,

$$\begin{aligned} \iiint p_1(x_1, y_1, t_1) w_2(x_1, y_1, t_1) dx_1 dy_1 dt_1 \\ = \iiint p_2(x_2, y_2, t_2) w_1(x_2, y_2, t_2) dx_2 dy_2 dt_2 \end{aligned} \quad (22)$$

where the two motions now follow the same path in reverse directions but are referred to the two sets of oppositely oriented axes satisfying the relations

$$x_1 = -x_2 + \xi, \quad y_1 = -y_2 + \eta, \quad z_1 = z_2, \quad t_1 = -t_2 + \tau \quad (23)$$

where ξ , η , and τ fix the relative positions of the two origins. Figure 3 indicates one possible orientation of the axes.

Equation (22) reduces to a much simpler form, provided further restrictions are imposed on the upwash functions

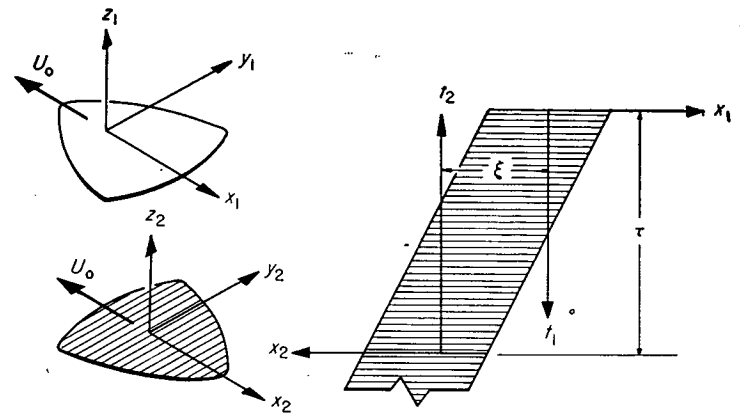


FIGURE 3.—Coordinate systems for three-dimensional wing in unsteady motion.

$w_2(x_1, y_1, t_1)$ and $w_1(x_2, y_2, t_2)$. In order to fix the idea, consider the case in which the wings have traveled a time T' , ($T' = T/a_0$) and a distance $U_0 T' = M_0 T$. Let the two systems of coordinate axes be placed such that $t_1 = 0$ sets the starting time of the forward motion and $t_2 = 0$ sets the starting time of the reverse motion; the two origins are furthermore oriented such that they are at opposite ends of the root chord of the common plan form. Equation (22) then becomes

$$\begin{aligned} \int_0^T dt_1 \int_{P_1(t_1)} p_1(x_1, y_1, t_1) w_2(x_1, y_1, t_1) dx_1 dy_1 \\ = \int_0^T dt_2 \int_{P_2(t_2)} p_2(x_2, y_2, t_2) w_1(x_2, y_2, t_2) dx_2 dy_2 \end{aligned}$$

where the functions w_1 and w_2 have an implicit dependence upon T . If w_1 and w_2 remain constant for $x_1 + M_0 t_1 = \text{const.}$ or $x_2 + M_0 t_2 = \text{const.}$, the expression

$$\begin{aligned} \int_{P_1(T)} p_1(x_1, y_1, T) w_2(x_1, y_1, T) dx_1 dy_1 \\ = \int_{P_2(T)} p_2(x_2, y_2, T) w_1(x_2, y_2, T) dx_2 dy_2 \end{aligned} \quad (24)$$

follows after taking a derivative with respect to T of the original equality. Equations (24) and (14) are now equivalent in form, with T taking the role of an auxiliary parameter. In this way, certain classes of unsteady motions can be treated simultaneously with steady motions.

In the applications to follow it will be convenient to introduce into equation (24) upwash functions of the indicial type; that is, functions that are zero up to a fixed time and, after experiencing a finite discontinuity, remain constant for all subsequent values of time. Such indicial or step variations can be assumed, say, for angle of attack, rate of pitch, and rate of roll since they satisfy the requirements underlying the derivation of equation (24). This choice of functions will prove to be advantageous in that the integrals of the responsive pressures will yield results relating the wing characteristics. Theorems to be given later will speak specifically of steady and indicial motions. It is to be understood, however, that the indicial results can be further extended when the same wing is assumed to be executing

identical motions in forward and reverse flight. Thus, by means of Duhamel's integral (see, e. g., ref. 20), if $f(t)$ is the response in the wing characteristic to a step variation in w at time $t=0$, the response to an arbitrary variation with time of w can be written

$$F(t) = \frac{\partial}{\partial t} \int_0^t f(t-\tau)w(\tau)d\tau \quad (25)$$

If it is known, for example, that the lift per unit angle of attack is the same at corresponding values of time for a wing experiencing an indicial angle-of-attack change in forward and reverse flight, it follows that the build-up of lift is the same at corresponding values of time for all forward and reverse motions, provided the time histories of the motions are the same. The equivalence of lift would thus be established, for instance, for oscillatory variations in angle of attack.

An alternative study of reverse-flow theorems for oscillatory motions could be based upon the modified wave equation

$$\chi_{xx} + \chi_{yy} + \chi_{zz} + \omega^2 \chi = 0$$

which results from setting $\psi = e^{i\omega t}\chi$ in equation (18). Such a study would corroborate the conclusions drawn from equations (24) and (25).

APPLICATIONS

The results of the foregoing analysis may be employed to determine a number of special theorems that are particularly useful in the calculation of the aerodynamic characteristics of twisted wings and of wings in nonuniform downwash fields. The theorems apply equally to wings acting either alone or, in certain cases, in combination with other wings or with cylindrical bodies having their generators aligned with the x axis. Moreover, they apply not only to wings in steady motion but also to wings performing unsteady motions of the indicial type, or unsteady motions derivable therefrom. For wings in more complex unsteady motions, however, it will be necessary to refer to the more general results of equation (22). Some problems of this nature will be described at the end of this section.

The applications to be included are exact within the framework of linear theory and involve no further restrictions on the wing plan form or Mach number except in certain indicated cases where it will be convenient to use results based on slender-wing theory. The examples are intended to be representative in nature.

REVERSAL THEOREMS—STEADY AND INDICIAL MOTIONS

Reversal theorems are defined here as relations between the aerodynamic characteristics of identical wings executing the same type of motions in forward and reverse flight. The results presented in this section apply not only to single wings in steady motion but also to combinations of wings, as in cascades or multiplanes, performing either steady motion or motions of the indicial type.

DRAG OF SYMMETRICAL NONLIFTING WINGS

The drag of a symmetrical, sharp-edged wing in linear theory may be determined by integrating over the plan form

the product of the pressure and the slope in the x direction of the wing surface; when the wing has blunt edges with slopes having square-root singularities, these singularities yield an added contribution (ref. 19). In general, therefore, the drag D of a symmetrical section is given by

$$D = D_e + 2 \iint_P p \left(\frac{dz}{dx} \right)_u dS \quad (26)$$

where D_e is the drag attributable to the edges.

If the subscripts 1 and 2 refer to the same wing in forward and reverse flow, respectively, and with the two systems of axes introduced in equation (23), local slopes are related as follows

$$\left. \frac{dz_1(x_1, y_1, T)}{dx_1} \right|_u = - \left. \frac{dz_2(x_2, y_2, T)}{dx_2} \right|_u \quad (27)$$

Equations (26) and (27), together with the reciprocal relation (24), yield

$$\begin{aligned} D_1 - (D_e)_1 &= 2 \iint_{P_1} p_1 \left(\frac{dz_1}{dx_1} \right)_u dS_1 = -2 \iint_{P_1} p_1 \left(\frac{dz_2}{dx_2} \right)_u dS_1 = \\ &= -2 \iint_{P_2} p_2 \left(\frac{dz_1}{dx_1} \right)_u dS_2 = 2 \iint_{P_2} p_2 \left(\frac{dz_2}{dx_2} \right)_u dS_2 = D_2 - (D_e)_2 \end{aligned} \quad (28)$$

Since the geometry of the wing is fixed, the edge contributions are the same,

$$(D_e)_1 = (D_e)_2 \quad (29)$$

and, consequently,

$$D_1 = D_2 \quad (30)$$

which confirms the relation stated in reference 9.

THEOREM: The pressure drag in steady or indicial motion of symmetrical nonlifting wings is the same in forward and reverse flight.

LIFT ON FLAT-PLATE WINGS

The lift L of a wing may be determined by integrating the differential pressure $\Delta p = p_l - p_u$ over the wing plan form, thus

$$L = \iint_P \Delta p dS \quad (31)$$

For flat-plate wings, the local angle of attack of the wing surface is a constant

$$\alpha_1(x_1, y_1, T) = \alpha_1 = \text{const.}, \quad \alpha_2(x_2, y_2, T) = \alpha_2 = \text{const.} \quad (32)$$

Application of equations (31), and (32), and (24) yields the following:

$$L_1 \alpha_2 = \iint_{P_1} \Delta p_1 \alpha_2 dS_1 = \iint_{P_2} \Delta p_2 \alpha_1 dS_2 = L_2 \alpha_1$$

or

$$L_1 / \alpha_1 = L_2 / \alpha_2 \quad (33)$$

THEOREM: The lift per unit angle of attack of flat-plate wings in steady or indicial motion is the same in forward and reverse flight.

This theorem generalizes the relation previously given by Brown (ref. 5) for steady motion.

DAMPING IN ROLL OF FLAT-PLATE WINGS

The rolling moment L' exerted on a wing, following the usual sign convention, is given by

$$L' = - \iint_P y \Delta p dS \quad (34)$$

The local angle of attack due to rotation about the x axis is

$$\alpha_1 = \frac{p'_1 y_1}{U_0}, \quad \alpha_2 = \frac{p'_2 y_2}{U_0} = - \frac{p'_2 y_1}{U_0} \quad (35)$$

where p' is the angular velocity of roll, assumed constant. Application of equations (34), (35), and (24) yields the following:

$$\begin{aligned} \frac{p'_2 L'_1}{U_0} &= - \frac{p'_2}{U_0} \iint_{P_1} y_1 \Delta p_1 dS_1 = \frac{p'_2}{U_0} \iint_{P_1} y_2 \Delta p_1 dS_1 \\ &= \frac{p'_1}{U_0} \iint_{P_2} y_1 \Delta p_2 dS_2 = - \frac{p'_1}{U_0} \iint_{P_2} y_2 \Delta p_2 dS_2 = \frac{p'_1 L'_2}{U_0} \end{aligned}$$

or

$$L'_1 / p'_1 = L'_2 / p'_2 \quad (36)$$

THEOREM: The rolling moment per unit angular rolling velocity of flat-plate wings in steady or indicial motion is the same in forward and reverse flight.

DAMPING IN PITCH OF FLAT-PLATE WINGS

Consider a wing, first, in forward flight and pitching with a uniform angular velocity q_1 about a lateral axis; second, in reverse flight and pitching with angular velocity q_2 about another lateral axis. Place each wing in a coordinate system such that the y axis coincides with the axis of rotation and designate the distances to the moment axes by x_0 with proper subscripts, as shown in figure 4. In such a coordinate system, the pitching moment M_{x_0} exerted on a wing, following the usual sign convention, is

$$M_{x_0} = - \iint_P (x - x_0) \Delta p dS \quad (37)$$

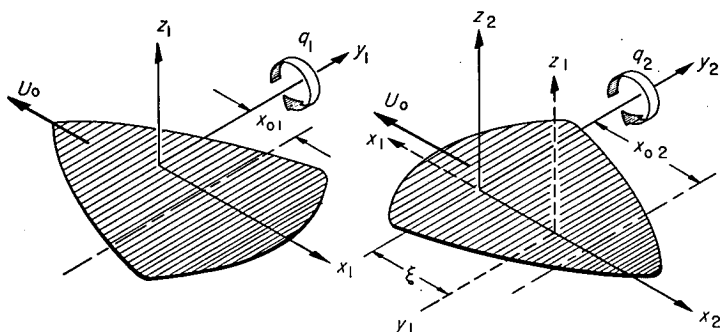


FIGURE 4.—Coordinate systems and symbols used in discussion of reverse-flow theorems for pitching wings.

The local angle-of-attack distributions due to rotations about the y axis are

$$\alpha_1 = \frac{q_1}{U_0} x_1, \quad \alpha_2 = \frac{q_2}{U_0} x_2 = \frac{q_2}{U_0} (\xi - x_1) \quad (38)$$

Application of equations (37), (38), and (24) yields the following:

$$\frac{q_2}{U_0} [(M_1)_{x_{01}} + (\xi - x_{01}) L_1] = \frac{q_2}{U_0} \iint_{P_1} [(x_{01} - x_1) + (\xi - x_{01})] \Delta p_1 dS_1 =$$

$$\frac{q_2}{U_0} \iint_{P_1} (\xi - x_1) \Delta p_1 dS_1 = \frac{q_2}{U_0} \iint_{P_1} x_2 \Delta p_1 dS_1 =$$

$$\frac{q_1}{U_0} \iint_{P_2} x_1 \Delta p_2 dS_2 = \frac{q_1}{U_0} [(M_2)_{x_{02}} + (\xi - x_{02}) L_2]$$

or

$$\frac{(M_1)_{x_{01}} + (\xi - x_{01}) L_1}{q_1} = \frac{(M_2)_{x_{02}} + (\xi - x_{02}) L_2}{q_2} \quad (39)$$

This equation indicates that the pitching moment due to pitching velocity is, in general, not the same for wings in forward and reverse flight. However, if $x_{01} = x_{02} = \xi$, the pitching moment per unit angular pitching velocity of flat-plate wings in steady or indicial motion is invariant.

SPECIAL RECIPROCAL THEOREMS AND APPLICATIONS

In the following section, several special reciprocal theorems will be derived and applications will be illustrated. Reciprocal theorems, in contrast to reversal theorems treated in the preceding section, are defined here as relations between the aerodynamic properties of wings in forward and reverse flight that have dissimilar camber, twist, and thickness distributions but have the same plan forms. The motions may or may not be similar, although it is assumed in this section that both wings are in either steady motion or unsteady motion of the indicial type. As noted in the preceding section, the results apply equally to wings acting alone or in combination.

SYMMETRIC NONLIFTING WINGS—STEADY MOTION

The problems of paramount interest in the application of the general relations are found from considerations of pressure integrals over lifting surface; such problems will be given detailed treatment later. In the present section, a brief indication is given of the manner in which useful results can be derived for symmetric wings. The discussion will be limited to steady-state, two-dimensional, subsonic pressure fields although fairly obvious extensions can be carried out.

If the geometry of a real symmetric airfoil is prescribed the theoretical pressure distribution exists and is unique. If, however, the pressure distribution is prescribed, a real airfoil does not necessarily exist, but by means of reciprocal relations it is possible to derive certain conditions of compatibility that need to be imposed. Consider, therefore, the two subsonic solutions

$$u_1(x_1) = 0, \quad w_1(x_1) = U_0 \sqrt{a^2 - x_1^2}; \quad -a < x_1 < a$$

and

$$u_2(x_2), \quad w_2(x_2)$$

The first solution has square-root singularities in w at each end of the airfoil and, correspondingly, singularities in u occur just ahead of the point $x = -a$ and just behind $x = a$. On the other hand, equation (14) certainly applies if w_2 is zero at $x = \pm a$. If the origins of the two systems of axes are at the same position, it follows from equation (14) that $u_2(x)$ must satisfy the relation

$$0 = \int_{-a}^a \frac{u_2(x_2) dx_2}{\sqrt{a^2 - x_2^2}} \quad (40)$$

This result is useful in the calculation of airfoil shapes involving a change in pressure distribution from that of a known reference profile. The restriction on w_2 at the nose and tail implies that the derived and reference profiles have the same slope and radius of curvature at those points. The restriction on u_2 , as given in equation (40), can be interpreted as a condition that must exist by virtue of the fact that the drag of an airfoil in two-dimensional potential flow is zero.

As a second example, consider the solutions

$$u_1(x_1) = \frac{tU_0}{2a}, \quad w_1(x_1) = \frac{-tx_1U_0}{2a\sqrt{a^2 - x_1^2}}; \quad -a < x_1 < a$$

that represent velocity and slope of a thin ellipse of thickness t and chord $2a$. If w_2 is chosen as above, such that it vanishes at the nose and tail of the airfoil, if u_2 is the corresponding velocity distribution, and if the two sets of axes are as before, equation (14) yields

$$\int_{-a}^a \frac{dz_2}{dx_1} dx_1 = - \int_{-a}^a \frac{u_2(x_2)x_2 dx_2}{\sqrt{a^2 - x_2^2}}$$

From this result, together with the general closure condition,

$$\int_{-a}^a \frac{dz}{dx} dx = \frac{1}{U_0} \int_{-a}^a w(x) dx = 0 \quad (41)$$

a necessary condition for the closure of the second airfoil is

$$\int_{-a}^a \frac{u_2(x_2)x_2 dx_2}{\sqrt{a^2 - x_2^2}} = 0 \quad (42)$$

As a final example, consider the solutions for $-a < x_1 < a$

$$u_1(x_1) = \frac{2tU_0}{3\sqrt{3}a^2} (a - 2x_1),$$

$$w_1(x_1) = \frac{2tU_0}{3\sqrt{3}a^2} \frac{2x_1^2 - ax_1 - a^2}{\sqrt{a^2 - x_1^2}},$$

representing velocity and slope of a thin Joukowski type airfoil. In this case, w_1 vanishes at the tail and the downwash distribution w_2 for the reverse wing may have a square-root singularity at the nose. The nose of the first wing is, however, blunt and for equation (14) to apply the second wing must have a cusped tail. Under these conditions, equation (14) yields

$$\begin{aligned} \frac{2tU_0^2}{3\sqrt{3}a^2} \int_{-a}^a (a - 2x_1) \frac{dz_2}{dx_2} dx_1 \\ = \frac{2tU_0}{3\sqrt{3}a^2} \int_{-a}^a \frac{2x_1^2 - ax_1 - a^2}{\sqrt{a^2 - x_1^2}} u_2(x_2) dx_2 \end{aligned}$$

Making the substitution $x_1 = -x_2$ in this equation and integrating the left side by parts, one has

$$\begin{aligned} U_0 \left\{ \left[(2x_1 - a)z_2 \right]_{-a}^a - 2 \int_{-a}^a z_2(x_1) dx_1 \right\} \\ = \int_{-a}^a \frac{2x_2^2 + ax_2 - a^2}{\sqrt{a^2 - x_2^2}} u_2(x_2) dx_2 \end{aligned}$$

For all real airfoils with cusped trailing edges, therefore, the area A_2 can be expressed as

$$A_2 = - \int_{-a}^a \frac{2x_2^2 + ax_2 - a^2}{\sqrt{a^2 - x_2^2}} \frac{u_2(x_2)}{U_0} dx_2 \quad (43)$$

LIFT—STEADY AND INDICIAL MOTION

The reciprocal theorems offer considerable advantage in the calculation of the lift of wings having a nonuniform angle-of-attack distribution or of wings in a stream having nonuniform flow directions. For these applications, it is convenient to consider a special form of the reciprocal theorem which relates the lift on a wing having arbitrary distribution of local angle of attack to that of the flat-plate wing of identical plan form in flight in the reverse direction. Since the solution of this latter problem is often known or can be found relatively easily, the solution of the original problem is facilitated in many instances.

Lift of arbitrarily cambered wings.—Consider two wings of identical plan form in flight in opposite directions, as shown in figure 5. Wing 1 is arbitrarily cambered and twisted and wing 2 is flat.

$$\alpha_1 = \alpha_1(x_1, y_1, T), \quad \alpha_2 = \text{const.} \quad (44)$$

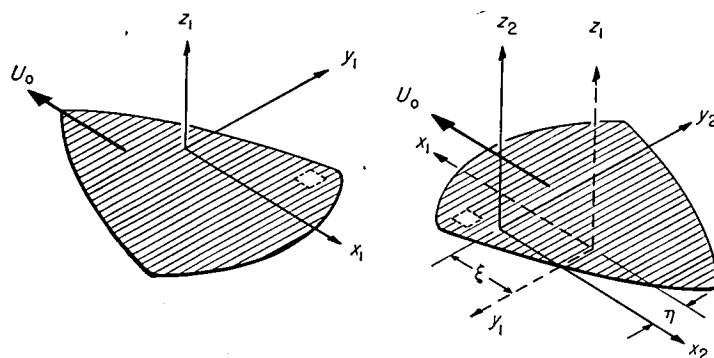


FIGURE 5.—Coordinate systems and symbols used in discussion of relation between lift of arbitrarily cambered wings and loading on flat-plate wings.

Application of equations (44) and (24) yields the following:

$$\alpha_2 L_1 = \iint_{P_1} \alpha_2 \Delta p_1 dS_1 = \iint_{P_2} \alpha_1 \Delta p_2 dS_2$$

or

$$L_1 = \iint_{P_2} \alpha_1 \left(\frac{\Delta p_2}{\alpha_2} \right) dS_2 \quad (45)$$

THEOREM: The lift in steady or indicial motion of a wing having arbitrary twist and camber is equal to the integral over the plan form of the product of the local angle of attack and the loading per unit angle of attack at the corresponding point of a flat-plate wing of identical plan form in flight in the reverse direction.

Equation (45) may be used to derive Munk's integral formula for the lift of an arbitrarily curved airfoil in subsonic flow. Consider airfoils 1 and 2 placed in their respective coordinate systems, as indicated in figure 6. The angle-of-attack distributions on the two airfoils are given by

$$\alpha_1 = -\frac{dz_1}{dx_1}, \quad \alpha_2 = \text{const.} \quad (46)$$

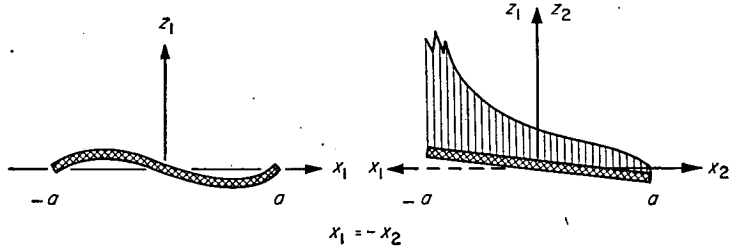


FIGURE 6.—Sketch of arbitrarily cambered airfoil illustrating symbols used in equations (46) through (50).

The loading per unit angle of attack on airfoil 2 is

$$\frac{\Delta p_2}{\alpha_2} = \frac{4q_0}{\beta} \sqrt{\frac{a-x_2}{a+x_2}} = \frac{4q_0}{\beta} \sqrt{\frac{a+x_1}{a-x_1}} \quad (47)$$

where q_0 is free-stream dynamic pressure ($\frac{1}{2}\rho_0 U_0^2$). Substitution into equation (47) yields the lift formula

$$L_1 = -\frac{4q_0}{\beta} \int_{-a}^a \frac{dz_1}{dx_1} \sqrt{\frac{a+x_1}{a-x_1}} dx_1 \quad (48)$$

The corresponding formula for the lift of a tier of curved airfoils may also be derived similarly from the expression for the loading on an equivalent tier of flat airfoils. Consider, for example, an unstaggered lattice of flat-plate airfoils arranged vertically. If the gap distance between the plates is h , the loading per unit angle of attack is

$$\frac{\Delta p_2}{\alpha_2} = \frac{4q_0}{\beta} \text{sech}\left(\frac{\pi a}{\beta h}\right) \sqrt{\frac{\sinh[\pi(a-x_2)/\beta h]}{\sinh[\pi(a+x_2)/\beta h]}}$$

The formula for the lift on one of a lattice of identically cambered airfoils² is therefore

$$L_1 = -\frac{4q_0}{\beta} \text{sech}\left(\frac{\pi a}{\beta h}\right) \int_{-a}^a \frac{dz_1}{dx_1} \sqrt{\frac{\sinh[\pi(a+x_1)/\beta h]}{\sinh[\pi(a-x_1)/\beta h]}}$$

The load distribution per unit angle of attack for a two-dimensional supersonic wing is

$$\frac{\Delta p_2}{\alpha_2} = \frac{4q_0}{\beta} \quad (49)$$

and, from equation (47), the lift is

$$L_1 = -\frac{4q_0}{\beta} \int_{-a}^a \frac{dz_1}{dx_1} dx_1 \quad (50)$$

The extension of this result to include supersonic-edged wings straight trailing edges leads to a result given originally by Lagerstrom and Van Dyke (ref. 21). If, as in figure 7, the

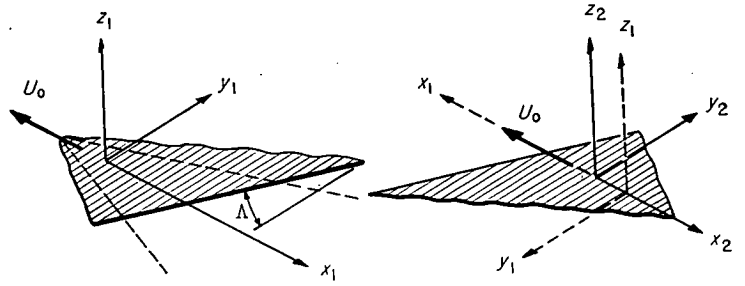


FIGURE 7.—Sketch of supersonic-edged wing illustrating symbols used in equations (51) and (52).

sweep angle of the straightedge is Λ , the load distribution per unit angle of attack of the reversed wing is

$$\frac{\Delta p_2}{\alpha_2} = \frac{4q_0 \cos \Lambda}{\sqrt{1 - M_0^2 \cos^2 \Lambda}} \quad (51)$$

and the lift is

$$L_1 = \frac{-4q_0}{\sqrt{\sec^2 \Lambda - M_0^2}} \iint_{P_1} \frac{dz_1}{dx_1} dx_1 dy_1 \quad (52)$$

A less obvious application yields the build-up of lift with time of an arbitrarily cambered two-dimensional supersonic airfoil starting impulsively from rest at a constant speed. Rewriting equation (45), lift is

$$L_1 = - \int_{-M_0 T}^{2a - M_0 T} \frac{dz_1(x_2, T)}{dx_1} \frac{\Delta p_2(x_2, T)}{\alpha_2} dx_2 \quad (53)$$

Figure 8 presents sketches showing the final positions of the

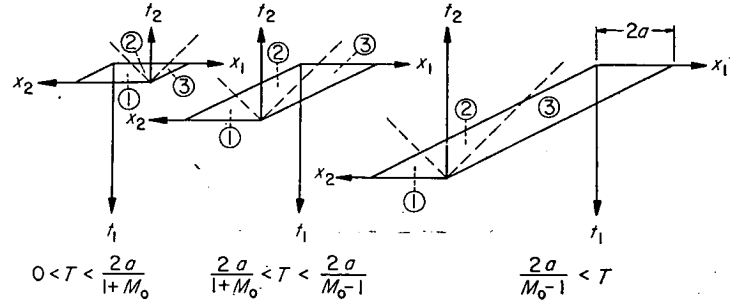


FIGURE 8.—Regions in which different forms of equation (54) apply.

airfoils relative to each other for various values of T . In reference 22 the expressions for the loading in equation (53) are given. Over the intervals denoted by ①, ②, and ③ in the sketch, these expressions are

$$\left. \begin{aligned} \text{Region ① } \frac{\Delta p_2}{\alpha_2} &= \frac{4q_0}{M_0} \\ \text{Region ② } \frac{\Delta p_2}{\alpha_2} &= \frac{4q_0}{\pi \sqrt{M_0^2 - 1}} \left[\arccos \frac{M_0 x + T}{x + M_0 T} + \frac{\sqrt{M_0^2 - 1}}{M_0} \left(\frac{\pi}{2} + \arcsin \frac{x}{T} \right) \right] \\ \text{Region ③ } \frac{\Delta p_2}{\alpha_2} &= \frac{4q_0}{\sqrt{M_0^2 - 1}} \end{aligned} \right\} \quad (54)$$

Lift on a wing in a nonuniform downwash field.—The reciprocal theorem of equation (45) can also provide a particularly good method of determining the lift on a wing in

² This result, as well as the detailed pressure distribution, has been derived by Mr. Paul F. Byrd of the Ames Aeronautical Laboratory by means of a direct inversion of the singular integral equation relating the aerodynamic pressure and geometry of the airfoil. His work has also been used in deriving equations (80) and (81).

certain nonuniform downwash fields of known structure. Such problems arise whenever a wing acts in the presence of other wings, bodies, or propellers but is always of prime concern in the determination of the lift on a tail acting in the downwash field of a wing. In most problems, the downwash velocities at the position of the tail may be considered to be constant in the longitudinal direction and to vary in the spanwise direction, thus

$$\alpha_1 = \alpha_1(y_1, T), \quad \alpha_2 = \text{const.} \quad (55)$$

and

$$L_1 = \iint_{P_2} \alpha_1 \left(\frac{\Delta p_2}{\alpha_2} \right) dS_2 = \int_{-s}^s \alpha_1 \left(\frac{l_2}{\alpha_2} \right) dy_2 \quad (56)$$

where l_2 is the span load distribution associated with the load distribution Δp_2 . Summarizing, the lift in steady or indicial motion of a wing in a downwash field which varies across the span is equal to the integral over the span of the product of the local angle of attack and the span loading per unit angle of attack at the corresponding spanwise station of a flat-plate wing of identical plan form in flight in the reverse direction. This statement generalizes the result given recently by Alden and Schindel (ref. 23) for steady flow about wings having supersonic leading and trailing edges and streamwise side edges.

As for example, consider the problem of determining the lift on a wing at a geometrical angle of attack of zero resulting from the presence of an infinite line vortex of strength Γ extending in the flight direction. The wing will be considered to have such a plan form that its span loading when in flight in the reverse direction is elliptic. The notation is as shown in figure 9. For this problem, therefore, the span

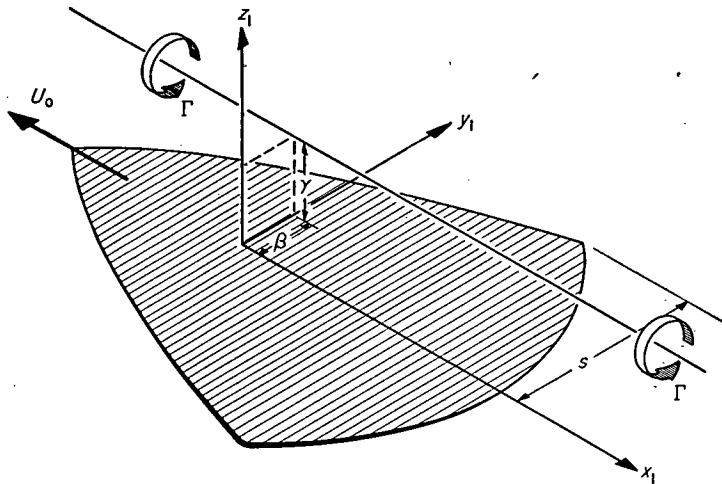


FIGURE 9.—View of wing and neighboring vortex.

loading of the wing in reverse flight is given by

$$\frac{l_2}{\alpha_2} = \frac{2L_2}{\pi s \alpha_2} \sqrt{1 - \frac{y_2^2}{s^2}} \quad (57)$$

The local angle of attack of the original wing due to the presence of the vortex is given by

$$\alpha_1 = \frac{\Gamma}{2\pi U_0} \frac{y_1 - \beta}{[(y_1 - \beta)^2 + \gamma^2]} \quad (58)$$

Substitution of equations (57) and (58) into equation (56) yields the following formula for the lift:

$$L_1 = \frac{\Gamma}{\pi U_0 s} \frac{L_2}{\alpha_2} \left\{ \frac{-\beta}{s} + \sqrt{\frac{1}{2} \left[\sqrt{\left(1 + \frac{\gamma^2}{s^2} - \frac{\beta^2}{s^2}\right)^2 + \frac{4\beta^2\gamma^2}{s^4}} - \left(1 + \frac{\gamma^2}{s^2} - \frac{\beta^2}{s^2}\right) \right]} \right\} \quad (59)$$

The lift on a wing in the vicinity of a number of such vortices may be found by superposition.

Lift due to deflection of a portion of the wing surface.—Let a portion P' of the surface of wing 1 be deflected a constant angle δ and the remainder of the wing be a flat plate aligned with the free-stream direction. Let wing 2 be a flat-plate wing inclined at an angle of attack α , thus

$$\alpha_1 = \begin{cases} \delta & \text{on } P' \\ 0 & \text{elsewhere} \end{cases} \quad \alpha_2 = \text{const.} \quad (60)$$

Substitution of equation (60) into (45) yields the following result:

$$\frac{L_1}{\delta} = \iint_{P_2'} \left(\frac{\Delta p_2}{\alpha_2} \right) dS_2 \quad (61)$$

The lift in steady or indicial motion per unit angular deflection of a portion of the wing surface is thus equal to the lift per unit angle of attack on the corresponding portion of a flat-plate wing in flight in the reverse direction. This generalizes a result given previously by Morikawa and Puckett (ref. 24) for steady flow about low-aspect-ratio wings.

This rule is very useful in the determination of the lift resulting from the deflection of a flap or control surface. This is particularly true for supersonic speeds since the loading on the related flat wing is often a constant over a large portion of the area of integration.

As a further example, consider the case of a low-aspect-ratio wing having a straight trailing edge and mounted on an infinite cylindrical body of revolution. The entire wing-body combination is at zero angle of attack except for the flaps on the rear of the wing that are deflected an angle δ . The problem is to determine the lift on the entire wing-body combination due to the deflection of the flaps. Slender-wing-theory results of reference 25 are to be used. The notation is indicated in figure 10.

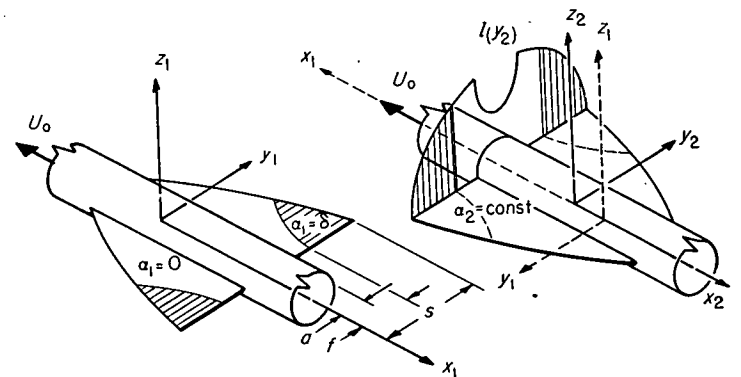


FIGURE 10.—View of lifting slender wing-body combinations.

$$\frac{L_1}{\delta} = \int \int_{flap} \left(\frac{\Delta p_2}{\alpha_2} \right) dS_2 = 2 \int_f^s \left[\frac{l_2(y_2)}{\alpha_2} \right] dy_2 \quad (62)$$
$$l_2(y_2) = 4q_0\alpha_1 a \sqrt{\left(1 - \frac{y^2}{s^2}\right) \left(\frac{s^2}{a^2} - \frac{a^2}{y^2}\right)} \quad (63)$$
$$L_{flat} = 4q_0 \delta s^2 \left\{ \left(1 - \frac{a^2}{s^2} \right)^2 \frac{\pi}{4} - \sqrt{\left(1 - \frac{f^2}{s^2} \right) \left(\frac{f^2}{s^2} - \frac{a^4}{s^4} \right)} + \right. \\ \left. \frac{1}{2} \left(1 + \frac{a^4}{s^4} \right) \arcsin \frac{1 - 2(f/s)^2 + (a/s)^4}{1 - (a/s)^4} + \right. \\ \left. \frac{a^2}{s^2} \arcsin \frac{[1 + (a/s)^4](f/s)^2 - 2(a/s)^4}{(f/s)^2 [1 - (a/s)^4]} \right\} \quad (64)$$

A graph showing the normalized flap deflection $(L/\delta)_{\text{flap}} / (L/a)_{W-B}$ on the y-axis versus the normalized flap length a/s on the x-axis. The y-axis ranges from 0 to 1.0 with major grid lines every 0.2. The x-axis ranges from 0 to 1.0 with major grid lines every 0.2. There are five curves representing different frequency ratios f/s :

- $f/s = .10$: A dashed line starting at (0, 1.0) and decreasing to approximately (0.2, 0.8).
- $f/s = .25$: A dashed line starting at (0, 0.85) and decreasing to approximately (0.4, 0.7).
- $f/s = .50$: A solid line starting at (0, 0.68) and decreasing to approximately (0.6, 0.6).
- $f/s = .75$: A solid line starting at (0, 0.4) and decreasing to approximately (0.75, 0.55).
- $f/s = 1.0$: A solid line starting at (0, 0.15) and decreasing to approximately (0.8, 0.5).

The curves show that as the frequency ratio f/s increases, the normalized flap deflection generally decreases for a given normalized flap length a/s .

$$(L/\alpha)_{W-B} = 2\pi q_0 s^2 \left(1 - \frac{a^2}{s^2}\right)^2 \quad (65)$$
$$\alpha_1 = \alpha_1(y_1, T), \quad \alpha_2 = \frac{p_2 y_2}{V_0} \quad (68)$$

The rolling moment of the first wing is then given by

$$L_1' = \iint_{P_2} \alpha_2 \left(\frac{\Delta p_2}{p_2'/U_0} \right) dS_2 = \int_{-s}^s \alpha_1 \left(\frac{l_2}{p_2'/U_0} \right) dy_2 \quad (69)$$

or, in words, the rolling moment in steady or indicial motion of a wing in a downwash field which varies across the span is equal to the integral over the span of the product of the local angle of attack and the span loading per unit (p_2'/U_0) at the corresponding spanwise station of a rolling flat-plate wing of identical plan form in flight in the reverse direction.

Rolling moment due to deflection of a portion of the wing surface.—Let a portion P' of the surface of a wing be deflected a constant angle δ and the remainder of the wing be a flat-plate aligned with the free-stream direction. The related wing is a flat-plate wing rolling with angular velocity p_2'

$$\alpha_1 = \begin{cases} \delta & \text{on } P' \\ 0 & \text{elsewhere} \end{cases} \quad \alpha_2 = \frac{p_2' y_2}{U_0} \quad (70)$$

Substitution from equation (70) into (67) yields the following result:

$$\frac{L_1'}{\delta} = \iint_{P_2'} \left(\frac{\Delta p_2}{p_2'/U_0} \right) dS_2 \quad (71)$$

Thus, the rolling moment in steady or indicial motion due to a given angular deflection of a portion of the wing surface is equal to the lift per unit (p_2'/U_0) on the corresponding portion of a rolling flat-plate wing of identical plan form in flight in the reverse direction.

As an example, consider a wing-body combination consisting of a low-aspect-ratio wing having a straight trailing edge mounted on an infinite circular cylinder, as shown in figure 13. The body is at zero angle of attack, the right wing

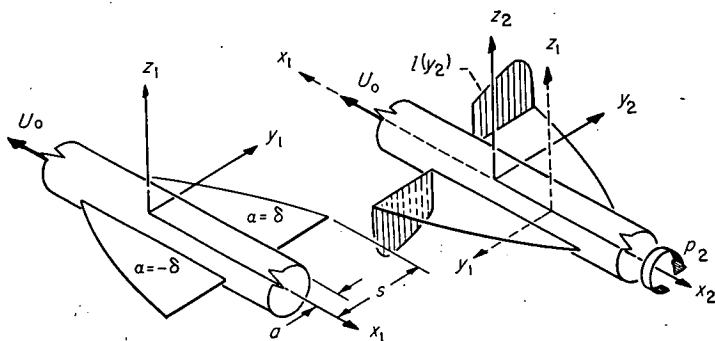


FIGURE 13.—Sketch illustrating symbols used in discussion of rolling moment resulting from differential deflection of wings of slender wing-body combination.

P' is deflected an angle δ , and the left wing P'' is deflected $-\delta$. The problem is to determine by means of slender-wing theory the rolling moment exerted on the entire wing-body combination. The notation is indicated in figure 13.

$$\alpha_1 = \begin{cases} -\delta & \text{on left wing, } P'' \\ 0 & \text{on body} \\ +\delta & \text{on right wing, } P' \end{cases} \quad \alpha_2 = \frac{p_2' y_2}{U_0} \quad (72)$$

Since slender-wing theory indicates that the loading on wing 2 is concentrated on the leading edge, the rolling moment of

wing 1 can be found by integrating the span loading on wing 2.

$$L_1' = \iint_{P_2'} \alpha_1 \left(\frac{\Delta p_2}{p_2'/U_0} \right) dS_2 = -\delta \int_{-s}^s \left(\frac{l_2}{p_2'/U_0} \right) dy_2 + \delta \int_a^s \frac{l_2}{p_2'/U_0} dy_2 = 2\delta \int_a^s \left(\frac{l_2}{p_2'/U_0} \right) dy_2$$

where, from reference 26,

$$\frac{l_2}{p_2'/U_0} = q_0 s^2 \left[\left(1 + \frac{2}{\pi} \arccos \frac{2as}{s^2 + a^2} \right) \left(y + \frac{a^2}{y} \right) \sqrt{s^2 + \frac{a^4}{s^2} - y^2 - \frac{a^4}{y^2}} + \frac{2}{\pi} \left(y - \frac{a^2}{y} \right)^2 \operatorname{arccosh} \frac{(y^2 + a^2)(s^2 - a^2)}{(y^2 - a^2)(s^2 + a^2)} \right] \quad (73)$$

The resulting expression for the rolling moment is

$$L_1' = 4q_0 \delta s^5 \left\{ F(\varphi, k) \left[\frac{R^2}{2} \left(1 - \frac{2R^2}{3} + R^4 \right) \left(1 + \frac{2}{\pi} \arccos \frac{2R}{1+R^2} \right) - \frac{2R^3}{3\pi} (1-R^2) \right] + E(\varphi, k) \left[\frac{1}{2} \left(\frac{1}{3} - 2R^2 + \frac{R^4}{3} \right) \left(1 + \frac{2}{\pi} \arccos \frac{2R}{1+R^2} \right) + \frac{2}{3\pi} R(1-R^2) \right] + \frac{8R^3}{3\pi} \ln \frac{2R}{1+R^2} + \frac{R^2}{3} (1-R^2) \left(1 + \frac{2}{\pi} \arccos \frac{2R}{1+R^2} \right) \right\} \quad (74)$$

where

$$R = a/s$$

$$\varphi = \arcsin \frac{1}{\sqrt{1+R^2}}$$

$$k = \sqrt{1-R^4}$$

A plot of the results is shown in figure 14. The rolling moment has been nondimensionalized by dividing by the value corresponding to that of the wing alone ($R=0$).

PITCHING MOMENT—STEADY AND INDICIAL MOTION

A number of useful relations regarding the pitching-moment characteristics of wings may be found through application of the reciprocal theorem. Since the general procedure is closely analogous to that of the preceding sections, the following discussion will be brief.

Pitching moment of arbitrarily cambered wing.—Consider the problem of determining the pitching moment M_1 about the origin of wing 1 possessing an arbitrary distribution of camber. The related wing in flight in the reverse direction, wing 2, is a flat-plate wing of identical plan form pitching about the moment axis of wing 1, as indicated in figure 15, thus

$$\alpha_1 = \alpha_1(x_1, y_1, T), \quad \alpha_2 = -\frac{q_2 x_1}{U_0} = \frac{q_2(x_2 - \xi)}{U_0} \quad (75)$$

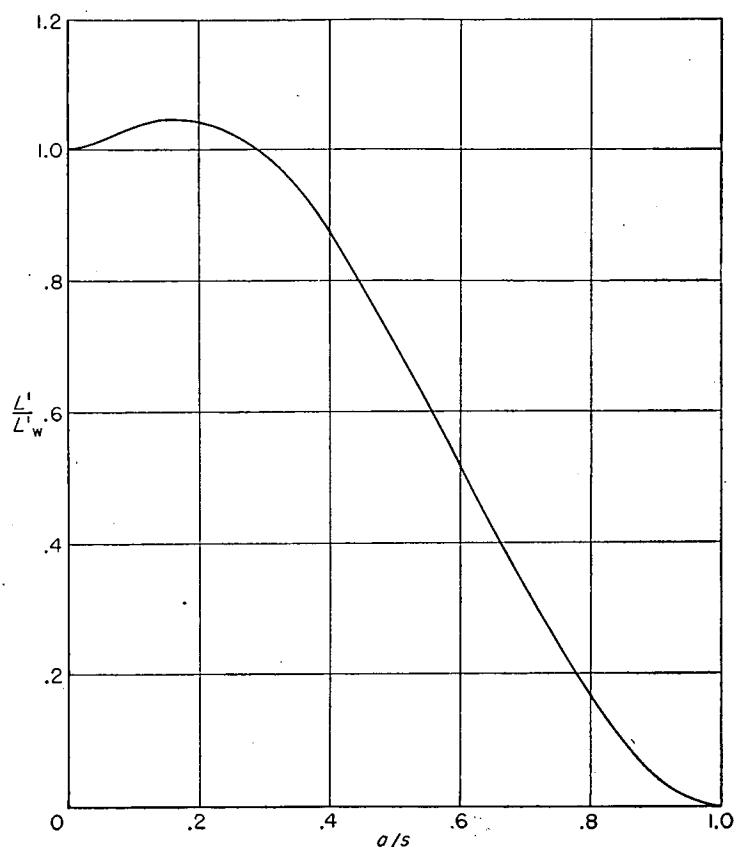


FIGURE 14.—Rolling moment of slender wing-body combination resulting from differential deflection of wings.

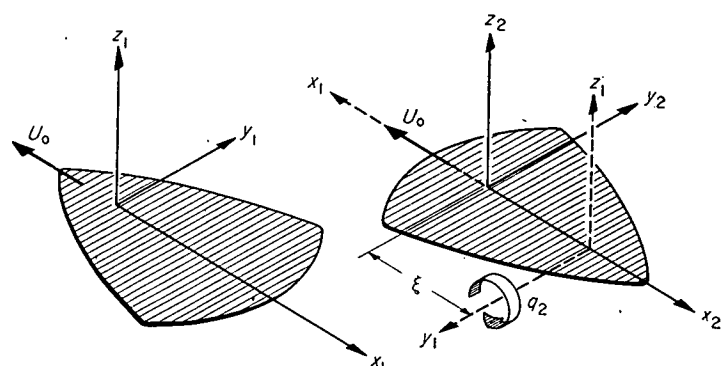


FIGURE 15.—Coordinate systems and symbols used in discussion of relation between pitching moment on arbitrarily cambered wings and the loading on pitching flat-plate wings.

The pitching moment of wing 1 is given by

$$M_1 = - \iint_{P_1} x_1 \Delta p_1 dS_1 = \iint_{P_1} \alpha_2 \left(\frac{\Delta p_1}{q_2/U_0} \right) dS_1 = \iint_{P_2} \alpha_1 \left(\frac{\Delta p_2}{q_2/U_0} \right) dS_2 \quad (76)$$

THEOREM: The pitching moment in steady or indicial motion of a wing having arbitrary twist and camber is equal to the integral over the plan form of the product of the local angle of attack and the loading per unit (q_2/U_0) at the corresponding point of a flat-plate wing of identical plan form in flight in the reverse direction and pitching about the moment axis of the first wing.

The necessity for pitching wing 2 about the moment axis of wing 1 may be removed by considering wing 2 to be re-expressed in terms of two component wings having angle-of-attack distributions given by

$$\alpha_{2'} = \frac{q_2(x_2' - x_{02}')}{U_0}, \quad \alpha_{2''} = \frac{q_2(x_{02}' - \xi)}{U_0} = \text{const.} \quad (77)$$

Wing 2' is thus pitching with angular velocity q_2 about an axis at $x_2' = x_{02}'$ and wing 2'' is a flat-plate wing at a constant angle of attack. The pitching moment on wing 1 is then given by

$$M_1 = \iint_{P_{2'}} \alpha_1 \left(\frac{\Delta p_{2'}}{q_2/U_0} \right) dS_{2'} + \iint_{P_{2''}} \alpha_1 \left(\frac{\Delta p_{2''}}{q_2/U_0} \right) dS_{2''} = \iint_{P_{2'}} \alpha_1 \left(\frac{\Delta p_{2'}}{q_2/U_0} \right) dS_{2'} + (x_{02}' - \xi) \iint_{P_{2''}} \alpha_1 \left(\frac{\Delta p_{2''}}{\alpha_{2''}} \right) dS_{2''} \quad (78)$$

Applications of pitching-moment theorem.—The application of equation (76) or (78) to problems analogous to those discussed in the preceding section can be carried out in a straightforward manner. Consider, first, unstaggered lattices on airfoils such that the airfoils in lattice 1 have arbitrary camber distributions and those in lattice 2 are flat plates pitching about their midchord positions. The angles of attack in the two lattice systems are

$$\alpha_1(x_1, z_1) = -\frac{dz_1}{dx_1}, \quad \alpha_2(x_2, z_2) = \frac{q_2 x_2}{U_0} \quad (79)$$

and the load distribution on each airfoil in lattice 2 is, in subsonic steady flow,

$$\Delta p_2(x_2) = q_0 \left(\frac{-4q_2 h}{\pi U_0} \right) \arccos \left[\text{sech} \left(\frac{\pi a}{\beta h} \right) \cosh \left(\frac{\pi x_2}{\beta h} \right) \right] \quad (80)$$

where $2a$ is chord length. Equation (76) yields, for pitching moment of the first airfoil about its midchord point, the result

$$M_1 = -\frac{4q_0 h}{\pi} \int_{-a}^a \frac{dz_1}{dx_1} \arccos \left[\text{sech} \left(\frac{\pi a}{\beta h} \right) \cosh \left(\frac{\pi x_1}{\beta h} \right) \right] dx_1 \quad (81)$$

A second example, illustrating unsteady effects, is the following: Let wing 1 be a flat-plate wing, then α_1 is constant, and equation (78) simplifies to

$$\frac{M_1}{\alpha_1} = \frac{L_{2'}}{q_2/U_0} + (x_{02}' - \xi) \frac{L_{2''}}{\alpha_{2''}} \quad (82)$$

where $L_{2'}$ is the lift on wing 2' pitching about $x_2' = x_{02}'$, and $L_{2''}$ is the lift on an inclined flat-plate wing. Equation (82) may be expressed in terms of conventional stability derivatives as follows:

$$(C_{m_a})_1 = (C_{L_a})_2 + \left(\frac{x_{02}' - \xi}{c_0} \right) (C_{L_a})_{2''} \quad (83)$$

An application of this result to unsteady-flow problems is indicated in figure 16 obtained from indicial-lift and pitching-moment results of reference 27. This figure shows the growth of lift and pitching moment on triangular wings with

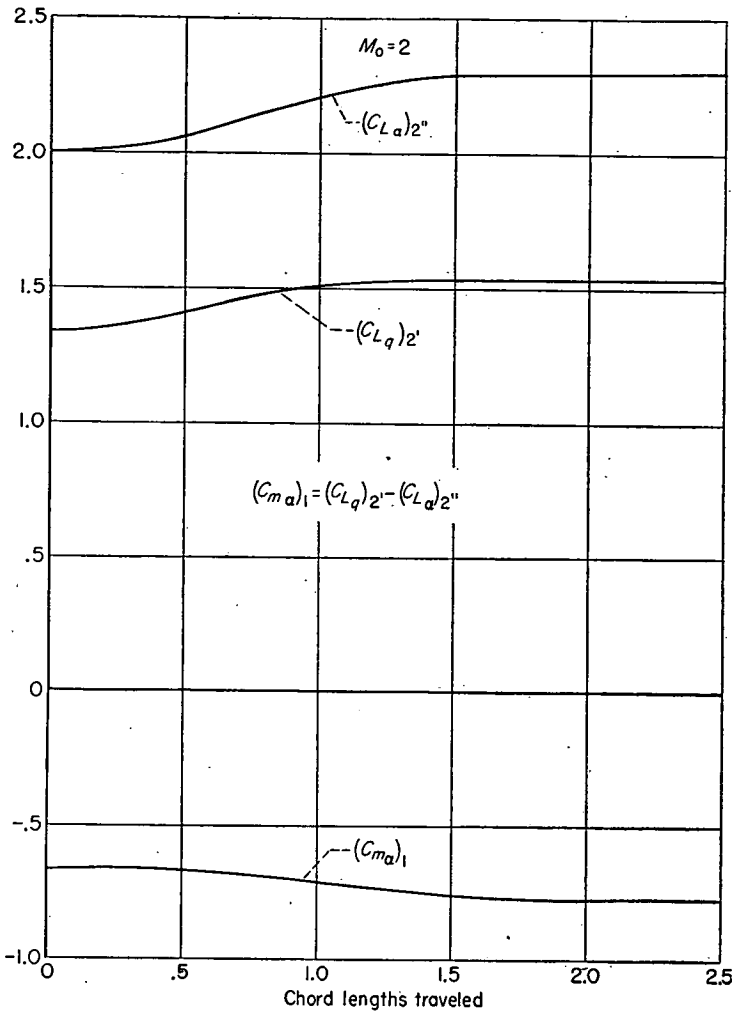


FIGURE 16.—Indicial lift and pitching moment on triangular wings with supersonic edges.

supersonic edges at a Mach number of 2 following indicial angle-of-attack and pitching-velocity changes. In these results, the rotation and moment axes are always at the leading edge or apex, therefore, $x_{02'} = 0$ and $\xi = c_0$. It may be seen that the three curves are related in the simple linear manner indicated by equation (83).

If α_1 is independent of x and varies only in the spanwise direction, that is, if $\alpha_1 = \alpha_1(y_1)$, the pitching moment on wing 1 is given by the following equation, analogous to equation (56) for lift:

$$M_1 = \int_{-s}^s \alpha_1 \left(\frac{l_{2'}}{q_2'/U_0} \right) dy_{2'} + (x_{02'} - \xi) \int_{-s}^s \alpha_1 \left(\frac{l_{2''}}{\alpha_{2''}} \right) dy_{2''} \quad (84)$$

If a portion P' of the surface of wing 1 is deflected a constant angle δ and the remainder of the wing is a flat plate aligned with the free-stream direction, the following relations hold:

$$\alpha_1 = \begin{cases} \delta & \text{on } P' \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\frac{M_1}{\delta} = \iint_{P_{2'}} \left(\frac{\Delta p_{2'}}{q_2'/U_0} \right) dS_{2'} + (x_{02'} - \xi) \iint_{P_{2''}} \left(\frac{\Delta p_{2''}}{\alpha_{2''}} \right) dS_{2''} \quad (85)$$

RECIPROCAL RELATIONS INVOLVING MOTION INTO A GUST

All previous applications that have been considered were derived from equation (24). In the present section, the more general equation (22) will be used to develop two theorems which relate the build-up of lift on a wing entering a gust and the build-up of circulation on the same wing moving indicially but in the opposite direction. The relations to be obtained hold for the Mach number range for which the wave equation applies. Under the special assumptions of incompressible flow, the results in two dimensions establish a direct connection between the circulation function calculated by Wagner (ref. 28) and the gust lift curve calculated by Küssner (ref. 29) and von Kármán and Sears (ref. 30). A proof of the connection between these functions for two-dimensional incompressible flow has been given by Sears (ref. 31).

TWO-DIMENSIONAL FLOW

A flat plate is assumed to be moving in two modes of motion: In the motion associated with the x_2, z_2, t_2 axes, the wing starts at time zero ($t_2 = 0$) and moves at a constant velocity U_0 and at a constant angle of attack; the motion associated with the x_1, z_1, t_1 axes starts at time zero ($t_1 = 0$) with the wing moving in the opposite direction at a velocity U_0 and entering a sharp-edged gust. The gust exists for all x_1 less than zero and has a vertical velocity $w_g = -\alpha_g U_0$. The two wings, therefore, have angles of attack as follows:

$$\alpha_2(x_2, t_2) = \alpha_2 = \text{const.} \quad \text{for } -M_0 t_2 < x_2 < 2a - M_0 t_2, \quad t_2 > 0$$

$$\alpha_1(x_1, t_1) = \alpha_g = \text{const.} \quad \text{for } \begin{cases} -M_0 t_1 < x_1 < 0 & 0 \leq t_1 \leq 2a/M_0 \\ -M_0 t_1 < x_1 < 2a - M_0 t_1 & 2a/M_0 \leq t_1 \end{cases}$$

The two-dimensional form of equation (22) yields

$$\alpha_2 \int_0^T dt_1 \int_{2a-M_0 t_1}^{-M_0 t_1} \Delta p_g(x_1, t_1) dx_1 = \rho_0 a_0 \alpha_g \iint_A \Delta \frac{\partial \varphi_2(x_2, t_2)}{\partial t_2} dS \quad (86)$$

where the region A is bounded by the lines $x_1 = 0$, $x_1 = 2a - M_0 t_1$, $t_1 = T$, and $x_1 = -M_0 t_1$. The integral on the right can be rewritten as a line integral by means of the identity

$$-\int P \cos(t, n) ds = \iint \frac{\partial P}{\partial t} dS$$

and equation (86) becomes

$$\alpha_2 \int_0^T L_g(t_1) dt_1 = \rho_0 a_0 \alpha_g \oint_C \Delta \varphi_2 \cos(t_2, n) ds$$

where the line integral extends around the boundary of the region A . Since $\Delta \varphi_2$ vanishes on the lines $x_2 = -M_0 t_2$ and $t_2 = 0$, the equation becomes

$$\alpha_2 \int_0^T L_g(t_1) dt_1 = \rho_0 a_0 \alpha_g \int_{2a-M_0 T}^{2a} \Delta \varphi_2 \left(x_2, \frac{2a-x_2}{M_0} \right) dx_2$$

Differentiation with respect to T yields

$$\alpha_2 L_g(T) = \rho_0 U_0 \alpha_g \Delta \varphi_2(2a - M_0 T, T)$$

The discontinuity in φ_2 is evaluated at the trailing edge at time T and is therefore equal to the circulation Γ_2 of the airfoil at time T . The equality thus becomes

$$\frac{L_g(T)}{\alpha_g} = \rho_0 U_0 \frac{\Gamma_2(T)}{\alpha_2} \quad (87)$$

THEOREM: The circulation per unit angle of attack of a flat plate moving indicially with a velocity U_0 is proportional to the lift per unit α_g of the plate entering a sharp-edged gust having a uniform vertical velocity equal to $w_g = -\alpha_g U_0$.

In figure 17, the time variation of these variables, as well as the lift of the indicial wing, is indicated for low speed and for flight Mach numbers equal to 0.8 and 1.46 as determined from references 22 and 27.

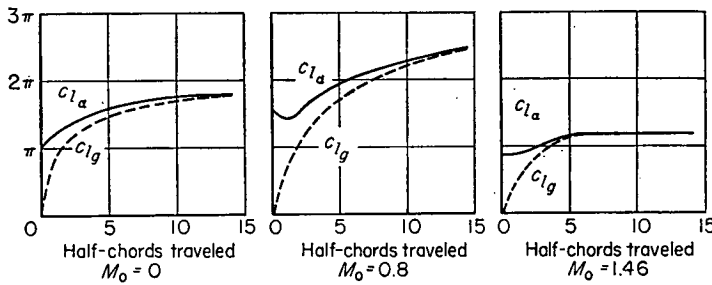


FIGURE 17.—Growth of $c_{l\alpha}$ and c_{lg} with chord lengths traveled.

THREE-DIMENSIONAL FLOW

The extension of the above results to three dimensions follows directly. The origin of the x_1, y_1, z_1, t_1 axes is assumed to be initially at the foremost point of the wing in reverse motion. The two wings have, respectively, angles of attack $\alpha_2 = \text{const.}$ over the reverse moving plan form for all values of time and $\alpha_g = \text{const.}$ over the region occupied simultaneously by the forward moving wing and the gust. Equation (22) gives

$$\alpha_2 \int_0^T dt_1 \iint_{P(t_1)} \Delta p_g(x_1, y_1, t_1) dx_1 dy_1 = \rho_0 a_0 \alpha_g \iiint \Delta \frac{\partial \varphi_2(x_2, y_2, t_2)}{\partial t_2} dx_2 dy_2 dt_2 \quad (88)$$

The integral on the right can be rewritten as a two-dimensional surface integral by means of the identity

$$-\iint P \cos(t, n) dS = \iiint \frac{\partial P}{\partial t} dx dy dt$$

and equation (88) becomes

$$\alpha_2 \int_0^T L_g(t_1) dt_1 = \rho_0 a_0 \alpha_g \iint \Delta \varphi_2 \cos(t_2, n) dS_2$$

where the integral on the right extends over the boundary of the volume in x_2, y_2, t_2 space occupied by the wing and the gust. The value of $\Delta \varphi_2$ must, of course, vanish on the leading edge of the wing and at $t_2 = 0$. In order to fix the limits of integration, suppose the wing is symmetrical about its longitudinal axis and let the leading edge of the forward

wing be given by the equation

$$x_1 = f(\pm y_1) - M_0 t_1 \quad \text{or} \quad y_1 = \pm s(x_1 + M_0 t_1)$$

For the reverse wing and its coordinate system, this edge, which is now the trailing edge, is

$$x_2 = c_r - M_0 t_2 - f(\pm y_2) \quad \text{or} \quad y_2 = \pm s(c_r - x_2 - M_0 t_2)$$

where c_r is the root chord. The reverse-flow integrals of equation (88) then become

$$\alpha_2 \int_0^T L_g(t_1) dt_1 = \rho_0 a_0 \alpha_g \int_{c_r - M_0 T}^{c_r} dx_2 \int_{-s(c_r - x_2)}^{s(c_r - x_2)} \Delta \varphi_2 \left[x_2, y_2, \frac{c_r - x_2 - f(\pm y_2)}{M_0} \right] dy_2$$

where $s(x_2)$ is the local half-span of the wing. Differentiation with respect to T yields

$$\alpha_2 L_g(T) = \rho_0 U_0 \alpha_g \int_{-s(M_0 T)}^{s(M_0 T)} \Delta \varphi_2 \left[c_r - M_0 T, y_2, T - \frac{f(\pm y_2)}{M_0} \right] dy_2 \quad (89)$$

The discontinuity in φ is thus to be integrated spanwise at the rearmost point of the indicial wing; this follows from the relation

$$\Delta \varphi_2 \left[x, y, T - \frac{f(\pm y_2)}{M_0} \right] = \Delta \varphi_2(x, y, t); \quad T - \frac{f(\pm y_2)}{M_0} < t < T$$

which fixes the vorticity in the wake of the wing once it is shed from the trailing edge.

It remains to mention the nature of the limits $\pm s(M_0 T)$. As shown in figure 18, the span width of the vortex wake at the trailing edge is, during the early stages of the motion, dependent on the local span width of the wing. The width $2s(M_0 T)$ of wake is, in fact, equal to the maximum width of the portion of the first wing that lies within the gust. After

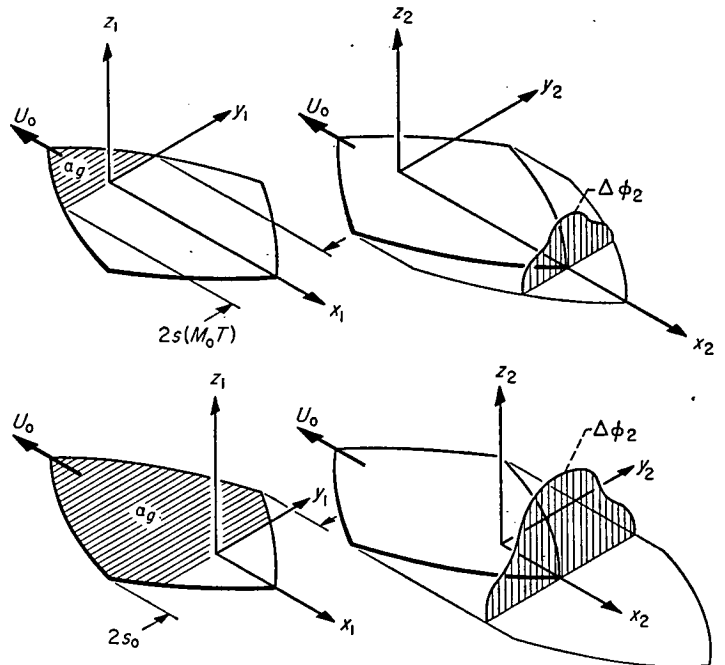


FIGURE 18.—Sketch illustrating nature of integration limits in equation (89).

sufficient time has passed for the vortex wake of the indicial wing to develop its full span width at the trailing edge, $s(M_0 T)$ becomes s_0 or semispan of the wing. From equation (89) one may conclude the following:

THEOREM: The lift per unit α_g of a flat-plate wing entering a sharp-edged gust having a uniform vertical velocity equal to $w_g = -\alpha_g U_0$ is proportional, at each instant of time, to the spanwise integral at the trailing edge of the vorticity shed by the same wing moving indicially in the reverse direction with a velocity U_0 .

As a direct example, this theorem has been used to confirm, from a knowledge of the indicial solution, the sharp-edged-gust lift of the rectangular-plan-form supersonic wing given by Miles in reference 32.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., Feb. 19, 1952.

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REPORT 1120

RELATIVE IMPORTANCE OF VARIOUS SOURCES OF DEFECT-PRODUCING HYDROGEN INTRODUCED INTO STEEL DURING APPLICATION OF VITREOUS COATINGS¹

By DWIGHT G. MOORE, MARY A. MASON, and WILLIAM N. HARRISON

SUMMARY

When porcelain enamels or vitreous-type ceramic coatings are applied to ferrous metals, there is believed to be an evolution of hydrogen gas both during and after the firing operation. At elevated temperatures rapid evolution may result in blistering while if hydrogen becomes trapped in the steel during the rapid cooling following the firing operation gas pressures may be generated at the coating-metal interface and flakes of the coating literally blown off the metal. This latter type of defect is known as fishscaling.

Although the behavior of hydrogen in the coating-steel system has received considerable study, the relative importance of the different possible sources of the hydrogen causing the defects has been principally a matter of conjecture. To determine experimentally the relative importance of the principal sources, a procedure was devised in which heavy hydrogen (deuterium) was substituted in turn for regular hydrogen in each of five possible hydrogen-producing operations in the coating process. The gas that was evolved when the coated steel specimens fishscaled after firing was collected and analyzed with the mass spectrometer. The content of the deuterium isotope in the total hydrogen gas evolved was then taken as a measure of the relative importance of the source under study.

The findings of the study were as follows:

(1) The principal source of the defect-producing hydrogen was the dissolved water present in the enamel frit that was incorporated into the coating. This water apparently reacts with the steel at elevated temperatures, releasing atomic hydrogen, some of which is dissolved by the steel. During fast cooling the steel becomes supersaturated with respect to hydrogen and the atomic hydrogen slowly diffuses to the interface where it collects in minute cavities or fissures, forming molecular hydrogen. This movement is irreversible and sufficient pressure builds up to cause fishscaling in the coating layer.

(2) The acid pickling, the milling water, the chemically combined water in the clay, and the quenching water were all minor sources of defect-producing hydrogen under the test conditions used.

Confirming experiments showed that fishscaling could be eliminated by using a water-free coating. Efforts to produce

a water-free frit, or glass, by using dehydrated raw materials in conjunction with electric-furnace smelting and water-free quenching showed that such a method has promise.

Other experiments indicated that reboiling (the blistering of a vitreous coating when reheated after the first firing) was also related to the dissolved water in the coating layer.

INTRODUCTION

Several investigations, especially over the past decade, have established that hydrogen is a major cause of coating defects when porcelain enamels are applied to a steel base (references 1 to 3). There have, however, been considerable uncertainty and difference of opinion as to the major source or sources of the defect-producing hydrogen.

The purpose of the present investigation was to study the possible sources and to evaluate their relative importance. With such information available it was believed likely that methods or procedures could be devised whereby coatings free of hydrogen defects² might consistently be obtained. It was also believed likely that the results obtained with porcelain enamel on steel would be applicable when vitreous-type ceramic coatings are applied to steel and to the so-called low-strategic alloys. Because of this connection with ceramic coatings and the growing importance of ceramic coatings to the aircraft industry, the investigation as herein described was performed as a part of a broad study on ceramic coatings being conducted at the National Bureau of Standards under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

The authors gratefully acknowledge the valuable assistance of the Mass Spectrometry Section of the National Bureau of Standards, under the direction of Dr. F. L. Mohler, for performing the mass-spectrometer analyses.

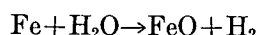
POSSIBLE SOURCES OF HYDROGEN IN ENAMELING PROCESS

Some hydrogen is usually present in the steel as received and additional hydrogen may be introduced at several stages of the enameling process. The first and most obvious source is the acid-pickling operation. During this treatment the acid reacts with the steel and part of the hydrogen thus formed may be readily occluded by the metal (reference 4).

¹ Supersedes NACA TN 2617, "Relative Importance of Various Sources of Defect-Producing Hydrogen Introduced into Steel During Application of Vitreous Coatings" by Dwight G. Moore, Mary A. Mason, and William N. Harrison, 1952.

² Coating defects that have been ascribed to hydrogen include fishscaling, primary boiling, reboiling, and blistering. Fishscaling is a localized fracture of the coating glass caused by excessive gas pressure at the coating-metal interface. Primary boiling is the rapid bubbling of the enamel layer during the early stages of firing caused by the escape of entrapped and evolved gases, while reboiling is the blistering at about 1050° F of a vitreous coating when it is reheated after the first firing.

The second possible source of hydrogen is a reaction of iron in the steel with any water that may be present in the enamel layer during the firing operation. When iron is heated to enameling temperature in the presence of steam the following reaction takes place (reference 5):



This reaction would proceed as shown if the hydrogen were continuously removed from the system. Such removal could occur in the enamel-metal system either by escape of the hydrogen through the enamel layer or by solution into the steel. If it passes into the steel, it then becomes capable of producing enamel defects.

The water that might come in contact with the hot steel during the firing operation could conceivably originate from one or more of the following sources:

- (1) The water vapor that might be present in the furnace during firing
- (2) The water used for milling and suspending the enamel slip
- (3) The chemically combined water in the clay used for suspending the enamel slip
- (4) The dissolved water in the frit

The water dissolved in the frit could, in turn, be derived from: (a) The water present in the batch materials, (b) the water present in the atmosphere of the smelting furnace, and (c) the water used in quenching the molten glass to form the small friable frit particles.

Some attention was given to all of the above-mentioned sources, but the effect of the water vapor present in the atmospheres of the smelting and enameling furnaces and the effect of water present in the raw batch materials used for preparing the frit were not studied by tracer techniques.

USE OF DEUTERIUM (HEAVY HYDROGEN) AS A TRACER

Deuterium is an isotope of hydrogen. It differs from protium (ordinary hydrogen) in that its nucleus contains a neutron as well as a proton whereas protium contains a proton only. Because of the difference in mass (deuterium, 2; protium, 1) the two isotopes can be readily separated with the mass spectrometer and, by using the proper techniques, the relative amounts of each that are present in any given sample can be determined quantitatively. Deuterium and protium have almost identical chemical properties and their physical properties are similar. Thus, deuterium can be used as a tracer in many reactions. The solubility of deuterium in iron has been determined by Sieverts, Zapf, and Moritz (reference 6). These data are plotted in figure 1.

In the present study, deuterium in the form of heavy water (D_2O) was substituted, in turn, for ordinary water (H_2O) in each of five different processes that are related to enameling operations. The gases given off when the resulting coated specimens fishscaled were collected and analyzed with the mass spectrometer. The percentage of deuterium in the hydrogen gas was then computed and this figure was taken as a measure of the importance of that particular source in producing hydrogen defects in the enamel coating.

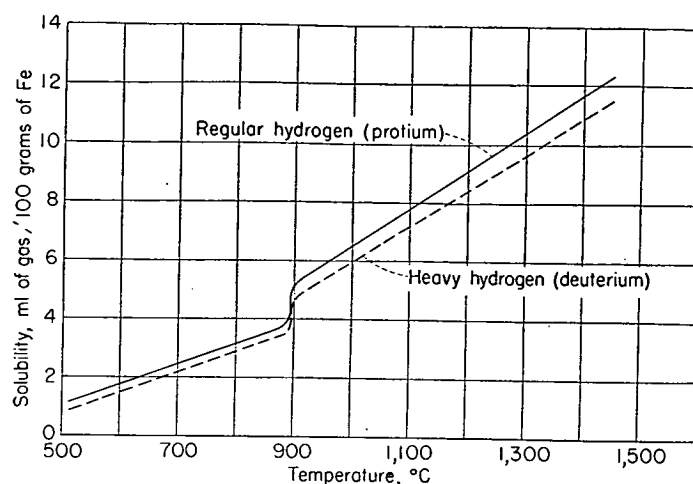


FIGURE 1.—Solubility-temperature curves for regular hydrogen and heavy hydrogen in ingot iron at atmospheric pressure as determined by Sieverts, Zapf, and Moritz (reference 6).

The methods used for substituting heavy water for regular water in the various operations are given in the section entitled "Methods of Introducing Heavy Water." The procedure used for collection of the gases during fishscaling and their subsequent analyses are given in the section "Test Equipment and Procedure."

MATERIALS

The deuterium oxide, which was obtained from a commercial source, contained 99.8 percent heavy water and 0.2 percent ordinary water. It was received in 100-gram sealed flasks.

Two steels were used in the principal part of the study. Steel A was 18-gage enameling iron while steel B was a 10-gage low-carbon steel that had given considerable trouble from delayed fishscaling in one enameling plant. In addition a titanium-bearing, low-carbon, 18-gage steel was used in a few experiments. This latter steel (steel C) is purported to have low susceptibility to hydrogen and it has been demonstrated (reference 7) that it tends to minimize defects when used as a base for application of porcelain enamels.

The chemical analyses of the three steels are given in table 1 and the determined gas contents before the enameling treatment are presented in table 2. Table 3 gives the batch and oxide compositions of the ground-coat enamel frit (109-0) used for most of the experiments. This frit was selected from an earlier study (reference 8) as being suitable for the preparation of a single-frit enamel ground coat. The 109-0 frit is the same as the frit E described in reference 8 except that the 109-0 contains no adherence oxides. The clay used in preparing the ground-coat enamel slips was a commercial grade of Florida kaolin.

TEST EQUIPMENT AND PROCEDURE

A schematic drawing of the collection cell, with heater, is shown in figure 2. Two cells of this type were used, one having a volume at 20° C of 152.3 cubic centimeters and the other, 173.3 cubic centimeters.

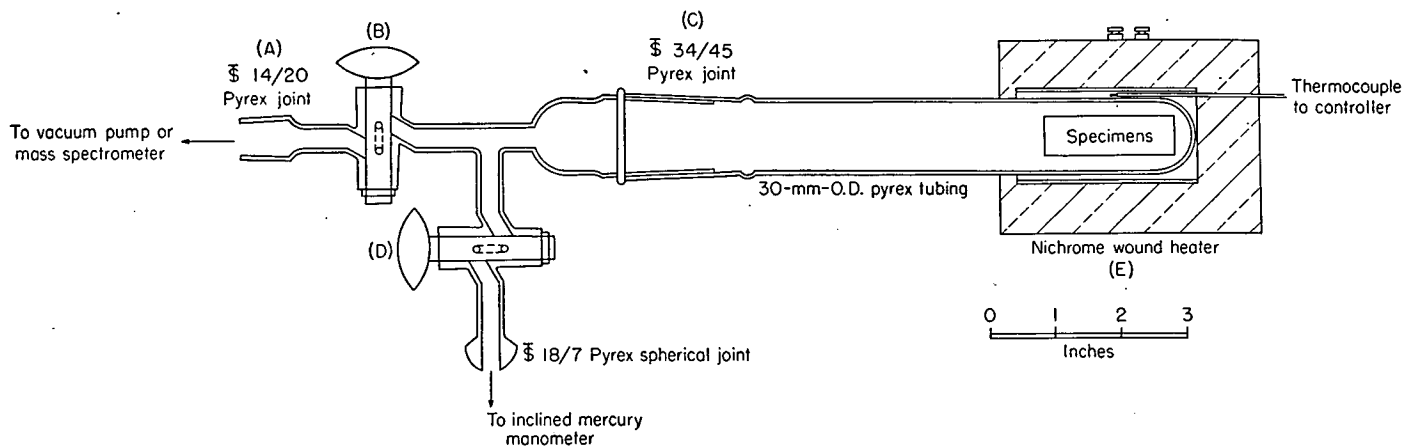


FIGURE 2.—Schematic drawing of collection cell.

The mass spectrometer was of the type described by Washburn, Wiley, Rock, and Berry (reference 9).

A standardized test procedure was used in each experiment for preparation of specimens, collection of gases given off during fishscaling, and analysis of the gas sample with the mass spectrometer. This procedure may be summarized as follows:

Before beginning a given experiment, 10 steel specimens, 2 inches by $\frac{3}{4}$ inch, were selected at random from a large supply that had been sheared from the same sheet. (In the case of specimens of steel B, which were of greater thickness, five specimens were used.) These specimens were degreased in an organic solvent and then pickled for 10 minutes in 5 percent by weight of sulfuric acid maintained at a temperature between 156° and 159° F. After removal from the acid the specimens were rinsed in hot tap water for 1 minute and then dried at 212° F for 15 minutes. Immediately after cooling to room temperature the specimens were coated by dipping in a slip (less than 48 hr old) of the E-1 coating that contained:

Constituent	Content (grams)
Frit 109-0.....	200
Florida kaolin.....	10
Water.....	85

and that was prepared in the following manner:

Milling time in 1-quart jar mill, hr.....	1½
Milling fineness, percent on 200-mesh sieve.....	6-8
Application thickness, mils.....	4-6

The specimens after dipping were dried at 230° F for 15 minutes, then fired for 4 minutes at 1550° F (4½ min for the heavier specimens of steel B). As soon as the specimens had cooled sufficiently for handling they were removed from the firing rack and quickly placed in the collection cell shown in figure 2. Joint C (see fig. 2) was sealed and the pressure in the cell was reduced by connecting through joint A to an 18-inch-diameter bell-jar reservoir which in turn was connected to an oil diffusion pump. The pressure in the bell jar before connecting in the cell was about 1×10^{-5} millimeter of mercury. After 2 minutes stopcock

B was closed at which time the pressure in the collection cell had usually dropped to about 0.2 millimeter of mercury. The heater E was then energized and the temperature of the specimens raised to 338° F (170° C) in approximately 30 minutes and maintained at this value within $\pm 4^\circ$ F for 17 hours. This temperature was the same as that used by Darken and Smith (reference 10) for removal of hydrogen from steel. After cooling the system to room temperature a final pressure measurement was recorded, stopcock D was closed, and the inclined mercury manometer was disconnected from the system. The collection cell with the specimens still in position then served as a sample tube for the mass spectrometer.

In the mass-spectrometer analysis the collection cell was first connected into the analyzer through joint A. When the pressure in the analyzer had reached approximately 1×10^{-5} millimeter of mercury, stopcock B was opened. The analysis of the various gases was then made by recognized procedures. The final result showed the content of each gas present in the sample in mole percent. From these data the mole percent of deuterium in the total hydrogen gas was computed.

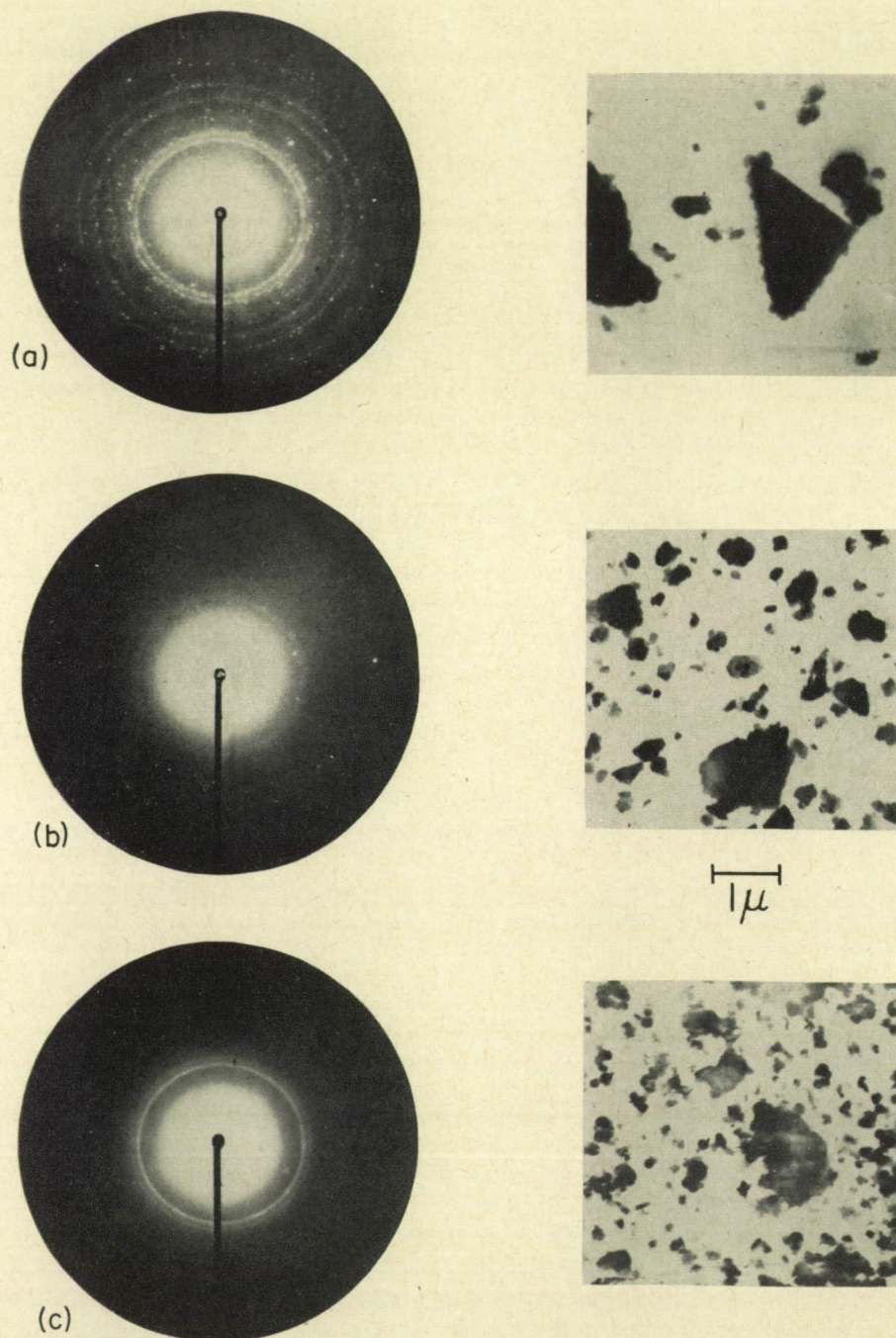
The volume of hydrogen gas at standard conditions per 100 grams of steel was computed from the following equation:

$$V = \left(\frac{Hv}{W} \right) \left(\frac{273}{760} \right) \left(\frac{P_2}{T_2} - \frac{P_1}{T_1} \right)$$

where W is the weight of the steel, H is the percent by volume of hydrogen in the gas, v is the volume of the cell not occupied by the specimens, P_2 is the final pressure in millimeters of mercury, P_1 is the initial pressure in millimeters of mercury, T_2 is the final absolute temperature, and T_1 is the initial absolute temperature.

METHODS OF INTRODUCING HEAVY WATER

In obtaining the data showing the relative importance of the several sources of hydrogen, the pickling treatment, the application of the enamel, the collection of gases after enameling, and the analysis of the gases with the mass spectrometer were all carried out in accordance with the



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(a) No treatment.
(b) After heating at 1000° F for 48 hours.
(c) After autoclaving dehydrated material for 300 hours at 250 pounds per square inch with heavy water.
FIGURE 3.—Electron diffraction patterns and electron micrographs of Florida kaolin.

standard procedures previously outlined. The only variation in these procedures came, for any given experiment, at that point in the process where it was necessary to substitute heavy water for regular water in order to evaluate the relative importance of a particular source. The following is a summary of the methods used to accomplish the substitutions. Only one substitution of heavy water for ordinary water was used in any experiment.

Acid pickling.—Heavy water was substituted for regular water in the 5-percent-sulfuric-acid solution. Deuterium sulfate, however, was not substituted for the sulfuric acid.³

Milling water.—Heavy water was substituted for the ordinary water used in preparing the enamel slip. The resulting slip was used within 48 hours after milling.

Quenching water.—A standard batch of the 109-0 frit was smelted by normal procedures in a small pot furnace. Part of this batch was quenched in regular water while a small part was quenched in heavy water. The part quenched in heavy water was then used in preparing the enamel slip.

Water in clay.—Only a few references could be found in the literature relative to methods for rehydrating clay from which the chemically combined water has been removed by heating. Most investigators who have accomplished the rehydration have used steam pressures of the order of 75 atmospheres. Schachtschabel (reference 11), however, reported that clay could be rehydrated if held at low pressures for long periods and this was the method adopted. The following procedure was used:

The Florida kaolin was first heated in shallow refractory boxes for 24 hours at 1000° F. This dehydrated clay was placed in platinum dishes and covered with heavy water. The dishes were then placed in an autoclave which contained additional heavy water and heated at 250 pounds per square inch (406° F) for 300 hours. The clay samples (a) before treatment, (b) after heating, and (c) after autoclaving were all examined by X-ray diffraction, by electron diffraction, and with the electron microscope. Weight-loss determinations were also made and these showed that the autoclave treatment accomplished an 80-percent rehydration of the clay molecule.⁴

Figure 3 shows the electron diffraction patterns and the electron micrographs. The electron diffraction patterns of untreated kaolin indicated a normal kaolinite structure while the heated sample gave a pattern typical of a dehydrated kaolinite. The rehydrated or deuterated clay, after removal from the autoclave, gave the normal kaolinite pattern but apparently had a smaller crystal size than the original material. This particular clay was unusual for a kaolin in that it showed small and poorly defined kaolinite crystals when examined with the electron microscope. A careful examination indicated practically no alteration in the appearance of the kaolinite fragments resulting from the

dehydration. Thus, the dehydrated crystals may be considered as pseudomorphs of the original kaolinite. This observation is in keeping with the findings of Eitel, Muller, and Radczewski (reference 12).

When the Florida kaolin containing the heavy water was used in the preparation of the mill batch, the amount added was increased by the ratio 100:80 to compensate for the lower water content of the rehydrated clay.

Dissolved water in frit.—Although it is well-known that the dissolved water in glasses may be removed by melting in vacuum, insofar as could be determined, no one has attempted to resaturate a glass after removal of the water. The method devised by the present authors was as follows:

A 300-gram quantity of the normally smelted 109-0 frit was placed in a clay crucible which in turn was placed in the furnace shown by the schematic drawing in figure 4. A vacuum of 4 millimeters of mercury was applied and the crucible heated to 1850° F in 20 minutes. After 30 minutes in the temperature range of 1850° to 1875° F, the power was shut off and the furnace allowed to cool to room temperature.

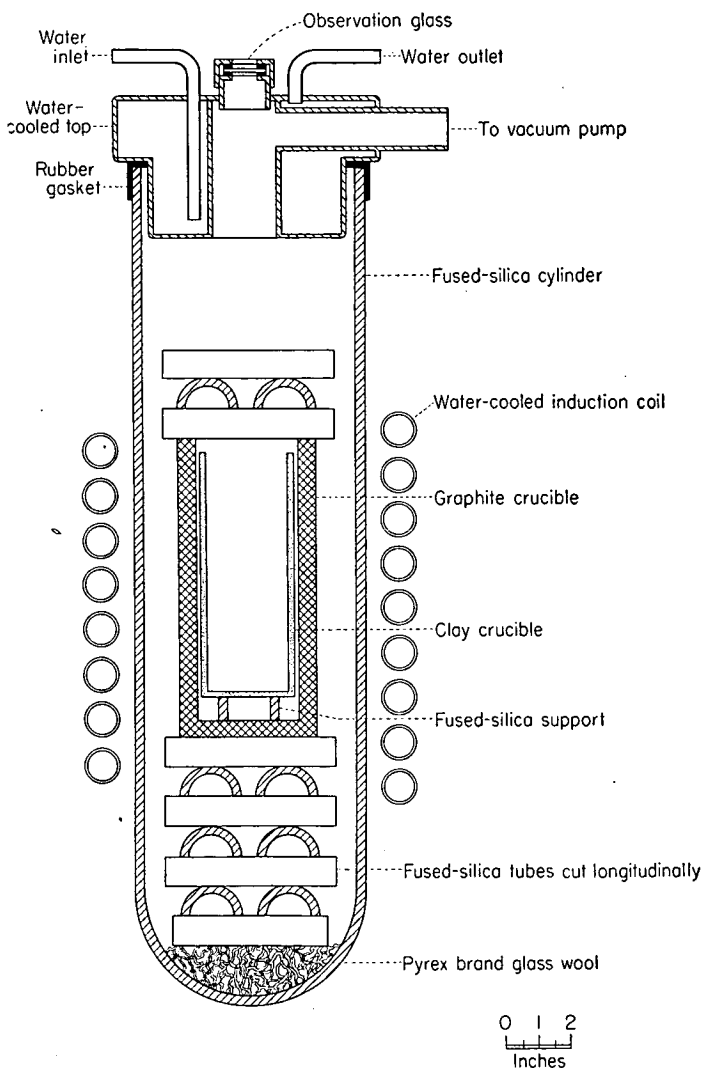


FIGURE 4.—Schematic drawing of vacuum melting furnace used for removing dissolved water from normally prepared frit.

³ Steel B, after the standard pickling treatment in the acid-heavy-water solution, was found to contain occluded hydrogen which consisted of 78 mole percent deuterium and 22 mole percent protium.

⁴ The heavy water content of the treated clay was obtained through analysis of a gas sample which was collected by heating the clay to 1000° F in a partial vacuum. The analysis showed that the water fraction contained 36.3 mole percent heavy water and 63.7 mole percent regular water.

The vacuum was then released. After removal of the crucible, the frit was cracked away from the crucible and pulverized. This pulverized material was placed in platinum dishes and covered with heavy water. The dishes were then placed in an autoclave with additional heavy water and heated for 3 hours at 406° F (250 lb/sq in.). The material was then dried for several hours at 230° F after which it was quickly charged into a clay crucible maintained at 2150° F in an electrically heated furnace. After 15 minutes' heating, the crucible was removed from the furnace to cool in air. The resulting frit, which was practically bubble-free, was separated from the crucible material, crushed to pass through a 16-mesh sieve, and then used as the frit ingredient in the E-1 mill batch.

RESULTS

Table 4 gives the results of several blank determinations. Table 5 lists the total volume of hydrogen collected in each experiment per 100 grams of steel, the mole percent of deuterium in the hydrogen gas collected, and the relative importance of each source.

Figure 5 is a chart showing the mole percent of deuterium in the hydrogen gas collected from each of the several experiments. The values used in preparing this chart are the averages of the individual values for steel A and steel B.

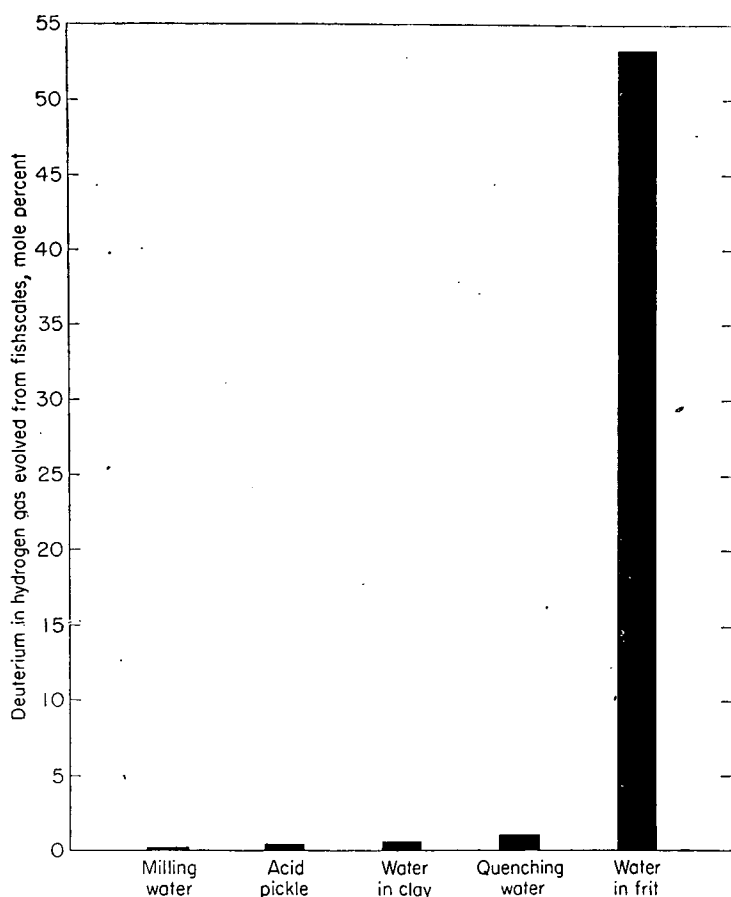


FIGURE 5.—Mole percent of deuterium in hydrogen (protium plus deuterium) evolved from ground-coated steel specimens. Experiments were designed so that heights of columns indicate relative importance of listed coating processes as sources of defect-producing hydrogen in steel.

CONFIRMING EXPERIMENTS

FISHSCALING AND LIFTING

The results with deuterium as a tracer showed that the dissolved water in the frit was by far the most important source of defect-producing hydrogen. For confirmation of this finding, a second means of evaluating the effect of the dissolved water was believed desirable. The method used was to apply the 109-0 frit to steel specimens (a) in the normally prepared condition and (b) with the dissolved water entirely or mostly removed. After firing, the coated specimens were examined for lifts⁵ and fishscales.

It should be emphasized that in these experiments the frit as such was applied without the use of the customary clay, water, and other mill additions. This was accomplished by first dry grinding the frit and then suspending the resulting powder in water-free Ethyl Cellosolve. The suspension was applied by spraying. The specimens, after drying, were fired for the normal time (4 min for steels A and C and 4½ min for steel B) at 1550° F.

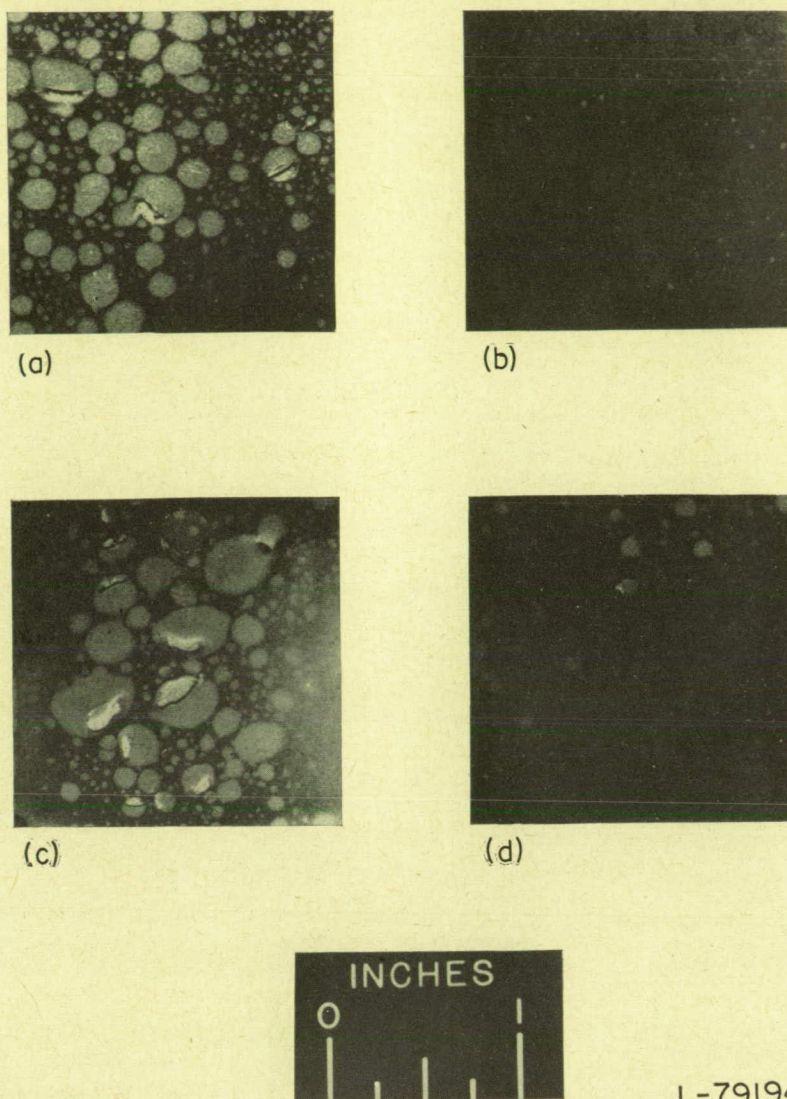
When the frit as normally prepared was applied to specimens of steels A and B by this procedure pronounced lifting and fishscaling resulted. Figure 6(a) shows the appearance of the surface for steel A while figure 7(a) represents the appearance for steel B. Specimens of steel C showed no lifting or fishscaling when this same frit was applied.

Figures 6(b) and 7(b) represent specimens of steels A and B, respectively, coated with frit 109-0 that had been vacuum-melted to removed dissolved water. It is significant that no lifting or fishscaling is evident in either steel. Inasmuch as pronounced lifting and fishscaling occurred when the normally prepared frit was applied, whereas neither of these hydrogen defects appeared when the dissolved water was removed from the frit by melting in vacuum, it seems safe to conclude that the dissolved water in the frit was responsible for the defects in figures 6(a) and 7(a). This conclusion is further substantiated by figures 6(c) and 7(c). In this case, the 109-0 frit that had been melted in vacuum was rehydrated with heavy water⁶ by the method previously outlined. The frit after this treatment again contained dissolved water and once again the pronounced lifting and fishscaling occurred.

In other experiments, an attempt was made to prepare the 109-0 frit in such a manner that it would initially contain little or no dissolved water. The raw batch materials were first weighed out and mixed and then slowly heated in a clay crucible to 500° F. After 20 hours at this temperature, which was believed sufficient to remove practically all of the water that was initially present in the raw materials, the mixture was charged without cooling into a crucible maintained at 2150° F in an electrically heated furnace.

⁵ In this paper a lift refers to a visible separation of the enamel layer from the underlying metal. These lifts (light gray patches in figs. 6 and 7) are gas pockets of hydrogen that form between the steel and the coating layer. As more gas comes out of the steel with continued aging the pressure in the pocket increases until the point is reached where the enamel fractures. Other investigators (references 1 to 3) have demonstrated that the gas coming from the fracture, or fishscale, is mostly hydrogen.

⁶ The reason frit rehydrated with heavy water was used was because such frit was already available from earlier experiments.



- (a) Normally prepared frit.
 (b) Same frit as shown in (a) after vacuum melting to remove dissolved water.
 (c) Vacuum-melted frit that was subsequently autoclaved with heavy water and then remelted.
 (d) Same frit as shown in (a) except prepared from dehydrated raw materials and melted in electric rather than gas-fired furnace.

FIGURE 6.—Specimens of enameling iron (steel A) coated with a ground-coat frit containing no adherence oxides. Light areas are separations at interface caused by hydrogen evolution.

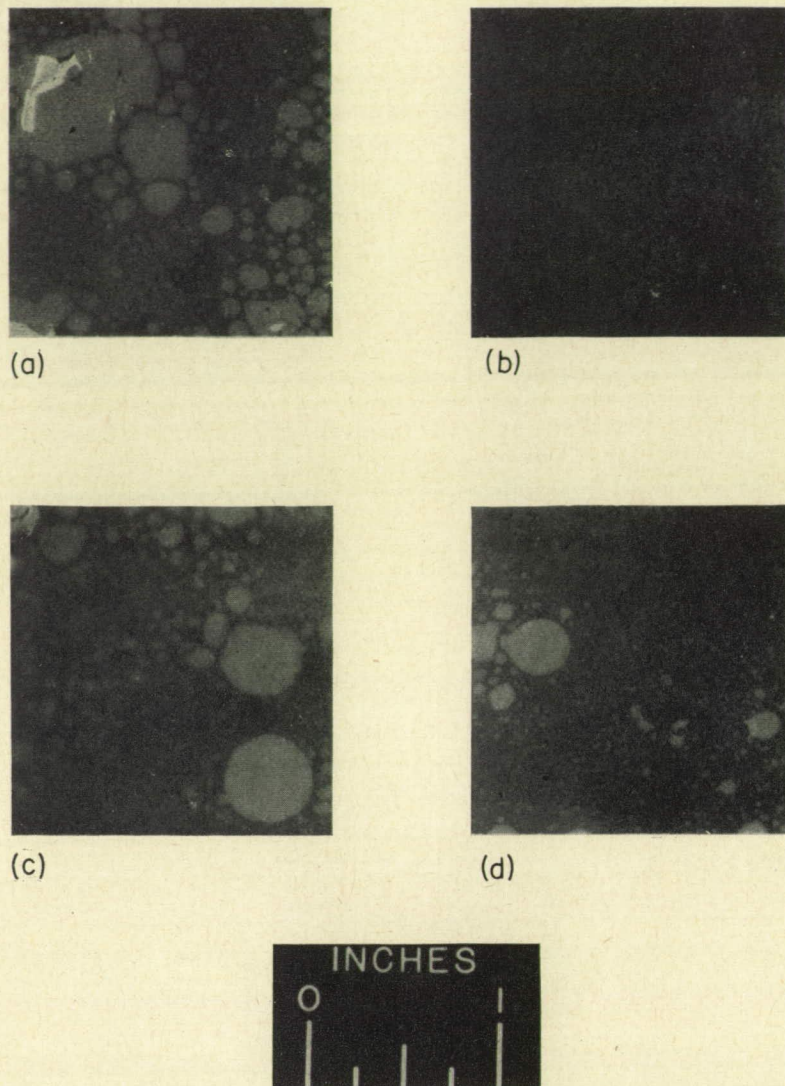
After a normal smelt, part of the batch was quenched on a heat-resistant alloy and the remainder was quenched in water. When the resulting frit was applied to the two steels, moderate lifting and fishscaling resulted. There was only a small difference between the frit quenched in water and the frit quenched on the heat-resisting alloy, the former showing a somewhat greater number of defects. Specimens prepared from frit quenched in water are illustrated in figures 6 (d) and 7 (d). It will be noted that lifting and fishscaling are not so common as on the specimens prepared from the normally smelted frit (figs. 6 (a) and 7 (a)). These results indicated that the special smelting treatment resulted in a frit of reduced water content. On the other hand, it is obvious from the photographs that the special

smelting treatment was not so effective in reducing the water content of the frit as was remelting in vacuum.

REBOILING

In the reboiling experiments frit 109-0 was prepared according to the formulation given in table 3 except that adherence oxides were added. After smelting, the frit contained 0.60 percent cobalt oxide and 1.94 percent manganese dioxide. Approximately 300 grams of this frit was then melted in vacuum with the same equipment and procedure as described earlier.

The resulting frit was dry-ground and applied to specimens of steels A and B by spraying from a Cellosolve suspension. After normal firing at 1550° F the specimens



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- (a) Normally prepared frit.
 (b) Same frit as shown in (a) after vacuum melting to remove dissolved water.
 (c) Vacuum-melted frit that was subsequently autoclaved with heavy water and then remelted.
 (d) Same frit as shown in (a) except prepared from dehydrated raw materials and melted in electric rather than gas-fired furnace.

FIGURE 7.—Specimens of low-carbon fishscaling steel B coated with a ground-coat frit containing no adherence oxides. Light areas are separations at interface caused by hydrogen evolution.

were placed over an oxygas flame and small areas heated beyond the reboil temperature (about 1050° F). Careful observation showed no reboil on either steel with this vacuum-melted frit while the untreated frit applied by the same procedure showed considerable reboil on steel B and slight reboil on steel A. When the vacuum-melted frit was prepared as a slip with clay and water and applied to the specimens, steel B showed slight reboil while steel A showed none.

DISCUSSION OF RESULTS

In the present study most attention was given to fish-scales. It should not be inferred from this fact that the authors necessarily consider the other phenomena caused by hydrogen to be of lesser importance but rather that the

experimental techniques were more adaptable to the fish-scale type of defect. If reboiling rather than fishscaling had been used as a tool to determine the relative importance of the various sources of hydrogen, it is believed that the results would have been substantially the same. That such may be the case was demonstrated by the reboil experiments.

The fishscaling tests with deuterium used as a tracer showed the dissolved water in the frit to be the principal source of hydrogen evolved from the coated steel. According to this finding, avoidance of the dissolved water in the frit should eliminate most of the hydrogen from the system and thus make it possible to prepare specimens that would not reboil. Tests described in this paper showed this to be the case.

The method devised for collecting the gas given off during fishscaling gave samples that were diluted both with residual air and with gases released by the enamel and by the Pyrex glass surfaces of the collection cell. In table 4 only the evolved gases are listed; nitrogen and oxygen were assumed to have originated from residual air and are not included in the analyses. The table shows that the absorbed gas given off at 338° F by the glass surfaces of the collection cell was very small. The gas consisted mostly of water. With chips of coating material in the cell, water vapor was again the principal gas but the carbon-dioxide content increased. A trace of hydrogen was also present.

In the tracer experiments, summarized in table 5, residual air and absorbed gases were also present to dilute the sample. The important data in these tests, however, were the values for the mole ratio of deuterium to protium and this ratio was not affected by the dilution.

The finding that the dissolved water in the frit is the major source of hydrogen is in keeping with the belief of Hoff and Klarding (reference 13). It also agrees with what might be predicted from the work of Miller and Sweo (reference 14), who experimented with fused masses of iron powder and porcelain enamel constituents. These authors found that partial removal of the dissolved water from the frit by heating in dry nitrogen to 350° C (662° F) decreased the gas (hydrogen) entrapped in the fused mass. The same authors also reported that the water removed by the dry nitrogen treatment was 0.30 percent by weight. This is equivalent to 372 cubic centimeters of water vapor at standard conditions per 100 grams of frit. Dalton (references 15 and 16) reports a maximum of 94 cubic centimeters per 100 grams for borosilicate glasses as determined by the vacuum fusion method, while the maximum reported by Hahner, Voight, and Finn (reference 17) for borosilicate optical glasses was 49 cubic centimeters per 100 grams of glass.

Dalton (reference 15) states that borosilicate glasses as a class contain more water than nonborosilicates. Ground-coat enamels are normally higher in boron oxide than borosilicate glasses; hence it is conceivable that ground-coat enamels could contain appreciable amounts of water. The data of Miller and Sweo (reference 14) point in this direction.

Calculations show that the total amount of enamel on the 10 specimens of steel A during any one experiment averaged about 6.5 grams. Assuming the water content of the enamel to be 0.3 percent by weight, 24 cubic centimeters of hydrogen could be formed if all of the dissolved water reacted with the steel. The fact that the maximum amount recovered in any of the tests was 0.8 cubic centimeter indicates the possibility that only a small part of the dissolved water reacts with the steel during a normal firing. Additional reaction could occur on successive firings and during each of these firings the steel could become recharged with hydrogen. Thus, reboiling might be expected to occur after each successive firing until the water content of the enamel had been depleted by reaction with the steel. King (reference 18) found that after 23 to 25 heatings no further reboil took

place. He also found if a ground-coated specimen is fired in vacuum it will not reboil in any subsequent firing. These results as well as the reboil experiments reported in this paper indicate that the dissolved water in the enamel is the major source of the hydrogen responsible for reboil.

Determination of the water content of the various frits was beyond the scope of this investigation. It was hoped that the vacuum melting would remove practically all of the original water from the 109-0 frit and that the autoclave treatment followed by rapid melting would successfully restore a comparable number of moles of deuterium oxide. Figures 6 and 7 show a comparable degree of lifting and fishscaling for the frit that had been rehydrated after vacuum melting and for the normal 109-0 frit. This would indicate that the water contents of the two frits were of the same general order of magnitude but gives no indication whether the regular water had been completely replaced with heavy water. That such a replacement may not have been accomplished is indicated by the data listed in table 5. These data show the pickling treatment, the milling water, and the clay to be very minor sources of hydrogen and of little significance when compared with the dissolved water in the frit. If this is the true picture then, when a frit was used in which regular water was replaced by heavy water, the resulting gas sample should have shown the hydrogen to consist almost entirely of the deuterium isotope. Actually, in the two experiments performed with the deuterated frit the mole percent of deuterium was 53.5 and 52.9. These results imply that the water in the frit had not been completely replaced with deuterium oxide.

To investigate whether such was the case, a sample of the deuterated frit was melted in vacuum in a closed system that had previously been degassed. A sample of the resulting gas was then reduced by use of a hot tungsten filament. This reduction was necessary because the mass spectrometer does not readily separate the two isotopes when they are present in the form of water. After the reduction only traces of water remained, the sample consisting, for the most part, of protium, deuterium, and carbon monoxide. The mass-spectrometer analysis when converted back to water, heavy water, and carbon dioxide showed the following mole percentages of these constituents in the deuterated frit:

H ₂ O-----	46.3
D ₂ O-----	37.7
CO ₂ -----	16.0

One explanation for this unexpectedly high content of regular water in the deuterated frit is the possible presence of residual water in the frit after the vacuum melting. Figures 6 and 7 show that the 109-0 frit when melted in vacuum did not cause lifting or fishscaling when applied to steels A and B. This implies a large reduction in the dissolved water content but does not necessarily prove that all of the dissolved water had been removed. There may be a critical value below which the water or hydroxyl group is so

tightly bound in the glass structure that no reaction will occur with the steel during the firing process. When deuterium oxide was introduced, the water content as such increased and the deuterium oxide became diluted by the residual water present in the sample.

Regardless of the contamination of deuterium oxide with residual water, the experiments showed the predominant importance of the dissolved water in the frit as compared with the other sources of defect-producing hydrogen that were investigated. In view of recent work on hydrogen in steel, this finding is not surprising. Darken and Smith (reference 10) found that heating at 170° C (338° F) removed 90 percent of the hydrogen that had been introduced into the steel during acid pickling. Thus, practically all of the hydrogen initially present in the steel, as well as that introduced during cleaning operations, would be expected to leave the steel before the enamel fuses over. Hydrogen introduced at firing temperatures, on the other hand, has a tendency to remain in the steel. Smith (reference 4) states that there is a possibility that metal which has been saturated at high temperatures may retain extra hydrogen, over and above the saturation limit (see fig. 1), if it has been rapidly cooled. Smith also states that this seems to occur in the case of iron.

A retention of this type is in keeping with the findings of Davis, Keeler, and Chu (reference 19). These authors found that when a ground-coated specimen was cooled slowly by any of several prescribed schedules no lifting or fishscaling resulted. When coated specimens are cooled slowly, they believe that the hydrogen comes out of the steel to react with iron oxide at the interface forming water and reduced iron and that the water in turn is dissolved by the enamel glass. They infer that this reduction reaction and the solution of the resulting water into the glass structure may occur at temperatures as low as 392° F. The results of the present study with enamel applied to the nonfish-scaling titanium-bearing steel indicate that such a hypothesis is unnecessary inasmuch as table 4 shows that hydrogen is capable of diffusing through the glass layer at temperatures as low as 338° F. An additional test was made in which coated specimens of steel B were placed in the collection cell while still hot. Practically no lifting or fishscaling resulted, yet a normal quantity of hydrogen was collected, again demonstrating that hydrogen is capable of diffusing through the glass under these conditions. Reports in the literature that hydrogen will not diffuse through glass in any significant quantity at such low temperatures (reference 20) refer to molecular hydrogen. It should be pointed out, however, that, in the enamel-steel system, the hydrogen diffusing through the enamel layer might be in the atomic or possibly even the ionic state.

Several authors (references 14 and 19) have ascribed considerable importance to the chemically combined water in the clay used for suspending the enamel slip. The present work indicates that the clay is only a minor source of hydrogen. The chemically combined water is released by the clay molecule at temperatures as low as 1000° F and, if the enamel is applied at a normal thickness, the water may be

mostly expelled before the enamel fuses. Before fusion the enamel structure is open and the steam that evolves from the clay escapes into the furnace atmosphere with very little chance of a high concentration at the steel surface. The large effect of clay noted by Miller and Sweo (reference 14) could have been due to the relatively large mass of their samples. In their work, the enamel undoubtedly fused before the clay molecule was completely dehydrated.

The water taken up by the enamel during milling is shown in table 5 and figure 5 to be of minor importance. The water that is picked up is possibly present as a thin gel film around each frit particle. This gel would become dehydrated in the early stages of firing without contributing significantly to the dissolved water content of the frit.

It was expected that the quenching water would have a larger effect than was actually found. The pouring of the molten glass into cold water should present ample opportunity for water absorption. The finding that this operation was of minor importance indicates that the molten glass may not be in contact with water for a sufficient time to absorb any appreciable quantity of water. If the glass was completely free of dissolved water at the time of pouring, however, the effect of the quenching water might be more pronounced.

In considering these data, it should be pointed out that according to the outlined procedure each variable was introduced into a system that already contained all of the other possible sources of hydrogen. For example, when the deuterated clay was introduced, regular hydrogen could have been contributed in the same experiment not only by the dissolved water in the frit but also by the quenching water, the milling water, and the pickling acid. This procedure was selected in order to obtain an estimation of the relative importance of the various sources. If a water-free frit had been used, however, the results might have been entirely different. For example, the present work shows clay to be of minor importance. If, on the other hand, the clay had been introduced to a system containing a frit free of dissolved water, then it is conceivable that the unsaturated frit could have a strong affinity for any water with which it came in contact. Under such conditions, the water given off by the clay might be assimilated by the semimolten frit and this water could in turn react with steel at the higher temperatures to form hydrogen. That some such mechanism might be active was indicated by the reboil experiments. When steel B was coated with frit 109-0 that had been melted in vacuum no reboiling occurred. Yet, when clay was added to this same frit and the batch milled with water, reboiling did occur. These results would indicate that, if an enamel is desired that is completely free of reboil and fishscaling tendencies, it may be necessary to remove all water from the system whatever its source.

Table 2 shows no hydrogen in steel C (titanium bearing) as received. This observation of itself does not prove that hydrogen cannot enter the structure of a titanium-bearing steel. It may mean only that this particular steel has an open rift structure. Because of this structure, the hydrogen

that enters the steel at elevated temperatures may come out readily during cooling rather than becoming dammed up in "lake rifts" of the types postulated by Davis, Keeler, and Chu (reference 19). That some hydrogen is introduced into the titanium-bearing steel during enameling operations is evident from the data in table 4, although the quantity of hydrogen expelled during the treatment at 338° F was much less for steel C than that given off by the two low-carbon steels A and B (see table 5).

The attempt to prepare a water-free frit by using dehydrated raw materials and smelting in an electric furnace was not completely successful. Some water may have been introduced into the frit through traces of residual water in the "dried" raw materials or through traces of water vapor in the atmosphere of the smelting furnace. Sufficient improvement was noted, however, to indicate that a method of this type shows promise.

The desirability of using a ground-coat frit free of dissolved water is believed obvious. The present data indicate that a coating prepared from such a frit and with all other water removed from the system should be free of reboil, fishscaling, and possibly primary boil. The use of a titanium-bearing steel appears to eliminate reboil and fishscaling, but primary boil if it is caused by hydrogen would be expected to occur in titanium-bearing steel just as it does in the others. The use of a water-free enamel should help materially to eliminate hydrogen-caused defects not only with porcelain enamels but also with vitreous-type ceramic coatings. In ceramic coatings complete coverage is of course essential for the optimum protection of the metal.

CONCLUSIONS

The relative importance of various sources of defect-producing hydrogen in the enameling process was determined by tracer experiments with a selected ground-coat enamel applied to several low-carbon steels. Additional experiments in which the presence or absence of lifting, fishscaling, and reboiling served as criteria confirmed the tracer work in general. From the results of this investigation, the following conclusions may be drawn:

1. At the conditions investigated, dissolved water in the frit is the principal source of defect-producing hydrogen.
2. In normal enameling operations, chemically combined water in the clay, milling water, quenching water, and acid pickling are minor sources of hydrogen. However, if frit free of dissolved water is used, these sources may increase in importance.

NATIONAL BUREAU OF STANDARDS,
WASHINGTON, D. C., July 12, 1951.

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TABLE 1
CHEMICAL ANALYSES OF THREE STEELS

Steel	Description	Source of analysis	Analysis in percent by weight of—				
			C	P	S	Mn	Ti
A	Enameling iron.....	(1)	0.012	0.005	0.025	0.017	-----
B	Fishscaling steel.....	(2)	.033	.003	.010	(3)	-----
C	Ti-bearing steel.....	(4)	.09	.040	.050	.50	0.2 to 0.5

¹ Representative analysis from reference 21.

² Analysis by Analytical Chemistry Section of NBS.

³ Not determined.

⁴ Nominal analysis listed by Inland Steel Co. in advertising literature published in 1949.

TABLE 2
GAS CONTENTS OF THREE STEELS

Steel	Description	Gas content by vacuum fusion ¹ in—					
		Weight percent			Cm ³ per 100 grams of steel		
		O ₂	N ₂	H ₂	O ₂	N ₂	H ₂
A	Enameling iron.....	0.0718	0.0073	0.00035	50.2	5.8	3.9
B	Fishscaling steel.....	.0215	.0053	.00014	15.0	4.2	1.6
C	Ti-bearing steel.....	.0047	.0333	Nil	3.3	26.6	Nil

¹ Determined by Chemical Metallurgy Section of NBS.

TABLE 3
BATCH AND COMPUTED OXIDE COMPOSITIONS OF
109-0 FRIT

(a) Batch composition

Constituent	Parts by weight
Potash feldspar.....	30.82
Borax (hydrated).....	44.25
Flint.....	30.50
Soda ash.....	9.16
Soda niter.....	5.15
Fluorspar.....	8.30
	128.18

(b) Computed oxide composition

Oxide	Percent by weight
SiO ₂	51.0
B ₂ O ₃	16.1
Al ₂ O ₃	5.7
Na ₂ O.....	15.4
K ₂ O.....	3.5
CaF ₂	8.3
	100.0

TABLE 4
VOLUME OF GAS COLLECTED AND ANALYSES OF EVOLVED GASES FROM DETERMINATIONS MADE WITH COLLECTION CELL EMPTY, WITH SPECIMENS OF COATED STEEL C IN CELL, AND WITH CHIPS OF COATING MATERIAL IN CELL

Test	Material in collection cell	Volume of gas collected at 338° F	Analysis of evolved gas (mole percent)		
			H ₂	H ₂ O	CO ₂
1	None—cell empty.....	Trace.....	0	98.7	1.3
2	Steel C with coating E-1.....	0.02 ^a	45.5	52.3	2.2
3	Coating chips ^b	Trace.....	.4	80.2	19.4

^a Cm³ of gas per 100 grams of steel.

^b Same total weight of E-1 coating as was present on specimens in test 2. Coating chips were obtained by flexing specimens of steel C immediately after firing.

TABLE 5
VOLUME OF HYDROGEN GAS COLLECTED AT 338° F, MOLE PERCENT OF DEUTERIUM IN HYDROGEN, AND RELATIVE IMPORTANCE OF INVESTIGATED SOURCES OF DEFECT-PRODUCING HYDROGEN IN STEEL

Source investigated	Steel base	Volume of hydrogen collected (cm ³ /100 grams of steel) ¹	Deuterium in hydrogen (mole percent)	Relative importance of source ²
Pickling acid.....	A	0.37	0.1	4
	B	(3)	.5	
Milling water.....	A	.24	.1	5
	B	.32	.1	
Water in clay.....	A	.25	.2	3
	B	.28	.8	
Quenching water.....	A	.80	.5	2
	B	.20	1.5	
Dissolved water in frit.....	A	.56	53.5	1
	B	.44	52.9	

¹ Computed from change in pressure in collection cell after 17 hr at 338° F and from hydrogen content of collected gas. Values are for gas at standard conditions.

² Number 1 refers to most important source and number 5 to least important source.

³ Through a misunderstanding, only the protium-to-deuterium ratio was determined on this sample, making it impossible to calculate the volume of hydrogen collected.

REPORT 1121

DIRECT MEASUREMENTS OF SKIN FRICTION¹

By SATISH DHAWAN

SUMMARY

A device has been developed to measure local skin friction on a flat plate by measuring the force exerted upon a very small movable part of the surface of the flat plate. These forces, which range from about 1 milligram to about 100 milligrams, are measured by means of a reactance device. The apparatus was first applied to measurements in the low-speed range, both for laminar and turbulent boundary layers. The measured skin-friction coefficients show excellent agreement with Blasius' and Von Kármán's results. The device was then applied to high-speed subsonic flow and the turbulent-skin-friction coefficients were determined up to a Mach number of about 0.8. A few measurements in supersonic flow were also made.

This paper describes the design and construction of the device and the results of the measurements.

INTRODUCTION

An object moving through a fluid experiences a drag force which can be decomposed into pressure drag and skin friction. This division is the same whether the body moves with supersonic or subsonic speeds. At present wave drag and induced drag are by far better understood both experimentally and theoretically than skin friction and boundary-layer separation. This is particularly true for supersonic velocities, but it is also curiously enough true that experimental investigations of skin friction in the subsonic range and in incompressible flow are exceedingly rare.

Recent advances in the design of high-speed aircraft and missiles have shown that a more exact knowledge of skin friction (and heat transfer, which is related) is of great importance. Theory of the laminar boundary layer and of laminar skin friction both in low-speed and high-speed flow has been worked out to a considerable extent in the course of the last decade. In spite of the lack of detailed experiments on laminar skin friction, it is generally felt that the theoretical results are adequate and trustworthy up to Mach numbers of the order of 4. Beyond this range, for hypersonic velocities, new effects such as dissociation, variable Prandtl number, and so forth appear and the present theory does not seem to be adequately explored.

Turbulent skin friction, on the other hand, presents a much more serious problem. The understanding of turbulent shear flow, even in incompressible flow, is inadequate and it is at present not possible to formulate a complete theory without recourse to empirical constants. Even if it

is assumed that low-speed turbulent skin friction is known, say from Von Kármán's logarithmic laws with empirically determined constants, the question of how to continue these laws to high velocities and to supersonic flow arises. This question was first discussed by Von Kármán in his Volta Congress paper in 1935 (reference 1) and has since then attracted a large number of investigations, for example, by Frankl and Voishel (reference 2), Ferrari (reference 3), Van Driest (reference 4), Li and Nagamatsu (reference 5), and so forth. The net result of all these investigations is a set of curves of skin friction against Mach number (at constant Reynolds number) which are bounded by Von Kármán's results for incompressible and compressible flows. That is, the general trend is a decrease of skin friction with Mach number as in laminar flow. However, this decrease may be larger or smaller depending upon the assumptions made in the analysis. None of the assumptions made have a theoretically convincing reason, because of the present lack of knowledge of turbulent flow. It turns out that Von Kármán's original assumption, which is also the simplest, gives so far the largest decrease in skin friction with Mach number; for example, the ratio of compressible to incompressible skin-friction coefficient at a Mach number of 2 and a Reynolds number of 1,000,000 is approximately 0.66 according to Von Kármán.

Hence, it is evident that the differences between various theories are anything but minor. In fact, the differences at Mach numbers of the order of 2 are so large that performance computations based upon one or the other of the theoretical skin-friction coefficients result in very large, and sometimes decisive, differences. It is clear, therefore, that, besides the fundamental interest in turbulent boundary layers, the determination of high-speed skin friction is of paramount technical importance today.

The present investigation was undertaken with the aim of developing an instrument capable of measuring skin friction in high-speed flow. The program consisted of four parts:

(a) A survey of possible methods of determining skin friction with the aim of selecting the most promising one for further development. The result of this study suggested direct measurement of local skin friction. This is carried out by measuring the shear force exerted upon a small movable part of a solid boundary by means of a reactance pickup.

¹ Supersedes NACA TN 2567, "Direct Measurements of Skin Friction" by Satish Dhawan, 1952.

(b) The design and construction of the device, including detailed studies of necessary tolerances and sensitivities.

(c) The application of the device to skin-friction measurements in incompressible flow where the results can be compared both with measurements by other means and with theory.

(d) The measurement of skin-friction coefficients in high-speed and supersonic flow.

The second part of (d) was, unfortunately, not completely accomplished. Only a few measurements in supersonic flow with doubtful flow conditions were carried out. This is due to purely external reasons and not due to a failure of the device. The main aim of the investigation was the development of the apparatus, a determination of its accuracy, and a demonstration of its applicability.

This work was conducted at GALCIT under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics as part of a long-range investigation of boundary-layer phenomena at high speeds.

SYMBOLS

B	total-head-tube reading $\left(p + \frac{1}{2} \rho u^2\right)$
c_f	local skin-friction coefficient
c_{f_i}	local skin-friction coefficient in incompressible flow
c_{f_0}	Blasius value for local skin-friction coefficient
$\Delta c_f = c_f - c_{f_0}$	
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
H	ratio of displacement to momentum thickness (δ^*/θ)
H_c	ratio of displacement to momentum thickness in compressible flow
H_i	ratio of displacement to momentum thickness in incompressible flow
k	heat conductivity of fluid
M	local Mach number
M_1	free-stream Mach number
Pr	Prandtl number ($\mu c_p/k$)
p	local static pressure
p_0	total head (impact pressure)
p_0'	local total head
q	dynamic pressure of free stream $\left(\frac{1}{2} \rho_\infty U^2\right)$
q_0	heat flow per unit time per unit area through surface
R	gas constant
Re	Reynolds number (Ux/ν)
T	static temperature of fluid
T_1	static temperature at outside edge of boundary layer
T_0	stagnation temperature
T_w	wall temperature
U	velocity at outside edge of boundary layer
u	velocity component in boundary layer parallel to surface
u_e	effective velocity obtained from Stanton tube reading
v	output of transformer, volts
x	coordinate along surface in direction of flow
y	coordinate normal to surface

γ	ratio of specific heats (c_p/c_v)
δ	boundary-layer thickness
δ^*	displacement thickness
ϵ	height above wall
η	Blasius' dimensionless coordinate normal to surface
	$\left(y/\sqrt{\frac{U}{\nu x}}\right)$
θ	momentum defect thickness
θ_i	momentum defect thickness in incompressible flow
μ	coefficient of viscosity
ν	kinematic coefficient of viscosity (μ/ρ)
ξ	coordinate along surface in vicinity of gap
ρ	fluid density
ρ_1	fluid density at outside edge of boundary layer
ρ_∞	fluid density of uniform flow
τ_0	shearing stress at surface (skin friction)
$()_w$	value at wall

EXPERIMENTAL METHODS OF SKIN-FRICTION MEASUREMENT

Under the assumption of continuum flow and no slip at the solid surface the intensity of skin friction τ_0 at any point on a solid surface in a flowing gas is given by

$$\tau_0 = \left(\mu \frac{\partial u}{\partial y} \right)_{y=0}$$

where

μ	coefficient of viscosity
u	velocity component in boundary layer parallel to surface
y	coordinate normal to surface

For the flow of gases at temperatures and pressures at which the mean free paths of the gas molecules are small compared with the physical dimensions of the system, the slip at the solid boundary is negligible. The following discussions will apply to cases where these conditions are realized.

The above expression for skin friction is valid for laminar and, because of the existence of the sublayer, also for turbulent boundary layers. In general, experimental attempts at measurement of the skin friction may be classified into two main types:

(a) Those which endeavor to determine the components of the right-hand side of the above equation

(b) Those which rely on a bulk measurement of the skin friction itself, or on some other physical quantity related to it.

SKIN FRICTION BY VELOCITY-PROFILE METHOD

In the first type of measurement of the skin friction the physical quantities to be measured are the viscosity μ and the velocity in the boundary layer u . Kinetic theory and experiment indicate the viscosity coefficient to be a function of the absolute temperature alone. Thus a local measurement of the wall temperature (with, say, a sensitive thermocouple) is sufficient to define $(\mu)_{y=0}$. The measurement of the velocity as a function of the normal coordinate y is, however, a more complicated procedure. Measurement of the local velocity in moving fluids is extremely difficult and up to the present time no really satisfactory direct methods exist for application to the observation of high-speed flows.

The only known published measurements of local velocity of interest to fluid mechanics are those of Fage and Townend (reference 6) who used an ultramicroscope with a rotating objective to observe boundary-layer flow in a water channel. For most aerodynamic investigations the local velocity is not found directly but related quantities like dynamic pressure, mass flow, density, and so forth are measured. To obtain velocity from such measurements one requires additional information about the flow.

For example, consider the three most extensively used instruments, the pitot tube, the hot-wire anemometer, and the interferometer. In order to obtain velocity from the reading of a pitot tube one must know the local static pressure and density. For the hot-wire anemometer the local density is required, while information about the local temperature is necessary for the interferometer. In the case of the boundary layer theoretical considerations, borne out by experiments, show that the variation of pressure normal to the surface is negligible. On this assumption the experimental determination of static pressure in the boundary layer is simplified to a local measurement on the surface. (Since the pressure is measured at the wall and its variation is small, the assumption of constant p through the boundary layer introduces only second-order errors in a skin-friction measurement.) However, for the instruments mentioned above, for each determination of velocity one would still have to make at least one additional measurement apart from the reading of the instrument itself.

In determining skin friction from the slope at the wall of a measured velocity profile, errors in measurement of the profile directly affect τ_o . A method which is not so sensitive to the portion of the velocity profile in the immediate neighborhood of the wall is provided in principle by Von Kármán's momentum integral. The basic consideration underlying this is the fact that the frictional force on the surface must appear as a momentum defect in the rest of the boundary layer. For the case of steady two-dimensional flow past a flat surface one obtains

$$\frac{\tau_o}{\rho_1 U^2} = \frac{d\theta}{dx} + \theta \left[\left(\frac{\delta}{\theta} + 2 \right) \frac{1}{U} \frac{dU}{dx} + \frac{1}{\rho_1} \frac{d\rho_1}{dx} \right]$$

where ρ_1 and U are the density and velocity just outside the boundary layer, θ and δ are defined by

$$\theta \quad \text{momentum defect thickness} \left(\int_0^\infty \frac{\rho u}{\rho_1 U} \left(1 - \frac{u}{U} \right) dy \right)$$

$$\delta \quad \text{displacement thickness} \left(\int_0^\infty \left(1 - \frac{\rho u}{\rho_1 U} \right) dy \right)$$

and x is the coordinate in the direction of the flow. Several profiles in the region of determination of τ_o would now have to be measured so that the derivative of θ with respect to x can be found. This method, while not sensitive to errors in u close to the wall, involves the added complication of knowing U , ρ_1 , and θ as functions of x . Its application is more practical in the case of uniform flow past a flat plate when the

pressure gradient in the x -direction is also zero and the above expression is reduced to

$$\frac{\tau_o}{\rho_\infty U^2} = \frac{d\theta}{dx}$$

The use of this method does not, however, eliminate the difficulties in the determination of velocity from a pressure or momentum measurement.

NOTES ON INSTRUMENTS USED IN MEASUREMENT OF VELOCITY PROFILES

Pitot (or total-head) tube.—The specialized form of the pitot tube usually used for boundary-layer measurements consists of a small bore tube (of the order of 0.05-in. diam.) with its mouth flattened out to form a narrow rectangular opening. The impact pressure at the mouth is proportional to ρu^2 and is usually measured on a mercury or alcohol manometer. As mentioned before, in order to obtain velocity from the reading of such an instrument one needs a knowledge of either ρ or T . For nondissipative flows with speeds below the speed of sound, the assumption of isentropic deceleration at the tube mouth is closely fulfilled and the velocity u is related to the impact pressure p by

$$u = \sqrt{2c_p T_o \left[1 - \left(\frac{p}{p_o} \right)^{\frac{\gamma-1}{\gamma}} \right]}$$

where

T_o stagnation temperature

p local static pressure

$\gamma = c_p/c_v$

c_p and c_v being the specific heats at constant pressure and constant volume, respectively.

The stagnation temperature T_o may be measured at some convenient reference point in the flow, for example, in the case of a wind tunnel, in the reservoir or settling chamber. At supersonic speeds a shock wave appears in front of the tube mouth and the entropy loss through this must be taken into account. By assuming the shape of the shock wave to be straight and normal to the flow direction the velocity can still be calculated by use of the well-known Rayleigh pitot-tube formula

$$\frac{p}{p_o'} = \frac{\left(\frac{2\gamma}{\gamma+1} M^2 - \frac{\gamma-1}{\gamma+1} \right)^{\frac{1}{\gamma-1}}}{\left(\frac{\gamma+1}{2} M^2 \right)^{\frac{\gamma}{\gamma-1}}}$$

where M is the local Mach number just ahead of the tube (in front of the shock wave) and p_o' , the impact-pressure reading of the tube (behind the normal shock wave). The two pitot-tube formulas become identical at $M=1$.

For a skin-friction determination with a pitot tube one would then have to know the following:

- (a) The impact pressure indicated by the tube as a function of y within the boundary layer
- (b) The static pressure at the point of measurement
- (c) The temperature distribution in the boundary layer

By assuming the total temperature to be constant through the boundary layer the temperature measurement is very much simplified, but such an assumption causes some error in the value of the shearing stress. This assumption amounts to taking the value of the Prandtl number $Pr \equiv \mu c_p / k = 1$. Since for actual gases $Pr \approx 1$ (for air 0.724), the error is not large at low speeds but increases rapidly with increasing Mach number. An approximate estimate of the error in τ_o caused by such an assumption may be readily obtained. For points in the boundary layer very close to the wall the flow may be considered incompressible so that the pitot tube reads $B = p + \frac{1}{2} \rho u^2$. The pressure p is measured on the surface and is approximately constant through the boundary layer. From this,

$$\frac{du}{dy} = \frac{1}{p} \sqrt{\frac{RT}{2\left(\frac{B}{p}-1\right)}} \frac{dB}{dy} + \sqrt{\frac{R}{2T} \left(\frac{B}{p}-1\right)} \frac{dT}{dy}$$

For $y \rightarrow 0$, $\left(\frac{B}{p}-1\right) \rightarrow 0$ and the second term on the right-hand side is negligible as compared with the first. Therefore, for small values of y ,

$$\frac{du}{dy} \propto \sqrt{T}$$

Also, the viscosity coefficient is approximately

$$\mu \propto \sqrt{T}$$

so that the shear stress at the wall

$$\tau_o = \left(\mu \frac{du}{dy} \right)_w \propto T_w$$

Now

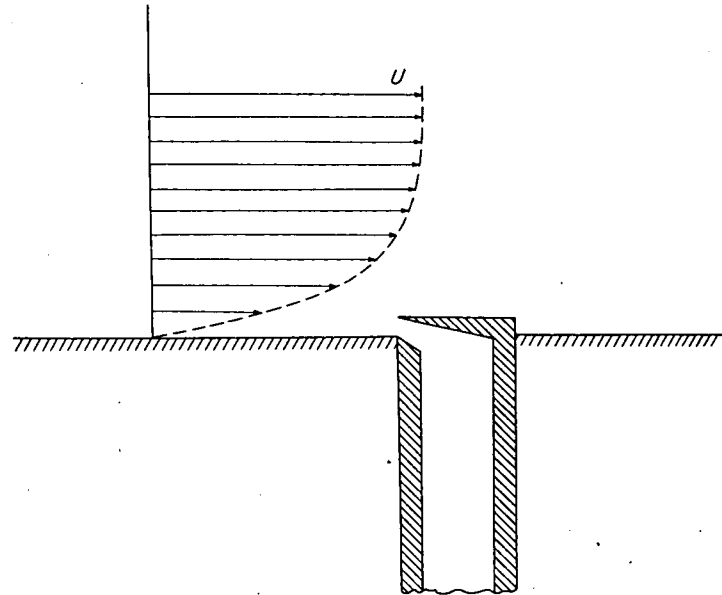
$$T_w = T_o - F(Pr) \frac{U^2}{2c_p}$$

where $F(Pr)$ is nearly $Pr^{1/2}$ for laminar and $Pr^{1/3}$ for turbulent boundary layers. Hence, by using $T_w = T_o$ the shearing stress would be directly in error. For the typical case of an insulated flat plate at $M \approx 1.5$ this error would amount to approximately 5 percent. At higher Mach numbers the errors would be even greater.

It is interesting to note the improvement in accuracy if an actual measured value of the wall temperature T_w is used for the determination of τ_o . The errors in $\left[\frac{du}{dy} \right]_w$ due to temperature are practically eliminated. The main limitation now is in the determination of $\left[\frac{dB}{dy} \right]_w$. Because of the finite size of the pitot tube, B cannot be measured right up to the wall. Another error introduced by the finite size of the pitot tube in boundary-layer measurements is due to the transversal total-pressure gradient in the boundary layer. The average indicated by the tube does not correspond to the velocity at the center of the tube but to that at some point which is shifted from the center toward the region of higher velocity. In a measurement of the velocity profile in a given boundary layer this error is emphasized for points close to

the wall. For a tube with a rectangular opening of 0.005 inch this shift is of the order of 0.001 inch in a velocity gradient $du/dy \approx 10^6$ feet per second per foot (representative of a typical laminar boundary layer at a Mach number of approximately 1.4 and a Reynolds number, based on a 10-cm length, of 10^6). In actual use the shift can be approximately accounted for (see reference 7 for details).

Stanton tube.—The Stanton tube is a modification of the pitot tube which gives an indication of the shearing stress on a surface. It consists of a small tube with a hole in its side (see sketch). The tube projects a small distance (of the order



of 0.005 in.) from the surface on which the friction is to be measured and the opening in the side of the tube faces the direction of flow. The end of the tube is ground flat to form a razor-sharp lip with the opening. For a distance ϵ very close to the surface (this would have to be within the laminar sublayer in flows with turbulent boundary layers) one has approximately

$$\tau_o \approx \mu_w \frac{u_\epsilon}{\epsilon} = (\rho_w u_\epsilon^2) \left(\frac{\nu_w}{\epsilon u_\epsilon} \right)$$

where u_ϵ is the effective velocity corresponding to the tube pressure reading and the subscript w denotes conditions at the wall. The surface tube may thus be usefully employed for indications of changes in τ_o (see reference 8 for example of use). However, because of the unprecise definition of the quantity u_ϵ as measured by the tube and the extreme sensitivity to ϵ , the height of the tube above the surface, this method is not recommended for absolute measurement of τ_o .

Hot-wire anemometer.—The hot-wire anemometer depends for its measurement of velocity on the heat loss from a very small diameter (of the order of 0.0005 in.) wire which is heated electrically. For a given temperature difference between the heated wire and the surrounding air the amount of heat transferred is proportional to the square root of the mass flow ρu . In order to find the velocity u from the mass flow one must know the density ρ . For the measurement of mean velocity in boundary layers at low speeds, where the effects of compressibility can be neglected, the hot-wire

anemometer is used as a calibrated instrument. The calibration is performed under controlled conditions, for example, in a wind tunnel with a low turbulence level. In this range the sensitivity and accuracy of this instrument are well-established. At high speeds the law of heat transfer deviates considerably from the one at low speeds. In particular, the effects of temperature and density gradients on the reading of the instrument under these conditions are not yet established (see reference 9 for some recent work on this subject). Since a hot-wire has to be calibrated under conditions which cannot be exactly reproduced during boundary-layer measurements, such effects may cause large changes in the calibration constants. Until the behavior of hot-wires in high-speed, particularly supersonic, flow is well-established, it is felt that this instrument is unsuitable for precise measurements of boundary-layer flow at high speeds.

Interferometer.—The interferometer measures directly the change in density with reference to some known value (see reference 10 for an account of the theory of operation of this instrument). Since the interferometer leaves the flow undisturbed, it seems, at first sight, to be the ideal instrument for two-dimensional boundary-layer measurements. There are, however, serious limitations to its use for such work. These are due to:

- (a) Errors arising from the refraction of the light traversing the boundary layer
- (b) Presence of side-wall boundary layers in wind tunnels
- (c) Transverse contamination of laminar boundary layers

The errors due to refraction are quite complex. As a ray of light traverses the boundary layer, it traces a curved path instead of a straight one because of the existence of density gradients in the boundary layer. Thus, two types of errors are introduced as a result of refraction. The first is in the optical path length of the ray and the second, in the position to which the indicated density corresponds. Further errors result from the fact that because of refraction a ray of light which enters the tunnel at right angles to the side wall leaves the test section at an angle to the exit window or wall. This causes refraction through the window, so that the location of the apparent origin of the ray is further in error. A more complete discussion of these errors may be found in references 11 and 12. It is sufficient to note here that investigations of the magnitudes of the refraction error show that the indicated density profiles in boundary-layer measurements with an interferometer may be in error to the extent of approximately 10 percent.

The second limitation, that of the presence of tunnel-wall boundary layers, affects the measured density since the interferometer beam traverses both the wall density field as well as the one being studied. One method of minimizing this error is to pass the reference beam of the interferometer also through the test section and thus cancel the effect of the wall boundary layers (reference 13).

The third limitation is of primary consequence in laminar-boundary-layer measurements. It is well-known (reference 14) that external disturbances can cause transition of laminar flow to the turbulent type. Observations of laminar boundary layers on flat plates show that because of transverse contamination of the sides there are always regions of

turbulent flow. Again the interferometer beam integrates the density in these regions as part of the laminar boundary layer and hence gives erroneous indications.

The above discussion shows that accurate measurement of density in a boundary layer with an interferometer is far from simple. Approximate methods of correction have been devised by various workers (references 11, 12, and 15). It is felt that the techniques so far developed are inadequate, particularly for taking the rather serious refraction error into account. In addition to these difficulties in the measurement of density one still has the basic problem, mentioned before, of obtaining velocity from the reading of the instrument. Knowing ρ , one still needs the distribution of the static temperature through the boundary layer for a calculation of u . One may proceed to use theoretically calculated temperature distributions in the boundary layer (e. g., in reference 16) or, alternatively, use a pitot tube to supplement the interferometer. In this connection the use of a total-head tube is considered to be an improvement over the first method of assuming theoretical distributions of temperature in the boundary layer and using the interferometer as a primary measuring instrument. The reading of a pitot tube (apart from errors due to finite size) is proportional to ρu^2 and, for all practical purposes, is independent of theoretical assumptions. Hence this, with an accurate independent measurement of ρ , should give a fairly reliable determination of the velocity.

MEASUREMENT OF SKIN FRICTION BY DIRECT METHODS

The preceding discussions show that experimental determination of skin friction by the indirect method of velocity-distribution measurement in thin boundary layers is subject to many errors. The shearing stress τ_0 has to be determined by differentiation, either of measured velocity profiles or of slowly varying parameters (the loss of momentum in the boundary layer). Even when the quantities to be differentiated are themselves measured comparatively accurately the results of differentiation can be quite inaccurate.

The second type of skin-friction measurement, relying on a direct or bulk measurement, may be performed in principle in two ways:

- (a) By a heat-transfer measurement
- (b) By a direct force determination

Skin friction by heat-transfer measurement.—The heat-transfer method depends on Reynolds' analogy between the transport of momentum and the transfer of heat. If q_0 is the heat flow per unit time from a unit area of the surface and k is the heat conductivity of the fluid, then

$$q_0 = k \left(\frac{\partial T}{\partial y} \right)_{y=0}$$

where T is the local temperature. The analogy with the expression for the intensity of skin friction is at once apparent. For the case of two-dimensional flow with the Prandtl number $c_p \mu / k = 1$ the temperature T is a parabolic function of the velocity u alone. In such a case the heat flow q_0 and the wall shearing stress τ_0 may be explicitly related by an expression (reference 17) of the form:

$$\frac{q_o}{\tau_o} = \frac{T_1}{U} \left(\frac{k}{\mu} \right)_{v=0} \left[\frac{\gamma-1}{2} M_1^2 - \left(\frac{T_w}{T_1} - 1 \right) \right]$$

where

- T_1 temperature of free stream
 T_w wall temperature
 U velocity of free stream
 M_1 Mach number of free stream

In the above relation assumptions of laminar flow and equal orders of magnitude for the viscous and thermal boundary layers are inherent. It is easily seen that estimates of skin friction can be obtained by measurement of q_o and the other quantities involved. For cases when the Prandtl number differs from unity but is constant and the flow is turbulent, similar relations between the heat transfer and the skin friction may be derived (reference 18). These relations, however, are not exact and, although they do indicate the existence of a relationship between q_o and τ_o , they are by themselves not reliable enough to form the basis of measurements for τ_o . In such cases calibration procedures have to be relied upon. Ludwig (reference 19) has successfully used this principle for the measurement of wall shearing stresses in turbulent flows at low speeds. His instrument consists of a small electrically heated element flush with the surface on which the measurement is to be made and thermally insulated from it. The heat flow q_o is measured by the amount of electrical energy supplied to the element. The temperatures of the element and the free stream are measured by thermocouples. It must be pointed out that in this case the general relation between q_o and τ_o must be modified to account for the fact that the thermal layer and the friction layer do not originate at the same point. Ludwig calibrated his instrument by measurements of q_o with known values of the friction force, so that his measurements are not affected by theoretical assumptions on the relation between τ_o and q_o . The use of this type of instrument for high-speed flow investigations has, however, one serious drawback. It is extremely difficult to provide known shearing stresses for calibration purposes. For instance, it is no longer possible to use the known relation at low speeds between pressure drop in a two-dimensional channel and the shearing stress on the walls. Furthermore, it is open to question whether a heat-transfer instrument calibrated in flows with fully developed turbulent boundary layers can be used for the measurement of skin friction in laminar flows, and vice versa. On account of these difficulties it was felt that a method measuring the shear force directly would be superior to the heat-transfer method.

Direct force measurement.—In principle the method of direct force measurement is very simple. The frictional force is allowed to move a small element of the surface in the direction of the flow and against some restoring force. This movement is calibrated to indicate the magnitude of the force. This method was used by early investigators such as Froude and Kempf in determining the fluid resistance of bodies in water. A more recent application is that of Schultz-Grunow (reference 20), who used it for measurements of τ_o in a low-speed wind tunnel. Since this method does not rely on any physical assumptions regarding the nature

of the boundary-layer flow, it is inherently very suitable for surface-friction measurements. There are, however, several difficulties and possible sources of error in the actual use of this principle, especially for applications to supersonic flow. The moving element has to be separated from the rest of the surface by small gaps. These may cause changes in the velocity and pressure distributions in their immediate vicinity and so distort the measurements. Again, for local measurements the force element has to be very small compared with the dimensions of the body on which the friction measurements are to be made. This is of special importance in cases where there are large pressure gradients in the direction of flow. The magnitudes of the friction forces on a small element are bound to be small so that the force-measuring mechanism has to be extremely sensitive. Investigations of these difficulties showed that, nevertheless, the direct force-measurement method was quite feasible and practical.

DEVELOPMENT AND DESIGN OF SKIN-FRICTION INSTRUMENT

Although the principle underlying the direct measurement of skin friction (as outlined in the last section) is quite straightforward, its actual application to flow measurements is far from simple. This is especially true for measurements at high speeds. An instrument based on this principle can be used with confidence only after rather extensive investigations and considerations of the features likely to introduce errors in the measurement. This section will be concerned with the investigations undertaken to evaluate the applicability of the direct skin-friction-measurement method and the design of an instrument based on the results of these investigations.

The main features to be investigated in determining the actual feasibility of the method may be briefly summed up as follows:

- (a) Effect of the gaps in the surface on the skin-friction measurement
- (b) The practicability of accurately measuring small forces with apparatus having extremely small physical dimensions
- (c) The effect of external vibrations and temperature changes on the instrument during measurement

EFFECT OF GAPS

That the gaps between the surface of the flat plate and the force element may have some effect on the force measured is quite clear from the fact that in the gaps the condition of no slip at the surface is no longer satisfied. The break in the continuity of the boundary may be regarded as a sudden disturbance in the slope of the velocity profile at the wall (i. e., the shearing stress τ_o). The magnitude of this disturbance evidently depends on the dimensions of the slot as well as the characteristics of the flow in its vicinity. The fact that for small surface gaps the disturbance in pressure is negligible forms the basis of the well-known method employed in the measurement of static pressures in moving fluids. Since pressure and velocity disturbances are related, it is reasonable to expect that for small gaps the associated velocity disturbance would also be small. In order to

substantiate this heuristic reasoning and to arrive at a somewhat more precise idea of the "smallness" of the gaps which would not disturb the velocity profile, tests were made on flat plates with slots in the surface. A slot width of 0.01 inch was used for the study in supersonic flow. This slot dimension was chosen as being a practical one for actual use in the instrument to be developed. The flow over the plate surface was observed by means of schlieren photographs. Figures 1 and 2 are typical schlieren photographs of the flow past the slots with laminar and turbulent boundary layer, respectively. It was found that the plate surface and the element had to be in the same plane to almost an optical degree of flatness before the shock waves disappeared on the schlieren. The laminar-boundary-layer photograph (fig. 1) indicates no detectable disturbances due to the presence of the slots. In the turbulent case (fig. 2) (there being a steeper velocity gradient at the plate surface) very faint waves can be seen originating at the gap locations. In order to estimate the strength of these waves, an attempt was made to measure the pressure rise through them by means of a 0.04-inch-diameter static-pressure probe and an alcohol manometer. No indication of pressure variation could be obtained. Since experience with similar probes has shown the instrument to

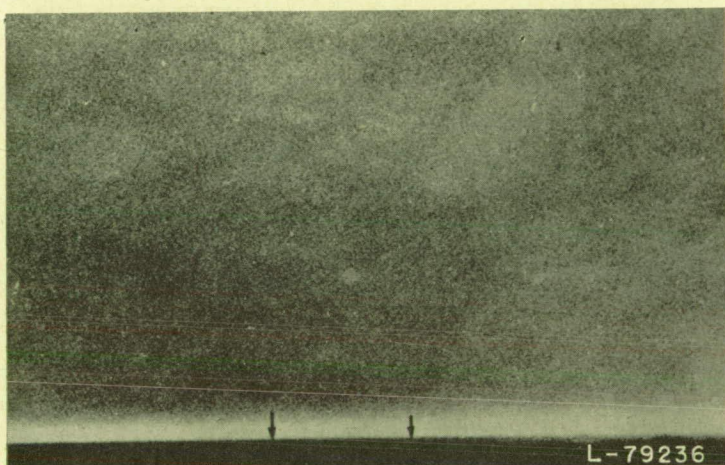


FIGURE 1.—Effect of slots in plate. Laminar boundary layer; $M \approx 1.4$; flow from left to right; arrows indicate position of slots.

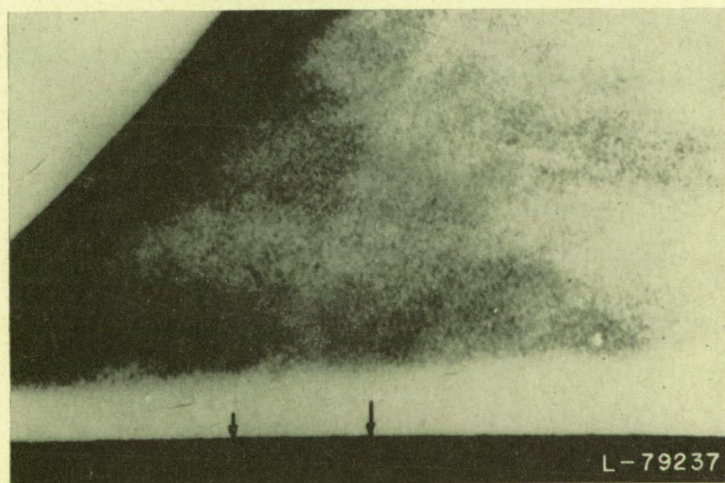


FIGURE 2.—Effect of slots in plate. Turbulent boundary layer; $M \approx 1.4$; flow from left to right; arrows indicate position of slots.

be an extremely sensitive one, it was concluded that the effect of the slots was negligible. This conclusion was further confirmed by a detailed exploration of velocity profiles in the vicinity of a sample slot. The profiles on a flat plate with a relatively large slot of about 0.2 centimeter were measured with a 0.0005-inch-diameter platinum hot-wire. Since the boundary layers at high speeds are too thin to allow accurate profile determination, these measurements were carried out in a low-speed wind tunnel. The boundary layers were relatively thick (about 1 cm) and the hot-wire measurements quite accurate. In order to retain some measure of dynamic similarity with the slot in the high-speed tunnel, the Reynolds number based on the gap width and free-stream velocity, density, and viscosity was kept approximately the same in the two cases. The results of these measurements are shown in figure 3. The slope and shape of the velocity profiles in the vicinity of the slot, and hence the shearing stress, are seen to be essentially unaltered by the presence of the slot. It is interesting to note the profile measured in the gap itself. A small finite velocity is indicated at the wall level, but the slope of the rest of the profile is not sensibly affected. The profile immediately behind the slot does not indicate any trace of the disturbance. The velocity profiles shown in figure 3 are for a turbulent boundary layer on the flat plate. The effect of the gap on a surface with laminar boundary layer would be even less.

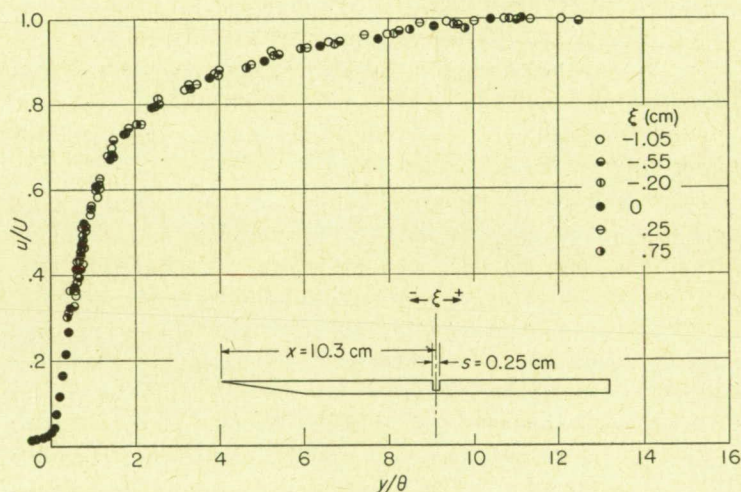


FIGURE 3.—Velocity profiles on flat plate with turbulent boundary layer and slot. $Re_x \approx 4 \times 10^4$; $Re_s \approx 1000$.

MEASUREMENT OF SMALL FORCES

Almost all the usual methods employed in the measurement of small forces have one common feature in that they all utilize a small linear or angular movement produced by the force's being measured against some mechanical restoring force. The principal methods available are:

- (a) Mechanical devices utilizing the torsion of a thin wire
- (b) Optical methods
- (c) Resistance wire gages
- (d) Reactance gages

The mechanical torsion wire is essentially very simple and is utilized extensively in laboratory measurements of small forces (e. g., surface tension of liquids). Considerable effort

was expended in an attempt to employ this method for the measurement of skin-friction forces. The test setup, however, showed that it could not be used to advantage without complicated lever systems which would destroy the simplicity and accuracy of the method.

Among the optical methods the use of interferometry for the accurate measurement of small displacements is well-known. The extreme sensitivity of this method, however, requires almost perfect vibration isolation of the optical elements and thus makes a practical application in a case like the present one extremely difficult. Two other optical methods were investigated. The first of these employed optical levers to magnify small angular changes, while the second used the fact that the width of the diffraction pattern from a narrow rectangular slit is proportional to the width of the slit. While extremely simple in their application, these methods were found to be not accurate enough.

The resistance strain gage, extensively used in experimental structural analysis and recently also in wind-tunnel balance systems, works on the principle that a change in strain in a thin conducting wire is accompanied by a change in electrical resistance of the wire. Since the percentage change in resistance is of the same order of magnitude as the percentage strain and the wire material cannot be stressed beyond its elastic limit, rather elaborate electrical equipment is necessary for precision measurements with this type of gage. Its use is further complicated by the necessity for temperature compensation, for changes in temperature would directly affect the resistance of the strain-gage wire. A test apparatus employing the wire strain gage showed that this method did not have the accuracy or stability required for the measurement of small forces of the order of 100 milligrams.

The reactance-type gage is similar to the resistance gage inasmuch as it also employs a mechanical movement to produce a corresponding change in an electrical quantity. In this case, however, it is possible for the percentage change in reactance to be several orders of magnitude larger than the percentage mechanical change. The change in reactance can thus be easily and accurately measured. The particular form of reactance gage or pickup chosen for the present application is a small $\frac{5}{16}$ -inch-diameter, $\frac{7}{16}$ -inch-long variable differential transformer manufactured commercially (Schae-vitz Engineering Co., Camden, N. J.). It consists (see fig. 4) essentially of three coaxial coils, one primary and two secondaries. A small iron core forms the controlling element of the transformer. The primary coil is energized with high-frequency (20-kc) alternating current so that the solenoidal

force on the core is effectively eliminated. The output of the secondary coils connected in opposition is a function of the core position (fig. 5). The sensitivity of the transformer is of the order of 0.03 volt per 0.001-inch core displacement at 5 volts input to the primary. This corresponds to a full-scale deflection on a Hewlett-Packard vacuum-tube voltmeter. The output change is large enough for measurement of core movements of less than $\frac{1}{10,000}$ inch to approximately 1-percent accuracy.

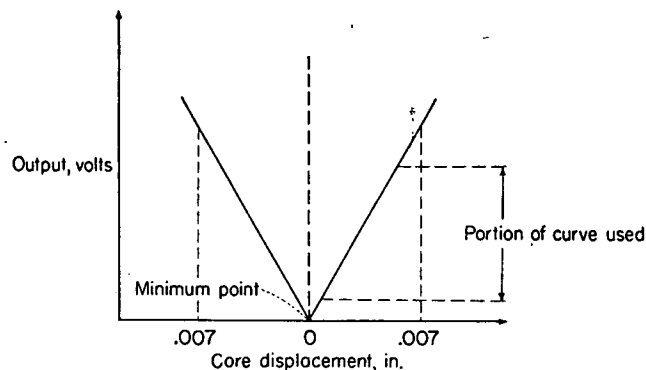


FIGURE 5.—Sketch of output characteristics of differential transformer.

EFFECT OF EXTERNAL VIBRATIONS AND TEMPERATURE CHANGES

Since the skin-friction instrument was intended for use in wind tunnels which are always subject to vibrations, actual measurements of the frequencies and amplitude of the tunnel vibrations were made with a standard vibration analyzer. These measurements showed that the predominant external frequencies were at approximately 200 cycles per second and 20 cycles per second. The maximum amplitude of vibration was about 0.001 inch at the lower frequency. In view of this the core suspension system of the instrument was designed with a natural frequency much lower (about 5 cps) than the external vibrations. Since the external vibrations actually have a continuous frequency spectrum rather than just discrete frequencies, it is impossible to eliminate entirely the effect of these vibrations. As a result the transformer core may oscillate to a small extent about some mean position or may be effectively displaced from its rest position. It is therefore important to operate on a part of the output curve (fig. 5) such that the oscillations do not make the core cross the minimum or "zero" point of the curve. The effect of the external vibrations was further minimized by viscous damping and distribution of the mass of the oscillating system so that its center of gravity and center of percussion were approximately coincident.

The transformer pickup was tested for temperature effects and it was found that temperature changes of $\pm 20^\circ \text{C}$ have no effect on the output. In the actual design one further precaution would be required. The link system carrying the core must be so designed that thermal changes leave the core position unaffected.

SUMMARY OF DESIGN REQUIREMENTS

To summarize the above investigations the following design requirements were set up for the shearing-stress instrument.

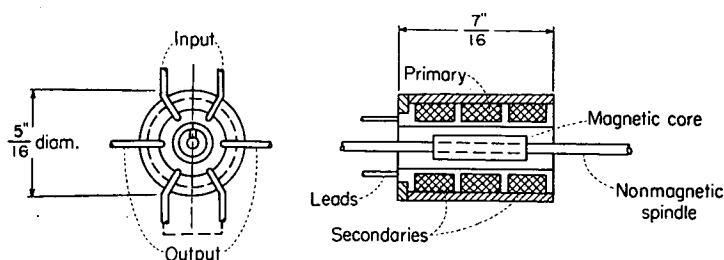


FIGURE 4.—Differential transformer.

(a) The instrument is to be used primarily for two-dimensional measurements in the GALCIT 4- by 10-inch transonic wind tunnel (see reference 21 for description). The small size of the test section demands that the instrument be of extremely small dimensions.

(b) The moving element and the flat plate must at all times be at the same level. Even small inclinations (of the order of $1/50^\circ$) of the element would introduce large errors in the measured forces. The gaps between the force element and the plate surface must be as small as possible.

(c) The force-measuring system must be sensitive enough to measure skin-friction forces in both laminar and turbulent boundary layers in the available range of Mach number and Reynolds number ($M \approx 0$ to 1.5 and $Re = 10^4$ to 10^6). This corresponds to shearing stresses ranging from approximately 1 milligram per square centimeter to about 1 gram per square centimeter.

(d) The instrument readings must be immune to vibrations from the tunnel and to temperature changes.

DESCRIPTION

The instrument developed is shown schematically in figure 6. The flat plate used for the measurements is supported from the ceiling of the tunnel. The small 0.2- by

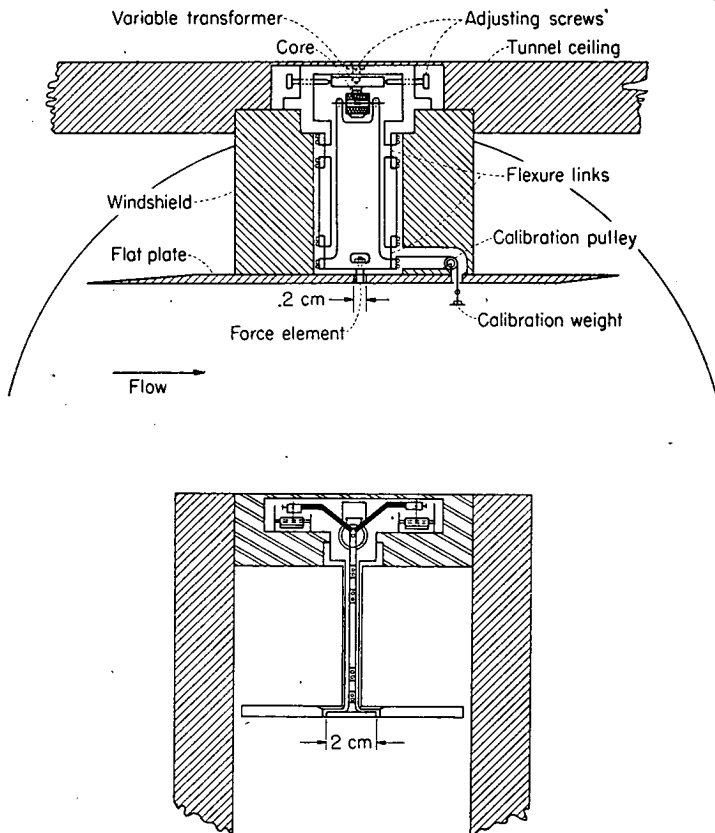


FIGURE 6.—Sketch of instrument for direct measurement of skin friction.

2.0-centimeter moving element is supported by a specially designed four-bar linkage which allows it to translate in the plane of the plate without rotation. The rectangular slot in the plate through which the force element is exposed to the flow has accurately lapped sides. The gaps at the leading and trailing edges of the element are approximately

0.004 inch and 0.008 inch, respectively. The element is lapped flush with the surface of the plate after assembly. The degree of flatness is checked by means of interference fringes. The flexure links in the suspension system can be replaced by links of various thickness (0.005 in., 0.01 in., 0.025 in.), thus providing a large range of variation of the restoring force. Movement of the force element is conveyed to the core of the transformer by a nonconducting rod and metal-yoke arrangement. The transformer is held in position by a brass cage which can be adjusted for position by means of set screws. The linkage assembly seats in recesses in the flat-plate support which also serves as a streamlined wind-shield. The damping of the force element is obtained by two dampers which are carried by the yoke of the flexure link assembly and are immersed in Dow Corning silicone fluid of 500-centistoke viscosity. The area of the surface of the dampers was adjusted to provide slightly less than critical damping to the oscillating system. The mass of the damping system is also utilized to obtain coincidence of the center of percussion and center of gravity of the linkage. The effectiveness of damping and vibration isolation can be seen from the sample calibration curve in figure 7.

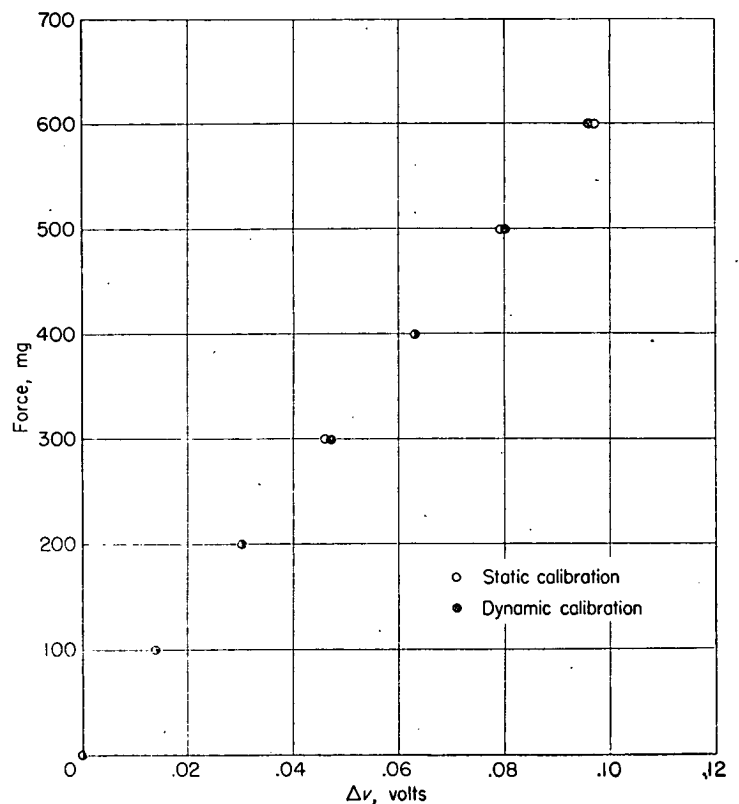


FIGURE 7.—Effect of vibrations on calibration of skin-friction instrument. Vibration frequency, approximately 25 cycles per second.

Calibration of the instrument is obtained by means of a pulley arrangement with jewel bearings. The pulley was constructed from magnesium (for lightness) and balanced in rotation so that it introduced no errors in calibration. The frictional torque of the jewel bearings is of the order of 10^{-6} milligram-centimeter and hence is negligible. A single-fiber nylon thread provided the means of attaching small fractional weights to the linkage during calibration. Sample

calibrations indicating the repeatability and linearity of the force-measuring system may be seen in figures 8 and 9. The entire force linkage and coil system is enclosed in the windshield which has a frontal area of $\frac{1}{4}$ inch by $2\frac{1}{2}$ inches. When in use, the inside of the windshield is sealed off from the flow, except for the gaps around the force element itself.

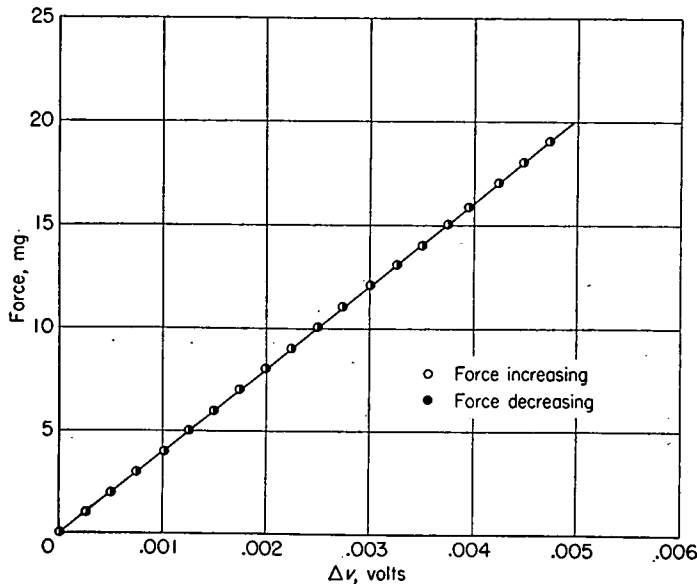


FIGURE 8.—Calibration of skin-friction instrument. Input, 5 volts at 20 kilocycles; 0.0005-inch by $\frac{1}{4}$ -inch links; $\tau_s = 39.25 \times \Delta$, dynes per square centimeter.

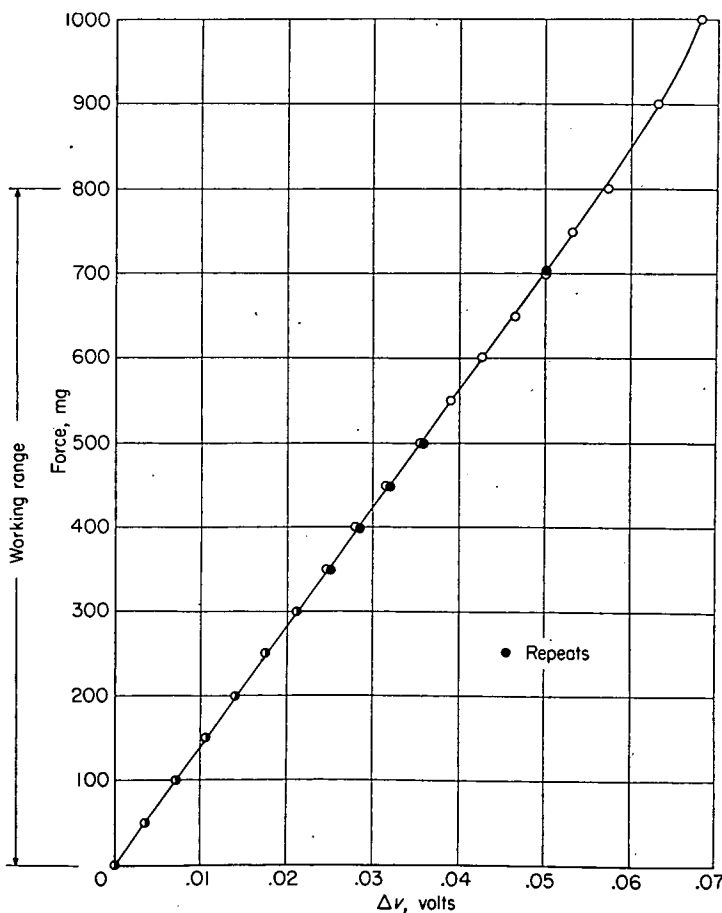


FIGURE 9.—Calibration curve of skin-friction instrument. 0.01-inch by $\frac{1}{4}$ -inch steel links; $\tau_s = 38.3 \times 10^3 \times \Delta$, dynes per square centimeter.

These gaps serve as vents for equalization of pressure between the inside and the outside of the force element. The electrical leads are likewise sealed. During operation the pressure across the gaps could be observed on an alcohol manometer. One lead of the manometer was connected to a static-pressure orifice on the plate just ahead of the element and the other indicated the pressure inside the windshield.

ELECTRONIC EQUIPMENT AND CIRCUIT DETAILS

A block diagram of the manner of use of the skin-friction instrument is shown in figure 10. A precision Hewlett-

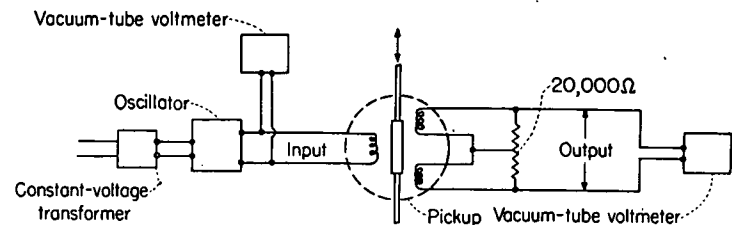


FIGURE 10.—Circuit used for skin-friction instrument.

Packard low-frequency oscillator acted as the power supply for the differential transformer. The output from the secondaries connected in opposition was read on a Hewlett-Packard vacuum-tube voltmeter to an accuracy of ± 1 percent. Since the output is directly proportional to the input, the latter was always kept at the same fixed value before taking a skin-friction reading. In order to avoid errors introduced by switching, a second vacuum-tube voltmeter was used to read the input to the primary coil of the transformer. The variable resistance of 20,000 ohms (shown in fig. 10) across the secondary coils of the transformer unit was found useful in reducing the magnitude of the output at the "balance point" (symmetrical core position) to approximately 0.001 volt. This effectively increased the range of readings on the voltmeter scale.

INCOMPRESSIBLE-FLOW MEASUREMENTS

At low speeds the assumption of incompressibility of the air is a close approximation and leads to considerable simplification in theoretical treatment of the boundary-layer equations. This is especially important for the case of laminar flow past a flat plate at zero incidence, for in this case an exact solution of the Prandtl equations is possible. This is the well-known Blasius solution. For the much more complicated case of turbulent flow, theoretical solutions of the equations are not possible. However, as shown by Prandtl, Von Kármán, and others, even for turbulent flow important theoretical deductions regarding the skin friction and nature of the boundary-layer flow can be made. The logarithmic law for skin friction at high Reynolds numbers is a well-known example.

In order to establish the general validity of the present method of measurement, the skin-friction instrument was adapted for measurements in low-speed flow. Two types of flow were investigated, the so-called Blasius flow and fully developed turbulent flow past a flat plate. It is believed

that the results of this investigation, apart from confirming the accuracy of the method of measurement, have some interest and value of their own.

EXPERIMENTAL EQUIPMENT AND SETUP

The general experimental arrangement for the low-speed measurements is shown in figure 11. The GALCIT "Correlation" tunnel was used for this investigation. This is a specially designed wind tunnel for turbulence measurements, with a very low free-stream turbulence level (u'/U of the order of 0.03 percent).

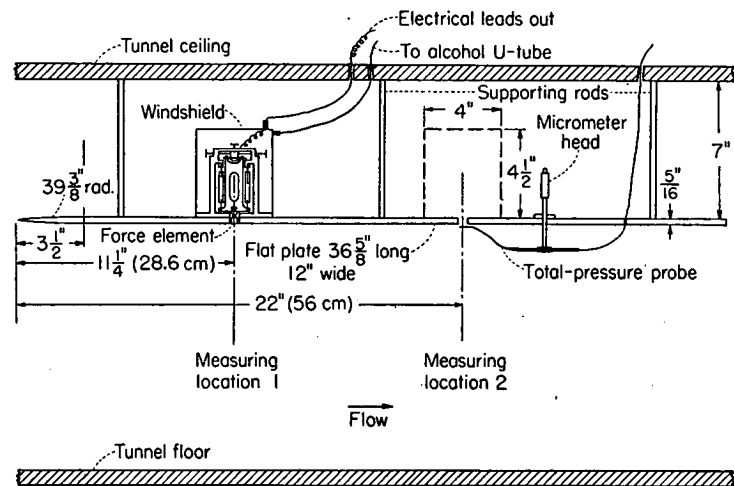


FIGURE 11.—Sketch of flat-plate installation in GALCIT 2½- by 2½-foot correlation tunnel.

Since the magnitude of the shearing stress τ_o at the low speeds and Reynolds numbers ($U \approx 400$ cm/sec to 1600 cm/sec and $Re \approx 6 \times 10^4$ to 6×10^5) is extremely small, being of the order of 1 milligram per square centimeter, the small (0.2- by 2.0-cm) element of the instrument was replaced by a larger one measuring 1.15 by 6.3 centimeters. Phosphor bronze 0.005-inch-thick by ¼-inch-wide flexure links were used instead of the regular 0.01-inch-thick links. This increased the sensitivity of the force-measuring system by a factor of approximately 4. The instrument was attached to the flat plate by means of a metal yoke, as shown in figure 11. The surface used for the low-speed boundary-layer measurements was a sharp-nosed, 12- by 36-inch flat plate made out of ⅛-inch-thick plastic material. The surface of this plate had a highly polished finish. There was, however, some irregularity present in the form of waviness in the surface. No attempt was made to remedy this because the amplitude of the surface waves was too small (of the order of 0.005 in. in a length of 2 ft) to be of any significance. The shearing-stress instrument could be located in two alternative positions on the plate. The change in position allowed an additional variation in the Reynolds number (i. e., in addition to the change made possible by free-stream velocity changes). The measuring element consisted of a small piece of the plastic (identical with the plate material) rectangular in shape and attached to the force linkage so that the lower surface of this element was flush with the surrounding plate surface. The element was separated from the rest of the surface (on all four sides) by a ½-inch-wide gap. That the element

was really flush with the plate surface could be checked accurately by means of an optical arrangement consisting of a reflecting surface and a magnifying lens. The force linkage system was shielded and sealed off from the flow by a streamlined brass shield. Total-head profiles could be measured at either location of the shear-stress measurement by means of a total-head tube and a portable micrometer head with a least count of 0.01 centimeter. The total-head-tube dimensions are shown in figure 12. A precision

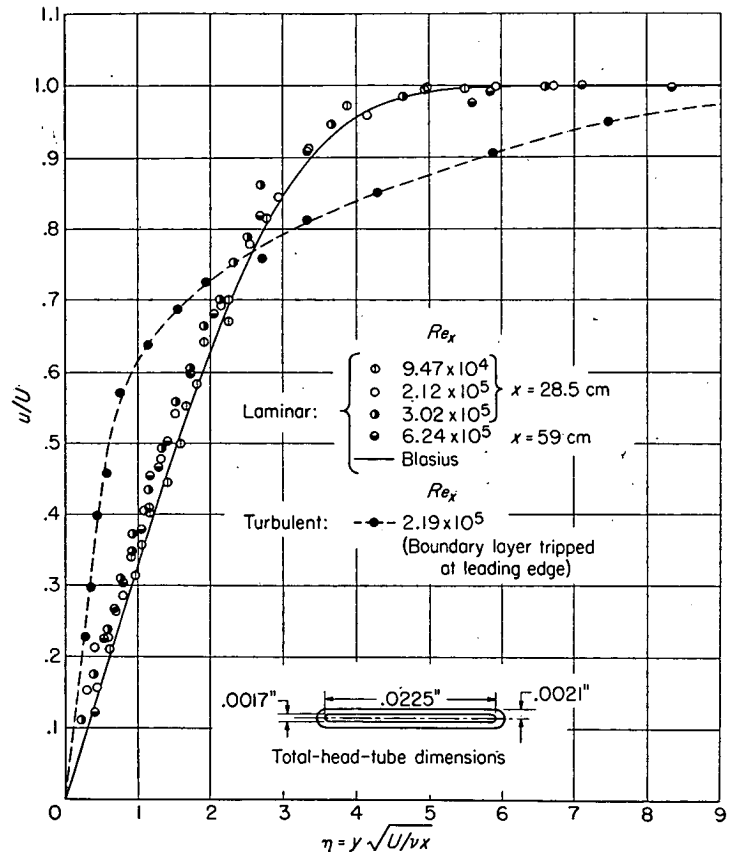


FIGURE 12.—Boundary-layer profiles on flat plate. Incompressible flow.

alcohol manometer of the Zahn type was used for the measurement of pressures. Total-head or static-pressure traverses could be made in the direction of flow by means of a motorized traverse.

EXPERIMENTAL METHOD

Laminar boundary layer.—In order to establish the type of boundary-layer flow, velocity profiles were measured on the surface of the plate for a range of velocities covering the Reynolds number range over which the shearing force was to be measured. These profiles (fig. 12) showed good agreement with the theoretical Blasius profile. That the flow was indeed laminar was further substantiated by observing laminar-boundary-layer oscillations (Tollmien-Schlichting waves) by means of a 0.0005-inch-diameter platinum hot-wire placed near the surface of the plate (approximately at a distance of 0.1 cm). The regular laminar oscillations could be changed to random turbulent fluctuations by introducing disturbances in the boundary layer ahead of the point of measurement. In addition to substantiating the

pressure-profile measurements, this precaution served another significant purpose: It removed any suspicion that the force of friction measured by the shearing-stress instrument was due to anything other than pure laminar flow. It is known that laminar flow does not break down at a definite (time-independent) distance from the leading edge. Transition to turbulent flow is evidenced by the appearance of "bursts" of turbulent fluctuations (see references 22 and 23) superimposed on the laminar oscillations. In case such bursts were present, the force measurements would not correspond to purely laminar friction.

The actual shearing-force measurement consisted of taking the difference of voltmeter readings with and without flow for each point measured. This process eliminated any possible errors due to zero shift in the measuring system. In the earlier stages of the experiments the instrument was calibrated before and after each set of shearing-force measurements. Since the calibration (fig. 8) did not perceptibly change, this process was later replaced by one of making periodic checks. The calibration remained constant during the entire duration of the measurements to better than 1 percent.

It was not possible to fulfill exactly the condition of zero pressure gradient required by Blasius flow. However, the deviation from $dp/dx=0$ was small ($\Delta p/q$ less than 1 percent) in the range of the measurements reported, and it was possible to adjust the plate in pitch to give approximately a constant favorable pressure gradient. This was actually measured and the force measurements were corrected accordingly when making the comparison with theory (see section "Laminar boundary layer").

Since the force-measuring system of the skin-friction instrument is essentially a suspension, it is affected by tilt of the instrument. In order to determine and account for the error introduced by changes in the attitude of the instrument due to deflection of the flat plate, the following procedure was adopted: The element was covered with a streamlined shield which left it free to move but shielded it completely from the flow. The difference between the "no flow" and "with flow" readings then gave the effect of the plate tilt. This error was determined for the entire range of velocities since the plate deflection is dependent on the air loads of the system. Figure 13 shows the effect of plate tilt on the instrument. The mean curve of figure 13 was then used to correct the instrument readings. The maximum correction was less than 10 percent. The scatter in figure 13, at about 1300-centimeter-per-second free-stream velocity, is due to excessive vibrations induced by resonance with a natural mode of oscillation of the tunnel. The above procedure of correction was necessary only for the measurements made at the forward location ($x=28.6$ cm) of the instrument. For the rear location ($x=58.0$ cm) the plate support was modified to increase its rigidity. This effectively eliminated any plate tilt.

As will be noticed from figure 14, the scatter of the experimental points is considerably less at the rear location than in front. During all the measurements precautions were taken to insure that there was no flow through the

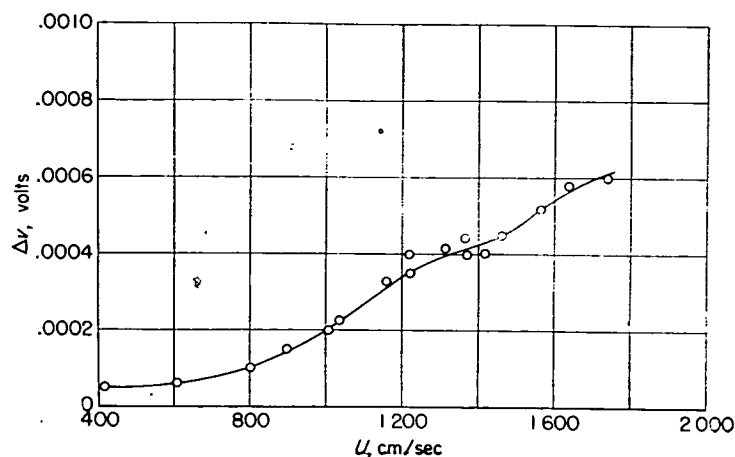


FIGURE 13.—Correction for deflection of flat plate.

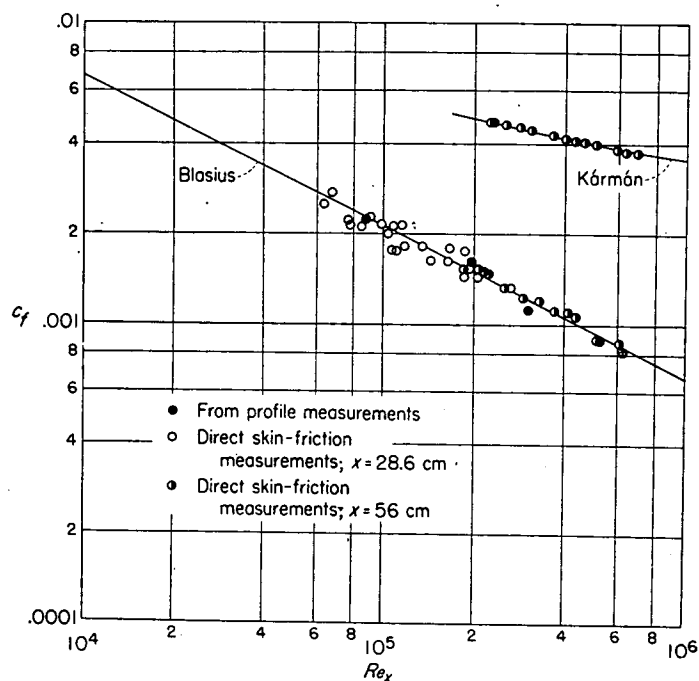


FIGURE 14.—Local skin friction in incompressible flow. $M=0$.

gaps between the force element and the plate. This was done by sealing the windshield onto the plate and checking the pressure difference between its inside and the plate surface on an alcohol U-tube.

Turbulent boundary layer.—The experimental procedure for the skin-friction measurements on the flat plate with turbulent boundary layer was essentially similar to that adopted for the laminar case. The boundary layer was tripped, that is, made turbulent, by a roughness element at the leading edge. The type of flow was identified by the shape of the velocity profile (see fig. 12) as well as by the random fluctuations shown by a hot-wire placed close to the flat-plate surface. That the particular manner in which the boundary layer was tripped was of no essential importance to the skin-friction measurements was ascertained by causing the transition in different ways (e. g., raising the free-stream turbulence level with the help of wire screens in the tunnel or, alternatively, by adjusting the plate orientation so that the

stagnation point on the leading edge was located on the side opposite to the surface of measurement). In all the measurements reported here the boundary layer was tripped by a single roughness element, a wire, at the leading edge of the flat plate. This method was adopted in preference to the others because it gave a convenient method of fixing the point of transition almost at the leading edge of the flat plate. Thus, any arbitrariness about the length entering into the calculation of the Reynolds number of the turbulent-boundary-layer flow is eliminated. That the boundary layer was really turbulent close to the leading edge was confirmed by a total-head-tube survey in the direction of flow with the tube touching the plate. The change from laminar to turbulent flow is then indicated as a sharp rise in total-head loss by the total-head tube.

RESULTS AND DISCUSSION

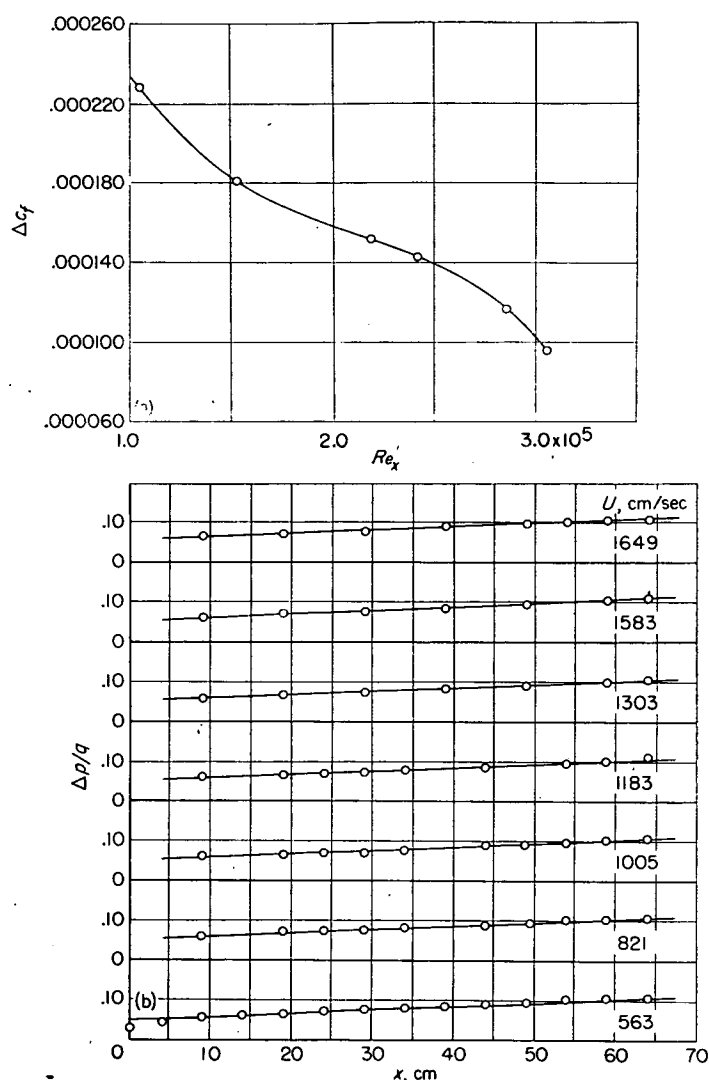
Laminar boundary layer.—The direct skin-friction measurements are shown in figure 14. They cover a range of Reynolds numbers from about 6×10^4 to 6×10^5 . In comparing the experimental values with theory, the values of the shearing stress, as indicated by the instrument, were corrected for experimental error due to plate deflection and the effect of pressure gradient. The method of correction for plate deflection has been discussed before. It applied only to the measurements at the forward location ($x=28.6$), since at the rear location there was no deflection of the plate and supports. The effect of pressure gradient was taken into account by use of Von Kármán's integral relation for the boundary layer. Since dp/dx was not exactly zero, its magnitude was ascertained by actual measurement. Figure 15 shows these pressure-gradient measurements. It will be noted from figure 15 that at any given value of free-stream velocity at the point of measurement dp/dx has very nearly a constant value over the entire length of the flat plate. Using this fact, the momentum integral relation may be written for a given point as:

$$\frac{\Delta c_f}{c_{f_0}} = \frac{H+2}{q} \frac{dp}{dx}$$

where

$$\Delta c_f = c_{f_0} - c_f$$

is the difference of the local skin-friction coefficient from the Blasius value c_{f_0} ; H , the ratio of displacement to momentum thickness; $q = \frac{1}{2} \rho_\infty U^2$, the dynamic pressure of the free stream; and dp/dx , the constant pressure gradient over the flat plate. Thus for fixed values of x and q , knowing the pressure gradient and using the Blasius value of $H=2.605$, the effect of pressure gradient can be estimated. That the use of the Blasius value for H is justified for the purpose of such calculations may be seen from the profiles of figure 12, which show that for the small values of dp/dx the departure from the Blasius profile is slight. All the measured values of c_f shown in figure 14 have been corrected for the effect of pressure gradient in the manner outlined above. The magnitude of the maximum correction was of the order of 8 percent in c_f . The agreement with the theoretical straight lines of



(a) Correction for pressure gradient; laminar boundary layer.
(b) Pressure distribution on flat plate; incompressible flow.

FIGURE 15.—Pressure-gradient measurements

Blasius is seen to be excellent. The experimental points shown in figure 14 are the results of several observations taken at different times and to some extent indicate the repeatability of the measurements. Also shown in the same figure are values of c_f obtained from the profiles of figure 12 by measuring the slope at the wall and correcting for the dp/dx effect.

From the discussion it is evident why the few scattered experiments on flat-plate skin friction often give too large values of c_f . The reason for this is the extreme sensitivity of the laminar layer to pressure gradients. If the pressure gradient is slightly negative, the layer is stable and laminar, but the skin friction is too high. If the pressure gradient is slightly positive, turbulent bursts are likely to appear and c_f is measured too large again.

Turbulent boundary layer.—The measurements of local skin friction with turbulent boundary layer on the plate were also corrected for pressure-gradient effect in a manner similar to the one adopted for the laminar case. For the purposes of this correction the values of H and θ were

obtained by assuming a power-law velocity profile in the boundary layer with a power index of $1/7$. The magnitude of the corrections in this case was of the order of 2 percent in c_f . Figure 14 shows the excellent agreement between the experimental values and the theoretical curve given by Von Kármán for the skin friction on a flat plate with turbulent boundary layer. This agreement is believed to be fortuitous to some extent, for the constants in Von Kármán's logarithmic relation

$$\frac{1}{\sqrt{c_f}} = A + B \log_{10} \sqrt{Re_x}$$

are not predicted by theory but have to be determined from experiment. The values of A and B , as found from the present experiments, were -0.91 and 5.06 , respectively, as compared with 1.7 and 4.15 in Von Kármán's formula. This difference in the constants is perhaps to be attributed to the fact that A and B are not absolute constants but depend somewhat on the conditions of experiment, for example, on Reynolds number and so forth. One can only conclude that the logarithmic formula of Von Kármán is a fair approximation for incompressible flow. Because of the limited range of the tunnel turbulent-boundary-layer flow at Reynolds numbers higher than 6×10^5 could not be studied.

COMPRESSIBLE-FLOW MEASUREMENTS

It was originally intended to investigate two simple representative types of viscous compressible flow, namely, the steady laminar and turbulent flows on an insulated flat plate with zero pressure gradient. It has already been pointed out that even in the incompressible-flow measurements considerable difficulties were encountered in obtaining experimental conditions which closely approximated the conditions forming the basis of theoretical analyses. In the case of the compressible-flow measurements the additional limitations imposed by the small size of the wind tunnel and by the effects accompanying compressibility were sometimes so severe that they resulted in either changing some of the original aims or even abandoning them. For instance, it was highly desirable to obtain skin-friction measurements for the case of a laminar boundary layer, for which rather extensive theoretical information is available. Experimentally, however, it was found impossible to maintain laminar flow without rather strong favorable pressure gradients (see section "Laminar-Boundary-Layer Measurements"). Hence, the measurements in the laminar case had to be restricted to a single set of measurements of skin friction with known pressure gradients. If the following pages indicate a lack of completeness and continuity it is largely a result of features inherent in the phenomenon being investigated.

RANGE OF WIND TUNNEL

The investigations were carried out in the 4- by 10-inch GALCIT transonic wind tunnel. This is a continuously operating closed-return type of tunnel equipped with air filter, silica-gel dehumidifier, and a flexible nozzle (reference 21). The Mach number range of this wind tunnel extends from about $M=0.25$ to $M=1.5$. Since all the

skin-friction measurements were carried out at a fixed location of the force-measuring element, the possible means of Reynolds number control were limited to:

- (a) Velocity changes
- (b) Stagnation-temperature changes
- (c) Stagnation-pressure changes

Of these, the last two leave the Mach number unaffected and hence are very suitable for isolating Reynolds number effects from Mach number effects. Unfortunately, the range of Reynolds number variations possible by these means was extremely limited in the present case, and for all practical purposes changes in Reynolds number were possible only through Mach number changes. The available range of Reynolds numbers was thus confined to approximately $Re=3 \times 10^5$ to $Re=1.2 \times 10^6$. Since experimental limitations severely limited the investigation on laminar boundary layers, the main part of the compressible-flow measurements was confined to turbulent-boundary-layer flow.

EXPERIMENTAL PROCEDURE—TURBULENT BOUNDARY LAYER

The size of the flat plate and the general arrangement are shown in figure 6. It will be noticed that the flat plate has a sharp leading edge, one side of the plate (the measuring surface) being parallel to the flow. For the case of the high-subsonic-flow measurements ($M=0.2$ to 0.8) this configuration served to produce turbulent boundary layers starting at the leading edge of the plate, the stagnation point in this range being on the surface opposite to the measuring surface. However, in order to remove any doubt regarding the nature of boundary-layer flow for most of the turbulent-boundary-layer measurements, a 0.005-inch-diameter steel wire cemented about 0.2 centimeter from the leading edge of the plate was used as a trip. Figure 16 shows that the experimental points with and without the wire trip agree very well. As shown in the sketch (fig. 6), the flat plate does not quite span the width of the wind tunnel. Previous experience in

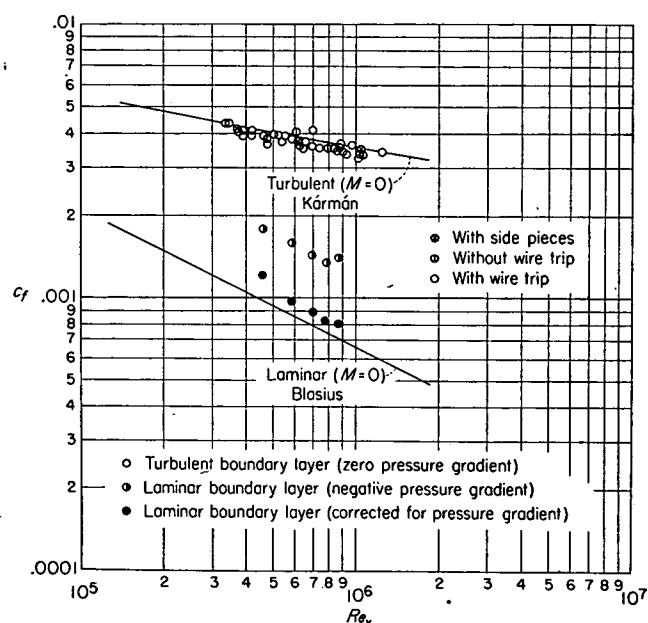


FIGURE 16.—Local skin friction on flat plate in subsonic flow. Mach number range, $M=0.2$ to 0.8 .

investigating high-speed flow phenomena had indicated that the side-wall boundary-layer effects were minimized by shortening the span of two-dimensional models an amount equal to the displacement thickness of the boundary layer. In the present case this amounted to approximately $\frac{1}{4}$ inch on either side of the flat plate. Detachable side pieces could be installed on the flat plate so that it effectively spanned the entire span of the tunnel. As a matter of precaution skin-friction measurements were made both with and without these side pieces. Figure 16 shows the agreement between these measurements. Figure 17 shows repre-

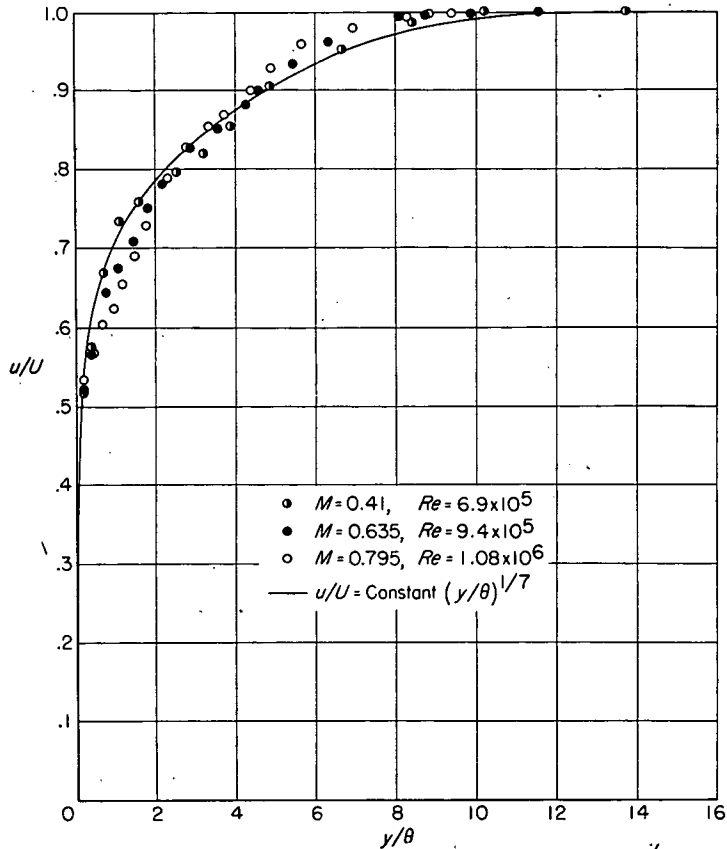


FIGURE 17.—Turbulent-boundary-layer velocity profiles on flat plates.

sentative velocity profiles measured with a total-head tube and computed on the basis of constant energy per unit mass through the boundary layer. The turbulent-boundary-layer skin-friction measurements shown in figure 16 are the results of some eight separate sets of observations. During these measurements every effort was made to eliminate or account for errors which might affect the reliability of the results. The following points were especially important.

Pressure gradients.—Figure 18 shows representative pressure distributions measured with a static-pressure probe and an alcohol micromanometer. Since the probe travel was limited, it was not possible to obtain the pressure distributions over the entire length of the plate. However, it is known from previous calibration checks of the flow in the tunnel that the flow is quite smooth, so that the pressure distributions shown in figure 18 extend continuously in either direction. As seen in figure 18 the pressure gradients are quite small but not zero. To estimate the effect of these small pressure gradients on the measured skin friction, use

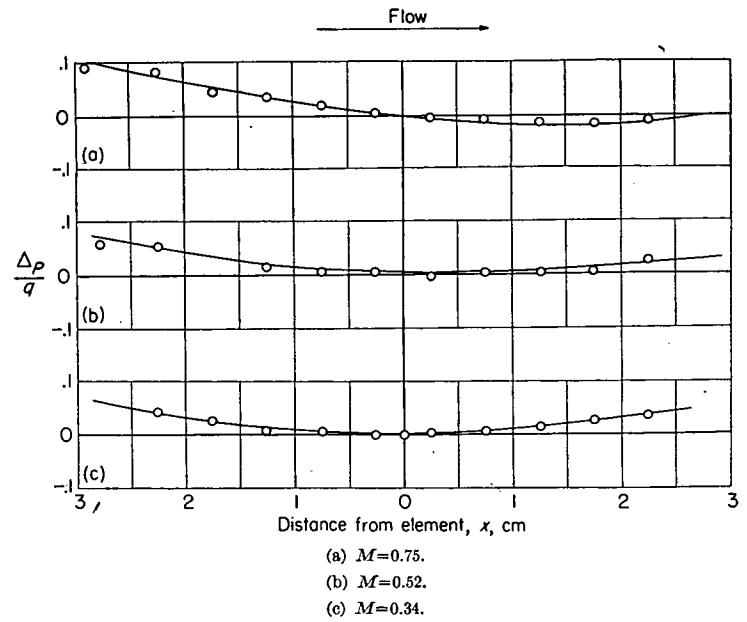


FIGURE 18.—Representative pressure distributions on flat plate with turbulent boundary layer. Supersonic flow.

was again made of the Von Kármán momentum integral. It was assumed for these computations that the turbulent-boundary-layer profiles could be represented by a power law. Justification for this is apparent from figure 17. The momentum integral relation for compressible flow may be written as

$$c_f = 2 \frac{d\theta}{dx} - \left[(2 - M^2) + \frac{\delta^*}{\theta} \right] \left(\frac{\theta}{q} \frac{dp}{dx} \right)$$

where the symbols have their usual meanings. It is well-known that as a first approximation θ , the momentum thickness, may be said to be unaffected by compressibility effects (equivalent to an assumption of $\rho u = \text{Constant}$). Using this and neglecting the effect of pressure gradients on θ and H one may write

$$H_c = H_i \left[1 + \frac{\gamma - 1}{2} M^2 \left(1 + \frac{1}{H_i} \right) \right]$$

where $H_c = (\delta^*/\theta)_{\text{compressible}}$ is the shape parameter for compressible flow and H_i , the same for incompressible flow. Assuming a $\frac{1}{7}$ -power velocity profile one can use the incompressible-flow values of θ_i and H_i given by

$$\theta_i = 0.036 \left(\frac{\nu}{U} \right)^{1/5} x^{4/5}$$

$$H_i = 1.286$$

The effect of pressure gradient then may be approximately determined by writing the integral relation as

$$\Delta c_f = - \left[(2 - M^2) + H_c \right] \frac{\theta_i}{q} \frac{dp}{dx}$$

where

$$\Delta c_f = c_f - 2 \frac{d\theta}{dx}$$

and using the approximate values for H_c and θ_i from above and $\frac{1}{q} \frac{dp}{dx}$ from figure 18. These calculations show that the maximum error in the local skin-friction coefficient is less than 1 percent.

Temperature equilibrium.—In order to realize the condition of zero heat transfer on the flat plate it is extremely important that thermal conditions be stable during operation of the wind tunnel. The method of supporting the flat plate along its center line by a thin strut minimized any losses due to conduction. In addition, the measurements were carried out with the stagnation temperature of the air in the neighborhood of the room temperature. The stagnation temperature could be maintained constant to approximately $\pm 1^\circ \text{C}$. The usual time taken for the skin-friction-instrument readings to become steady was of the order of 2 minutes or so after the tunnel flow and the stagnation temperature had stabilized. That conditions on the flat plate had really reached the state of zero heat transfer was confirmed by a calibrated thermocouple probe touching the surface of the plate. The thermocouple output was read by a precision Leeds and Northrup potentiometer on which temperature changes of about 0.01°C could be easily detected. It was found that, if any changes in stagnation temperature were made, thermal equilibrium on the plate was attained after 2 or 3 minutes of stabilization. During the period of stabilization the effect on the skin friction was observed to be strikingly large. It is for this reason that these precautions are strongly stressed here. As a point of interest, the temperature recovery on the plate, as indicated by the thermocouple probe, is shown in figure 19. Also shown for comparison is the usual empirical relation for temperature recovery on a flat plate with a turbulent boundary layer. The results in figure 19 are to be regarded only as of a qualitative nature, since no account is taken of the effect of the probe itself on the temperature recovery. The main reason for these measurements was to confirm that thermal equilibrium had been reached on the plate. On the basis of these experiments the usual procedure followed was to let conditions stabilize for over 10 minutes or so before recording the skin-friction readings.

Gap flows.—During all the measurements reported here there was no flow through the gaps separating the force element from the plate. This was ascertained by means of an alcohol U-tube which, for all the reported measurements, indicated less than 0.5 centimeter of alcohol difference in pressure across the gaps between the inside of the shield and the outside flow. In order to determine the degree to which the skin-friction readings were affected when there was flow through the gaps, a test setup was arranged. Any desired pressure difference could be maintained across the

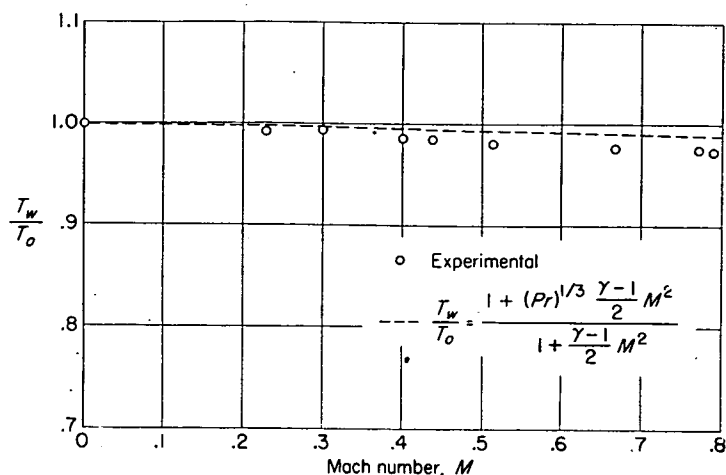


FIGURE 19.—Temperature recovery on flat plate with turbulent boundary layer. Zero heat transfer.

gaps by means of a suction-pump arrangement connected to the inside of the sealed windshield of the skin-friction instrument. It was found that, with no flow in the tunnel, pressure differences of about 5 centimeters of alcohol caused an error of about 3 percent in the reading of the instrument. This was true regardless of the direction of the pressure difference. Of course these values would be somewhat different under actual flow conditions. But they certainly would not change by an order of magnitude and, since during the measurements the pressure difference was always less than $\frac{1}{2}$ centimeter of alcohol, only negligible errors could have been caused.

Errors due to deflections and vibrations.—The effects due to deflections and vibrations were checked for in a manner similar to one adopted during the incompressible-flow measurements. The force element was covered up with a streamlined shield protecting it from the flow but leaving it completely free to move. If any deflection of the structure supporting the instrument occurred it would show up as a reading. A similar effect would be produced in the case of imperfect vibration isolation. It was found that the installation of the instrument in the wind tunnel was perfectly rigid and that the vibration isolation and damping were quite adequate, so that no errors were introduced on these accounts.

Occurrence of transonic regions and shock waves.—The upper limit (approx. $M=0.8$) of Mach number for the high-subsonic-flow measurements reported here was set by the appearance of shock waves near the leading edge of the flat plate. It was found that with the appearance of a transonic shock at the leading edge of the plate no sudden changes could be observed in the skin friction at the point of measurement. Nevertheless, in order to retain clean experimental

conditions it was decided to restrict the measurements to flow without shock waves.

Alinement of force element.—During the course of the experiments reported here the alinement of the force element with the rest of the plate surface was periodically checked optically by means of interference fringes. During the entire period of measurement the alinement remained the same to within $\pm 11 \times 10^{-6}$ inch (corresponding to a fringe movement of one fringe width of cadmium light).

RESULTS OF MEASUREMENTS FOR TURBULENT BOUNDARY LAYER AT HIGH-SUBSONIC SPEEDS

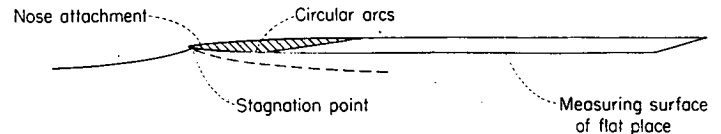
The experimental values of local skin-friction coefficient as measured by the skin-friction instrument are shown in figure 16. In computing the skin-friction coefficient and the Reynolds number, free-stream values of density and viscosity have been used. As seen, the experimental points lie quite close to (but definitely below) the curve for incompressible flow, showing that the effect of compressibility in the range of Mach numbers up to $M=0.8$ is quite small but in the expected direction. The experimental scatter is of the order of ± 5 percent about the mean.

As briefly mentioned before, the effect of heat transfer on the skin friction was found to be quite important. Quantitative measurements of this effect were not possible with the present equipment, but in general the following effect was observed: If after equilibrium conditions had been attained in the wind tunnel the stagnation temperature in the settling chamber was lowered, the skin-friction reading showed a marked increase. The magnitude of this increase seemed to depend on the rate of heat transfer for it could be controlled to some extent by controlling the rate of cooling of the air in the settling chamber of the tunnel. Roughly speaking (for no precise measurements could be obtained), during the period of heat transfer the skin friction changed by about 50 percent of the steady-state (zero-heat-transfer) value. Raising the stagnation temperature resulted in lower skin-friction readings. Upon reattainment of steady conditions, the skin friction always regained its zero-heat-transfer value. As indicated by these observations, the resultant effect of heat transfer from or to the flat plate is the opposite of what one expects from theoretical considerations. No explanation has been found for this anomalous behavior.

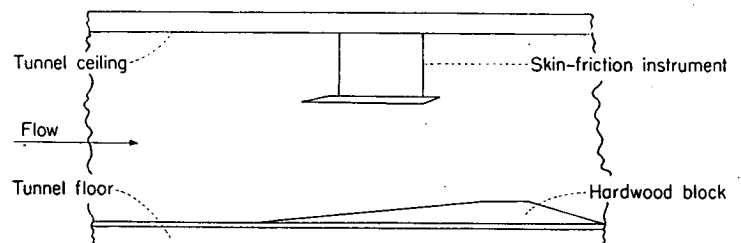
LAMINAR-BOUNDARY-LAYER MEASUREMENTS

It has been mentioned before that because of experimental difficulties it was not possible to obtain laminar boundary layers of sufficient extent without favorable pressure gradients. There are several reasons for this. Laminar boundary layers are notoriously sensitive to leading-edge conditions. For supersonic-flow measurements it is necessary to use the type of "half-wedge" leading edge employed in the present experiments (see reference 24). This type of leading edge is not very suitable for subsonic work, for it requires the stagnation point to be on the side of the surface of measurement. In the present setup this would require the flat plate to be at a relatively large positive angle of

attack. This was not feasible. For the case of supersonic flow with an attached bow wave this difficulty would be obviated. (Actually, in the supersonic-flow measurements other effects prevented any reliable measurements of skin friction with laminar boundary layers. See section "Compressible-Flow Measurements.") As an alternative for obtaining laminar boundary layers on the flat plate, a special attachment was made for modifying the nose shape. As seen in the accompanying sketch, this served the earlier-



mentioned purpose of fixing the stagnation point where desired. With this attachment laminar boundary layers could be obtained on the flat plate at low-subsonic Mach numbers (about $M=0.3$). At higher subsonic Mach numbers, however, the disturbance caused by the joint between the nose attachment and the leading edge of the flat plate was sufficient to cause transition ahead of the point of measurement. In spite of all efforts to make the joint smooth, laminar boundary layers could not be maintained past the force element. The only way laminar boundary layers could be obtained with this configuration was with the help of stabilizing pressure gradients in the direction of flow. The favorable pressure gradient was produced by altering the shape of the wind-tunnel wall opposite the flat plate with the help of a tapered hardwood block. The scheme is shown in the accompanying sketch. The pressure gradients



prevailing over the flat plate were then measured for each Mach number at which the skin friction was measured. Representative pressure distributions on the flat plate are shown in figure 20. The skin-friction measurements are presented in figure 16. As a point of interest, the skin-friction coefficients are also shown after approximately allowing for the measured pressure gradient; these points are seen to be approximately 15 percent above the Blasius curve for $M=0$. The calculation of the dp/dx effect was carried out in a manner similar to that for the turbulent case. Although these measurements cannot be used for comparison with the cases computed by theory, they are believed to be of some value. It will be noticed that, in spite of the large deviation from the condition of zero pressure gradient and the crude method for allowing for pressure-gradient

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REPORT 1122

SURVEY OF PORTIONS OF THE CHROMIUM-COBALT-NICKEL-MOLYBDENUM QUATERNARY SYSTEM AT 1,200° C¹

By SHELDON PAUL RIDEOUT and PAUL A. BECK

SUMMARY

A survey was made of portions of the chromium-cobalt-nickel-molybdenum quaternary system at 1,200° C by means of microscopic and X-ray diffraction studies. Since the face-centered cubic (alpha) solid solutions form the matrix of almost all practically useful high-temperature alloys, the solid solubility limits of the quaternary alpha phase were determined up to 20 percent molybdenum. The component cobalt-nickel-molybdenum, chromium-cobalt-molybdenum, and chromium-nickel-molybdenum ternary systems were also studied. The survey of these systems was confined to the determination of the boundaries of the face-centered cubic (alpha) solid solutions and of the phases coexisting with alpha at 1,200° C.

INTRODUCTION

In the development of technologically useful alloys it is usually of considerable help if the phase relationships and solid solubility limits are known. At the Metallurgy Department of the University of Notre Dame a project has been in progress for some years to determine the phase relationships in alloy systems involving chromium, cobalt, nickel, iron, and molybdenum; the transition elements of greatest importance in high-temperature alloys.

The determination of phase diagrams for systems of four or more components is an extremely laborious task. The problem must be approached in a systematic manner in order to avoid becoming hopelessly lost. The best method of attack is to begin by establishing the phase relationships in systems of two or three components and then continue by adding one new element at a time. The problem of presenting quantitative phase relationships diagrammatically for systems of three or more components necessitates holding one or more thermodynamic variables constant. For example, a ternary phase diagram may be presented as a series of isothermal sections or as a series of sections in each of which the amount of one component is held constant. For a quaternary system it is necessary to hold both temperature and the amount of one component constant in order to obtain two-dimensional diagrams. The temperature 1,200° C was chosen as that at which an initial isothermal survey could be most profitably made. This temperature is of immediate interest because it lies within the range of solution treatment for most high-temperature alloys now in use and also because here diffusion rates are fast enough to allow equilibrium conditions to be approached in reasonably short annealing periods. At lower temperatures, such as 800° C, the determination of these phase diagrams within extensive

composition ranges would be too time-consuming and therefore expensive. Work of this kind is planned only for limited important composition ranges.

Thus far two reports have been issued covering work done on this project. In the first report (reference 1) the 1,200° C isothermal section of the chromium-cobalt-nickel ternary system was presented. The second report (reference 2) gave a survey of the chromium-cobalt-nickel-iron quaternary system at 1,200° C in the composition ranges near the face-centered cubic solid-solution phase. This report also presented a study of some features of the chromium-cobalt-nickel and the chromium-cobalt-iron ternary systems at lower temperatures.

The present report presents a survey of portions of the chromium-cobalt-nickel-molybdenum quaternary system at 1,200° C. The face-centered cubic (alpha) solid solutions have, by far, the greatest practical importance in high-temperature alloys. For this reason the solid solubility limits of the quaternary alpha phase were determined up to 20 percent molybdenum.

It was also necessary to investigate the component cobalt-nickel-molybdenum, chromium-cobalt-molybdenum, and chromium-nickel-molybdenum ternary systems, since these systems had not been adequately explored by previous investigators. (See "Literature Survey" for discussion of previous studies of the binary and ternary systems comprising the chromium-cobalt-nickel-molybdenum quaternary system.)

At the beginning of this work it became apparent that a definite and consistent nomenclature for phase designation would have to be adopted in order to avoid confusion and contradiction. For example, the iron-chromium sigma phase is isomorphous with the chromium-cobalt gamma phase and the iron-molybdenum zeta phase. In order to avoid contradiction in ternary and quaternary systems of these elements, where the isomorphous phases form uninterrupted solid solutions, it was decided to give all these phases the same designation. The designation sigma phase was selected as one most generally associated with this particular structure. Since this work began with a nonferrous system, the face-centered cubic solid solutions in the systems involving chromium, cobalt, nickel, and molybdenum have all been designated as alpha phase. For the sake of consistency this nomenclature was extended even to the chromium-cobalt-nickel-iron quaternary system, although this phase in ferrous systems has been customarily referred to as gamma. On the other hand, the body-centered cubic solid solutions, which are usually named alpha in ferrous systems, are here referred

¹ Supersedes NACA TN 2683, "Survey of Portions of the Chromium-Cobalt-Nickel-Molybdenum Quaternary System at 1200° C" by Sheldon Paul Rideout and Paul A. Beck, 1952

to as epsilon phase in conformity with the nomenclature used for the chromium-rich alloys in the chromium-cobalt binary system. In the cobalt-molybdenum binary system the intermediate phase which coexists with alpha at 1,200° C has been earlier designated as epsilon. In the present investigation this phase was renamed mu since, as stated above, epsilon had already been used to designate the chromium-rich body-centered cubic phase. The hitherto unknown ternary phases discovered in the 1,200° C isothermal sections of the chromium-cobalt-molybdenum and the chromium-nickel-molybdenum ternary systems have been named R and P, respectively.

This work was conducted at the University of Notre Dame under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. The authors wish to thank Mr. Francis Pall for doing some of the work relating to the chromium-cobalt-molybdenum and chromium-nickel-molybdenum systems. The assistance of Messrs. C. Patrick Sullivan and Robert Hochman is also appreciated.

LITERATURE SURVEY

The chromium-cobalt-nickel-molybdenum quaternary system comprises the following six binary and four ternary systems: Chromium-cobalt, chromium-nickel, cobalt-nickel, chromium-molybdenum, cobalt-molybdenum, nickel-molybdenum, chromium-cobalt-nickel, chromium-cobalt-molybdenum, chromium-nickel-molybdenum, and cobalt-nickel-molybdenum. All of the above binary systems were investigated previously. The chromium-cobalt system was investigated by Elsea, Westerman, and Manning (reference 3), and the most recent chromium-nickel phase diagram was reported by Jenkins, Bucknall, Austin, and Mellor (reference 4). The "Metals Handbook" (reference 5) gives the accepted cobalt-nickel diagram, which was thoroughly investigated by several workers.

The chromium-molybdenum binary system is also given in the "Metals Handbook" (reference 5). In recent work on molybdenum-rich alloys Kessler and Hansen (reference 6) confirm the results of other investigators that chromium and molybdenum are completely soluble in the solid state. The cobalt-molybdenum binary system was investigated by Sykes and Graff (reference 7). The transformation from face-centered cubic to hexagonal in pure cobalt above 1,000° C indicated in their diagram was not found by several later investigators (references 8 to 10). Henglein and Kohsok (reference 11) recently pointed out that the cobalt-molybdenum intermediate phase Co_7Mo_8 is isomorphous with the iron-molybdenum, cobalt-tungsten, and iron-tungsten intermediate phases Fe_7Mo_8 , Co_7W_8 , and Fe_7W_8 . The crystal structure of the Co_7Mo_8 phase can be described as hexagonal or rhombohedral (reference 11). The nickel-molybdenum binary system, as determined by Ellinger, is given in reference 5.

Of the four ternary systems involved in the chromium-cobalt-nickel-molybdenum quaternary system, two were previously investigated. The 1,200° C isothermal section of the chromium-cobalt-nickel ternary system was investi-

gated by Manly and Beck (reference 1). This diagram was slightly modified in the chromium-cobalt sigma-phase-field region by Kamen and Beck (reference 2). Siedschlag (reference 12) investigated the chromium-nickel-molybdenum ternary system, but the results of that investigation proved to be of no help in the present work. The nickel-rich alloys were examined from a technological point of view only, and phase relationships were not studied in detail. It was, therefore, necessary to investigate the three ternary systems chromium-cobalt-molybdenum, chromium-nickel-molybdenum, and cobalt-nickel-molybdenum. The survey of these systems was confined to the determination of the boundaries of the face-centered cubic (alpha) solid solutions and of the phases coexisting with the alpha phase at 1,200° C.

The brittle intermetallic sigma phase, which occurs in many systems involving the transition elements, has recently become the subject of much interest and investigation. Sully and Heal (reference 13) have pointed out that the iron-chromium sigma phase and the chromium-cobalt gamma phase are isomorphous, and Goldschmidt (reference 14) found that the iron-molybdenum zeta phase is also isomorphous with the iron-chromium sigma phase. Beck and Manly (reference 15) investigated the chromium-cobalt-iron and the chromium-cobalt-nickel ternary systems. They proved that the iron-chromium sigma phase and the chromium-cobalt gamma phase form an uninterrupted series of solid solutions across the chromium-cobalt-iron ternary isothermal section at 800° C. They also found that the chromium-cobalt gamma phase extends deep into the chromium-cobalt-nickel ternary system at 1,200° C in a manner suggesting that nickel atoms and cobalt atoms substitute for each other in forming the sigma phase, while the chromium content of the phase remains essentially unchanged. On the basis of these results and of the existence of the sigma phase in the iron-vanadium system, Beck and Manly (reference 15) suggested that the sigma phase should also occur in the cobalt-vanadium and the nickel-vanadium systems. Indeed, this was confirmed by Duwez and Baen (reference 16) who also formally postulated the criterion that a face-centered and a body-centered cubic metal are required in an alloy system before the sigma phase will form. In the present investigation special attention was given to the occurrence of the sigma phase, and further confirmation was obtained for the earlier observation that atoms of elements of like structure can substitute for each other in the formation of the sigma phase. A criterion for the formation of the sigma lattice in terms of electron vacancy concentration in the 3d sub-band was derived in order to rationalize the above observations.

EXPERIMENTAL PROCEDURE

In this work the phase boundaries were established by microscopic examination of carefully homogenized alloys. The phases were identified by means of X-ray diffraction. Details relating to the equipment used and to the melting and homogenizing procedures followed throughout the work were reported by Manly and Beck (reference 1). Molyb-

denum in the form of a 1/8-inch-diameter rod was used to make up the alloys. The lot analyses of the electrolytic chromium, of the cobalt rondelles, and of the nickel used are given in table I.

Most of the alloys were melted in Alundum crucibles, except for 34 alloys melted in zirconia and stabilized zirconia crucibles at the beginning of the work. The type of crucible used to melt each alloy is listed in table II. Early in the work it was found that molybdenum-bearing alloys were susceptible to zirconium pickup. This was confirmed by semiquantitative spectrographic analysis. The use of zirconia and stabilized zirconia crucibles was, therefore, discontinued, and all subsequent alloys were melted in Alundum crucibles. The zirconium pickup is treated in greater detail under "Discussion."

The ingots were generally found to be free of excessive segregation, except in a few isolated cases discussed later. Specimens for homogenization and subsequent microscopic and X-ray analysis were taken from the bottom section of each ingot. Immediately adjacent examples were used for chemical analysis. All specimens which consisted mainly of the face-centered cubic (alpha) solid solution were double-forged prior to homogenization. This double-forging treatment consisted of heating the specimen at 1,200° C for 1/2 hour, forging, reheating at 1,200° C for 1/2 hour, and forging again. With this preliminary treatment alpha alloys were easily homogenized by annealing at 1,200° C for 48 hours. Alpha alloys containing more than about 20 percent of any second phase were too brittle to be forged. Specimens from such alloys were homogenized for 95 to 150 hours at 1,200° C. All specimens were quenched directly into cold tap water. It is very important that the oxygen be removed from the furnace atmosphere in which alloys containing molybdenum are being annealed, because of the extremely rapid rate of oxidation of these alloys at high temperatures. It is believed that the homogenizing treatments used gave very nearly equilibrium conditions, except in a few cases to be discussed later, because continued annealing, which in some cases was extended up to 200 hours, resulted in no detectable further changes in the microstructures.

After homogenization a powder for X-ray analysis was taken from each specimen, by either filing or crushing, depending on the brittleness of the alloy, and the remaining piece was prepared for microscopic examination to detect the presence or absence of a second phase. The phase boundaries were thus determined by the disappearing phase method.

It was found necessary to vary the etching procedure considerably, according to the composition of the alloy.

(1) The following etchants were used successfully to differentiate between the various phases:

Hydrochloric acid, milliliters.....	8. 0
Nitric acid, milliliter.....	0. 5
Glycerin, milliliters.....	2. 0
Cupric chloride, milligrams.....	50 to 150

The freshly polished specimen was either immersed in the etchant for several minutes, or, for more rapid attack,

swabbed with cotton saturated with the etchant. This etchant was used only to reveal the structure of the alpha phase; grain boundaries, annealing twins, and transformation striations became apparent, and any second-phase particles present were clearly delineated. It was not possible, however, to identify which second phase was present by using this etchant alone.

(2) The following electrolytic etching and staining procedure was used with alloys in the chromium-nickel-molybdenum ternary system:

Oxalic acid, grams.....	8
Distilled water, milliliters.....	92
Cathode.....	Stainless steel
Voltage, volts (d-c.).....	6
Electrode spacing, inch.....	1
Temperature, °C.....	20 to 30
Time, seconds.....	8 to 10

The specimen was removed from the etching bath and immediately immersed for 10 to 20 seconds into a staining solution consisting of 5 grams of potassium permanganate and 5 grams of sodium hydroxide dissolved in 90 milliliters of distilled water. This etching and staining method worked well with the epsilon phase. Grain boundaries became evident and the characteristic Widmanstätten precipitate of sigma in the epsilon phase, when present, became clearly recognizable. The epsilon phase was stained a very light tan color, or a darker brown when the Widmanstätten precipitate was present. The alpha phase always remained unattacked and unstained. The chromium-nickel-molybdenum ternary sigma phase always stained, and a faint structure was sometimes brought out in this phase. The color of the stain on the sigma phase varied from bright green or red to purple. The grain boundaries of the ternary P phase were attacked and this phase, too, was always stained, the color varying from green to red and orange. The delta phase was generally very unevenly attacked and either did not stain or stained an uneven brownish color. Grain boundaries were revealed in the delta phase. With chromium-nickel-molybdenum alloys great success was experienced in differentiating between epsilon, sigma, and alpha when these phases co-existed. As a result of this, it was possible to locate the corner of the three-phase alpha-plus-epsilon-plus-sigma field on the alpha boundary by metallographic means. Even when present as small particles in an alpha matrix, sigma consistently stained a brilliant color, whereas the epsilon phase never stained more than a light tan color. It was difficult to distinguish between minor amounts of the sigma and the P phases in an alpha matrix because of their similar staining characteristics. However, when sigma and P were adjacent in the same alloy, it was possible to distinguish one phase from the other. Final identification, of course, was made by means of X-ray analysis.

(3) The electrolytic oxalic-acid etchant described above proved unsatisfactory for chromium-cobalt-molybdenum alloys. The following electrolytic etchant, used in conjunction with the previously described staining solution, was used with satisfactory results:

Concentrated phosphoric acid, milliliters	5
Distilled water, milliliters	95
Cathode	Stainless steel
Voltage, volts (d-c.)	6
Electrode spacing, inch	1
Temperature, °C.	20 to 30
Time, seconds	10 to 15

The alpha phase was lightly attacked but remained unstained. The sigma phase was lightly attacked and stained colors varying from orange to blue or purple. A fine Widmanstätten precipitate was sometimes observed in the sigma phase. The chromium-cobalt-molybdenum ternary R phase was unattacked, but this phase stained yellow to rusty brown. The mu phase was attacked and usually stained pale blue or did not stain at all. Grain boundaries and tiny annealing twins were observed in the mu phase. Sigma, R, and alpha were easily distinguished from each other when these phases occurred together in the same alloy, but small particles of sigma were indistinguishable from small particles of either the R or the mu phase in an alpha matrix. The mu phase was readily differentiated from R.

(4) It was necessary to decrease the acidity of the phosphoric-acid electrolyte and increase the alkalinity of the alkaline permanganate staining solution in order to differentiate successfully between the delta and the mu phases in cobalt-nickel-molybdenum alloys. The following solution was used:

Concentrated phosphoric acid, milliliters	2.5
Distilled water, milliliters	97.5

Etching conditions were the same as those for procedure 3. The specimen was then immersed into a staining solution containing 20 grams of sodium hydroxide and 5 grams of potassium permanganate dissolved in 75 milliliters of distilled water. The alpha phase was lightly attacked but remained unstained. The delta phase was unevenly attacked, grain boundaries being revealed, and was stained dark green or blue. The mu phase was more evenly attacked; grain boundaries and occasional twins were evident. The mu phase stained a number of different colors, apparently depending upon the orientation, each grain or twin showing only one color. With this etching procedure it was possible to detect small amounts of the mu phase in a matrix of delta but, because of the wide variation of color from grain to grain in the mu phase, it was not possible to identify small amounts of delta in a matrix of the mu phase. This difficulty of microscopically identifying small amounts of the delta phase in a matrix of mu was circumvented by the use of an X-ray method to determine the solubility limit of nickel in the mu phase. In a plot of the d values for one high-angle diffraction line in a series of mu alloys as a function of the nickel content, the solubility limit is indicated where the slope of the curve becomes zero.

The powders for X-ray analysis were sealed under vacuum in fused-quartz capsules and annealed at 1,200° C for ½ hour. The capsules were then quenched directly into cold tap water. The powders were mounted on 1- by 1-inch cards with a colloidal glue, and a diffraction pattern was taken in an asymmetrical focusing camera of 20-centimeter diameter,

using unfiltered chromium radiation at 8 milliamperes and 30 kilovolts. Thus, the identity of the phases present in each homogenized alloy was confirmed by X-ray diffraction. However, the X-ray method of identification was found relatively insensitive to small amounts of a second phase, especially in alloys having an alpha matrix. This difficulty was surmounted partially by careful preparation of the powder. A large quantity of filings was collected from the specimen to insure obtaining a representative sample. The minor phase was generally much more brittle than the alpha matrix and had a much smaller particle size. It was concentrated by sieving the powder to 200 mesh. Only the finest powder from the sample was used to prepare the X-ray specimen. In handling this fine powder precautions were necessary in order to avoid losing any of the minor phase. For example, the original filings had to be collected on a smooth tracing paper or Celluloid, to which the powder will not adhere and from which the fine powder may be easily recovered.

An X-ray method of locating three-phase-field corners on the alpha phase boundary, as described previously by Manly and Beck (reference 1), was used in this work to supplement the microscopic results. The method consists of plotting the lattice parameters of the saturated alpha alloys as a function of composition. The three-phase-field corner is usually indicated by a rather sharp change in the slope of the curve. The lattice parameters for these plots were determined by taking back-reflection diffraction patterns from the surface of microspecimens known to be saturated alpha. Flat film and a collimating system of 1-millimeter diameter were used. By using unfiltered chromium radiation and a specimen-to-film distance of 29.54 millimeters, the chromium $K\alpha$ reflections from the (220) planes and the chromium $K\beta$ reflections from the (311) planes of the alpha phase were recorded. Smooth circles were obtained by rotating the specimen about an axis parallel to that of the collimating system but offset from it. The lattice parameter determined from each of these lines was plotted against $\cos^2\theta(2\cos^2\theta-1)$ and extrapolated to $\theta=90^\circ$ in order to eliminate systematic errors.

EXPERIMENTAL RESULTS

The experimental data for all alloys investigated are given in tables III to VI. The 1,200° C isothermal sections for the cobalt-nickel-molybdenum, chromium-cobalt-molybdenum, and chromium-nickel-molybdenum ternary systems and the solubility limits of the alpha phase in the 2.5-, 5-, 10-, and 20-percent-molybdenum quaternary sections, as shown in figures 1 to 8, were drawn in accordance with these data. The amount of each phase, as estimated microscopically in the homogenized structure, is tabulated together with corresponding X-ray diffraction data for each alloy. The amounts of the phases, corresponding to the phase diagrams, are also given for comparison. For alloys which were chemically analyzed the content in acid insoluble material was determined and found to vary from nil to a few tenths of 1 percent. The compositions of chemically analyzed alloys reported in the tables were corrected to 100-percent-metal content.

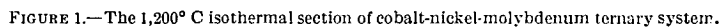


FIGURE 1.—The 1,200° C isothermal section of cobalt-nickel-molybdenum ternary system.

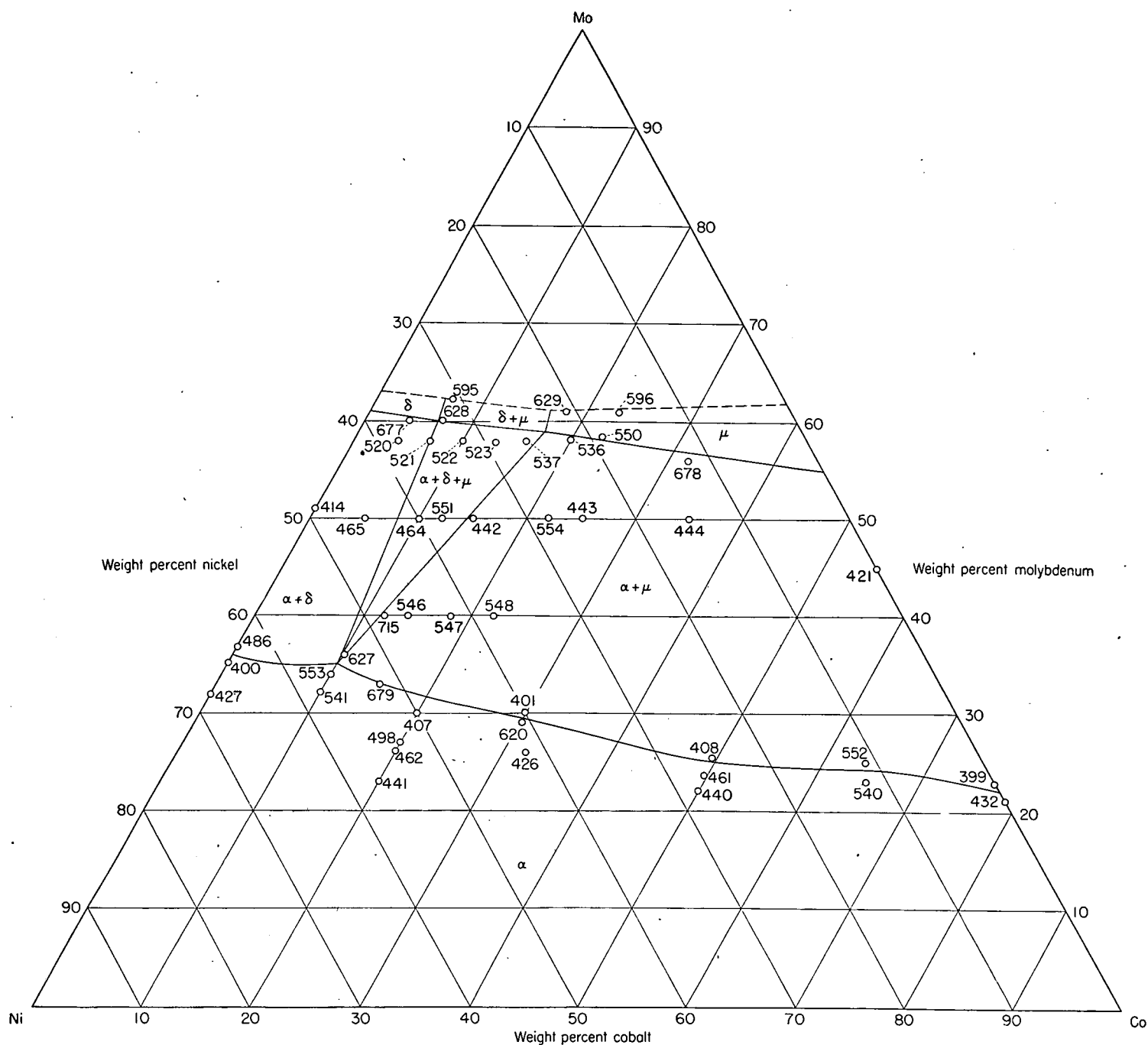


FIGURE 2.—The 1,200° C isothermal section of cobalt-nickel-molybdenum ternary system with alloy compositions indicated.

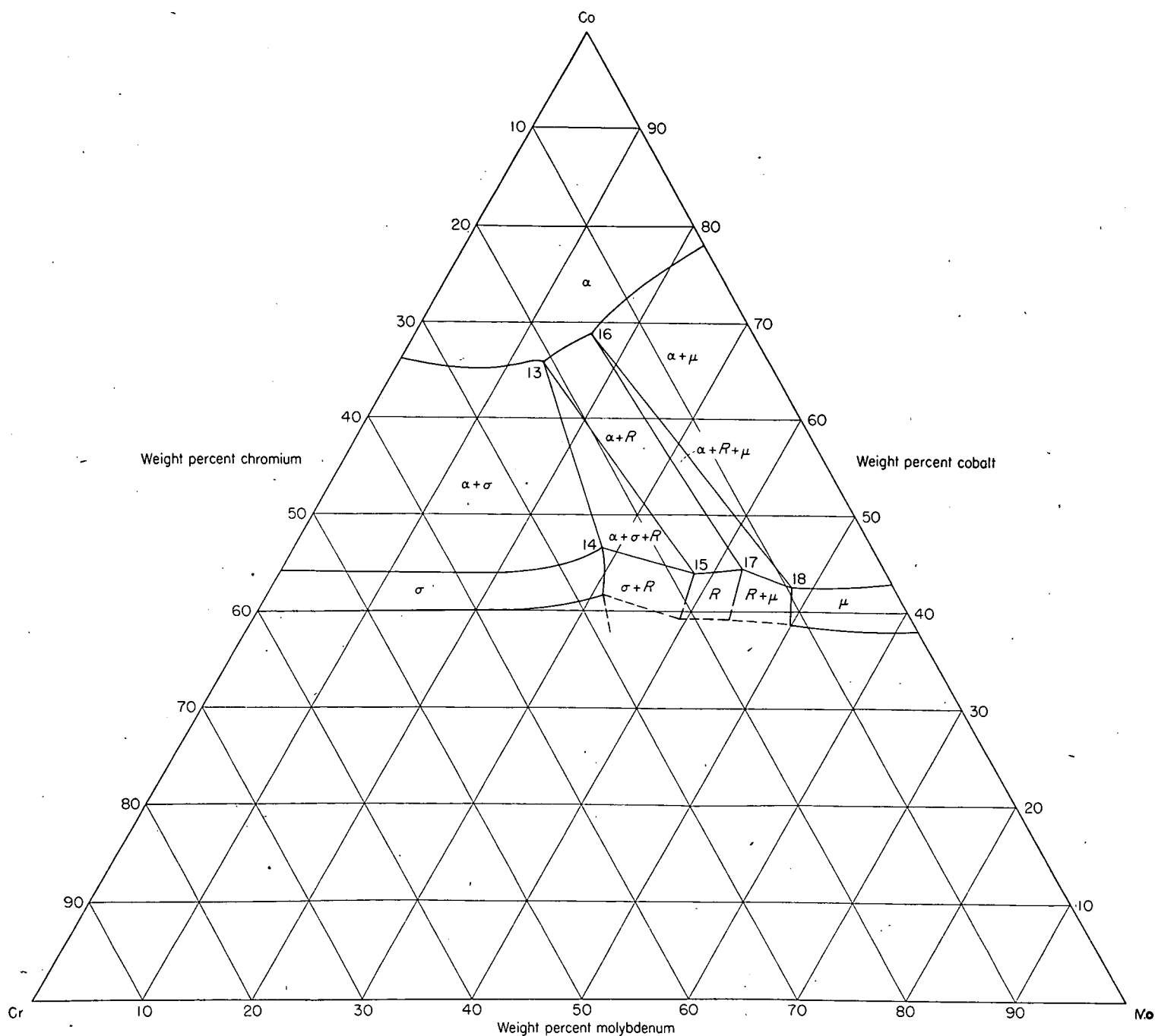


FIGURE 3.—The 1,200° C isothermal section of chromium-cobalt-molybdenum ternary system.

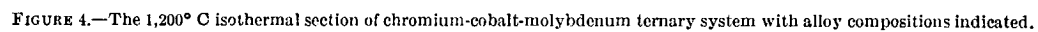


FIGURE 4.—The 1,200° C isothermal section of chromium-cobalt-molybdenum ternary system with alloy compositions indicated.

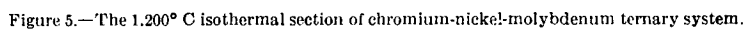
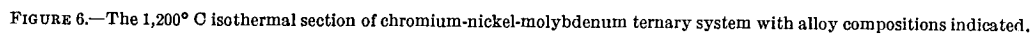


Figure 5.—The 1,200° C isothermal section of chromium-nickel-molybdenum ternary system.



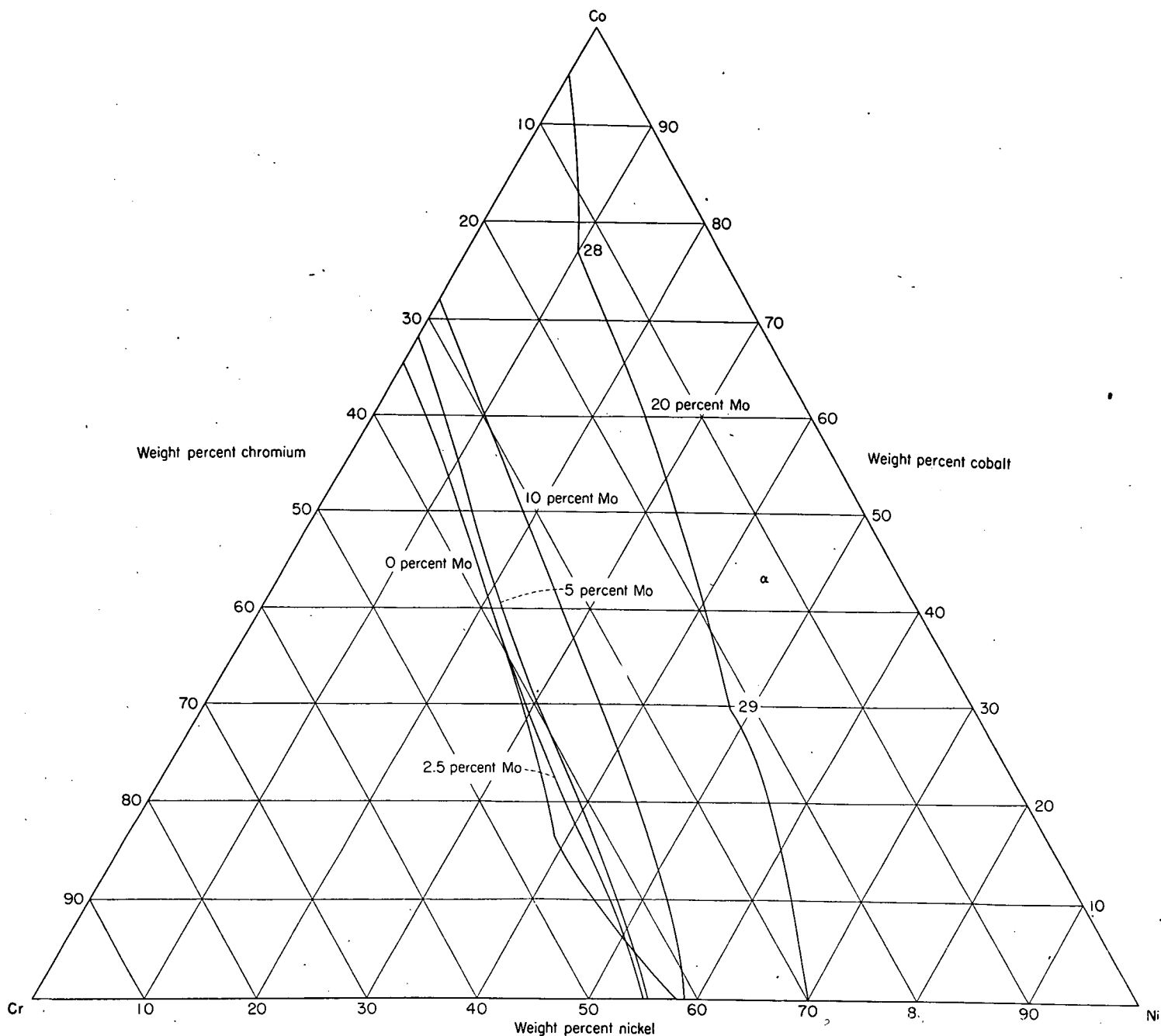


FIGURE 7.—Alpha-phase-field boundary at 1,200° C in chromium-cobalt-nickel-molybdenum quaternary system for constant molybdenum contents of 2.5, 5, 10, and 20 percent

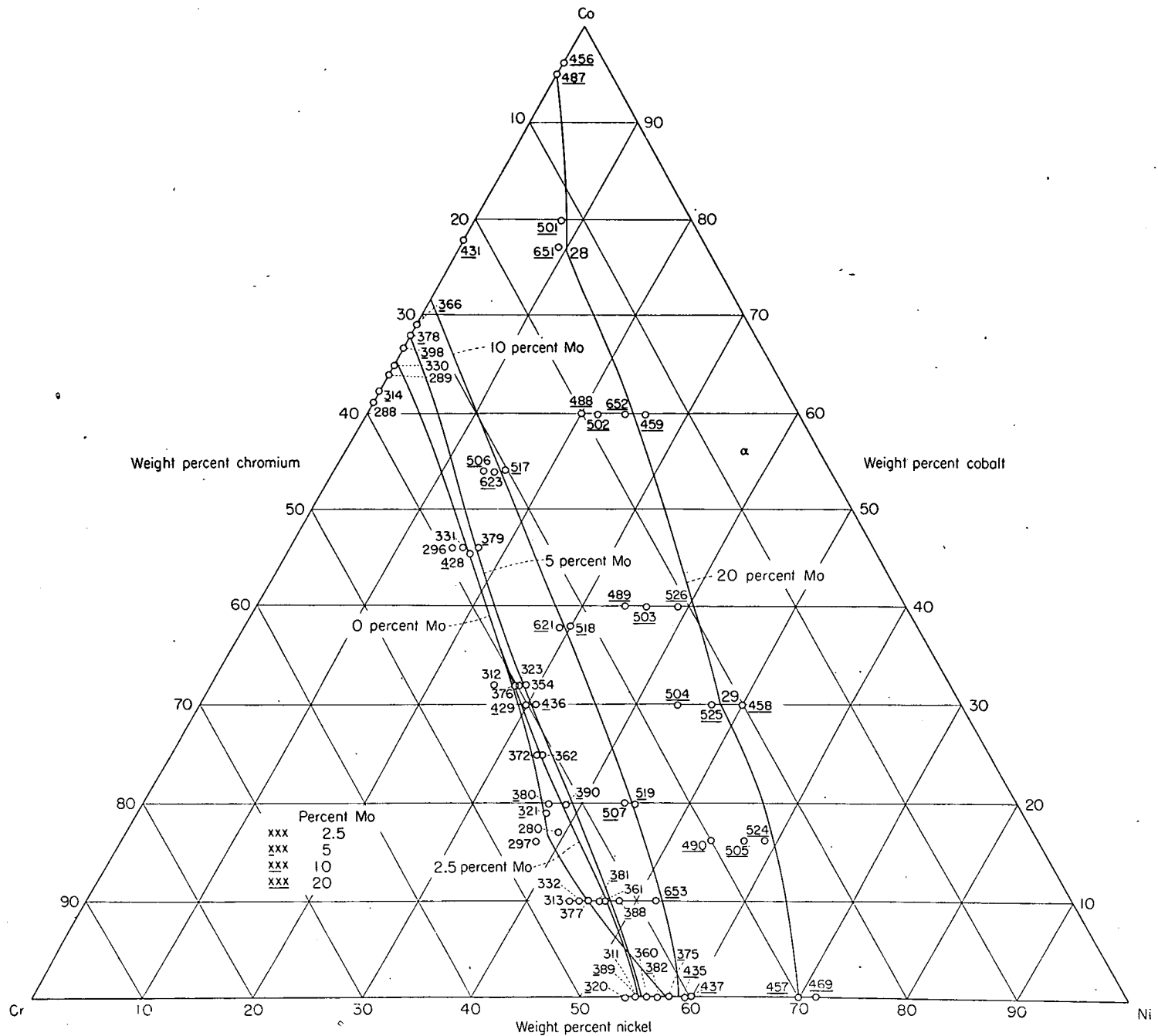


FIGURE 8.—Alpha phase-field boundary at 1,200° C in chromium-cobalt-nickel-molybdenum quaternary system for constant molybdenum contents of 2.5, 5, 10, and 20 percent with alloy compositions indicated.

PHASES

The various phases identified in this investigation and the metallographic characteristics of each phase are described in the following paragraphs.

Alpha.—The alpha phase, which forms the matrix of most of the practically important high-temperature alloys, is based on solid solutions of the face-centered cubic elements cobalt and nickel. This phase, being relatively soft and ductile, was easily hot-forged, and alpha alloys rich in nickel could be cold-worked to some extent. Reagent 1 was used to reveal the microstructure of the alpha phase. The structure is typical of most face-centered cubic solid solutions, showing equiaxed grains and numerous annealing twins. In alpha alloys containing small amounts of a second phase a great grain-size contrast was sometimes seen. Such "duplex" structures undoubtedly occur as a result of the inhibition effect of small second-phase particles on grain growth (reference 17). Upon cooling, pure cobalt is known to transform from the face-centered cubic to the hexagonal close-packed structure by a martensitic transformation (reference 8). In cobalt-rich alpha solid solutions the beginning of this transformation is manifested in the microstructure by the presence of striations, as illustrated in figure 9. These transformation striations were also observed in saturated cobalt-molybdenum and cobalt-chromium alpha alloys and in quaternary alpha alloys rich in cobalt. Additions beyond about 15 percent nickel to cobalt-molybdenum and 30 percent nickel to cobalt-chromium saturated alpha alloys seem to suppress the transformation, as evidenced by the absence of striations in the microstructure.

Microspecimens of alloys containing alpha in addition to any other phases exhibited a pronounced relief effect in the as-polished and unetched condition. The relief is due to the large difference in hardness between alpha and the phases which coexist with it at 1,200° C. In unetched microspecimens the alpha phase had a slightly amber or yellowish

tint, while the other phases appeared white. Figure 10 shows a banded structure which was observed in alloys consisting of approximately equal amounts of the alpha and mu phases. Figure 11 illustrates the general shape and distribution of minor amounts of alpha in a matrix of the sigma phase. This is typical of the distribution of minor amounts of alpha in all phases. Alloys which consisted of larger amounts of alpha coexisting with any of the other phases exhibited microstructures similar to the one shown in figure 12. Data from a typical X-ray diffraction pattern of the alpha phase are given in table VII. Evidence that preferred orientation may occur in forged and annealed alpha alloys was found in some of the back-reflection pictures taken from the surface of microspecimens.

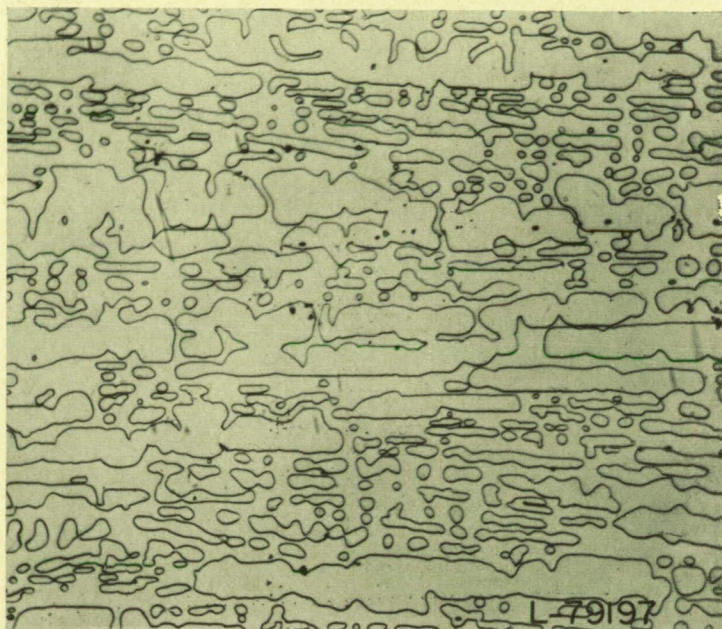


FIGURE 10.—Alloy 443 containing 25 percent cobalt, 25 percent nickel, and 50 percent molybdenum. Etched according to procedure 4 but stain omitted; alpha phase arranged in banded dendritic pattern in matrix of mu phase; X250.

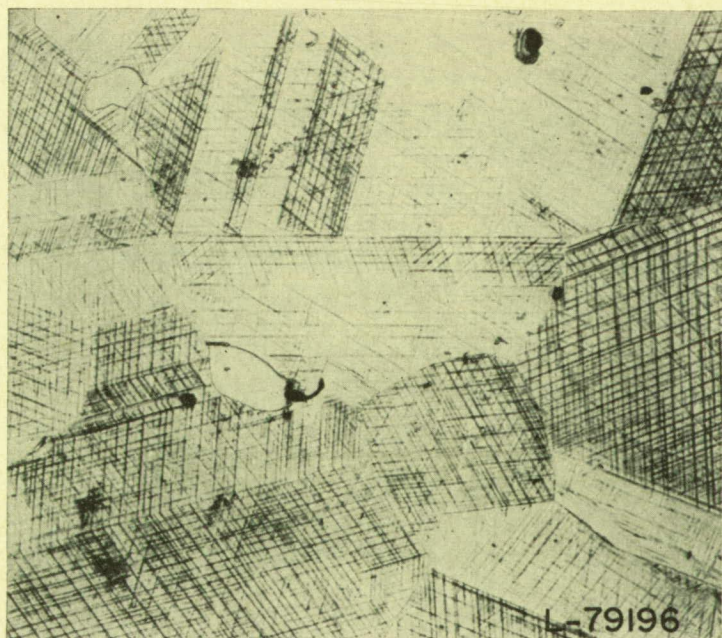


FIGURE 9.—Alloy 296 containing 38.02 percent chromium, 44.85 percent cobalt, 14.63 percent nickel, and 2.5 percent molybdenum. Etched according to procedure 1; small particles of second phase in an alpha matrix; transformation striations revealed in alpha phase; X500.

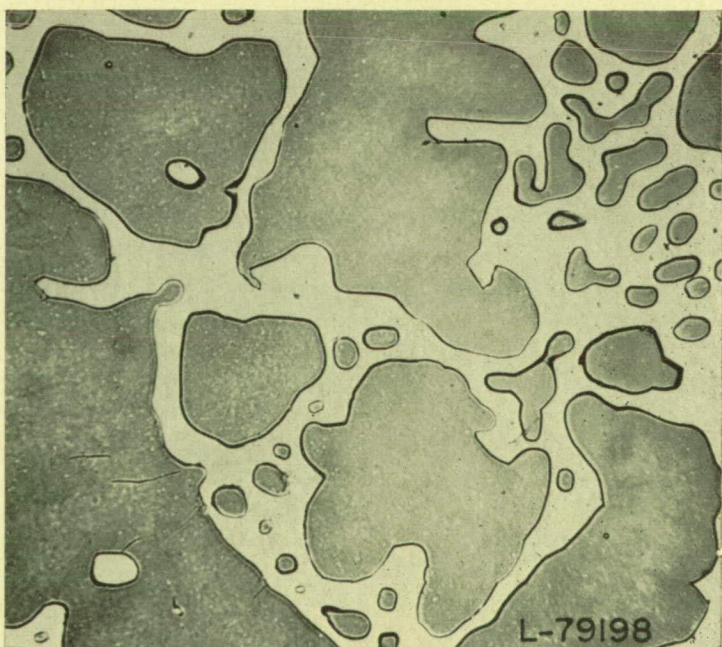


FIGURE 11.—Alloy 608 containing 23.5 percent chromium, 39 percent nickel, and 37.5 percent molybdenum. Etched and stained according to procedure 2; minor amounts of unstained alpha in matrix of stained sigma phase; X250.



FIGURE 12.—Alloy 467 containing 10 percent chromium, 55 percent nickel, and 35 percent molybdenum. Etched according to procedure 2 but stain omitted; typical distribution of minor amounts of P phase in matrix of alpha; X250.

Epsilon.—The body-centered cubic epsilon phase at 1,200° C is based on solid solutions of chromium and molybdenum. Within the composition ranges investigated in this work, the epsilon phase was encountered only in the chromium-nickel-molybdenum ternary isothermal section where it coexists with the alpha and sigma phases and in quaternary alloys containing less than about 3 percent molybdenum near the chromium-nickel side of the diagram, where it coexists with alpha. When etched and stained according to procedure 2, the microstructure of the epsilon phase was clearly revealed. This phase stained a light tan color and, in this work, a characteristic Widmanstätten type of precipitate was always observed in the large equiaxed grains of the epsilon phase, with the exception of epsilon coexisting with alpha in alloys near the chromium-nickel binary side of the diagram. This Widmanstätten precipitate was previously found in chromium-rich alloys in the epsilon phase fields of both the chromium-cobalt-nickel and the chromium-cobalt-iron ternary isothermal sections at 1,200° C (references 1 and 2) and was identified in both cases as being the sigma phase. In the present work, too, it has been concluded that the Widmanstätten precipitate consists of the sigma phase. In epsilon alloys very near to the corner of the three-phase alpha-epsilon-sigma field in the chromium-nickel-molybdenum ternary isothermal section, the Widmanstätten precipitate in the epsilon phase is extremely heavy. (See fig. 13.) The epsilon phase was easily distinguished from alpha or sigma, either by the precipitate in epsilon or by its tan color, when stained (procedure 2), since the alpha phase remains unstained and the sigma phase stains to a brilliant color. Figure 14 is an example of this, where the three phases are seen to coexist. This alloy was slightly deformed prior to microscopic examination. Note the severe cracks in the heavily stained sigma and the few cracks in the lightly stained epsilon. Table VIII gives the

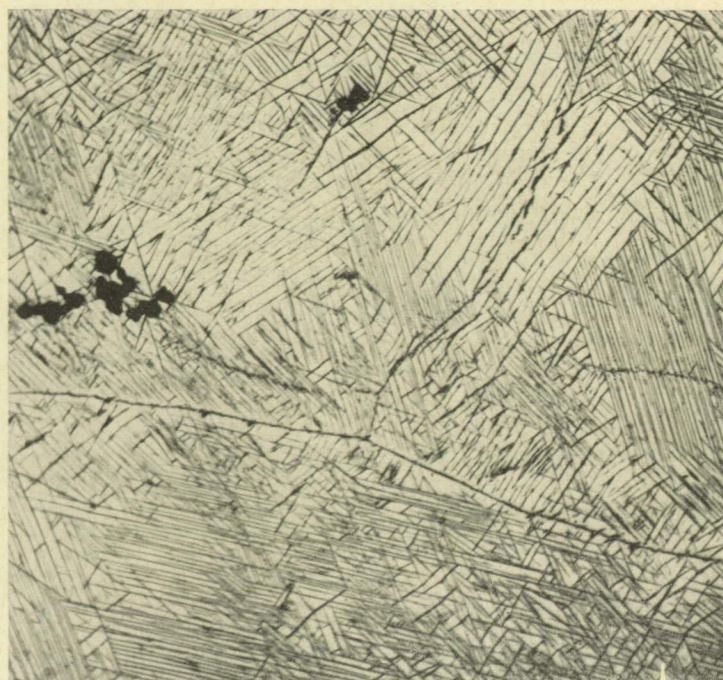


FIGURE 13.—Alloy 599 containing 65 percent chromium, 22 percent nickel, and 13 percent molybdenum. Etched and stained according to procedure 2; heavy Widmanstätten precipitate of sigma in epsilon phase; dark oxide inclusions and grain boundaries shown in epsilon phase; X500.



FIGURE 14.—Alloy 600 containing 48.5 percent chromium, 45.4 percent nickel, and 6.1 percent molybdenum. Etched and stained according to procedure 2; heavily stained and severely cracked particles of sigma plus lightly stained epsilon with few cracks in matrix of unstained alpha phase; X250.

data from a typical X-ray diffraction pattern of the epsilon phase.

Sigma.—The sigma phase is an extremely brittle intermetallic phase which, in this investigation, occurred in the 1,200° C isothermal sections of the chromium-cobalt-molybdenum and chromium-nickel-molybdenum ternary systems, as well as in chromium-cobalt-nickel-molybdenum quaternary alloys containing up to 20 percent molybdenum. The sigma phases in these, as in other systems, are isomorphous. Recent investigations on single crystals of the

throughout the specimen were preferentially attacked, giving one the impression that the alloy contained two phases when actually there was only one. (This was proved repeatedly by careful X-ray diffraction studies.) After experimenting with various etching techniques, all with the same result, it was concluded that the preferential attack is an orientation effect. Figure 21 shows the structure of a typical mu alloy. This specimen was etched according to procedure 4 but the stain was omitted.

The mu phase coexists with the delta and R phases in randomly distributed particles. A banded structure in alloys of mu plus alpha has been discussed and illustrated previously in figure 10. Table XI gives the data from a typical X-ray diffraction pattern of the mu phase.

P phase.—The P phase was discovered in the 1,200° C isothermal section of the chromium-nickel-molybdenum ternary system. This phase is not known to occur in any of the three binary systems at any temperature. The crystal structure of the P phase was not determined. Alloys of P were hard and brittle, having much the same physical characteristics as alloys of the chromium-nickel-molybdenum ternary sigma phase. Numerous cracks were observed under the microscope, as in the case of the ternary sigma phase. Grain boundaries in the P phase were attacked when etched by procedure 2, and this etchant also produced stain colors varying from green to red and orange. Because of the similarity in microstructures and staining characteristics of the P and sigma phases, it was difficult to distinguish between them microscopically. In figure 22, minor amounts of sigma and P are seen in an alpha matrix. Sigma is stained darker than P, but the contrast is slight. Small particles of the delta or alpha phases could easily be identified in a matrix of the P phase. Data from a typical X-ray diffraction pattern of the P phase are given in table XII.

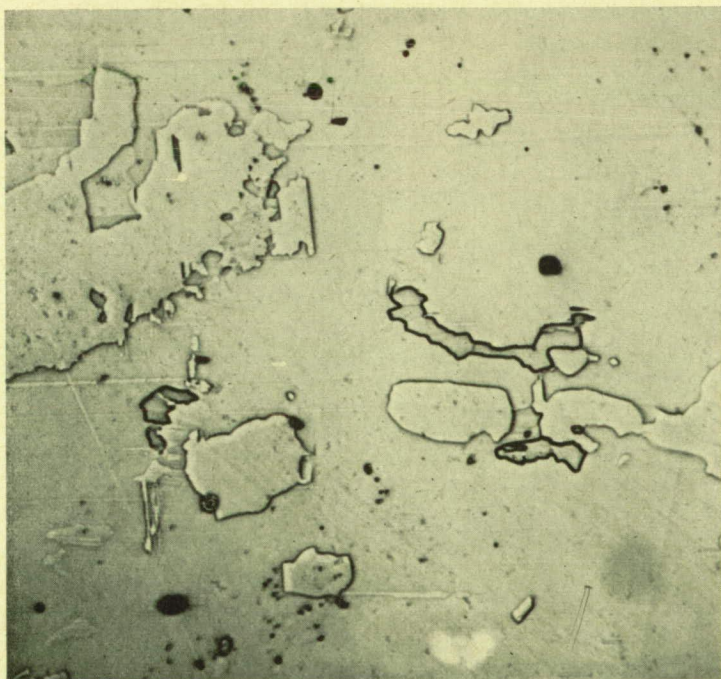


FIGURE 21.—Alloy 550 containing 22.74 percent cobalt, 18.77 percent nickel, and 58.49 percent molybdenum. Etched according to procedure 4 but stain omitted; structure of mu phase; preferential attack probably due to orientation effect; X500.

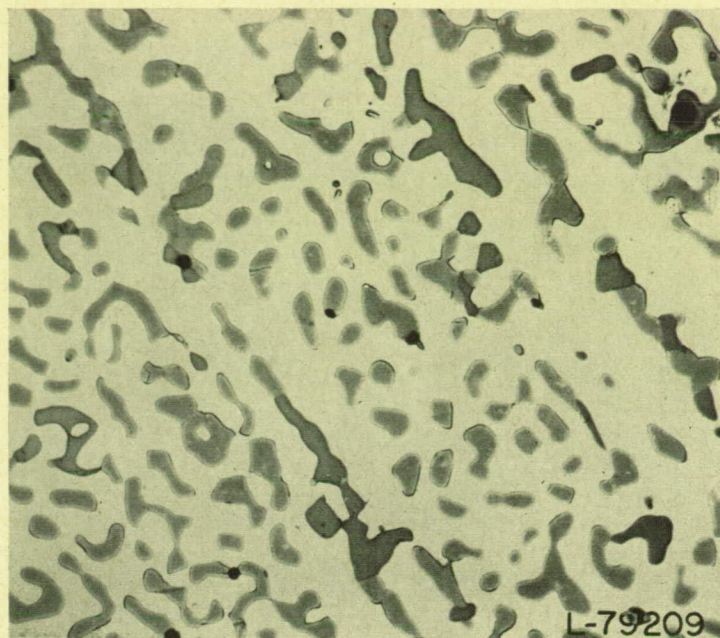


FIGURE 22.—Alloy 669 containing 22 percent chromium, 49 percent nickel, and 29 percent molybdenum. Etched and stained according to procedure 2; minor amounts of sigma and P phases (both stained) in unstained matrix of alpha phase; slight contrast between sigma and P particles; X500.

R phase.—The R phase was discovered in the 1,200° C isothermal section of the chromium-cobalt-molybdenum ternary system. The crystal structure of the R phase was not determined. Ingots of alloys which consisted of the R phase had about the same physical characteristics as delta and mu alloy ingots. When etched and stained according to procedure 3, the R phase was unattacked but stained yellow to rusty brown. Small quantities of the sigma, mu, or alpha phases in a matrix of R were easily identified microscopically. Figure 23 shows alpha and a few particles of the sigma phase in a matrix of the R phase. The sigma



FIGURE 23.—Alloy 605 containing 18.5 percent chromium, 45.5 percent cobalt, and 36 percent molybdenum. Etched and stained according to procedure 3; small particles of stained sigma plus minor amounts of unstained (white) alpha phase in stained matrix of R phase; X500.

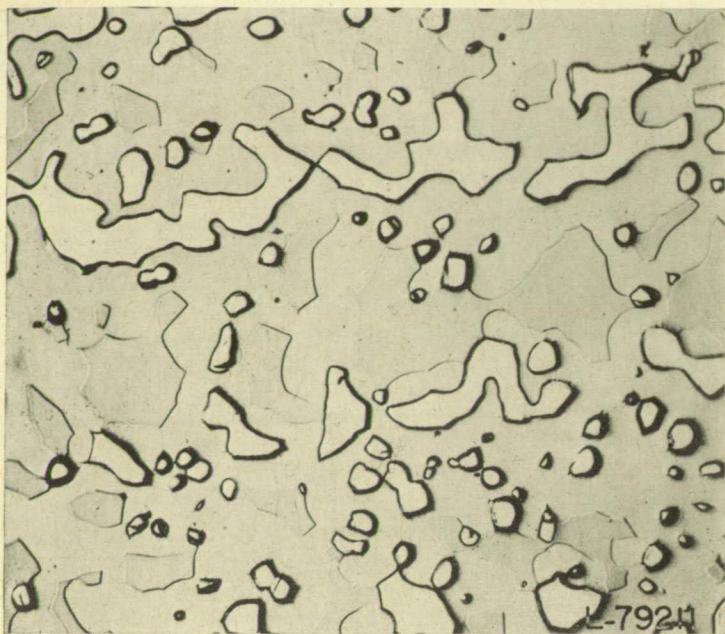


FIGURE 24.—Alloy 513 containing 19.88 percent chromium, 49.78 percent cobalt, and 30.34 percent molybdenum. Etched and stained according to procedure 3; phase contrast shown in three-phase alpha-sigma-R alloy; alpha is unstained (white), sigma is stained (darkest), and R is lightly stained; X500.

particles were stained but not so clearly delineated as unstained alpha particles. The contrast achieved by procedure 3 is further illustrated in figure 24, where larger amounts of alpha and sigma coexist in a matrix of the R phase. Data from a typical X-ray diffraction pattern of the R phase are presented in table XIII.

Impurity phases.—Three main impurity phases were observed in the microstructures of alloys investigated in this work. Metallic oxide inclusions (mostly chromium oxide) were present in the alloys in varying amounts. The chromium-rich epsilon alloys showed the highest number of oxide inclusions, while alloys rich in nickel and molybdenum showed the least. Metallic oxide inclusions are easily recognized in as-polished and unetched microstructures. They are generally unattacked by etchants. Chromium-oxide particles are visible in figure 13 which is an epsilon alloy etched to bring out the Widmanstätten precipitate of sigma.

Another impurity phase appeared in some alloys which were melted in zirconia or stabilized zirconia crucibles. The amount of this impurity phase was observed to be especially high in an alloy which had been slightly overheated during melting in a stabilized zirconia crucible. The specimen had been forged and homogenized at 1,200° C for 48 hours. The microstructure showed small amounts of sigma second phase associated with a dark etching phase in a network outlining the grains of alpha, suggesting a eutectic origin. This structure is illustrated in figure 25. This alloy was analyzed by quantitative spectrographic analysis and was found to contain more than 1.00 percent zirconium. It was concluded that the impurity phase resulted from the reaction between the molybdenum-containing liquid metal and the crucible.

Minute amounts of a bright yellow impurity phase were found in all alloys which were melted in zirconia or stabilized

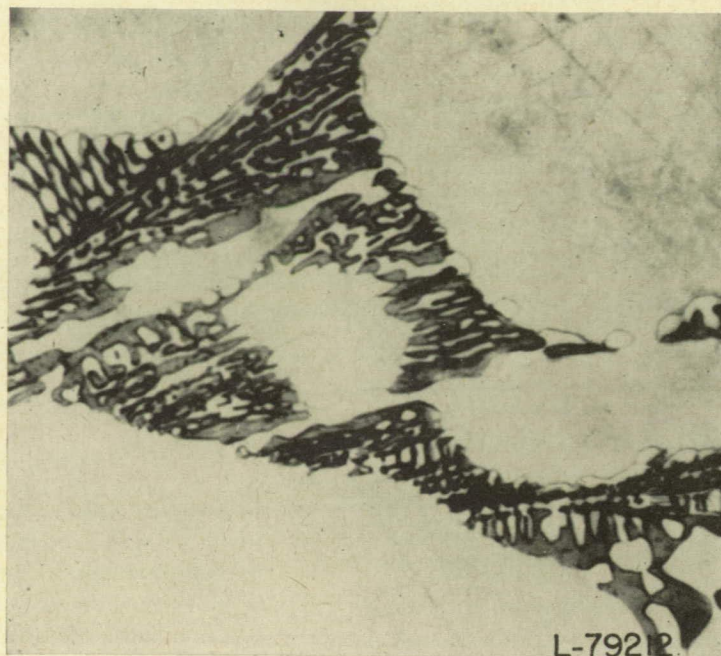


FIGURE 25.—Alloy 376 containing 38.76 percent chromium, 31.2 percent cobalt, 27.54 percent nickel, and 2.5 percent molybdenum. Etched lightly according to procedure 1; dark etching zirconium impurity phase associated with small second-phase particles of sigma in matrix of alpha phase; X2,000.

zirconia crucibles. It is believed that this impurity, too, resulted from zirconium pickup by the alloy. Spectrographic analysis of an alloy melted in an Alundum crucible revealed no aluminum pickup. Microscopic observation did not show any impurity connected with aluminum in any alloy melted in an Alundum crucible. All alloys in the latter part of this work were melted in Alundum crucibles.

PHASE DIAGRAMS

The various phase diagrams determined in the present work are described in the following paragraphs.

Cobalt-nickel-molybdenum ternary system at 1,200° C.—The 1,200° C isothermal section of the cobalt-nickel-molybdenum ternary system, as drawn from alloy data listed in table III, is presented in figure 1. For convenience, the same diagram with the alloy compositions indicated is shown in figure 2. Within the composition ranges investigated, only solid solutions of the phases known from the binary systems were found; no ternary phases occur. About two-thirds of the diagram is covered by the alpha phase field and the two-phase alpha-plus-mu field. The mu phase field penetrates deep into the ternary isothermal section, roughly parallel to the cobalt-nickel side. Limited solid solutions of delta coexist with the mu and alpha phases in two narrow two-phase fields. Alpha, delta, and mu coexist in a three-phase field pointing toward the nickel corner of the diagram.

The face-centered cubic solid solutions based on the cobalt-nickel alpha phase extend to 36 percent molybdenum on the nickel-molybdenum side and to 22.5 percent molybdenum on the cobalt-molybdenum side of the diagram. This is in good agreement with the published nickel-molybdenum and cobalt-molybdenum binary systems. The solubility limit of the alpha phase in the ternary isothermal section is slightly concave, running from the cobalt-molybdenum side

to the corner of the three-phase alpha-delta-mu field (point 10, fig. 1), and almost a straight line from the corner of the three-phase field to the nickel-molybdenum binary diagram. The corner of the three-phase field on the alpha boundary (point 10, fig. 1) is placed at 10 percent cobalt, 35 percent molybdenum, and 55 percent nickel. This point was located by combined microscopic and X-ray data. Alloy 627 contained very small amounts of both the delta and mu phases. In order to check the microscopic findings, the lattice parameters of saturated alpha alloys along the boundary were plotted as a function of cobalt content. The corner of the three-phase field is usually indicated by a sudden change in the slope of the parameter curve. This plot is presented in figure 26 from data listed in table XIV. It is evident from figure 26 that the general effect of cobalt going into solid solution in the saturated nickel-molybdenum alpha phase is to contract the lattice. Unfortunately, however, there were not enough points on the parameter curve to locate the cusp accurately. The point at which the parameter curve suddenly changes slope was definitely located by determining the parameter of the alpha participating in the three-phase field. Since the composition of the alpha is constant anywhere in the three-phase field, the lattice parameter is also constant. The composition corresponding to this point on the parameter curve (point 10, fig. 26) gives the location of the three-phase-field corner. Within the limits of experimental accuracy, this point lies between 9.50 and 10.50 percent cobalt. This result is in excellent agreement with the microscopic findings. Tests were made on the reproducibility of parameter measurements on the same alloy. The lattice parameter of an alpha alloy which gave diffraction rings that were particularly difficult to read (intensity maximums due to preferred orientation) was determined

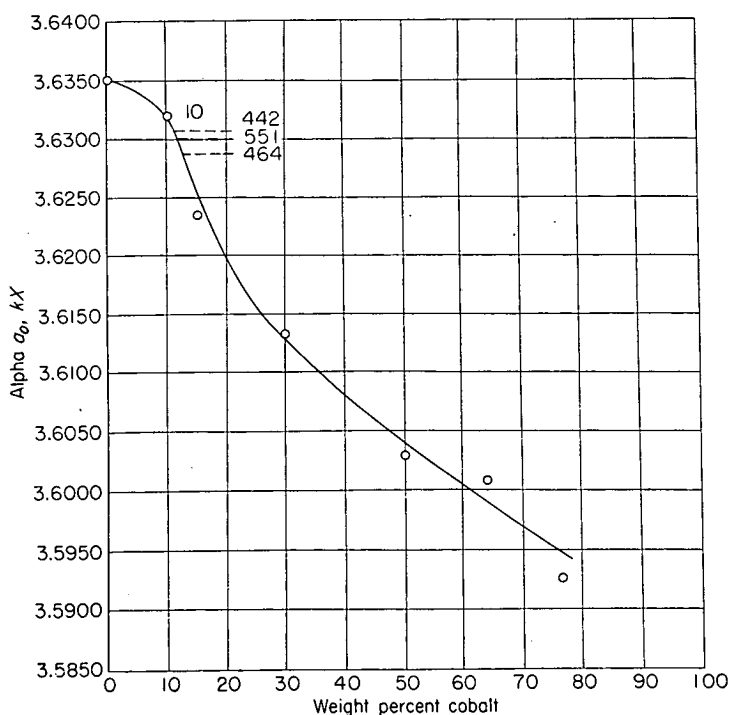


FIGURE 26.—Variation of lattice parameter at 1,200° C of alpha boundary alloys in cobalt-nickel-molybdenum ternary system plotted as function of cobalt content.

from five different films, each one taken after replacing the specimen on the camera. The maximum variation in these five measurements was 0.001 kX unit. A similar check on an alloy which produced sharp, smooth diffraction rings gave a maximum variation in five measurements of only 0.0003 kX unit.

In alloy 407 a large discrepancy was found between the intended melted analysis and the reported chemical analysis. Both analyses are reported in table III. The reported chemical analysis, however, moves the location of the alloy almost parallel to the alpha boundary so that the course of the boundary is not changed. The intended analysis is plotted in figure 2.

Solid solutions based on the mu phase extend to at least 24 percent nickel (point 11, fig. 1) in the cobalt-nickel-molybdenum 1,200° C isothermal ternary section. This was confirmed by X-ray investigation. As nickel goes into solid solution, the lattice of the mu phase expands gradually, becoming constant in the two-phase delta-plus-mu field and in the three-phase field. The d values for a high-angle X-ray line in a series of mu alloys were plotted as a function of nickel content, as shown in figure 27. The solubility limit is indicated at 24 percent nickel (point 11, fig. 27). The d values for the same line in patterns of mu for alloys located in the three-phase field were determined as a further check. As seen from the data given in table XV, the agreement was good.

The alpha and mu phases coexist in a large two-phase field extending from the cobalt-molybdenum side of the diagram to the corners of the three-phase field on the alpha and mu phase boundaries. The corner of the three-phase field on the mu phase boundary (point 11, fig. 1) is placed at 17 percent cobalt, 24 percent nickel, and 59 percent molybdenum.

The delta phase extends to only 6.5 percent cobalt in the isothermal section and coexists with alpha in a two-phase field which extends from the nickel-molybdenum side of the diagram to the corners of the three-phase field on the alpha and delta phase boundaries. The corner of the three-phase field on the delta phase boundary (point 12, fig. 1) was found to be at 6.5 percent cobalt, 33.5 percent nickel, and 60 percent molybdenum.

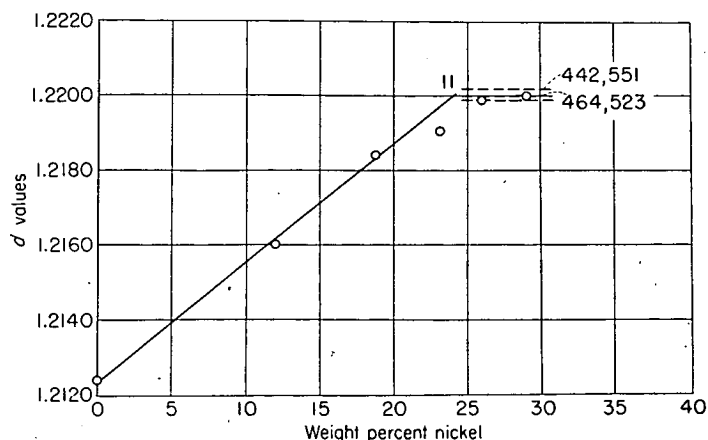


FIGURE 27.—Variation at 1,200° C of d values of twenty-eighth X-ray diffraction line of mu phase in cobalt-nickel-molybdenum ternary system plotted as function of nickel content.

The composition range of the three-phase alpha-delta-mu field is outlined by the three corners (points 10, 11, and 12, fig. 1) given above. The obtuse angle of this triangular field is at the delta corner.

Chromium-cobalt-molybdenum ternary system at 1,200° C.—Figure 3 is the 1,200° C isothermal section of the chromium-cobalt-molybdenum ternary system, drawn in accordance with the alloy data listed in table IV. The same diagram is presented in figure 4 with alloy compositions indicated. In addition to solid solutions of the phases known from the binary systems, a new ternary phase, not known in any of the binary systems, was discovered. This new ternary phase was designated as the R phase. Solid solutions of the face-centered cubic alpha phase surround the cobalt corner of the diagram. The chromium-cobalt sigma phase extends approximately halfway across the isothermal section in a field roughly parallel to the chromium-molybdenum side of the diagram, and solid solutions based on the cobalt-molybdenum mu phase occur in a small field, also approximately parallel to the chromium-molybdenum side. The R phase field is located between the ends of the sigma and mu phase fields. The extension of the mu and R phase fields in the direction parallel to the cobalt-molybdenum side of the diagram was not investigated in detail. Alpha coexists with the sigma and mu phases in two wide two-phase fields and with the R phase in a long, narrow two-phase field. The R phase coexists with the sigma and mu phases in two narrow two-phase fields. The alpha, sigma, and R phases coexist in a three-phase field between the alpha-sigma and alpha-R two-phase fields. Alpha, R, and mu coexist in a three-phase field between the two-phase fields of alpha and R and of alpha and mu.

The solubility of chromium in cobalt at 1,200° C was determined by Manly and Beck (reference 1) at 34 percent. In the present work the solubility of molybdenum in cobalt was found to be 22.5 percent. This value is approximately 1.00 percent lower than that indicated by the cobalt-molybdenum diagram reported by Sykes and Graff (reference 7). The difference is of the order of magnitude of experimental error. The alpha phase boundary in the 1,200° C isothermal ternary section is a convex curve from the chromium-cobalt binary to the corner of the three-phase alpha-sigma-R field (point 13, fig. 3) and almost a straight line from the cobalt-molybdenum binary to the corner of the alpha-sigma-R three-phase field. There is a slight cusp on the alpha boundary at the corner of the three-phase alpha-R-mu field (point 16, fig. 3). The alpha-sigma-R three-phase-field corner (point 13, fig. 3) is placed at 21.0 percent chromium, 66 percent cobalt, and 13 percent molybdenum, while the corner of the alpha-R-mu three-phase field (point 16, fig. 3) is at 16 percent chromium, 69 percent cobalt, and 15 percent molybdenum. In figure 28, the lattice parameters of saturated alpha alloys along the alpha phase boundary are plotted as a function of chromium content. Data for this plot are given in table XVI. The three-phase-field corners on the alpha phase boundary are clearly indicated by sharp breaks in the parameter curve. The breaks occur at approximately 14.5 and 21.0 percent chromium (points 13

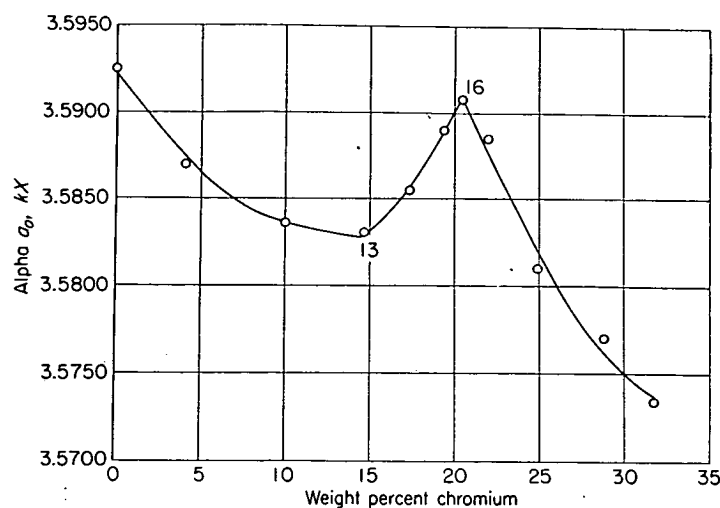


FIGURE 28.—Variation of lattice parameter at 1,200° C of alpha boundary alloys in chromium-cobalt-molybdenum ternary system plotted as function of chromium content.

and 16, fig. 28), agreeing well with the microscopic work. (The reason for the sudden decrease and increase in parameters at the corners is explained under "Discussion.")

The sigma phase field is a narrow band, from 40 to 45 percent cobalt, which penetrates to a maximum of 31 percent molybdenum in the 1,200° C isothermal ternary section. The corner of the three-phase alpha-sigma-R field on the sigma phase boundary (point 14, fig. 3) is at 25 percent chromium, 46.5 percent cobalt, and 28.5 percent molybdenum. Evidence was found in the microstructure of alloy 531 that another three-phase-field corner occurs on the sigma phase boundary at approximately 27 percent chromium, 42 percent cobalt, and 31 percent molybdenum. This may be a corner of a three-phase sigma-epsilon-R field. It was observed from X-ray diffraction patterns of the sigma phase that the lattice expands as the amount of molybdenum in solid solution increases. No quantitative plot of d values against composition was made to check the solubility limit, however, because the end of the sigma phase field was easily located by microscopic means.

The two-phase alpha-plus-sigma field extends from the chromium-cobalt side of the diagram to the corners of the three-phase alpha-sigma-R field on the alpha and sigma phase boundaries.

The third corner of the alpha-sigma-R three-phase field (point 15, fig. 3) is on the boundary of the R phase at 18 percent chromium, 44 percent cobalt, and 38 percent molybdenum. The obtuse angle in this triangular field is at the sigma corner.

The solubility limits of the R phase in the direction parallel to the chromium-molybdenum side of the diagram were located at approximately 14.5 and 19.5 percent chromium. The alpha-R-mu three-phase-field corner on the R boundary (point 17, fig. 3) is placed at 13.5 percent chromium, 44 percent cobalt, and 42.5 percent molybdenum. The other corner of this very narrow three-phase field (point 18, fig. 3) is on the boundary of the mu phase at 9.5 percent chromium, 42.5 percent cobalt, and 48 percent molybdenum. The obtuse angle in this triangular field is at the corner on the R phase boundary.

The alpha-plus-R two-phase field is located between the three-phase fields of alpha, sigma, and R and of alpha, R, and mu.

The mu phase takes a maximum of approximately 11 percent chromium into solid solution and coexists with alpha in a large two-phase field extending from the cobalt-molybdenum binary to the corners of the three-phase alpha-R-mu field on the alpha and mu phase boundaries.

Chromium-nickel-molybdenum ternary system at 1,200° C.—The 1,200° C isothermal section of the chromium-nickel-molybdenum ternary system was drawn from data given in table V and is presented in figure 5. The same diagram with the alloy compositions indicated is shown in figure 6. Besides the phases known from the binary systems two new ternary phases were found to occur in the composition ranges investigated. One of these ternary phases has the familiar sigma structure and was, therefore, designated sigma. The crystal structure of the other new ternary phase, designated as the P phase, is unknown. About three-quarters of the area of the diagram, within the range of compositions investigated, is covered by solid solutions of the alpha phase around the nickel corner and by two-phase fields and three-phase fields in which alpha coexists with the epsilon, sigma, P, and delta phases. The large two-phase fields of alpha and sigma and of alpha and P are separated by a very narrow three-phase alpha-sigma-P field. A narrow three-phase alpha-P-delta field lies between the alpha-P and alpha-delta two-phase fields. The alpha and epsilon phases coexist in a two-phase field along the chromium-nickel side of the diagram. The three-phase alpha-sigma-epsilon field, which separates the alpha-epsilon and alpha-sigma two-phase fields, is relatively large in comparison with the other three-phase fields. The ternary sigma phase coexists with epsilon and P in 2 two-phase fields. Delta and P coexist in a small two-phase field.

At 1,200° C solid solutions of face-centered cubic nickel in the chromium-nickel binary system extend to 57.5 percent chromium and in the nickel-molybdenum binary system, to 36 percent molybdenum. The alpha phase boundary in the ternary isothermal section has a sharp peak at the alpha corner of the three-phase alpha-epsilon-sigma field (point 19, fig. 5). This point was located by microscopic means at 43 percent chromium, 53.5 percent nickel, and 3.5 percent molybdenum. Except for a slight cusp at the alpha corner of the other three-phase fields (points 22 and 25, fig. 5), the alpha boundary is fairly straight from the nickel-molybdenum binary to the alpha corner of the three-phase alpha-epsilon-sigma field. From data listed in table XVII, the lattice parameters of saturated alpha alloys along the alpha boundary were plotted in figure 29 as a function of chromium content. The alpha-epsilon-sigma three-phase-field corner on the alpha boundary (point 19, fig. 5) is very clearly indicated at about 43 percent chromium by the drastic break in the curve (point 19, fig. 29). The curve is almost vertical beyond this point because the chromium content of saturated alpha coexisting with the epsilon phase varies between only 43 and 42.5 percent. The break in the parameter curve at 23 percent chromium (point 22, fig. 29),

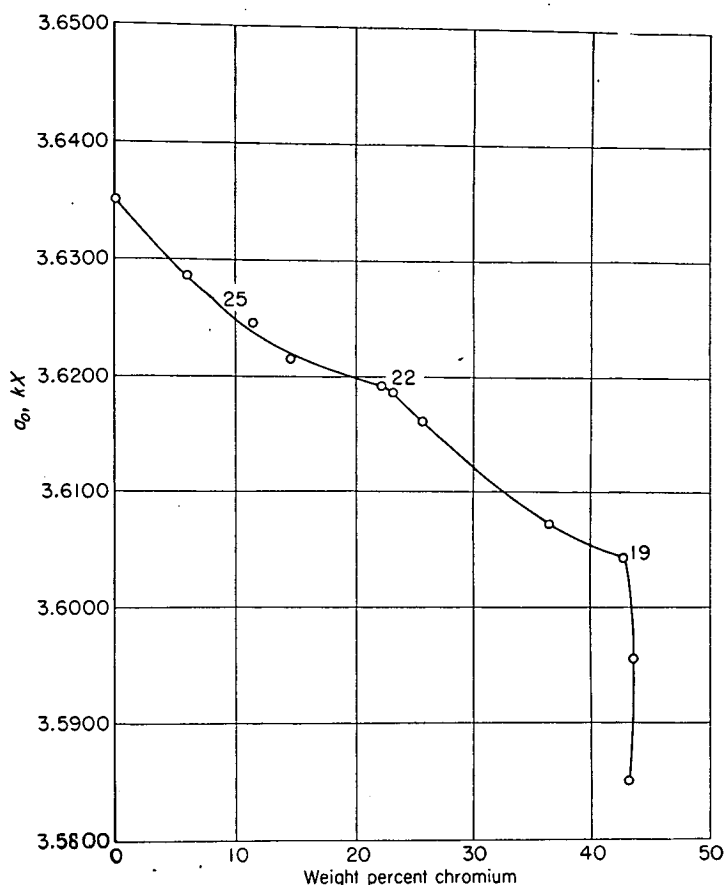


FIGURE 29.—Variation of lattice parameter at 1,200° C of alpha boundary alloys in chromium-nickel-molybdenum ternary system plotted as function of chromium content.

though not so drastic, is also in good agreement with the location of the three-phase alpha-sigma-P field corner (point 22, fig. 5) from microscopic findings. The alpha corner of the three-phase alpha-P-delta field (point 25, fig. 5) is not manifested by a pronounced cusp and is almost imperceptible. This point was located at 6 percent chromium, 61.5 percent nickel, and 32.5 percent molybdenum by carefully determining the boundaries of the three-phase alpha-P-delta field. Extrapolated segments of the parameter curve indicate a break at approximately 7.5 percent chromium (point 25, fig. 29), in fair agreement with the above observations.

The epsilon boundary was determined only along the two-phase alpha-epsilon field. This boundary extends from the chromium-nickel side of the diagram to the epsilon corner of the three-phase alpha-epsilon-sigma field (point 20, fig. 5). As shown in figure 5, alpha, epsilon, and sigma coexist in a three-phase field whose boundaries are as follows: The alpha corner (point 19, fig. 5) at 43 percent chromium, 53.5 percent nickel, and 3.5 percent molybdenum; the sigma corner (point 21, fig. 5) at 49 percent chromium, 28 percent nickel, and 23 percent molybdenum; and the epsilon corner (point 20, fig. 5) at 60 percent chromium, 29.5 percent nickel, and 10.5 percent molybdenum.

The sigma phase field is approximately parallel to the chromium-molybdenum side of the diagram, comprising a range of about 23 to 49 percent chromium. As molybdenum replaces chromium in the sigma structure, the lattice expands.

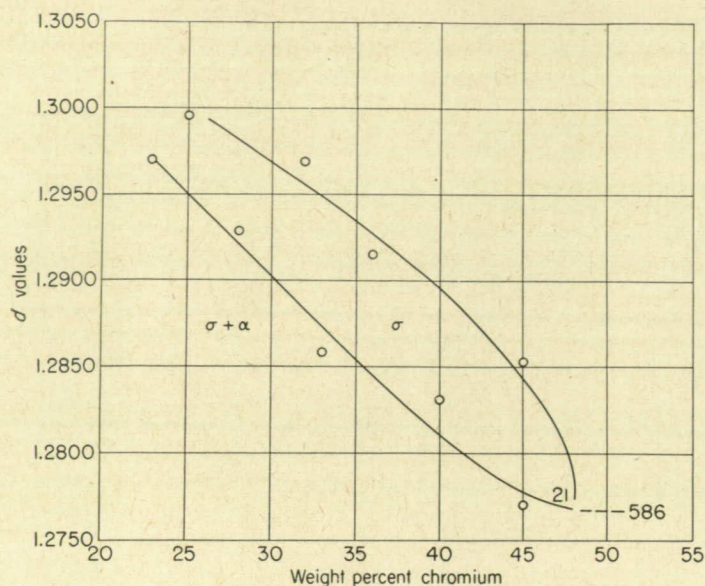


FIGURE 30.—Variation at 1,200° C of d values of twenty-third X-ray diffraction line of sigma phase in chromium-nickel-molybdenum ternary system plotted as function of chromium content.

This is clearly illustrated in figure 30, which was plotted from the data given in table XVIII. The variation of one high-angle d value for a series of saturated sigma alloys along both sides of the sigma field is shown as a function of chromium content. The two curves meet at approximately 48 percent chromium (point 21, fig. 30), corresponding closely to the sigma corner of the three-phase alpha-epsilon-sigma field (point 21, fig. 5).

Solid solutions of the new ternary P phase exist between 8 and 19 percent chromium. The two-phase fields of P and sigma and of P and delta are very narrow, each one extending over a range of only about 4 percent chromium.

The corners of the three-phase alpha-sigma-P field are located at the following points: The alpha corner (point 22, fig. 5) at 23 percent chromium, 57.5 percent nickel, and 19.5 percent molybdenum; the sigma corner (point 23, fig. 5) at 22 percent chromium, 31 percent nickel, and 47 percent molybdenum; and the P corner (point 24, fig. 5) at 19 percent chromium, 32.5 percent nickel, and 48.5 percent molybdenum.

At 1,200° C the nickel-molybdenum delta phase dissolves only about 4 percent chromium. This phase coexists with alpha in a long, narrow, almost rectangular two-phase field along the nickel-molybdenum binary system.

The boundaries of the extremely narrow three-phase alpha-P-delta field are located by the following points: The alpha corner (point 25, fig. 5), as described above; the P corner (point 26, fig. 5) at 8 percent chromium, 34.5 percent nickel, and 57.5 percent molybdenum; and the delta corner (point 27, fig. 5) at 4 percent chromium, 37.5 percent nickel, and 58.5 percent molybdenum.

Some discrepancies were noted in a few alloys of chromium, nickel, and molybdenum near the three-phase alpha-epsilon-sigma field. The microstructure of alloys 475, 569, and 597 showed only two phases, alpha and sigma, although, according to the intended compositions of these alloys, they should contain three phases. Chemical analysis of these alloys

moved 569 and 597 into the two-phase alpha-sigma field, thus removing the discrepancy for these two alloys. Alloy 475, however, still remained in the three-phase alpha-epsilon-sigma field. In drawing the phase boundary between the two-phase alpha-sigma and the three-phase alpha-epsilon-sigma fields, alloy 475 was neglected. The microstructure of alloy 640 showed small amounts of the epsilon phase. The chemical analysis of this alloy, however, would place it in the sigma phase field. Both the melted and chemical analyses are given in table V, but the intended melted analysis is plotted in figure 6. Alloys 598 and 600 contained more sigma than would be expected from the phase diagram. This is probably a result of the difficulty in obtaining equilibrium in chromium-rich alloys.

Small amounts of a phase not explainable from the phase diagram were observed in alloys 622 and 447. The identity of this phase is not known. It may have resulted from nonequilibrium or segregation. Small particles of this unknown phase are shown in figure 31, where second-phase particles of alpha in a sigma matrix are also seen.

Alpha boundaries in constant-molybdenum sections of chromium-cobalt-nickel-molybdenum quaternary system at 1,200° C.—Figure 7 is a 1,200° C isothermal diagram in which the alpha phase boundaries for the 2.5-, 5-, 10-, and 20-percent-molybdenum sections through the chromium-cobalt-nickel-molybdenum quaternary system are plotted. Figure 8 is the same diagram with the alloy compositions from table VI indicated.

At constant temperature a quaternary phase diagram is represented by a tetrahedron, each corner point of which corresponds to 100 percent of one component. The four faces of the tetrahedron, then, are isothermal sections of four ternary systems, and the six edges are isothermal binary sections. In the chromium-cobalt-nickel-molybdenum qua-



FIGURE 31.—Alloy 447 containing 40 percent chromium, 30 percent nickel, and 30 percent molybdenum. Etched and stained according to procedure 2; small particles of phase not identifiable from figure 6 plus unstained alpha in matrix of sigma phase; X750.

ternary system at 1,200° C sections of constant molybdenum content are triangular sections through the tetrahedron parallel to the chromium-nickel-cobalt face. As the amount of molybdenum increases, the size of the equilateral triangle becomes smaller. Because of this change in size, these sections cannot be superimposed upon each other for the purpose of showing, in a two-dimensional diagram, the limits of a phase in the quaternary system. In order to facilitate the diagrammatic presentation of results and to allow direct comparison of the effect of increasing molybdenum content, on the alpha boundary, the following method was used: Each section of constant molybdenum content was projected onto the chromium-cobalt-nickel ternary isothermal section by using the molybdenum corner of the tetrahedron as the center of projection. This projection may be accomplished arithmetically by increasing proportionately the percentages of chromium, cobalt, and nickel in the quaternary composition to add up to 100 percent and then plotting quaternary compositions as if they were ternary chromium-cobalt-nickel compositions. The correction factor for each section is the inverted fraction of the total amount of chromium, cobalt, and nickel in the quaternary composition. For example, in a quaternary alloy containing 20 percent molybdenum, 10 percent chromium, 10 percent cobalt, and 60 percent nickel, the chromium, cobalt, and nickel represent eight-tenths of the total composition. Multiplying the quaternary percentages of chromium, cobalt, and nickel by the inverted fraction, that is, $10/8$, gives the projection of this quaternary point on the chromium-cobalt-nickel ternary diagram. While the molybdenum content of alloys used to fix the alpha boundaries was actually never exactly 2.5, 5, 10, or 20 percent, the corresponding boundaries were corrected by factors of $10/9.75$, $10/9.5$, $10/9$, and $10/8$, respectively. The results are shown in figure 7, using this method of plotting. An alternative method was used in figure 32 where the actual solubility of chromium in the alpha phase at the various constant molybdenum levels is plotted as a function of actual cobalt content. The data for these curves were interpolated from figure 7 and are tabulated in table XIX. The alpha phase boundary in the 1,200° C isothermal section of the chromium-cobalt-nickel ternary system was determined by Manly and Beck (reference 1) and is included for comparison.

As seen from figures 7 and 32, increasing the molybdenum content decreases the solubility of chromium in the alpha phase, thereby decreasing the extent of the alpha field at 1,200° C. At the chromium-nickel side of the diagram the solubility of chromium in alpha first increases slightly and then rapidly decreases beyond about 5 percent molybdenum.

No attempt was made to identify the phases which coexist with the alpha phase in the quaternary system, but some general statements can be made from observations of the chromium-cobalt-molybdenum and chromium-nickel-molybdenum ternary systems at 1,200° C. On the chromium-cobalt side of the diagram the sigma phase coexists with alpha from 0 to approximately 12.5 percent molybdenum. Between 12.5 and approximately 16 percent molybdenum the

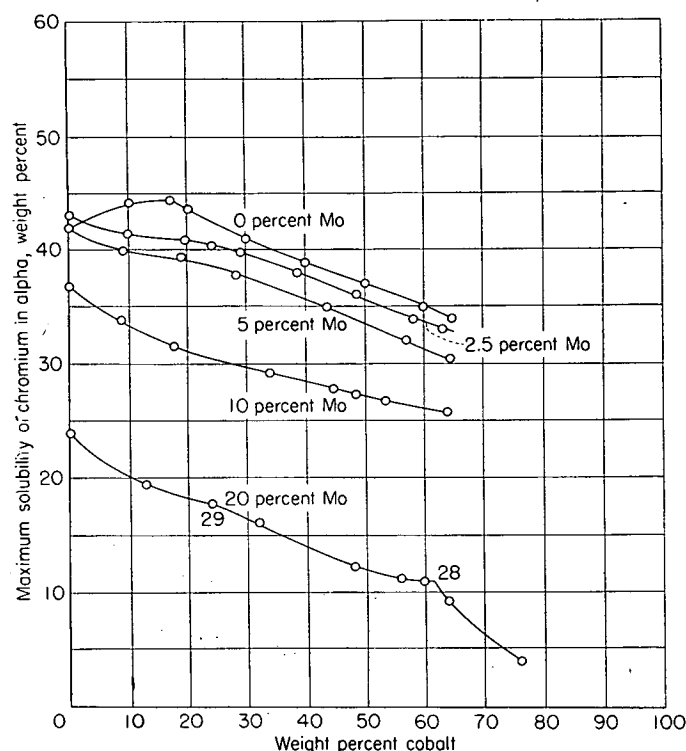


FIGURE 32.—Maximum solubility at 1,200° C of chromium in alpha phase of chromium-cobalt-nickel-molybdenum alloys plotted as function of actual cobalt content for constant molybdenum contents of 2.5, 5, 10, and 20 percent.

ternary R phase coexists with the alpha phase at the chromium-cobalt side of the diagram. Above 16 percent molybdenum the mu phase coexists with alpha. On the chromium-nickel side of the diagram the alpha phase coexists with epsilon from 0 to approximately 3 percent molybdenum and with the ternary sigma phase between 3 and about 19 percent molybdenum. The ternary P phase coexists with alpha on the chromium-nickel side between 19 and 33 percent molybdenum. The delta and quaternary alpha phases coexist over very limited composition ranges, above 33 percent molybdenum. It is not known whether the chromium-cobalt-molybdenum and chromium-nickel-molybdenum sigma phases form an uninterrupted series of solid solutions through the quaternary system at 1,200° C, although there is some basis for expecting it, as stated under "Discussion."

In the 2.5-, 5-, and 10-percent-molybdenum alpha boundaries no sharp breaks were found from the metallographic data, and X-ray parameter plots are not available to check this. However, in figure 32, there is some indication of a cusp on the 2.5- and 5-percent alpha boundaries at approximately 20 percent actual cobalt content. Evidence of two corners on the 20-percent-molybdenum alpha phase boundary was found, one being at approximately 61 percent actual cobalt content (point 28, fig. 32) and the other, at approximately 24 percent actual cobalt content (point 29, fig. 32). The lattice parameters of saturated alpha alloys along this boundary were plotted as a function of actual cobalt content in figure 33. The data for this plot are given in table XX. Two distinct breaks occur in the curve of parameter against composition at approximately 24 and 62 percent actual

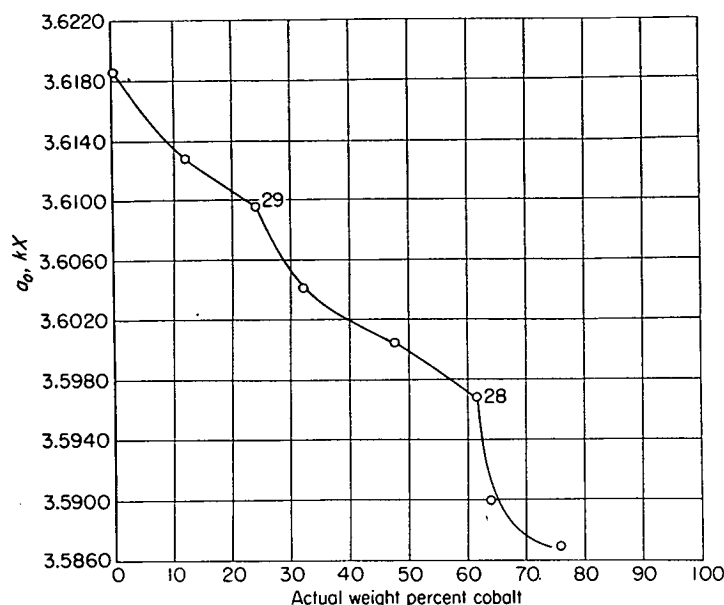


FIGURE 33.—Variation of lattice parameter at 1,200° C of alpha boundary alloys in 20-percent-molybdenum section of chromium-cobalt-nickel-molybdenum quaternary system plotted as a function of actual cobalt content.

cobalt content (points 28 and 29, fig. 33). This is in good agreement with the microscopic results.

DISCUSSION

In view of the fact that the face-centered cubic' (alpha) solid solutions form the matrix of almost all practically useful high-temperature alloys, the extent of the alpha phase field in the chromium-cobalt-nickel-molybdenum quaternary system and in the four adjoining ternary systems at 1,200° C is of great interest. The alpha phase field in the 1,200° C isothermal section of the chromium-cobalt-nickel ternary system was shown previously to be very broad (reference 1). In the cobalt-nickel-molybdenum ternary system, also, the alpha phase field is quite large. As the cobalt content increases, the solubility of molybdenum in the alpha phase decreases from 36 percent in the nickel-molybdenum binary system at 1,200° C to 22.5 percent molybdenum in the cobalt-molybdenum system. Both the delta and mu phases, which coexist with alpha in the cobalt-nickel-molybdenum ternary system at 1,200° C, are very hard, although their presence in alpha alloys does not appear to have nearly so drastic an effect on toughness as that of the sigma phase.

Solid solutions of the alpha phase surround the cobalt corner of the chromium-cobalt-molybdenum ternary isothermal section and the nickel corner of the chromium-nickel-molybdenum ternary isothermal section. In both of these systems the extent of the alpha phase field is small in comparison with the chromium-cobalt-nickel and the cobalt-nickel-molybdenum alpha fields. It is interesting to note that in the chromium-nickel-molybdenum isothermal section the sigma phase coexists with alpha above 3.5 percent molybdenum. The fact that the brittle sigma phase may precipitate out of supersaturated alpha containing more than 3.5 percent molybdenum is of great practical importance, since considerable amounts of the sigma phase may seriously impair the forging characteristics of alpha alloys.

In the chromium-cobalt-nickel-molybdenum quaternary system the extent of the alpha phase field at 1,200° C becomes smaller as the molybdenum content increases. In quaternary alloys above 36 percent molybdenum alpha probably does not occur in a single phase field, although it coexists with other phases up to approximately 61 percent molybdenum.

Some interesting features were observed in the variation of lattice parameter of alpha with composition. Taylor (reference 19) has shown that the general effect of increasing cobalt content in nickel-cobalt binary alloys is to increase the lattice parameter of the alpha phase. Figure 26 shows that the lattice parameter of saturated nickel-molybdenum alpha decreases with increasing cobalt content. Also, from figure 33, along the alpha boundary in the 20-percent constant-molybdenum quaternary section, the lattice parameter again decreases with increasing cobalt content. Molybdenum going into binary solid solution increases the lattice parameter of both nickel and cobalt. Although nickel itself has a smaller a_0 than cobalt, the lattice parameter of saturated nickel-molybdenum alpha is larger than that of saturated cobalt-molybdenum alpha. Apparently, chromium has the same general effect as molybdenum. The peculiar variation of the lattice parameter along the alpha boundary in the 1,200° C isothermal section of the chromium-cobalt-molybdenum ternary system (fig. 28) may be interpreted as shown in figure 34, which is a schematic sketch of the ternary section with isoparameter lines drawn in the alpha phase field. The decrease of lattice parameter beyond the corner of the three-phase alpha-sigma-R field results from the intersection of lower and lower isoparameter lines with the alpha phase boundary. At the alpha-R-mu three-phase-field corner, the lattice parameter reaches a minimum value, and beyond this point the alpha boundary again intersects successively higher isoparameter curves until the cobalt-molybdenum side of the diagram is reached.

The accuracy of the location of the alpha phase boundaries is estimated to about ± 1 percent of any component. Chemical analysis of all definitive boundary alloys gave generally good correlation with intended compositions. Any large discrepancies were usually found to be errors in chemical analysis, although, in some instances referred to in the description of results, the alloy compositions appeared to be in fact different from the intended compositions.

The occurrence of the brittle intermetallic sigma phase in alloy systems involving the transition elements is interesting from both the fundamental and practical viewpoints. The presence of sigma, even in small amounts, in high-temperature alloys is undesirable because of the brittleness which it imparts to alloys. The composition ranges over which sigma exists are, therefore, important. In the cobalt-nickel-molybdenum ternary system at 1,200° C the sigma phase does not occur. However, solid solutions of the chromium-cobalt sigma phase penetrate deep into the 1,200° C isothermal section of the chromium-cobalt-molybdenum ternary system, and in this system sigma coexists with alpha below 13 percent molybdenum (fig. 3). In the present investigation a ternary sigma phase was found in the 1,200°

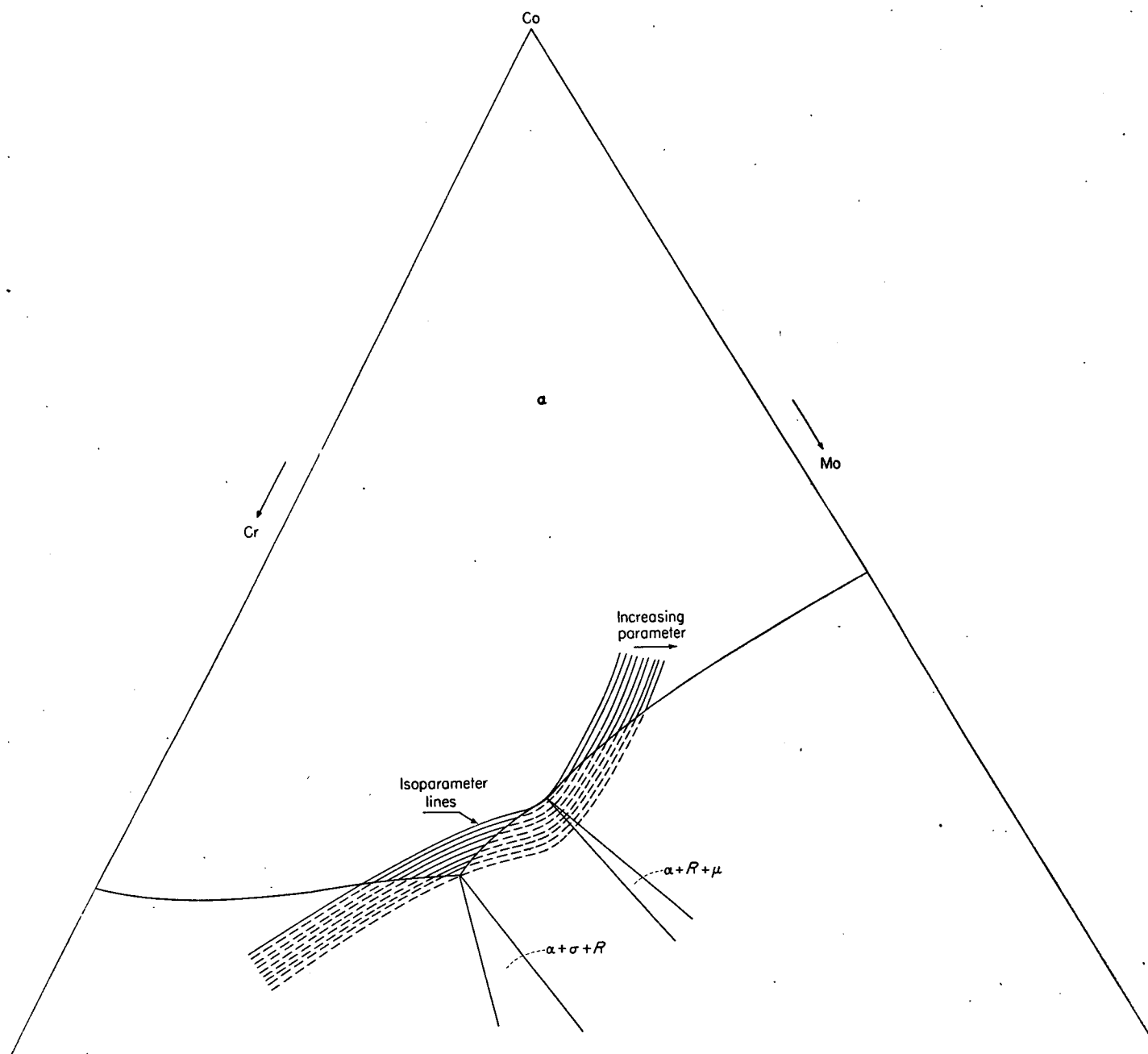


FIGURE 34.—Schematic sketch of alpha-phase-field boundary at 1,200° C in chromium-cobalt-molybdenum ternary system with hypothetical isoparameter lines shown.

C isothermal section of the chromium-nickel-molybdenum ternary system. The sigma phase in this system was also found independently by Putman, Grant, and Bloom (reference 20). In a discussion of their paper, Beck (reference 20) presented a tentative diagram for the chromium-nickel-molybdenum ternary system at 1,200° C. This diagram was refined in the present investigation and is presented in figure 5. At 1,200° C the sigma phase coexists with alpha in chromium-nickel-molybdenum alloys containing from 3.5 to 19 percent molybdenum.

The sigma phases in the chromium-cobalt-molybdenum and chromium-nickel-molybdenum ternary systems may form a continuous series of solid solutions in the chromium-cobalt-nickel-molybdenum quaternary system at 1,200° C. If such quaternary sigma solid solutions do occur, the sigma phase might very likely coexist with alpha in the 1,200° C isothermal quaternary section over wide ranges of the alpha

phase boundary in alloys containing 3.5 to 13 percent molybdenum. Some indication that the sigma phases do form an uninterrupted series of quaternary solid solutions is found in the 10-percent constant-molybdenum section. The alpha boundary in this section has no pronounced break suggestive of a three-phase-field corner, and sigma does coexist with alpha at both ends of the phase boundary.

The shape and location of the sigma phase field in the chromium-cobalt-molybdenum and chromium-nickel-molybdenum isothermal sections give further support to the previous observation (reference 15) that atoms of transition elements having the same crystal structure can substitute for each other in the formation of the sigma phase. In the chromium-cobalt-molybdenum isothermal section (fig. 3) the sigma phase field extends in a direction parallel to the chromium-molybdenum side of the diagram, and approximately half of the chromium is replaced by molybdenum.

Again in the chromium-nickel-molybdenum ternary isothermal section (fig. 5) the ternary sigma phase occurs in a long, narrow field parallel to the chromium-molybdenum side of the diagram. In these two systems body-centered cubic transition elements are substituting for each other. In the chromium-cobalt-nickel ternary system at 1,200° C the sigma phase field is parallel to the cobalt-nickel side of the diagram, indicating that nickel is replacing cobalt (both face-centered cubic) while the chromium content remains essentially unchanged (references 1 and 15).

As the molybdenum content of the sigma phase increases, the lattice expands in both the chromium-cobalt-molybdenum and the chromium-nickel-molybdenum systems. It is interesting that an X-ray diffraction pattern of a chromium-cobalt-molybdenum sigma alloy saturated with molybdenum is almost identical to a pattern of chromium-nickel-molybdenum sigma of the lowest molybdenum content.

The existence of a ternary sigma phase in the chromium-nickel-molybdenum ternary system is rather unexpected, as none of the adjoining binaries are known to have a sigma phase. From the results of the investigation of the 1,200° C isothermal section of the chromium-cobalt-nickel ternary system, Beck and Manly (reference 15) suggested that the nickel-chromium system, which is not reported to include any intermediate phase, might contain a sigma phase at some lower temperature, and Duwez and Baen (reference 16) note that, according to the criteria postulated by them, the nickel-chromium system would be expected to have a sigma phase. The ternary sigma phase found in the chromium-nickel-molybdenum system may have some connection with the suspected nickel-chromium binary sigma phase. Attempts were made to confirm this by annealing for long periods of time an alloy of 38 percent nickel and 62 percent chromium at temperatures below 600° C. These experiments were, however, not successful. This does not rule out the possibility of the existence of sigma in this system at a temperature so low that the diffusion rates are prohibitively low.

From the experimental results of this and other investigations of alloy systems in which the sigma phase occurs, a criterion for the formation of the sigma phase was developed in terms of electron vacancy concentration in the 3*d* sub-band. Details of this development are given in reference 21. Briefly, the procedure is to calculate the electron vacancy concentration N_v in any alloy from the 3*d* sub-band electron vacancies in each component of the alloy, and from the atomic percent concentration of the components. Electron vacancy numbers for chromium, cobalt, and nickel were given by Pauling (reference 22; see also reference 23). Molybdenum was assumed to have the same number of vacancies in the 4*d* sub-band as chromium has in the 3*d* sub-band, on the basis that their electronic structure is similar in the outermost shells. The formula for calculating N_v is given as

$$N_v = 4.66(\text{Cr} + \text{Mo}) + 3.66(\text{Mn}) + 2.66(\text{Fe}) + 1.71(\text{Co}) + 0.61(\text{Ni})$$

It was found possible to define fairly well the composition

ranges over which the sigma phases occur in terms of electron vacancy concentration calculated from this formula. Almost all of the binary and ternary sigma phase fields were found to fall within a rather narrow range of electron vacancy concentration. The favorable range of concentration seems to be from 3.15 to 3.65 N_v . It appears that the composition ranges of other intermetallic phases occurring in alloy systems of transition elements can also be defined in terms of electron vacancy concentration, although some overlapping may occur.

There are some interesting features connected with the solid solutions designated mu in the present investigation. For example, the mu phase, commonly referred to as epsilon in the cobalt-molybdenum binary system, is isomorphous with the corresponding intermediate phase in the iron-molybdenum, cobalt-tungsten, and iron-tungsten binary systems (reference 11). Köster and Tonn (reference 24) reported that the mu phases in the iron-molybdenum and cobalt-molybdenum binary systems form a continuous series of solid solutions across the 1,300° C isothermal section of the iron-molybdenum-cobalt ternary system. In the iron-molybdenum-cobalt system the mu phase field extends in a direction approximately parallel to the iron-cobalt side of the diagram, so that the face-centered cubic elements iron and cobalt substitute for each other while the molybdenum content remains unchanged. As shown in the present work, solid solutions of the mu phase extend more than halfway across the 1,200° C isothermal section of the cobalt-nickel-molybdenum ternary system (fig. 1). Here, again, the face-centered cubic elements cobalt and nickel replace each other, while the molybdenum content remains essentially unchanged. The body-centered cubic elements chromium and molybdenum seem to replace each other in forming the mu phase in the 1,200° C isothermal section of the chromium-cobalt-molybdenum ternary system (fig. 3), with the cobalt content remaining constant. In agreement with these observations one would expect that solid solutions of the mu phase in ternary systems of iron, cobalt, molybdenum, and tungsten would also extend in such a manner that atoms of elements having the same crystal structure could substitute for each other. The intermediate nickel-molybdenum delta phase forms only limited solid solutions in the cobalt-nickel-molybdenum and chromium-nickel-molybdenum isothermal sections (figs. 1 and 5).

In addition to the ternary sigma phase, another new ternary phase P was discovered in the 1,200° C isothermal section of the chromium-nickel-molybdenum ternary system. The P phase field is located between the ends of the sigma and delta phase fields (fig. 5). In physical characteristics and microstructure the P and sigma phases are very much alike, but the X-ray diffraction patterns of the two phases are quite different. The differentiation between P and sigma provides a good example of the need to supplement microscopic work with X-ray diffraction to identify the phases. In alloys consisting wholly of P or sigma it was impossible to identify the phase present by any of the microscopic techniques used in this investigation, even though it was possible to differentiate microscopically between sigma and P when they occurred side by side in the same alloy.

The 1,200° C isothermal section of the chromium-cobalt-molybdenum ternary system was also found to contain a ternary phase, located between the ends of the sigma and mu phase fields (fig. 3). This phase R has a distinctive X-ray diffraction pattern (table XIII) and it is easily differentiated microscopically from the sigma or mu phases when these phases occur in the same alloy. However, as in the case of chromium-nickel-molybdenum ternary sigma and P phases, chromium-cobalt-molybdenum alloys consisting wholly of the R or sigma phases could not be differentiated microscopically (except possibly by microhardness). Final identification was made in all cases by X-ray diffraction.

CONCLUSIONS

A survey at 1,200° C of portions of the chromium-cobalt-nickel-molybdenum quaternary system and of the component cobalt-nickel-molybdenum, chromium-cobalt-molybdenum, and chromium-nickel-molybdenum ternary systems indicated the following conclusions:

1. A portion of the 1,200° C isothermal section of the cobalt-nickel-molybdenum ternary system was investigated, using 47 vacuum-melted alloys with molybdenum contents up to a maximum of 62 percent. The system features an extensive range of face-centered cubic (alpha) solid solutions based on the cobalt-nickel binary system, solid solutions based on the nickel-molybdenum binary delta phase, and solid solutions based on the cobalt-molybdenum binary mu phase. The narrow delta and mu phase fields penetrate into the ternary system to 6.5 percent cobalt and 24 percent cobalt, respectively, and coexist in a narrow two-phase delta-plus-mu field. The alpha phase coexists with the delta and mu phases in two wide two-phase fields and an alpha-delta-mu three-phase field.

2. A portion of the 1,200° C isothermal section of the chromium-cobalt-molybdenum ternary system was determined, using 85 vacuum-melted alloys with molybdenum contents up to a maximum of 53 percent and cobalt contents of 35 percent and more. The phases present in this composition range were found to be the face-centered cubic (alpha) solid solutions around the cobalt corner, solid solutions based on the chromium-cobalt binary sigma phase, solid solutions based on the cobalt-molybdenum binary mu phase, and a new ternary phase R, of fairly small composition range, which does not appear in any of the binary systems. The sigma phase was found to extend in an elongated field from the chromium-cobalt binary into the ternary system along a line of approximately constant cobalt content of 45 percent, taking molybdenum into solution up to a maximum of 31 percent, so that up to about half of the chromium is replaced by molybdenum. The R phase, comprising a range of chromium contents from about 14.5 to 19.5 percent, is located between the ends of the sigma phase field and of the mu phase field, and it coexists with the sigma and mu phases in 2 two-phase fields. The cobalt-molybdenum binary mu phase extends into the ternary system up to approximately 11 percent chromium. The alpha face-centered cubic solid solutions coexist with the sigma, R, and mu phases in three rather wide two-phase fields and two narrow three-phase fields of alpha, sigma, and R and of alpha, R, and mu.

3. The 1,200° C isothermal section of the chromium-nickel-molybdenum ternary system was investigated in the composition range of 0 to 59 percent molybdenum and 18 to 100 percent nickel, using 119 vacuum-melted alloys. The following phases were located: The face-centered cubic (alpha) solid solutions around the nickel corner, the body-centered cubic (epsilon) solid solutions based on chromium, solid solutions based on the nickel-molybdenum binary delta phase, the ternary sigma solid solutions which, at 1,200° C, are not connected with any of the binary systems, and another ternary phase P, which has not been encountered in any of the binary systems. The ternary sigma phase is located in a long, narrow field between about 21 and 49 percent chromium and approximately 23 to 30 percent molybdenum, which extends approximately parallel to the chromium-molybdenum side of the ternary diagram, as found also in the chromium-cobalt-molybdenum ternary system. Here, too, the shape and location of the sigma field indicate that chromium and molybdenum can substitute for each other in forming the sigma phase. Successive pairs of the epsilon, sigma, P, and delta phases coexist in 3 two-phase fields. The alpha solid solution coexists with the epsilon, sigma, P, and delta phases in 4 two-phase fields and 3 three-phase fields of alpha, epsilon, and sigma; alpha, sigma, and P; and alpha, P, and delta. The two-phase alpha-plus-sigma field covers an extensive area, starting at molybdenum contents as low as 3.5 percent.

4. The alpha solid-solution phase boundary in the chromium-cobalt-nickel-molybdenum quaternary system at 1,200° C was determined for constant molybdenum contents of 2.5, 5, 10, and 20 percent. The over-all effect of molybdenum additions to the chromium-cobalt-nickel ternary system at 1,200° C is to decrease the extent of the face-centered cubic (alpha) solid solutions, that is, to shift the alpha phase boundary toward lower chromium contents. At 1,200° C the alpha phase boundary in the 2.5-percent-molybdenum section is essentially the same as that in the chromium-cobalt-nickel ternary system, except near the chromium-nickel side, where it is moved toward higher chromium contents. The phases which coexist with the alpha phase in the 2.5-percent-molybdenum section are sigma and epsilon. In the 5-percent constant-molybdenum section the alpha-phase-boundary shift becomes more evident, although the magnitude of the shifting is still relatively small. The sigma phase is seen to coexist with the alpha solid solutions near both the chromium-cobalt and the chromium-nickel sides of the diagram. This is due to the presence of the ternary sigma phase in the chromium-nickel-molybdenum ternary system at 1,200° C. The alpha phase also coexists with epsilon and sigma in a three-phase field near the chromium-nickel side of the diagram. The shift of the alpha phase boundary toward lower chromium contents is more pronounced in the 10-percent-molybdenum section of the isothermal quaternary system. Again the sigma phase is seen to coexist with the alpha solid solutions at both the chromium-cobalt and the chromium-nickel ends of the alpha phase boundary. The alpha phase also coexists with epsilon and sigma in a three-phase field near the chromium-nickel side of the diagram. The alpha phase bound-

ary in the 20-percent-molybdenum section is shifted to 4 percent chromium on the chromium-cobalt side of the diagram and to 30 percent chromium on the chromium-nickel side. The mu phase coexists with the alpha solid solutions near the chromium-cobalt side of the quaternary section, whereas the ternary P phase coexists with the alpha phase at the chromium-nickel end of the alpha phase boundary.

5. The results of this investigation give further support to the conclusion that atoms of both face-centered and body-centered cubic transition elements are required to form the sigma phase and that elements having the same crystal structure can substitute for each other. It appears that the same criterion may hold for ternary solid solutions of the cobalt-molybdenum intermediate mu phase, which is isomorphous with the corresponding intermediate phases in the iron-molybdenum, cobalt-tungsten, and iron-tungsten binary systems. The composition ranges over which the sigma phases exist can be described fairly well in terms of electron vacancy concentration in the $3d$ sub-band. This suggests a criterion for predicting whether or not the sigma phase may occur within any given composition range in alloy systems involving the transition elements.

UNIVERSITY OF NOTRE DAME,
NOTRE DAME, IND., January 31, 1951.

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TABLE I

LOT ANALYSES BY WEIGHT PERCENT

Element or compound	Cobalt rondelles	Chromium	Nickel	Molybdenum
C.....	0.17	0.01	-----	0.003 max.
CaO.....	.12	-----	-----	-----
Co.....	Bal.	-----	0.6 to 0.8	-----
Cr.....	-----	Bal.	-----	-----
Cu.....	.02	Trace	.01	-----
Fe.....	.12	.06	.01	.005 max.
H.....	-----	.045	-----	-----
MgO.....	.04	-----	-----	-----
Mn.....	.06	-----	-----	Bal.
Mo.....	-----	-----	-----	-----
N.....	-----	.010	-----	-----
Ni.....	.46	-----	Bal.	-----
O.....	-----	.51	-----	.003 approx.
Pb.....	-----	.001	-----	-----
S.....	.009	.012	.001	-----
SiO ₂	.13	-----	-----	-----

TABLE II

CRUCIBLES USED FOR MELTING ALLOYS

Zirconia	Stabilized zirconia	Alundum
273	323	272
311 to 314	330 to 332	280
320	354	288
322	360 to 362	289
402 to 405	366	296 to 298
414 to 419	376 to 383	321
		370 to 372
		375
		384 to 401
		407 to 413
		420 to 715

TABLE III
DATA FOR THE PHASE DIAGRAM OF THE COBALT-NICKEL-MOLYBDENUM SYSTEM AT 1,200° C

Alloy	Analysis (weight percent)			Phases from figure 2 (percent)			Phases (percent) found from—				Annealing time (hr)
							Microstructure			X-ray	
	Co	Ni	Mo	Alpha	Delta	Mu	Alpha	Delta	Mu		
Alpha boundary alloys *											
b 399	77.08	-----	22.92	98	----	2	95	----	5	Alpha.....	48
400	-----	65	35	100	----	-----	100	----	-----	Alpha.....	48
401	30	40	30	98	----	2	100	----	Trace	Alpha.....	48
b 407	20	50	30	100	----	-----	100	----	-----	Alpha.....	56
	17.4	50	32.6	-----	-----	-----	-----	-----	-----	-----	-----
b 408	49.62	24.95	25.43	99	----	1	100	----	Trace	Alpha.....	65
426	32	42	26	100	----	-----	100	----	-----	Alpha.....	48
427	-----	68	32	100	----	-----	100	----	-----	Alpha.....	48
432	79	-----	21	100	----	-----	100	----	-----	Alpha.....	48
440	50	28	22	100	----	-----	100	----	-----	Alpha.....	48
441	20	57	23	100	----	-----	100	----	-----	Alpha.....	48
b 461	49.89	26.47	23.64	100	----	-----	100	----	-----	Alpha.....	48
462	20	54	26	100	----	-----	100	----	-----	Alpha.....	48
b 486	-----	62.92	37.08	97	----	3	95	5	-----	Alpha.....	65
498	20	53	27	100	----	-----	100	----	-----	Alpha.....	48
540	65	12	23	100	----	-----	100	----	-----	Alpha.....	48
541	10	58	32	100	----	-----	100	----	-----	Alpha.....	48
552	64	11	25	98	----	2	96	-----	4	Alpha.....	48
553	10	56	34	100	----	-----	100	----	-----	Alpha.....	48
b 620	30.24	40.84	28.92	100	----	-----	100	----	-----	Alpha.....	48
627	10.2	53.72	36.08	98	Trace	2	96	1	3	Alpha.....	48
679	15	62	33	100	----	Trace	100	----	Trace	Alpha.....	48
Alpha-plus-mu alloys *											
421	55	-----	45	30	----	70	25	----	75	Alpha plus mu.....	95
442	15	35	50	40	----	60	50	----	50	Alpha plus mu.....	95
443	25	25	50	30	----	70	25	----	75	Alpha plus mu.....	95
444	35	15	50	20	----	80	25	----	75	Alpha plus mu.....	95
b 536	18.28	24.12	57.60	1	----	99	5	-----	95	Mu.....	95
546	14	46	40	75	----	25	80	-----	20	Alpha plus mu.....	95
547	18	42	40	70	----	30	75	-----	25	Alpha plus mu.....	95
548	22	38	40	70	----	30	75	-----	25	Alpha plus mu.....	95
554	22	28	50	35	----	65	5	-----	95	Mu.....	95
715	12	48	40	80	----	20	75	-----	25	Alpha plus mu.....	95
Mu alloys *											
b 550	22.74	18.77	58.49	----	----	100	----	----	100	Mu.....	95
596	23	16	61	----	----	100	----	----	100	Mu.....	95
629	20	20	60	----	----	100	----	----	100	Mu.....	95
Alpha-plus-delta alloys *											
414	-----	49	51	40	60	----	50	50	----	Alpha plus delta.....	95
465	5	45	50	45	55	----	50	50	----	Alpha plus delta.....	95
520	4	38	58	10	90	----	15	85	-----	Delta.....	95
521	7	35	58	10	90	Trace	12	85	3	Delta.....	95
677	4	36	60	1	99	----	Trace	100	----	Delta.....	95
Delta-plus-mu alloys *											
b 595	7.05	30.78	62.17	----	95	5	----	95	5	Delta.....	95
b 628	7.22	32.65	60.13	----	98	2	1	95	4	Delta.....	95
Alpha-plus-delta-plus-mu alloys *											
464	10	40	50	35	35	30	40	30	30	Alpha plus delta plus mu.....	95
522	10	32	58	10	60	30	15	70	15	Delta plus mu.....	95
523	13	29	58	10	25	65	10	10	80	Delta plus mu.....	95
537	16	26	58	5	10	85	5	-----	95	Mu.....	95
551	12	38	50	35	30	35	50	10	40	Alpha plus delta plus mu.....	95

* All forged prior to annealing.

b Chemical composition from analysis. Composition given for other alloys is the intended melted analysis.

• Not forged prior to annealing.

TABLE IV
DATA FOR THE PHASE DIAGRAM OF THE CHROMIUM-COBALT-MOLYBDENUM SYSTEM AT 1,200° C

Alloy	Analysis (weight percent)			Phases from figure 4 (percent)				Phases found from— (percent)				X-ray	Anneal- ing time (hr)
								Microstructure					
	Cr	Co	Mo	Alpha	Sigma	R	Mu	Alpha	Sigma	R	Mu		
Alpha boundary alloys ^a													
272	33.15	64.35	2.5	90	10	---	---	85	15	---	---	Alpha.....	48
366	29.45	65.55	5	100	---	---	---	100	---	---	---	Alpha.....	48
^b 378	30.31	65.96	3.73	100	---	---	---	100	---	---	---	Alpha.....	48
^b 399	---	77.08	22.92	98	---	---	2	95	---	---	5	Alpha.....	48
413	10	72	18	95	---	---	5	90	---	---	10	Alpha.....	48
430	10	76	14	100	---	---	---	100	---	---	---	Alpha.....	48
431	20	70	10	100	---	---	---	100	---	---	---	Alpha.....	48
432	---	79	21	100	---	---	---	100	---	---	---	Alpha.....	48
453	20	68	12	100	---	---	---	100	---	---	---	Alpha.....	48
455	10	74	16	100	---	---	---	100	---	---	---	Alpha.....	48
456	3	77	20	100	---	---	---	100	---	---	---	Alpha.....	48
^b 471	28.83	64.8	6.37	98	2	---	---	98	2	---	---	Alpha.....	48
487	4	76	20	100	---	---	Trace	100	---	---	Trace	Alpha.....	48
^b 499	25.61	65.4	8.99	100	---	---	---	100	---	---	---	Alpha.....	48
606	22.5	64.5	13	90	10	---	---	90	10	---	---	Alpha.....	48
^b 619	14.58	69.78	15.64	100	---	---	---	100	---	---	---	Alpha.....	48
^b 632	9.88	72.99	17.13	100	---	---	---	100	---	---	---	Alpha.....	48
644	25	64.5	10.5	92	---	---	---	100	---	---	---	Alpha.....	48
^b 645	17.42	68.01	14.57	100	---	---	---	95	5	---	---	Alpha.....	48
^b 665	14.7	68.65	16.65	95	---	---	5	98	---	---	2	Alpha.....	48
Alpha-plus-sigma alloys ^c													
273	31.28	60.72	8	75	25	---	---	65	35	---	---	Alpha.....	96
288	38.02	59.48	2.5	65	35	---	---	60	40	---	---	Alpha plus sigma.....	96
314	36.1	58.9	5	62	38	---	---	60	40	---	---	Alpha plus sigma.....	96
322	36.8	55.2	8	50	50	---	---	50	50	---	---	Alpha plus sigma.....	105
^b 330	34.16	63.41	2.43	87	13	---	---	90	10	---	---	Alpha.....	56
384	42.3	51.7	6	35	65	---	---	40	60	---	---	Sigma plus alpha.....	96
^b 398	31.7	63.09	5.21	87	13	---	---	90	10	---	---	Alpha.....	96
470	29	62	9	85	15	---	---	80	20	---	---	Alpha.....	96
^b 474	32.22	47.57	20.21	15	85	---	---	15	85	---	---	Sigma.....	145
512	27	53	20	45	55	---	---	50	50	---	---	Alpha plus sigma.....	96
538	28.5	51.5	20	40	60	---	---	50	50	---	---	Sigma plus alpha.....	96
539	38	48	14	20	80	---	---	25	75	---	---	Sigma.....	96
543	28	46	26	5	95	---	---	5	95	---	---	Sigma.....	96
555	41	45	14	5	95	---	---	10	90	---	---	Sigma.....	145
^b 557	23.3	56.08	20.62	55	45	---	---	50	50	Trace	---	Sigma.....	150
^b 616	33.75	44.98	21.27	5	95	---	---	5	95	---	---	Alpha plus sigma.....	96
^b 682	7.2	45.6	47.2	8	92	---	---	10	90	---	---	Sigma.....	150
Alpha-plus-R alloys ^c													
^b 412	19.36	65.29	15.35	90	---	10	---	90	---	10	---	Alpha.....	76
533	16	52	32	40	---	60	---	50	---	50	---	Alpha plus R.....	96
^b 558	19.81	57.9	22.29	55	---	45	---	50	---	50	---	Alpha plus R.....	96
560	16	60	24	60	---	40	---	60	---	40	---	Alpha plus R.....	96
592	18	44	38	2	---	98	---	2	---	98	---	R.....	150
Alpha-plus-mu alloys ^c													
^b 534	11.6	54.27	34.13	45	---	---	55	50	---	---	50	Alpha plus mu.....	96
535	8	56	36	50	---	---	50	50	---	---	50	Alpha plus mu.....	96
^b 544	7.86	43.05	49.09	5	---	---	95	5	---	---	95	Mu.....	96
^b 573	12.7	61.31	25.99	65	---	---	35	75	---	---	25	Alpha plus mu.....	96

^a All forged prior to annealing.^b Chemical composition from analysis. Composition given for other alloys is the intended melted analysis.^c Not forged prior to annealing.

TABLE IV—Concluded

DATA FOR THE PHASE DIAGRAM OF THE CHROMIUM-COBALT-MOLYBDENUM SYSTEM AT 1,200° C

Alloy	Analysis (weight percent)			Phases from figure 4 (percent)				Phases found from— (percent)				X-ray	Anneal- ing time (hr)	
								Microstructure						
	Cr	Co	Mo	Alpha	Sigma	R	Mu	Alpha	Sigma	R	Mu			
Alpha-plus-sigma-plus-R alloys ^c														
^b 513	19.88	49.78	30.34	20	5	75	----	20	10	70	----	R plus alpha.....	96	
556	24	50	26	18	80	2	----	20	80	Trace	----	Sigma.....	96	
605	18.5	45.5	36	8	2	90	----	9	1	90	----	R.....	96	
633	22.5	46.5	31	5	70	25	----	10	70	20	----	Sigma plus R.....	96	
649	24.5	47	28.5	5	90	5	----	5	90	5	----	Sigma.....	150	
Alpha-plus-R-plus-mu alloys ^c														
^b 559	13.67	53.12	33.21	40	----	50	1	40	----	60	Trace	R plus alpha.....	96	
^b 585	13.76	60.99	25.25	70	----	10	20	75	----	10	15	Alpha plus mu.....	96	
680	11.5	44	44.5	2	----	43	55	2	----	48	50	Mu plus R.....	96	
^b 681	9.82	43.03	47.15	4	----	Trace	96	2	----	Trace	98	Mu.....	96	
Sigma alloys ^c														
392	52.6	41.4	6	----	100	----	----	----	100	----	----	Sigma.....	150	
393	56.4	37.6	6	----	----	----	----	----	80	----	----	Sigma.....	150	
423	34	36	30	----	----	----	----	----	75	----	----	Sigma.....	150	
^b 472	39.97	40.56	19.47	----	100	----	----	----	100	----	----	Sigma.....	150	
473	37	43	20	----	100	----	----	----	100	----	----	Sigma.....	150	
^b 485	33.99	42.94	23.07	----	100	----	----	----	100	----	----	Sigma.....	150	
^b 500	41.3	38.56	20.14	----	----	----	----	----	95	----	----	Sigma.....	150	
530	32	42	26	----	100	----	----	----	100	----	----	Sigma.....	150	
^b 531	28.37	40.69	30.94	----	----	----	----	----	^d 95	----	----	Sigma.....	150	
543	28	46	26	----	----	----	----	----	95	----	----	Sigma.....	150	
590	46	44	10	----	100	----	----	Trace	100	----	----	Sigma.....	150	
^b 593	27.1	42.4	30.5	----	100	----	----	----	100	Trace	----	Sigma.....	150	
594	30	42	28	----	100	----	----	----	100	----	----	Sigma.....	150	
624	28.5	44.5	27	----	100	----	----	----	100	----	----	Sigma.....	150	
625	27	45.5	27.5	----	100	----	----	Trace	100	----	----	Sigma.....	150	
Sigma-plus-R alloys ^c														
^b 422	24.42	45.82	29.76	----	85	15	----	----	80	20	----	Sigma.....	150	
^b 497	19.79	41.95	38.26	----	10	90	----	----	10	90	----	R.....	96	
532	24	42	34	----	60	40	----	----	50	50	----	Sigma plus R.....	96	
650	19	44	37	----	15	85	----	----	25	75	----	R.....	96	
R alloys ^c														
^b 514	15.71	41.69	42.60	----	----	100	----	----	----	100	----	R.....	96	
^b 591	13.78	43.89	42.33	----	----	100	----	----	----	100	----	R.....	96	
R-plus-mu alloys ^c														
515	12	41	47	----	----	40	60	----	----	50	50	Mu plus R.....	96	
666	16	40	44	----	----	98	2	----	----	95	5	R.....	96	
Mu alloys ^c														
516	6	41	53	----	----	----	100	----	----	----	100	Mu.....	96	
545	11	38	51	----	----	----	99	----	----	----	96	Mu.....	96	
^b 574	8.51	40.37	51.12	----	----	----	100	----	----	----	100	Mu.....	96	
^b 630	9.4	42.8	47.8	2	----	----	98	2	----	----	98	Mu.....	96	
631	10	42	48	----	----	2	98	----	----	2	98	Mu.....	96	

^b Chemical composition from analysis. Composition given for other alloys is the intended melted analysis.^c Not forged prior to annealing.^d Also two second phases.

TABLE V
DATA FOR THE PHASE DIAGRAM OF THE CHROMIUM-NICKEL-MOLYBDENUM SYSTEM AT 1,200° C

Alloy	Analysis (weight percent)			Phases from figure 6 (percent)					Phases (percent) found from—					Annealing time (hr)	
									Microstructure						X-ray
	Cr	Ni	Mo	Alpha	Epsilon	Sigma	P	Delta	Alpha	Epsilon	Sigma	P	Delta		
Alpha alloys *															
360	42.51	55.08	2.41	100					100					Alpha	48
375	39.9	55.1	5	100					100					Alpha	48
382	40.56	54.92	4.52	100					100					Alpha	48
400		65	35	100					100					Alpha	48
427		68	32	100					100					Alpha	48
433	10	64	26	100					100					Alpha	48
434	26	58	16	100					100					Alpha	48
437	35.99	54.28	9.73	100					100		Trace			Alpha	48
438	25.48	57.31	17.21	100		Trace			100		1			Alpha	48
454	10	62.5	27.5	100					100					Alpha	48
486		62.92	37.08	100					100					Alpha	48
688	5	61.75	33.25	100				Trace	100				Trace	Alpha	48
Alpha-plus-epsilon alloys *															
298	56.33	41.17	2.5	55	45				50	50				Alpha plus epsilon	96
311	43.37	54.17	2.46	95	5				95	5				Alpha	48
387	53.86	40.28	5.86	50	50				50	50				Alpha plus epsilon	96
404	70.5	23.6	6	5	95				10	90				Epsilon	96
449	60	30	10	5	95				10	90				Epsilon	96
562	60	35	5	35	65				40	60				Epsilon plus alpha	96
563	57.5	35	7.5	30	70				30	70				Epsilon plus alpha	96
614	43.5	53	3.5	98	2				95	5				Alpha	48
Alpha-plus-sigma alloys *															
389	42.75	52.25	5	95		5			95		5			Alpha	48
410	26	54	20	85		15			90		10			Alpha	56
411	40	50	10	85		15			80		20			Alpha	56
418	45	35	20	25		75			25		75			Sigma plus alpha	96
435	36.25	53.63	10.12	98		2			95		5			Alpha	48
445	28	49	23	75		25			75		25			Alpha plus sigma	96
446	32	38	30	35		65			25		75			Sigma plus alpha	96
447	40	30	30	5		95			5		95			Sigma	96
451	36	46	18	70		30			60		40			Alpha plus sigma	96
452	43	44	13	65		35			60		40			Alpha plus sigma	96
476	40	40	20	45		55			50		50			Sigma plus alpha	96
477	35	40	25	40		60			50		50			Sigma plus alpha	96
478	48	32	20	20		80			25		75			Sigma	96
479	41	33	26	20		80			20		80			Sigma	96
480	36	34	30	20		80			25		75			Sigma	96
482	28	32	40	10		90			10		90			Sigma	96
483	33	32	35	10		90			5		95			Sigma	96
484	27	38	35	30		70			40		60			Sigma plus alpha	96
491	45	30	25	10		90			15		85			Sigma	96
569	45.98	37.87	16.5	40		60			50		50			Sigma plus alpha	96
570	46.5	36	17.5	30		70			40		60			Sigma plus alpha	96
597	44.35	44.67	10.98	65		35			60		40			Alpha plus sigma	96
608	23.5	39	37.5	30		70			25		75			Sigma plus alpha	96
615	48.5	30.5	21	10		90			15		85			Sigma	96
617	24.5	32	43.5	7		93			5		95			Sigma	96
634	23.3	48	28.7	65		35			70		30			Alpha plus sigma	96
641	22.76	31.6	45.64	5		95			5		95			Sigma	96
654	35	30	35	1		99			Trace		100			Sigma	150
663	22.5	48.5	29	75		25			75		25			Alpha plus sigma	96
684	22	32	46	5		95			5		95			Sigma	96
687	29.5	30.5	40	2		98			1		99			Sigma	96
Alpha-plus-P alloys *															
409	10	60	30	95			5		93			7		Alpha	48
415	20	40	40	35			65		30			70		P plus alpha	96
457	22.9	56.45	20.65	90			10		95			5		Alpha	48
460	14.28	59	26.72	95			30		98			2		Alpha	48
466	20	50	30	75			5		80			20		Alpha plus P	96
467	10	55	35	80			20		75			25		Alpha plus P	96
468	10	45	45	35			65		50			50		P plus alpha	96
469	22.09	57.33	20.58	95			5		95			5		Alpha	48
510	10	35	55	5			95		3			97		P	96
579	15.07	33.72	51.21	3			97		5			95		P	96
612	8.5	35	56.5	3			97		5			95		P	96
636	11.28	59.46	29.26	93			7		95			5		Alpha	48
664	21.25	48.23	30.52	65			35		70			30		Alpha plus P	96
689	6.41	53.48	40.11	65			35		70			30		Alpha plus P	96
Alpha-plus-delta alloys *															
414		49	51	40					50					Alpha plus delta	95
691	4.5	54.5	41	70					30					Alpha plus delta	96
692	4.5	46	49.5	40					60					Delta plus alpha	96

* All forged prior to annealing.

† Chemical composition from analysis. Composition given for other alloys is the intended melted analysis.

• Not forged prior to annealing.

TABLE V—Concluded

DATA FOR THE PHASE DIAGRAM OF THE CHROMIUM-NICKEL-MOLYBDENUM SYSTEM AT 1,200° C

Alloy	Analysis (weight percent)			Phases from figure 6 (percent)					Phases (percent) found from—						Annealing time (hr)
									Microstructure					X-ray	
	Cr	Ni	Mo	Alpha	Epsilon	Sigma	P	Delta	Alpha	Epsilon	Sigma	P	Delta		
Alpha-plus-epsilon-plus-sigma alloys *															
320	43.7	51.3	5	90	Trace	10	---	---	90	1	9	---	---	Alpha.....	75
^b 448	48.66	31.25	20.09	15	5	80	---	---	10	10	80	---	---	Sigma.....	150
^b 450	49.82	37.19	12.99	30	30	40	---	---	35	25	40	---	---	Sigma plus epsilon.....	150
475	47	40	13	45	5	50	---	---	50	---	50	---	---	Sigma plus alpha.....	150
564	54.5	35	10.5	35	50	15	---	---	25	55	20	---	---	Epsilon plus sigma.....	150
^b 565	52.5	34.54	12.96	20	40	40	---	---	30	35	35	---	---	Epsilon plus sigma.....	150
566	55	30	15	10	50	40	---	---	20	50	30	---	---	Epsilon plus sigma.....	150
586	51	29	20	5	20	75	---	---	5	20	75	---	---	Sigma.....	150
^b 588	52.6	28.73	18.67	2	38	60	---	---	5	20	75	---	---	Sigma.....	150
^b 598	55.85	35.1	9.05	23	75	2	---	---	30	50	20	---	---	Epsilon plus alpha plus sigma.....	150
^b 600	48.5	45.4	6.1	65	34	1	---	---	50	30	20	---	---	Alpha plus epsilon plus sigma.....	150
622	50	28	22	Trace	10	90	---	---	5	10	85	---	---	Sigma.....	150
655	43.5	52.25	4.25	94	1	5	---	---	95	Trace	5	---	---	Alpha.....	75
672	59.5	29	11.5	2	93	5	---	---	Trace	80	20	---	---	Epsilon plus sigma.....	150
Alpha-plus-sigma-plus-P alloys *															
568	20	35	45	30	---	30	40	---	30	---	30	40	---	P plus sigma.....	96
^b 613	19.3	32.5	48.2	5	---	5	90	---	10	---	15	75	---	P.....	96
642	21.25	32	46.75	5	---	95	Trace	---	5	---	95	Trace	---	Sigma.....	96
669	22	49	29	70	---	15	15	---	75	---	10	15	---	Alpha.....	96
Alpha-plus-P-plus-delta alloys *															
635	7	45	48	38	---	---	60	2	50	---	---	50	Trace	P plus alpha.....	96
^b 670	7.1	35.8	57.1	5	---	---	70	25	8	---	---	87	5	P.....	96
690	5.5	54	40.5	70	---	---	10	20	70	---	---	10	20	Alpha plus delta.....	96
Epsilon alloys *															
397	76.2	17.8	6	---	100	---	---	---	---	100	---	---	---	Epsilon.....	150
599	65	22	13	---	100	---	---	---	---	100	---	---	---	Epsilon.....	150
618	61	27	12	---	98	2	---	---	---	100	Trace	---	---	Epsilon.....	150
676	65	25.75	9.25	---	100	---	---	---	Trace	100	---	---	---	Epsilon.....	150
Epsilon-plus-sigma alloys *															
419	61	19	20	---	---	---	---	---	---	65	35	---	---	Sigma plus epsilon.....	150
567	52.5	25	22.5	---	35	65	---	---	---	30	70	---	---	Sigma.....	150
^b 587	51.99	27.19	20.82	---	25	75	---	---	---	30	70	---	---	Sigma.....	150
^b 640	47.6	27.2	25.2	---	2	98	---	---	---	5	95	---	---	Sigma.....	150
643	49	27.5	23.5	---	---	---	---	---	---	---	---	---	---	Epsilon plus sigma.....	150
643	59	28	13	---	80	20	---	---	---	60	40	---	---	Epsilon plus sigma.....	150
Sigma alloys *															
416	34	26	40	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
481	37	28	35	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
493	47	23	30	---	---	---	---	---	---	---	^d 90	---	---	Sigma.....	96
508	32	23	45	---	---	---	---	---	---	---	^d 90	---	---	Sigma.....	96
^b 509	25.06	26.12	48.82	---	---	98	---	---	---	---	^d 95	---	---	Sigma.....	96
511	38	22	40	---	---	---	---	---	---	---	^d 90	---	---	Sigma.....	96
527	31	29	40	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
577	31.5	24.5	44	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
578	30.5	26.5	43	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
602	25	27.5	47.5	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
603	36	24	40	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
604	45	25	30	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
685	22.5	30	47.5	---	---	100	---	---	---	---	100	---	---	Sigma.....	96
Sigma-plus-P alloys *															
528	22	28	50	---	---	50	50	---	---	---	50	50	---	Sigma plus P.....	96
686	23.5	27.5	49	---	---	80	20	---	---	---	90	10	---	Sigma.....	96
492	20	30	50	---	---	10	90	---	---	---	4	96	---	P.....	96
P alloys *															
580	15	30	55	---	---	---	100	---	---	---	---	100	---	P.....	96
Delta alloys *															
529	5	36	59	---	---	---	30	70	---	---	---	40	60	Delta plus P.....	96
^b 561	2.81	36.67 ^a	60.52	---	---	---	---	100	---	---	---	---	100	Delta.....	96
626	4	36.5	59.5	---	---	---	2	98	---	---	---	Trace	100	Delta.....	96
671	4	37.25	58.75	1	---	---	---	99	Trace	---	---	---	100	Delta.....	96

* Chemical composition from analysis. Composition given for other alloys is the intended melted analysis.

^b Not forged prior to annealing.^d Also 10 percent of a second phase.

* Also two second phases.

TABLE VI

DATA FOR THE PHASE DIAGRAM OF THE CHROMIUM-COBALT-NICKEL-MOLYBDENUM SYSTEM AT 1,200° C.

Alloy	Analysis (weight percent)				Phases from figure 8 (percent)			Phases (percent) found from—				Annealing time (hr)
	Cr	Co	Ni	Mo	Alpha	Sigma	Epsilon	Microstructure			X-ray	
								Alpha	Sigma	Epsilon		
2.5-percent-molybdenum quaternary alloys ^a												
280	42.9	16.58	38.02	2.5	---	---	---	^b 99	---	---	Alpha plus sigma	48
288	38.02	59.48	---	2.5	65	35	---	60	40	---	Alpha	96
289	35.1	62.4	---	2.5	85	15	---	90	10	---	Alpha	96
296	38.02	44.85	14.63	2.5	---	---	---	^b 95	---	---	---	48
297	44.35	15.6	37.05	2.5	---	---	---	^b 95	---	---	---	48
^c 311	43.37	---	54.17	2.46	95	---	5	95	---	5	Alpha	48
312	40.95	31.2	25.35	2.5	---	---	---	^b 85	---	---	---	48
313	44.85	9.75	42.9	2.5	---	---	---	^b 90	---	---	---	48
323	39	31.2	27.3	2.5	---	---	---	^b 98	---	---	---	48
^c 330	34.16	63.41	---	2.43	87	13	---	90	10	---	Alpha	48
^c 331	36.84	45.13	15.61	2.42	---	---	---	^b 99	---	---	---	48
332	43.38	9.75	43.87	2.5	---	---	---	^b 98	---	---	---	48
^c 354	37.34	31.90	28.08	2.68	100	---	---	100	---	---	Alpha	48
^c 360	42.51	---	55.08	2.41	100	---	---	100	---	---	Alpha	48
^c 361	41.94	9.51	46.12	2.43	100	---	---	100	---	---	Alpha	48
362	39.98	24.38	33.15	2.5	100	---	---	100	---	---	Alpha	48
372	40.46	24.38	32.66	2.5	---	---	---	^b 98	---	---	---	48
376	38.76	31.2	27.54	2.5	---	---	---	^b 99	---	---	---	48
377	43.39	9.75	44.36	2.5	---	---	---	^b 99	---	---	---	48
5-percent-molybdenum quaternary alloys ^a												
314	36.1	58.9	---	5	62	38	---	60	40	---	Alpha plus sigma	96
320	43.7	---	51.3	5	90	10	Trace	90	9	1	Alpha	75
321	43.7	19	32.3	5	---	---	---	^b 90	---	---	---	48
366	29.45	65.55	---	5	100	---	---	100	---	---	Alpha	48
375	39.9	---	55.1	5	100	---	---	100	---	---	Alpha	48
^c 378	30.31	65.96	---	3.73	100	---	---	100	---	---	Alpha	48
^c 379	34.62	42.49	18.72	4.17	100	---	---	100	---	---	Alpha	48
^c 380	40.37	16.8	37.84	4.99	---	---	---	^b 95	---	---	---	48
^c 381	39.69	9.34	45.78	5.19	---	---	---	^b 98	---	---	---	48
^c 382	40.56	---	54.92	4.52	100	---	---	100	---	---	Alpha	48
^c 388	38.84	10	46.56	4.60	100	---	---	100	---	---	Alpha	48
389	42.75	---	52.25	5	95	5	---	95	5	---	Alpha	48
^c 390	39.06	17.14	38.99	4.81	100	---	---	^d 100	---	---	---	48
^c 398	31.7	63.09	---	5.21	87	13	---	90	10	---	Alpha	96
^c 428	35.17	42.45	17.16	5.22	---	---	---	^b 98	---	---	---	48
^c 429	37.64	27.83	29.62	5.27	---	---	---	^b 95	---	---	---	48
436	37.05	28.5	29.45	5	100	---	---	100	---	---	Alpha	48
10-percent-molybdenum quaternary alloys ^a												
431	20	70	---	10	100	---	---	100	---	---	Alpha	48
^c 435	36.25	---	53.63	10.12	98	2	---	98	5	---	Alpha	48
^c 437	35.99	---	54.28	9.73	100	---	---	100	---	---	Alpha	48
506	28.8	48.6	12.6	10	---	---	---	^b 90	---	---	---	48
507	34.2	18	37.8	10	---	---	---	^b 85	---	---	---	48
^c 517	26.75	48.65	14.3	10.3	100	---	---	100	---	---	Alpha	48
^c 518	28.72	34.2	26.87	10.21	100	---	---	100	---	---	Alpha	48
^c 519	31.5	18	40.5	10	100	---	---	100	---	---	Alpha	48
^c 621	29.48	33.90	26.42	10.2	---	---	---	^b 95	---	---	---	48
^c 623	27.84	48.25	13.48	10.43	---	---	---	^b 93	---	---	---	48
^c 653	33.4	9.2	46.7	10.7	---	---	---	^b 92	---	---	---	48
20-percent-molybdenum quaternary alloys ^a												
					Alpha	P	Mu	Alpha	P	Mu		
456	3	77	---	20	100	---	---	100	---	---	Alpha	48
^c 457	22.9	---	56.45	20.65	90	10	---	95	5	---	Alpha	48
458	16	24	40	20	100	---	---	100	---	---	Alpha	48
459	11.2	48	20.8	20	100	---	---	100	---	---	Alpha	48
^c 469	22.09	---	57.33	20.58	95	5	---	95	5	---	Alpha	48
487	4	76	---	20	100	---	Trace	100	---	Trace	Alpha	48
488	16	48	16	20	---	---	---	^b 80	---	---	---	48
489	20.8	32	27.2	20	---	---	---	^b 80	---	---	---	48
490	24	12.8	43.2	20	---	---	---	^b 80	---	---	---	48
501	9.6	64	6.4	20	---	---	---	^b 98	---	---	---	48
502	14.8	48	17.2	20	---	---	---	^b 90	---	---	---	48
503	19.2	32	28.8	20	---	---	---	^b 88	---	---	---	48
504	20.8	24	35.2	20	---	---	---	^b 85	---	---	---	48
505	21.6	12.8	45.6	20	---	---	---	^b 92	---	---	---	48
524	20	12.8	47.2	20	---	---	---	^d 100	---	---	---	48
525	18.4	24	37.6	20	---	---	---	^b 95	---	---	---	48
526	16.8	32	31.2	20	---	---	---	^b 92	---	---	---	48
651	14	61.6	7.2	20	---	---	---	^b 98	---	---	---	48
^c 652	12.4	47.7	19.3	20.6	---	---	---	^d 100	---	---	---	48

^a All forged prior to annealing.^b Remaining percentage is unidentified second phase.^c Chemical composition from analysis. Composition given for other alloys is the intended melted analysis.^d Also trace of unidentified second phase.

TABLE VII

TYPICAL X-RAY DIFFRACTION PATTERN OF
FACE-CENTERED CUBIC ALPHA PHASE

Line	Estimated intensity	θ	d	Radiation Cr K	hkl
1	Medium.....	29.86	2.089	Beta.....	111
2	Strong.....	33.17	2.089	Alpha.....	111
3	Very weak.....	35.21	1.804	Beta.....	200
4	Medium.....	39.30	1.805	Alpha.....	200
5	Very weak.....	54.84	1.272	Beta.....	220
6	Weak.....	63.85	1.273	Alpha ₁	220
7	Weak.....	64.01	1.273	Alpha ₂	220
8	Very weak.....	73.53	1.085	Beta.....	311

TABLE VIII

TYPICAL X-RAY DIFFRACTION PATTERN OF
BODY-CENTERED CUBIC EPSILON PHASE

Line	Estimated intensity	θ	d	Radiation Cr K	hkl
1	Medium strong.	30.69	2.039	Beta.....	110
2	Strong.....	34.13	2.037	Alpha.....	110
3	Weak.....	46.43	1.435	Beta.....	200
4	Weak.....	52.75	1.435	Alpha ₁	200
5	Very weak.....	52.87	1.435	Alpha ₂	200
6	Medium weak.	62.64	1.171	Beta.....	211
7	Strong.....	77.42	1.171	Alpha ₁	211
8	Medium strong.	77.89	1.171	Alpha ₂	211

TABLE IX

TYPICAL X-RAY DIFFRACTION PATTERNS OF CHROMIUM-NICKEL-MOLYBDENUM AND CHROMIUM-COBALT-MOLYBDENUM SIGMA PHASES

Line	Estimated intensity	θ	d	Radiation Cr K	Line	Estimated intensity	θ	d	Radiation Cr K
Cr-Ni-Mo sigma phase determined from alloy 587									
1	Medium.....	25.32	2.672	Alpha.	20	Very weak.....	58.02	1.347	Alpha ₁ .
2	Medium weak.....	26.66	2.318	Beta.	21	Very very weak.....	58.14	1.347	Alpha ₂ .
3	Medium weak.....	27.38	2.485	Alpha.	22	Weak.....	61.04	1.306	Alpha.
4	Medium weak.....	28.25	2.416	Alpha.	23	Medium.....	63.52	1.276	Alpha ₁ .
5	Medium weak.....	28.61	2.387	Alpha.	24	Medium weak.....	63.72	1.276	Alpha ₂ .
6	Medium.....	29.51	2.320	Alpha.	25	Weak.....	64.77	1.263	Alpha ₁ .
7	Medium weak.....	30.40	2.056	Beta.	26	Very very weak.....	64.94	1.263	Alpha ₂ .
8	Medium weak.....	31.25	2.004	Beta.	27	Weak.....	65.57	1.255	Alpha ₁ .
9	Medium.....	31.76	2.172	Alpha.	28	Very weak.....	65.72	1.255	Alpha ₂ .
10	Medium weak.....	31.96	2.160	Alpha.	29	Weak.....	65.95	1.251	Alpha ₁ .
11	Medium.....	32.87	1.917	Beta.	30	Very weak.....	66.15	1.251	Alpha ₂ .
12	Medium strong.....	33.75	2.057	Alpha.	31	Medium weak.....	66.86	1.243	Alpha ₁ .
13	Strong.....	34.81	2.002	Alpha.	32	Weak.....	67.03	1.243	Alpha ₂ .
14	Strong.....	35.60	1.964	Alpha.	33	Weak.....	68.74	1.226	Alpha ₁ .
15	Medium.....	36.58	1.918	Alpha.	34	Very weak.....	68.95	1.226	Alpha ₂ .
16	Very weak.....	37.78	1.864	Alpha.	35	Weak.....	69.67	1.218	Alpha ₁ .
17	Very very weak.....	39.03	1.815	Alpha.	36	Very weak.....	69.87	1.219	Alpha ₂ .
18	Very very weak.....	39.70	1.789	Alpha.	37	Very weak.....	74.68	1.185	Alpha ₁ .
19	Medium weak.....	43.15	1.672	Alpha.	38	Very very weak.....	75.04	1.184	Alpha ₂ .
Cr-Co-Mo sigma phase determined from alloy 530									
1	Very weak.....	25.53	2.651	Alpha.	21	Very weak.....	58.28	1.343	Alpha ₁ .
2	Very weak.....	26.7	2.314	Beta.	22	Very very weak.....	58.47	1.343	Alpha ₂ .
3	Weak.....	27.52	2.473	Alpha.	23	Medium strong.....	63.87	1.273	Alpha ₁ .
4	Medium weak.....	28.31	2.410	Alpha.	24	Medium.....	64.05	1.273	Alpha ₂ .
5	Medium.....	28.68	2.382	Alpha.	25	Medium.....	65.27	1.258	Alpha ₁ .
6	Medium.....	29.58	2.315	Alpha.	26	Medium weak.....	65.46	1.258	Alpha ₂ .
7	Weak.....	30.44	2.053	Beta.	27	Medium strong.....	65.89	1.252	Alpha ₁ .
8	Medium weak.....	31.35	1.999	Beta.	28	Medium weak.....	66.05	1.252	Alpha ₂ .
9	Medium strong.....	31.88	2.164	Alpha.	29	Weak.....	66.42	1.247	Alpha ₁ .
10	Medium.....	32.11	2.150	Alpha.	30	Very weak.....	66.65	1.247	Alpha ₂ .
11	Medium strong.....	32.95	1.913	Beta.	31	Medium weak.....	67.23	1.239	Alpha ₁ .
12	Medium.....	33.85	2.052	Alpha.	32	Medium weak.....	67.42	1.239	Alpha ₂ .
13	Strong.....	34.86	2.000	Alpha.	33	Weak.....	67.71	1.235	Alpha ₁ .
14	Strong.....	35.69	1.960	Alpha.	34	Very weak.....	67.95	1.235	Alpha ₂ .
15	Strong.....	36.65	1.915	Alpha.	35	Medium strong.....	69.31	1.221	Alpha ₁ .
16	Medium.....	37.87	1.863	Alpha.	36	Medium.....	69.6	1.221	Alpha ₂ .
17	Medium weak.....	39.81	1.786	Alpha.	37	Weak.....	70.4	1.213	Alpha ₁ .
18	Very very weak.....	43.09	1.674	Alpha.	38	Weak.....	70.63	1.213	Alpha ₂ .
19	Very weak.....	55.16	1.392	Alpha ₁ .	39	Weak.....	75.44	1.181	Alpha ₁ .
20	Very weak.....	55.33	1.391	Alpha ₂ .	40	Very weak.....	75.73	1.181	Alpha ₂ .

TABLE X

TYPICAL X-RAY DIFFRACTION PATTERN OF DELTA PHASE DETERMINED FROM ALLOY 522

Line	Estimated intensity	θ	d	Radiation Cr K	Line	Estimated intensity	θ	d	Radiation Cr K
1	Weak	25.24	2.679	Alpha.	23	Very weak	38.24	1.847	Alpha.
2	Very weak	25.46	2.660	Alpha.	24	Very very weak	38.91	1.820	Alpha.
3	Weak	28.09	2.428	Alpha.	25	Very very weak	40.85	1.748	Alpha.
4	Weak	28.27	2.196	Beta.	26	Very weak	57.77	1.351	Alpha.
5	Very very weak	28.64	2.384	Alpha.	27	Very very weak	57.89	1.351	Alpha.
6	Very weak	29.03	2.144	Beta.	28	Very weak	58.90	1.334	Alpha.
7	Weak	29.90	2.086	Beta.	29	Very very weak	59.08	1.334	Alpha.
8	Medium weak	30.37	2.058	Beta.	30	Weak	63.94	1.272	Alpha.
9	Weak	30.70	2.038	Beta.	31	Very weak	64.15	1.272	Alpha.
10	Strong	31.23	2.204	Alpha.	32	Very weak	65.85	1.252	Alpha.
11	Weak	31.42	2.192	Alpha.	33	Very very weak	65.99	1.253	Alpha.
12	Very weak	31.62	1.983	Beta.	34	Very weak	66.36	1.247	Alpha.
13	Very very weak	32.01	2.156	Alpha.	35	Very very weak	66.52	1.248	Alpha.
14	Medium weak	32.25	2.142	Alpha.	36	Weak	66.73	1.244	Alpha.
15	Very very weak	32.77	2.113	Alpha.	37	Very weak	66.93	1.244	Alpha.
16	Very very weak	33.01	1.910	Beta.	38	Very very weak	67.51	1.236	Alpha.
17	Medium strong	33.24	2.084	Alpha.	39	Very weak	67.74	1.237	Alpha.
18	Strong	33.77	2.056	Alpha.	40	Weak	68.02	1.232	Alpha.
19	Medium	34.14	2.037	Alpha.	41	Very weak	68.22	1.232	Alpha.
20	Medium	35.19	1.983	Alpha.	42	Weak	70.64	1.211	Alpha.
21	Medium weak	36.35	1.929	Alpha.	43	Weak	70.89	1.211	Alpha.
22	Very weak	36.72	1.912	Alpha.	44	Very weak	74.01	1.189	Alpha.
					45	Very very weak	74.38	1.188	Alpha.

TABLE XI

TYPICAL X-RAY DIFFRACTION PATTERN OF MU PHASE DETERMINED FROM ALLOY 678

Line	Estimated intensity	θ	d	Radiation Cr K	Line	Estimated intensity	θ	d	Radiation Cr K
1	Medium weak	26.00	2.608	Alpha.	17	Weak	36.92	1.903	Alpha.
2	Very weak	28.65	2.169	Beta.	18	Very weak	38.98	1.817	Alpha.
3	Medium strong	28.83	2.370	Alpha.	19	Very very weak	39.76	1.787	Alpha.
4	Very weak	29.26	2.339	Alpha.	20	Very weak	56.73	1.366	Alpha.
5	Very very weak	29.71	2.306	Alpha.	21	Very very weak	56.91	1.366	Alpha.
6	Medium weak	30.19	2.069	Beta.	22	Very weak	59.64	1.324	Alpha.
7	Very very weak	30.57	2.045	Beta.	23	Very very weak	59.80	1.324	Alpha.
8	Weak	31.00	2.020	Beta.	24	Weak	61.43	1.301	Alpha.
9	Medium	31.83	2.166	Alpha.	25	Very weak	61.62	1.301	Alpha.
10	Medium	32.52	2.125	Alpha.	26	Weak	62.51	1.288	Alpha.
11	Very very weak	33.17	1.901	Beta.	27	Very weak	62.68	1.288	Alpha.
12	Strong	33.52	2.070	Alpha.	28	Medium weak	70.04	1.216	Alpha.
13	Medium weak	33.98	2.045	Alpha.	29	Weak	70.26	1.216	Alpha.
14	Medium	34.48	2.019	Alpha.	30	Medium weak	74.83	1.184	Alpha.
15	Very very weak	34.91	1.998	Alpha.	31	Weak	75.19	1.184	Alpha.
16	Very weak	35.83	1.952	Alpha.					

TABLE XII

TYPICAL X-RAY DIFFRACTION PATTERN OF P PHASE DETERMINED FROM ALLOY 580

Line	Estimated intensity	θ	d	Radiation Cr K	Line	Estimated intensity	θ	d	Radiation Cr K
1	Very weak	25.54	2.650	Alpha.	25	Medium weak	35.14	1.985	Alpha.
2	Very very weak	25.82	2.388	Beta.	26	Weak	36.44	1.925	Alpha.
3	Very very weak	25.9	2.381	Beta.	27	Weak	36.76	1.910	Alpha.
4	Very very weak	26.51	2.329	Beta.	28	Weak	37.07	1.897	Alpha.
5	Weak	28.35	2.191	Beta.	29	Very weak	38.69	1.829	Alpha.
6	Medium weak	28.55	2.391	Alpha.	30	Medium weak	60.2	1.317	Alpha.
7	Medium weak	28.69	2.381	Alpha.	31	Weak	60.39	1.317	Alpha.
8	Medium weak	29.11	2.137	Beta.	32	Weak	60.99	1.307	Alpha.
9	Weak	29.36	2.332	Alpha.	33	Very weak	61.1	1.307	Alpha.
10	Medium weak	29.65	2.102	Beta.	34	Weak	61.5	1.300	Alpha.
11	Weak	29.98	2.287	Alpha.	35	Weak	61.62	1.301	Alpha.
12	Weak	30.21	2.067	Beta.	36	Weak	62.71	1.285	Alpha.
13	Weak	30.65	2.040	Beta.	37	Very weak	62.88	1.286	Alpha.
14	Weak	30.81	2.031	Beta.	38	Medium weak	63.48	1.277	Alpha.
15	Strong	31.40	2.190	Alpha.	39	Medium weak	65.06	1.260	Alpha.
16	Weak	31.7	2.175	Alpha.	40	Weak	65.24	1.269	Alpha.
17	Medium weak	32.32	2.137	Alpha.	41	Medium weak	67.34	1.238	Alpha.
18	Medium	32.93	2.102	Alpha.	42	Weak	67.51	1.239	Alpha.
19	Medium	32.97	2.101	Alpha.	43	Medium strong	73.17	1.194	Alpha.
20	Medium	33.36	2.078	Alpha.	44	Medium weak	73.47	1.194	Alpha.
21	Medium	33.58	2.067	Alpha.	45	Weak	74.01	1.189	Alpha.
22	Medium weak	33.85	2.052	Alpha.	46	Very weak	74.28	1.189	Alpha.
23	Medium	34.04	2.041	Alpha.	47	Medium weak	76.42	1.175	Alpha.
24	Medium strong	34.26	2.031	Alpha.	48	Weak	76.83	1.175	Alpha.

TABLE XIII

TYPICAL X-RAY DIFFRACTION PATTERN OF R PHASE DETERMINED FROM ALLOY 592

Line	Estimated intensity	θ	d	Radiation Cr K	Line	Estimated intensity	θ	d	Radiation Cr K
1	Weak	25.45	2.659	Alpha.	22	Medium strong	35.53	1.966	Alpha.
2	Very strong	25.94	2.612	Alpha.	23	Medium	35.81	1.954	Alpha.
3	Very strong	26.18	2.592	Alpha.	24	Very weak	37.22	1.890	Alpha.
4	Very weak	26.43	2.536	Beta.	25	Weak	37.42	1.881	Alpha.
5	Very very weak	26.87	2.529	Alpha.	26	Very weak	38.20	1.849	Alpha.
6	Weak	27.52	2.473	Alpha.	27	Medium weak	40.24	1.770	Alpha.
7	Medium weak	28.59	2.173	Beta.	28	Very weak	62.20	1.291	Alpha.
8	Medium weak	28.81	2.159	Beta.	29	Very weak	62.22	1.293	Alpha.
9	Weak	29.29	2.337	Alpha.	30	Weak	63.23	1.280	Alpha.
10	Medium weak	29.59	2.107	Beta.	31	Weak	63.42	1.280	Alpha.
11	Weak	29.91	2.291	Alpha.	32	Medium weak	64.27	1.268	Alpha.
12	Medium weak	30.45	2.053	Beta.	33	Weak	64.49	1.268	Alpha.
13	Very weak	30.71	2.238	Alpha.	34	Very weak	64.81	1.263	Alpha.
14	Weak	31.27	2.004	Beta.	35	Very weak	66.51	1.246	Alpha.
15	Weak	31.51	2.186	Alpha.	36	Medium weak	66.69	1.246	Alpha.
16	Strong	31.76	2.171	Alpha.	37	Weak	68.18	1.231	Alpha.
17	Strong	31.97	2.159	Alpha.	38	Medium weak	68.39	1.231	Alpha.
18	Medium strong	32.87	2.106	Alpha.	39	Weak	69.62	1.219	Alpha.
19	Medium strong	33.85	2.052	Alpha.	40	Medium weak	69.90	1.219	Alpha.
20	Strong	34.76	2.005	Alpha.	41	Weak	71.24	1.207	Alpha.
21	Medium strong	35.11	1.987	Alpha.	42	Weak	71.47	1.207	Alpha.

TABLE XIV

DETERMINATION OF ALPHA THREE-PHASE-FIELD CORNER BY X-RAY DIFFRACTION IN COBALT-NICKEL-MOLYBDENUM SYSTEM

[Data for fig. 26]

Alloy	Weight percent cobalt	Extrapolated value of a_0 (kX)
399	77	3.5925
552	64	3.6008
408	50	3.6028
401	30	3.6133
679	15	3.6235
627	10	3.6320
486	0	3.6351
464	Three-phase alloy	3.6287
551	Three-phase alloy	3.6300
442	At boundary of three-phase field.	3.6307

TABLE XVI

DETERMINATION OF ALPHA THREE-PHASE-FIELD CORNERS BY X-RAY DIFFRACTION IN CHROMIUM-COBALT-MOLYBDENUM SYSTEM

[Data for fig. 28]

Alloy	Weight percent chromium	Extrapolated value of a_0 (kX)
399	0	3.5925
487	4	3.5870
413	10	3.5834
665	14.7	3.5832
645	17.42	3.5855
412	19.36	3.5890
606	21.98	3.5886
644	24.9	3.5810
471	28.83	3.5771
398	31.7	3.5733

TABLE XV

DETERMINATION OF SOLUBILITY LIMIT IN COBALT-NICKEL-MOLYBDENUM MU PHASE BY X-RAY DIFFRACTION

[Data for fig. 27]

Alloy	Weight percent nickel	θ	$\sin \theta$	d
421	0	70.45	0.94235	1.2124
678	12	69.94	.93953	1.2160
550	18.77	69.67	.93771	1.2184
536	24.12	69.59	.93724	1.2190
537	26	69.47	.93649	1.2199
523	29	69.46	.93642	1.2200
442	At boundary of three-phase field.	69.45	.93637	1.2202
464	Three-phase alloy	69.48	.93650	1.2200
551	Three-phase alloy	69.45	.93637	1.2202

TABLE XVII

DETERMINATION OF ALPHA THREE-PHASE-FIELD CORNERS BY X-RAY DIFFRACTION IN CHROMIUM-NICKEL-MOLYBDENUM SYSTEM

[Data for fig. 29]

Alloy	Weight percent chromium	Extrapolated value of a_0 (kX)
486	0	3.6351
688	5.5	3.6276
636	11.3	3.6245
460	14.3	3.6213
469	22.1	3.6189
457	23	3.6186
438	25.5	3.6162
435	36.25	3.6073
389	42.75	3.6045
614	43.5	3.5956
*89	43	3.5850

*From reference 1; a_0 redetermined in the present investigation.

TABLE XVIII

VALUES OF d OF CHROMIUM-NICKEL-MOLYBDENUM
TERNARY SIGMA PHASE AS FUNCTION OF CHROMIUM
CONTENT

[Data for fig. 30]

Alloy	Weight percent chrom- ium	d value ¹
509	25.06	1.2996
508	32.0	1.2969
641	22.76	1.2970
482	28.0	1.2928
603	36.0	1.2916
483	33.0	1.2857
604	45.0	1.2853
447	40.0	1.2832
491	45.0	1.2769
586	51.0	1.2768

¹ d values determined from twenty-
third line of sigma patterns.

TABLE XX

DETERMINATION OF THREE-PHASE-FIELD CORNERS
ON 20-PERCENT-MOLYBDENUM QUATERNARY ALPHA
PHASE BOUNDARY

[Data for fig. 33]

Alloy	Actual weight percent cobalt	Extrapolated value of α_s (kX)
457	0	3.6186
524	12.2	3.6128
525	24.0	3.6097
526	32.0	3.6042
652	47.6	3.6004
651	61.6	3.5969
501	64.0	3.5900
487	76.0	3.5870

TABLE XIX

DATA USED FOR PLOTTING ACTUAL SOLUBILITY OF
CHROMIUM IN ALPHA PHASE AT VARIOUS CONSTANT
MOLYBDENUM CONTENTS AS FUNCTION OF ACTUAL
COBALT CONTENT

[Data for fig. 32]

Weight percent molyb- denum	Weight percent cobalt	Maximum weight per- cent chro- mium in alpha solid solution
0	0	42.2
0	10	44.3
0	18	44.5
0	20	43.4
0	30	40.8
0	40	38.6
0	50	37.0
0	60	35.0
0	65	34.0
2.5	0	43.3
2.5	9.75	41.4
2.5	19.5	40.9
2.5	24.3	40.5
2.5	29.25	39.9
2.5	39.0	38.0
2.5	48.75	36.0
2.5	58.6	34.0
2.5	63.4	33.2
5	0	42.2
5	9.5	40.0
5	19.0	39.4
5	28.5	37.6
5	43.5	35.0
5	57.0	32.0
5	64.5	30.5
10	0	36.9
10	9.0	33.75
10	18.0	31.5
10	34.2	29.25
10	45.0	27.9
10	48.6	27.0
10	54.0	26.6
10	64.4	25.7
20	0	24.0
20	12.8	19.2
20	24.0	17.6
20	32.0	15.6
20	48.0	12.0
20	56.0	11.2
20	60.0	10.8
20	61.5	10.5
20	64.0	9.2
20	72.0	5.6
20	76.0	4.0

REPORT 1123

A STUDY OF INVISCID FLOW ABOUT AIRFOILS AT HIGH SUPERSONIC SPEEDS¹

By A. J. EGGERS, Jr., CLARENCE A. SYVERTSON, and SAMUEL KRAUS

SUMMARY

Steady flow about curved airfoils at high supersonic speeds is investigated analytically. With the assumption that air behaves as an ideal diatomic gas, it is found that the shock-expansion method may be used to predict the flow about curved airfoils up to arbitrarily high Mach numbers, provided the flow deflection angles are not too close to those corresponding to shock detachment. This result applies not only to the determination of the surface pressure distribution, but also to the determination of the whole flow field about an airfoil. Verification of this observation is obtained with the aid of the method of characteristics by extensive calculations of the pressure gradient and shock-wave curvature at the leading edge, and by calculations of the pressure distribution on a 10-percent-thick biconvex airfoil at 0° angle of attack.

An approximation to the shock-expansion method for thin airfoils at high Mach numbers is also investigated and is found to yield pressures in error by less than 10 percent at Mach numbers above 3 and flow deflection angles up to 25°. This slender-airfoil method is relatively simple in form and thus may prove useful for some engineering purposes.

Effects of caloric imperfections of air manifest in disturbed flow fields at high Mach numbers are investigated, particular attention being given to the reduction of the ratio of specific heats. So long as this ratio does not decrease appreciably below 1.3, it is indicated that the shock-expansion method, generalized to include the effects of these imperfections, should be substantially as accurate as for ideal-gas flows. This observation is verified with the aid of a generalized shock-expansion method and a generalized method of characteristics employed in forms applicable for local air temperatures up to about 5,000° Rankine.

The slender-airfoil method is modified to employ an average value of the ratio of specific heats for a particular flow field. This simplified method has essentially the same accuracy for imperfect-gas flows as its counterpart has for ideal-gas flows.

An approximate flow analysis is made at extremely high Mach numbers where it is indicated that the ratio of specific heats may approach close to 1. In this case, it is found that the shock-expansion method may be in considerable error; however, the Busemann method for the limit of infinite free-stream Mach number and specific-heat ratio of 1 appears to apply with reasonable accuracy.

INTRODUCTION

Small-disturbance potential-flow theories have been employed widely, and for the most part successfully, for pre-

dicting the pressures (and velocities) at the surface of an airfoil in steady motion at low supersonic speeds. Thus the linear theory of Ackeret (ref. 1) has proven particularly useful in studying the flow about relatively thin, sharp-nosed airfoils at small angles of attack, while the second-order theory of Busemann (ref. 2) has found application when thicker airfoils at larger angles of attack were under consideration. At high free-stream Mach numbers the range of applicability of any potential theory is seriously limited, however, due to the production of strong shocks by even the relatively small flow deflections caused by thin airfoils. The assumption of potential flow is invalidated, of course, by the pronounced entropy rises occurring through these shocks.

This limitation on potential theories was early recognized and led to the adoption (see ref. 3) of what is now commonly called the shock-expansion method. The latter method derives its advantage over potential theories, principally, by accounting for the entropy rise through the oblique shock emanating from the leading edge of a sharp-nosed airfoil. Consequently, so long as the disturbed air behaves essentially like an ideal gas, and so long as entropy gradients normal to the streamlines (due to curvature of the surface) do not significantly influence flow at the surface, the shock-expansion theory should predict the pressures at the surface of an airfoil with good accuracy—it is tacitly assumed, of course, that the flow velocity is everywhere supersonic, and that the Reynolds number of the flow is sufficiently large to minimize viscous effects on surface pressures.

The departure of the behavior of air from that of an ideal gas at the temperatures encountered in flight at high supersonic speeds has been the subject of some investigation in the case of flows through oblique shock waves. In reference 4, the effects of thermal and caloric imperfections on the pressure rise across an oblique shock wave were investigated at sea-level Mach numbers of 10 and 20, and it was found that these effects decreased the rise by less than 5 percent for maximum temperatures up to 3,000° R. (corresponding to flow deflection angles up to 24°). This decrease was found to be due almost entirely to caloric imperfections, or changes in vibrational heat capacities of the air passing through the shock wave. The changes in temperature and density of the air passing through the wave were affected to a considerably greater extent. Subsequently, an investigation was carried out by Ivey and Cline up to Mach numbers as high as 100 (ref. 5), using the results for normal shock waves obtained by Bethe and Teller considering effects of dissociation (ref.

¹ Supersedes NACA TN 2646, "Inviscid Flow About Airfoils at High Supersonic Speeds" by A. J. Eggers, Jr., and Clarence A. Syvertson, 1952, and NACA TN 2729 "An Analysis of Supersonic Flow in the Region of the Leading Edge of Curved Airfoils, Including Charts for Determining Surface-Pressure Gradient and Shock-Wave Curvature" by Samuel Kraus, 1952.

6). As would be expected, the pressures were found to be affected to a somewhat greater extent at the higher Mach numbers.

The extent to which flow in the region of the leading edge of an airfoil departs from the simple Prandtl-Meyer type has also been investigated at high supersonic airspeeds. If the surface is curved, for example, to give an expanding flow downstream of the leading edge, expansion waves from the surface will interact with the nose shock wave, thereby curving it and yielding a nonisentropic flow field. This flow field may be characterized not only by disturbances emanating from the surface but also by disturbances reflecting to some extent from the shock wave back toward the surface. The manner in which these phenomena dictate shock-wave curvature and surface pressure gradient in ideal-gas flows at the leading edge has been treated by Crocco (ref. 7) and more recently by Schaefer (ref. 8), Munk and Prim (ref. 9), Thomas (ref. 10), and others. In the cases considered by Munk and Prim, it was found that surface pressure gradients were less (in absolute value) than those obtained assuming Prandtl-Meyer flow at the higher Mach numbers (i. e., Mach numbers greater than about 3) although, generally, by no more than about 10 percent. Since curved airfoils are likely to be of fundamental interest at high flight speeds (see, e. g., ref. 11), the effects of reflected disturbances would appear to merit further investigation, both at the leading edge and as regards their influence on the whole flow field. In addition, it would appear desirable to consider effects of gaseous imperfections through the field.

Such an investigation has therefore been undertaken in the present report, using the method of characteristics to obtain accurately flow fields and as a basis for obtaining the more approximate methods of analysis. The method is employed in a generalized form which allows caloric imperfections, as well as entropy gradients, in the flow to be considered at temperatures up to the order of 5,000° R.—thermal imperfections are neglected (see ref. 4). A 10-percent-thick biconvex airfoil is treated at Mach numbers from 3.5 to infinity, and the results are compared with the predictions of the shock-expansion method, including a simplified form of the method applicable to slender airfoils at high Mach numbers and a generalized form of the method including effects of caloric imperfections. In addition, flow in the region of the leading edge of curved airfoils is considered in some detail. Values of the surface pressure gradient and shock-wave curvature are presented for a wide range of Mach numbers and flow deflection angles.

SYMBOLS

a	local speed of sound
c	chord
C_1, C_2	characteristic coordinates (C_1 positively inclined and C_2 negatively inclined with respect to the local velocity vector)
c_d	section drag coefficient
c_l	section lift coefficient
c_m	section moment coefficient (moment taken about leading edge)

C_p	pressure coefficient, $\frac{p-p_0}{q_0}$
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
K	curvature
M	Mach number (ratio of local velocity to local speed of sound)
P	pressure ratio, $\frac{p}{p_0}$
p	static pressure
q	dynamic pressure
R	gas constant
s, n	rectangular coordinates (in streamline direction and normal to streamline direction, respectively)
T	temperature, °R.
t	time
V	resultant velocity
W	distance measured from leading edge along airfoil surface
x, y	rectangular coordinates
α	angle of attack, radians unless otherwise specified
β	Mach angle, $\arcsin\left(\frac{1}{M}\right)$, radians
γ	ratio of specific heats, $\frac{c_p}{c_v}$ (Average value of γ is γ_a)
δ	flow deflection angle, radians unless otherwise specified
ζ	angle between shock wave and flow direction just downstream of shock wave, radians
θ	molecular vibrational energy constant, °R. (5,500° R. for air)
κ	ratio of shock-wave curvature to that given by the shock-expansion method
ρ	mass density
σ	shock-wave angle, radians
ψ	ratio of surface pressure gradient to that given by the shock-expansion method
ω	ray angle for Prandtl-Meyer flow, radians

SUBSCRIPTS

0	free-stream conditions
A, B, C, D	conditions at different points in flow field
i	ideal-gas quantities
N	conditions at the leading edge immediately downstream of the shock wave
S	conditions on streamline
w	conditions along airfoil surface
σ	conditions along shock wave

SUPERSCRIPT

-	vector quantities
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DEVELOPMENT OF METHODS OF ANALYSIS

GENERAL METHODS

Method of characteristics.—Two-dimensional rotational supersonic flows have been treated by numerous authors with

the aid of the method of characteristics, and various adaptations of the method have been found which are especially suited for studying particular types of such flows. In the case of steady flows in which atmospheric air does not behave as an ideal diatomic gas, a very familiar and simple form of the compatibility equations may be employed. To illustrate, consider the Euler equation

$$\rho \frac{d\bar{V}}{dt} = -\text{grad } p \quad (1)$$

the continuity equation

$$\text{div } (\rho \bar{V}) = 0 \quad (2)$$

and the equation for the speed of sound (evaluated at constant entropy)

$$a^2 = \frac{dp}{d\rho} \quad (3)$$

Rewriting equations (1) and (2) in the form of partial differential equations and transforming the resulting expressions to the characteristic, or C_1, C_2 , coordinate system, there is obtained, upon combination with equation (3), the following relations for steady flow:

$$\frac{\cot \beta}{\rho V^2} \left(\frac{\partial p}{\partial C_1} - \frac{\partial p}{\partial C_2} \right) + \left(\frac{\partial \delta}{\partial C_1} + \frac{\partial \delta}{\partial C_2} \right) = 0 \quad (4)$$

and

$$\frac{\cot \beta}{\rho V^2} \left(\frac{\partial p}{\partial C_1} + \frac{\partial p}{\partial C_2} \right) + \left(\frac{\partial \delta}{\partial C_1} - \frac{\partial \delta}{\partial C_2} \right) = 0 \quad (5)$$

A simple addition or subtraction of equations (4) and (5) then yields the compatibility equations (see, e. g., ref. 12)

$$\frac{\partial p}{\partial C_1} = -\rho V^2 \tan \beta \frac{\partial \delta}{\partial C_1} \quad (6)$$

and

$$\frac{\partial p}{\partial C_2} = \rho V^2 \tan \beta \frac{\partial \delta}{\partial C_2} \quad (7)$$

Now, in reference 4 both caloric and thermal imperfections of air were considered, and it was found that the latter imperfections² have a negligible effect on shock processes in atmospheric air. It may easily be shown that this conclusion also applies to expansion processes and, for this reason, caloric imperfections only are considered in detail in the present paper. These imperfections become significant in air at temperatures greater than about 800° R. and first manifest themselves as changes in the vibrational heat capacities with temperature. Thus, the specific heats c_p and c_v and their ratio γ for the gas also change. The equation of state remains, however,

$$p = \rho RT \quad (8)$$

and the specific heats are still related to the gas constant by the expression

$$c_p - c_v = R \quad (9)$$

Furthermore, it readily follows from the differential energy equation and these expressions that the speed of sound is given by the simple relation

$$a^2 = \gamma RT \quad (10)$$

Combining equations (8) and (10) and noting that $\sin \beta = a/V$ there is then obtained

$$\rho V^2 = \frac{\gamma p}{\sin^2 \beta} \quad (11)$$

Hence, on combining this equation with equations (6) and (7), it is apparent that the familiar compatibility equations

$$\frac{\partial p}{\partial C_1} = \frac{-2\gamma p}{\sin 2\beta} \frac{\partial \delta}{\partial C_1} \quad (12)$$

and

$$\frac{\partial p}{\partial C_2} = \frac{2\gamma p}{\sin 2\beta} \frac{\partial \delta}{\partial C_2} \quad (13)$$

also hold for the more general type of flow under consideration. These equations are basic, of course, to two-dimensional characteristics theory, and, as will be shown later, form a convenient starting point for developing simpler theories of two-dimensional supersonic flow.

In order to apply equations (12) and (13), it is evident that the manner in which γ and β or M are connected to p or δ must be known. Relations implicitly connecting these variables at temperatures up to the order of 5000° R may be readily obtained from the results of reference 4 by simply eliminating the terms therein accounting for thermal imperfections. Thus we have as a function of the local static temperature and free-stream conditions

$$\gamma = \gamma_t \left[\frac{1 + \left(\frac{\gamma_t - 1}{\gamma_t} \right) \left(\frac{\theta}{T} \right)^2 \frac{e^{\theta/T}}{(e^{\theta/T} - 1)^2}}{1 + (\gamma_t - 1) \left(\frac{\theta}{T} \right)^2 \frac{e^{\theta/T}}{(e^{\theta/T} - 1)^2}} \right] \quad (14)$$

and

$$M^2 = \frac{2}{\gamma} \left(\frac{T_0}{T} \right) \left[\frac{\gamma_0 M_0^2}{2} + \frac{\gamma_t}{\gamma_t - 1} \left(1 - \frac{T}{T_0} \right) + \frac{\theta}{T_0} \left(\frac{1}{e^{\theta/T_0} - 1} - \frac{1}{e^{\theta/T} - 1} \right) \right] \quad (15)$$

For isentropic flow along a streamline, the pressure is related to the temperature by the expression

$$\frac{p}{p_\sigma} = \frac{\Lambda(T_\sigma)}{\Lambda(T)} \quad (16)$$

where

$$\Lambda(T) = \frac{e^{\theta/T} - 1}{e \left(\frac{\theta}{T} \frac{e^{\theta/T}}{e^{\theta/T} - 1} \right)} \left(\frac{1}{T} \right)^{\frac{\gamma_t}{\gamma_t - 1}} \quad (17)$$

If there is a shock wave in the flow,³ in particular, a nose or leading-edge shock, then the following additional relations obtained with equations (8), (10), and (15) and the conditions

² Thermal imperfections usually appear in the form of intermolecular forces and molecular-size effects and may be accounted for with additional terms in the equation of state.

³ If there are no shock waves, then the subscript σ in equation (16) can, of course, be replaced with the subscript 0.

for continuity of flow and conservation of momentum along a streamline through the shock are also required:

$$\frac{p_0}{p_\sigma} = \frac{1}{2} \left\{ (1 + \gamma_\sigma M_\sigma^2) - \frac{T_0}{T_\sigma} (1 + \gamma_0 M_0^2) + \sqrt{\left[(1 + \gamma_\sigma M_\sigma^2) - \frac{T_0}{T_\sigma} (1 + \gamma_0 M_0^2) \right]^2 + 4 \frac{T_0}{T_\sigma}} \right\} \quad (18)$$

$$\sin^2 \sigma = \frac{[(\gamma_\sigma T_\sigma M_\sigma^2)/(\gamma_0 T_0 M_0^2)] - 1}{(\rho_0/\rho_\sigma)^2 - 1} \quad (19)$$

and

$$\tan \delta = \frac{1}{\tan \sigma} \frac{1}{\frac{\gamma_0 M_0^2}{(p_\sigma/p_0) - 1} - 1} \quad (20)$$

By use of the local static temperature as a parameter, the term $2\gamma p/\sin 2\beta$ in equations (12) and (13) may now be evaluated with equations (14) through (17). Equations (18) through (20) define the initial conditions downstream of a leading edge or other shock wave in the flow field. Thus, equations (12) through (20) provide all the information necessary to calculate the flow about an airfoil by means of the method of characteristics. As described in detail in Appendix A, the calculations are of three general types: (1) determination of conditions at a point in the flow field between the shock and the surface; (2) determination of conditions at a point on the surface; and (3) determination of conditions at a point just downstream of the shock. Case (1) entails the use of both compatibility equations, while case (2) entails the use of the compatibility equation for a second-family characteristic line in combination with the equation of the airfoil surface, and case (3) involves the compatibility equation for a first-family line in combination with the oblique-shock equations. With the aid of the three general types of calculations, the entire flow field about an airfoil can be built up numerically using a computing procedure working from the leading edge downstream. In cases where changes in the vibrational heat capacities with temperature are neglected, the calculations are, of course, simplified since γ of the gas can be considered constant, and temperature, pressure, and density ratios are simply the ideal-gas functions of Mach number.

Shock-expansion method.—This method of calculating supersonic flow of an ideal gas at the surface of an airfoil is well known, entailing simply the calculation of flow at the nose with the oblique-shock equations and flow downstream of the nose with the Prandtl-Meyer equations. Determination of airfoil characteristics in this manner requires only a small amount of time, of course, compared to that involved when the method of characteristics is used, hence, the advantage of the former method. The questions arise, however, as to exactly what the simplifying assumptions underlying the shock-expansion method are, and what form the method takes (for calculative purposes) when the gas displays varying vibrational heat capacities.

The matter of simplifying assumptions may perhaps best be considered by employing equations (12) and (13), the basic compatibility equations. If these expressions are

resolved into the streamline direction and combined, noting that

$$\frac{\partial p}{\partial s} = \frac{1}{2 \cos \beta} \left(\frac{\partial p}{\partial C_1} + \frac{\partial p}{\partial C_2} \right) \quad (21)$$

and

$$\frac{\partial \delta}{\partial s} = \frac{1}{2 \cos \beta} \left(\frac{\partial \delta}{\partial C_1} + \frac{\partial \delta}{\partial C_2} \right) \quad (22)$$

there is then obtained the relation

$$\frac{\partial p}{\partial s} = \left[\frac{1 - \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}}{1 + \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}} \right] \frac{2\gamma p}{\sin 2\beta} \frac{\partial \delta}{\partial s} \quad (23)$$

defining the gradient of p along s . If flow along streamlines downstream of the nose is of the simple Prandtl-Meyer type, however, we have

$$\frac{\partial p}{\partial s} = \frac{2\gamma p}{\sin 2\beta} \frac{\partial \delta}{\partial s} \quad (24)$$

Hence, it is evident that the requirement for this type flow is

$$\left| \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2} \right| \ll 1 \quad (25)$$

Equation (25) is, of course, simply an approximate statement of a well-known property of Prandtl-Meyer flows; namely, that flow inclination angles are essentially constant along first-family Mach lines. It follows from equation (12) that if equation (25) holds, then the pressures will also be essentially constant along these lines. It does not follow, however, that the Mach number will be constant or, for that matter, that the first-family characteristic lines will be straight (as is the case for isentropic expansion flows about a corner). In fact, it may easily be shown that the Mach number gradient along C_1 is proportional to the local entropy gradient normal to the streamlines and that the C_1 lines are curved according to the change in M . Thus we see that there is really only one basic assumption underlying the shock-expansion method; namely, disturbances incident on the nose shock (or, for that matter, any other shock) are consumed almost entirely in changing the direction of the shock. In this regard, it is interesting to note that the assumption of Thomas (ref. 13), that pressure is a function only of flow deflection angle and entropy, is equivalent to this assumption. It follows, of course, that the most general solution obtainable with Thomas' series representation of the pressure is that given by the shock-expansion method.

With the assumption that all disturbances incident on the shock wave are consumed, it is evident that the shock-expansion method provides a relatively simple means for calculating the whole flow field about an airfoil, including the effects of shock-wave curvature. The details of such calculations are presented in Appendix B. In general, of course, the validity of this assumption and the accuracy of the shock-expansion method can only be checked by com-

parison of calculations using this method with those using the method of characteristics.

The shock-expansion method for a calorically imperfect, diatomic gas is readily deduced from the equations previously obtained. For example, flow conditions at the leading edge of an airfoil can be evaluated with the oblique-shock-wave expressions (eqs. (18) through (20)) and the expression for conservation of energy (eq. (15)). The variation of flow inclination angle with pressure along the surface is then obtained by graphically integrating equation (24); namely,

$$\delta_S - \delta_N = \int_{p_N}^{p_S} \frac{\sin 2\beta}{2\gamma p} dp \quad (26)$$

where the variables γ , p , and β are evaluated using equations (14) through (17), employing the static temperature as a parameter. When extreme accuracy is not essential, this rather tedious calculation can be avoided, and a relatively simple algebraic solution of the flow downstream of the nose can be employed.⁴ The details of this solution are presented in Appendix C. In the special case of flow at high supersonic speeds about slender airfoils, the whole calculation becomes particularly simple and warrants special attention.

If it is assumed that the local surface slopes are small compared to 1 and, in addition, that the free-stream Mach number is large compared to 1, it follows that σ and β are everywhere small compared to 1. In this case, equation (24) takes on the approximate form

$$\left. \frac{dp}{d\delta} \right|_S = \gamma p M \quad (27)$$

Furthermore, if it is assumed that γ is constant at an average value γ_a for a particular flow field (this assumption appears reasonable since, in the temperature range up to 5,000° R, the change in γ is less than about 15 percent), then the Mach number and pressure may be related by the simple expression

$$M = M_N \left(\frac{p_N}{p} \right)^{\frac{\gamma_a - 1}{2\gamma_a}} \quad (28)$$

Equations (27) and (28) combine to yield the differential equation

$$\frac{1}{\gamma_a M_N} \left(\frac{p}{p_N} \right)^{-\frac{(\gamma_a + 1)}{2\gamma_a}} d \left(\frac{p}{p_N} \right) = d\delta \quad (29)$$

which readily integrates (between N and S) to the form

$$\frac{p_S}{p_N} = \left[1 - \left(\frac{\gamma_a - 1}{2} M_N \delta_N \right) \left(1 - \frac{\delta_S}{\delta_N} \right) \right]^{\frac{2\gamma_a}{\gamma_a - 1}} \quad (30)$$

now denoting

$$\frac{\gamma_a - 1}{2} M_N \delta_N = f(M_0 \delta_N) \quad (31)$$

and

$$\frac{p_N}{p_0} = g(M_0 \delta_N) \quad (32)$$

there is obtained from the oblique-shock equations, simplified to conform with this analysis,

$$f(M_0 \delta_N) = \frac{M_0^2 \sigma_N^2 - 1}{\sqrt{\left(M_0^2 \sigma_N^2 + \frac{2}{\gamma_a - 1} \right) \left(\frac{2\gamma_a}{\gamma_a - 1} M_0^2 \sigma_N^2 - 1 \right)}} \quad (33)$$

and

$$g(M_0 \delta_N) = \frac{2\gamma_a M_0^2 \sigma_N^2 - (\gamma_a - 1)}{\gamma_a + 1} \quad (34)$$

where

$$M_0 \sigma_N = \frac{\gamma_a + 1}{4} M_0 \delta_N + \sqrt{1 + \left(\frac{\gamma_a + 1}{4} M_0 \delta_N \right)^2} \quad (35)$$

With equations (30) through (35), the pressures on the surface of an airfoil may easily be obtained. In terms of pressure coefficient we have

$$C_p = \frac{2}{\gamma_a M_0^2} \left[\left(\frac{p_N}{p_0} \right) \left(\frac{p_S}{p_N} \right) - 1 \right] \quad (36)$$

or

$$C_p = \frac{2}{\gamma_a M_0^2} \left\{ g(M_0 \delta_N) \left[1 - f(M_0 \delta_N) \left(1 - \frac{\delta_S}{\delta_N} \right) \right]^{\frac{2\gamma_a}{\gamma_a - 1}} - 1 \right\} \quad (37)$$

The advantage of these slender-airfoil expressions lies, of course, in their relative simplicity and, thus, the ease of calculation which is inherent to them. It may be noted in this regard that the functions $f(M_0 \delta_N)$ and $g(M_0 \delta_N)$ can be calculated once and for all with equations (33), (34), and (35), provided the variation of γ_a with $M_0 \delta_N$ is known. This calculation has been carried out for a constant value of γ equal to 1.4 and for average values of γ , assuming $T_0 = 500^\circ$ R.⁵ The results are presented in table I.

It should also be noted that the slender-airfoil expressions of the shock-expansion method satisfy the hypersonic similarity law for airfoils first deduced by Tsien (ref. 15).⁶ A necessary condition for the validity of these expressions is thus satisfied; however, the accuracy of the shock-expansion method, whether for slender airfoils or otherwise, remains to be investigated.

METHODS FOR CALCULATING THE FLOW IN THE REGION OF THE LEADING EDGE

As was pointed out previously, pressure disturbances emanating from the surface of a curved airfoil interact with, and thereby curve, the leading-edge shock wave. The geometry of this phenomenon near the leading edge of a convex airfoil is illustrated in figure 1. The pressure disturbances from the airfoil (expansion waves for a convex airfoil) travel along first-family Mach lines C_1 . In addition to changing the inclination of the shock wave, the interaction between these disturbances and the shock produces another system of disturbances which travel along second-family Mach lines C_2 from the shock wave to the surface.

Method of characteristics.—An exact solution for the surface pressure gradient and shock-wave curvature at the leading

⁴ For a given value of T_0 , T_N , to the accuracy of this analysis, is the ideal-gas function of $M_0 \delta_N$. Thus, knowing T_N , γ_N can be determined. The average value of γ used is $\gamma_a = \gamma_N (M_0 \delta_N) = (\gamma_N + \gamma_i)/2$.

⁶ This fact was employed by Linnell (ref. 16) to obtain an expression for pressure coefficient equivalent to equation (37) for the case of constant γ and to obtain explicit solutions for the lift, drag, and pitching-moment coefficients of several airfoils at hypersonic speeds.

⁴ The tabulated results of Noyes (ref. 14) may also prove useful in this case for Mach numbers up to 3.

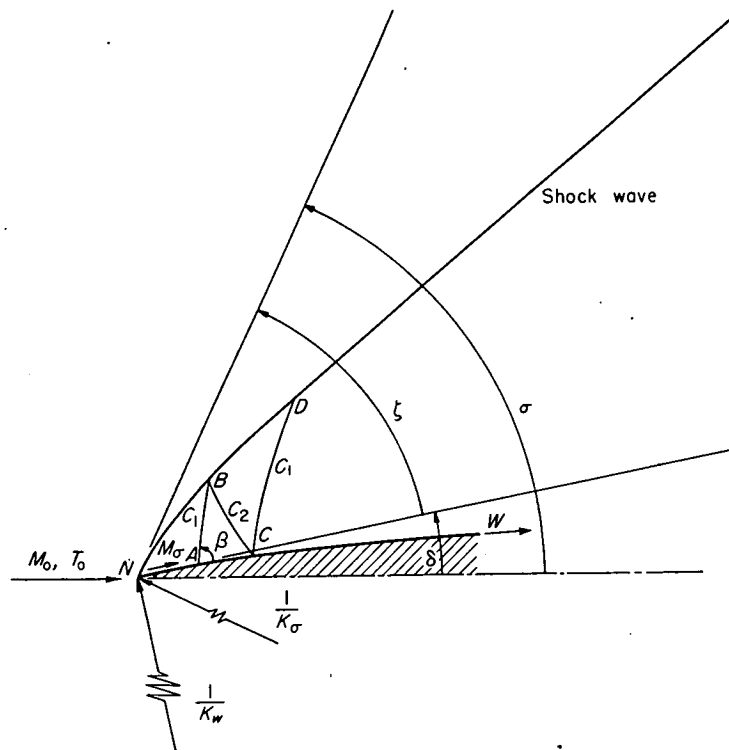


FIGURE 1.—Schematic diagram of supersonic flow past a curved sharp-nose airfoil.

edge may be determined in the following manner. It is clear, referring to figure 1, that the change in flow angle between points A and C given by the compatibility equations (see eqs. (12) and (13)) along the path ABC must equal the change determined by the airfoil surface from A to C. Similarly, the difference in pressure between points B and D given by the compatibility equations along the path BCD must equal that determined by the change in shock-wave inclination between B and D. In reference 9, these conditions were employed at the leading edge to obtain equations, in simple parametric form, for determining the surface pressure gradients and shock-wave curvatures. These equations can be written in the form

$$\frac{1}{K_w} \frac{dP}{dW} = \left[\frac{1 - \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}}{1 + \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}} \right] \frac{2\gamma P}{\sin 2\beta} \quad (38)$$

for the surface pressure gradient, and

$$\frac{K_\sigma}{K_w} = \frac{\sin(\beta - \sigma + \delta) + \left(\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2} \right) \sin(\beta + \sigma - \delta)}{\left(1 + \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2} \right) \left(\frac{\partial \delta}{\partial \sigma} \right)_\sigma \sin \beta} \quad (39)$$

for the shock-wave curvature, where

$$\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2} = \left[\frac{\frac{2\gamma P}{\sin 2\beta} - \left(\frac{dP}{d\delta} \right)_\sigma}{\frac{2\gamma P}{\sin 2\beta} + \left(\frac{dP}{d\delta} \right)_\sigma} \right] \left[\frac{\sin(\beta - \sigma + \delta)}{\sin(\beta + \sigma - \delta)} \right] \quad (40)$$

It should be pointed out that equation (38) is, of course, equivalent to equation (23).

A procedure for evaluating equations (38) and (39) for a calorically imperfect, diatomic gas, as well as for an ideal gas, is presented in Appendix D. Since they are exact for two-dimensional, steady, inviscid flow, values of the surface pressure gradient and shock-wave curvature at the leading edge, determined using these equations, may be put to two uses. First, the accuracy of approximate methods of calculating the flow field about a curved airfoil can be evaluated at the leading edge by comparing the values of the surface pressure gradient and shock-wave curvature predicted by these approximate methods to the values obtained using equations (38) and (39). Second, the pressure gradient and shock-wave curvature can be used to calculate the initial points of a characteristic solution for the flow about an airfoil.

Shock-expansion method.—One approximate solution for the surface pressure gradient and shock-wave curvature has already been indicated in the previous discussion of the shock-expansion method. The requirement for the application of the shock-expansion method is given by equation (25). Hence, it is apparent that under this condition the expression for the surface pressure gradient (eq. (38)) reduces to

$$\frac{1}{K_w} \frac{dP}{dW} = \frac{2\gamma P}{\sin 2\beta} \quad (41)$$

Similarly, the expression for the shock-wave curvature (eq. (39)) reduces to

$$\frac{K_\sigma}{K_w} = \frac{\sin(\beta - \sigma + \delta)}{\left(\frac{d\delta}{d\sigma} \right)_\sigma \sin \beta} \quad (42)$$

It should be realized that the flow field is determined by the basic flow equations in conjunction with the shock wave and airfoil surface as boundary conditions. Thus, the additional requirement for this shock-expansion method of zero pressure gradient along first-family Mach lines means that one of the flow relations cannot be satisfied exactly (i. e., the flow field is overdetermined). Equations (41) and (42) satisfy the shock relations and the airfoil surface as boundary conditions; however, the compatibility equations are only approximately satisfied. (See Appendix B.)

The error in surface pressure gradient associated with neglecting the reflected disturbances might be expected to be largest in the region of the leading edge of a curved airfoil due to the close proximity of the shock wave and the surface. The magnitude of the error in this region may be deduced, of course, from the ratios of values of surface pressure gradient and shock-wave curvature given by the characteristics method to those given by the shock-expansion method. The surface-pressure-gradient ratio and shock-wave-curvature ratio can be written (using eqs. (38) and (41))

$$\psi = \frac{1 - \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}}{1 + \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}} \quad (43)$$

and (using eqs. (39) and (42))

$$\kappa = \frac{\sin(\beta - \sigma + \delta) + \left(\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2} \right) \sin(\beta + \sigma - \delta)}{\left(1 + \frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2} \right) \sin(\beta - \sigma + \delta)} \quad (44)$$

respectively. A procedure for evaluating equations (43) and (44) for the flow of a calorically imperfect gas, as well as for an ideal gas, is presented in Appendix D. (The application of eq. (43) for an ideal gas has already been given in ref. 9.)

As was stated previously, the pressure gradient and shock-wave curvature can be used to calculate the initial points of a characteristic solution for the flow about an airfoil. It is apparent that the initial points for a shock-expansion solution can be found in a similar manner. With either type of solution, flow conditions at an initial point on the surface downstream of the leading edge can be calculated with the aid of the appropriate value of the surface pressure gradient. Similarly, the flow conditions at an initial point on the shock wave can be obtained with the aid of the corresponding value of the shock-wave curvature. Additional points can be obtained between these two by linear interpolation. If the two initial points are chosen as the ends of a first-family Mach line, there is sufficient information available to determine the curvature of this Mach line. Therefore, if detailed knowledge of the flow in the region of the leading edge is required, the surface, shock wave, and first-family Mach line can be approximated by circular arcs. (See ref. 17.)

INVESTIGATION OF THE FLOW ABOUT AIRFOILS AND DISCUSSION OF RESULTS

This study is divided into two sections: first, an investigation of the flow in the region of the leading edge of curved

airfoils; and second, an investigation of the complete flow field about an example airfoil. Each of these sections is further subdivided into a consideration of the effects of Mach number, assuming air behaves as an ideal gas, and into a consideration of the combined effects of Mach number and gaseous imperfections. In the latter regard, the principal emphasis is placed on the caloric imperfections previously discussed.

FLOW IN THE REGION OF THE LEADING EDGE OF CURVED AIRFOILS

Ideal-gas flow.—The results of the calculations (using eq. (D4)) of the surface pressure gradient are presented in table II and figure 2.⁷ The values presented in the table are for a range of Mach numbers from 1.5 to ∞ and for leading-edge deflection angles from 0° to 45° . Where no value appears in the table, the flow behind the shock wave is subsonic. Corresponding results of the calculations of surface-pressure-gradient ratio are presented in table III and figure 3. From these results it is seen that except near shock detachment, the surface-pressure-gradient ratio varies only from 0.98 to 1.02 for Mach numbers less than 4. Therefore, very little error will result from the use of the shock-expansion method for the surface pressure gradient at these lower Mach numbers. For Mach numbers greater than 4 (even

⁷ Charts were also presented for surface pressure gradient, surface-pressure-gradient ratio, and shock-wave curvature for ideal-gas flows in reference 9; however, the results given in the present report are somewhat more extensive. These results were also presented in cross-plotted forms in reference 17.

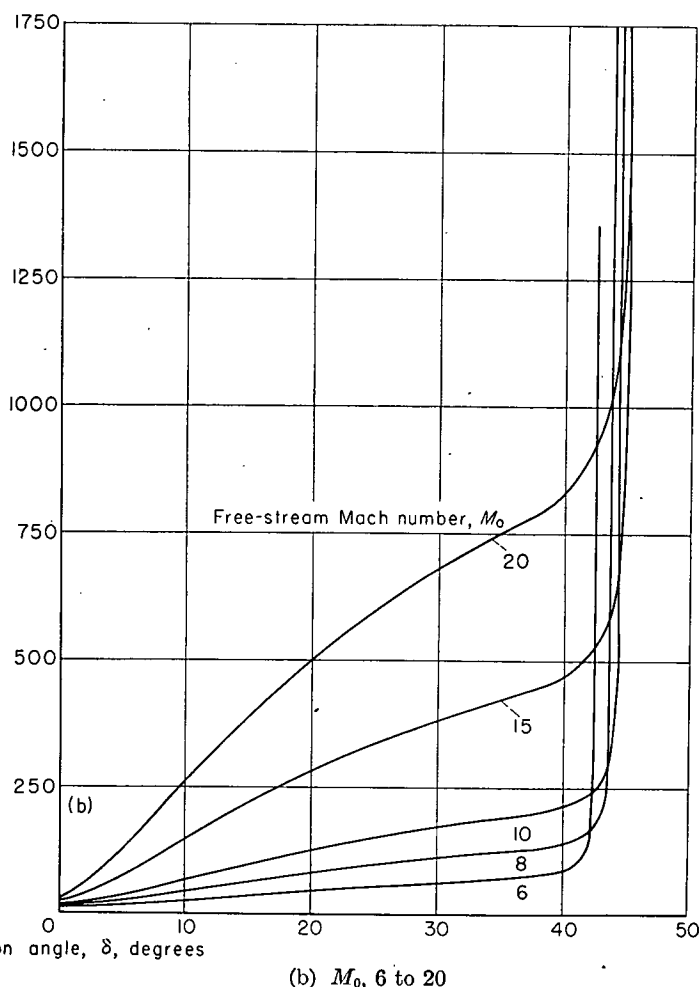
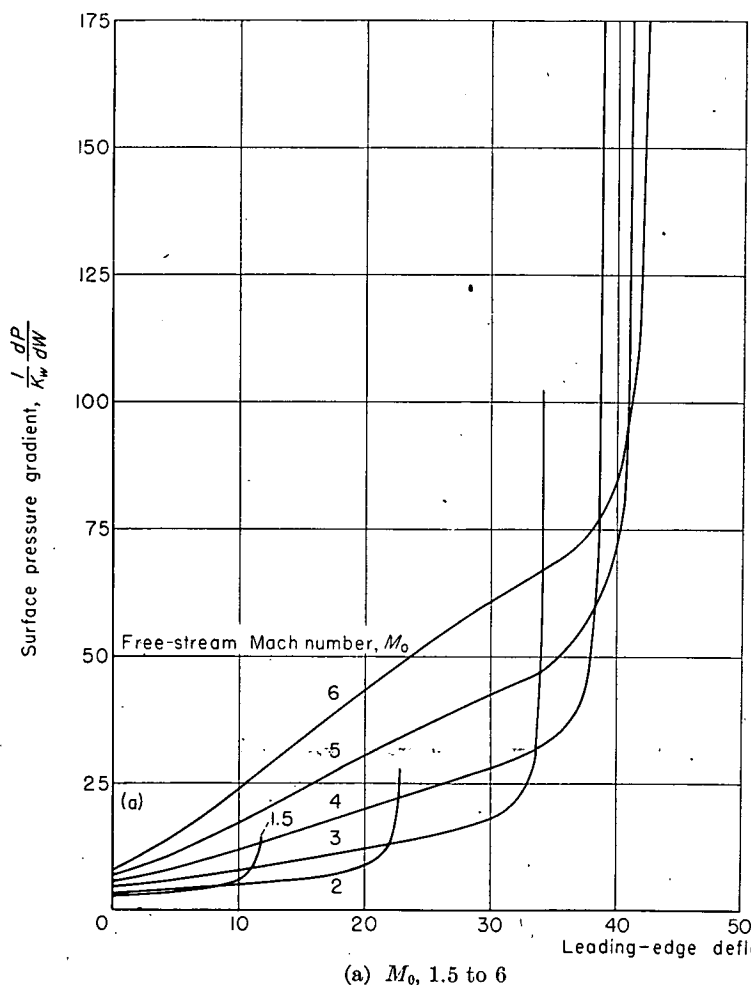


FIGURE 2.—Variation of surface pressure gradient with leading-edge deflection angles for various free-stream Mach numbers.

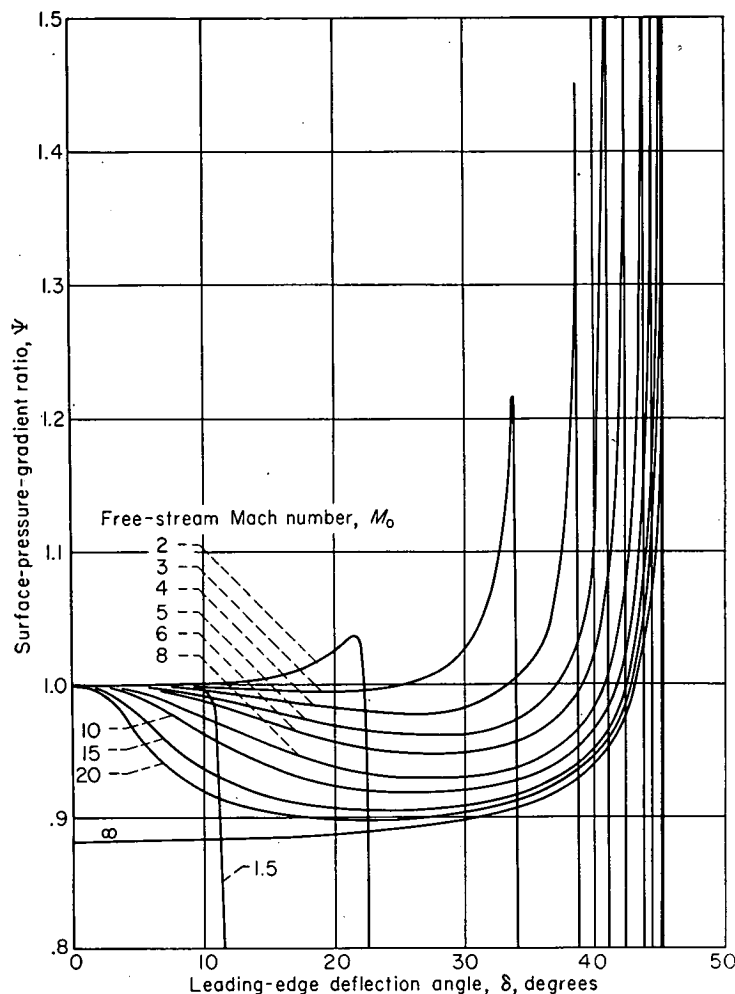


FIGURE 3.—Variation of surface-pressure-gradient ratio with deflection angle for various free-stream Mach numbers.

including ∞),⁸ the ratio only varies between about 0.9 and 1.1 (again except near detachment) and, therefore, only small error might be expected from the use of the shock-expansion method. Near detachment, at the higher Mach numbers, the surface-pressure-gradient ratio attains a very large range of values and the maximum value increases with Mach number. For these conditions, then, the use of the shock-expansion method for calculating the surface pressure gradient at the leading edge would result in appreciable error.

The flow along the surface is isentropic. Hence, it can be shown that ψ , the ratio of surface pressure gradient to that given by the generalized shock-expansion method, is also the velocity-gradient ratio, the Mach number gradient ratio, the Mach angle gradient ratio, the density-gradient ratio, and the temperature-gradient ratio. Any of these gradients may be found, then, by calculating the gradient, using the shock-expansion method and applying the appropriate value of ψ . This property of the ratio ψ makes it useful in the application of the method of characteristics with any of the coordinate systems commonly employed in the compatibility equations.

The results of the calculations of shock-wave curvature

⁸ For the particular case of infinite free-stream Mach number and zero deflection angle, the pressure-gradient ratio is double valued. From equation (D6), it is apparent that ψ is unity or zero deflection. Yet, at infinite Mach number, ψ approaches $\frac{3}{2\gamma-1} \sqrt{\frac{\gamma(\gamma-1)}{2}}$ as the deflection angle approaches zero.

(eq. (D5)) are presented in table IV and figure 4.⁹ Similarly, the results of the calculations of shock-wave-curvature ratio (eq. (44)) are presented in table V and figure 5. Except near detachment, the curvature ratio varies from 0.92 to 1.08 for all Mach numbers including ∞ .¹⁰ Thus, only small errors would result from using the value of the shock-wave curvature given by the shock-expansion method for all flow conditions except near shock detachment.

Calorically-imperfect-gas flows.—With increasing Mach number and leading-edge slope, the temperature ratio across an oblique shock wave increases as shown in figure 6. As the temperature behind the shock wave increases, the behavior of air diverges from that of an ideal gas as discussed previously. Below 800° R., the divergence is not significant, and the equations for ideal-gas flow can be applied with only minute errors resulting. Above 800° R., the energy of the vibrational degrees of freedom of the gas molecules is appreciable and becomes greater with increasing temperature. For these conditions, the specific heats and their ratio vary significantly with temperature. The equations developed previously consider these effects and permit the extension of the solution for surface pressure gradient and shock-wave curvature to the case of calorically imperfect gases. These

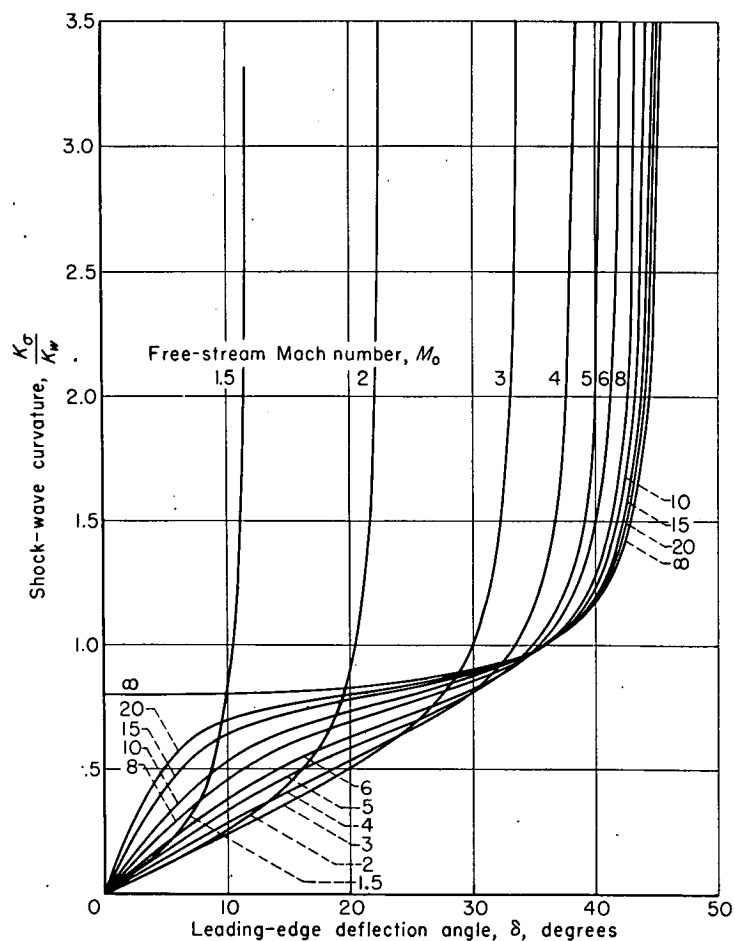


FIGURE 4.—Variation of shock-wave curvature with leading-edge deflection angle for various free-stream Mach numbers.

⁹ As in the case of ψ , $K₀/K∞$ has the double values of 0 and $\frac{(\gamma+1)^2}{4(2\gamma-1)}$ for $M₀ = \infty$, $\delta = 0$.

¹⁰ As in the case of ψ and $K₀/K∞$, ψ has the double values of 1 and $\frac{\gamma+1}{2(2\gamma-1)[1-\sqrt{(\gamma-1)/2\gamma}]}$ for $M₀ = \infty$, $\delta = 0$.

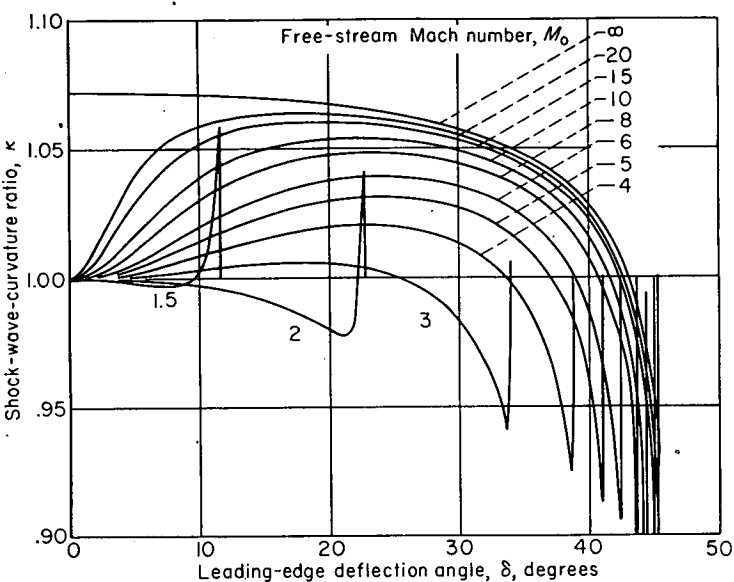


FIGURE 5.—Variation of shock-wave-curvature ratio with leading-edge deflection angle for various free-stream Mach numbers.

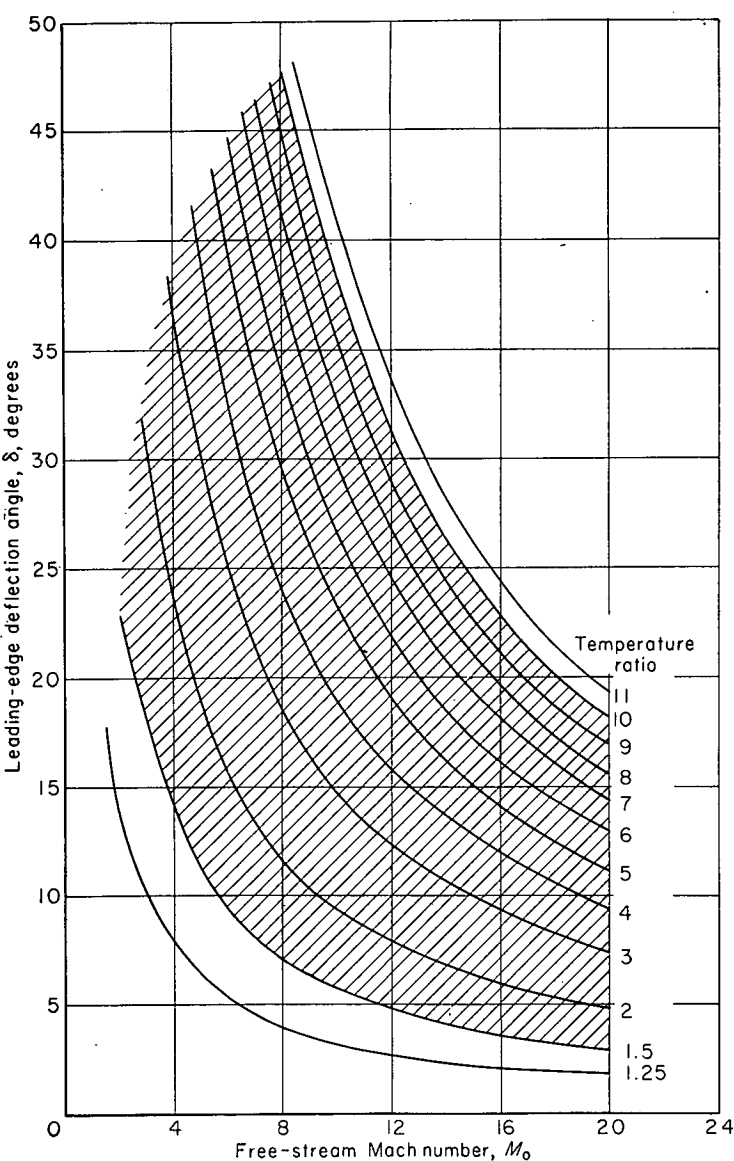


FIGURE 6.—Variation of leading-edge deflection angle with free-stream Mach number for various values of the temperature ratio.

equations are valid for temperatures up to the order of 5,000° R. For a free-stream temperature of 500° R., therefore, the shaded area between lines of constant temperature ratio of 1.5 and 10, in figure 6, represents the range of conditions for which the method developed in this report for the flow of a diatomic, calorically imperfect gas would apply.

The excitation of the vibrational degrees of freedom of the gas molecules requires a finite number of collisions, causing the well-known heat-capacity lag discussed in references 5 and 6. The flow distance (i. e., along the streamline) required to establish equilibrium is usually small in dense air and will be considered infinitesimal in this report. Also, the dissociation of air (see ref. 6) will not be considered here.

Since the free-stream static temperature is an additional parameter in calculations of flow of imperfect gases, only a limited number of calculations of $(1/K_w)(dP/dW)$, ψ , K_w/K_w , and κ were made. The purpose of these calculations is to compare the variations of these quantities with the values as given by the ideal-gas-flow computations. The calculations followed the procedure described in Appendix D. A free-stream static temperature of 500° R. was used. The results of these calculations are presented in table VI for various Mach numbers and leading-edge deflection angles.

The surface pressure gradients for an ideal gas and for a calorically imperfect, diatomic gas are compared in figure 7. In all cases calculated, the gradient is smaller for the imperfect gas and diverges gradually, with increasing free-stream Mach number and deflection angle, from the value of the gradient for an ideal gas. This divergence is consistent with the increasing effects of the caloric imperfections due to the increasing temperature behind the shock wave.

The surface-pressure-gradient ratio for the imperfect gas is compared in figure 8 with the ratio for an ideal gas. The surface-pressure-gradient ratio is smaller for imperfect-gas flows than for ideal-gas flows indicating that the effects of shock-wave and expansion-wave interaction are greater. This result is attributed, in part, to the fact that the angle between the shock wave and the airfoil surface is smaller for

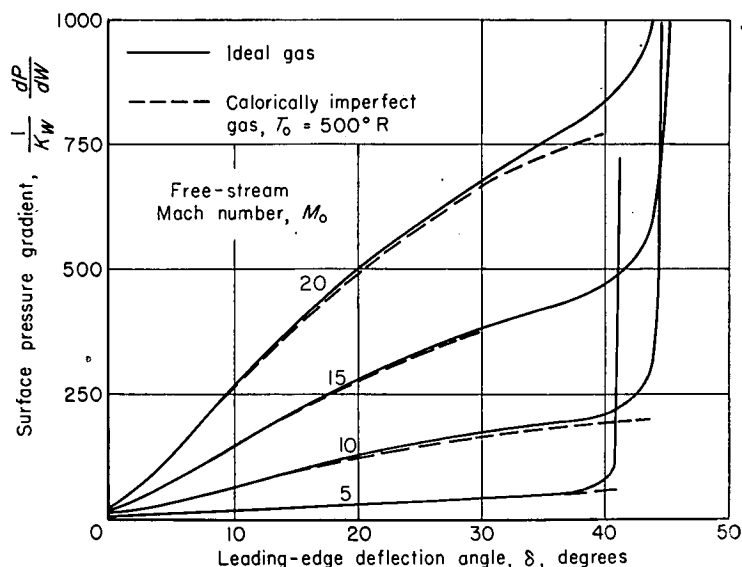


FIGURE 7.—Comparison of the variation of surface pressure gradient with leading-edge deflection angle for various free-stream Mach numbers for an ideal gas and for a calorically imperfect, diatomic gas.

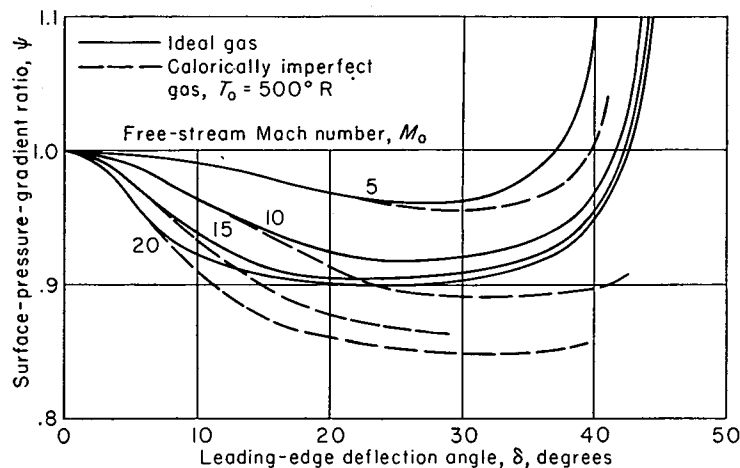


FIGURE 8.—Comparison of the variation of surface-pressure-gradient ratio with leading-edge deflection angle for various free-stream Mach numbers for an ideal gas and for a calorically imperfect, diatomic gas.

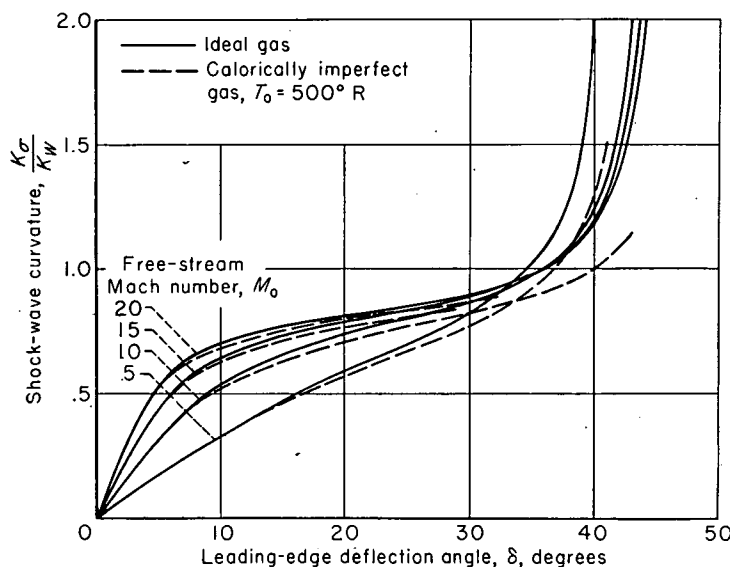


FIGURE 9.—Comparison of the variation of the shock-wave curvature with the leading-edge deflection angle for various free-stream Mach numbers for an ideal gas and for a calorically imperfect, diatomic gas.

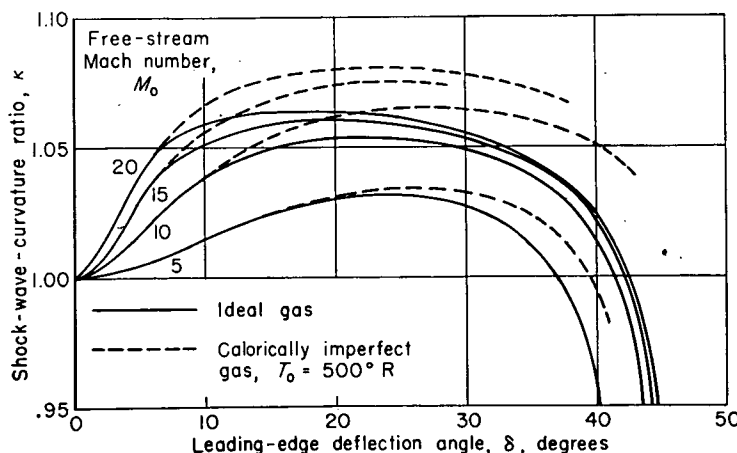


FIGURE 10.—Comparison of the variation of shock-wave-curvature ratio with leading-edge deflection angle for various free-stream Mach numbers for an ideal gas and for a calorically imperfect, diatomic gas.

imperfect-gas flows than for ideal-gas flows. The difference between the imperfect- and ideal-gas calculations increases with increasing Mach number and deflection angle, as might be expected. In the range of Mach numbers and flow deflection angles investigated, however, the extreme values of ψ differ only by about 5 percent (see fig. 8). Thus, it is apparent that while the shock-expansion method will not be quite as accurate for calorically-imperfect-gas flows as for ideal-gas flows, the method will not be expected to be invalidated.

A divergence with Mach number and deflection angle is also apparent in figure 9 in which the shock-wave curvatures for an ideal gas and a calorically imperfect, diatomic gas are compared. This divergence is compatible with the change in surface pressure gradient due to the caloric imperfections of the gas. The shock-wave-curvature ratio for a calorically imperfect, diatomic gas and this ratio for an ideal gas are shown in figure 10. Again, it is seen that the effect of caloric imperfection is to increase the effects of shock-wave and expansion-wave interaction.

COMPLETE FLOW FIELDS

Ideal-gas flows.—The effects of Mach number of primary interest here are, of course, those which result from the interaction between the leading-edge (or other) shock wave and small disturbances originating on the surface of an airfoil. Although the reflected disturbances that are the product of this interaction will have the largest effect on the flow near the leading edge, their influence on the complete flow field about an airfoil also warrants investigation. Further insight into these effects can be obtained in the region just downstream of the shock wave without regard for the shape of the airfoil producing the shock. To this end, it is convenient to consider the ratio $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2} = -\frac{\partial p / \partial C_1}{\partial p / \partial C_2}$ (see eq. (25))

which may be termed "the disturbance strength ratio" since, in the region under consideration, it is a measure of the ratio of strengths of disturbances reflected from the shock wave to disturbances incident on the wave. This ratio may be evaluated with equation (40). This calculation has been carried out for Mach numbers from 3.5 to ∞ ($\gamma = 1.4$) and flow deflection angles approaching those corresponding to shock detachment (i. e., $M_\sigma \approx 1$), and the results are presented in figure 11. It is evident that except near $M_\sigma = 1$,

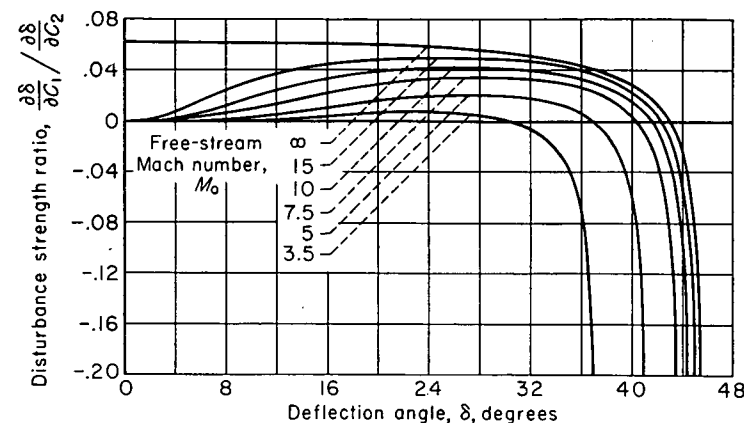


FIGURE 11.—Variation with deflection angle of the disturbance strength ratio behind an oblique shock wave for various free-stream Mach numbers ($\gamma = 1.4$).

the ratio is small (in absolute value) compared to 1 throughout the entire range considered—this observation also applies, of course, at lower supersonic Mach numbers. Thus it is indicated that almost all of an incident disturbance is generally absorbed in the shock wave, provided the air behaves like an ideal diatomic gas. This result is substantially the same, of course, as that which is assumed in deriving the shock-expansion method of calculating flows about airfoils, and therefore yields additional credence in the method for high Mach number as well as low Mach number applications. It should also be noted that this result is contrary to that obtained by Lighthill (ref. 18), who reports that for hypersonic flows ($M_0 \delta \gg 1$) a disturbance is reflected from a shock wave with opposite sign but essentially undiminished strength. Lighthill's conclusion is based on an incorrect evaluation of his results for the case of very high Mach numbers.

As an over-all check on the shock-expansion method, surface pressure distributions calculated thereby are compared in figure 12 with those obtained with the method of characteristics for a 10-percent-thick parabolic-arc biconvex airfoil ($\alpha=0^\circ$) operating at free-stream Mach numbers of 3.5, 10, and ∞ . (Additional calculations presented in ref. 19 were also performed for $M_0=5, 7.5$, and 15.) Predictions of the slender-airfoil approximation to the former method for high supersonic speeds are also shown. There is no apparent difference between the pressure distributions given by the method of characteristics and the shock-expansion method at a Mach number of 3.5; at a Mach number of 10, and more so at infinite Mach number, however, the latter method predicts pressures which are slightly low downstream of the nose. This result would be deduced from figure 11 where it is observed that, at the Mach numbers under consideration, expansion waves incident on the nose shock wave are reflected back toward the surface as compression waves of relatively small but increasing strength with increasing Mach number. The effect of these waves does not become pronounced even at infinite Mach number (see fig. 12 (c)) and it can be seen, upon comparison of these results with those presented previously for the pressure-gradient ratio (see fig. 3) that the effect of the reflected waves dissipates somewhat downstream of the nose. The shock-expansion method is thus further substantiated as being a reliable simplified method for predicting the flow about airfoils at high supersonic speeds, again, so long as the air behaves as an ideal diatomic gas. The further simplified slender-airfoil method also appears to be a good approximation over the entire range of Mach numbers,¹¹ although, as would be expected from the assumptions made in its development, it is in somewhat greater error than the shock-expansion method at lower Mach numbers.

The shape of the shock wave given by the shock-expansion method, as presented in Appendix B, is compared in figure 13 with the shape given by the method of characteristics for the biconvex airfoil at $M_0=\infty$. (These shock waves correspond to the pressure distributions given in fig. 12 (c).) The shock-expansion method gives a reasonably good

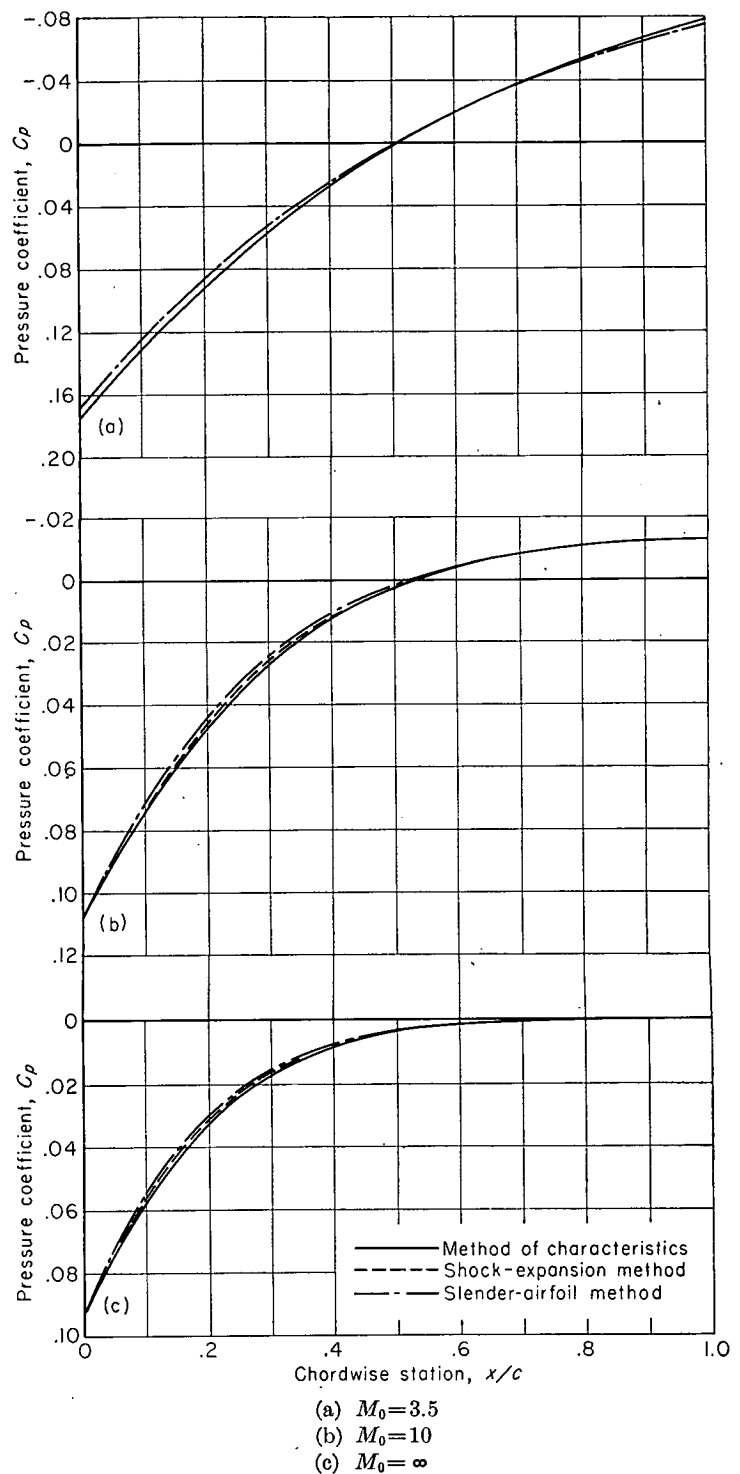


FIGURE 12.—Pressure distribution on 10-percent-thick biconvex airfoil section for various free-stream Mach numbers at $\alpha=0^\circ$.

approximation to the shock-wave shape though, as would be expected from the results given in figure 5, the curvature is somewhat too small. (A procedure for getting a closer approximation to the shock-wave shape is also given in Appendix B.) It can be seen, however, that this method of determining the shock-wave shape is far better than the assumption of a straight shock wave that is often associated with the shock-expansion method. Evidently, then, the shock-expansion method can also be used to calculate the flow in regions away from the airfoil surface. (See Appendix B.)

¹¹ The hybrid expression for pressure coefficient obtained by Ivey and Cline (ref. 5) gives reasonably good results also, although not as accurate as the slender-airfoil method at the higher Mach numbers under consideration.

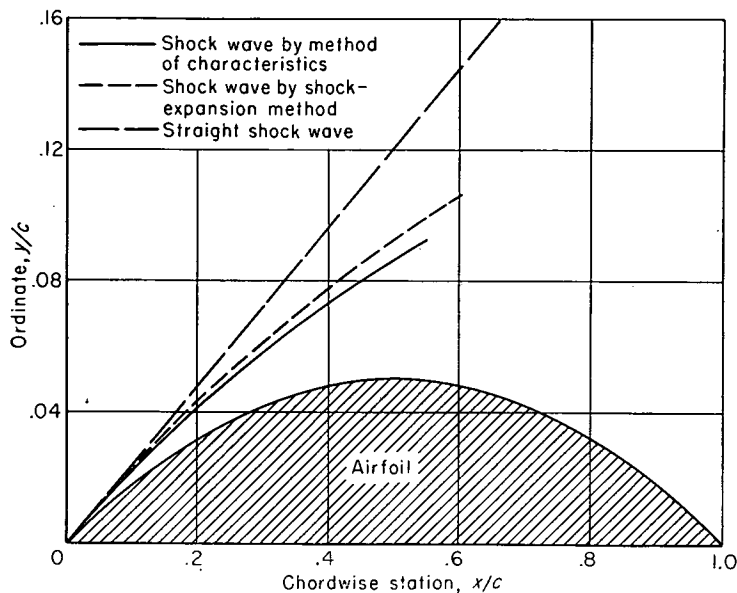


FIGURE 13.—Shape of the shock wave for 10-percent-thick biconvex airfoil section at $\alpha=0^\circ$ and $M_0=\infty$.

The relative accuracy at high Mach numbers of the slender-airfoil method and linear and second-order potential theories may be seen in figure 14. As might be expected, the slender-airfoil method is more accurate than linear theory at both $M_0=5$ and 15 and more accurate than second-order theory at $M_0=15$. It is perhaps surprising to note, however, that at the lower Mach number of 5, the slender-airfoil method is also somewhat superior to the second-order theory.

The pressure distributions of figures 12 and 14 have been employed to calculate the zero-lift drag of the biconvex airfoil, and the results of these calculations, along with additional predictions of the different methods, are shown in figure 15. Predictions of the shock-expansion method are, of course, in best agreement with those of the method of characteristics; while the slender-airfoil method, although slightly less accurate than the shock-expansion method, is apparently superior to both linear and second-order theories at Mach numbers above 3.5.

The preceding findings verify that, so long as the disturbance strength ratio is small compared to 1, the flow along streamlines is essentially of the Prandtl-Meyer type. If we choose, on the basis of these findings, a maximum absolute value for $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}$ of 0.06 (note the maximum value of $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}$ for the cases presented in fig. 12 was approximately 0.06 at $M_0=\infty$), the region in which the shock-expansion method is applicable can readily be obtained from figure 11. The upper boundary line of this region is shown in figure 16, and it is evident that it lies only slightly below (about 1° , in general) the line corresponding to shock detachment given approximately by the $M_0=1.0$ line. Almost the entire region of completely supersonic (ideal gas) flow is then covered by the method. (See shaded area of fig. 16.) The range of applicability shown in figure 16 is appreciably larger than that indicated by Rand (ref. 20) who required

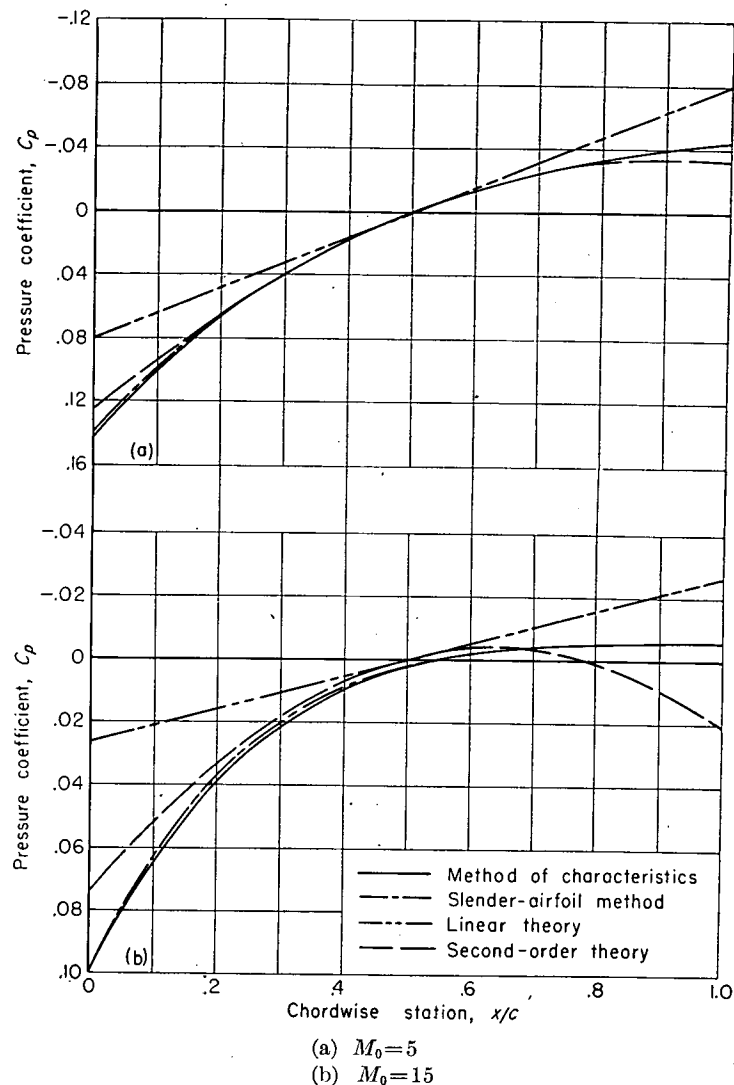


FIGURE 14.—Pressure distribution on 10-percent-thick biconvex airfoil section at $\alpha=0^\circ$.

that the entire flow field be of the true Prandtl-Meyer type (i. e., that all flow properties be constant along first-family Mach lines and not just δ and p). The results presented in

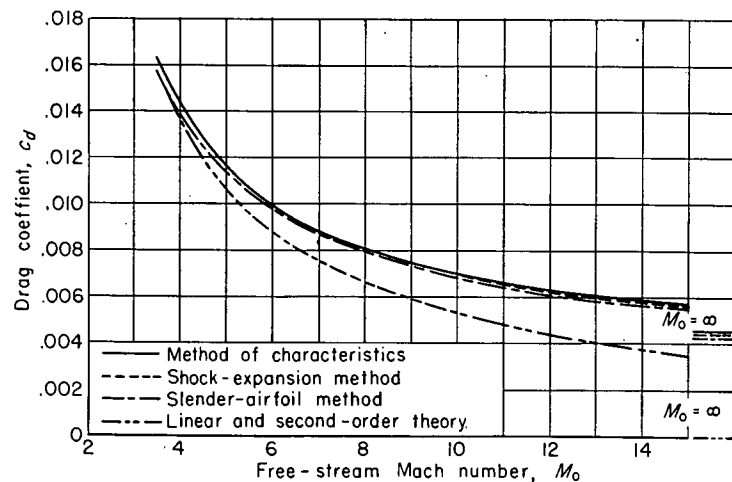


FIGURE 15.—Variation of drag coefficient with free-stream Mach number for 10-percent-thick biconvex airfoil section at $\alpha=0^\circ$ ($\gamma=1.4$).

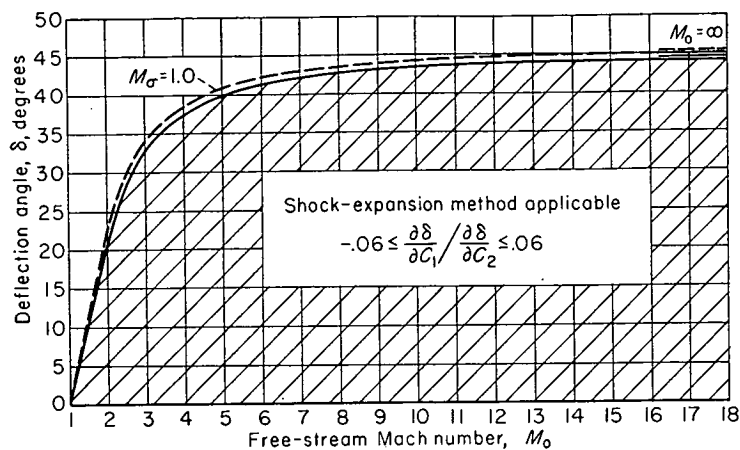
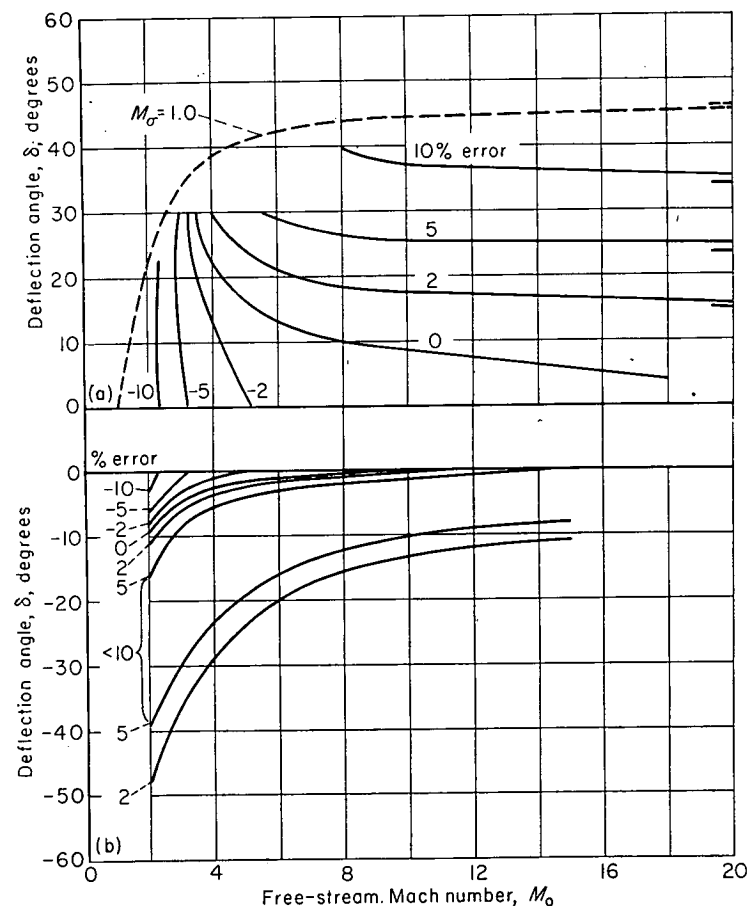


FIGURE 16.—Range of applicability of shock-expansion method ($\gamma=1.4$).

figures 12 and 13 show, however, that this restriction is not necessary.

The question naturally arises concerning the corresponding range of applicability of the slender-airfoil method. This question may be answered, in part, by comparing separately the predictions of the method for oblique-shock flows and expansion flows with those of the exact oblique-shock equations and Prandtl-Meyer equations. Such a comparison is shown in figure 17 in terms of the percentage



(a) Oblique-shock-wave flows.
(b) Expansion flows.

FIGURE 17.—Accuracy of slender-airfoil method in predicting pressure coefficients ($\gamma=1.4$).

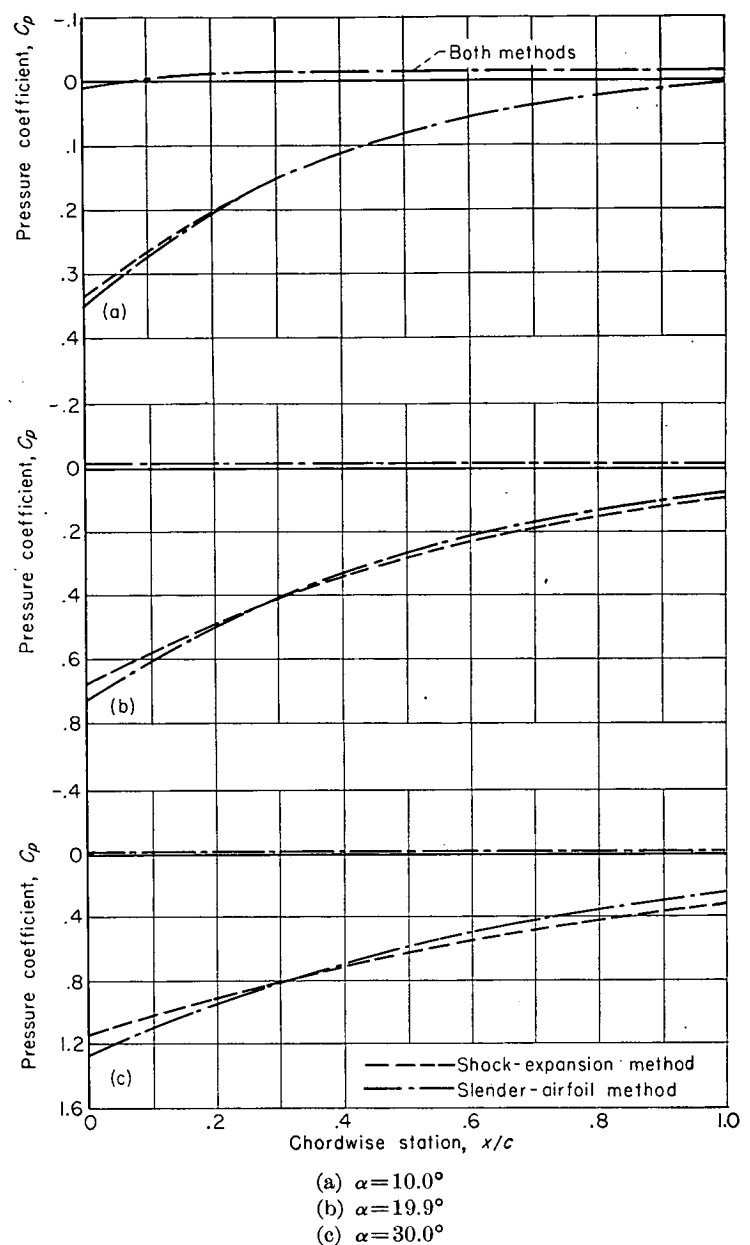


FIGURE 18.—Pressure distribution on 10-percent-thick biconvex airfoil section for various angles of attack at $M_0=10$ ($\gamma=1.4$).

error in the pressure coefficients predicted by the slender-airfoil method. As would be expected, this method does not exhibit good accuracy over the wide range of applicability of the shock-expansion method; however, it is indicated that it should predict pressure coefficients with less than 10-percent error down to Mach numbers as low as 3 for airfoils producing flow deflections up to as high as 25° .

As a further check on the utility of the slender-airfoil method, the pressure coefficients on the 10-percent-thick biconvex airfoil have been calculated with this method and the shock-expansion method at a Mach number of 10 and angles of attack up to about 30° .¹² The results of this calculation are shown in figure 18 (see fig. 12 (b) for $\alpha=0^\circ$)

¹² These conditions are within the range of applicability of the shock-expansion method as defined in figure 16; hence, the use of the method as a base of comparison seems justified. Since the shock-expansion method is far less tedious to apply than the method of characteristics, it will be employed as such a base in subsequent calculations whenever the conditions being investigated have been determined to be within its range of applicability.

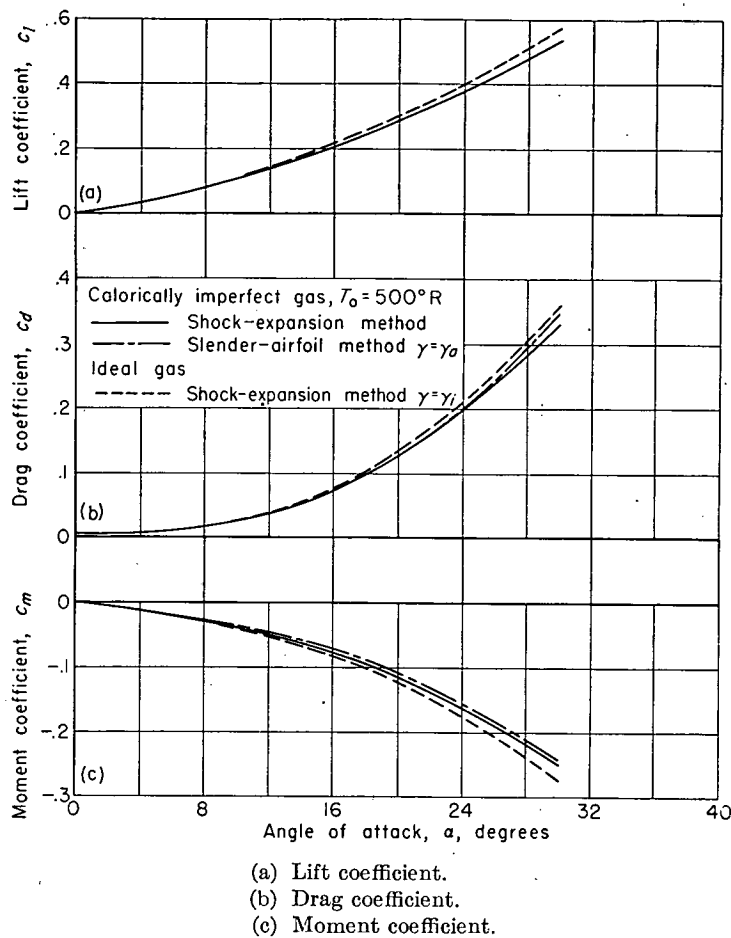


FIGURE 19.—Variation of force and moment coefficients with angle of attack for 10-percent-thick biconvex airfoil section at $M_0=10$ ($\gamma=1.4$).

where it is seen that the agreement is reasonably good, even at the highest angle of attack. This fact is reflected in figure 19 showing the force and moment coefficients for the airfoil as a function of angle of attack. Little difference is observed in the force coefficients as calculated by the two methods, while the moment coefficients display more pronounced but, nevertheless, small differences at the higher angles of attack.

From these and previous considerations, the ranges of applicability of the shock-expansion and slender-airfoil methods for supersonic ideal-gas flows are reasonably well established. It remains now to determine the manner and extent to which gaseous imperfections in the flow at higher supersonic speeds may alter these ranges, and the reasons therefor.

Imperfect-gas flows.—As a first step toward investigating the effects of gaseous imperfections on the high Mach number flows under consideration, it is convenient to extend our consideration of the disturbance strength ratio $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}$.

It is recalled that when air exhibits a constant value of γ equal to 1.4 (the value for an ideal diatomic gas), the disturbance strength ratio is small at arbitrarily large Mach numbers, provided the flow deflection angles are not too close to those for shock detachment. One of the most important effects of gaseous imperfections is, however, to decrease γ of the disturbed air below this value due to the excitation of

additional degrees of freedom (e. g., vibrational) in the molecules at the high temperatures encountered at high Mach numbers. Indeed, at arbitrarily high Mach numbers it might be expected that γ of the disturbed air would approach 1, since the number of degrees of freedom may effectively become very large (see, e. g., refs. 3 and 6). In this case, however, the extent of the disturbance flow field is decreased to a layer at the surface of the body which is negligibly thin compared to that for the case of ideal-gas flow. Thus, it is apparent that significant changes in the flow about airfoils at high Mach numbers may result from decreases in γ of the disturbed air; hence, the effects of such decreases on the disturbance strength ratio would appear to warrant attention.

A detailed analysis of these effects is impractical at the present time, due to the limited range over which the variation of γ with temperature is accurately known. Even in the range where this variation is so known, the additional complication required to consider the effects of variable γ and the addition of another independent parameter (free-stream temperature) make extensive calculations of the disturbance strength ratio impractical. However, some knowledge of these effects can be gained by performing the calculations for one free-stream Mach number and temperature. Such calculations have been carried out at a Mach number of 10 for a free-stream temperature of 500° R. and the results are presented in figure 20. The curve for a calorically imperfect gas cannot be extended to shock detachment because the temperature behind the shock wave exceeds that for which the calorically-imperfect-gas equations are valid. It can be seen that the effect of the caloric imperfection of air is to increase the value of the disturbance strength ratio and that the effect increases with increasing temperature or decreasing γ . However, it appears that if γ does not decrease appreciably below 1.3, as in this case, the disturbance strength ratio is still small compared to unity. It might be expected, therefore (as previously found for flow in the region of the leading edge), that the shock-expansion method would continue to predict the flow about complete airfoils with reasonable accuracy. This point has been checked with the methods developed previously for analyzing the flow of a calorically imperfect, diatomic gas at local air temperatures up to about 5,000° R. (note γ has a value only

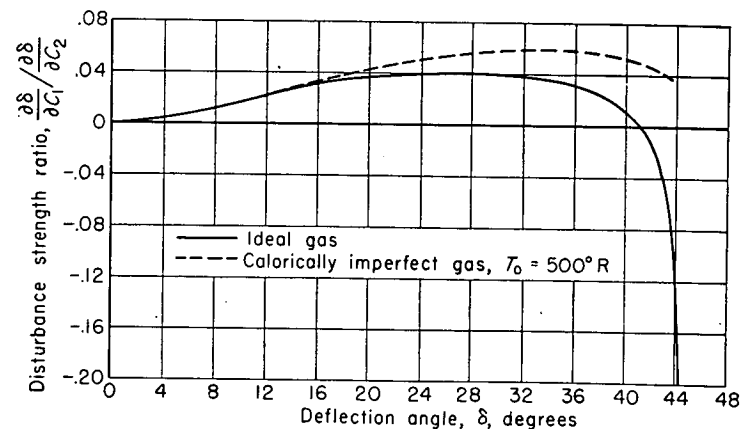


FIGURE 20.—Effect of the caloric imperfections of air on the disturbance strength ratio at $M_0=10$ and $T_0=500^\circ \text{R}$.

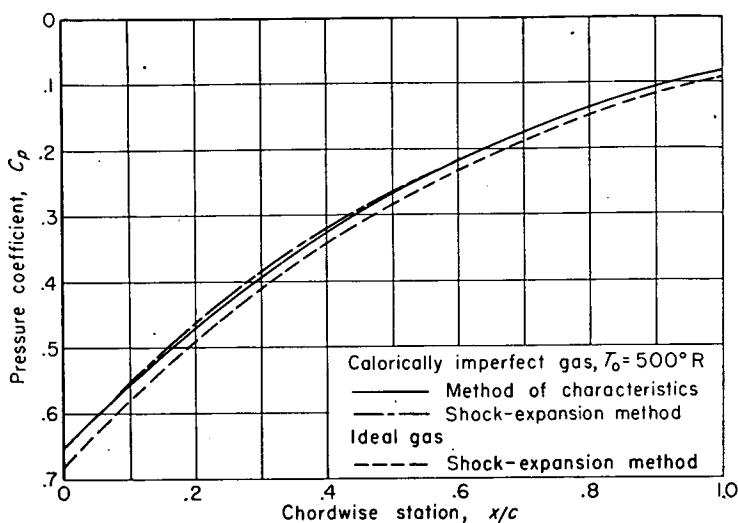


FIGURE 21.—Pressure distribution on lower surface of 10-percent-thick biconvex airfoil section at $M_0=10$, $T_0=500^\circ \text{R}$, $\alpha=19.9^\circ$.

slightly less than 1.3 at this temperature). In particular, the pressure distribution on the lower surface of the biconvex airfoil at $M_0=10$, $\alpha=19.9^\circ$, and $T_0=500^\circ \text{R}$. ($T_N \approx 4000^\circ \text{R}$.) has been calculated with both the method of characteristics and the shock-expansion method.¹³ The results of these calculations are presented in figure 21, and it would appear that the conclusions drawn from figure 20 are substantiated. (Due to the lower temperatures, pressures in the expansion flow about the upper surface are not influenced by caloric imperfections and, hence, are the same as shown in figure 18 (b).) For these same conditions, the shapes of the shock waves given by the shock-expansion method and by the method of characteristics (both for a calorically imperfect gas) are compared in figure 22. Just as in the case of ideal-gas flows, the shock-expansion method gives a good approximation to the shock-wave shape, far better than the assumption of a straight shock wave. Thus, it is seen that the conclusion drawn from figure 20 should also apply for the use of the shock-expansion method to calculate the flow field away from the airfoil surface.

Shown also in figure 21 is the pressure distribution obtained by the shock-expansion method for an ideal gas ($\gamma_i=1.4$). It is apparent, on comparing this pressure distribution with the other distributions, that although the effect of caloric imperfections on the disturbance strength ratio is small, the pressures are appreciably reduced by the increase in specific heats. The extent of this reduction is more completely illustrated in figure 23 where the lower-surface pressure distributions on the biconvex airfoil are presented for $M_0=10$ and $T_0=500^\circ \text{R}$, at $\alpha=0^\circ$, 10° , 19.9° , and 30° . As one might expect, the reduction in pressures increases with angle of attack (due to the corresponding increase in static temperature of the disturbed air). The pressure coefficients calculated with consideration of the imperfections in the gas are less on the lower surface (up to 6 percent at the leading edge and 15 percent at the trailing edge) than those calculated assuming the gas behaves ideally. The upper-surface pres-

ures are again unaffected by the caloric imperfections of air in all the cases presented (except at $\alpha=0^\circ$) since this surface experiences lower pressures and, hence, lower temperatures. They are therefore the same as shown in figure 18. Shown also in figure 23 are the pressure distributions calculated with the slender-airfoil method for $\gamma=\gamma_a$. The accuracy of this simplified method is substantially the same as was previously observed for the corresponding method in the case of ideal-gas flows, although the local error may be greater than the reduction in pressure coefficients due to the caloric imperfections of air.

The force and moment coefficients, corresponding to the lower-surface pressure distributions shown in figure 23 and the upper-surface distributions of figure 18, are presented in figure 24. The reduction in the lower-surface pressures leads, of course, to a general reduction in all three coefficients (up to about 10 percent for $\alpha=30^\circ$). The slender-airfoil method again predicts these coefficients with surprising accuracy.

In order to further assess the accuracy of the slender-airfoil method, some additional calculations were carried out for the biconvex airfoil at $\alpha=0^\circ$ and $M_0=20$ and 30. The pressure distributions for these cases were calculated by the shock-expansion method, slender-airfoil method ($\gamma=\gamma_a$), and slender-airfoil method ($\gamma=\gamma_i$). These results are presented in figure 25, and it is observed that the use of γ_a rather than γ_i improves the accuracy of the slender-airfoil method. The extent of this improvement in the case of drag coefficient is shown in figure 26; it would appear that predictions of the slender-airfoil method ($\gamma=\gamma_a$) and shock-expansion method are in as good agreement as for ideal-gas flows (see fig. 15). On the basis of these and previous results, it may be concluded, then, that not only does the shock-expansion method retain its range of applicability when air exhibits caloric imperfections, provided γ of the disturbed air is not appreciably less than 1.3, but also the slender-airfoil method ($\gamma=\gamma_a$) retains its range of applicability.

It would be surprising indeed, however, if this conclusion continued to apply as γ of the disturbed fluid approached 1

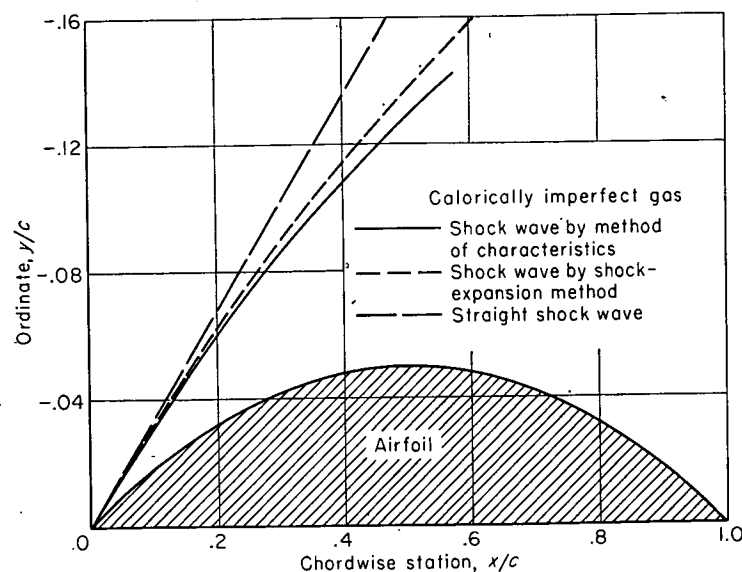


FIGURE 22.—Shape of the shock wave for 10-percent-thick biconvex airfoil section at $\alpha=19.9^\circ$, $M_0=10$, $T_0=500^\circ \text{R}$.

¹³ For added ease of calculation the expansion method of Appendix C was employed. This method is also employed in all subsequent calculations of this type since it has been found to yield results differing by less than 1 percent from those obtained by the more tedious graphical integration method.

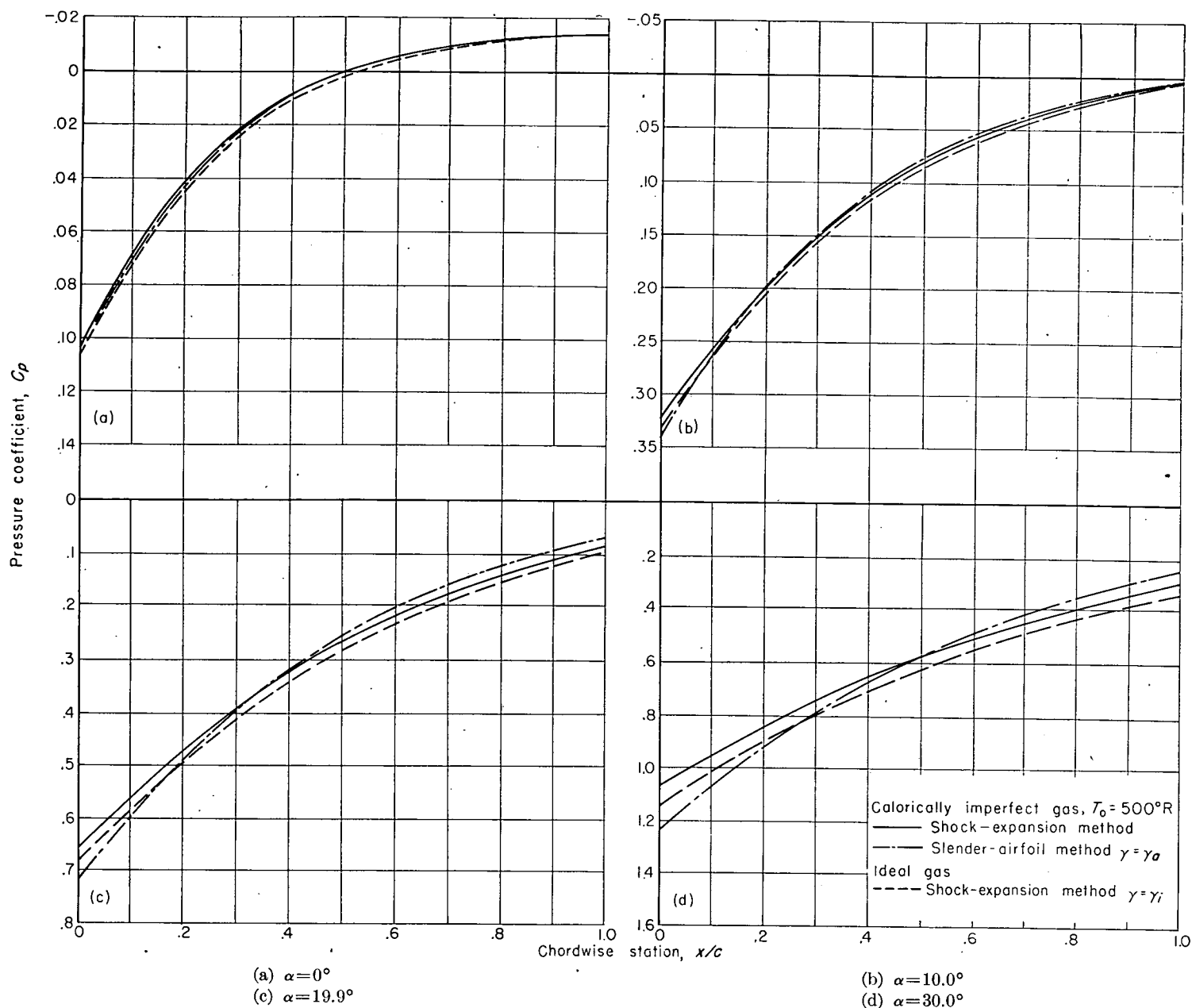


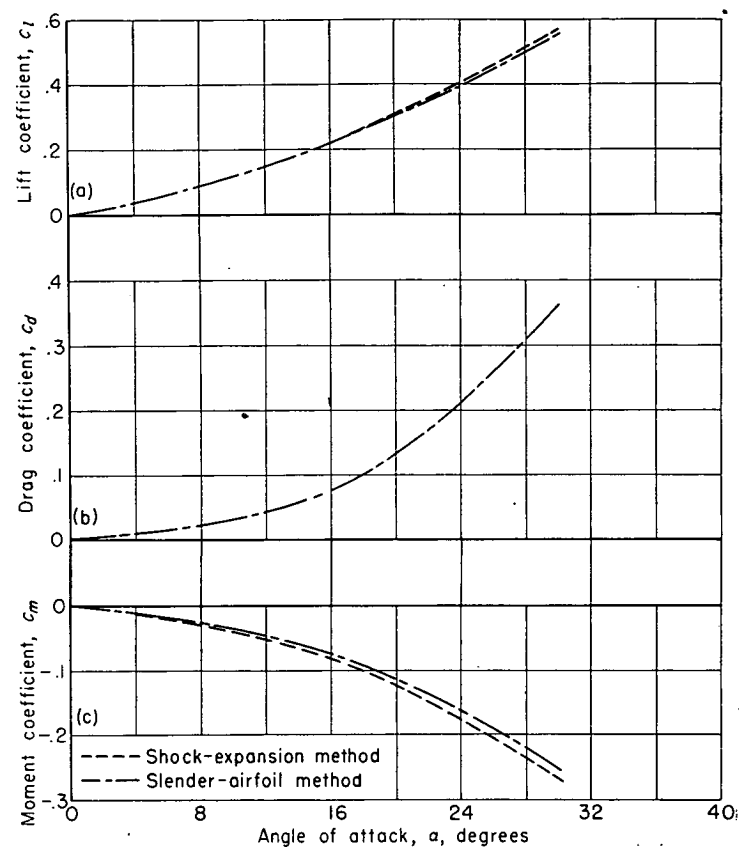
FIGURE 23.—Pressure distribution on lower surface of 10-percent-thick biconvex airfoil section for various angles of attack at $M_0 = 10$ and $T_0 = 500^\circ\text{R}$.

since, as indicated previously, $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}$ increases with decreasing γ . Although the manner in which γ varies with temperature is not known in this range, some knowledge of these effects can be gained by repeating the ideal-gas calculations for constant values of γ between 1.4 and 1.0.¹⁴ Such calculations have been carried out for infinite Mach number since, in this case, $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}$ has its maximum value for a given γ , and the results are presented in figure 27. It is seen that except near detachment, the disturbance strength ratio increases with decreasing γ . This increase is slow at first; for example, the value of $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}$ is still less than 0.1 at $\gamma = 1.3$. This result is in agreement with the previous con-

¹⁴ Since the enthalpy is negligibly small compared to the mass kinetic energy of the undisturbed fluid at the high Mach numbers of interest and, hence, γ of this fluid does not influence the flow, this approach corresponds to employing an average value of γ for the disturbed fluid.

clusion regarding cases where γ is greater than 1.3. However, $\frac{\partial \delta / \partial C_1}{\partial \delta / \partial C_2}$ continues to increase as γ decreases, and, in fact, approaches 1 as γ approaches 1. The effect on pressure distributions of this increase in the strength of the reflected disturbances may also be investigated by using the ideal-gas relationships in combination with appropriate values of γ .

The limiting case of infinite free-stream Mach number and $\gamma = 1.0$ (for the disturbed fluid, see footnote 14) has already been investigated by Busemann (ref. 21) and more recently by Ivey, Klunker, and Bowen (ref. 22). In this case, as pointed out previously, the shock wave emanating from the leading edge remains attached to the surface downstream of the leading edge (this is easily verified with the oblique-shock-wave equations), and the disturbance flow field is confined to an infinitesimally thin layer adjacent to the surface. In addition, the velocity along a streamline downstream of the shock is constant, as may easily be shown with



(a) Lift coefficient.
(b) Drag coefficient.
(c) Moment coefficient.

FIGURE 24.—Variation of force and moment coefficients with angle of attack for 10-percent-thick biconvex airfoil section at $M_0=10$, $T_0=500^\circ \text{ R}$.

the compatibility equations. Surface pressures therefore become a simple function of airfoil geometry.

$$C_p = 2 \sin^2 \delta_s + 2 \cos \delta_s \frac{d\delta_s}{dx} \int_0^x \sin \delta_s dx \quad (45)$$

varying, to a first approximation, directly with the square of the component of free-stream velocity normal to the surface (i. e., the flow is approximately of the Newtonian corpuscular type). With this theory, then, and the method of characteristics, we can get an idea of both the extent to which changes of γ from 1.4 toward 1 will alter surface pressures, and the accuracy with which the shock-expansion theory predicts the alterations. To this end, figure 28 is presented showing the pressure distributions about the biconvex airfoil at $M_0=\infty$ as calculated by the several methods for different values of γ . It is observed that, whereas the shock-expansion method agrees very closely with the method of characteristics for $\gamma=1.4$, there is a large difference at $\gamma=1.05$. This, of course, is precisely what one would expect from the previous discussion of the disturbance strength ratio. On the other hand, if the two characteristics solutions and the Busemann method are considered in order of decreasing γ , it is indicated that the characteristics solutions approach the Busemann theory as

γ approaches 1. For $\gamma=1.0$ and $M_0=\infty$ the shock-expansion method, in turn, predicts a discontinuous pressure distribution with a pressure coefficient equal to that of the Busemann theory at the leading edge but a pressure coefficient of zero at all points downstream of the leading edge. Hence, it may be concluded that when the free-stream Mach number approaches infinity and γ approaches 1, the Busemann method rather than the shock-expansion method for calculating the flow about airfoils should be employed.

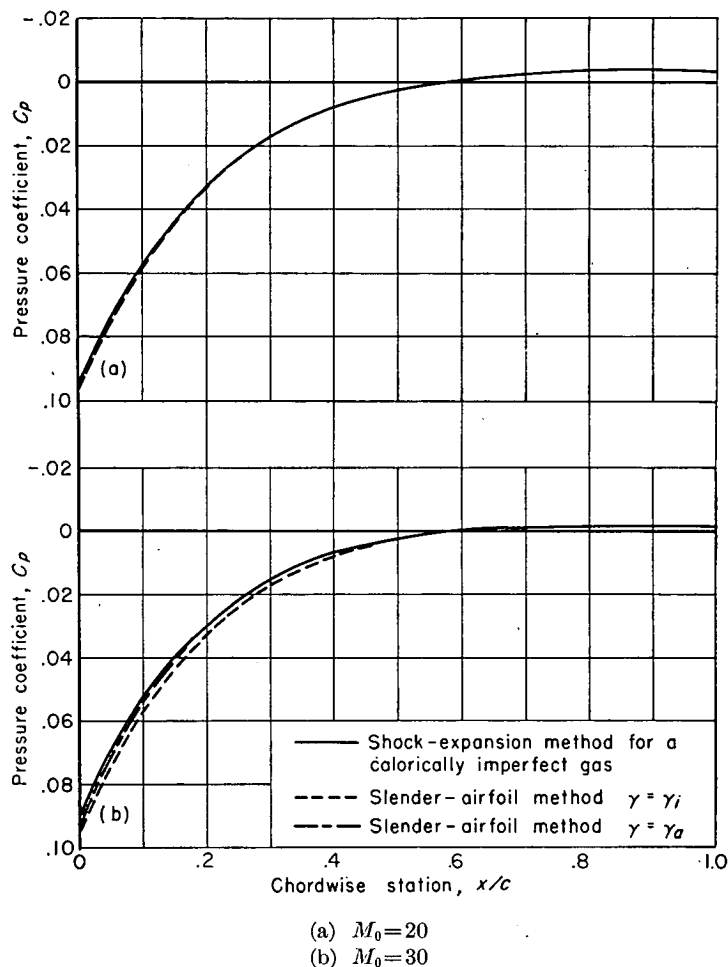


FIGURE 25.—Pressure distribution on 10-percent-thick biconvex airfoil section at $\alpha=0^\circ$ and $T_0=500^\circ \text{ R}$.

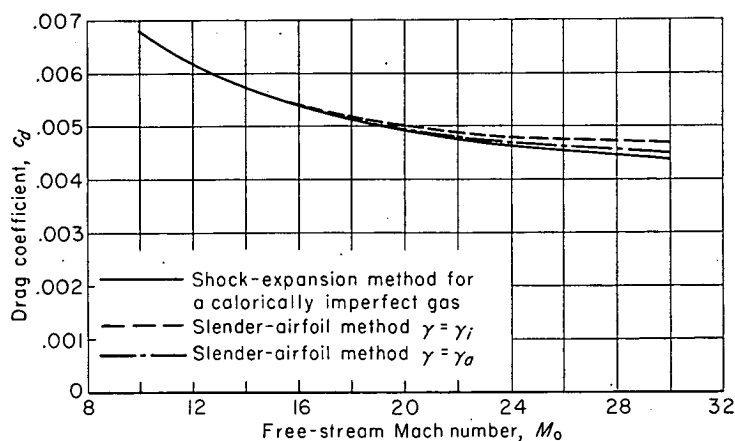


FIGURE 26.—Variation of drag coefficient with Mach number for 10-percent-thick biconvex airfoil section for $\alpha=0^\circ$ and $T_0=500^\circ \text{ R}$.

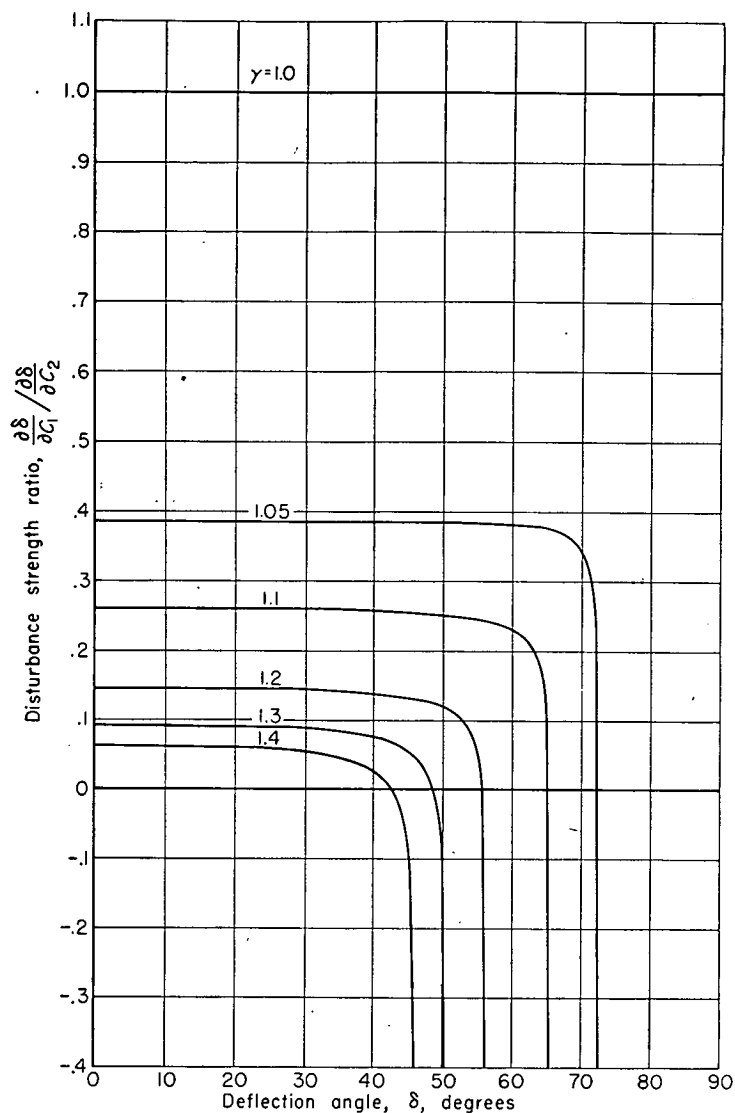


FIGURE 27.—Variation of disturbance strength ratio with deflection angle at infinite free-stream Mach number for various values of γ .

CONCLUSIONS

Inviscid flow about curved airfoils at high supersonic speeds was investigated analytically, first assuming air behaves as an ideal gas, and then assuming it behaves as a thermally perfect, calorically imperfect gas. This study has led to the following conclusions:

1. So long as air behaves as an ideal gas, the shock-expansion method may be used with good accuracy to predict the flow about a curved airfoil up to arbitrarily high Mach numbers, provided the flow deflection angles are about 1° or more below those corresponding to shock detachment. This conclusion applies not only to the determination of surface pressure distributions, but also to the determination of the whole flow field about the airfoil.

2. An approximation to the shock-expansion method, applicable to ideal-gas flows about slender airfoils at high

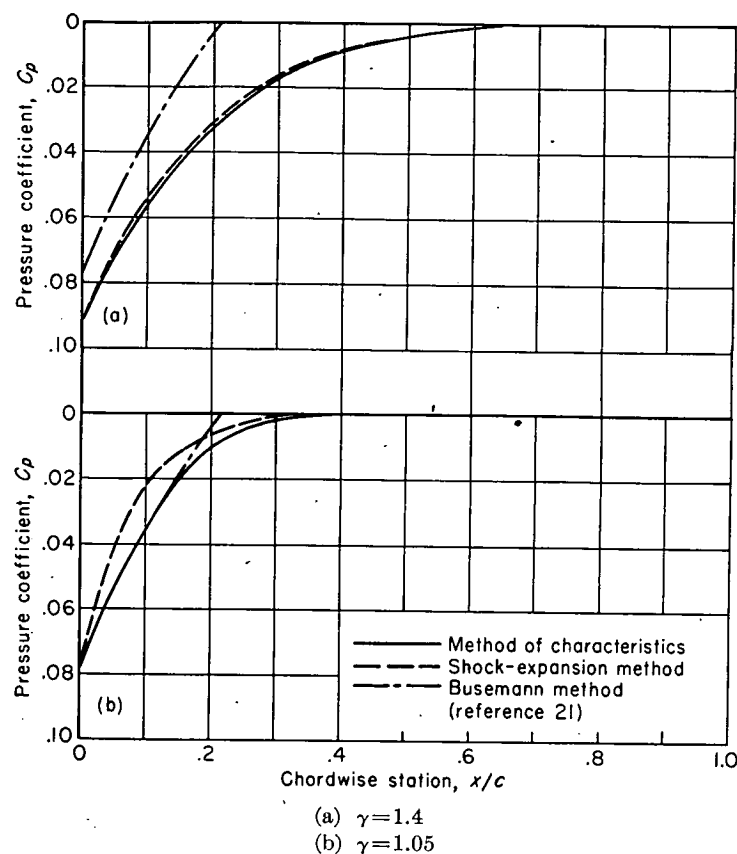


FIGURE 28.—Pressure distribution on 10-percent-thick biconvex airfoil section at $M_0 = \infty$ and $\alpha = 0^\circ$.

Mach numbers, predicts pressure coefficients in error by less than 10 percent for Mach numbers above 3 and flow deflection angles up to 25° .

3. So long as caloric imperfections of air do not decrease the ratio of specific heats appreciably below 1.3 (corresponding to air temperatures up to the order of $5,000^\circ \text{R.}$), the shock-expansion method, generalized to include the effects of these imperfections, is substantially as accurate as for ideal-gas flows. The principal effect of caloric imperfections is to reduce pressure coefficients by as much as 15 percent. The slender-airfoil method can also be made as accurate as for ideal-gas flows by employing an average value of the ratio of specific heats.

4. If the ratio of specific heats approaches 1, as it may at extremely high Mach numbers, the shock-expansion method can be in considerable error. In this case, the Busemann method for flow in the limit of infinite Mach number and specific-heat ratio of 1 applies with reasonable accuracy.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., January 9, 1952.

APPENDIX A

METHOD OF CHARACTERISTICS FOR TWO-DIMENSIONAL FLOW OF A CALORICALLY IMPERFECT GAS

In the application of the method of characteristics for a calorically imperfect, diatomic gas to the particular problem of analyzing the flow about curved two-dimensional airfoils, many of the calculations are identical to those encountered in the solution of any problem where characteristics theory is employed. Since the details of these calculations are well known and well reported (see, e. g., ref. 12), they will not be repeated here.

A lattice-point system with an initial-value, numerical computing procedure will be used. The form of the compatibility equations to be employed was developed previously;¹⁵ however, it is convenient for purposes of calculation to substitute the pressure ratio, p/q_0 , into these equations and to rewrite them as difference equations. Equations (12) and (13) are thus reduced to the following forms:

$$(p/q_0)_C - (p/q_0)_A = -\lambda_A(\delta_C - \delta_A) \quad (A1)$$

and

$$(p/q_0)_C - (p/q_0)_B = \lambda_B(\delta_C - \delta_B) \quad (A2)$$

where

$$\lambda = \frac{2\gamma(p/q_0)}{\sin 2\beta} \quad (A3)$$

It is also convenient to employ several reference curves. These curves can be divided into two groups. The general reference curves consist of γ and $\Lambda(T)$ as a function of temperature, T . Equations (14) and (17) are used to determine these curves. A second set of shock-wave reference curves consisting of p/q_0 , σ , and δ as a function of temperature, T , are determined by use of equations (18) through (20)—the values of T_0 and M_0 are presumed known.

In the computations three types of points are encountered. These are (1) a point in the flow field between the shock wave and the airfoil surface, (2) a point on the airfoil surface, and (3) a point just downstream of the shock wave. Each one of these types of points requires a slightly different computing procedure and they will be considered in order.

POINT IN THE FLOW FIELD BETWEEN THE SHOCK WAVE AND THE AIRFOIL SURFACE

Figure 29 (a) shows a schematic diagram of the system of points to be considered in these calculations. Point C is the unknown point at the intersection of the first-family characteristic line passing through point A and the second-family characteristic line passing through point B. Six quantities are known at both points A and B, and the problem is to calculate these same quantities at point C. These

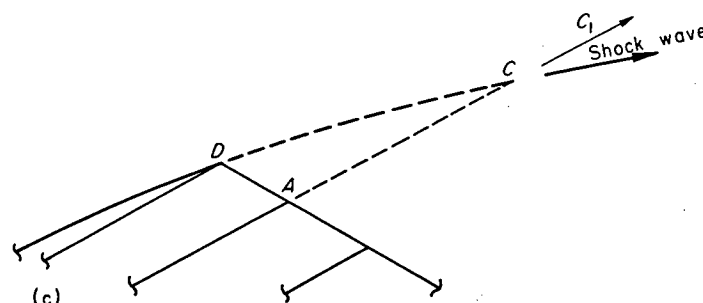
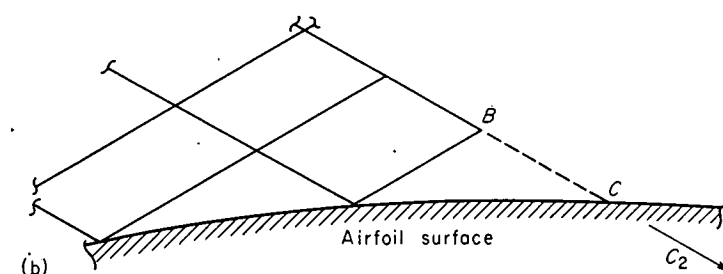
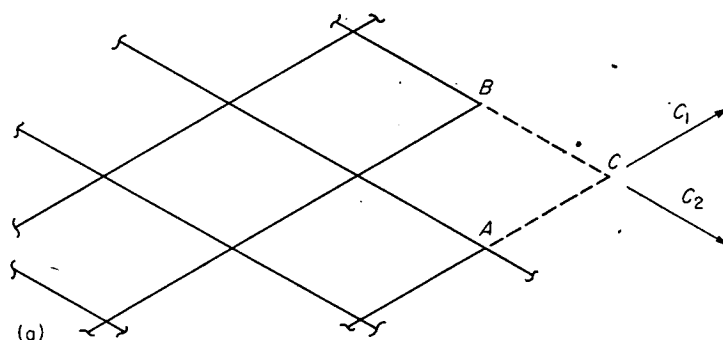
quantities are x , y , δ , p/q_0 , T , and T_σ . The first five quantities are of obvious significance. The sixth, T_σ , is defined as the static temperature, just downstream of the shock wave, on the streamline passing through the point C.

The physical coordinates of the point $C(x_C, y_C)$ may be determined by standard procedures such as those given in reference 12. In order to determine the quantity δ_C , it is necessary to solve equations (A1) and (A2) simultaneously, thus,

$$\delta_C = \frac{\lambda_A \delta_A + \lambda_B \delta_B + (p/q_0)_A - (p/q_0)_B}{\lambda_A + \lambda_B} \quad (A4)$$

Equation (A1) or (A2) is then used to obtain $(p/q_0)_C$.

There remains only the problem of determining T_C and T_{σ_C} at point C. The temperature T_{σ_C} is obviously constant



- (a) Point in field.
- (b) Point on surface.
- (c) Point on shock wave.

FIGURE 29.—Diagram of point system in the method of characteristics for the two-dimensional flow of a calorically imperfect gas.

¹⁵ This form of the compatibility equations (in p and δ coordinates) was also used in obtaining some of the characteristics solutions for ideal-gas flows. The majority of these solutions were carried out, however, with the compatibility equations in β , δ , and entropy coordinates, since it was found that greater accuracy was usually obtained for a given net size. In general, the net size employed yielded pressures at from 30 to 35 surface points on an airfoil with a maximum error in the corresponding pressure coefficients equal to less than 1 percent of the pressure coefficient at the leading edge.

along the streamline through C. This quantity may therefore be calculated in the same manner as the entropy is calculated in similar flow fields for ideal-gas processes (see, e. g., ref. 12). Furthermore, since the flow along streamlines downstream of the shock wave is isentropic, equation (16) may be applied in the form

$$\frac{(p/q_0)_C}{(p/q_0)_{\sigma_C}} = \frac{\Lambda(T_{\sigma_C})}{\Lambda(T_C)} \quad (A5)$$

The pressure, $(p/q_0)_{\sigma_C}$, is defined in a manner analogous to T_{σ_C} , and may thus be determined using the shock-wave reference curves and the known value of T_{σ_C} . Similarly, $\Lambda(T_{\sigma_C})$ may be determined from the general reference curves. The only unknown in equation (A5) then is $\Lambda(T_C)$ which may now be calculated. Once $\Lambda(T_C)$ is determined, T_C may be determined by again using the general reference curves. All six quantities, x_C , y_C , δ_C , $(p/q_0)_C$, T_C , and T_{σ_C} , have now been determined.

POINT ON THE AIRFOIL SURFACE

Figure 29(b) shows a schematic diagram of the points to be considered in these calculations. The physical coordinates of point C (x_C , y_C) are first calculated by solving simultaneously the equation of the second-family Mach line passing through point B and the equation of the airfoil surface. When x_C and y_C have been determined, δ_C is readily obtained from the equation of the airfoil surface. Equation (A2) is then applied to determine $(p/q_0)_C$.

Since the airfoil surface is a streamline, T_{σ_C} is constant along the surface and may be evaluated at the leading edge. The temperature, T_C , may then be determined using equation (A5) and the previously described procedure. All six quantities, x_C , y_C , $(p/q_0)_C$, δ_C , T_C , and T_{σ_C} , are thus determined.

In the special case of the first point on the airfoil surface downstream of the leading edge, the pressure ratio is calculated using the pressure gradient evaluated at the leading edge. (See section Methods for Calculating the Flow in

the Region of the Leading Edge, and Appendix D.) Additional initial points in this region may be calculated by the procedure previously described.

POINT ON THE SHOCK WAVE

Figure 29 (c) shows a schematic diagram of the points to be considered in these calculations. The physical coordinates of point C (x_C , y_C) are first calculated by solving simultaneously the equation of the first-family Mach line passing through point A and the equation of the shock wave linearized at point D, the last known point on the wave. The variation of p/q_0 with δ along the shock wave may be approximated by the relation

$$(p/q_0)_C - (p/q_0)_D = \left[\frac{d(p/q_0)}{d\delta} \right]_{\sigma} (\delta_C - \delta_D) \quad (A6)$$

In this equation $\left[\frac{d(p/q_0)}{d\delta} \right]_{\sigma}$ is the rate of change of p/q_0

with δ along the downstream side of the shock wave evaluated at point D. Because of the complicated nature of the shock-wave equations, it is generally easier to evaluate $\left[\frac{d(p/q_0)}{d\delta} \right]_{\sigma}$ graphically or numerically from the shock-wave reference curves; however, this derivative may also be evaluated from the equations given in Appendix D. Equations (A1) and (A6) are solved simultaneously for δ_C , thus,

$$\delta_C = \frac{\lambda_A \delta_A + \left[\frac{d(p/q_0)}{d\delta} \right]_{\sigma} \delta_D + (p/q_0)_A - (p/q_0)_D}{\lambda_A + \left[\frac{d(p/q_0)}{d\delta} \right]_{\sigma}} \quad (A7)$$

when δ_C has been calculated, T_C and, in turn, $(p/q_0)_C$ may be determined from the shock-wave reference curves. Since point C in this case is just downstream of the shock wave, T_C and T_{σ_C} are identical. The six quantities, x_C , y_C , $(p/q_0)_C$, δ_C , T_C , and T_{σ_C} have now been determined.

APPENDIX B

SHOCK-EXPANSION METHOD FOR CALCULATING THE FLOW FIELD ABOUT AN AIRFOIL

An initial-value procedure which is similar to, although markedly simpler than, that associated with the method of characteristics may be employed to carry out this calculation.¹⁶ To illustrate, consider figure 30. With the oblique-shock-wave and expansion equations, all fluid properties at points A, B, D, and so forth, on the airfoil surface may be

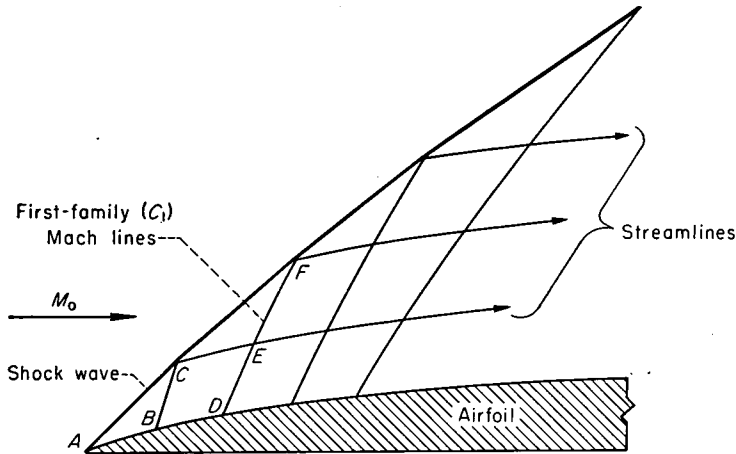


FIGURE 30.—Schematic diagram of shock-expansion method for calculating the flow field about an airfoil.

calculated in the usual manner. If point B is chosen close to point A, the first-family (C_1) Mach line connecting B to point C on the shock wave may be considered straight and inclined at an angle to the free stream equal to $\beta_B + \delta_B$. Similarly, the segment of the shock wave AC may be considered straight and inclined to the free-stream direction at an angle of σ_A . Thus, the physical coordinates of point C may be easily calculated. Since δ is assumed constant along first-family Mach lines, δ_C is equal to δ_B . All fluid properties at point C may be calculated from this known value of δ_C with the oblique-shock-wave equations. In a similar manner, the segment DE of the next first-family Mach line is considered straight and inclined at $\beta_D + \delta_D$, and the streamline joining points C and E is considered straight and inclined at δ_C . The physical coordinates of point E are therefore easily obtained. Since the flow along streamlines downstream of the shock wave is isentropic and since the pressure is also assumed constant along the first-family Mach lines (i. e., $P_E = P_D$), the fluid properties at point E are readily obtained from the known properties at point C, using the isentropic flow relationships. The construction of the remainder of the flow field follows in a similar manner.

As was discussed previously, the assumption that δ is constant along first-family Mach lines is an additional con-

¹⁶ It is clear, of course, that an average-value procedure could also be employed. In fact, for the shock-expansion method, an average-value procedure requires very little additional computation since the fluid properties for a system of points can be obtained independently of their physical coordinates. (The difference in the average- or initial-value procedures appears only in computing the physical coordinates.) Thus, the slopes at both ends of a line segment are known before the line is added to the construction. No iteration, as required in characteristic solutions, is necessary in this case.

dition which, in general, overdetermines the flow field. It is possible, therefore, to calculate two values of the shock-wave angle at each point on the shock, one assuming δ is constant and one assuming the pressure is constant. These two values will differ slightly and it can be shown that they will bracket the correct value that would be given by the method of characteristics. It is also relatively easy to show that if the change in δ (or p) along C_1 given by the corresponding characteristic solution is small,¹⁷ the true value of the shock-wave angle lies just midway between the two values given by the different assumptions. It is apparent then that a closer approximation to the shock-wave shape can be easily obtained by simply averaging the values of shock-wave angle determined by assuming δ is constant and by assuming p is constant. The increase in accuracy is illustrated in figure 31 for the biconvex airfoil at $M_0 = \infty$. The averaging procedure gives a shock-wave shape that is closer to that given by the method of characteristics by 60 to 80 percent. The increase in computation time (at least for ideal-gas flows) is negligible. In this regard, it is interesting to note that the shock-expansion solutions require less than 20 percent the computation time of the characteristic solutions.

The shock-expansion method is applicable to the determination of the flow not only in the region adjacent to the airfoil (whether concave or convex), but also in the region downstream of the airfoil; hence, it may, for example, prove useful in downwash studies and the like.

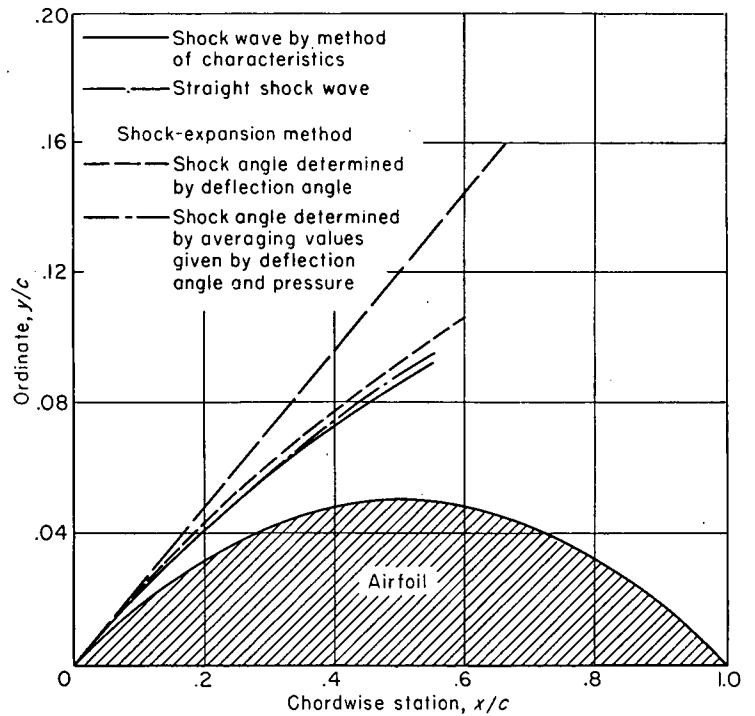


FIGURE 31.—Shape of the shock wave for 10-percent-thick biconvex airfoil section at $\alpha=0^\circ$ and $M_0=\infty$.

¹⁷ The necessary assumption here is that $\left(\frac{dP}{d\delta}\right)_{C_1}$ and $\left(\frac{dP}{d\delta}\right)_s$ are nearly equal. This is equivalent to the condition that $\frac{\partial\delta/\partial C_1}{\partial\delta/\partial C_2}$ is much less than one. (See eq. (40).)

APPENDIX C

APPROXIMATE SOLUTION FOR PRANDTL-MEYER FLOW OF A CALORICALLY IMPERFECT GAS

The following solution is obtained with an analysis similar to that used in Meyer's original paper (ref. 23). A schematic diagram of the subject flow field is shown in figure 32. It is

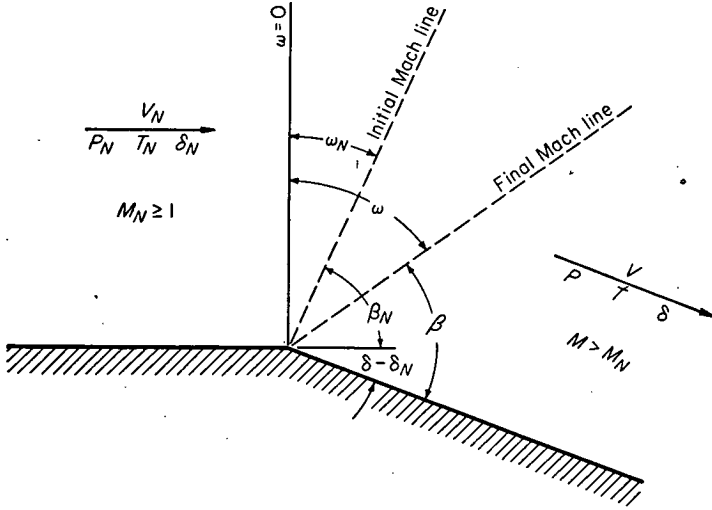


FIGURE 32.—Schematic diagram of Prandtl-Meyer flow around a corner.

evident that the change in flow-inclination angle for Prandtl-Meyer flow can be written as follows:

$$\delta_N - \delta = (\beta_N - \beta) + (\omega_N - \omega) \quad (C1)$$

Since the flow is isentropic, a given value of the local pressure will determine the Mach angle, β . The problem, then, is to evaluate the angle, ω . To this end, the velocity components tangential and normal to the first-family Mach lines may be expressed in the usual manner in terms of a potential φ , thus,

$$u = \frac{\partial \varphi}{\partial r} \quad (C2)$$

$$a = \frac{1}{r} \frac{\partial \varphi}{\partial \omega} \quad (C3)$$

It is clear, however, that these components are functions of ω only; hence, it is convenient to define a new velocity potential which is a function of ω alone. Such a potential is

$$\Phi(\omega) = \frac{\varphi}{r} \quad (C4)$$

The velocity components may then be written in terms of this new potential.

$$u = \Phi \quad (C5)$$

$$a = \Phi_\omega \quad (C6)$$

The resultant velocity is given by the expression

$$V^2 = \Phi^2 + \Phi_\omega^2 \quad (C7)$$

Equation (15), for conservation of energy, may be written in terms of the local temperature as follows:

$$V^2 + 2 \left(\frac{\gamma_t}{\gamma_t - 1} \right) RT + 2RT \left(\frac{\theta/T}{e^{\theta/T} - 1} \right) = A^2 \quad (C8)$$

The constant, A , is evaluated at the conditions existing upstream of the expansion region; namely,

$$A^2 = V_N^2 + \left(\frac{2\gamma_t}{\gamma_t - 1} \right) RT_N + 2RT_N \left(\frac{\theta/T_N}{e^{\theta/T_N} - 1} \right) \quad (C9)$$

Equations (C7) and (C8) are then combined to yield

$$\Phi^2 + \Phi_\omega^2 = -2RT \left(\frac{\gamma_t}{\gamma_t - 1} + \frac{\theta/T}{e^{\theta/T} - 1} \right) + A^2 \quad (C10)$$

It was shown previously, however, that

$$a^2 = \gamma RT \quad (C11)$$

Equations (C6), (C10), and (C11) may therefore be combined to obtain the following relationship:

$$\Phi^2 + \Phi_\omega^2 \left[1 + \frac{2}{\gamma} \left(\frac{\gamma_t}{\gamma_t - 1} + \frac{\theta/T}{e^{\theta/T} - 1} \right) \right] = A^2 \quad (C12)$$

or

$$\Phi^2 + \Phi_\omega^2 \left\{ \frac{\gamma_t + 1}{\gamma_t - 1} + \frac{2}{\gamma_t - 1} \left[\frac{\gamma_t}{\gamma} - 1 + \frac{\gamma_t - 1}{\gamma_t} \left(\frac{\gamma_t}{\gamma} \right) \frac{\theta/T}{e^{\theta/T} - 1} \right] \right\} = A^2 \quad (C13)$$

From the imperfect-gas relationship for γ we have

$$\frac{\gamma_t}{\gamma} = \frac{1 + (\gamma_t - 1) \left(\frac{\theta}{T} \right)^2 \frac{e^{\theta/T}}{(e^{\theta/T} - 1)^2}}{1 + \frac{\gamma_t - 1}{\gamma_t} \left(\frac{\theta}{T} \right)^2 \frac{e^{\theta/T}}{(e^{\theta/T} - 1)^2}} \quad (C14)$$

By substitution of this relation into equation (C13) there is then obtained

$$\Phi^2 + \Phi_\omega^2 \left[\frac{\gamma_t + 1}{\gamma_t - 1} + \frac{2}{\gamma_t} F \left(\frac{\theta}{T} \right) \right] = A^2 \quad (C15)$$

where

$$F \left(\frac{\theta}{T} \right) = \frac{\theta/T}{e^{\theta/T} - 1} \left\{ \frac{1 + (\gamma_t - 1) \frac{(\theta/T) e^{\theta/T}}{e^{\theta/T} - 1} \left[1 + \frac{\theta/T}{e^{\theta/T} - 1} \right]}{1 + \frac{\gamma_t - 1}{\gamma_t} \frac{(\theta/T)^2 e^{\theta/T}}{(e^{\theta/T} - 1)^2}} \right\} \quad (C16)$$

Now

$$\frac{\gamma T}{\gamma_t \theta} = \frac{a^2}{\gamma_t R \theta} \quad (C17)$$

For every value of T/θ there is thus a particular value of $a^2/\gamma_t R \theta$. The function $F(\theta/T)$ is therefore uniquely determined for any value of a^2 since $\gamma_t R \theta$ is, of course, a constant. With this point in mind, let

$$F(\theta/T) = G(a^2/\gamma_t R \theta) \quad (C18)$$

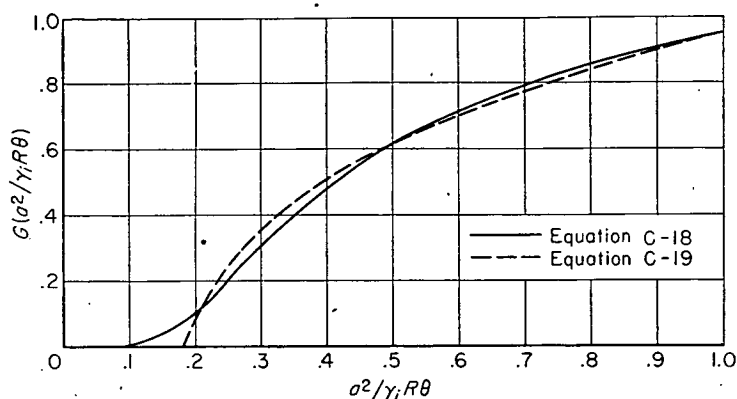


FIGURE 33.—Accuracy of approximation used in Appendix C to obtain solution for Prandtl-Meyer flow of a calorically imperfect gas.

Figure 33 shows $G(a^2/\gamma_i R\theta)$ plotted as a function of $a^2/\gamma_i R\theta$. This curve is approximated with the following simple relation:

$$G(a^2/\gamma_i R\theta) = 0.38 \frac{a^2}{\gamma_i R\theta} + 0.71 - \frac{0.14}{a^2/\gamma_i R\theta} \quad (C19)$$

for $0.18 < \frac{a^2}{\gamma_i R\theta} < 1.0$

and

$$G(a^2/\gamma_i R\theta) = 0 \quad (C20)$$

for $0 < \frac{a^2}{\gamma_i R\theta} < 0.18$

Equation (C19) is also plotted in figure 33 to show the accuracy of this approximation. Consider first the case when G is given by equation (C19) which is written in the form

$$G(a^2/\gamma_i R\theta) = \xi a^2 + \mu + \eta \frac{1}{a^2} \quad (C21)$$

where, obviously,

$$\left. \begin{aligned} \xi &= 0.38/\gamma_i R\theta \\ \mu &= 0.71 \\ \eta &= -0.14 (\gamma_i R\theta) \end{aligned} \right\} \quad (C22)$$

Equation (C21) is substituted into equation (C15) and, with equations (C18) and (C6), the following expression results:

$$\frac{2\xi}{\gamma_i} \Phi_\omega^4 + \left(\frac{\gamma_i + 1}{\gamma_i - 1} + \frac{2\mu}{\gamma_i} \right) \Phi_\omega^2 + \Phi^2 + \frac{2\eta}{\gamma_i} - A^2 = 0 \quad (C23)$$

In order to simplify this equation the following substitutions are made:

$$D^2 = \left(\frac{\gamma_i + 1}{\gamma_i - 1} + \frac{2\mu}{\gamma_i} \right) + \frac{8}{\gamma_i} \xi \left(A^2 - \frac{2}{\gamma_i} \eta \right) \quad (C24)$$

$$\sin^2 \nu = \frac{\frac{8\xi}{\gamma_i} \left(A^2 - \frac{2}{\gamma_i} \eta \right)}{D^2} \quad (C25)$$

$$\sin^2 \tau = \frac{\frac{8\xi}{\gamma_i} \Phi^2}{D^2} \quad (C26)$$

and

$$\Phi_\omega^2 = \frac{D^2 \cos^2 \tau (\tau_\omega)^2}{\frac{8}{\gamma_i} \xi} \quad (C27)$$

Equation (C23) then reduces to

$$\cos^4 \tau (\tau_\omega)^4 + \frac{4 \cos \nu}{D} \cos^2 \tau (\tau_\omega)^2 + \frac{4 \sin^2 \tau}{D^2} - \frac{4 \sin^2 \nu}{D^2} = 0 \quad (C28)$$

This equation is solved for τ_ω , thus,

$$\tau_\omega = \sqrt{\frac{2}{D}} \frac{1}{\cos \tau} (\cos \tau - \cos \nu)^{1/2} \quad (C29)$$

or

$$d\omega = \sqrt{\frac{D}{2}} \frac{\cos \tau d\tau}{(\cos \tau - \cos \nu)^{1/2}} \quad (C30)$$

This expression is readily integrated to obtain the following equation relating ω to the local velocity:

$$\omega - \omega_N = \sqrt{D} \{ 2 [E(k, z) - E(k, z_N)] - [F(k, z) - F(k, z_N)] \} \quad (C31)$$

where

E elliptic integral of the second kind

F elliptic integral of the first kind

k $\sin \frac{\nu}{2}$ (modulus)

z $\sin^{-1} \left(\frac{\sin \tau/2}{\sin \nu/2} \right)$ (amplitude)

The procedure for calculating corresponding values of the pressure, p , and the deflection angle, δ , is straightforward with the aid of the preceding equations and may be summarized as follows:

1. Calculate A^2 (eq. (C9))
2. Calculate D^2 (eq. (C24))
3. Calculate ν (eq. (C25))
4. Assume a value of T , less than T_N
5. Calculate p (eqs. (16) and (17))
6. Calculate V^2 (eq. (C8))
7. Calculate γ (eq. (C14))
8. Calculate a^2 (or Φ_ω^2) (eq. (C11))
9. Calculate M and, in turn, β from V and a
10. Calculate u^2 (or Φ^2 (eq. (C7))
11. Calculate τ (eq. (C26))
12. Calculate ω (eq. (C31))
13. Calculate δ (eq. (C1))

This procedure is followed so long as the quantity $a^2/\gamma_i R\theta$ is greater than 0.18. (This is equivalent to the temperature being greater than approximately 1,000° R.) For values of $a^2/\gamma_i R\theta$ less than or equal to 0.18 (or temperatures less than about 1,000° R.), G is set equal to zero (see eq. (C20)) In this case, equation (C15) reduces to the same form as for an ideal gas and, therefore, the well-known ideal-gas relationships can be used.

APPENDIX D

EVALUATION OF $(1/K_w)(dP/dW)$, K_w/K_w , ψ , AND κ

For an ideal gas, the Mach number, Mach angle, shock-wave angle, and pressure ratio can be calculated at the leading edge, using the standard Rankine-Hugoniot shock relations and utilizing the free-stream Mach number and leading-edge deflection angle. With these flow parameters known, the only terms to be determined in equations (38), (39), (40), (43), and (44) are $(\frac{dP}{d\delta})_\sigma$ and $(\frac{d\delta}{d\sigma})_\sigma$. These derivatives are easily obtained from the shock relations. (See ref. 9.)

$$\left(\frac{dP}{d\delta}\right)_\sigma = \frac{(dP/d\sigma)_\sigma}{(d\delta/d\sigma)_\sigma} \quad (D1)$$

where

$$\left(\frac{dP}{d\sigma}\right)_\sigma = \frac{2\gamma M_0^2 \sin 2\sigma}{\gamma + 1} \quad (D2)$$

and

$$\left(\frac{d\delta}{d\sigma}\right)_\sigma = 1 - \frac{2(\gamma-1)}{\gamma+1} \cos^2(\sigma-\delta) + \sin 2(\sigma-\delta) \cot 2\sigma \quad (D3)$$

By again using the standard forms of the Rankine-Hugoniot shock relations, it is possible to transform equations (38), (39), and (43) (given in the analysis) into

$$\frac{1}{K_w} \frac{dP}{dW} = \frac{\gamma \tan \zeta}{\sin^2 \beta} \left[\frac{1 + \frac{1}{2} \left(\frac{\cos^2 \beta}{\cos^2 \zeta} + \frac{1}{M_0^2 \sin^2 \sigma} \right)}{\frac{\tan^2 \zeta}{\tan^2 \beta} + \frac{1}{2} \left(\frac{\cos^2 \beta}{\cos^2 \zeta} + \frac{1}{M_0^2 \sin^2 \sigma} \right)} \right] \times \left[\frac{2\gamma M_0^2 \sin^2 \sigma - (\gamma-1)}{\gamma+1} \right] \quad (D4)$$

for the surface pressure gradient,

$$\frac{K_\sigma}{K_w} = \frac{\gamma+1}{4 \cos \zeta} \left[\frac{1 - \frac{\tan^2 \zeta}{\tan^2 \beta}}{\frac{\tan^2 \zeta}{\tan^2 \beta} + \frac{1}{2} \left(\frac{\cos^2 \beta}{\cos^2 \zeta} + \frac{1}{M_0^2 \sin^2 \sigma} \right)} \right] \quad (D5)$$

for the shock-wave curvature, and

$$\psi = \left[\frac{1 + \frac{1}{2} \left(\frac{\cos^2 \beta}{\cos^2 \zeta} + \frac{1}{M_0^2 \sin^2 \sigma} \right)}{\frac{\tan^2 \zeta}{\tan^2 \beta} + \frac{1}{2} \left(\frac{\cos^2 \beta}{\cos^2 \zeta} + \frac{1}{M_0^2 \sin^2 \sigma} \right)} \right] \frac{\tan \zeta}{\tan \beta} \quad (D6)$$

for the surface-pressure-gradient ratio. These equations are similar to those given by Schaefer in reference 8 and require less work to compute than equations (38), (39), and (43).

For calorically-imperfect-gas flows, the standard shock relations are obviously not applicable. In this case, the flow parameters at the leading edge can be evaluated using the oblique-shock-wave equations previously presented. (See eqs. (14), (15), (18), (19), and (20).) Since the primary variable in these equations is temperature, the required

derivatives are most easily determined by employing the temperature as a parameter.¹⁸ Thus,

$$\left(\frac{dP}{d\delta}\right)_\sigma = \frac{(dP/dT)_\sigma}{(d\delta/dT)_\sigma} \quad (D7)$$

and

$$\left(\frac{d\delta}{d\sigma}\right)_\sigma = \frac{(d\delta/dT)_\sigma}{(d\sigma/dT)_\sigma} \quad (D8)$$

Differentiation of equation (18) yields

$$\left(\frac{dP}{dT}\right)_\sigma = \frac{P^2}{T_\sigma} \frac{T_0(P-1-\gamma_0 M_0^2) - M_\sigma T_\sigma^2 \left(M_\sigma \frac{d\gamma}{dT} + 2\gamma_\sigma \frac{dM}{dT} \right)}{T_\sigma + T_0 P^2} \quad (D9)$$

where from equation (15)

$$\frac{dM}{dT} = \left(\frac{-1}{\gamma_\sigma T_\sigma M_\sigma} \right) \left[\frac{M_\sigma^2 T_\sigma}{2} \frac{d\gamma}{dT} + \frac{\gamma_\sigma M_\sigma^2}{2} + e^{\theta/T_\sigma} \left(\frac{\theta/T_\sigma}{e^{\theta/T_\sigma} - 1} \right)^2 + \frac{\gamma_t}{\gamma_t - 1} \right] \quad (D10)$$

and from equation (14)

$$\frac{d\gamma}{dT} = \frac{(\gamma_t - \gamma)(\gamma - 1)}{T_\sigma(\gamma_t - 1)} \left(2 - \frac{\theta}{T_\sigma} \frac{e^{\theta/T_\sigma} + 1}{e^{\theta/T_\sigma} - 1} \right) \quad (D11)$$

The derivative $(\frac{d\delta}{dT})_\sigma$ can be evaluated from equation (20)

$$\left(\frac{d\delta}{dT}\right)_\sigma = \sin^2 \delta \left[\frac{\gamma_0 M_0^2 \tan \sigma \left(\frac{dP}{dT} \right)_\sigma}{(P-1)^2} - \left(\frac{\gamma_0 M_0^2}{P-1} - 1 \right) \sec^2 \sigma \left(\frac{d\sigma}{dT} \right)_\sigma \right] \quad (D12)$$

where from equation (19)

$$\left(\frac{d\sigma}{dT}\right)_\sigma = \frac{\csc 2\sigma \left(\frac{\gamma_\sigma M_\sigma^2 T_\sigma}{\gamma_0 M_0^2 T_0} \right) \left(\frac{2}{M_\sigma} \frac{dM}{dT} + \frac{1}{T_\sigma} + \frac{1}{\gamma_\sigma} \frac{d\gamma}{dT} \right) + \tan \sigma \frac{\rho_0^2}{\rho_\sigma^3} \left(\frac{d\rho}{dT} \right)_\sigma}{\left(\frac{\rho_0}{\rho_\sigma} \right)^2 - 1} \quad (D13)$$

and from equation (8)

$$\left(\frac{d\rho}{dT}\right)_\sigma = \frac{1}{T_\sigma} \left[\rho_0 T_0 \left(\frac{dP}{dT} \right)_\sigma - \rho_\sigma \right] \quad (D14)$$

The procedure for calculating $\frac{1}{K_w} \frac{dP}{dW}$, $\frac{K_\sigma}{K_w}$, ψ , and κ is straightforward with the aid of the preceding equations and may be summarized as follows:

1. For any M_0 , T_0 , δ_N , and K_w choose a value of T_σ/T_0 .

¹⁸ The forms of some of the derivatives presented have been somewhat simplified from the forms originally presented in reference 17.

(Fig. 6 or the ideal-gas relations will provide an initial estimate.)

2. Calculate γ_* (eq. (14))
3. Calculate M_* (eq. (15))
4. Calculate P_* (eq. (18))
5. Calculate ρ_0/ρ_* (eq. (8))
6. Calculate σ (eq. (19))
7. Calculate δ_* (eq. (20))

If this value of δ_* is not close enough to the desired value of δ_N , iterate, choosing a new T_*/T_0 .

8. Calculate $\frac{d\gamma}{dT}$ (eq. (D11))
9. Calculate $\frac{dM}{dT}$ (eq. (D10))
10. Calculate $\left(\frac{dP}{dT}\right)_*$ (eq. (D9))
11. Calculate $\left(\frac{d\rho}{dT}\right)_*$ (eq. (D14))
12. Calculate $\left(\frac{d\sigma}{dT}\right)_*$ (eq. (D13))
13. Calculate $\left(\frac{d\delta}{dT}\right)_*$ (eq. (D12))
14. Calculate $\left(\frac{dP}{d\delta}\right)_*$ (eq. (D7))
15. Calculate $\frac{\partial\delta/\partial C_1}{\partial\delta/\partial C_2}$ (eq. (40))
16. Calculate $\frac{1}{K_w} \frac{dP}{dW}$ (eq. (38))
17. Calculate ψ (eq. (43))
18. Calculate $\left(\frac{d\delta}{d\sigma}\right)_*$ (eq. (D8))
19. Calculate $\frac{K_\sigma}{K_w}$ (eq. (39))
20. Calculate κ (eq. (44))

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TABLE I.—FUNCTIONS FOR SLENDER-AIRFOIL METHOD

$M_0 \delta_N$	$\gamma = \gamma_i$		γ_a	$\gamma = \gamma_a$		$M_0 \delta_N$	$\gamma = \gamma_i$		γ_a	$\gamma = \gamma_a$	
	$g(M_0 \delta_N)$	$f(M_0 \delta_N)$		$g(M_0 \delta_N)$	$f(M_0 \delta_N)$		$g(M_0 \delta_N)$	$f(M_0 \delta_N)$		$g(M_0 \delta_N)$	$f(M_0 \delta_N)$
0	1.000	0	1.400	1.000	0						
.05	1.072	.009901	1.400	1.072	.009897	5.2	47.56	.3487	1.348	44.91	.3275
.10	1.148	.01961	1.400	1.148	.01959	5.4	51.13	.3506	1.347	48.23	.3292
.15	1.230	.02912	1.400	1.230	.02909	5.6	54.82	.3523	1.347	51.68	.3309
.20	1.316	.03845	1.400	1.316	.03841	5.8	58.66	.3539	1.346	55.26	.3323
.25	1.406	.04760	1.400	1.406	.04755	6.0	62.62	.3553	1.346	58.96	.3337
.30	1.502	.05656	1.399	1.502	.05649	6.2	66.73	.3566	1.346	62.79	.3350
.35	1.604	.06537	1.399	1.603	.06524	6.4	70.96	.3578	1.345	66.77	.3363
.40	1.710	.07393	1.399	1.710	.07380	6.6	75.33	.3589			
.45	1.823	.08235	1.399	1.822	.08220	6.8	79.83	.3600			
.50	1.941	.09058	1.399	1.940	.09040	7.0	84.47	.3610			
.55	2.065	.09863	1.399	2.064	.09841	7.2	89.24	.3618			
.60	2.195	.1071	1.399	2.194	.1062	7.4	94.14	.3626			
.65	2.332	.1142	1.399	2.330	.1139	7.6	99.19	.3633			
.70	2.474	.1217	1.399	2.473	.1213	7.8	104.4	.3640			
.75	2.624	.1290	1.398	2.622	.1286	8.0	109.7	.3647			
.80	2.780	.1362	1.398	2.777	.1356	8.5	123.5	.3661			
.85	2.943	.1431	1.398	2.939	.1425	9.0	138.2	.3673			
.90	3.112	.1499	1.398	3.108	.1492	9.5	153.8	.3684			
.95	3.289	.1565	1.398	3.284	.1557	10.0	170.2	.3693			
1.00	3.473	.1630	1.397	3.466	.1620	10.5	187.4	.3701			
1.1	3.862	.1753	1.396	3.852	.1740	11.0	205.4	.3707			
1.2	4.280	.1869	1.396	4.266	.1853	11.5	224.3	.3714			
1.3	4.728	.1978	1.395	4.708	.1958	12.0	244.1	.3719			
1.4	5.206	.2064	1.393	5.179	.2056	12.5	264.7	.3723			
1.5	5.715	.2178	1.392	5.679	.2147	13.0	286.1	.3727			
1.6	6.256	.2269	1.391	6.207	.2232	13.5	308.3	.3731			
1.7	6.827	.2354	1.389	6.764	.2309	14.0	331.4	.3735			
1.8	7.431	.2433	1.388	7.349	.2381	14.5	355.4	.3738			
1.9	8.066	.2507	1.386	7.962	.2447	15.0	380.2	.3740			
2.0	8.734	.2577	1.384	8.605	.2508	16.0	432.3	.3745			
2.1	9.411	.2646	1.382	9.274	.2564	17.0	487.7	.3749			
2.2	10.17	.2702	1.380	9.973	.2616	18.0	546.5	.3752			
2.3	10.93	.2759	1.378	10.70	.2664	19.0	608.6	.3755			
2.4	11.73	.2812	1.376	11.46	.2709	20.0	674.2	.3757			
2.5	12.56	.2862	1.374	12.24	.2749	22.0	815.3	.3761			
2.6	13.42	.2908	1.372	13.06	.2788	24.0	969.8	.3764			
2.7	14.32	.2951	1.370	13.90	.2823	26.0	1138	.3766			
2.8	15.25	.2992	1.369	14.77	.2856	28.0	1319	.3768			
2.9	16.21	.3030	1.367	15.68	.2887	30.0	1514	.3770			
3.0	17.21	.3066	1.365	16.61	.2916	35.0	2060	.3772			
3.2	19.30	.3130	1.362	18.56	.2968	40.0	2690	.3774			
3.4	21.52	.3187	1.360	20.64	.3015	45.0	3404	.3775			
3.6	23.88	.3237	1.358	22.83	.3057	50.0	4202	.3776			
3.8	26.37	.3280	1.356	25.16	.3094	60.0	6050	.3777			
4.0	29.00	.3322	1.354	27.60	.3128	70.0	8233	.3778			
4.2	31.76	.3358	1.352	30.17	.3158	80.0	10750	.3778			
4.4	34.65	.3389	1.351	32.86	.3186	90.0	13610	.3779			
4.6	37.68	.3418	1.350	35.68	.3211	100.0	16800	.3779			
4.8	40.84	.3443	1.349	38.63	.3234	∞	∞	.3780			
5.0	44.14	.3466	1.349	41.70	.3255						

TABLE II.—SURFACE PRESSURE GRADIENT, $\frac{1}{K_w} \frac{dP}{dW}$

M_0 δ_N	1.5	2.0	3.0	4.0	5.0	6.0	8.0	10.0	15.0	20.0	∞
0°	2.817	3.233	4.455	5.784	7.144	8.519	11.29	14.07	21.05	28.04	∞
2.0°	3.089	3.530	5.038	6.809	8.757	10.87	15.59	20.99	37.78	59.81	∞
5.0°	3.615	4.014	6.008	8.577	11.64	15.20	23.94	35.02	74.01	130.2	∞
7.5°	4.298	4.458	6.896	10.24	14.41	19.45	32.30	49.10	109.3	195.9	∞
10.0°	5.878	4.950	7.849	12.05	17.46	24.12	41.43	64.23	145.7	261.6	∞
15.0°		6.185	9.908	15.96	24.00	34.06	60.32	94.76	216.1	387.1	∞
20.0°		8.765	12.12	19.99	30.60	43.87	78.34	123.2	280.2	500.7	∞
25.0°			14.56	23.94	36.80	52.88	94.44	148.3	336.2	599.6	∞
30.0°			18.25	27.94	42.54	60.91	108.3	169.7	383.4	682.9	∞
35.0°				33.89	48.83	68.80	120.9	188.5	423.8	753.5	∞
40.0°					71.76	84.90	139.2	212.8	471.1	833.8	∞
45.0°								1727.	1711.		∞

TABLE III.—SURFACE-PRESSURE-GRADIENT RATIO, ψ

M_0 δ_N	1.5	2.0	3.0	4.0	5.0	6.0	8.0	10.0	15.0	20.0	∞
0°	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.8820
2.0°	1.000	1.000	1.000	1.000	.9999	.9998	.9996	.9993	.9976	.9948	.8820
5.0°	1.000	1.000	.9998	.9994	.9987	.9978	.9950	.9910	.9773	.9619	.8822
7.5°	1.001	1.000	.9995	.9981	.9961	.9935	.9864	.9777	.9550	.9362	.8825
10.0°	.9978	1.001	.9988	.9959	.9920	.9870	.9754	.9631	.9369	.9198	.8830
15.0°		1.006	.9970	.9895	.9807	.9715	.9538	.9392	.9158	.9041	.8846
20.0°		1.025	.9958	.9824	.9696	.9578	.9386	.9253	.9071	.8991	.8871
25.0°			.9997	.9783	.9623	.9493	.9309	.9194	.9052	.8994	.8911
30.0°			1.025	.9825	.9622	.9482	.9305	.9203	.9087	.9042	.8979
35.0°				1.016	.9787	.9603	.9407	.9307	.9199	.9158	.9104
40.0°					1.114	1.028	.9835	.9672	.9523	.9472	.9407
45.0°									1.737	1.325	1.188

TABLE IV.—SHOCK-WAVE CURVATURE, K_s/K_w

M_0 δ_N	1.5	2.0	3.0	4.0	5.0	6.0	8.0	10.0	15.0	20.0	∞
0°	0	0	0	0	0	0	0	0	0	0	0.8000
2.0°	.06597	.04547	.04805	.05767	.06891	.08080	.1054	.1305	.1931	.2537	.8003
5.0°	.1951	.1175	.1212	.1450	.1728	.2017	.2598	.3155	.4357	.5264	.8018
7.5°	.3698	.1827	.1827	.2170	.2566	.2969	.3733	.4406	.5653	.6417	.8042
10.0°	.7988	.2553	.2447	.2869	.3354	.3829	.4676	.5356	.6460	.7042	.8077
15.0°		.4487	.3704	.4183	.4737	.5244	.6044	.6597	.7345	.7677	.8181
20.0°		9.191	.5051	.5399	.5894	.6332	.6970	.7371	.7863	.8064	.8352
25.0°			.6756	.6634	.6956	.7271	.7723	.7997	.8318	.8445	.8621
30.0°			1.014	.8222	.8168	.8294	.8536	.8696	.8885	.8961	.9064
35.0°				1.168	1.021	.9890	.9774	.9779	.9818	.9840	.9875
40.0°					2.040	1.492	1.293	1.240	1.202	1.191	1.179
45.0°									7.668	3.829	2.811

TABLE V.—SHOCK-WAVE-CURVATURE RATIO, κ

M_0 δ_N	1.5	2.0	3.0	4.0	5.0	6.0	8.0	10.0	15.0	20.0	∞
0°	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.072
2.0°	.9997	.9998	1.000	1.000	1.001	1.001	1.002	1.003	1.007	1.012	1.072
5.0°	.9984	.9992	1.001	1.002	1.004	1.006	1.011	1.017	1.030	1.040	1.072
7.5°	.9970	.9983	1.002	1.005	1.009	1.013	1.021	1.029	1.044	1.053	1.071
10.0°		.9968	1.003	1.009	1.014	1.020	1.030	1.038	1.052	1.059	1.071
15.0°		.9912	1.005	1.015	1.024	1.031	1.042	1.049	1.059	1.063	1.069
20.0°		.9795	1.005	1.019	1.029	1.037	1.047	1.053	1.060	1.063	1.067
25.0°			1.000	1.019	1.031	1.039	1.048	1.053	1.058	1.060	1.063
30.0°			.9830	1.013	1.027	1.035	1.044	1.048	1.053	1.055	1.057
35.0°				.9909	1.012	1.023	1.034	1.039	1.044	1.045	1.048
40.0°					.9587	.9879	1.008	1.016	1.023	1.025	1.028
45.0°									.8854	.9242	.9475

TABLE VI.—SURFACE PRESSURE GRADIENT, SURFACE-PRESSURE-GRADIENT RATIO, SHOCK-WAVE CURVATURE, AND SHOCK-WAVE-CURVATURE RATIO FOR A CALORICALLY IMPERFECT, DIATOMIC GAS
[$T_0=500^\circ$ Rankine]

M_0	δ°	T_s/T_0	$\frac{1}{K_w} \frac{dP}{dW}$	ψ	K_s/K_w	κ
3	30.48	2.01200	18.36	1.026	1.021	0.9828
5	8.042	1.32750	15.04	.9954	.2732	1.010
5	15.36	1.75357	24.38	.9796	.4773	1.025
5	20.48	2.13800	30.94	.9675	.5866	1.031
5	26.01	2.62500	37.39	.9577	.6908	1.034
5	31.09	3.12700	42.63	.9549	.7986	1.031
5	37.30	3.81250	49.50	.9725	1.041	1.015
5	41.01	4.30000	60.76	1.042	1.571	.9827
10	10.01	2.10800	63.86	.9621	.5265	1.040
10	10.38	2.17000	66.07	.9598	.5373	1.041
10	20.52	4.40000	123.6	.9112	.7132	1.062
10	27.61	6.45000	156.2	.8932	.7922	1.065
10	31.27	7.64000	169.8	.8886	.8363	1.064
10	40.18	* 10.87500	195.0	.8969	1.018	1.050
10	42.98	* 12.00000	204.1	.9128	1.151	1.039
15	5.230	1.77500	76.83	.9750	.4454	1.032
15	10.49	3.25000	150.5	.9283	.6363	1.057
15	16.31	5.60000	229.5	.8936	.7250	1.070
15	22.26	8.70000	301.4	.8733	.7834	1.074
15	28.65	* 12.75000	364.5	.8630	.8391	1.074
20	5.122	2.15000	132.4	.9594	.5242	1.042
20	10.67	4.70000	274.2	.9027	.6923	1.068
20	16.53	8.70000	417.0	.8706	.7648	1.077
20	20.98	* 12.60000	514.4	.8579	.8011	1.079
20	32.51	* 25.80000	705.0	.8490	.8914	1.075
20	39.15	* 35.00000	771.1	.8571	.9861	1.066

* Values of the temperature greater than 5,000° R. downstream of the shock wave.

REPORT 1124

DISPLACEMENT EFFECT OF A THREE-DIMENSIONAL BOUNDARY LAYER¹

By FRANKLIN K. MOORE

SUMMARY

A method is described for determining the "displacement surface" of a known three-dimensional compressible boundary-layer flow in terms of the mass-flow defects associated with the profiles of the two velocity components parallel to the surface. The result is a generalization of the plane flow concept of displacement thickness introduced in order to describe how a thin boundary layer distorts the outer nonviscous flow.

The height of the displacement surface above the body surface for flow about a yawed infinite cylinder is shown to be equal to the height characterizing the mass-flow defect of the chordwise velocity profile. The displacement-surface height is shown to differ, in general, from that associated with the resultant mass-flow defect, even at stagnation points of the secondary flow. Numerical values are found for the known three-dimensional boundary-layer flow about a cone at a small angle of attack to a supersonic stream.

INTRODUCTION

The boundary layer established in the flow of a slightly viscous fluid about a body is normally considered an isolated region wherein the effects of viscosity predominate and outside of which the motion of the fluid is governed by the laws of nonviscous motion. For large Reynolds numbers, the boundary layer is assumed to be so thin that the nonviscous portion of the flow occurs as though there were no boundary layer. This assumption is strictly correct in the limit of infinite Reynolds number. For large but finite Reynolds numbers, the growth of the boundary layer causes the stream to be deflected away from the body surface.

This displacement effect of the boundary layer on the nonviscous flow may properly be determined from the behavior of the boundary layer itself, as established either by experiment or by solution of the Prandtl boundary-layer equations for laminar flow.

It does not follow, however, that this revised outer flow may properly be used in conjunction with the Prandtl equations to yield an improvement in the boundary-layer calculation. Such an improvement may be obtained only by use of a new set of equations that take into account the variation of pressure across the boundary layer. This variation is neglected in the Prandtl equations. (See Alden's iterative solution for incompressible flat-plate flow, ref. 1.)

The customary definition of displacement thickness (ref. 2) is applicable to two-dimensional flow and is expressed in the following equation:

$$\int_0^h \rho u \, dy = \rho_1 u_1 (h - \delta^*) \quad (1)$$

where h is some location well outside the boundary layer (fig. 1) at which $\rho u = \rho_1 u_1$ and beyond which the flow may be considered nonviscous. (A list of symbols is provided in the appendix.) Under the assumption of an extremely thin boundary layer, h is so small that $\rho_1 u_1$ may be taken as the evaluation at the body surface ($y=0$) of the nonviscous flow obtained by neglecting the presence of the boundary layer. Equation (1) equates the actual mass flow near the surface with the mass flow which would be associated with a nonviscous flow that terminates at δ^* rather than extending to the wall. Thus, the nonviscous portion of the flow behaves as if it occurred in the presence of a solid boundary given by the displacement thickness $\delta^*(x)$. Equation (1) may be solved for δ^* :

$$\delta^* = \int_0^h \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy$$

Ordinarily, theoretical boundary-layer solutions for $\rho u / \rho_1 u_1$ asymptotically approach 1 for large values of Reynolds number based on y . Therefore, displacement thickness is often defined as follows:

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy$$

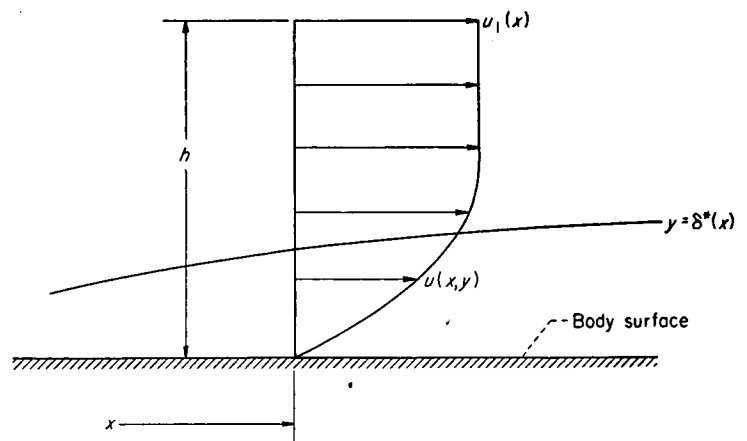


FIGURE 1.—Plane boundary-layer flow.

¹ Supersedes NACA TN 2722, "Displacement Effect of a Three-Dimensional Boundary Layer," by Franklin K. Moore, 1952.

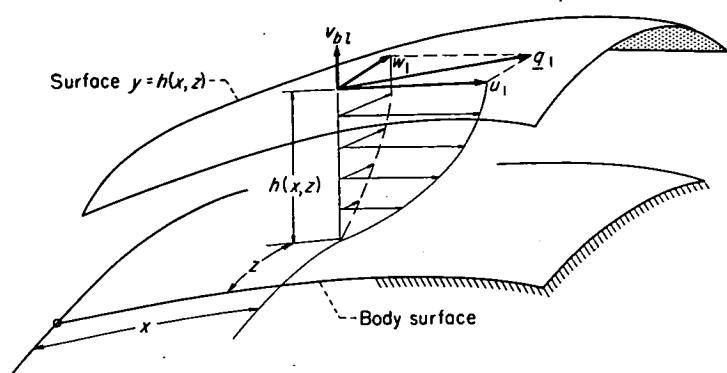


FIGURE 2.—Three-dimensional boundary-layer flow (Cartesian coordinates).

For three-dimensional boundary-layer flows, two lengths characterizing mass-flow defects may be defined in terms of the profiles of the two velocity components tangential to the surface (fig. 2),

$$\begin{aligned}\delta_x &\equiv \int_0^h \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy \\ \delta_z &\equiv \int_0^h \left(1 - \frac{\rho w}{\rho_1 w_1}\right) dy\end{aligned}\quad (2)$$

and it is not clear which, if either, defines a "displacement surface" that properly describes the extent to which the nonviscous flow is deflected by the boundary layer.

Of course, it is expected that there does exist a displacement surface for such flows. The analysis that follows shows that such a surface may be described by a defining equation more fundamental than equation (1).

The velocity and density profiles are assumed to be known for the three-dimensional flows under consideration. Cases for which this is true include: the laminar boundary layer on yawed infinite cylinders, treated by Prandtl (ref. 3), R. T. Jones (ref. 4), and Sears (ref. 5); and the laminar boundary layer on a cone at a small angle of attack to a supersonic stream (ref. 6). The displacement effect of these flows will be treated specifically.

The investigation was conducted at the NACA Lewis laboratory in February 1952.

THEORY

DEFINING RELATION FOR DISPLACEMENT SURFACE

The boundary-layer solution (assumed known) yields a certain distribution of velocity $v_{bl}(x, z)$ normal to the body surface at the outer edge $h(x, z)$ of the boundary layer, where ρ, u , and w may be taken essentially equal to ρ_1, u_1 , and w_1 (see fig. 2). Under the assumption that the nonviscous flow is altered only slightly by displacement, the most direct way to compute this effect would be to suppose that the nonviscous equations hold for $y \geq h(x, z)$ and to impose the following boundary condition on the normal velocity v_{of} in the outer flow:

$$v_{of} = v_{bl}(x, z) \text{ at } y = h(x, z) \quad (3)$$

However, since the boundary conditions usually encountered in nonviscous flow specify an impermeable surface, it is convenient to recast the boundary condition given in equation (3) in answer to the question: What impermeable surface $y = \Delta(x, z)$ would deflect a nonviscous fluid in such a way as to produce a normal velocity satisfying condition (3)? This fictitious surface may be called the displacement surface.

Because $\Delta(x, z)$ is imagined to be a stream surface, it is necessary to specify that at $y = \Delta$ the resultant velocity vector (u, v, w) be tangent to the surface $y = \Delta$. Thus, at $y = \Delta$ the ratio of v to the magnitude of $\underline{q}$, which is defined as the vector (u, w) , must be set equal to the slope of the surface $y = \Delta$, measured in the direction of the vector $\underline{q}$, or, equivalently, equal to the component in the direction of $\underline{q}$ of the vector $\text{grad } \Delta$. In vector notation, therefore, the normal velocity v which would be produced in a nonviscous fluid at an impermeable surface $y = \Delta$ is $\underline{q} \cdot \text{grad } \Delta$. The vector $\underline{q}$ may be obtained by evaluating the velocity vector of the unrevised nonviscous flow at the body surface ($y = 0$), under the related assumptions that the velocity vector varies only slightly over distances of the order of the actual boundary-layer thickness and that the revision required to take account of displacement is slight. Thus, at $y = \Delta$ (see fig. 3),

$$v_{of} = \underline{q}_1 \cdot \text{grad } \Delta$$

The increment in v_{of} between Δ and h is approximately $(h - \Delta) \partial v_{of} / \partial y$; again, a thin boundary layer is assumed and only the first term in a Taylor's series is used.

To the order of approximation contemplated in this analysis, $\partial v_{of} / \partial y$ may be obtained from the unrevised nonviscous flow evaluated at $y = 0$. Thus, the fictitious impermeable surface Δ would produce, at $y = h$, a normal velocity (see fig. 3)

$$v_{of} = \underline{q}_1 \cdot \text{grad } \Delta + (h - \Delta) \left(\frac{\partial v_{of}}{\partial y} \right)_{y=0}$$

Introducing this result into the boundary condition given in equation (3) yields the defining relation for $\Delta(x, z)$

$$(v_{bl})_{y=h} = (h - \Delta) \left(\frac{\partial v_{of}}{\partial y} \right)_{y=0} + \underline{q}_1 \cdot \text{grad } \Delta \quad (4)$$

The boundary-layer solution yields v_{bl} .

EXPRESSION OF Δ IN TERMS OF MASS-FLOW DEFECTS

The displacement surface Δ may be related to the mass-flow defects (eqs. (2)), which characterize the boundary layer, as follows: In a Cartesian coordinate system (fig. 2), the equation of continuity for both the boundary-layer and nonviscous flow is

$$\frac{\partial \rho v}{\partial y} = - \frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z} \quad (5)$$

Under the Prandtl boundary-layer assumptions, the Cartesian equations of motion, and hence equation (5), may be applied in an orthogonal curvilinear coordinate system in which the

surface of the body is given by $y=0$, provided the radius of curvature of the body is everywhere large as compared with the boundary-layer thickness. Integrating equation (5) across the boundary layer yields

$$\begin{aligned} (\rho v_{bl})_{y=h} &= - \int_0^h \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} \right) dy \\ &= \int_0^h \left[\frac{\partial}{\partial x} (\rho_1 u_1 - \rho u) + \frac{\partial}{\partial z} (\rho_1 w_1 - \rho w) \right] dy - \\ &\quad \int_0^h \left(\frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 w_1}{\partial z} \right) dy \end{aligned}$$

or, inasmuch as h is outside the boundary layer where $\rho u = \rho_1 u_1$ and $\rho w = \rho_1 w_1$,

$$\rho_1 (v_{bl})_{y=h} = -h \left(\frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 w_1}{\partial z} \right) + \frac{\partial}{\partial x} (\rho_1 u_1 \delta_x) + \frac{\partial}{\partial z} (\rho_1 w_1 \delta_z) \quad (6)$$

where δ_x and δ_z are the mass-flow defects defined in equations (2). For the nonviscous flow, $v_{or}=0$ at the body surface ($y=0$), and equation (5) becomes

$$\rho_1 \left(\frac{\partial v_{or}}{\partial y} \right)_{y=0} = - \left(\frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 w_1}{\partial z} \right) \quad (7)$$

Introducing equations (6) and (7) into equation (4) yields

$$\frac{\partial}{\partial x} [\rho_1 u_1 (\Delta - \delta_x)] + \frac{\partial}{\partial z} [\rho_1 w_1 (\Delta - \delta_z)] = 0 \quad (8a)$$

In a Cartesian coordinate system, the displacement surface is related to the mass-flow defects by equation (8a). When cases arise for which other coordinate systems must be used, the following generalization in vector form of equation (8a) may be used:

$$\text{div} \left[\rho_1 \underline{q}_1 \Delta - \int_0^h (\rho_1 \underline{q}_1 - \rho \underline{q}) dy \right] = 0 \quad (8b)$$

where y is the distance normal to the body surface and where the divergence operator involves differentiation only with respect to the two coordinates parallel to the body surface.

EXAMPLES

Plane flow.—Equation (8a) may be integrated to yield, for plane flow ($\partial/\partial z=0$),

$$\Delta = \delta_x + \frac{K}{\rho_1 u_1} \quad (9)$$

where K is a constant of integration.

The appearance of this constant means, in general, that the revised boundary condition on the nonviscous flow near the wall may be applied at any surface in the boundary-layer region; for example, along the wall itself. Of course, if there is a stagnation point on the body where u_1 vanishes, then K must be taken equal to zero and the revised boundary

condition must be imposed at the location $\Delta = \delta_x$, at least near the stagnation point.

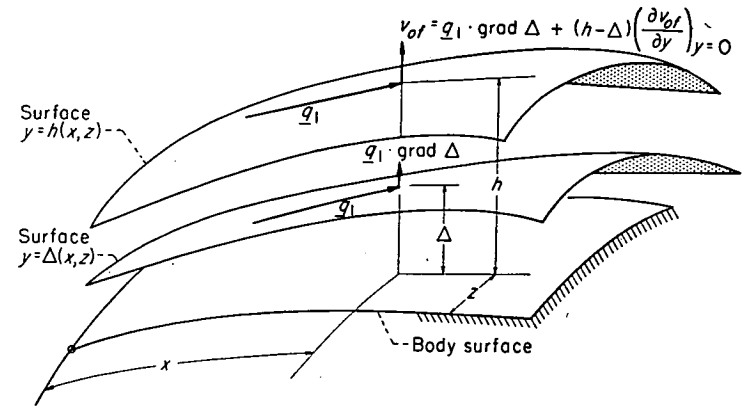


FIGURE 3.—Displacement surface in three-dimensional flow (Cartesian coordinates).

Stagnation point of secondary flow.—In the vertical plane of symmetry of the flow about a body of revolution at an angle of attack, the circumferential velocity component vanishes in the boundary layer as well as in the nonviscous outer region. Such stagnation of the secondary motion would occur in a variety of cases, in particular, wherever an essentially three-dimensional flow has a plane of symmetry. When w_1 is designated as the component of secondary flow (circumferential velocity for a body of revolution) at the outer edge of the boundary layer, and equation (8a) is written in the form

$$\frac{\partial}{\partial x} [\rho_1 u_1 (\Delta - \delta_x)] + w_1 \frac{\partial}{\partial z} [\rho_1 (\Delta - \delta_z)] + \rho_1 (\Delta - \delta_z) \frac{\partial w_1}{\partial z} = 0 \quad (10)$$

it is clear that Δ cannot equal δ_z , in general, even if both w_1 and the boundary-layer profile of w vanish, unless $\partial w_1/\partial z$ also vanishes.

Flow about yawed infinite cylinder.—In the flow about a yawed infinite cylinder, there is a spanwise boundary-layer velocity profile and an associated spanwise mass-flow defect. (See refs. 3 to 5.) If x is taken in the chordwise direction (fig. 4), the entire flow depends only on x . Thus, derivatives with respect to the spanwise coordinate z vanish; and, hence, from equation (8a), the plane-flow result (eq. (9)) applies. Accordingly, the spanwise mass-flow defect represented by δ_z does not enter into the determination of the displacement surface.

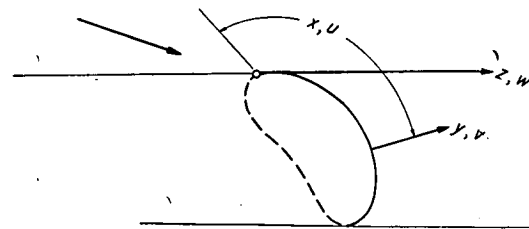


FIGURE 4.—Coordinate system for yawed infinite cylinder.

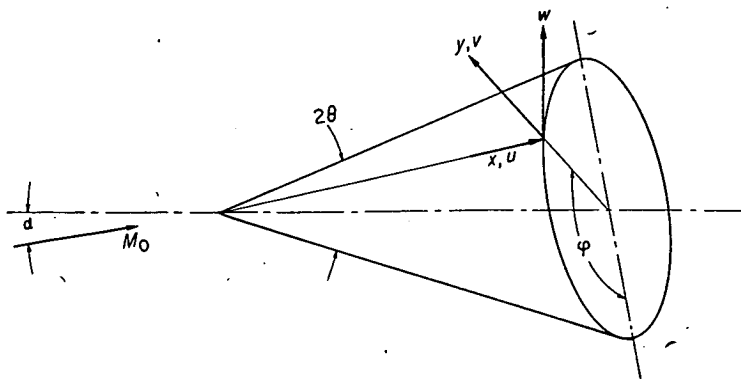


FIGURE 5.—Coordinate system for cone at angle of attack.

Supersonic flow about cone at small angle of attack.—When the coordinate system shown in figure 5 is used, equation (8b) becomes

$$\sin \theta \frac{\partial}{\partial x} [\rho_1 u_1 x (\Delta - \delta_x)] + \frac{\partial}{\partial \varphi} [\rho_1 w_1 (\Delta - \delta_\varphi)] = 0 \quad (11)$$

where

$$\begin{aligned} \delta_x &= \int_0^h \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy \\ \delta_\varphi &= \int_0^h \left(1 - \frac{\rho w}{\rho_1 w_1}\right) dy \end{aligned} \quad (12)$$

Because the outer flow is conical, ρ_1 , u_1 , and w_1 are functions only of φ ; and equation (11) may be written

$$\rho_1 u_1 \sin \theta \frac{\partial}{\partial x} [x (\Delta - \delta_x)] + \frac{\partial}{\partial \varphi} [\rho_1 w_1 (\Delta - \delta_\varphi)] = 0 \quad (13)$$

In the case of conical outer flow, the associated boundary-layer profiles show similarity of the Blasius type in meridional planes (see ref. 7 or 8). Thus, in a meridional plane, Δ , δ_x , and δ are proportional to $\sqrt{x}$. Incorporation of this information into equation (13) gives

$$\frac{3}{2} \rho_1 u_1 \sin \theta (\Delta - \delta_x) + \frac{\partial}{\partial \varphi} [\rho_1 w_1 (\Delta - \delta_\varphi)] = 0 \quad (14)$$

For a cone at small angle of attack α , u_1 is nearly equal to $\bar{u}$, the velocity on the cone surface at zero angle of attack. The quantities ρ_1 , Δ , δ_x , and δ_φ vary only slightly with angle of attack, whereas

$$w_1 = \alpha \bar{u} A_2 \sin \varphi \quad (15)$$

where A_2 depends only on the cone vertex angle and the flow Mach number and is defined in reference 6 as follows:

$$A_2 = -\frac{z}{\bar{u}} - \frac{2x}{\bar{u} \sin \theta}$$

The quantities z , x , and $\bar{u}$ are in the notation of reference 9, wherein they are tabulated.

To the first order in α , the substitution of equation (15) into equation (14) yields the result

$$\Delta = \delta_x + \frac{2}{3} \frac{\alpha}{\sin \theta} A_2 (\delta_\varphi - \delta_x) \cos \varphi \quad (16)$$

The analysis of reference 6 yields the values of δ_x and δ_φ .

Clearly, Δ differs from δ_x in the plane of symmetry $\varphi = 0, \pi$, where the circumferential velocity w vanishes. It might,

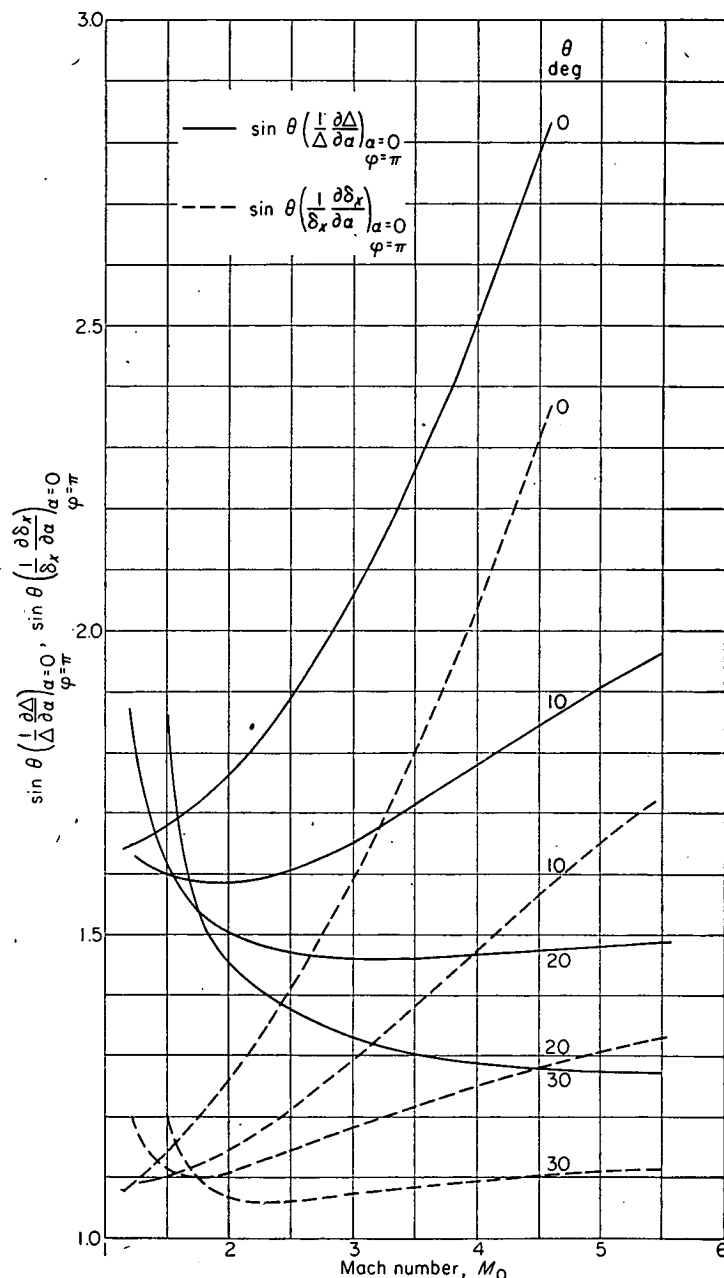


FIGURE 6.—Proportional rate of increase of displacement thickness with angle of attack on cone.

however, be noted that in the plane $\varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \Delta = \delta_z$. Figure 6 shows the proportional rate of increase of displacement thickness Δ with angle of attack in the plane of symmetry $\varphi = \pi$ at zero angle of attack. The corresponding rate of increase of the mass-flow defect δ_z is shown for comparison. These curves are obtained from equation (16) and the results of reference 6. The sine of the semivertex angle is introduced as a factor primarily to permit presentation of the case $\theta = 0$ as the limit of an indeterminate form. For a stream Mach number of 2, the change in displacement thickness appears to be of the order of 50 percent larger than the change in the meridional mass-flow defect.

CONCLUDING REMARKS

The foregoing analysis deals only with the displacement effect of a known boundary layer on the nonviscous outer flow, and hence can be applied only if the boundary-layer behavior has been determined either theoretically or experimentally. The latter approach might possibly find application in the correction of nozzle contours for boundary-layer development. In this connection, perhaps, it should be noted that the analysis is not restricted to laminar flows.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, March 6, 1952

APPENDIX—SYMBOLS

The following symbols are used in this report:

A_2	function of cone angle and Mach number (eq. (15))
h	height above body surface at which $\rho, u, w = \rho_1, u_1, w_1$, and beyond which nonviscous equations apply (eq. (3))
M_o	stream Mach number
q	velocity vector composed of components parallel to body surface u, w
u	velocity component in x -direction
$\bar{u}$	meridional velocity component at surface of cone at zero angle of attack
v	velocity component in y -direction
v_{bl}	boundary-layer solution for velocity normal to surface, evaluated at outer edge of boundary layer
v_{of}	nonviscous solution for velocity normal to surface, evaluated near the surface
w	velocity component in z -direction

x	coordinates in body surface
z	
y	coordinate normal to surface
α	angle of attack
Δ	height above body of displacement surface
δ^*	displacement thickness in plane flow
δ_x	length characterizing mass-flow defect of u -profile (eqs. (2) and (12))
δ_z	length characterizing mass-flow defect of w -profile (eq. (12))
δ_φ	length characterizing mass-flow defect of w -profile (eq. (12))
θ	semivertex angle of cone
ρ	density
φ	angular coordinate (fig. 5)

Subscript 1 denotes evaluation of nonviscous flow at body surface, taken equivalent to conditions at outer edge of boundary layer of infinitesimal thickness.

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REPORT 1125

DYNAMICS OF MECHANICAL FEEDBACK-TYPE HYDRAULIC SERVOMOTORS UNDER INERTIA LOADS¹

By HAROLD GOLD, EDWARD W. OTTO, and VICTOR L. RANSOM

SUMMARY

An analysis of the dynamics of mechanical feedback-type hydraulic servomotors under inertia loads is developed and experimental verification is presented. This analysis, which is developed in terms of two physical parameters, yields direct expressions for the following dynamic responses: (1) the transient response to a step input and the maximum cylinder pressure during the transient and (2) the variation of amplitude attenuation and phase shift with the frequency of a sinusoidally varying input. The validity of the analysis is demonstrated by means of recorded transient and frequency responses obtained on two servomotors. These data, which were obtained over a wide range of inertia loads, input magnitudes, and pressure differentials, are presented along with the analytically determined responses. In all cases the calculated responses are in close agreement with the measured responses. The relations presented are readily applicable to the design as well as to the analysis of hydraulic servomotors.

INTRODUCTION

The servomotor dealt with in this paper is a power-amplifying, positioning device of the type used in such applications as control-valve positioners, gun-turret positioners, flight controls, and power-steering devices. The hydraulic servomotor as a device has been known for approximately 100 years. Its application to high-speed machinery, however, appears to be relatively recent. There is, consequently, very little published literature on the dynamics of this servomotor in spite of its long history. Nevertheless, when properly designed, the hydraulic servomotor is particularly suited for high-speed service because of the extremely high force-mass ratios that can be obtained and because the device inherently is heavily damped.

A differential equation for the response to a step input of the hydraulic servomotor with mechanical feedback under an inertia load is available in the literature (ref. 1). This equation (a form of which is derived in the present paper) can be considered to be exact over a fairly representative portion of the response but is not valid in the early part of the transient. Furthermore, under a heavy inertia load the fluid on the driving side of the piston may cavitate, in which case the response cannot be described by a single equation. It is therefore necessary to treat the response of the servomotor in distinct phases.

The basic technique employed in this paper in the analysis of the servomotor is the approximation by one or more linear systems whose individual responses match the behavior of the actual system in definable phases of the response. The several linear systems are then correlated by relating each to the same physical parameters of the system. In this instance, two parameters are all that are required for the correlations. One of these parameters is a direct function of the dimensions of the servomotor and the hydraulic pressure drop across the motor. The second parameter is a function of the magnitude of the disturbance and the mass of the load. By means of this method, analytical expressions are obtained for the following dynamic responses of the servomotor: (1) the transient response to a step input and the maximum cylinder pressure during the transient and (2) the variation of amplitude attenuation and phase shift with the frequency of a sinusoidally varying input.

The validity of the analysis is demonstrated by means of recorded transient and frequency responses that were obtained on both a straight-line and a rotary type of servomotor. These data, which were obtained over a wide range of inertia loads, input magnitude, and pressure differential, are presented along with the analytically determined responses. The investigation was conducted at the NACA Lewis laboratory.

SYMBOLS

The following symbols are used in this analysis:

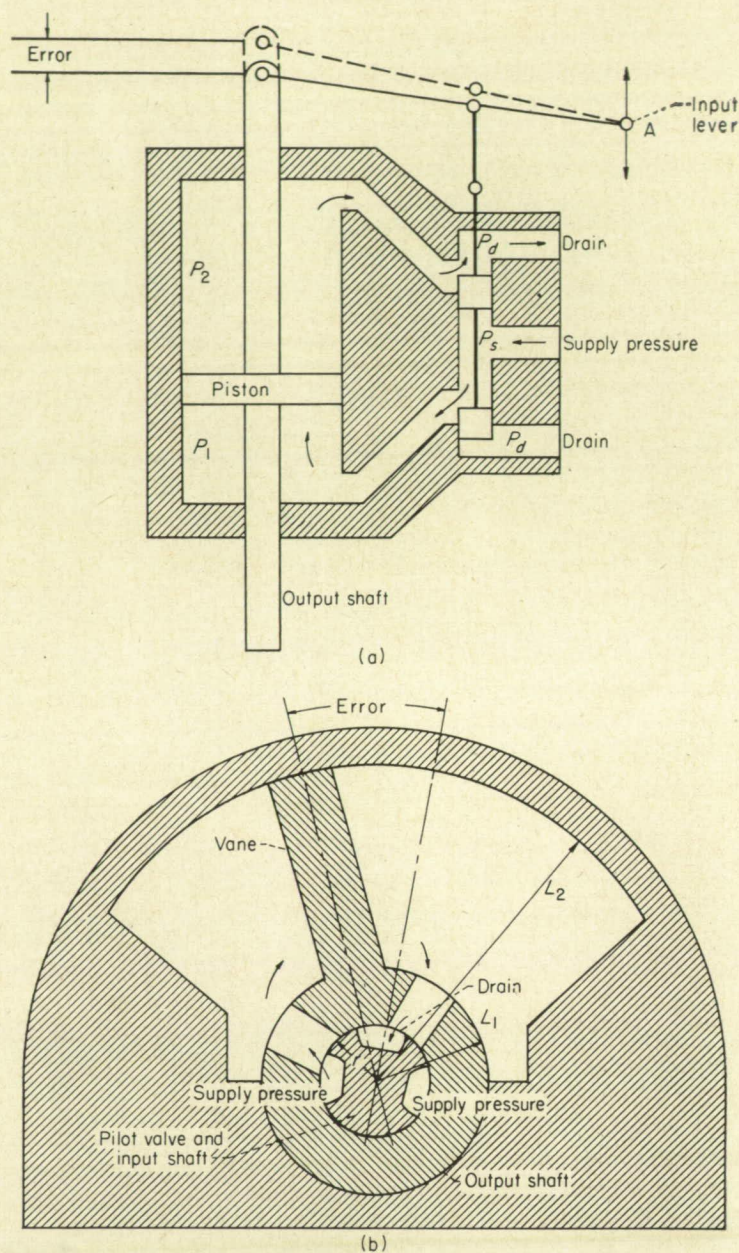
A	ratio of output amplitude at a given frequency to output amplitude at zero frequency
A_p	piston area, sq in.
A_v	open area of pilot valve (inlet or discharge side), sq in.
a	constant
b	constant
C	dimensional constant in fluid-flow equation ($95.1 \frac{\text{sq in.}}{\text{sec} \sqrt{\text{lb}}}$ based on specific gravity of 0.851 and flow coefficient of 0.59)
c	constant
D	constant
E	inertia index (transient response)
E'	inertia index (frequency response)
F, F_1, F_2	functions

¹ Supersedes NACA TN 2767, "Dynamics of Mechanical Feedback-Type Hydraulic Servomotors Under Inertia Loads" by Harold Gold, Edward W. Otto, and Victor L. Ransom, 1952.

f_1	low-frequency-band break frequency, cps
f_2	high-frequency-band break frequency, cps
f_3	cross-over frequency, cps
H	constant
h	width of vane, in.
i	$\sqrt{-1}$
J	polar moment of inertia, (lb-in.) (sec ²)/radians
L_1	inner vane radius of rotary servomotor, in.
L_2	outer vane radius of rotary servomotor, in.
M	load mass, (lb) (sec ²)/in.
P_1	upstream cylinder pressure, lb/sq in. abs
P_2	downstream cylinder pressure, lb/sq in. abs
P_d	drain pressure, lb/sq in. abs
P_s	supply pressure, lb/sq in. abs
ΔP_p	pressure drop across piston, lb/sq in.
ΔP_v	valve pressure drop, lb/sq in.
$\Delta P_{v,d}$	discharge-valve pressure drop, lb/sq in.
$\Delta P_{v,i}$	inlet-valve pressure drop, lb/sq in.
Q	shaft torque, lb-in.
q	flow through valve, cu in./sec
R	ratio of valve travel to piston travel at fixed input, in./in.
r	ratio of valve travel to vane shaft rotation at fixed input, in./radians
S	magnitude of step (measured at output), in.
S'	amplitude of output sine wave at zero frequency, in.
T	no-load time constant, sec
t	time from start of transient, sec
t_1	value of t at inflection point of transient, sec
t_2, t_3	value of t at phase limits in transient, sec
W	width of valve port (measured perpendicular to line of valve travel), in.
x	instantaneous position of output measured from position at $t=0$, in.
x_1	value of x at inflection point of transient response, in.
x_2, x_3	value of x at phase limits in transient, in.
x_m	value of x at point of maximum deceleration in transient response, in.
α	instantaneous position of output measured from position at $t=0$, radians
θ	magnitude of step (measured at output), radians
θ'	amplitude of output sine wave at zero frequency, radians
φ	phase shift, radians
ω	angular frequency, radians/sec
ω_1	low-frequency-band break frequency, radians/sec
ω_2	high-frequency-band break frequency, radians/sec
ω_3	cross-over frequency, radians/sec

DEFINITIONS AND INITIAL ASSUMPTIONS

Straight-line servomotor.—The elements of the straight-line hydraulic servomotor are shown schematically in figure 1(a). In the neutral position, the spool member of the pilot valve closes the passages to the piston. When the spool member is displaced from the neutral position by movement of the input lever at point A, the flow of fluid through the



(a) Straight-line servomotor.

(b) Rotary servomotor.

FIGURE 1.—Schematic drawings of two types of hydraulic servomotor with mechanical feedback.

pilot valve causes the piston to move in the direction which returns the spool to the neutral position. It follows from the geometry of the linkage that for every position of the linkage point A there is a corresponding equilibrium position of the piston. The description of several other forms of pilot valving and feedback linkage is available in the literature.

Rotary servomotor.—The rotary servomotor is shown schematically in figure 1(b). Rotation of the pilot valve with respect to the output shaft opens a pressure passage to one side of the vane and a drain passage to the opposite side of the vane. The vane is thereby caused to rotate in the same direction as the pilot valve. In the neutral position of the valve the passages to either side of the vane are closed.

Initial assumptions.—The analysis which follows is developed with the following initial assumptions:

- (1) The area of opening of the pilot valve varies linearly with the motion of the load.
- (2) At all positions of the pilot valve the inlet and discharge openings are equal.
- (3) At fixed input, the ratio of pilot-valve travel to piston travel is constant.
- (4) The supply and drain pressures are constant.
- (5) Structure and linkage are rigid.
- (6) The compressibility and mass of the hydraulic fluid are negligible.
- (7) Mechanical friction forces are negligible.
- (8) Leakage is negligible.
- (9) Fluid friction losses in the motor passages are negligible.

TRANSIENT RESPONSE TO A STEP INPUT

The transient response is analyzed for the no-load case as well as for the inertia-load case. The analysis of the response at no load yields an important parameter used in the analysis of the response under an inertia load.

NO-LOAD RESPONSE

Basic character of response.—Under the conditions of zero load on the output shaft and negligible piston and shaft mass, the pressure drop across the piston will be zero during the transient as well as in steady state. In the transient state, therefore, the fluid flow through the cylinder is essentially unobstructed. On the basis of the initial assumptions and on the further assumption of constant flow coefficient of the pilot valve, the flow of fluid is then proportional to the valve opening and hence proportional to the position error of the piston. The velocity of the piston is therefore proportional to the error. This relation between the piston velocity and the error may be expressed by the following equation:

$$T\dot{x} = (S - x) \quad (1)$$

The solution of equation (1) is:

$$x = S \left(1 - e^{-\frac{t}{T}} \right) \quad (2)$$

In the no-load case the transient response is therefore defined by the time constant T .

Determination of time constant from servomotor dimensions (straight-line servomotor).—In the no-load case the sum of the pressure drops across the inlet and discharge ports is equal to the pressure difference across the servomotor. From the initial assumptions it therefore follows that the pressure drops across the two valves are the same and hence equal to half the pressure difference across the servomotor:

$$\Delta P_v = \frac{P_s - P_d}{2} \quad (3)$$

If the flow coefficient of the pilot valve is considered constant, the rate of fluid flow into the cylinder is given by the

relation

$$q = C A_v \sqrt{\frac{P_s - P_d}{2}} \quad (4)$$

The area of opening of the valves is proportional to the error and may be written

$$A_v = (S - x) R W \quad (5)$$

The velocity of the piston is determined by the flow rate through the valves and is related by the following expression:

$$A_p \dot{x} = q \quad (6)$$

Equations (4), (5), and (6) may be combined to form the differential equation of the response

$$A_p \dot{x} = \left(C R W \sqrt{\frac{P_s - P_d}{2}} \right) (S - x) \quad (7)$$

Equation (7) is of the same form as equation (1), from which it follows that

$$T = \frac{\sqrt{2} A_p}{C R W \sqrt{P_s - P_d}} \quad (8)$$

Determination of time constant from motor dimensions (rotary servomotor).—The area of opening of the valves as a function of the error may be written

$$A_v = (\theta - \alpha) r W \quad (9)$$

The angular velocity of the output shaft may be related to the flow rate through the valves by the following expression:

$$\frac{h}{2} (L_2^2 - L_1^2) \dot{\alpha} = q \quad (10)$$

Equations (4), (9), and (10) may be combined to form the differential equation of the response

$$\frac{h}{2} (L_2^2 - L_1^2) \dot{\alpha} = \left(C r W \sqrt{\frac{P_s - P_d}{2}} \right) (\theta - \alpha) \quad (11)$$

From equation (11) the time constant is

$$T = \frac{h (L_2^2 - L_1^2)}{\sqrt{2} C r W \sqrt{P_s - P_d}} \quad (12)$$

Experimental responses.—A typical response of a hydraulic servomotor to a step input at no load is presented in figure 2. The servomotor used in this run is of the rotary type. The data are plotted as the logarithm of the characteristic term $\left(1 - \frac{\alpha}{\theta} \right)^{-1}$ against time. In a response described by equation (2) (rewritten in terms of α and θ), the term $\log \left(1 - \frac{\alpha}{\theta} \right)^{-1}$ varies linearly with time. The data as shown fall essentially along a straight line and are in close agreement with the calculated response based on the calculated time constant. The calculated response is based on the value of time constant computed by means of equation (12). The dimensions

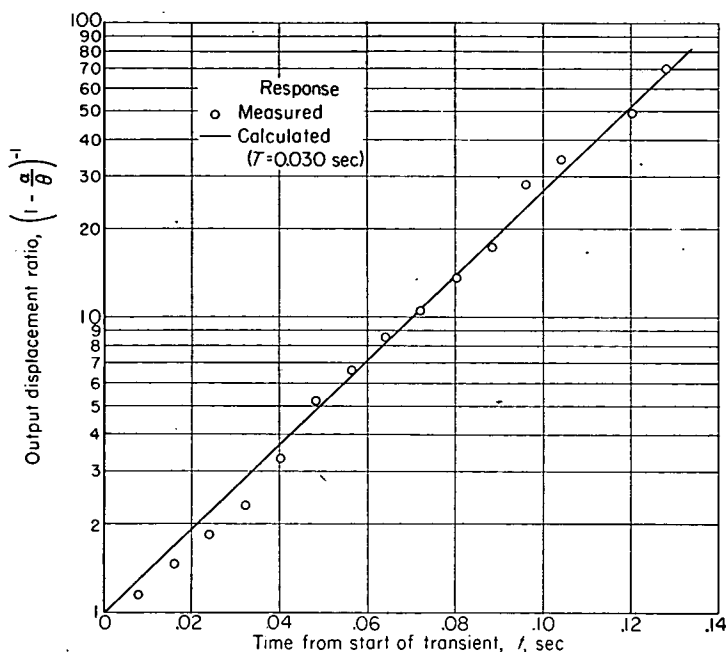


FIGURE 2.—Response of hydraulic servomotor to step input under negligible inertia load. Rotary servomotor; torque-inertia ratio, 3,500,000 radians per second per second; supply pressure, 1000 pounds per square inch; total shaft displacement, 20°.

of the servomotor necessary for the application of equation (12) are given in appendix A; also described are the experimental methods used to obtain the data.

In figure 2 the deviation of the data points from the theoretical straight line is the greatest in the early part of the transient where the effect of the internal servomotor mass is greatest. The response in the later part of the transient is less affected by the internal mass and is therefore indicative of the theoretical no-load response. The close agreement of the points with the theoretical straight line over the entire transient can be attributed to the relatively small internal mass of this servomotor. The ratio of static torque to the moment of inertia of the motor in this case was 3,500,000 radians per second per second.

TRANSIENT RESPONSE UNDER INERTIA LOAD

General characteristics of response.—Under the condition of an inertia load on the output shaft, the pressure drop across the piston will be proportional to the acceleration of the load. The general nature of the variation of the pressure drop across the piston along with the corresponding output shaft response is shown in figure 3. In the figure the following relations exist among the cylinder pressures P_1 and P_2 and the pressure drops across the piston ΔP_p , the inlet valve $\Delta P_{v,i}$, and the discharge valve $\Delta P_{v,d}$:

$$\Delta P_p = P_1 - P_2$$

$$\Delta P_{v,i} = P_s - P_1$$

$$\Delta P_{v,d} = P_2 - P_d$$

In the steady state, the pressure drop across the piston is zero. The cylinder pressures are equal and their magnitude

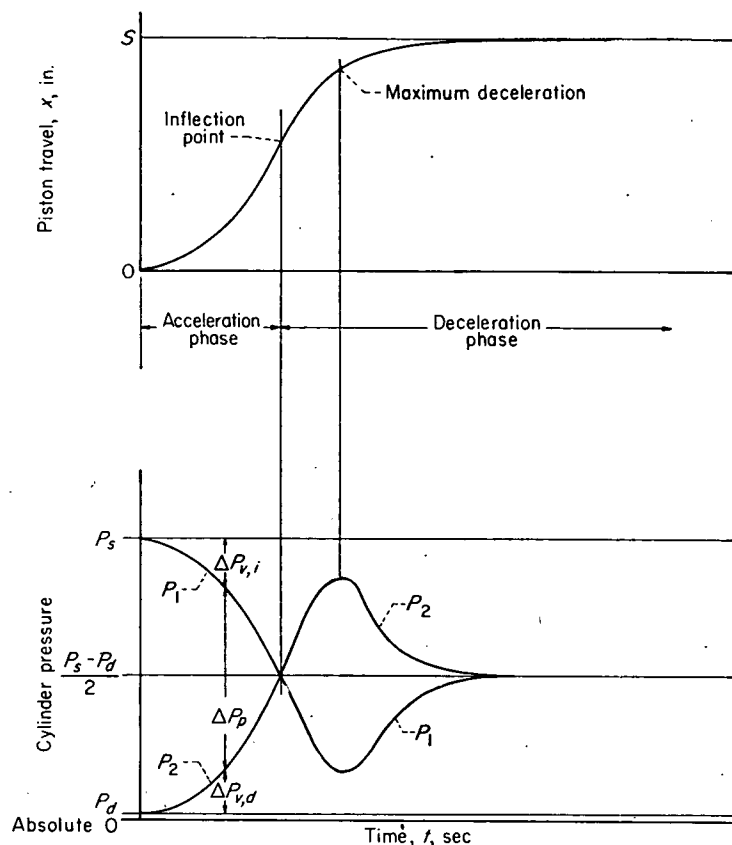


FIGURE 3.—Characteristic pressure variations during transient response of hydraulic servomotor with mechanical feedback. Step input and inertia load (cylinder pressures not limited).

is a function of the leakage areas around the valves. If the leakage areas around the valves are equal, the cylinder pressures will be equal to $(P_s - P_d)/2$. This condition is assumed in figure 3.

In response to a step input, P_1 immediately rises to the supply pressure P_s , and P_2 immediately drops to the drain pressure P_d . The accelerating pressure differential is then initially $(P_s - P_d)$. As the piston accelerates, the flow of fluid through the valve ports increases and at the same time the valve-port areas decrease. This action causes P_1 to decrease and P_2 to increase. The two curves ($P_1 = F_1(t)$ and $P_2 = F_2(t)$) are mirror images and therefore intersect at the value of $(P_s - P_d)/2$. At the intersection, the pressure differential across the piston is zero and the transient is therefore at the inflection point. Beyond the point of intersection of the two pressure curves the momentum of the load causes P_1 to continue to decrease and P_2 to continue to increase, which action results in a decelerating pressure differential across the piston. The deceleration causes a reduction in the rate of fluid flow through the valves and a consequent reduction in the rate of change of P_1 and P_2 . The pressures P_2 and P_1 therefore pass through maximum and minimum values, respectively. The deceleration continues until the error is reduced to zero. The magnitude of the maximum and minimum values of the cylinder pressures during the deceleration phase is a function of the value of error and of momentum at the inflection point. Based

on these factors alone, the value of the maximum and minimum is finite but not limited. The pressure P_1 , however, is physically limited at absolute zero. The effect of P_1 limited at absolute zero is treated in a later section. In the analysis that follows, the minimum value of P_1 is not limited.

In the transient response treated in this section, P_1 and P_2 vary as mirror images throughout the entire transient. In this case

$$\Delta P_{v,i} = \Delta P_{v,d}$$

The sum of the valve pressure drops may be written

$$\Delta P_{v,i} + \Delta P_{v,d} = 2\Delta P_v$$

The pressure drop across the piston may be written

$$\Delta P_p = P_s - P_d - 2\Delta P_v \quad (13)$$

The pressure drop across the piston is related to the acceleration by the following expression:

$$\Delta P_p = \frac{M}{A_p} \ddot{x} \quad (14)$$

From equations (13) and (14)

$$\Delta P_v = \frac{1}{2} \left(P_s - P_d - \frac{M}{A_p} \ddot{x} \right) \quad (15)$$

With the flow coefficient of the pilot valve considered constant, the equation of flow through the valve ports is

$$A_p \dot{x} = \left[CRW \sqrt{\frac{1}{2} \left(P_s - P_d - \frac{M}{A_p} \ddot{x} \right)} \right] (S - x) \quad (16)$$

Equation (16) cannot be integrated to x except by numerical or graphical methods. Some solutions of equation (16) are given in reference 1.

Under an inertia load the piston is accelerated from zero velocity. There is consequently an initial period in the response during which the flow through the valve ports is laminar. As a result of this, the flow coefficient of the pilot valve is not constant but is subject to wide variation. The net effect of the variation in flow coefficient is that of a marked reduction, which results in a slower initial acceleration rate than is indicated by equation (16). This effect is apparent in the comparison between measured responses and responses calculated by a form of equation (16) shown in reference 1.

At the conclusion of the transient the piston velocity again approaches zero, but in this part of the transient the valve areas also approach zero so that high fluid velocity is maintained in the valve ports. The flow coefficient may therefore be considered constant except in the initial acceleration phase. In the no-load case the assumption of a constant-flow coefficient is valid because the piston velocity is a maximum at the start of the transient.

In spite of the complex nature of the response there are basically only two phases in the transient, the acceleration phase and the deceleration phase. This conclusion, particularly with reference to a continuous deceleration phase

without overshoot or oscillation, is based on the assumption of rigid oil and structure and zero leakage. Figure 4 shows an oscillographic record of the response of a servomotor to a step input under a relatively heavy inertia load. The characteristic acceleration phase and dead-beat deceleration phase are quite clearly demonstrated.

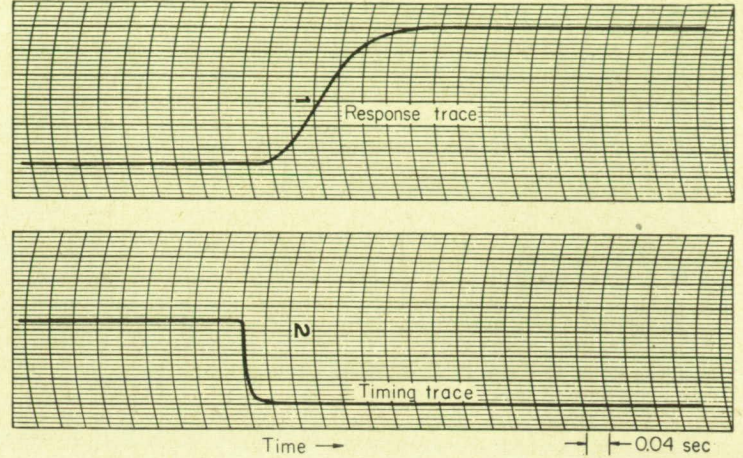


FIGURE 4.—Oscillographic record of response to step input of hydraulic servomotor under an inertia load.

Linear system for approximation of acceleration phase of transient response.—It is indicated by the measured responses of hydraulic servomotors under inertia loads that the acceleration phase may be approximated by a linear second-order system. The general form of a second-order differential equation with constant coefficients may be written

$$a\ddot{x} + b\dot{x} + x = c \quad (17)$$

The constants a , b , and c are now evaluated to match the physical system.

The equilibrium value of x in the physical system has been defined by the symbol S ; hence,

$$c = S$$

At no load the servomotor responds as a first-order system. Equation (17) should therefore reduce to equation (1) for the inertialess case. Therefore,

$$b = T$$

The constant a can be determined from the initial conditions:

$$x = 0$$

$$\dot{x} = 0$$

$$\ddot{x} = \frac{(P_s - P_d) A_p}{M} \quad (\text{see fig. 3})$$

The substitution of these values in equation (17) yields

$$a = \frac{MS}{(P_s - P_d) A_p} \quad (18)$$

The differential equation of the linear system that approximates the acceleration phase is then

$$\frac{MS}{(P_s - P_d) A_p} \ddot{x} + T\dot{x} + x = S \quad (19)$$

Evaluation of coefficients for rotary servomotors.—The stalled torque of the rotary servomotor is given by the following expression:

$$Q = \frac{(P_s - P_d)(L_2^2 - L_1^2)h}{2} \quad (20)$$

Hence, when

$$t = 0$$

$$\ddot{\alpha} = \frac{(P_s - P_d)(L_2^2 - L_1^2)h}{2J}$$

The term A_p/M , which occurs in the case of the straight-line servomotor, is replaced in the case of the rotary servomotor by the term $h(L_2^2 - L_1^2)/2J$. Replacing terms in equation (18) yields

$$a = \frac{2J\theta}{h(L_2^2 - L_1^2)(P_s - P_d)} \quad (21)$$

The differential equation of the linear system that approximates the acceleration phase in the case of the rotary servomotor is

$$\left[\frac{2J\theta}{h(L_2^2 - L_1^2)(P_s - P_d)} \right] \ddot{\alpha} + T\dot{\alpha} + \alpha = \theta \quad (22)$$

Linear system for approximation of deceleration phase of transient response.—In the deceleration phase of the transient the flow through the valve ports is turbulent; consequently the flow coefficient remains constant and equation (16) may be directly applied.

Rearranging terms of equation (16) and dividing both sides by the term $\sqrt{P_s - P_d}$ yield

$$\left(\frac{\sqrt{2} A_p}{CRW\sqrt{P_s - P_d}} \right) \frac{\dot{x}}{S - x} = \sqrt{1 - \frac{M\ddot{x}}{A_p(P_s - P_d)}} \quad (23)$$

Substituting equation (8) in equation (23) yields

$$T \left(\frac{\dot{x}}{S - x} \right) = \sqrt{1 - \frac{M\ddot{x}}{A_p(P_s - P_d)}} \quad (24)$$

At the start of the deceleration phase the value of $\ddot{x}$ is zero and consequently the right-hand side of equation (24) equals unity. As the transient continues, the value of $\ddot{x}$ increases to a maximum value and then returns to zero. For small inertia loads, the peak deceleration pressure difference across the piston will not exceed the value of the term $(P_s - P_d)$ (see fig. 3). In a transient in which the maximum decelerating pressure difference across the piston equals the difference $(P_s - P_d)$, the right-hand side of equation (24) has a maximum value of $\sqrt{2}$. In even extremely severe transients the maximum value of this term will not exceed 2. High values of the maximum deceleration are associated with short durations. The decelerating pressure differential will

therefore have a small effect on the integrated solutions. In treating the deceleration phase of the position response of the servomotor, therefore, the variation in pressure drop across the valve ports may be neglected. Equation (24) may therefore be reduced to

$$T \left(\frac{\dot{x}}{S - x} \right) = 1 \quad (25)$$

Equation (25) is the same as equation (1). In this linearization, therefore, the deceleration phase of the transient is approximated by an exponential decay.

Application of equations.—Equations (19) and (22), which are used in this analysis to approximate the acceleration phase of the transient, are linear second-order differential equations and may be integrated in terms of several parameters. The no-load time constant T will be employed as a parameter in the integrated solution because this quantity is a direct function of the physical dimensions of the servomotor. The second parameter that will be used is the reciprocal of the damping ratio. This quantity is herein designated the inertia index E . The new term is employed in this paper because the quantity is later applied to equations in which the term "damping ratio" would have no meaning.

Equation (19) expressed in terms of the parameters T and E may be written

$$\frac{T^2 E^2}{4} \ddot{x} + T \frac{\dot{x}}{S} + \frac{x}{S} = 1 \quad (26)$$

The value of E may be obtained directly from the dimensions of the servomotor, the load mass, and the initial error. Equating like coefficients in equations (19) and (26) gives

$$E = \frac{2}{T} \sqrt{\frac{MS}{(P_s - P_d) A_p}} \quad (27)$$

With the substitution of equation (8) in equation (27) the general expression for E is obtained:

$$E = \frac{\sqrt{2} CRW \sqrt{MS}}{A_p^{3/2}} \quad (28)$$

With the same procedure followed in the case of the rotary servomotor, the inertia index is

$$E = \frac{4CrW\sqrt{\theta J}}{[h(L_2^2 - L_1^2)]^{3/2}} \quad (29)$$

The integrated forms of equation (26) are as follows:

When $E = 1$,

$$\frac{x}{S} = 1 - e^{-\left(\frac{2t}{T}\right)} \left[1 + \left(\frac{2}{T}\right)t \right] \quad (30)$$

When $E < 1$,

$$\frac{x}{S} = 1 - \frac{1}{2} \left(1 + \frac{1}{\sqrt{1 - E^2}} \right) e^{-\left(\frac{2}{E^2 T} - \frac{2}{E^2 T} \sqrt{1 - E^2}\right)t} - \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 - E^2}} \right) e^{-\left(\frac{2}{E^2 T} + \frac{2}{E^2 T} \sqrt{1 - E^2}\right)t} \quad (31)$$

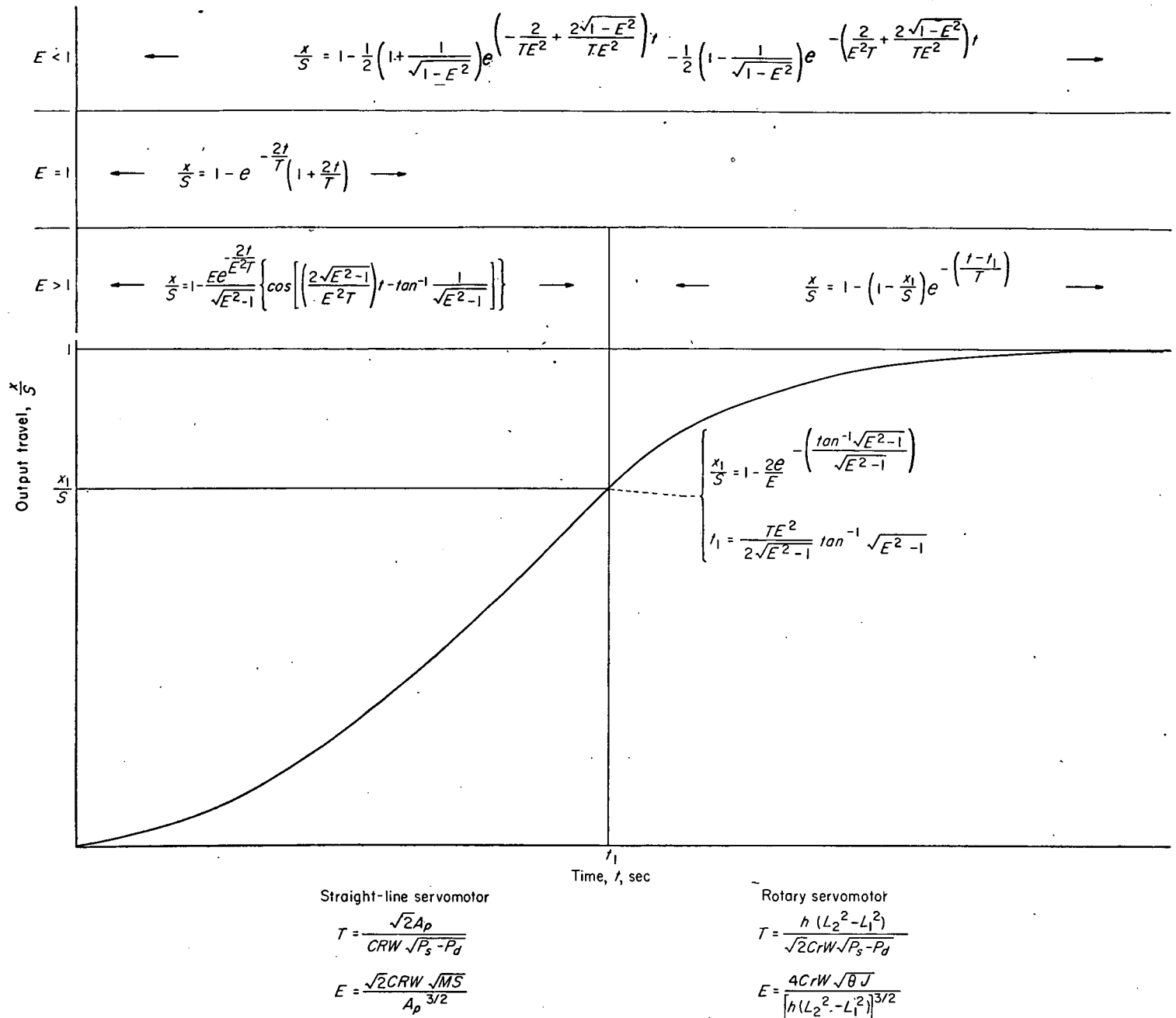


FIGURE 5.—Summary of linear relations for transient response of hydraulic servomotors with mechanical feedback.

When $E > 1$,

$$\frac{x}{S} = 1 - \frac{Ee^{-\left(\frac{2t}{E^2T}\right)}}{\sqrt{E^2-1}} \left\{ \cos \left[\left(\frac{2\sqrt{E^2-1}}{E^2T} \right) t - \tan^{-1} \frac{1}{\sqrt{E^2-1}} \right] \right\} \quad (32)$$

Equations (30), (31), and (32) apply specifically to the acceleration phase of the transient. In this analysis the deceleration phase is approximated by an exponential decay as defined by equation (25). There is, however, very little difference between the values of $\frac{x}{S}$ as defined by equation (30) or equation (31) beyond the inflection point and as defined by the integrated form of equation (25). When $E \leq 1$, the corresponding equations (30) or (31) may therefore be applied

in evaluating $\frac{x}{S} = F(t)$ in the deceleration phase of the transient as well as the acceleration phase. When $E > 1$, equation (32), which applies to the acceleration phase, deviates markedly from a first-order response in the deceleration phase. Equation (32) may therefore be applied only up to the inflection point. The time at which the inflection point occurs as evaluated from equation (32) is

$$t_1 = \frac{TE^2}{2\sqrt{E^2-1}} (\tan^{-1} \sqrt{E^2-1}) \quad (33)$$

Equation (32) is therefore solved for values of $\frac{x}{S}$ for values of t between zero and t_1 .

Values of $\frac{x}{S}$ for values of $t > t_1$ are obtained by integrating equation (25) with the initial conditions

$$t = t_1$$

$$x = x_1$$

which yield

$$\frac{x}{S} = 1 - \left(1 - \frac{x_1}{S}\right) e^{-\left(\frac{t-t_1}{T}\right)} \quad (34)$$

The relations defined in this section are summarized in figure 5 along with the expressions for T and E .

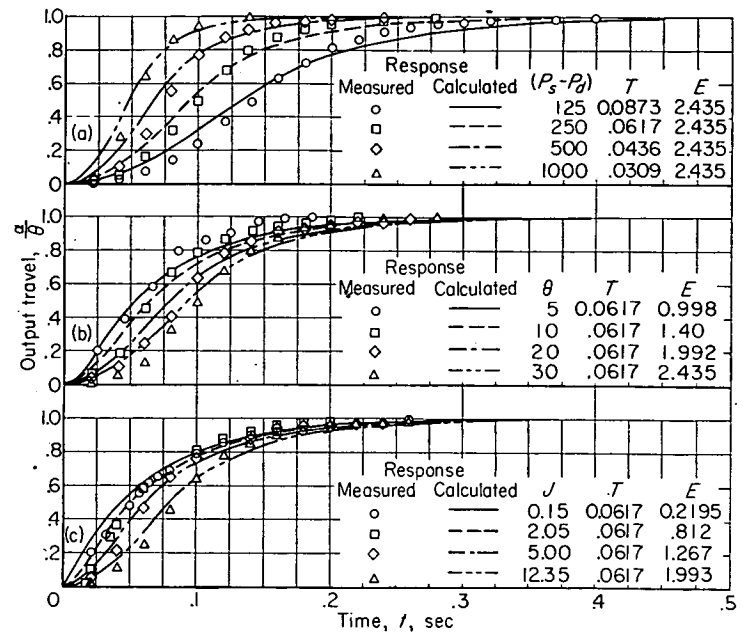
Experimental responses.—As derived in this analysis, the transient response of the servomotor is characterized dynamically by an acceleration phase that is approximately described by a linear second-order differential equation and a deceleration phase that is approximately described by a linear first-order differential equation. The coefficients of the equations for both phases are determined by the two parameters T and E . The parameter T is a function of the motor dimensions and the pressure difference across the motor. The parameter E is a function of the motor dimensions, the load mass (or moment of inertia), and the magnitude of the input step. Figure 6 shows the characteristic agreement between calculated and measured responses in a series of runs in which the factors that determine the parameters T and E have been varied. The data shown were obtained on a rotary servomotor. The servomotor and the experimental procedure are described in appendix A.

In figure 6(a) is shown the agreement between calculated and measured responses at various pressure differences across the motor. This set of runs was made at a fixed step magnitude and a fixed load moment of inertia. Figure 6(b) shows the agreement obtained in a series of runs in which the magnitude of the step was varied while pressure difference and load moment of inertia were held constant. Figure 6(c) shows the agreement obtained in a series of runs made at constant pressure difference and step magnitude in which the load moment of inertia was varied.

As can be seen in figure 6, the calculated responses have provided a close approximation of the actual responses over a very wide range of conditions. It may be of particular interest to note that the effect of the magnitude of the input step predicted by the approximating equations is evident in the measured responses.

DETERMINATION OF PEAK CYLINDER PRESSURE DURING TRANSIENT RESPONSE UNDER AN INERTIA LOAD

It has been indicated in the previous section that the pressure difference across the piston during the deceleration phase does not cause the motor response to deviate significantly from a response characterized by an exponential decay. The linear equation (eq. (25)) that is therefore adequate to describe the deceleration phase of the position response neglects the variation in deceleration rate and cannot be used to obtain an indication of the peak cylinder pressure during the transient. In the analysis that follows



(a) Effect of pressure differential. Step input, 30°; moment of inertia of load, 12.35 pound inches per second per second.
 (b) Effect of magnitude of step. Moment of inertia of load, 12.35 pound inches per second per second; pressure differential, 250 pounds per square inch.
 (c) Effect of load inertia. Step input, 20°; pressure differential, 250 pounds per square inch.

FIGURE 6.—Responses of hydraulic servomotor to step input under inertia load. Rotary servomotor.

a method will be developed by which an equation similar to equation (16) can be utilized by purely analytical means to determine the peak cylinder pressure that occurs in the deceleration phase.

Initial assumptions.—In the construction of high-speed, high-output hydraulic servomotors, it is usual to employ high supply-pressure differences across the motor. In such instances, the drain pressure P_d is, relative to the supply pressure P_s , close to absolute zero. Under this condition, a severe deceleration, resulting from a heavy inertia load, which causes the downstream pressure P_2 to rise above P_s , will drive the upstream pressure P_1 to its limit at essentially absolute zero. In the analysis that follows this condition is assumed to hold. The characteristic pressure variation during such a transient is presented in figure 7.

As shown in figure 7, the pressure transient is divided into three phases. In phases I and III the two pressure curves ($P_1 = F_1(t)$ and $P_2 = F_2(t)$) are mirror images. In phase II, P_1 is considered constant at absolute zero. The calculation of the maximum value of P_2 in phase II is based on the determination of the maximum value of deceleration. In order to evaluate the maximum deceleration, it will be necessary to determine the output position and velocity at the beginning of phase II. The symbols to be used in defining the initial conditions for each of the three phases are shown on the upper curve of figure 7.

Determination of initial conditions for phase II.—Up to the inflection point, phase I is identical with the acceleration.

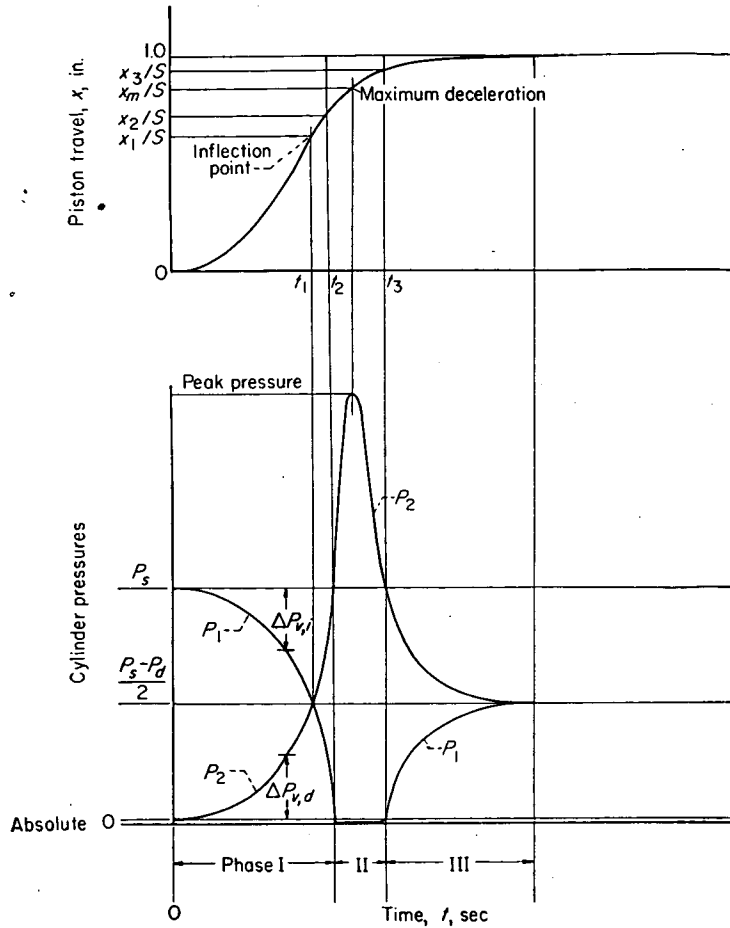


FIGURE 7.—Characteristic pressure variations during transient response of hydraulic servomotor with mechanical feedback. Step input and inertia load (upstream cylinder pressure limited at absolute zero).

phase previously treated. It is an assumption that the transients that result in high decelerating pressures will be of the type in which the inertia index E is large; therefore, only the solution to equation (26) for $E > 1$ need be considered. Equation (32) therefore describes the function $x = F(t)$ up to the inflection point. As defined in figure 7, phase I extends beyond the inflection point. The coordinates of the junction of phases I and II are

$$x = x_2$$

$$t = t_2$$

At this point, by definition,

$$P_1 \cong P_d = 0$$

$$P_2 = P_s$$

$$\Delta P_{v,i} = \Delta P_{v,d} = P_s$$

At any point in the transient the piston velocity is related to the flow through the valves by equation (16). Thus, at

the junction of phases I and II

$$\dot{x}_2 = \frac{CRW}{A_p} \sqrt{P_s}(S - x_2) \quad (35)$$

From equation (8) the following relation may be written:

$$\frac{CRW\sqrt{P_s}}{A_p} = \frac{\sqrt{2}}{T} \quad (36)$$

Substituting equation (36) in equation (35) yields

$$\dot{x}_2 = \sqrt{2} \frac{(S - x_2)}{T} \quad (37)$$

From equation (37) it is seen that the velocity at the junction of phases I and II is the velocity corresponding to the inertialess case multiplied by $\sqrt{2}$. At the inflection point the velocity corresponds exactly to the inertialess case. Thus,

$$\dot{x}_1 = \frac{S - x_1}{T}$$

Based on the consideration that

$$(S - x_1) > (S - x_2)$$

the following approximation is made:

$$(S - x_1) \cong \sqrt{2} (S - x_2)$$

Hence,

$$\dot{x}_2 \cong \dot{x}_1$$

From this the conclusion is drawn that the piston moves from the inflection point to x_2 with substantially the velocity at the inflection point.

The expression for the term $\dot{x}_1$ can be found by differentiating equation (32) and setting $t = t_1$, where t_1 is given by equation (33). This yields

$$\dot{x}_2 = \dot{x}_1 = \frac{2Se^{-\left(\frac{\tan^{-1}\sqrt{E^2-1}}{\sqrt{E^2-1}}\right)}}{ET} \quad (38)$$

The term x_2 is determined by substituting the value of $\dot{x}_2$ (as determined by eq. (38) in eq. (37)).

Differential equation for phase II of response.—As shown in figure 7, the following relations exist in phase II:

$$P_1 = 0$$

$$\Delta P_p = -P_2$$

$$\Delta P_{v,d} = P_2$$

The pressure drop across the piston is related to the acceleration by the following expression:

$$\Delta P_p = \frac{M}{A_p} \ddot{x}$$

From the condition specified above,

$$\Delta P_{v,d} = -\frac{M}{A_p} \ddot{x}$$

The equation of flow through the valve ports is

$$A_p \dot{x} = \left(CRW \sqrt{-\frac{M}{A_p} \ddot{x}} \right) (S-x)$$

Squaring and rearranging terms give

$$\left(\frac{A_p^3}{C^2 R^2 W^2 M} \right) \left(\frac{\dot{x}}{S-x} \right)^2 + \ddot{x} = 0 \quad (39)$$

From equation (28),

$$\frac{2S}{E^2} = \frac{A_p^3}{C^2 R^2 W^2 M}$$

Hence, equation (39) may be written in terms of the inertia index

$$\frac{2}{E^2} \left[\frac{\frac{\dot{x}}{S}}{\left(1 - \frac{x}{S}\right)} \right]^2 + \frac{\ddot{x}}{S} = 0 \quad (40)$$

Determination of maximum value of deceleration.—Differentiating equation (40) and setting $\ddot{x}=0$ yield

$$\ddot{x}(S-x) + (\dot{x})^2 = 0 \quad (41)$$

Eliminating $\ddot{x}$ between equation (40) and equation (41) gives

$$\frac{2S}{E^2} \frac{(\dot{x})^2}{(S-x)} - (\dot{x})^2 = 0$$

from which

$$\frac{x_m}{S} = 1 - \frac{2}{E^2} \quad (42)$$

Substituting equation (42) in equation (40) yields

$$\ddot{x}_{max} = -\frac{E^2 (\dot{x})^2}{2S} \quad (43)$$

The value of $\dot{x}$ at $x=x_m$ is found by integrating equation (40). This integration is shown in appendix B. By inserting this value of $\dot{x}$ in equation (43), the value of $\ddot{x}_{max}$ is obtained. Based on the consideration that P_1 equals zero, the relation between the maximum downstream cylinder pressure and the maximum value of the deceleration is

$$P_{2,max} = \frac{M}{A_p} \ddot{x}_{max} \quad (44)$$

It is further shown in appendix B that the ratio $P_{2,max}/P_s$ can be expressed as a function solely of the inertia index. This relation is given below:

$$\frac{P_{2,max}}{P_s} = \frac{E^2 e^{2F(E)}}{14.77} \quad (45)$$

where

$$F(E) = \frac{\sqrt{2}}{E} e^{\left(\frac{\tan^{-1} \sqrt{E^2-1}}{\sqrt{E^2-1}} \right)} - \frac{\tan^{-1} \sqrt{E^2-1}}{\sqrt{E^2-1}}$$

It is shown in appendix B that equation (45) has real values for all values of $E > 2.38$.

Comparison of experimental and analytical values of peak cylinder pressure.—Equation (45) is plotted in figure 8 for values of E from 2.38 to 6.5. Also shown in the figure are experimental values obtained on the rotary servomotor described in appendix A. The experimental technique used to obtain the data is also described in appendix A. It can be noted that the experimental values are slightly lower than the analytical curve at low values of E and are in close agreement with the curve at higher values of E . The value of E equal to 5.7, which is the highest experimental value shown, was the highest value that was practicably obtainable with the test equipment. In general, values of E in excess of 6 represent very heavy inertia loads and large step magnitudes.

Effect of high decelerating cylinder pressure on transient response.—In the derivation of the equations that describe

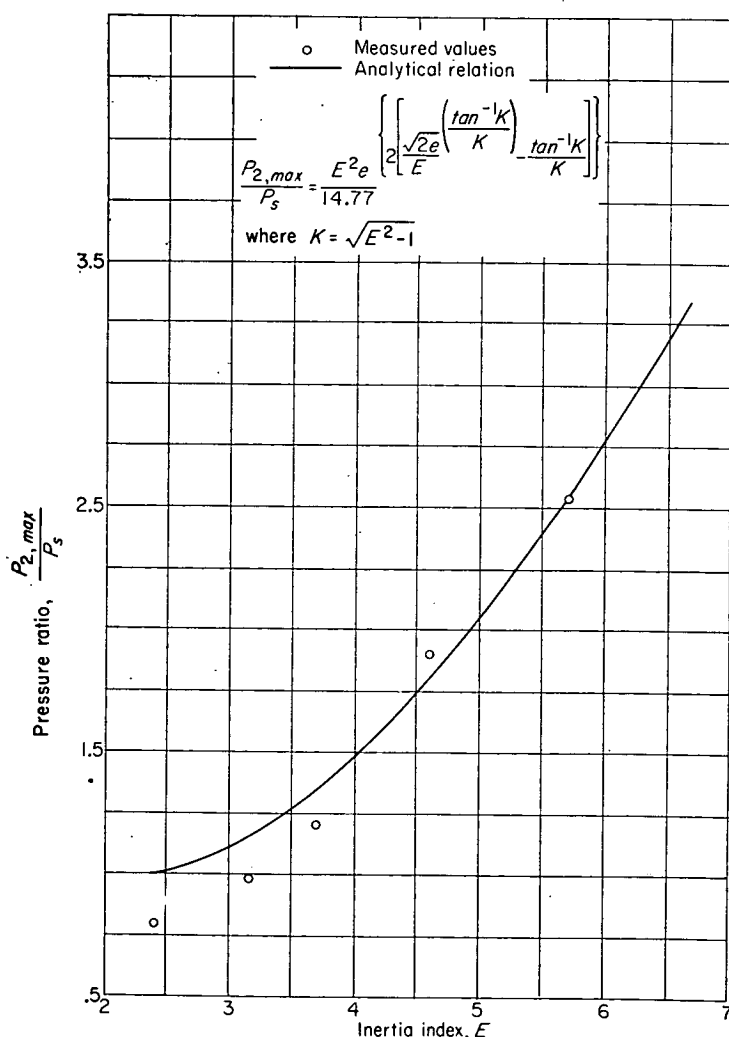


FIGURE 8.—Ratio of peak transient cylinder pressure to supply pressure as function of inertia index. Hydraulic servomotor with mechanical feedback.

the transient response of the servomotor, it was shown that the deceleration phase of the transient response could be approximated by an exponential decay, in which case the variations in the cylinder pressures are neglected. This method of approximation is outlined in figure 5. For transients in which the upstream cylinder pressure is driven to absolute zero, the relations that have been derived for the determination of the peak decelerating cylinder pressure can be used for a more precise determination of the transient response than is afforded by the method of figure 5. The application of these relations to the transient response is presented in appendix B and is outlined in figure 9. A comparison of the method of figure 5 and the method of figure 9 with an experimental response is shown in figure 10. The agreement between the measured response and the response calculated by the method of figure 9 is extremely close. The value of the inertia index in this response was 4.49; hence, from figure 8, the ratio $P_{2,max}/P_s$ equals 1.75. Even with this high decelerating pressure the method of figure 5 provides a fair approximation of the response. The calculations involved in the application of the method outlined in figure 9 are many times longer than those required with the method outlined in figure 5. For this reason the method of figure 9 should be applied only when the need for increased accuracy justifies the longer calculation.

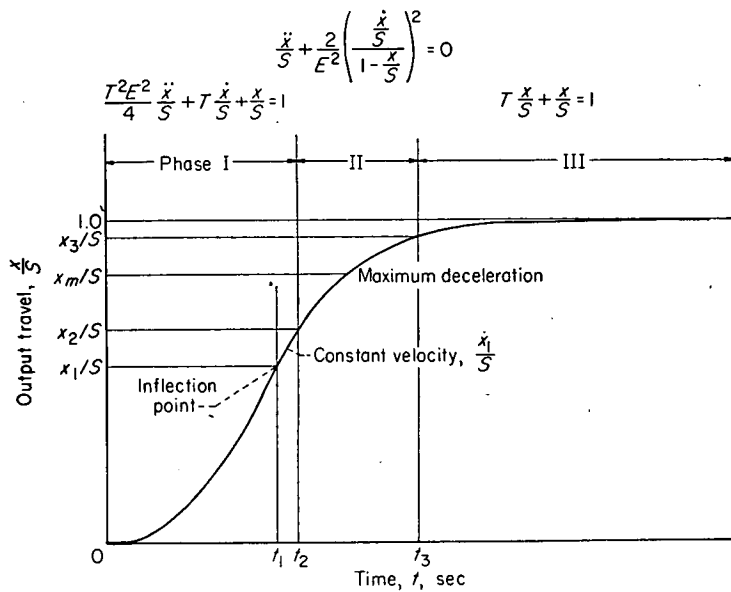


FIGURE 9.—Analytical relations for approximating transient response of hydraulic servomotor with mechanical feedback. Step input and inertia load (upstream cylinder pressure limited at absolute zero).

RESPONSE TO A SINUSOIDAL INPUT

The analysis of the frequency response at no load yields an important parameter used in the analysis of the response under an inertia load. For this reason, both the no-load and the inertia-load cases are treated.

NO-LOAD RESPONSE

Basic character of response.—The basic character of the zero mass response is defined by the linear proportionality

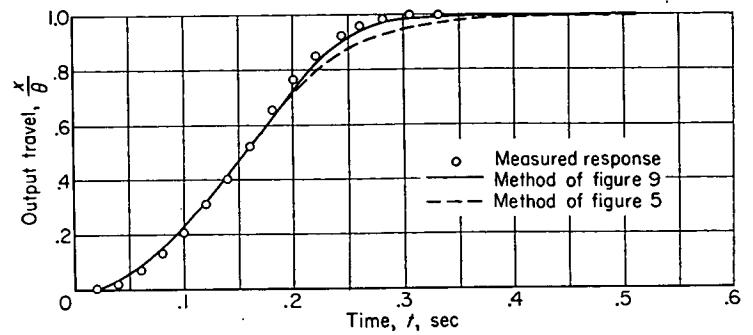


FIGURE 10.—Comparison of method of figures 5 and 9 with measured response. Transient response of hydraulic servomotor with mechanical feedback. Rotary servomotor; step input, 30°; moment of inertia of load, 41.75 pound inches per second per second; pressure differential, 250 pounds per square inch. Inertia index E , 4.49; no-load time constant T , 0.0617 second.

between the output velocity and the pilot-valve opening (or position error). The proportionality constant between the velocity and the error is the no-load time constant T . For a sinusoidally varying input the instantaneous output velocity is then

$$\dot{x} = \frac{S' \cos(\omega t) - x}{T} \quad (46)$$

The solution of equation (46) is

$$\frac{x}{S'} = A e^{i(\omega t + \varphi)} \quad (47)$$

Substituting equation (47) and its derivative in equation (46) gives

$$A e^{i\varphi} = \frac{1}{1 + i\omega T} \quad (48)$$

The term $A e^{i\varphi}$ is a vector quantity having an amplitude A and a phase angle φ . From equation (48),

$$A = \frac{1}{\sqrt{1 + T^2 \omega^2}} \quad (49)$$

and

$$\varphi = -\tan^{-1} T\omega \quad (50)$$

For large values of ω

$$A \rightarrow \frac{1}{T\omega}$$

Hence, the asymptote of the response is given by

$$A = \frac{1}{T\omega} \quad (51)$$

The intersection of the asymptotic line and $A=1$ yields the break frequency and orients the asymptote

$$\omega_1 = \frac{1}{T} \quad (52)$$

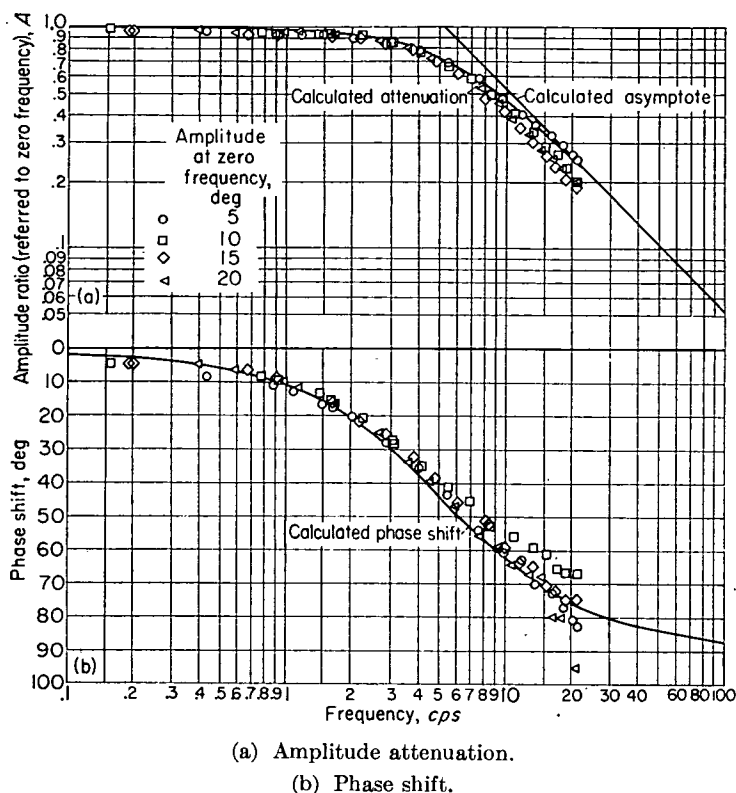


FIGURE 11.—Frequency response of hydraulic servomotor under negligible inertia load. Rotary servomotor; torque-inertia ratio, 3,500,000 radians per second per second; supply pressure, 1000 pounds per square inch.

Experimental responses.—Figure 11 shows the correlation between the analytical first-order frequency response and the measured frequency response of a servomotor at no load. The servomotor used was the rotary motor described in appendix A. The techniques of instrumentation and experiment are also described in appendix A. The close agreement between the calculated response and the measured response for the wide range of input amplitudes used characterizes the basic linearity of the response of the servomotor at no load.

RESPONSE UNDER AN INERTIA LOAD

It has been shown that under an inertia load the transient response of the servomotor is nonlinear. In the transient response the basic character of the response varied with time. It is therefore to be expected that in the frequency response the basic character of the response will vary with frequency.

Low-frequency amplitude attenuation.—At low frequencies the forces that act on the mass of the system are small and hence the response in this frequency range will be similar to the no-load response. The attenuation may therefore be described by equations (49) and (51). In the log-log plot of amplitude ratio against frequency (fig. 12), an asymptote may then be considered to exist with unity slope and a break frequency of $1/T$. The break frequency of the low-frequency asymptote expressed in cycles per second is

$$f_1 = \frac{1}{2\pi T} \quad (53)$$

High-frequency amplitude attenuation.—At no load the piston velocity is at all times proportional to the valve opening. Therefore, in the response to a sinusoidal input at no load the pilot-valve area is zero at the ends of the output travel (the velocity being zero). Under an inertia load the piston velocity is not proportional to the pilot-valve opening, and hence in the response to a sinusoidal input the valve area is not necessarily zero at the ends of the output travel. If at a given frequency the response of the servomotor is assumed to be essentially sinusoidal, the maximum acceleration can be considered to occur at the limits of the output travel and hence when the piston velocity is zero. Under the condition of negligible mass of the hydraulic fluid, the pressure difference across the piston at any instant, when the piston velocity is zero and the pilot-valve area is greater than zero, is the pressure difference across the servomotor. Above some frequency the system may then be approximated by a linear system wherein the pressure difference across the piston varies sinusoidally with an amplitude of $(P_s - P_d)$ and with the frequency of the input. On the basis of this approximation the acceleration of the piston is

$$\ddot{x} = \frac{(P_s - P_d)A_p}{M} \sin \omega t \quad (54)$$

Integrating equation (54), introducing the condition that x varies about zero, and neglecting the change in sign yield

$$x = \frac{(P_s - P_d)A_p}{M\omega^2} \sin \omega t \quad (55)$$

After both sides of equation (55) are divided by the output amplitude at zero frequency, the equation relating the amplitude ratio and the frequency is

$$A = \frac{(P_s - P_d)A_p}{S'M\omega^2} \quad (56)$$

In the log-log plot of amplitude ratio against frequency (fig. 12) an asymptote may therefore be considered to exist having a slope of 2 decades per decade. The break frequency of the high-frequency asymptote is found from equation (56) by setting $A=1$:

$$\omega_2 = \sqrt{\frac{(P_s - P_d)A_p}{S'M}} \quad (57)$$

The expression for the value of ω_2 may be made independent of the type of servomotor by relating ω_2 to the no-load time constant and the dimensionless quantity previously defined as the inertia index. The inertia index is defined for the frequency response by replacing the term magnitude of step S with the term amplitude of output sine wave at zero frequency S' .

Rewriting equation (28) and introducing the symbol S' in place of S give

$$E' = \frac{\sqrt{2} CRW\sqrt{MS'}}{A_p^{3/2}}$$

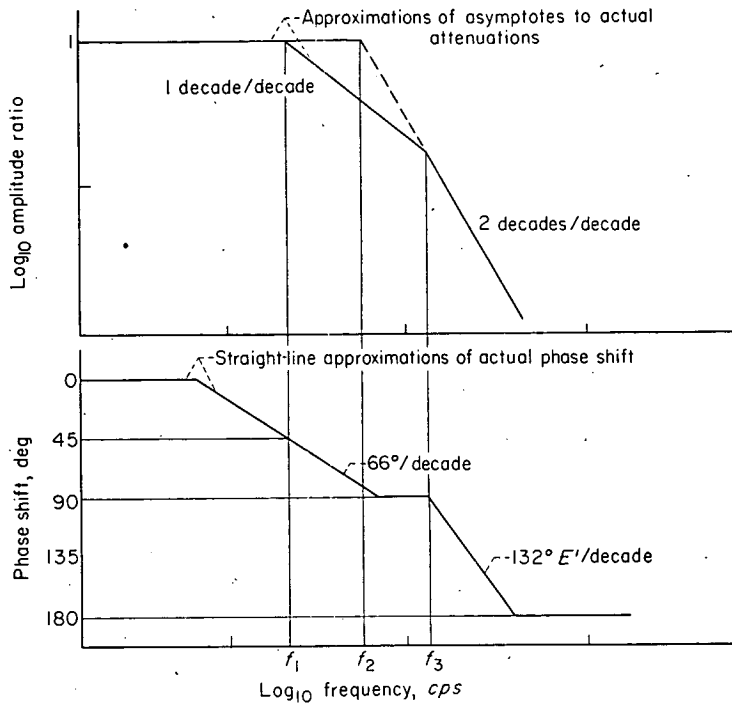


FIGURE 12.—Summary of linear relations for frequency response of hydraulic servomotors with mechanical feedback.

From equation (8)

$$T = \frac{\sqrt{2} A_p}{CRW\sqrt{P_s - P_d}}$$

Combining these two relations yields

$$E'T = 2\sqrt{\frac{MS'}{A_p(P_s - P_d)}} \quad (58)$$

Substituting equation (58) in equation (57) gives

$$\omega_2 = \frac{2}{TE'} \quad (59)$$

The break frequency of the high-frequency asymptote expressed in cycles per second is

$$f_2 = \frac{1}{\pi TE'} \quad (60)$$

The amplitude ratio may also be expressed in terms of T and E' . Substituting equation (58) in equation (56) gives

$$A = \frac{4}{(E')^2 T^2 \omega^2} \quad (61)$$

Cross-over frequency.—The intersection of the low- and high-frequency asymptotes defines the limit of the low-frequency band and the start of the high-frequency band. For $f_2 > f_1$ this intersection is found by equating the amplitude ratios as defined by equations (51) and (61):

$$\frac{1}{T\omega} = \frac{4}{(E')^2 T^2 \omega^2}$$

from which

$$\omega_3 = \frac{4}{T(E')^2} \quad (62)$$

The cross-over frequency expressed in cycles per second is

$$f_3 = \frac{2}{\pi T(E')^2} \quad (63)$$

For $f_2 < f_1$ the cross-over frequency occurs at f_2 ; hence

$$f_3 = f_2$$

Correlation of frequency response and transient response.—It has been shown that the derived attenuation asymptotes of the frequency response are functions of the same parameters that govern the derived characteristics of the transient response. It has been shown further that the analytical relation that governs the characteristics in the low-frequency band is the same as the analytical relation that governs the characteristics of the deceleration phase of the transient response (fig. 5). It can also be shown that the linear system used to approximate the acceleration phase of the transient response attenuates along the same asymptote as has been derived for the high-frequency band.

Equation (26) expresses the dynamic equilibrium in the acceleration phase of the response to a step input. Equation (26) rewritten for a sinusoidal input is

$$\frac{T^2(E')^2}{4} \ddot{x} + T\dot{x} + x = S'e^{i\omega t} \quad (64)$$

The solution to equation (64) is

$$\frac{x}{S'} = Ae^{i(\omega t + \varphi)} \quad (65)$$

Substituting equation (65) and its derivatives in equation (64) yields

$$Ae^{i\omega} = \frac{1}{\left[1 - \frac{\omega^2 T^2 (E')^2}{4}\right] + i\omega T} \quad (66)$$

The term $Ae^{i\omega}$ is a vector quantity having an amplitude A and a phase angle φ :

$$A = \frac{1}{\sqrt{\left[1 - \frac{\omega^2 T^2 (E')^2}{4}\right]^2 + \omega^2 T^2}} \quad (67)$$

$$\varphi = -\tan^{-1} \left[\frac{\omega T}{1 - \frac{\omega^2 T^2 (E')^2}{4}} \right] \quad (68)$$

At high frequencies the relation of equation (67) approaches the asymptote

$$A = \frac{4}{(E')^2 T^2 \omega^2} \quad (69)$$

Equation (69) is identical with equation (61). It is therefore shown that the linear system described by equation (64) attenuates along the same asymptote as the linear system described by equation (26).

Phase shift.—The correlation of the amplitude attenuation with a linear first-order system in the low-frequency band and with a linear second-order system in the high-frequency band provides a basis for the description of the phase shift of the servomotor. The phase shift of linear systems can be represented by straight lines on the coordinates of phase shift against log frequency. The characteristic slope of the straight line for a first-order system is the slope $\frac{d\varphi}{d\omega}$ of the phase-shift-frequency relation at $\varphi=45^\circ$. The characteristic slope of the straight line for a second-order system is the slope $\frac{d\varphi}{d\omega}$ of the phase-shift-frequency relation at $\varphi=90^\circ$.

The orientation of these lines and the relations for the slopes are shown in figure 12. The derivation of the relations shown in figure 12 is presented in the following paragraphs.

Based on the correlation of the low-frequency-amplitude attenuation with the no-load response, the phase shift is, from equation (50),

$$\varphi = -\tan^{-1} T\omega$$

and, from equation (52), the break frequency in the low-frequency band is

$$\omega_1 = \frac{1}{T}$$

Substituting equation (52) in equation (50) yields

$$\varphi_1 = -\tan^{-1} 1 = -45^\circ$$

Differentiating equation (50) with respect to ω and setting $\omega = \frac{1}{T}$ yield the slope $\frac{d\varphi}{d\omega}$ at $\alpha=45^\circ$:

$$\frac{d\varphi}{d\omega} = \frac{T}{2} \quad (70)$$

The straight line on the semilog plot may be written

$$\varphi = K \log_{10} \omega \quad (71)$$

Differentiating equation (71) with respect to ω and solving for the constant K give

$$K = 2.3\omega \frac{d\varphi}{d\omega} \quad (72)$$

Substituting the values of $\frac{d\varphi}{d\omega}$ and ω at $\varphi=45^\circ$ yields the characteristic slope of the first-order system on the semilog plot

$$K_1 = 1.15 \quad (73)$$

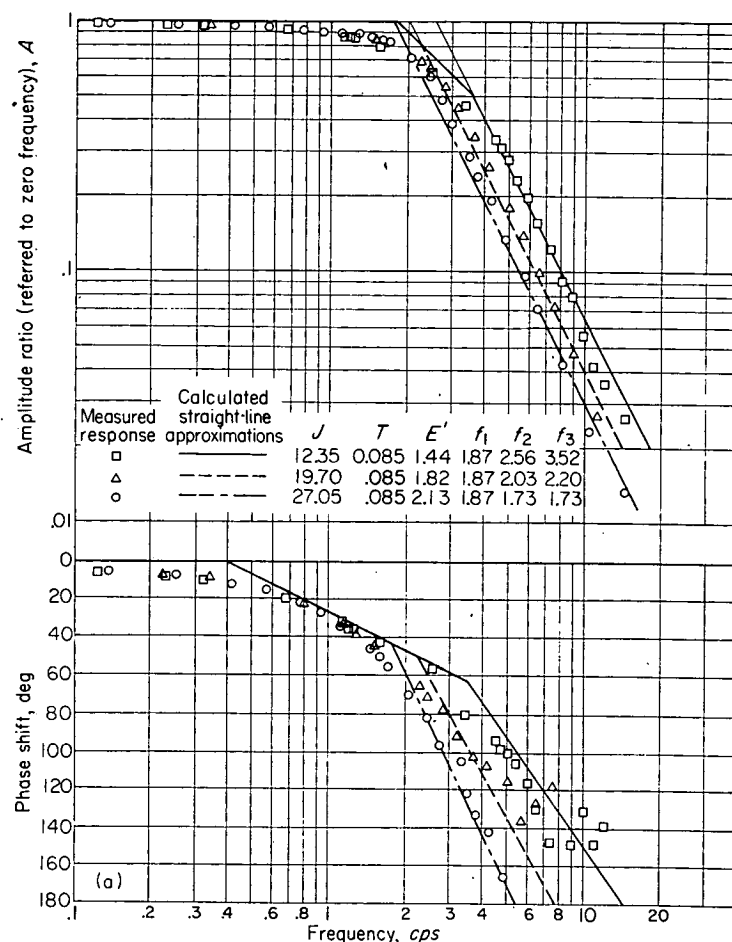
The phase shift in the low-frequency band may therefore be represented by a straight line on the semilog plot having a slope of 1.15 radians per decade ($66^\circ/\text{decade}$) and passing through the point $\omega = \frac{1}{T}$, $\varphi=45^\circ$.

Based on the correlation of the amplitude attenuation in the high-frequency band with the acceleration phase of the response to a step input, the phase shift in the high-frequency band is characterized by the relation expressed in equation (68). The characteristic slope $\frac{d\varphi}{d\omega}$ of equation (68) is found by differentiating equation (68) and setting ω equal to $\frac{2}{TE'}$ ($\varphi=90^\circ$):

$$\frac{d\varphi}{d\omega} = \frac{T(E')^2}{2} \quad (74)$$

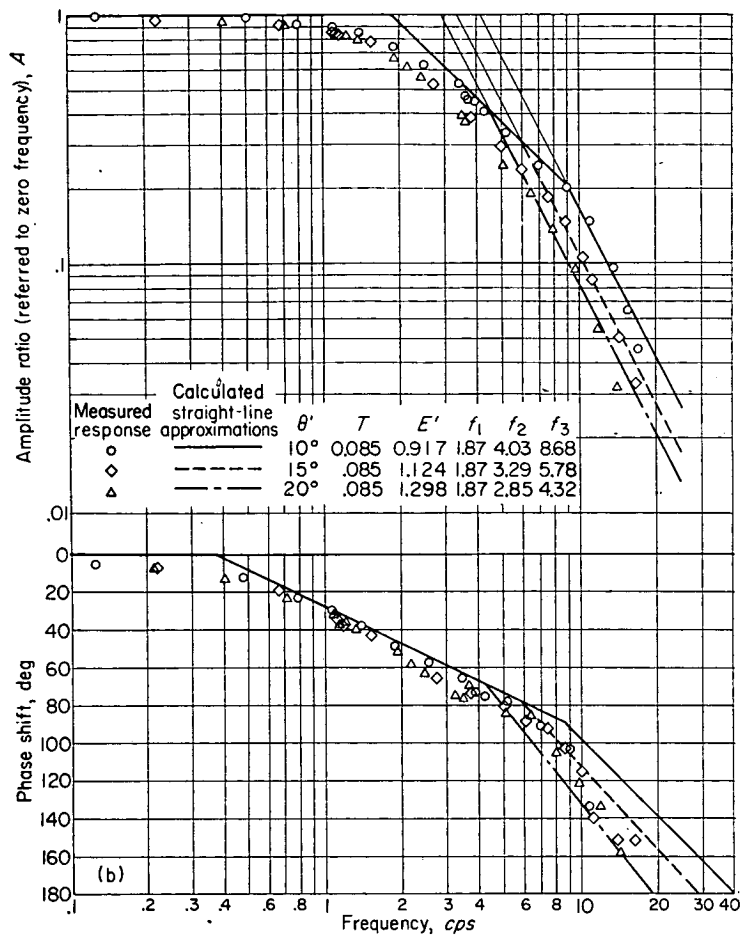
After the substitution of equation (74) and $\omega = \frac{2}{TE'}$ in equation (72), the characteristic slope of the second-order system on the semilog plot is

$$K_2 = 2.3E' \quad (75)$$



(a) Effect of load inertia J . Rotary servomotor; amplitude at zero frequency, 10°; pressure differential, 125 pounds per square inch.

FIGURE 13.—Frequency responses of hydraulic servomotors under inertia load.

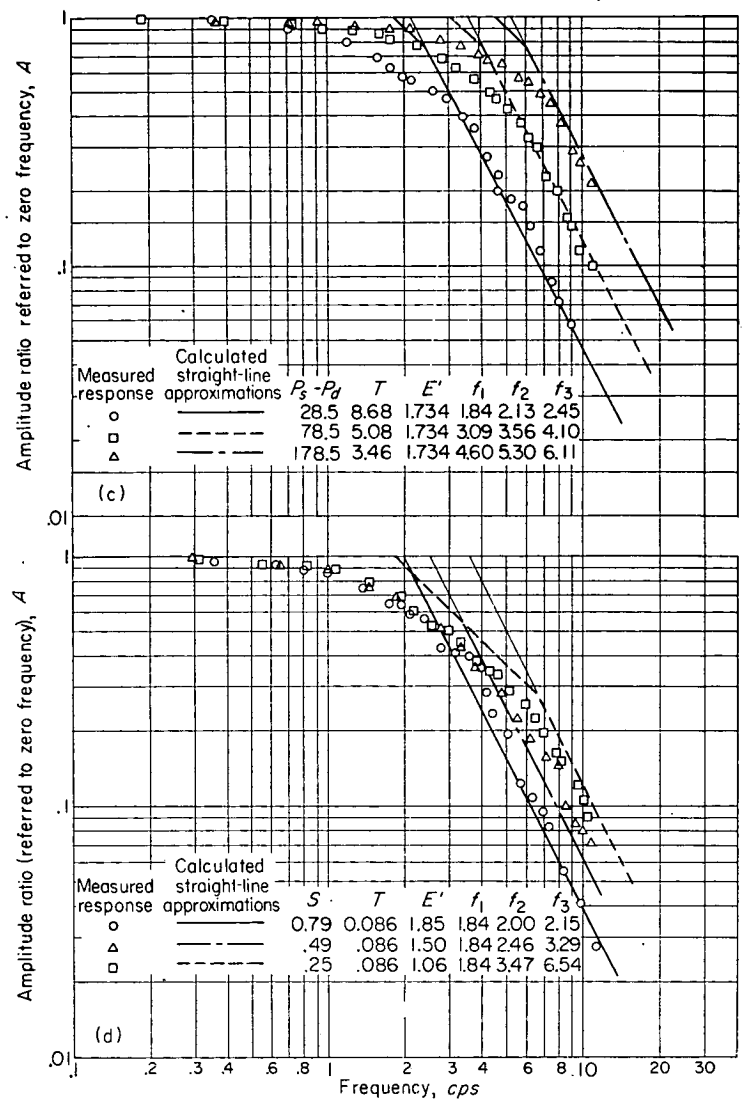


(b) Effect of amplitude θ' . Rotary servomotor; moment of inertia of load, 5 pound inches per second per second; pressure differential, 125 pounds per square inch.

FIGURE 13.—Continued. Frequency responses of hydraulic servomotors under inertia load.

It is a fundamental precept that in representing the response of the servomotor in two frequency bands the relations used to approximate the response shall yield equal amplitude ratios and equal phase shifts at the cross-over frequency. The cross-over frequency has already been defined for equal amplitude ratios. The phase-shift line in the low-frequency band has been previously oriented. The phase-shift line in the high-frequency band therefore intersects the low-frequency phase-shift line at the cross-over frequency and has a slope of $2.3E'$ radians per decade ($132^\circ E'$ decade). It should be noted that the low-frequency phase-shift line is limited at 90° and the high-frequency-band phase-shift line is limited at 180° . In figure 12 the cross-over frequency is shown to occur after the low-frequency phase-shift line has reached the 90° limit. The orientation of the high-frequency phase-shift line for other relative locations of the cross-over frequency is shown in conjunction with the experimental responses.

Experimental responses.—Figure 13 shows the experimentally and analytically determined effect on the frequency response of the hydraulic servomotor of the parameters: load inertia, input amplitude, and pressure. Examples are



(c) Effect of pressure differential ($P_s - P_d$). Straight-line servomotor; load mass, 1.08 pounds per second per second per inch; amplitude at zero frequency, 0.65 inch.

(d) Effect of amplitude S' . Straight-line servomotor; load mass, 1.08 pounds per second per second per inch; pressure differential, 28.5 pounds per square inch.

FIGURE 13.—Concluded. Frequency responses of hydraulic servomotors under inertia load.

presented for both the rotary and straight-line types of motor.

In figure 13(a) is shown the effect of load inertia on the amplitude attenuation and on the phase shift of a rotary servomotor. An increase in load inertia results in a reduction in the frequency at which the attenuation becomes rapid. In the analytical expression developed in this paper (summarized in fig. 12) this effect is evident in the increased value of E' with increasing load inertia and the consequent reduction in the values of f_2 and f_3 .

The experimental and calculated frequency responses of the same rotary servomotor at various input amplitudes are shown in figure 13(b). The amplitudes given in the figure correspond to the term θ' and consequently are half the total

output stroke at zero frequency. In the case of this particular servomotor, this amplitude corresponds exactly to the amplitude of the input sine wave. The increase in input amplitude is seen to have an effect similar to that of increasing load inertia. This effect is made evident in the analysis by equation (57).

The agreement between experimental and analytical responses for a straight-line servomotor is shown in figures 13(c) and (d). Phase shift could not be measured in this installation. Figure 13(c) shows the effect of the pressure difference across the motor. Increased pressure results in increasing the frequency at which the motor begins to attenuate rapidly. In the analytical expressions, the increase in pressure results in a decrease in T and a consequent increase in f_1 , f_2 , and f_3 . The inertia index E' is independent of the pressure difference and therefore the effect of pressure on f_2 and f_3 is not as great as the effect on f_1 . The effect of amplitude shown in figure 13(d) is similar to that already shown in the case of the rotary servomotor in figure 13(b).

In both amplitude attenuation and phase shift the agreement between the measured response and the analytical straight-line approximation is, in general, well within the experimental accuracy. The slopes of the attenuation and phase data clearly demonstrate the first-order characteristics of the response in the low-frequency band and the second-order characteristics of the response in the high-frequency band. The transition from first-order to second-order characteristics at the calculated cross-over frequency is quite pronounced. The values of T and E' shown in figure 13 are based on a value of C of 95.1, on the dimensions of the servomotors as given in appendix A, and on the conditions stated on each plot. In figures 13(c) and (d), the pressure differences given are not the actual pressure differences across the motor but are reduced values based on a pressure necessary to overcome friction in the loading carriage. This

reduction is discussed in appendix A. In all the other calculated results presented in this paper, no correction whatever was applied to the measured pressure difference across the motor.

CONCLUDING REMARKS

Application to analysis.—The dynamic relations presented in this paper can be directly applied to the analysis of a given servomotor. The dimensions of the servomotor and the operating pressure difference across the motor determine the no-load time constant T . The inertia index E is then determined from the load inertia and the magnitude or amplitude of the input disturbance. With these two constants determined, the relations for the transient response, the peak cylinder pressure in the transient, and the frequency response can be applied.

The validity of the analysis has been demonstrated by means of recorded transient and frequency responses obtained on two servomotors. In all cases the calculated responses are in close agreement with the measured responses.

Application to design.—The optimum combination of servomotor dimensions to meet specific dynamic requirements involves further discussion of physical considerations that are beyond the scope of this paper. It is nevertheless apparent that, based on this analysis, procedures can be established for the rational design of hydraulic servomotors. In general, the procedures will involve the inversion of the analytical expressions in order that the dimensional parameters (such as A_p , R , and W) may be expressed in terms of the analytical parameters T and E , and the establishment of means of specifying the desired response in terms of the analytical parameters. The application of this analysis to the design of servomotors is given in reference 2.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, May 19, 1952.

APPENDIX A

APPARATUS AND EXPERIMENTAL PROCEDURES

SERVOMOTOR DIMENSIONS

The two servomotors used in this investigation had the following dimensions:

Straight-line servomotor:

Piston area, A_p , sq in.	4.4
Ratio of valve travel to piston travel at fixed inputs, R , in./in.	0.1062
Width of valve port, W , in.	1.33

Rotary servomotor:

Width of vane, h , in.	2.000
Inner vane radius, L_1 , in.	0.760
Outer vane radius, L_2 , in.	2.267
Ratio of valve travel to vane shaft rotation at fixed input, r , in./radians	0.3125
Width of valve port, W , in.	0.2225

TRANSIENT RESPONSE

Position recorder.—Input and output shaft positions were recorded by means of direct-writing oscillographs. The

oscillographs were driven by amplifiers. The amplifiers, in turn, received their signal from potentiometers coupled to the servomotor shafts. The frequency response of the amplifier-oscillograph combination was essentially flat over a frequency range from 0 to 80 cycles per second.

Pressure recorder.—Cylinder pressures were also recorded by means of direct-writing oscillographs. The pressure pickups used were of the strain-gage type. The signal developed across the strain-gage bridge was amplified by suitable amplifiers which, in turn, drove the oscillographs. The frequency response of this amplifier-oscillograph combination was essentially flat over a frequency range from 0 to 80 cycles per second. The natural frequency of the pressure pickup was 1000 cycles per second.

Step-input apparatus.—In order to introduce a step change in a mechanical system such as the hydraulic servomotor, it is necessary to accelerate and decelerate a finite mass (such as the input shaft) at very high rates. The time constant of the servomotor to be tested was about 0.03 second. The very high accelerations that would be required of the input mechanism for the transient to be negligible did not appear to be reasonably attainable in this case. It was therefore decided to use a step input that is obtained by restraining the output. This procedure should be made clear by figure 14.

As can be seen in the photograph, the output shaft is held

in position by a wire suitably anchored. The wire used was music wire stressed to approximately 150,000 pounds per square inch. With the output so restrained, the input lever is advanced for the desired magnitude of step. The transient is then triggered by cutting the highly stressed wire. In the transient runs, the output motion was recorded directly. The input motion, which has no meaning in this case, was not recorded. The start of the transient was recorded by placing the restraining wire in the signal circuit of one recorder. A change occurred in the signal voltage when the wire was parted.

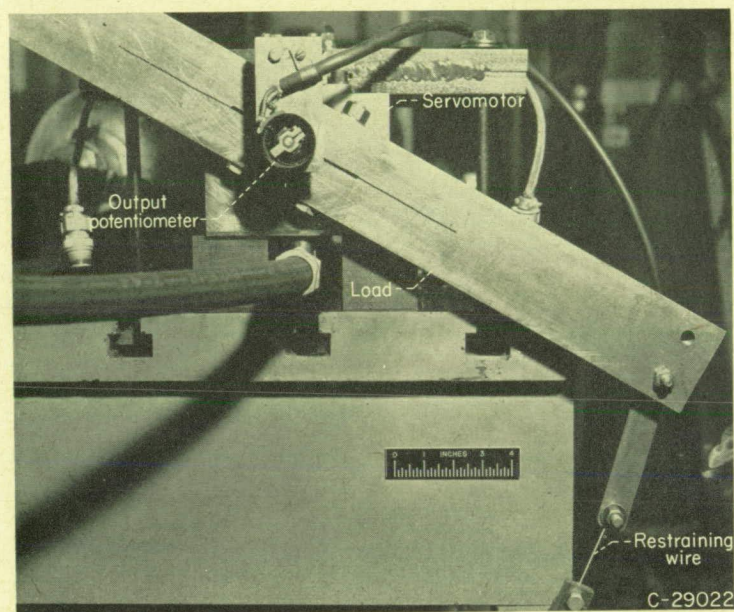


FIGURE 14.—Rotary hydraulic servomotor instrumented for recording transient response to a step input.

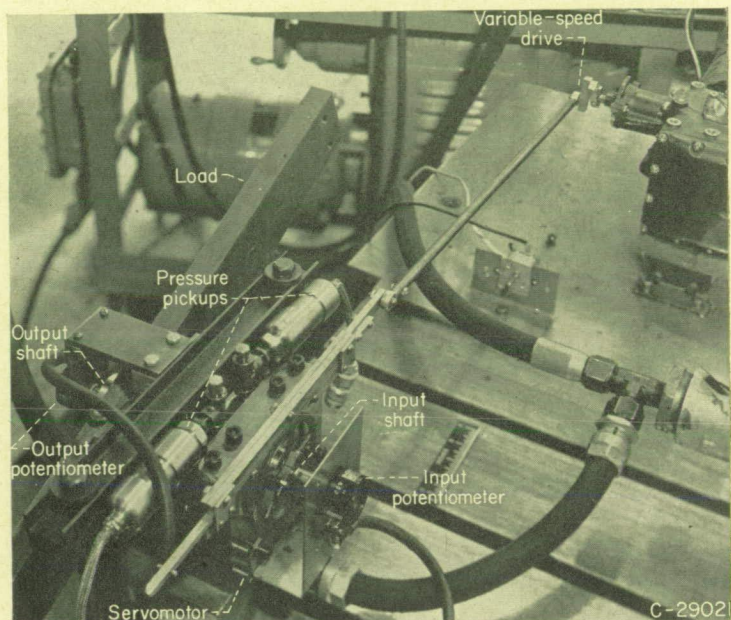


FIGURE 15.—Rotary hydraulic servomotor instrumented for recording response to a sinusoidally varying input.

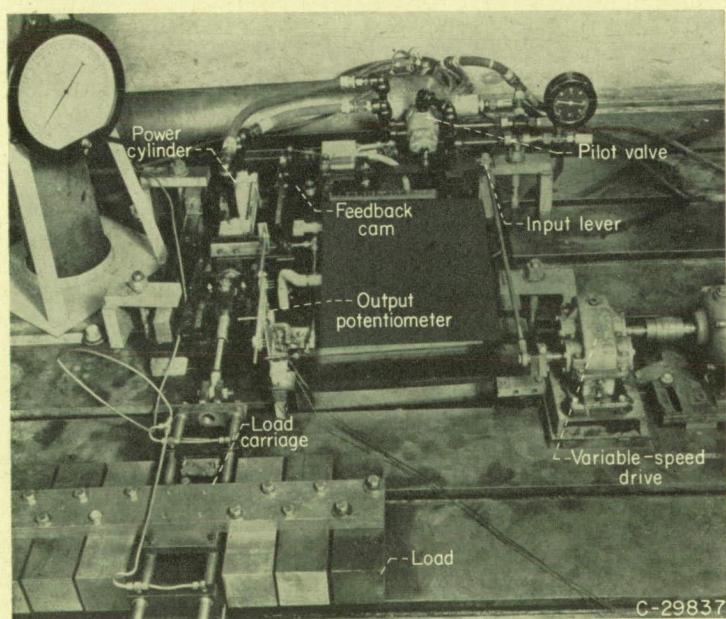


FIGURE 16.—Straight-line hydraulic servomotor instrumented for recording response to a sinusoidally varying input.

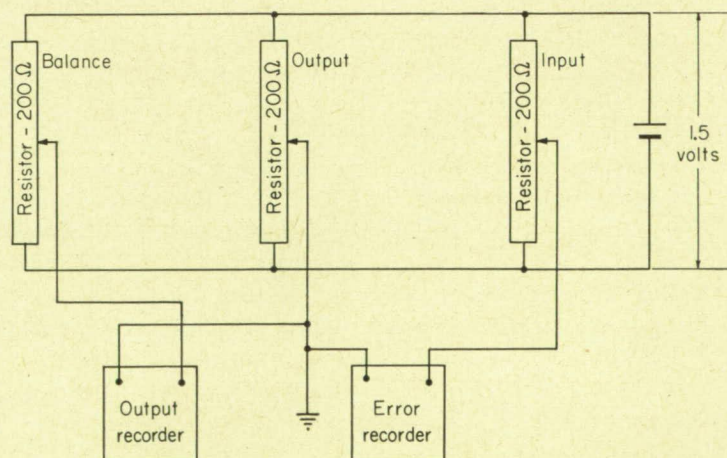


FIGURE 17.—Instrumentation for determining frequency-response characteristics of hydraulic servomotor.

The moment of inertia of the load was varied by bolting additional weights to the ends of the bar that was fastened to the output shaft of the servomotor. For the no-load runs a light-weight arm was used in place of the bar that is shown in figure 14.

FREQUENCY-RESPONSE APPARATUS

Drive apparatus.—The rotary-servomotor setup for frequency-response measurement is shown in figure 15. In the photograph of figure 15 the servomotor input shaft is on the right-hand side. A rack and gear assembly is coupled to the input shaft. The rack is connected to a variable-stroke crank that is driven by a variable-speed transmission. The drive had a range from 0.1 to 20 cycles per second.

The straight-line-servomotor setup for frequency-response measurements is shown in figure 16. The input lever is linked directly to a variable-speed, variable-stroke drive. The output potentiometer is coupled to the output shaft by means of a rack and pinion assembly. The servomotor is loaded by means of weights that are bolted to a sliding carriage. Input motion was not measured in this apparatus. The variable-speed drive had a range from 0.1 to 11 cycles per second.

Output and phase-angle measurement.—Phase angle was measured only in the case of the rotary servomotor. The circuit diagram showing the method of connecting the potentiometers to the recorder amplifiers is shown in figure 17. By means of the arrangement shown, the output motion and the error between the output and input shaft position are recorded. In the diagram the potentiometers marked input and output are the two that are visible in figure 15 and are coupled directly to the input and output shaft, respectively. The balance potentiometer is uncoupled.

The output attenuation ratio is obtained directly from the oscillograph traces. The phase angle is obtained by means of the graphical construction shown in figure 18. The input amplitude is laid out to an arbitrary scale. With the use of this scale, the output-amplitude ratio is swung as an arc from the starting point of the input vector. The error-amplitude ratio, referred to amplitude at infinite frequency (obtained by locking output), is swung as an arc from the opposite end of the input vector. The triangle thus formed yields the phase angle.

The particular advantage of this procedure lies in the relatively greater accuracy with which the amplitude of a wave can be measured compared with the determination of the exact point in the cycle at which the maximum height of the wave occurs.

Friction determination.—In the frequency-response runs made with the straight-line servomotor, the limitations imposed by the sinusoidal drive and pumping equipment restricted the range of frequencies to a maximum of 11 cycles

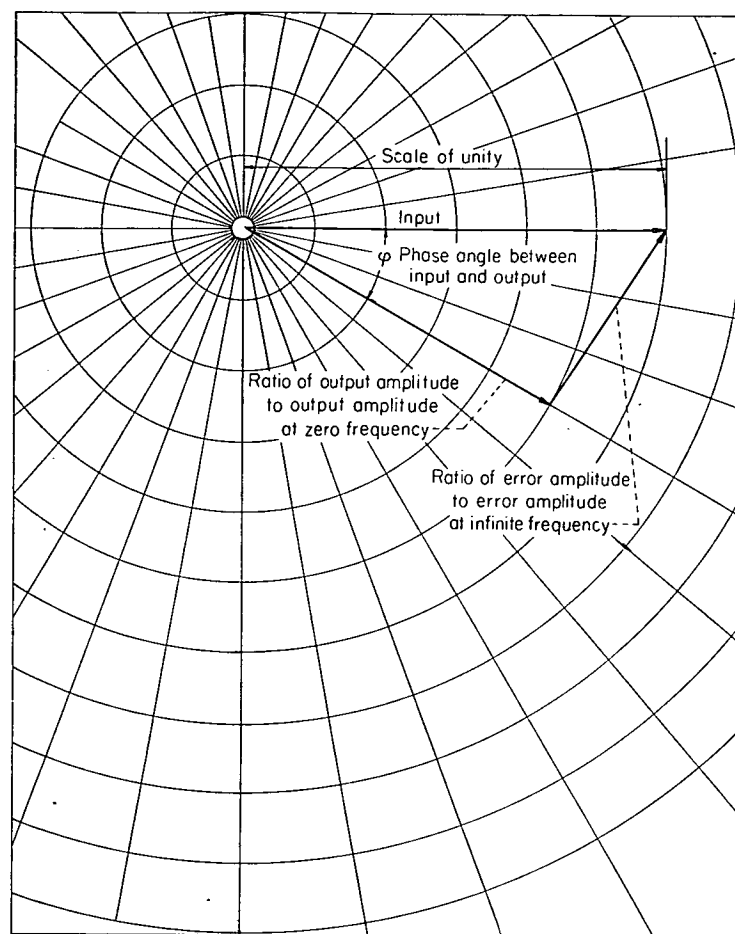


FIGURE 18.—Diagram for determination of phase angle from frequency-response data.

per second. In order to obtain a significant range of amplitude ratios below 11 cycles, it was necessary to make these runs at low pressure differences across the motor. The pressure necessary to overcome friction in the servomotor was approximately 2 pounds per square inch and therefore could be neglected in the calculations. The pressure necessary to overcome friction in the loading carriage was as high as 22.5 pounds per square inch at maximum load. This pressure was defined as the pressure necessary to maintain a steady oscillation with a given load on the carriage. The pressure was found to be substantially independent of the frequency in the range of frequency up to 11 cycles per second. The calculated asymptotes shown in figures 13(c) and (d) were made with the friction pressure subtracted from the measured pressure difference across the motor.

The procedure previously outlined applies only to figures 13(c) and (d). In all the other runs shown, no correction whatever was applied to the measured pressure difference across the motor.

APPENDIX B

DERIVATION OF EQUATIONS FOR TRANSIENT RESPONSE IN WHICH UPSTREAM CYLINDER PRESSURE IS LIMITED AT ABSOLUTE ZERO

In the following sections the formal mathematical operations employed in the derivations of the expressions for the peak cylinder pressure and for the position response are presented.

PEAK CYLINDER PRESSURE

Integration of differential equation for limited pressure phase (eq. (40)).—The differential equation for the phase of the transient in which the upstream cylinder pressure is limited at absolute zero is, from equation (40),

$$\frac{2S}{E^2} \left(\frac{\dot{x}}{S-x} \right)^2 + \ddot{x} = 0 \quad (B1)$$

The order of the equation may be reduced by means of the following general relation:

$$\ddot{x} = \frac{d(\dot{x})}{dt} \left(\frac{dx}{dx} \right) = \dot{x} \frac{d(\dot{x})}{dx} \quad (B2)$$

With the substitution of equation (B2) in equation (B1), the reduced equation is obtained:

$$\frac{2S}{E^2} \left(\frac{\dot{x}}{S-x} \right)^2 + \dot{x} \frac{d(\dot{x})}{dx} = 0$$

Rearranging terms yields

$$\frac{2S}{E^2} \frac{dx}{(S-x)^2} + \frac{d(\dot{x})}{\dot{x}} = 0$$

Integrating each term gives

$$\frac{2S}{E^2} \frac{1}{(S-x)} + \ln \dot{x} = D \quad (B3)$$

from which

$$\dot{x} = e^{\left[D - \frac{2S}{E^2} \frac{1}{(S-x)} \right]} \quad (B4)$$

With use made of the symbols defined in figure 7, the initial conditions are

$$t = t_2$$

$$x = x_2$$

$$\dot{x} = \dot{x}_2$$

Introducing these values in equation (B3) yields

$$D = \frac{2S}{E^2} \frac{1}{(S-x_2)} + \ln \dot{x}_2 \quad (B5)$$

Relation between ratio of peak pressure to supply pressure and inertia index.—From equation (43),

$$\ddot{x}_{max} = \frac{E^2}{2S} (\dot{x})^2 \quad (B6)$$

From equation (42) the value of x when $\ddot{x}$ is at the maximum value is given by the following relation:

$$S - x_m = \frac{2S}{E^2} \quad (B7)$$

With the substitution of equation (B7) into equation (B4), an expression is obtained for the piston velocity when the deceleration is at the maximum value:

$$\dot{x}_m = e^{(D-1)} \quad (B8)$$

Substituting equation (B8) in equation (B6) yields

$$\ddot{x}_{max} = \frac{E^2}{2S} \frac{e^{2D}}{e^2} \quad (B9)$$

Based on the consideration that P_1 equals zero, the relation between the downstream cylinder pressure P_2 and the deceleration is

$$A_p P_2 = M \ddot{x} \quad (B10)$$

Combining equations (B9) and (B10) and dividing by P_s yield

$$\frac{P_{2,max}}{P_s} = \frac{ME^2 e^{2D}}{2A_p S e^2 P_s} \quad (B11)$$

Based on the consideration that P_1 and P_a are zero, the following relation is derived from equation (27):

$$P_s = \frac{4MS}{E^2 T^2 A_p} \quad (B12)$$

Substituting equation (B12) in the right-hand side of equation (B11) gives

$$\frac{P_{2,max}}{P_s} = \frac{E^4 T^2 e^{2D}}{8S^2 e^2} \quad (B13)$$

Inserting the numerical value of e^2 gives

$$\frac{P_{2,max}}{P_s} = \frac{E^4 T^2 e^{2D}}{59.1 S^2} \quad (B14)$$

The exponent D may be expressed in terms of the relations that have been derived for x_2 and $\dot{x}_2$. From equation (37),

$$S - x_2 = \frac{T \dot{x}_2}{\sqrt{2}} \quad (B15)$$

and from equation (38),

$$\dot{x}_2 = \frac{2S e^{-\left(\frac{\tan^{-1} K}{K}\right)}}{TK} \sin(\tan^{-1} K) \quad (B16)$$

where $K = \sqrt{E^2 - 1}$.

By use of the trigonometric identity

$$\sin a = \frac{\tan a}{\sqrt{1 + \tan^2 a}}$$

it can be shown that

$$\sin(\tan^{-1} K) = \frac{K}{\sqrt{1 + K^2}}$$

Furthermore,

$$\sqrt{1+K^2}=E$$

Hence,

$$\sin(\tan^{-1} K) = \frac{K}{E}$$

and equation (B16) may be written

$$\dot{x}_2 = \frac{2Se^{-(\frac{\tan^{-1} K}{K})}}{TE} \quad (B17)$$

Substituting equation (B17) in equation (B15) yields

$$(S-x_2) = \frac{\sqrt{2}Se^{-(\frac{\tan^{-1} K}{K})}}{E} \quad (B18)$$

Substituting equations (B17) and (B18) in equation (B5) gives

$$D = \frac{\sqrt{2}e^{(\frac{\tan^{-1} K}{K})}}{E} - \frac{\tan^{-1} K}{K} + \ln \frac{2S}{TE} \quad (B19)$$

Let

$$\frac{\sqrt{2}e^{(\frac{\tan^{-1} K}{K})}}{E} - \frac{\tan^{-1} K}{K} = F(E)$$

Then

$$D = F(E) + \ln \frac{2S}{TE}$$

and

$$e^{2D} = \frac{4S^2 e^{[2F(E)]}}{T^2 E^2} \quad (B20)$$

Substituting equation (B20) in equation (B14) gives

$$\frac{P_{2,max}}{P_s} = \frac{E^2 e^{[2F(E)]}}{14.77} \quad (B21)$$

It can be seen from figure 7 that equation (B21) will yield real values only if the following relation exists:

$$\frac{x_m}{S} > \frac{x_2}{S}$$

Because $\frac{x_m}{S}$ and $\frac{x_2}{S}$ are both functions solely of E , the value of E when $\frac{x_m}{S} = \frac{x_2}{S}$ represents the limiting value of E for real values of $\frac{P_{2,max}}{P_s}$.

From equation (B7),

$$\frac{x_m}{S} = 1 - \frac{2}{E^2} \quad (B22)$$

From equation (B18)

$$\frac{x_2}{S} = 1 - \frac{\sqrt{2}e^{-(\frac{\tan^{-1} \sqrt{E^2-1}}{\sqrt{E^2-1}})}}{E} \quad (B23)$$

Equating equations (B7) and (B18) gives

$$E = \sqrt{2}e^{(\frac{\tan^{-1} \sqrt{E^2-1}}{\sqrt{E^2-1}})} \quad (B24)$$

The value of E that satisfies this relation is 2.38.

Because $\frac{x_m}{S}$ approaches unity more rapidly than $\frac{x_2}{S}$ as E increases, $\frac{P_{2,max}}{P_s}$ as defined by equation (B21) has real values for $E > 2.38$. The validity of this proof is demonstrated by the evaluation of equation (B21) at $E = 2.38$. Inserting this value of E in equation (B21) yields

$$\frac{P_{2,max}}{P_s} = 1$$

POSITION RESPONSE

Determination of initial coordinates of phase II.—From equation (37),

$$\frac{x_2}{S} = 1 - \frac{T}{\sqrt{2}} \frac{\dot{x}_2}{S} \quad (B25)$$

From equation (38),

$$\frac{\dot{x}_2}{S} = \frac{2e^{-(\frac{\tan^{-1} \sqrt{E^2-1}}{\sqrt{E^2-1}})}}{ET} \quad (B26)$$

Substituting equation (B26) in equation (B25) yields

$$\frac{x_2}{S} = 1 - \frac{\sqrt{2}e^{-(\frac{\tan^{-1} \sqrt{E^2-1}}{\sqrt{E^2-1}})}}{E} \quad (B27)$$

The velocity is constant from the point $\frac{x_1}{S}$ to the point $\frac{x_2}{S}$; hence,

$$t_2 = t_1 + \left[\frac{\frac{x_2}{S} - \frac{x_1}{S}}{\frac{\dot{x}}{S}} \right] \quad (B29)$$

Integration of differential equation for phase II.—The first step in this integration is presented in the previous section in which equation (B1) is integrated to $\dot{x}$ (eq. (B4)). Let

$$\frac{2S}{E^2} \frac{1}{(S-x)} = u$$

$$du = \frac{2S}{E^2} (S-x)^{-2} dx$$

$$dx = \frac{2S}{E^2 u^2} du$$

Making these substitutions in equation (B4) and rearranging terms yield

$$dt = \left(\frac{2Se^{-D}}{E^2} \right) \frac{e^u}{u^2} du \quad (B30)$$

$$\int \frac{e^u}{u^2} du = \ln u - \frac{e^u}{u} + \sum_{n=1}^{\infty} \frac{u^n}{n \times n!} \quad (B31)$$

The integrated equation is then

$$t = \left(\frac{2Se^{-D}}{E^2} \right) \left(\ln u - \frac{e^u}{u} + \sum_{n=1}^{\infty} \frac{u^n}{n \times n!} + H \right) \quad (B32)$$

The constant H is evaluated by introducing the initial conditions

$$t = t_2$$

$$x = x_2$$

Hence,

$$H = t_2 - \frac{2S}{E^2} e^{-D} \left\{ \ln z - \frac{e^z}{z} + \sum_{n=1}^{\infty} \frac{z^n}{n \times n!} \right\} \quad (\text{B33})$$

where

$$z = \frac{2S}{E^2} \frac{1}{(S - x_2)}$$

Determination of initial conditions for phase III (fig. 7).—

As defined in figure 7, the coordinates of the junction of phases II and III are

$$x = x_3$$

$$t = t_3$$

As further defined, the following conditions exist at $t = t_3$:

$$P_1 = P_2 \cong 0$$

$$P_2 = P_s$$

$$\Delta P_p = -P_s$$

$$\Delta P_{v,i} = \Delta P_{v,d} = P_s$$

Hence,

$$\ddot{x}_3 = -\frac{A_p P_s}{M}$$

By equation (7)

$$\dot{x}_3 = -\frac{CRW}{A_p} \sqrt{P_s} (S - x_3) \quad (\text{B34})$$

Substituting equation (36) yields

$$\dot{x}_3 = \frac{\sqrt{2}}{T} (S - x_3) \quad (\text{B35})$$

From equation (B4) a second velocity-position relation is obtained:

$$\dot{x}_3 = e^{\left(D - \frac{2S}{E^2} \frac{1}{(S - x_3)} \right)} \quad (\text{B36})$$

Combining equations (B35) and (B36) yields

$$D = \frac{2S}{E^2} \frac{1}{(S - x_3)} + \ln \frac{\sqrt{2}}{T} (S - x_3) \quad (\text{B37})$$

The constant D is evaluated by means of equation (B19) and x_3 is determined graphically from equation (B37). The coordinate t_3 is evaluated by means of x_3 and equation (B32).

Equation for phase III of response.—Phase III is identical with the deceleration phase of figure 3. Therefore, following the derivation of equation (34), the equation of the response in phase III is

$$\frac{x}{S} = 1 - \left(1 - \frac{x_3}{S} \right) e^{-\left(\frac{t - t_3}{T} \right)} \quad (\text{B38})$$

REFERENCE

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REPORT 1126

THE EFFECT OF BLADE-SECTION THICKNESS RATIOS ON THE AERODYNAMIC CHARACTERISTICS OF RELATED FULL-SCALE PROPELLERS AT MACH NUMBERS UP TO 0.65¹

By JULIAN D. MAYNARD and SEYMOUR STEINBERG

SUMMARY

The results of an investigation of two 10-foot-diameter, two-blade NACA propellers are presented for a range of blade angles from 20° to 55° at airspeeds up to 500 miles per hour. These results are compared with those from previous investigations of five related NACA propellers in order to evaluate the effects of blade-section thickness ratios on propeller aerodynamic characteristics.

The envelope efficiencies of all the NACA propellers are high at the lower rotational speeds at which the adverse effects of compressibility are small. The highest efficiencies, about 93 percent at a helical tip Mach number of 0.9 and 84 percent at a helical tip Mach number of 1.1, reflect the importance of using thin, efficient airfoil sections throughout the blade. For propeller operation at constant rotational speed and power at helical tip Mach numbers below 0.8, a reduction in blade-section thickness from 12 to 8 percent at the 0.7-radius station, or approximately one-third all along the radius, results in gains in propeller efficiency up to 10 percent.

The maximum efficiency of a propeller operating at a helical tip Mach number of 1.1 and Mach number of advance of 0.625 may be increased approximately 20 percent by reducing the blade-section thickness from 12 to 5 percent at the 0.7-radius station. At this same condition of operation for propellers having blade-section thicknesses between 12 and 8 percent at the 0.7-radius station, the maximum efficiency increases approximately 3 percent for each decrease in thickness of 1 percent at this station. For blade-section thicknesses between 8 and 5 percent at the 0.7-radius station, the rate of increase in propeller efficiency with reductions in blade-section thickness is smaller, but further reductions in thickness may still improve the maximum efficiency of propellers operating at high forward speeds with helical tip Mach numbers as high as 1.1.

INTRODUCTION

A general investigation of the aerodynamic characteristics of a series of full-scale 10-foot-diameter propellers at airspeeds up to 500 miles per hour has been made in the Langley 16-foot high-speed tunnel. The purpose of this investigation was to determine the combined influence of propeller-design

parameters and air compressibility upon propeller performance. The blade designs embody variations in shank form, blade airfoil section, design lift coefficient or camber, blade width, and blade thickness ratio. Most of the blades have the high-critical-speed NACA 16-series airfoil sections (ref. 1) and have been designed, without consideration of compressibility effects, for a minimum induced-energy loss when operating as four-blade propellers at an advance ratio of 2.1 and a blade angle of 45° at the 0.7-radius station.

The primary effects of blade-section camber on propeller performance have been presented in reference 2, and the data showing the characteristics of other related propellers in the series have been presented in references 3 to 9. This report presents the aerodynamic characteristics of two propellers and extends the investigation of related propellers to include those having thickness ratios as low as 0.05 at the 0.7-radius station. The purpose of the report is to make a comparison of the performance data for these two propellers with the data contained in references 3, 5, 7, and 8 in order to afford an evaluation of the effects of blade-section thickness ratio on propeller aerodynamic characteristics.

The thickness ratio of propeller blade sections is of increasing importance in the design of propellers for high speeds because of the compromise which must be made between structural requirements and the requirements for thin high-critical-speed sections which are necessary to avoid excessive compressibility losses. Compressibility effects have long been known to cause radical changes in the characteristics of the sections along a propeller blade, and a lack of suitable airfoil section characteristics at the present time has made it necessary to evaluate by means of propeller tests the effect of blade-section thickness ratio upon propeller performance.

SYMBOLS

B	number of blades
b	blade width, ft
C_P	propeller power coefficient, $P/\rho n^3 D^5$
C_T	propeller thrust coefficient, $T/\rho n^2 D^4$
c_{l_d}	blade-section design lift coefficient
D	propeller diameter, ft

¹ Supersedes the recently declassified NACA RM L9D29, "The Effect of Blade-Section Thickness Ratios on the Aerodynamic Characteristics of Related Full-Scale Propellers at Mach Numbers up to 0.65" by Julian D. Maynard and Seymour Steinberg, 1949.

h	blade-section maximum thickness, ft
J	propeller advance ratio, V/nD
M	Mach number of advance
M_t	helical tip Mach number, $M\sqrt{1+\left(\frac{\pi}{J}\right)^2}$
n	propeller rotational speed, rps
P	power absorbed by propeller, ft-lb/sec
r	radius at any blade section, ft
R	propeller tip radius
T	propeller thrust, lb
V	velocity of advance, fps
x	fraction of propeller tip radius, r/R
β	blade angle, deg
$\beta_{0.75R}$	blade angle at 0.75-radius station, deg
η	propeller efficiency, $J \frac{C_T}{C_P}$
η_i	induced efficiency
ρ	mass density of air, slugs/cu ft
σ	solidity, $B \frac{b}{D} / \pi x$

APPARATUS

PROPELLER DYNAMOMETER

Photographs of the 2,000-horsepower dynamometer in the test section of the Langley 16-foot high-speed tunnel with the tunnel open and closed are shown in figures 1 and 2, respectively. A diagram showing the important dimensions of the propeller dynamometer and its location with respect to the test section is presented as figure 3. A detailed description of all the test apparatus and the methods of measuring thrust and torque are presented in reference 3. The fairing profile was calculated from a distribution of sources and sinks to produce a body of revolution with uniform axial velocity in the plane of the propeller. This axial-velocity distribution has been checked experimentally and found to be uniform within 1 percent. The gap between the propeller blade and the spinner surface at the propeller blade-spinner juncture is very small (fig. 1) but is not sealed.

PROPELLER BLADES

The two propellers for which data are presented in this report are the NACA 10-(3)(062)-045A and NACA 10-(3)(05)-045. The NACA design numbers are descriptive of the shape, size, and aerodynamic characteristics of the blades used in this investigation. The first group of digits represents the propeller diameter in feet, and the remaining groups of digits indicate the design lift coefficient, thickness ratio, and solidity per blade at the 0.7-radius station. The following table shows the blade design numbers of the various propellers discussed in this report and also shows the significance of the groups of digits in the number designation:

NACA design number	c_{l_d} at 0.7R	h/b at 0.7R	σ/B at 0.7R
10-(3)(08)-03	0.3	0.08	0.03
10-(3)(08)-03R	.3	.08	.03
10-(3)(12)-03	.3	.12	.03
10-(3)(05)-045	.3	.05	.045
10-(3)(062)-045	.3	.062	.045
10-(3)(062)-045A	.3	.062	.045
10-(3)(08)-045	.3	.08	.045

The suffix R indicates a blade having conventional round shank sections, and the suffix A indicates a blade with modified shank sections. The NACA 16-series blade sections were used for all the propellers listed in the table, and, with the exception of the NACA 10-(3)(08)-03R blade, wide airfoil sections extend to the spinner. The spinner has a diameter equal to 21.7 percent of the diameter of a 10-foot propeller.

Figure 4 shows the blade-form curves for the NACA propellers having a solidity of 0.03 per blade at the 0.7-radius station, and figure 5 shows a comparison of the blade sections at two radial stations for the same group of propellers. The blade designs are closely related, but two of the propellers of this group differ not only in thickness but also in distribution of section design lift coefficient, blade width, and pitch distribution. These differences between the NACA 10-(3)(08)-03 and NACA 10-(3)(08)-03R blades are the result of an effort to maintain the loading for minimum induced-energy loss as far inboard as possible on the round-shanked blade. The NACA 10-(3)(12)-03 blade has the same radial distribution of blade-section design lift coefficient, the same blade width, and approximately the same pitch distribution as the NACA 10-(3)(08)-03 blade, but its thickness is greater at all radii.

Figure 6 shows the blade-form curves and figure 7 shows the blade-section comparisons for the group of propellers having a solidity of 0.045 per blade at the 0.7-radius station. The propellers of this group have the same radial distribution of blade-section design lift coefficient, the same blade width, and approximately the same pitch distribution. The NACA 10-(3)(062)-045 design has the thickest shank sections of this group, although its outboard sections are considerably thinner than those of the NACA 10-(3)(08)-045 design. The NACA 10-(3)(062)-045A design was made by simply thinning the shank sections of the NACA 10-(3)(062)-045 blade until they had the same thickness as the NACA 10-(3)(08)-045 design at the spinner; the thickness of the sections between the spinner and the 0.7-radius station was obtained from a faired line between these two stations. The NACA 10-(3)(05)-045 design was made by thinning the sections of the NACA 10-(3)(062)-045 blade until the sections at the spinner and tip had the same thickness as the NACA 10-(3)(062)-045A blade, but the sections between these two stations were made thinner.

TESTS AND REDUCTION OF DATA

Thrust, torque, and rotational speed were measured during tests at fixed blade angles of 20°, 25°, 30°, 35°, 40°, 45°, 50°, and 55° at the 0.75-radius (45-in.) station. A constant rotational speed was used for most of the tests, and a range of advance ratio was covered by changing the tunnel airspeed, which could be varied from about 60 to 500 miles per hour. The range of blade angles covered at the various rotational speeds used in the tests of the NACA 10-(3)(062)-045A and NACA 10-(3)(05)-045 propellers, together with figure numbers, is shown in table I. Similar information is also shown in table I for the other propellers as taken from references 3, 5, 7, and 8. For the higher blade angles, the complete range of advance ratio could not be covered at the higher rotational

speeds because of power limitations. In order to obtain propeller characteristics at maximum tunnel airspeeds, a blade angle of 45° was chosen, for which the peak-efficiency operating condition could be attained when the tunnel airspeed was at or near the maximum and the dynamometer was operating at its maximum power and rotational speed. For these tests at a blade angle of 45° , the rotational speed was varied to obtain data from the peak-efficiency conditions to the zero-torque operating condition.

The test data have been corrected for tunnel-wall interference and for forces acting on the spinner by the methods described in reference 3 and are presented in the form of the usual thrust and power coefficients and propeller efficiency. Propeller thrust, as used herein, is defined as the shaft tension caused by the spinner-to-tip part of the blade rotating in the air stream. Tests were frequently repeated during the investigation, and, for purposes of comparison, the data are considered accurate within 1 percent and the faired envelopes are believed to be accurate within much closer limits.

RESULTS AND DISCUSSION

Faired curves of thrust coefficient, power coefficient, and propeller efficiency plotted against advance ratio are presented in figures 8 to 16 for the two-blade NACA 10-(3)(062)-045A propeller and in figures 17 to 25 for the two-blade NACA 10-(3)(05)-045 propeller. Test points are shown on the figures giving thrust and power coefficients. The variation of Mach number of advance and helical tip Mach number with advance ratio is shown in the figures giving propeller efficiency.

EFFECT OF BLADE-SECTION THICKNESS RATIO ON ENVELOPE EFFICIENCY

Figure 26 presents a comparison of the envelope efficiencies of NACA propellers 10-(3)(08)-03, 10-(3)(08)-03R, and 10-(3)(12)-03 (refs. 3, 5, and 8) at the various rotational speeds. The thinnest blade of this group, NACA 10-(3)(08)-03, maintains an envelope efficiency of over 0.90 throughout the range of advance ratio of the tests at rotational speeds of 1,140, 1,350, and 1,600 rpm. At these rotational speeds and at the design value of advance ratio, 2.1, the NACA 10-(3)(08)-03 propeller is from 2 to 3 percent more efficient than the round-shanked propeller, NACA 10-(3)(08)-03R, and from 1.5 to 5.5 percent more efficient than the NACA 10-(3)(12)-03 propeller. At the higher rotational speeds (2,000 and 2,160 rpm) the envelope efficiencies of all three propellers are reduced. The propeller with the thinnest blade sections suffers the least reduction in envelope efficiency, whereas the propeller with the thickest outboard blade sections suffers the greatest loss in envelope efficiency. At 2,160 rpm and an advance ratio of 0.90, the envelope efficiencies of the NACA 10-(3)(08)-03R and NACA 10-(3)(12)-03 propellers are lower by 4 and 12 percent, respectively, than the envelope efficiency of the NACA 10-(3)(08)-03 propeller. These differences in envelope efficiency at the higher rotational speeds may be attributed to compressibility effects, which generally lower the lift-drag ratios of thick blade sections at high section Mach numbers.

In figure 26 (b) the envelope efficiencies of the NACA propellers in the 0.03-solidity group are compared with the induced efficiency of a two-blade propeller with the Betz

loading for minimum induced-energy loss. This curve of optimum efficiency was calculated by a method neglecting all profile-drag losses (ref. 10) for a two-blade propeller operating at the same values of power coefficient as were obtained with the NACA 10-(3)(08)-03 propeller. Although the power coefficients for maximum efficiency are slightly different for the three propellers in this group, the values of induced efficiency may be considered accurate within about 1 percent for all three propellers. At the design value of advance ratio, 2.1, the induced losses amount to about 4 percent and the profile-drag losses amount to 3 percent for the NACA 10-(3)(08)-03 propeller. At this same design value of advance ratio, the profile-drag losses of the NACA 10-(3)(08)-03R and NACA 10-(3)(12)-03 propellers are about twice as great as those of the propeller having the thinner blade sections. The higher efficiency of the NACA 10-(3)(08)-03 propeller, about 93 percent at a helical tip Mach number of 0.9 and 84 percent at a helical tip Mach number of 1.1, reflects the importance of using thin, efficient airfoil sections throughout the blade.

A comparison of the envelope efficiencies of the NACA propellers in the 0.045-solidity group is shown in figure 27 for various rotational speeds. Again, the thinnest blade, NACA 10-(3)(05)-045, of the group has the highest efficiency, and the envelope efficiencies of all the propellers are reduced at the higher rotational speeds. Only small differences exist in the envelope efficiencies of the NACA 10-(3)(05)-045 and NACA 10-(3)(062)-045A propellers, although the blade sections of the latter propeller are thicker at all radial stations except at the shank and at the tip. The fact that the thinner blade sections of the NACA 10-(3)(05)-045 propeller do not improve its efficiency much above that of the NACA 10-(3)(062)-045A propeller may possibly be explained by the slightness of the improvements in the lift-drag ratios of the thinner blade sections. Figure 14 of reference 11 shows that there is little difference in the lift-drag ratios of 6- and 9-percent-thick sections (16-3xx airfoils) at lift coefficients up to 0.4. Since a large portion of the radial load is perhaps carried by the blade sections of the NACA propellers having a thickness of 9 percent or less, a reduction in thickness from 6.2 to 5 percent at the 0.7-radius station might be expected to cause only small changes in the propeller efficiency. This result is perhaps true for the conditions of operation under which the tests were made; however, the differences in efficiency between the two propellers may be greater at Mach numbers of advance higher than those used in these tests. If higher Mach numbers of advance and lower rotational speeds had been used to attain the helical tip Mach numbers shown in the figures, greater portions of the blades would be subjected to the effects of compressibility. Since the airfoil data in reference 11 show that, in general, the thinner sections have the higher lift-drag ratios at the higher Mach numbers, a reasonable assumption would be that the propeller having the thinner blade sections along the radius would have smaller efficiency losses due to compressibility. The envelope efficiencies of the NACA 10-(3)(062)-045 and NACA 10-(3)(08)-045 propellers, which had the thickest blade sections, are from 1.5 to 4 percent lower than the

envelope efficiencies of the thinnest propeller in the 0.045-solidity group.

The induced efficiency has been calculated from the values of power coefficient obtained for each of the NACA propellers in the 0.045-solidity group, and the curve showing induced efficiency in figure 27 (b) may be considered accurate within about 1 percent for all four propellers. At the design value of advance ratio, 2.1, the curves in figure 27 (b) show that the induced losses amount to about 5 percent and the profile-drag losses amount to only 2 percent for the NACA propeller having the thinnest blade sections. The NACA 10-(3)(08)-045 propeller, which has the thickest outboard blade sections in this group, suffers the greatest profile-drag loss (about 4 percent).

The envelope efficiencies of all the NACA propellers in both solidity groups are very high, and the differences in efficiency between the various propellers of each group are small and difficult to analyze for some conditions of operation. Where the differences in efficiency are small, the relative differences in thrust and power coefficients are also small, and it is difficult to draw any general conclusion as to whether a loss in efficiency is caused by a loss of thrust or an increase of power, or both.

EFFECT OF THICKNESS RATIO ON CONSTANT-POWER PROPELLER OPERATION

Airplane propellers often operate over an extensive range of advance ratio at constant rotational speed and torque. Since blade-section thickness ratio affects the power-absorption qualities of a propeller to some extent, the data for the NACA propellers have been compared in figure 28 for operation at a constant power coefficient of 0.15 and a rotational speed of 1,140 rpm. This condition of operation may be considered representative for two-blade, 10-foot-diameter propellers and provides a reasonable basis of comparison. An interesting observation in figure 28 (a) is that the drop in efficiency occurs at a lower value of advance ratio but is more gradual for the round-shanked propeller than for the NACA 10-(3)(12)-03 propeller, which has thinner shanks but thicker outboard blade sections than the round-shanked propeller. At advance ratios above 3.2, the thicker outboard sections of the NACA 10-(3)(12)-03 propeller appear to cause greater efficiency losses than the thick shank sections of the NACA 10-(3)(08)-03R propeller. However, the gain in efficiency of about 10 percent at an advance ratio of 3.2 ($M=0.54$), which may be realized by using the thinner blade sections of the NACA 10-(3)(08)-03 propeller, should be emphasized. A reduction in blade-section thickness from 12 to 8 percent at the 0.7-radius station, or approximately one-third all along the radius, resulted in gains in propeller efficiency up to 10 percent; also, a reduction in thickness of only the inboard blade sections (from 30 to 13 percent at the 0.3-radius station) resulted in gains in propeller efficiency up to 10 percent.

The efficiencies for constant-power operation of the propellers in the 0.045-solidity group are compared in figure 28 (b). The differences in efficiency of the propellers in this group do not amount to more than 4 percent, and these efficiencies appear to be about equal at both the lowest and highest values of advance ratio. The single exception is

the propeller which has the thickest shank sections, NACA 10-(3)(062)-045; at the highest value of advance ratio (3.5), the efficiency of this propeller is about 2 percent less than the efficiency of the other three propellers. The curves in figure 28 (b) show that a reduction in blade-section thickness of only the outboard blade sections (from 8 to 5 percent at the 0.7-radius station) resulted in gains in propeller efficiency up to 4 percent. These improvements in efficiency appear to be limited to the range of advance ratio for which the propeller is designed. The helical tip Mach number of the NACA propellers, however, did not exceed 0.8 for the conditions of operation shown in figure 28.

Figures 29 and 30 have been prepared to emphasize the importance of blade-section thickness in the design of propellers to operate at airspeeds where the tip Mach numbers are below the critical value. Figure 29 shows the effect of airspeed on the difference in efficiency between the NACA 10-(3)(08)-03 and NACA 10-(3)(12)-03 two-blade propellers when operating at a constant power coefficient of 0.15 and a constant propeller rotational speed of 1,140 rpm. The thinner blade sections of the NACA 10-(3)(08)-03 propeller effect an increase in efficiency of 10 percent with an increase in airspeed from about 260 to 420 miles per hour. The corresponding change in helical tip Mach number is from 0.63 to 0.76 as shown in figure 29. Figure 30 shows the effect of airspeed on the difference in efficiency between the NACA 10-(3)(05)-045 and NACA 10-(3)(08)-045 two-blade propellers when operating at a constant power coefficient of 0.15 and a constant rotational speed of 1,140 rpm. The thinner outboard blade sections of the NACA 10-(3)(05)-045 propeller effect an increase in efficiency of 4 percent with an increase in airspeed up to 220 miles per hour, but, from 220 to 460 miles per hour, the beneficial effects of the thinner blade sections are gradually lost. Apparently, the thicker outboard blade sections of the NACA 10-(3)(08)-045 propeller can carry their loads just as efficiently as the thinner sections of the NACA 10-(3)(05)-045 propeller at an airspeed of 460 miles per hour. The helical tip Mach number at this airspeed is only 0.8, and beneficial effects of the thinner blade sections might possibly appear for different radial distributions of section Mach number. None of the NACA propellers was designed to operate at advance ratios as high as 3.2; consequently, higher efficiencies might be expected at the higher values of advance ratio because of different pitch distributions. Since the superiority of the thinner outboard blade sections appears principally in the range of advance ratio for which the propeller was designed, substantial gains in efficiency through the use of thinner outboard blade sections may not be realized unless care is used in selecting the radial pitch distribution.

EFFECT OF THICKNESS RATIO AND COMPRESSIBILITY ON PROPELLER CHARACTERISTICS

The effect of compressibility on the maximum efficiency of NACA propellers having different blade-section thicknesses is shown in figure 31 for a blade angle of 45° at the 0.75-radius station. Figure 31 (a) shows that the NACA 10-(3)(12)-03 propeller, which has the thickest outboard blade sections in the 0.03-solidity group, suffers the greatest efficiency losses at the higher tip Mach numbers. These

losses begin for this propeller at a helical tip Mach number of about 0.825, and the loss amounts to 26 percent at a helical tip Mach number of 1.1. The loss in efficiency due to compressibility is more gradual for the NACA 10-(3)(08)-03R propeller, and the serious losses do not begin until a helical tip Mach number of about 0.875 is reached. For this propeller the loss in efficiency due to compressibility amounts to about 19 percent at a helical tip Mach number of 1.1. The maximum efficiency of the NACA 10-(3)(08)-03 propeller, which has the thinnest blade sections in the 0.03-solidity group, is about 2 percent higher than the maximum efficiency of the other two propellers in the range of helical tip Mach numbers below the critical value. The critical value of tip Mach number is perhaps slightly higher for the NACA 10-(3)(08)-03 propeller than for the NACA 10-(3)(08)-03R propeller, and the loss in maximum efficiency due to compressibility amounts to about 16 percent at a helical tip Mach number of 1.1.

Figure 31 (a) shows that a reduction in blade-section thickness from 12 to 8 percent at the 0.7-radius station, or approximately one-third all along the radius, resulted in a gain in propeller efficiency of about 12 percent at a helical tip Mach number of 1.1. At this same helical tip Mach number, a reduction in blade-section thickness of only the inboard blade-sections (from 30 to 13 percent at the 0.3-radius station) resulted in a gain in propeller efficiency of about 5 percent.

Figure 31 (b) shows the effect of compressibility on the maximum efficiency of the NACA propellers in the 0.045-solidity group. Again, the propeller having the thickest outboard blade sections, NACA 10-(3)(08)-045, suffers the greatest efficiency losses at the higher tip Mach numbers. From a helical tip Mach number of 0.90 to 1.1, the efficiency loss due to compressibility for this propeller amounts to 18 percent. Over this same range of helical tip Mach number, the propeller having the thinnest outboard blade sections, NACA 10-(3)(05)-045, has a loss in efficiency due to compressibility of only 9 percent. The NACA 10-(3)(062)-045 propeller, which has the thickest shank sections, shows a loss in efficiency due to compressibility of 13 percent at a helical tip Mach number of 1.1. These losses at the higher tip Mach numbers are more gradual for the NACA 10-(3)(062)-045 propeller than for the NACA 10-(3)(08)-045 propeller, which has thinner blade sections at the shank but thicker outboard blade sections. The critical values of helical tip Mach number are approximately the same for all the propellers in the 0.045-solidity group, and the differences in maximum efficiency are small at tip Mach numbers below the critical value. The maximum efficiency of the NACA 10-(3)(062)-045A propeller is the same as for the thinner NACA 10-(3)(05)-045 propeller except at the lower values of helical tip Mach number, where the thinner blade perhaps has a slight advantage.

The curves in figure 31 (b) show that a reduction in blade-section thickness of only the outboard blade sections (from 8 to 5 percent at the 0.7-radius station) resulted in a gain in propeller efficiency of about 12 percent at a helical tip Mach number of 1.1. At this same helical tip Mach number, a reduction in blade-section thickness of only the inboard blade

sections (from 16.6 to 13 percent at the 0.3-radius station) resulted in a gain in propeller efficiency of about 6 percent. A small reduction in thickness (from 6.2 to 5 percent at the 0.7-radius station) of the blade sections between the shank and the tip had little effect on the maximum propeller efficiency for the conditions of operation tested.

An examination of the thrust and power coefficients of the propellers operating when the effects of compressibility are present may provide a better understanding of the results. In figure 32 the thrust and power coefficients for maximum efficiency are shown plotted against helical tip Mach number for the test propellers at a blade angle of 45° at the 0.75-radius station. The scarcity of data prevents a definite establishment of the critical Mach numbers, but the curves in figure 32 illustrate the trends indicated by the data. The curves are somewhat similar to plots of airfoil lift coefficient against Mach number for constant angles of attack and show that increases in thrust and power coefficient occur before the critical Mach number is reached. The critical Mach number is higher for the propellers having the thinner blade sections. After the critical Mach number is reached, there is a marked decrease in both thrust and power coefficients up to a helical tip Mach number of approximately 1.0, at which the power coefficients begin to increase again and the thrust coefficients either level off or, in the case of the thinner blades, begin to increase again. These changes in thrust and power coefficients which occur with changes in helical tip Mach number are, with one exception, less abrupt for the propellers having the thicker blade sections. The single exception is the NACA 10-(3)(08)-045 propeller, which has relatively thin shank sections but thick outboard blade sections. The curves in figures 31 and 32 show that the radial distribution of blade-section thickness ratios has a pronounced effect on the characteristics of propellers operating at helical tip Mach numbers above the critical value.

Any efficiency comparisons of the propellers in the 0.03-solidity group with those in the 0.045-solidity group in order to study the effects of thickness ratio will include the effects of solidity; however, these effects are small (of the order of 2 percent) and some generalization may be permitted despite the variation in solidity. The NACA propellers discussed in this report are closely related, and the assumption may be made that the radial distribution of camber, solidity, and blade-section thickness is a reasonable optimum for the better propellers. With this generalization in mind, the data for the NACA propellers may serve to indicate the compressibility losses to be expected for propellers having various blade-section thickness ratios.

The curves in figure 33 show the variation of maximum propeller efficiency with thickness ratio at the 0.7-radius station for constant values of helical tip Mach number. At a helical tip Mach number of 0.900 and Mach number of advance of 0.520, the maximum efficiency of a propeller may be increased approximately 7 percent by reducing the blade-section thickness from 12 to 5 percent at the 0.7-radius station. For this Mach number gradient along the blade, the rate of change of maximum propeller efficiency with blade-section thickness is small for thicknesses up to 12 percent at the 0.7-radius station, and figure 33 indicates

that reductions in blade-section thickness below 5 percent at the 0.7-radius station will probably increase the maximum efficiency very little. However, at a helical tip Mach number of 1.1 and a Mach number of advance of 0.625, the maximum efficiency of a propeller may be increased approximately 20 percent by reducing the blade-section thickness from 12 to 5 percent at the 0.7-radius station. For this higher Mach number gradient along the blade, the rate of change of maximum propeller efficiency with blade-section thickness is greater, and, for thicknesses between 12 and 8 percent at the 0.7-radius station, the maximum efficiency increases approximately 3 percent for each decrease in thickness of 1 percent at this station. For thicknesses between 8 and 5 percent at the 0.7-radius station, the rate of increase in propeller efficiency with reductions in blade-section thickness is smaller. Figure 33 indicates that further reductions in thickness may still improve the maximum efficiency of propellers operating at helical tip Mach numbers as high as 1.1, particularly for conditions of operation where the Mach number of advance is high so that large portions of the blades are subjected to the effects of air compressibility.

CONCLUSIONS

An investigation of a series of 10-foot-diameter two-blade NACA propellers differing in blade-section thickness has been completed for a range of blade angles from 20° to 55° at airspeeds up to 500 miles per hour. The results of these investigations have been compared to afford an evaluation of the effects of blade-section thickness ratios on propeller aerodynamic characteristics, and the following conclusions may be drawn:

1. The envelope efficiencies of all the NACA propellers are high at the lower Mach numbers at which the adverse effects of compressibility are small. The higher envelope efficiencies, however, are attained by the propellers having the thinner blade sections. The highest efficiencies, about 93 percent at a helical tip Mach number of 0.9 and 84 percent at a helical tip Mach number of 1.1, reflect the importance of using thin, efficient airfoil sections throughout the blade.

2. For propeller operation at constant rotational speed (1,140 rpm) and power (propeller power coefficient, 0.15) at helical tip Mach numbers below 0.8,

(a) A reduction in blade-section thickness from 12 to 8 percent at the 0.7-radius station, or approximately one-third all along the radius, results in gains in propeller efficiency up to 10 percent.

(b) A reduction in blade-section thickness of only the inboard blade sections (from 30 to 13 percent at the 0.3-radius station) results in gains in propeller efficiency up to 10 percent.

(c) A reduction in blade-section thickness of only the outboard blade sections (from 8 to 5 percent at the 0.7-radius station) results in gains in propeller efficiency up to 4 percent.

3. For operation at a blade angle of 45° at the 0.75-radius station and a helical tip Mach number of 1.1, the loss in maximum propeller efficiency due to compressibility amounts to 26 percent for the NACA propeller having a blade-section thickness of 12 percent at the 0.7-radius station. The corresponding loss in maximum propeller efficiency amounts to only

9 percent for the NACA propeller having a blade-section thickness of 5 percent at the 0.7-radius station.

4. At a helical tip Mach number of 0.900 and Mach number of advance of 0.520, the rate of change of maximum propeller efficiency with blade-section thickness is small for thicknesses up to 12 percent at the 0.7-radius station, and reductions in blade-section thickness below 5 percent at this station will probably increase the maximum efficiency very little.

5. At a helical tip Mach number of 1.1 and Mach number of advance of 0.625, the maximum efficiency of a propeller may be increased approximately 20 percent by reducing the blade-section thickness from 12 to 5 percent at the 0.7-radius station. For blade-section thicknesses between 12 and 8 percent at the 0.7-radius station, the maximum efficiency increases approximately 3 percent for each decrease in thickness of 1 percent. For blade-section thicknesses between 8 and 5 percent at the 0.7-radius station, the rate of increase in propeller efficiency with reductions in blade-section thickness is smaller, but further reductions in thickness may still improve the maximum efficiency of propellers operating at high forward speeds with helical tip Mach numbers as high as 1.1.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., April 25, 1949.

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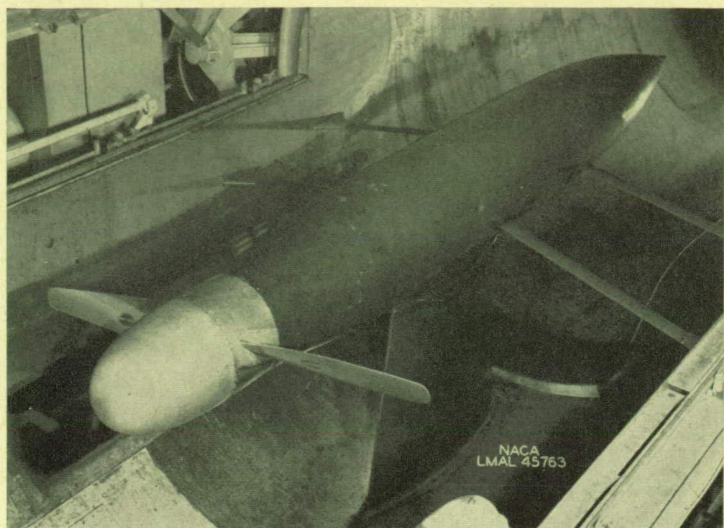


FIGURE 1.—Langley 2,000-horsepower propeller dynamometer in test section with tunnel open.

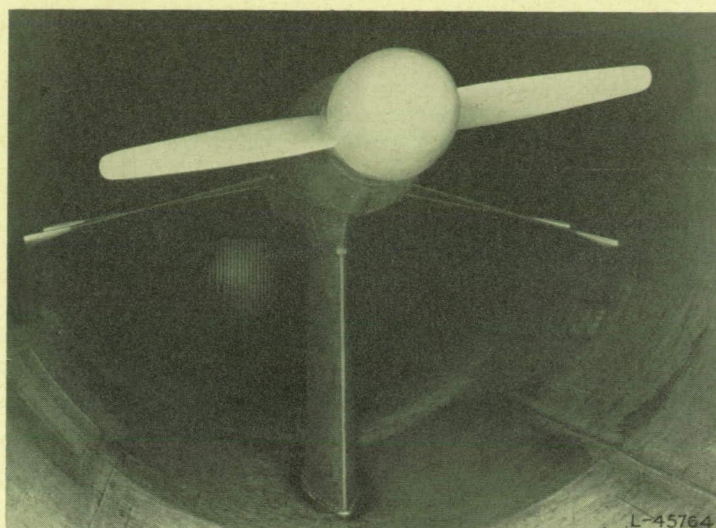


FIGURE 2.—Langley 2,000-horsepower propeller dynamometer in test section with tunnel closed.

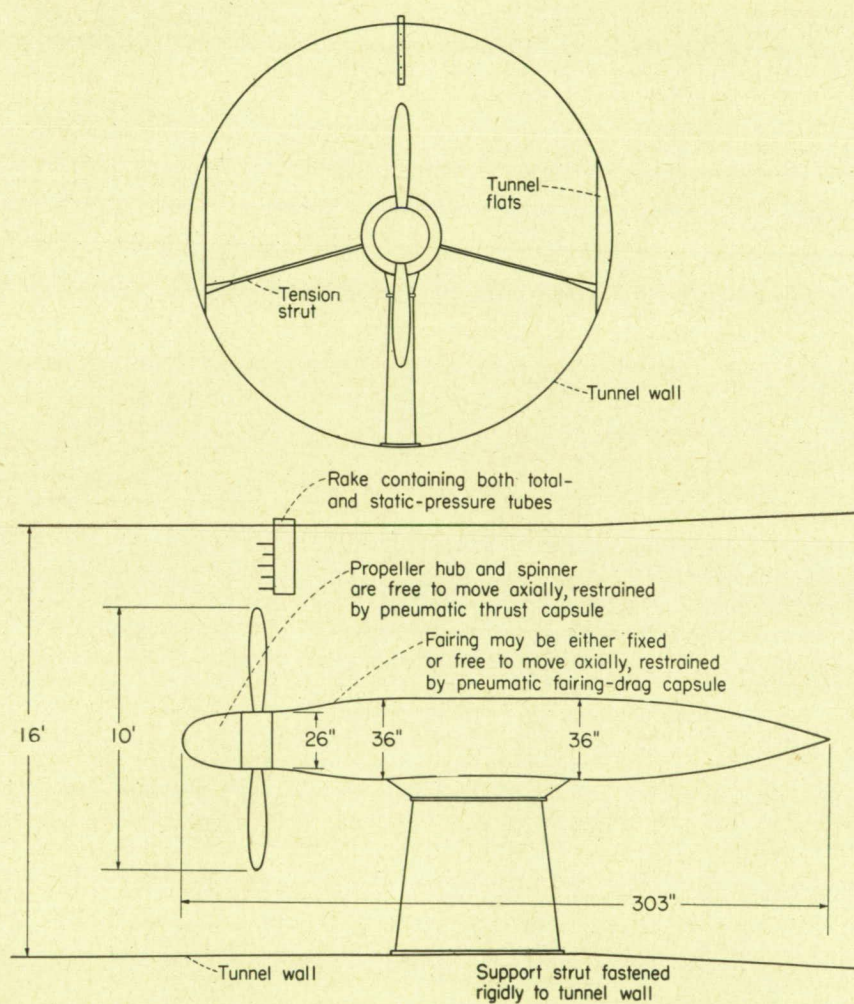


FIGURE 3.—Configuration of 2,000-horsepower dynamometer for tests of propellers in the Langley 16-foot high-speed tunnel.

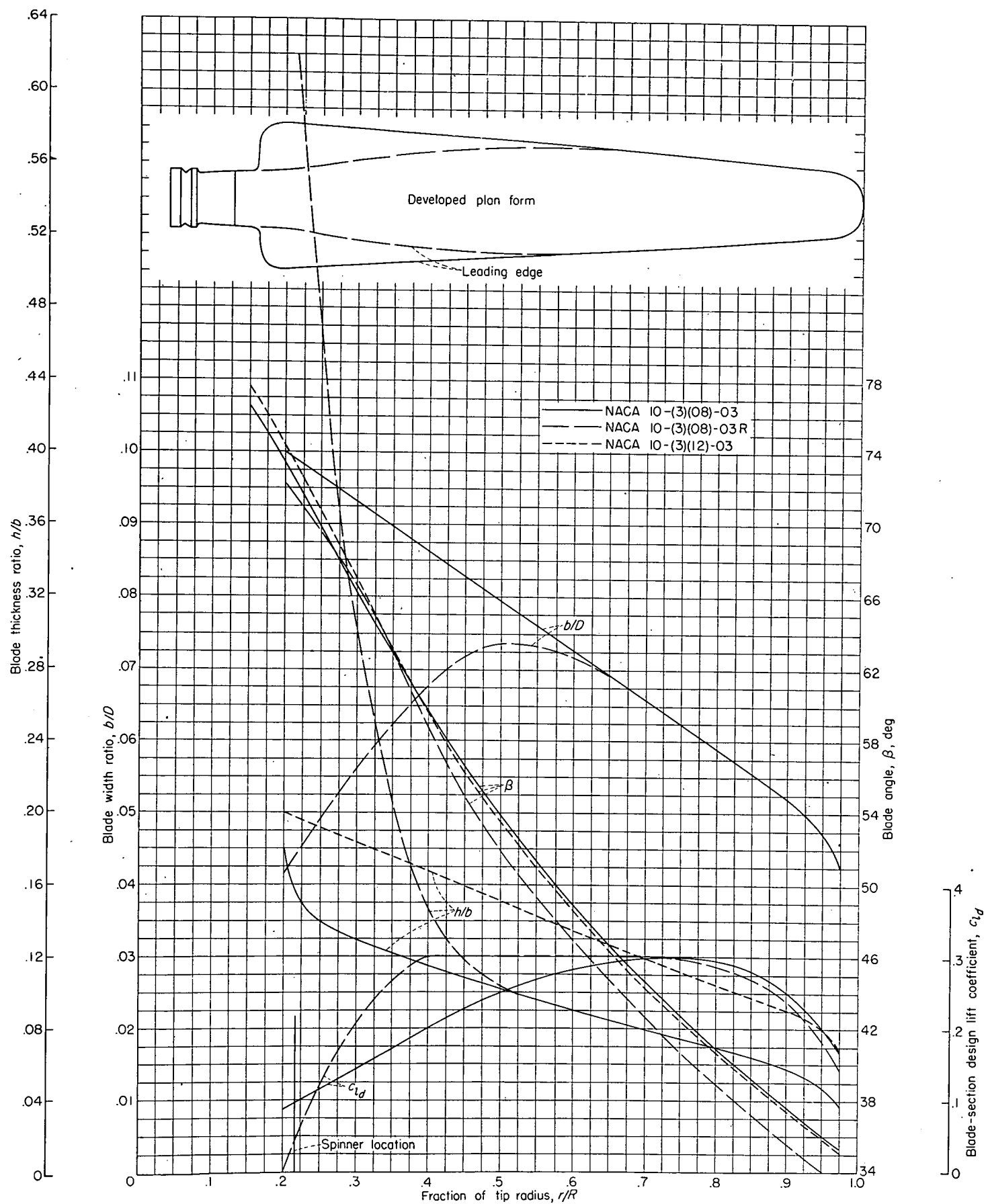


FIGURE 4.—Blade-form curves for NACA propellers having a solidity of 0.03 per blade at the 0.7-radius station.

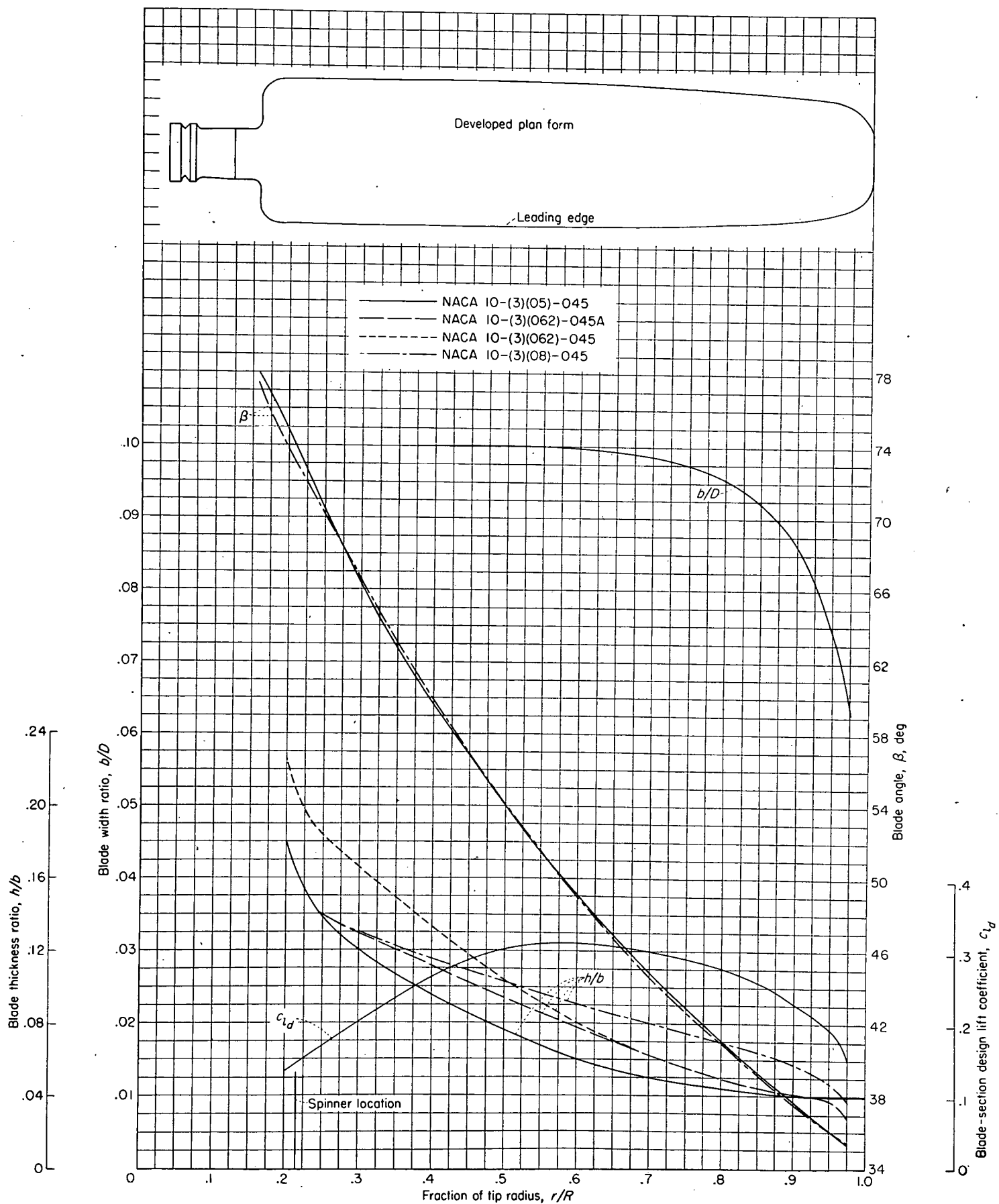


FIGURE 6.—Blade-form curves for NACA propellers having a solidity of 0.045 per blade at the 0.7-radius station.

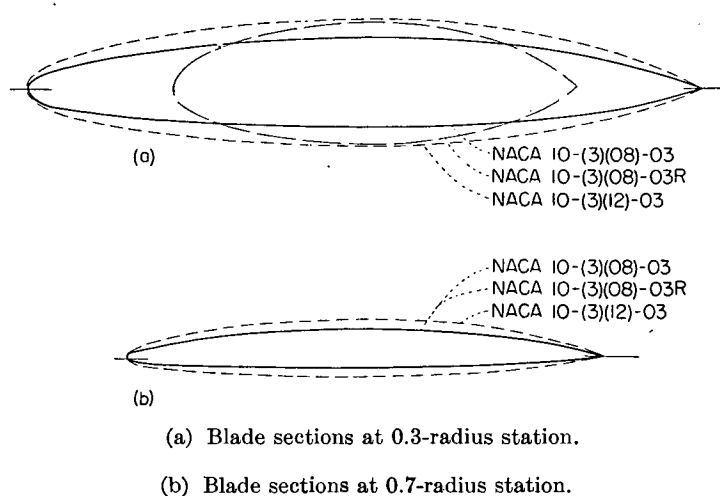


FIGURE 5.—Comparison of blade sections at two radial stations for NACA propellers having a solidity of 0.03 per blade at the 0.7-radius station.

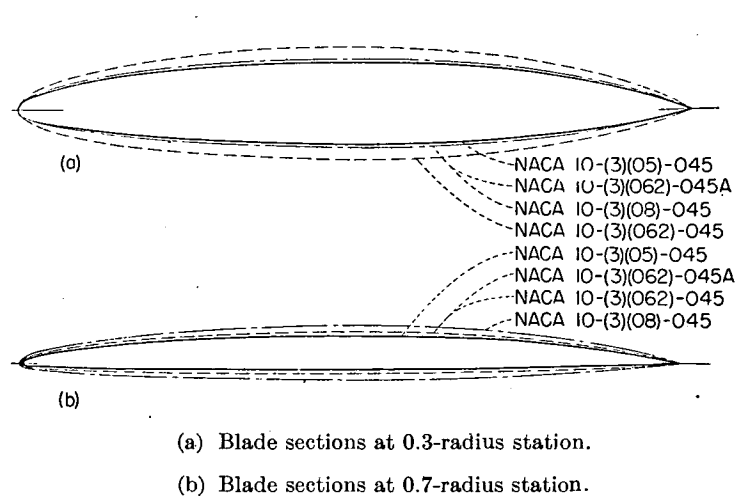


FIGURE 7.—Comparison of blade sections at two radial stations for NACA propellers having a solidity of 0.045 per blade at the 0.7-radius station.

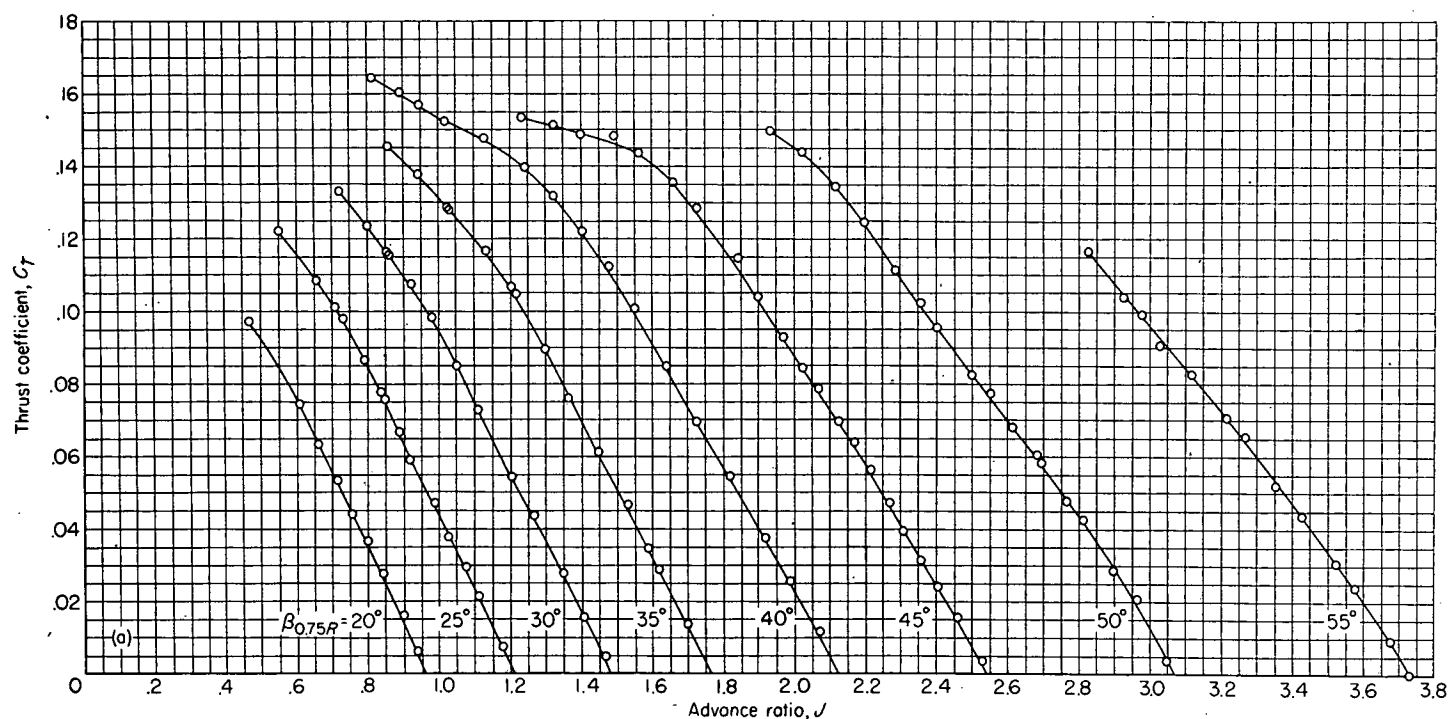
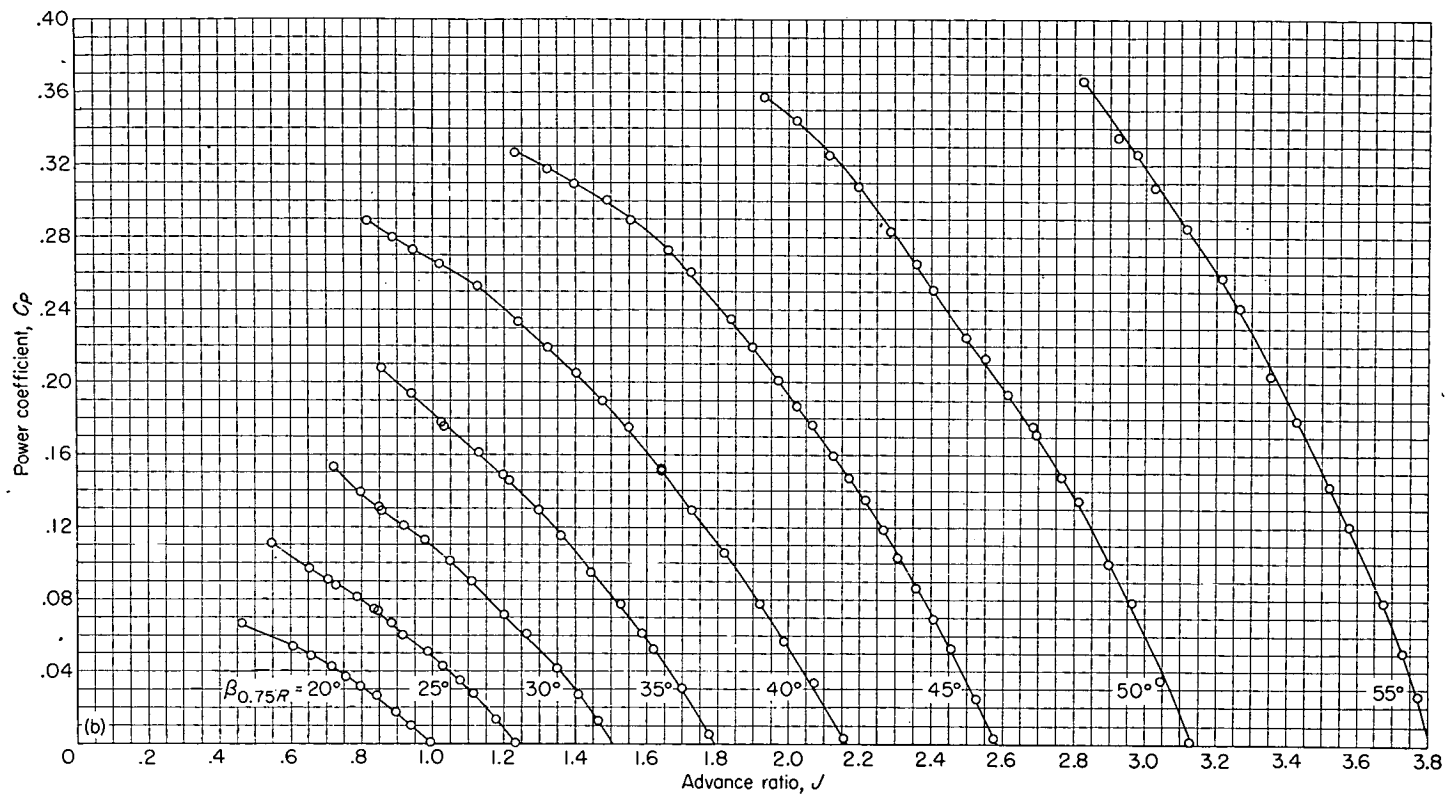
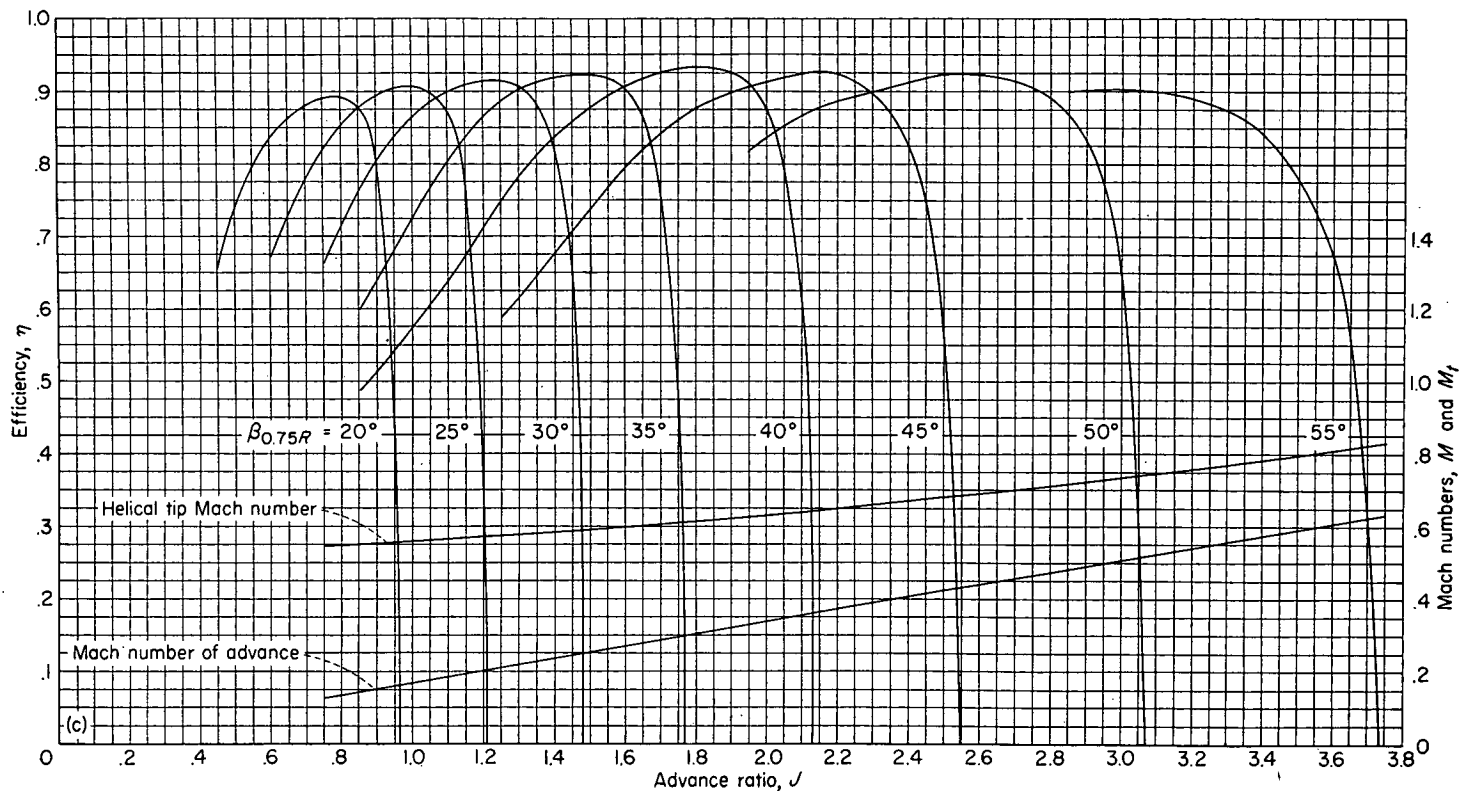


FIGURE 8.—Characteristics of NACA 10-(3)(062)-045A propeller. Rotational speed, 1,140 rpm.

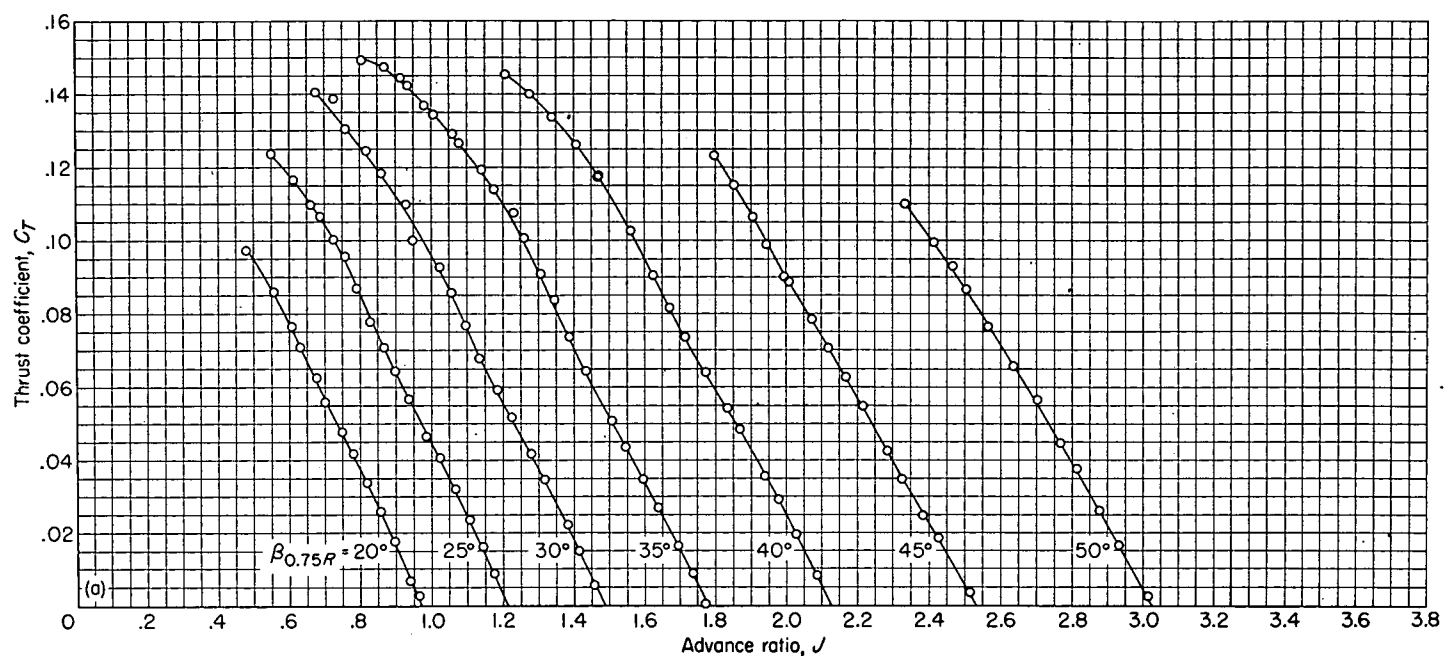


(b) Propeller power coefficient.

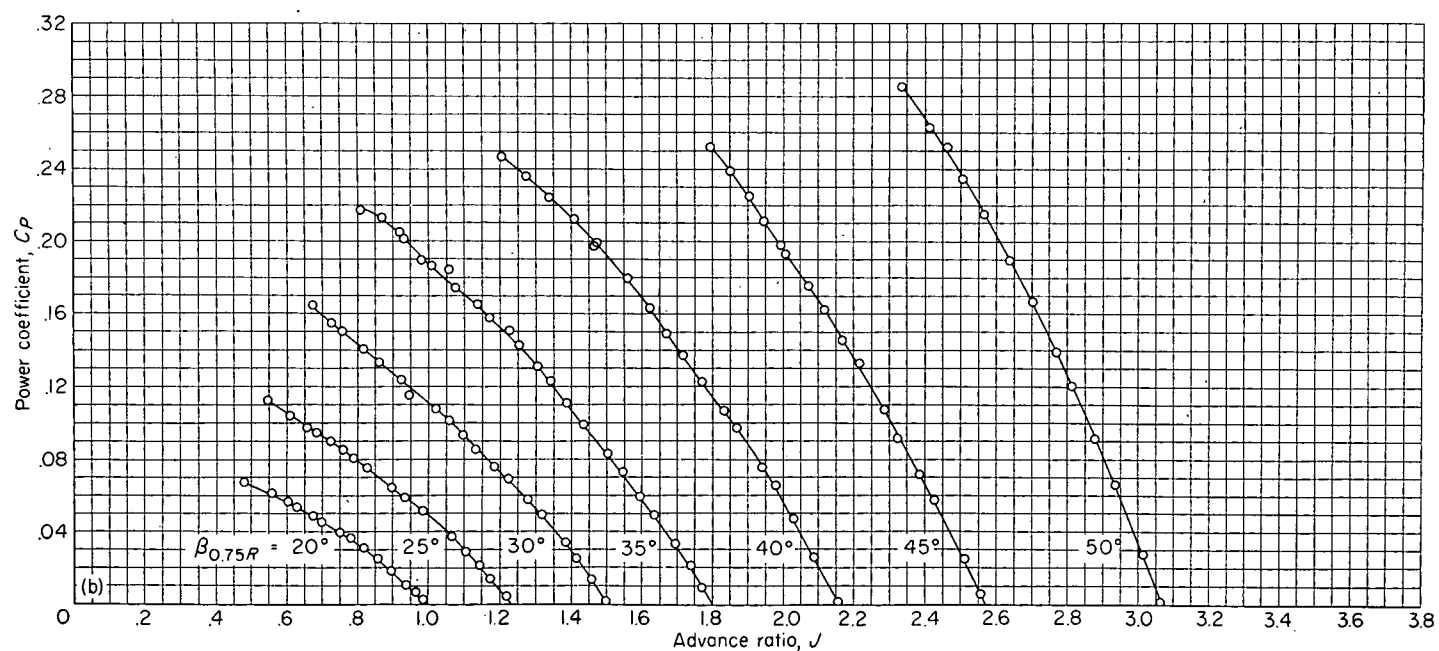


(c) Propeller efficiency.

FIGURE 8—Concluded. Rotational speed, 1,140 rpm.

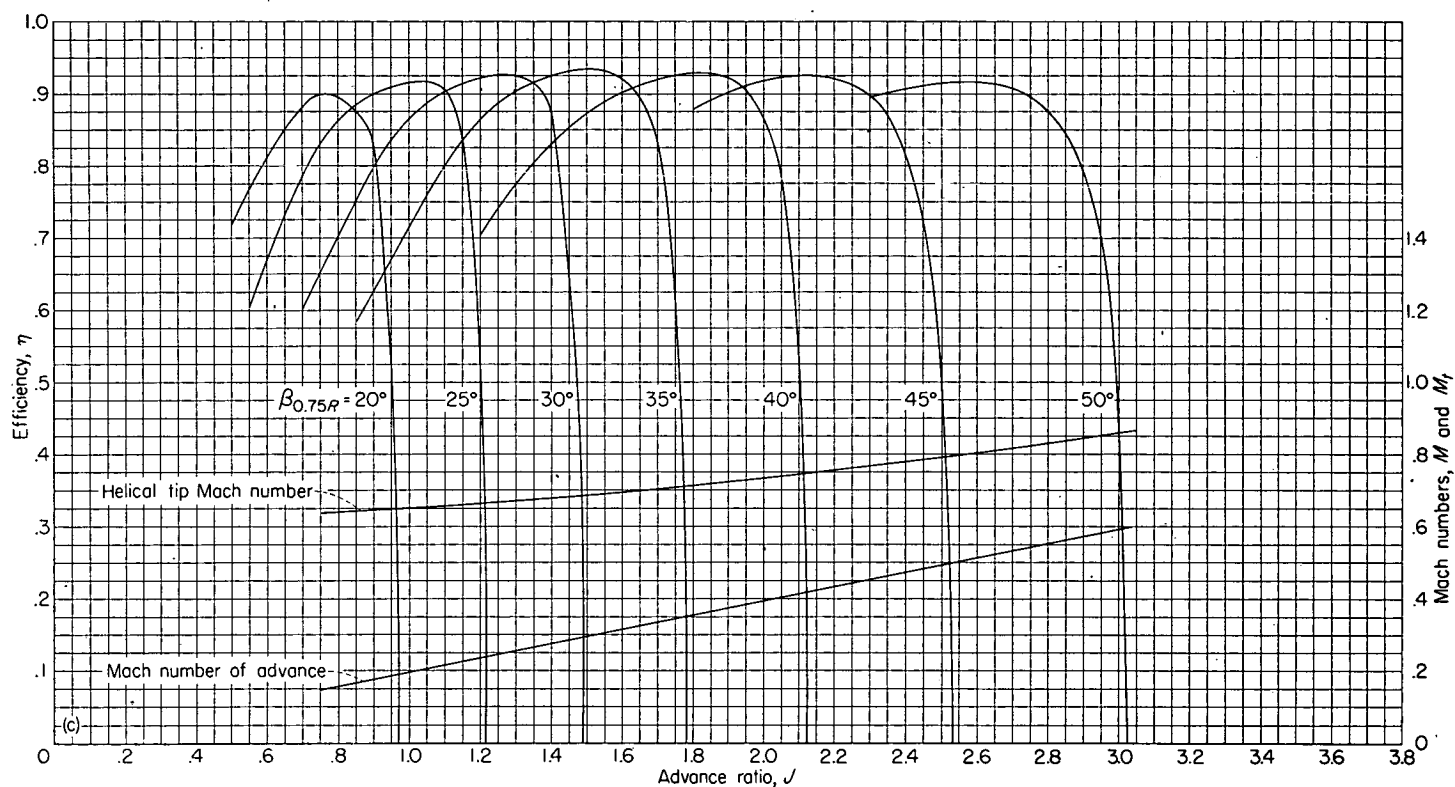


(a) Propeller thrust coefficient.



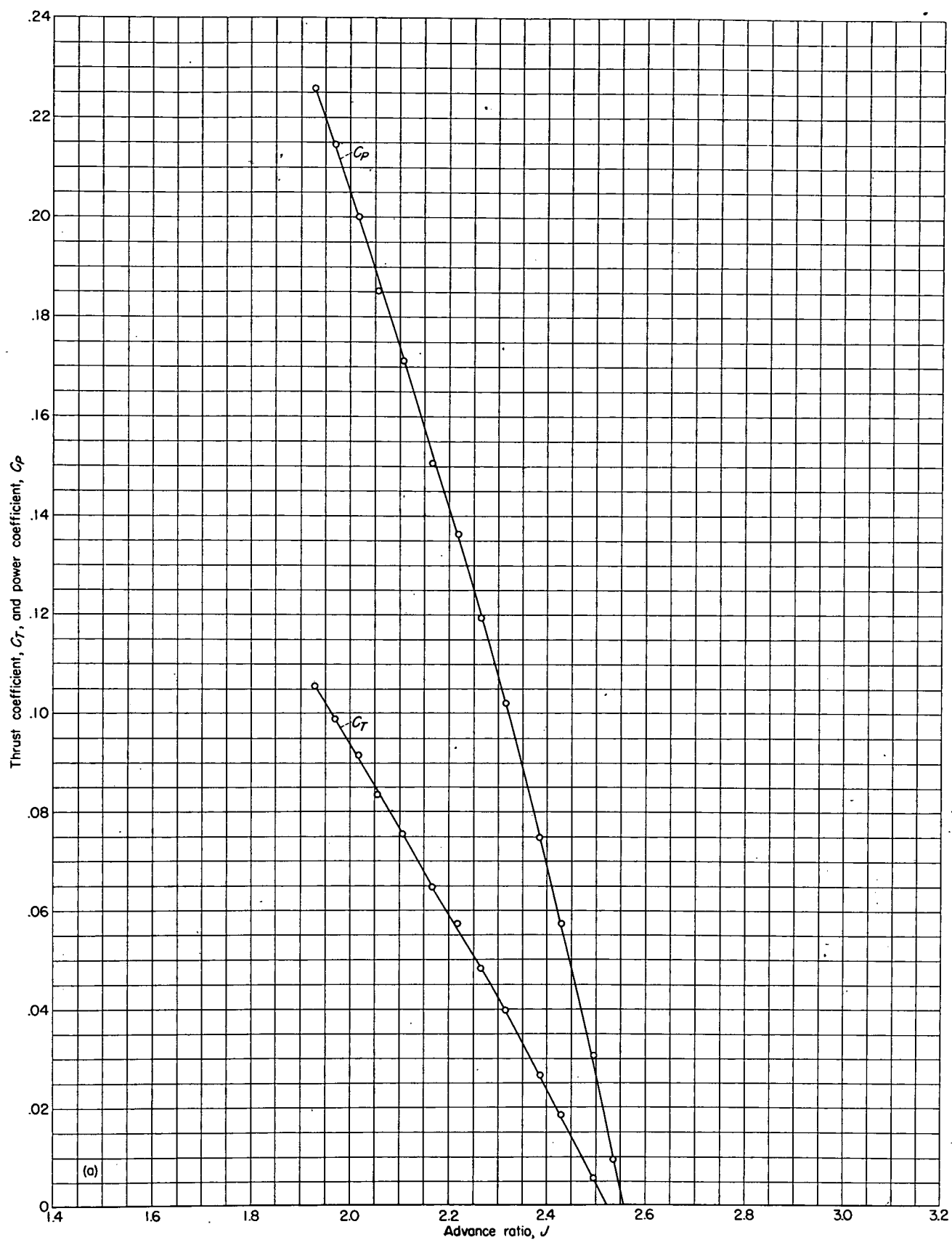
(b) Propeller power coefficient.

FIGURE 9.—Characteristics of NACA 10-(3)(062)-045A propeller. Rotational speed, 1,350 rpm.



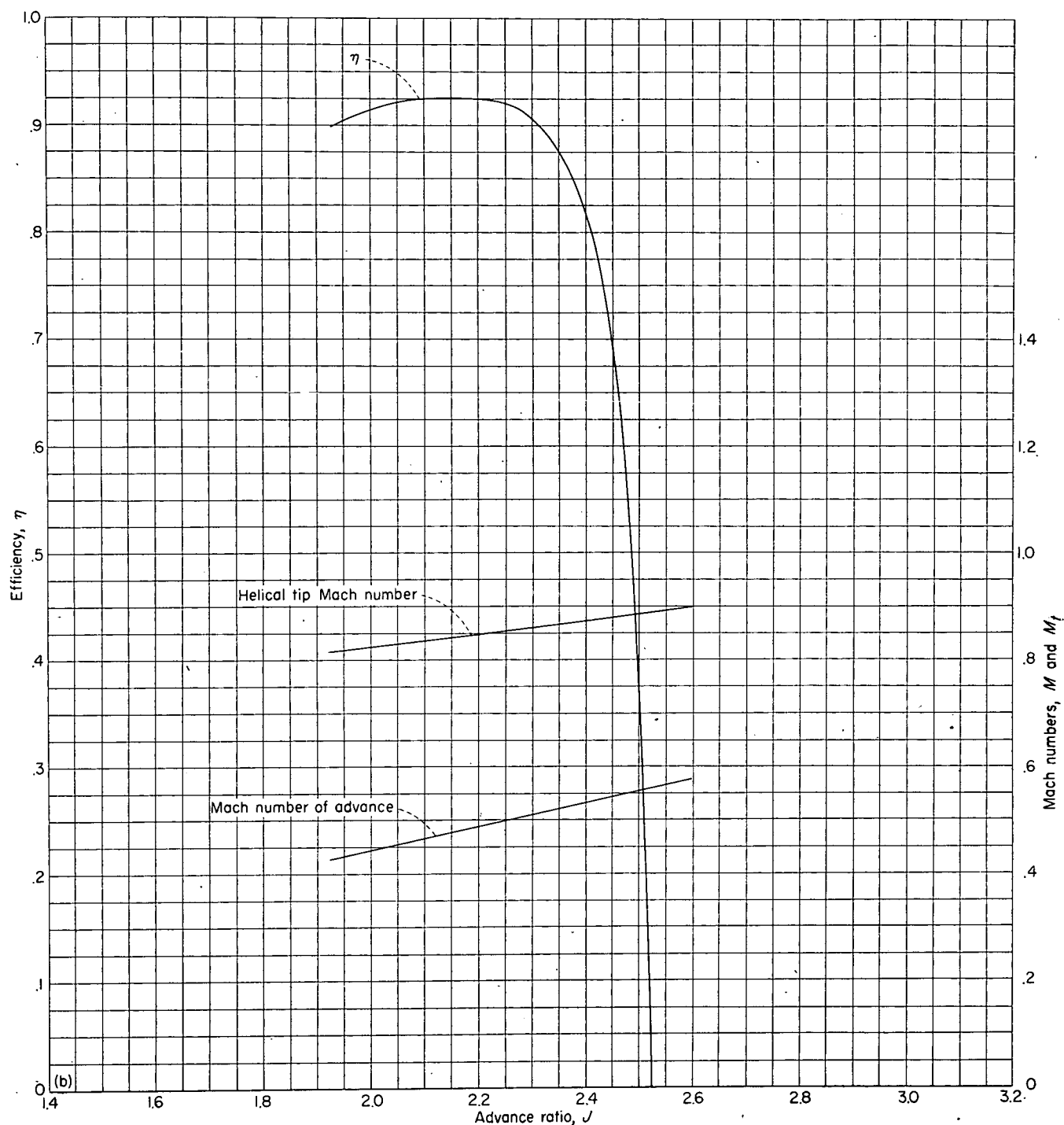
(c) Propeller efficiency.

FIGURE 9.—Concluded. Rotational speed, 1,350 rpm.



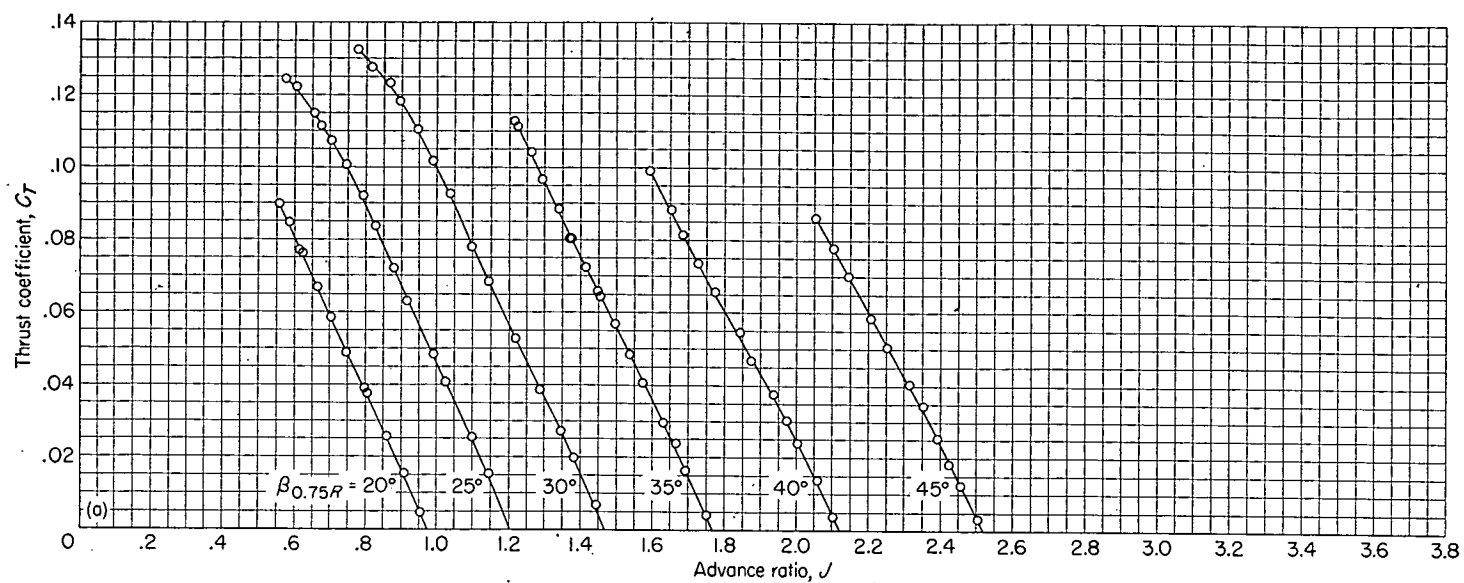
(a) Propeller thrust and power coefficients.

FIGURE 10.—Characteristics of NACA 10-(3)(062)-045A propeller. Rotational speed, 1,500 rpm; $\beta_{0.75R}=45^\circ$.

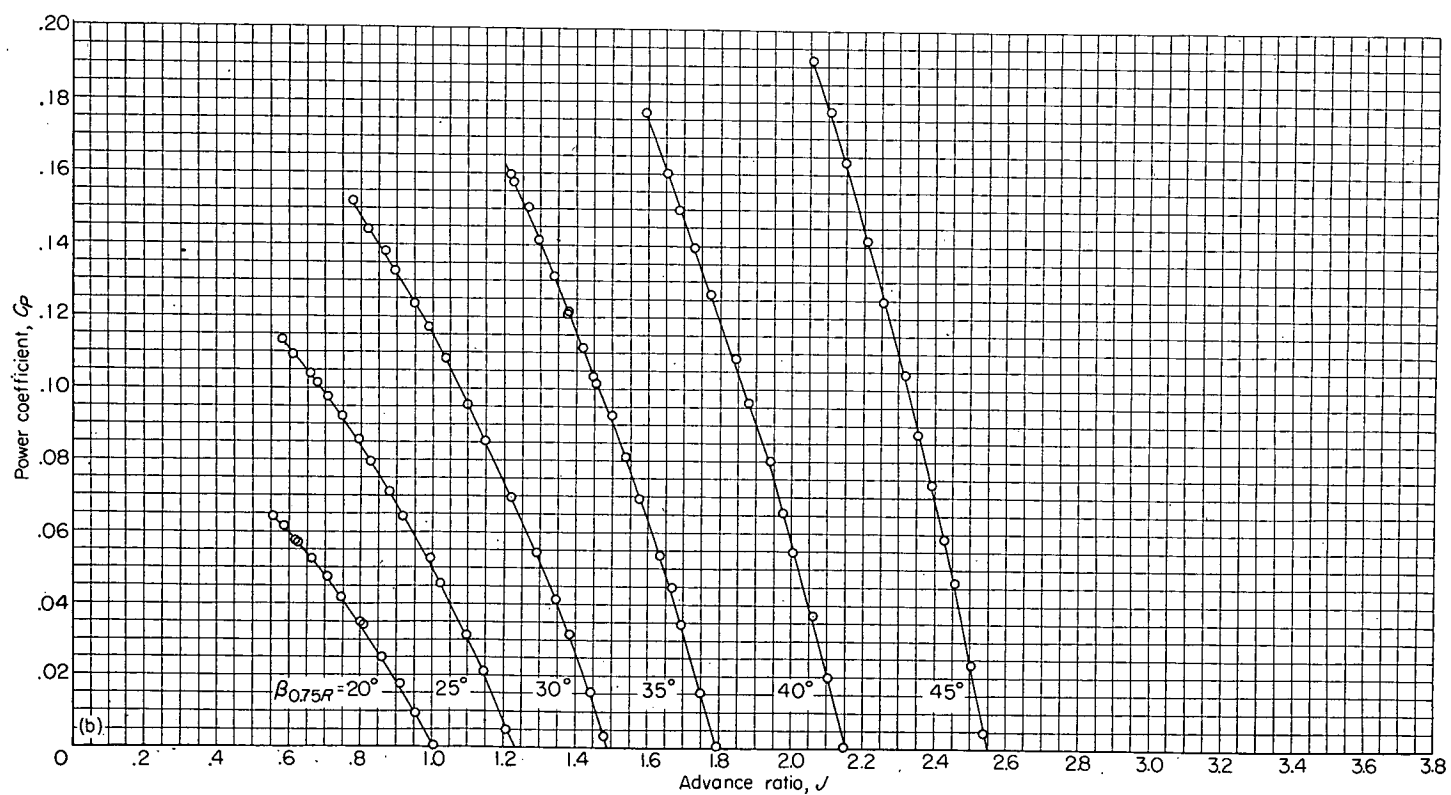


(b) Propeller efficiency.

FIGURE 10.—Concluded. Rotational speed, 1,500 rpm.



(a) Propeller thrust coefficient.



(b) Propeller power coefficient.

FIGURE 11.—Characteristics of NACA 10-(3)(062)-045A propeller. Rotational speed, 1,600 rpm.

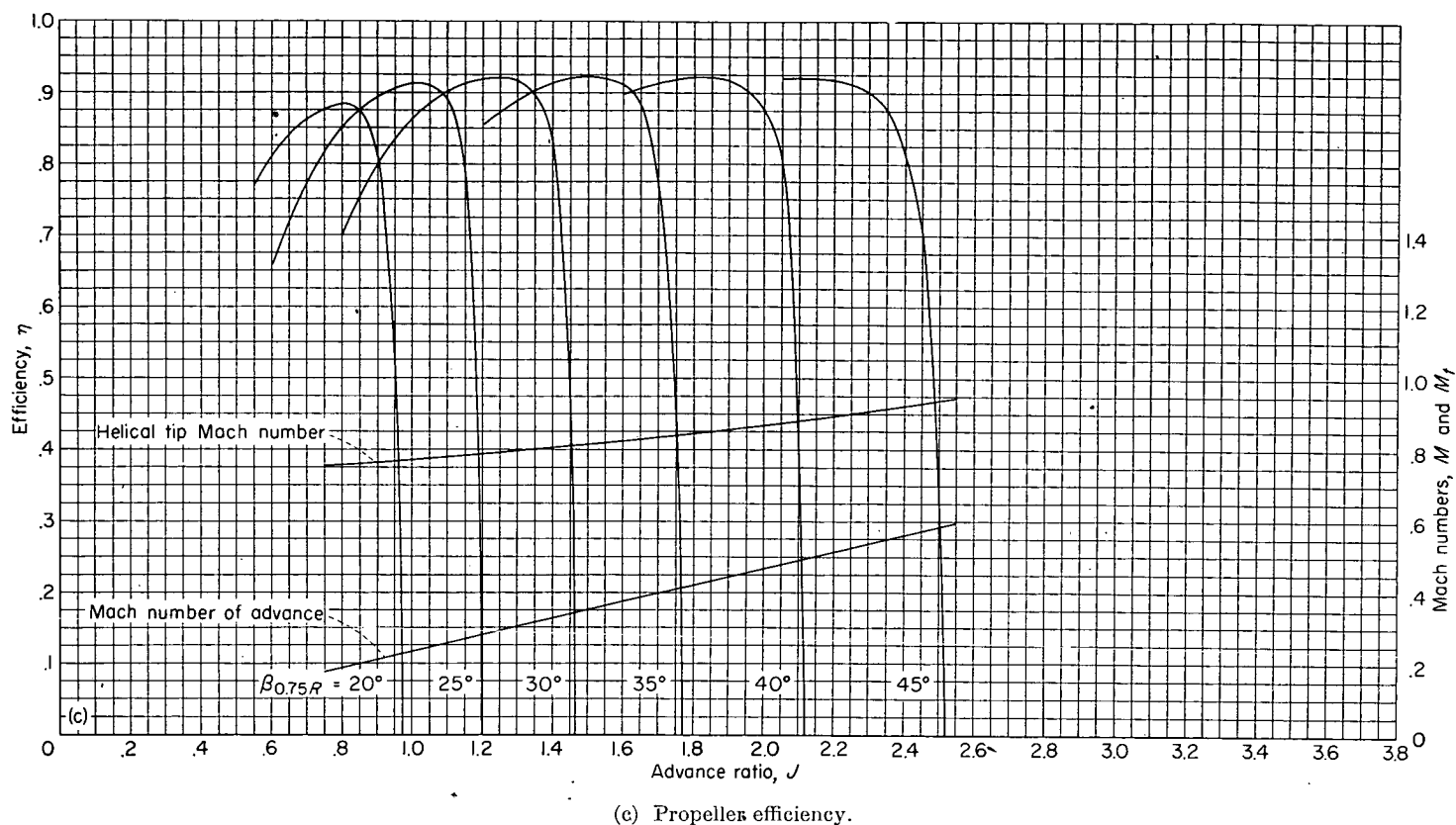


FIGURE 11.—Concluded. Rotational speed, 1,600 rpm.

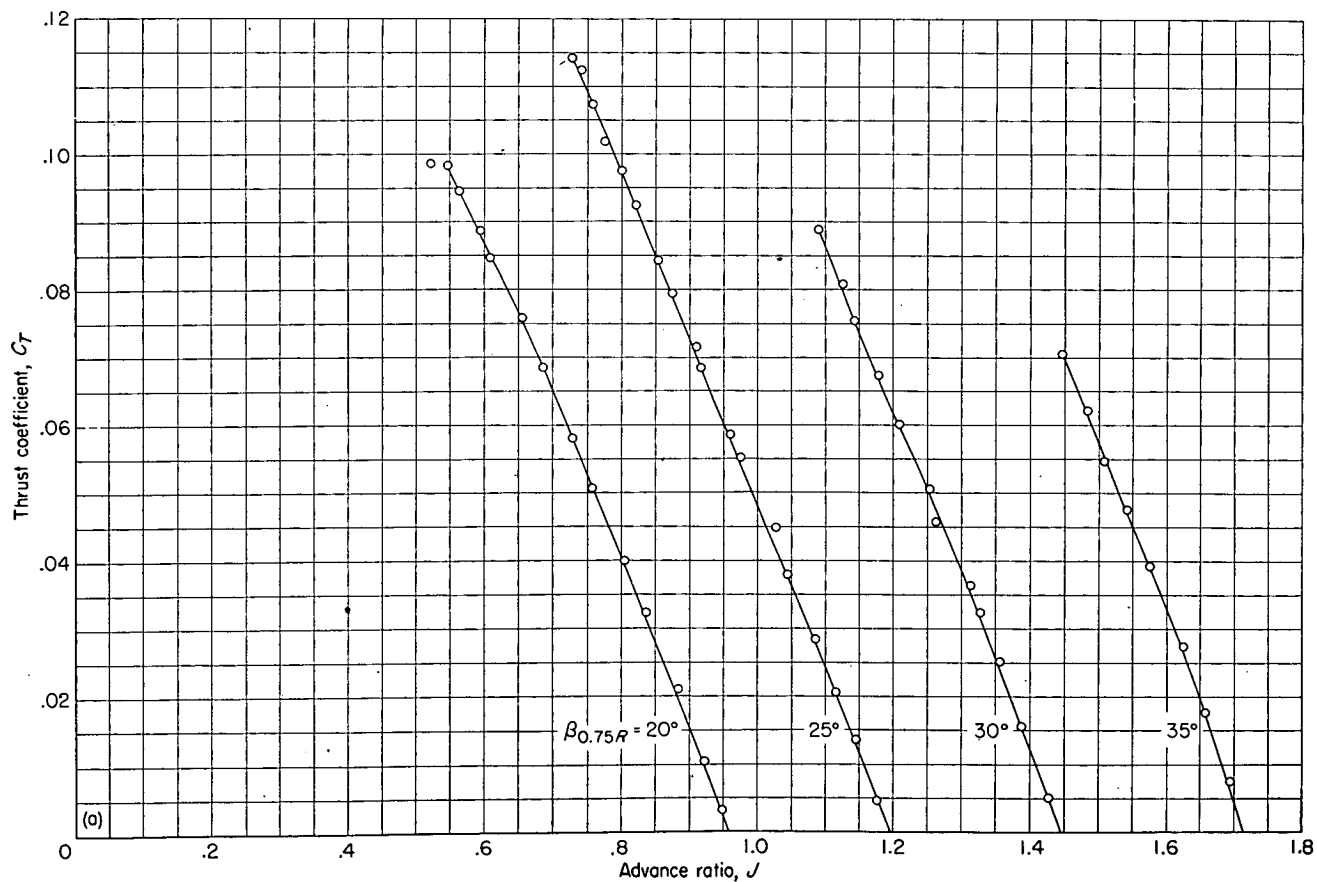
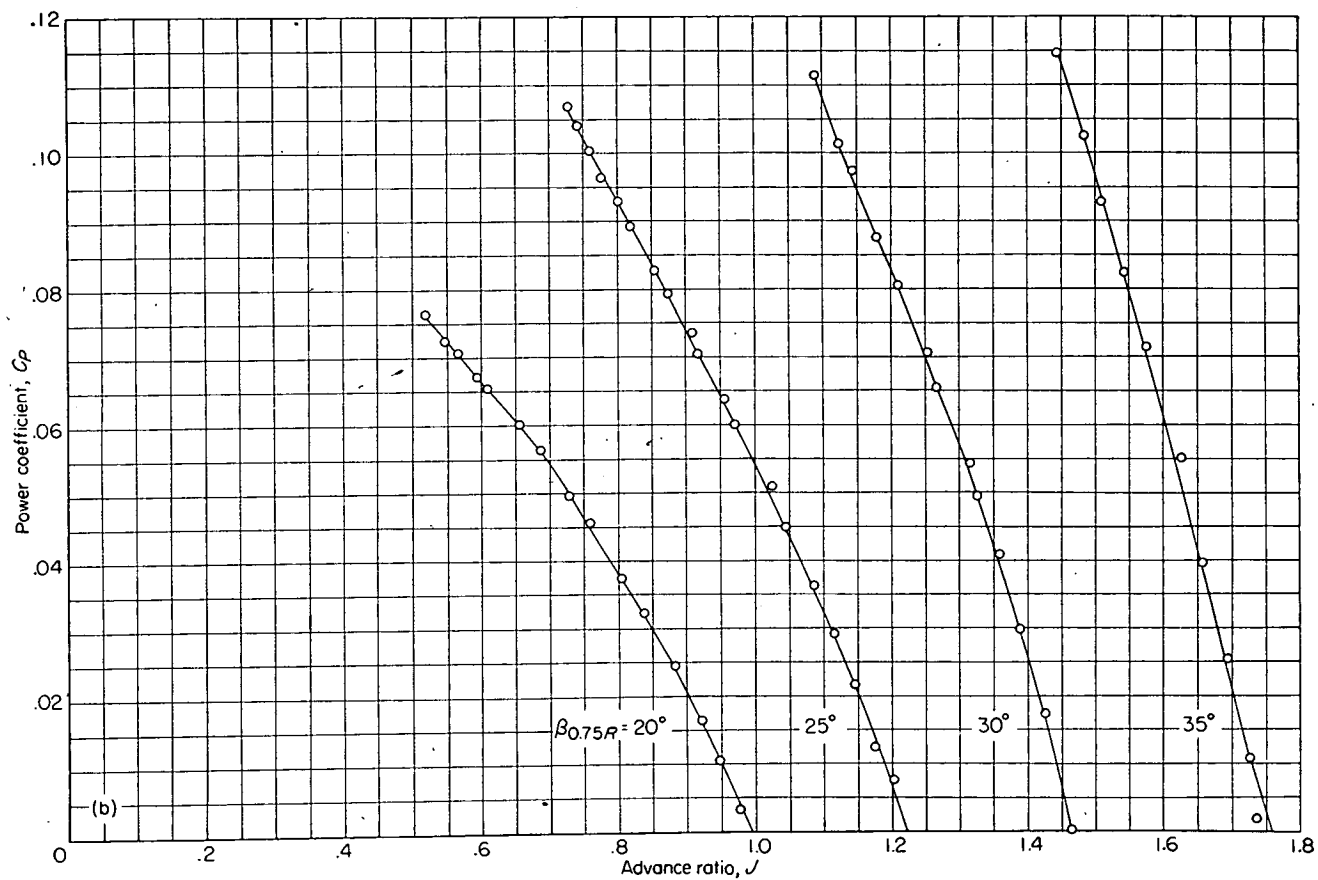
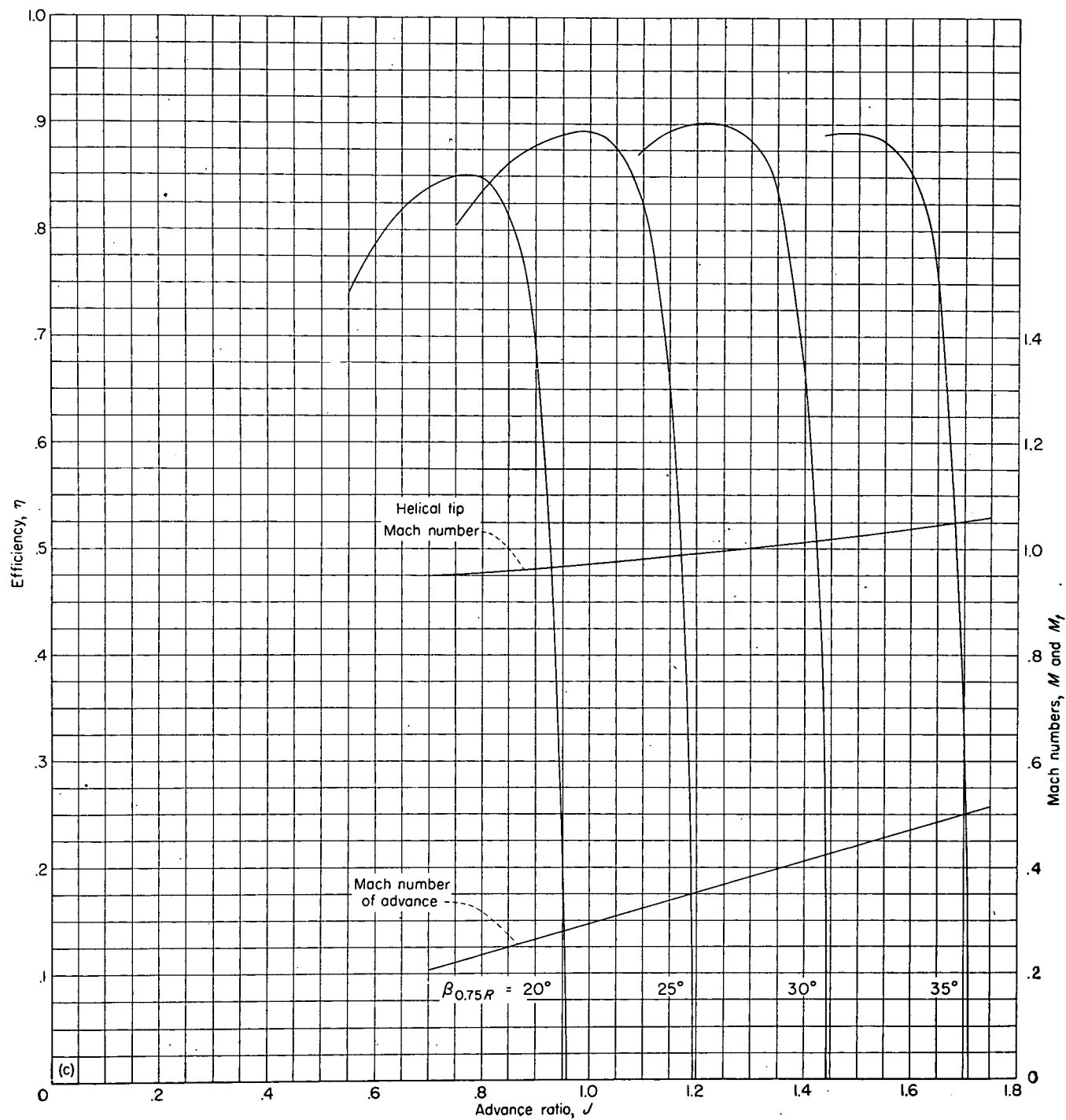


FIGURE 12.—Characteristics of NACA 10-(3)(062)-045A propeller. Rotational speed, 2,000 rpm.



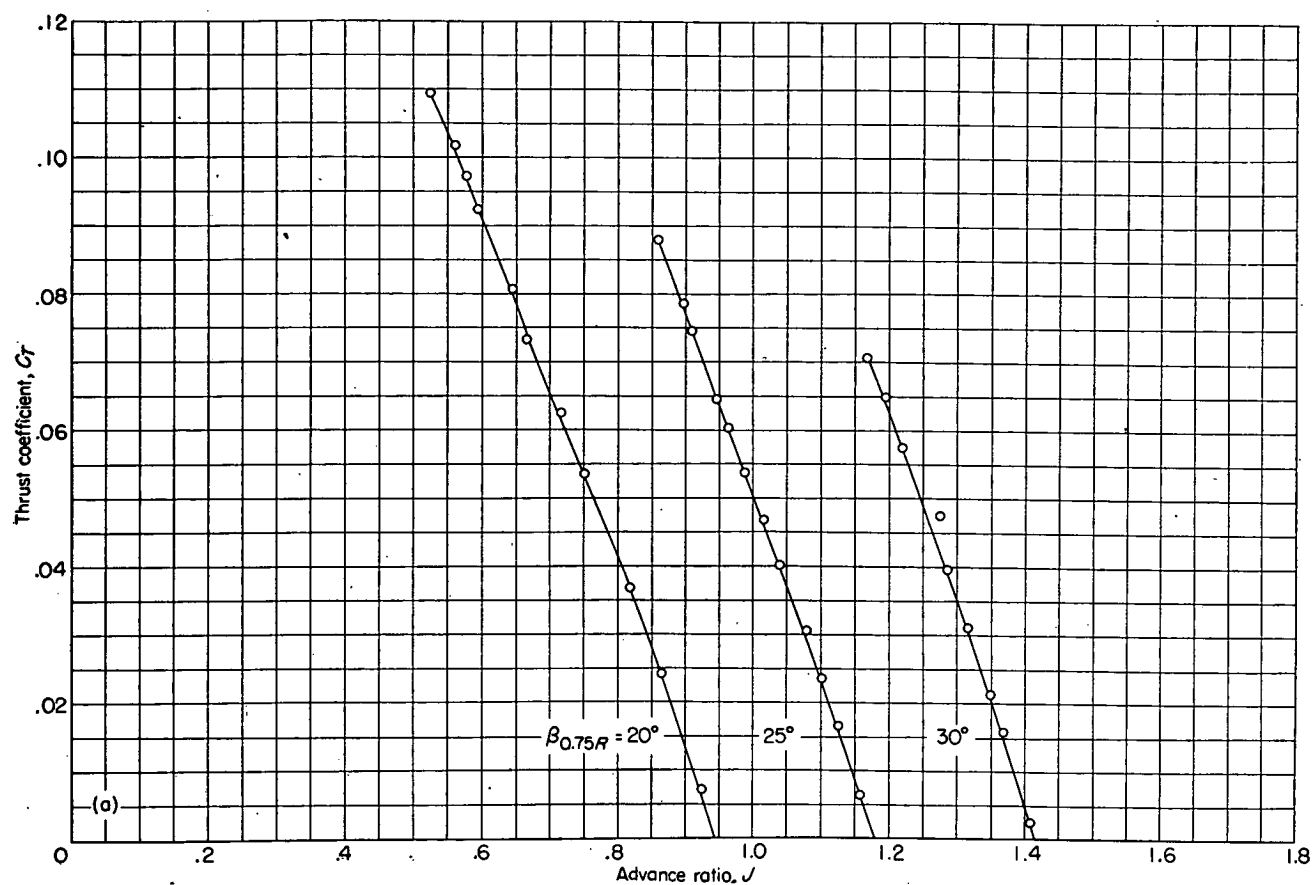
(b) Propeller power coefficient.

FIGURE 12.—Continued. Rotational speed, 2,000 rpm.

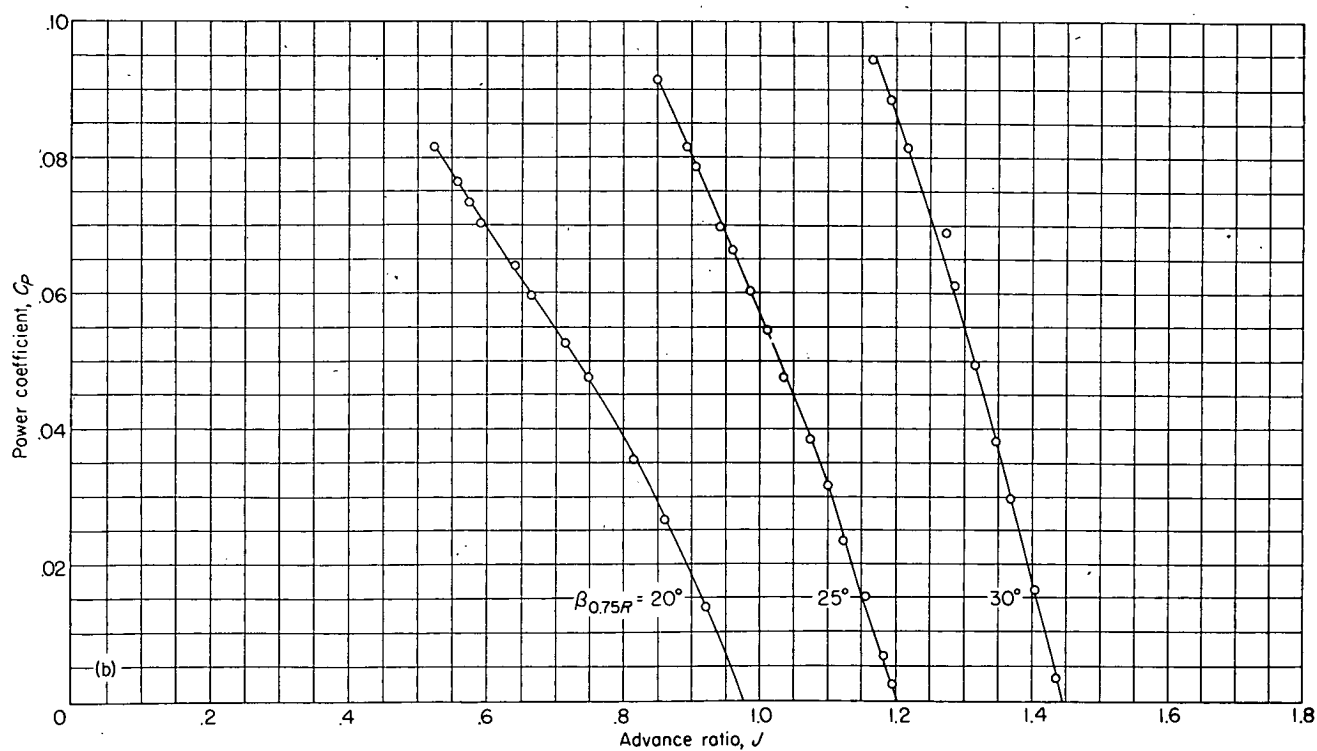


(c) Propeller efficiency.

FIGURE 12.—Concluded. Rotational speed, 2,000 rpm.

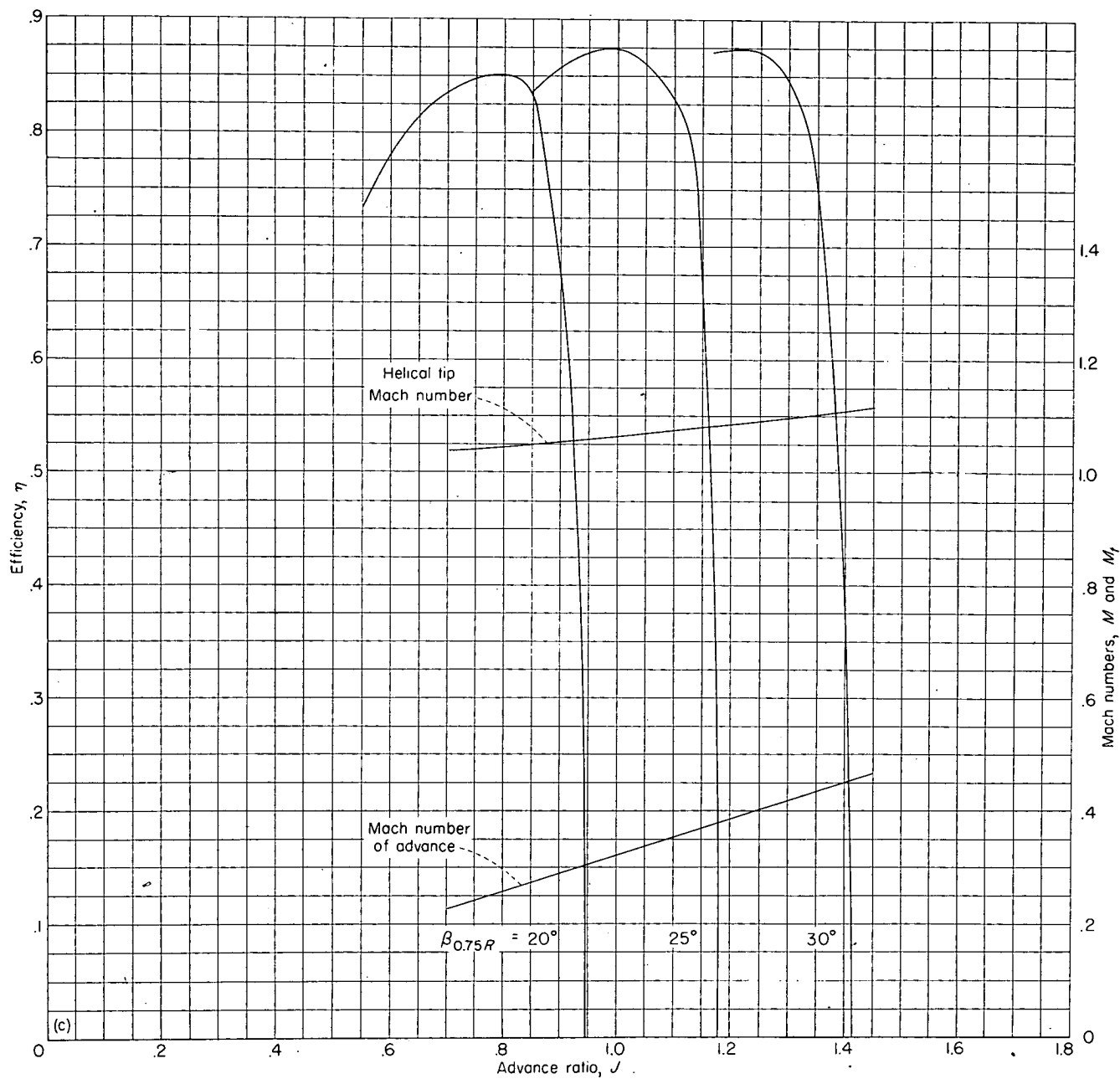


(a) Propeller thrust coefficient.



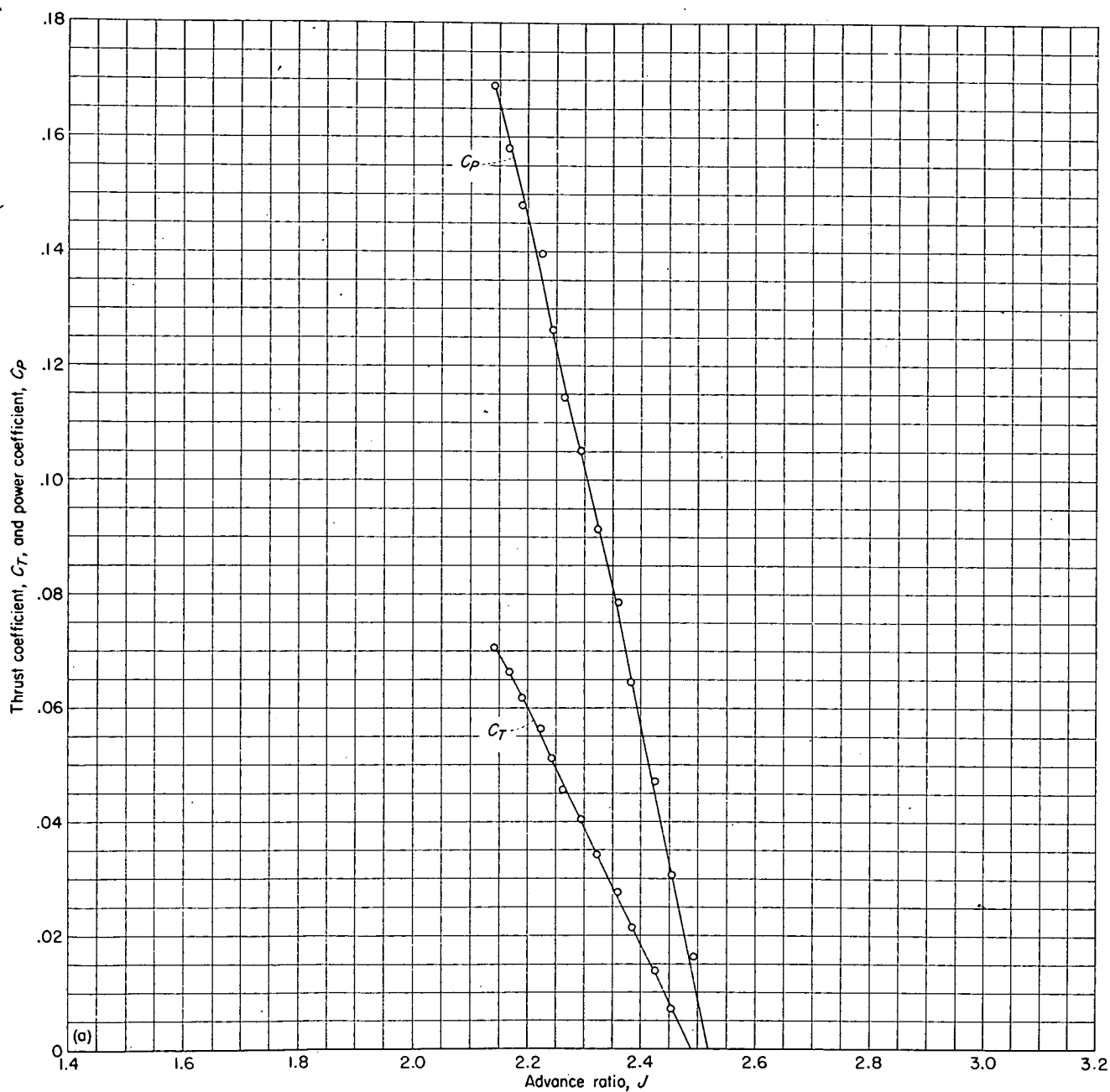
(b) Propeller power coefficient.

FIGURE 13.—Characteristics of NACA 10-(3)(062)-045A propeller. Rotational speed, 2,160 rpm.



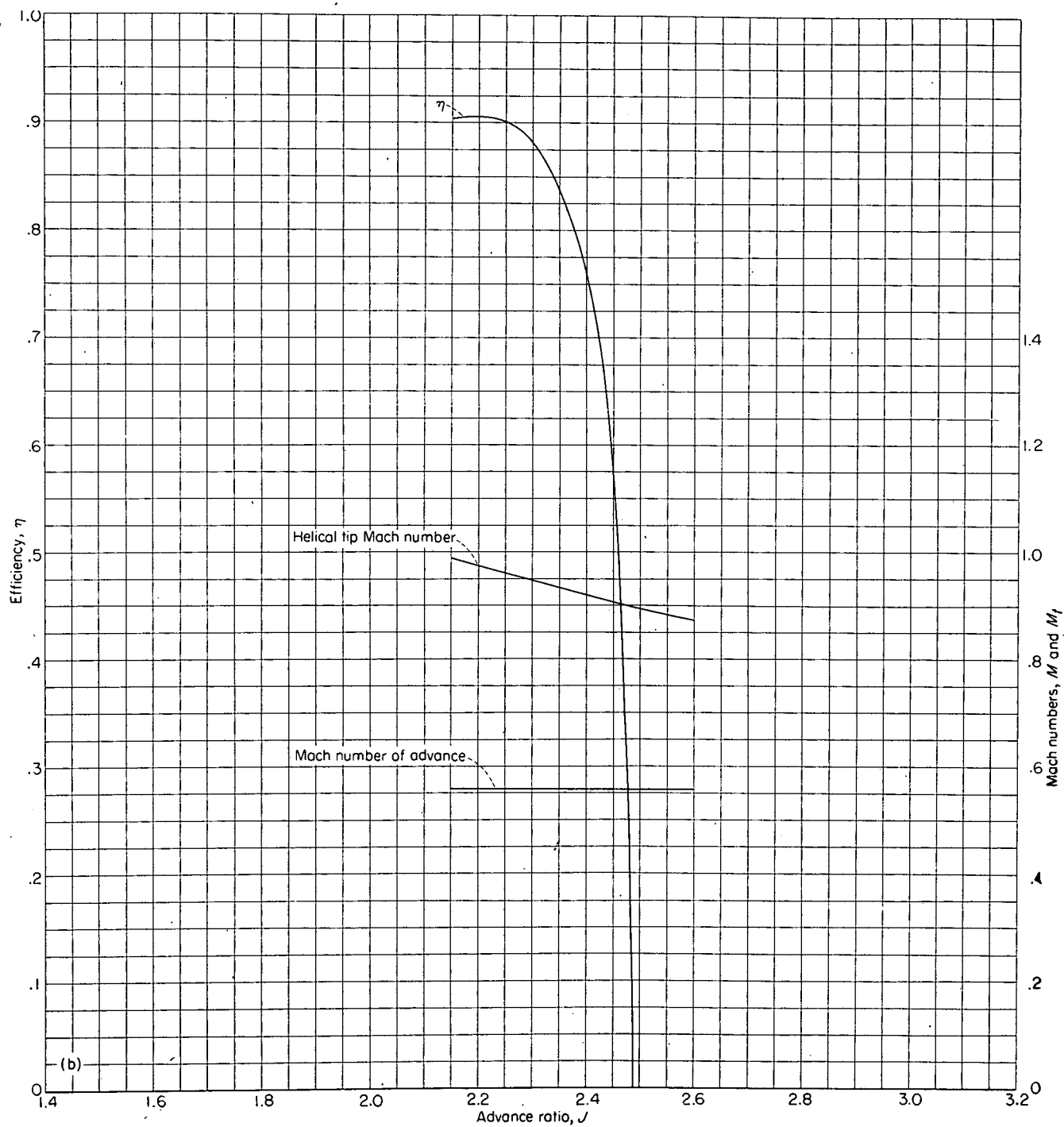
(c) Propeller efficiency.

Figure 13.—Concluded. Rotational speed, 2,160 rpm.



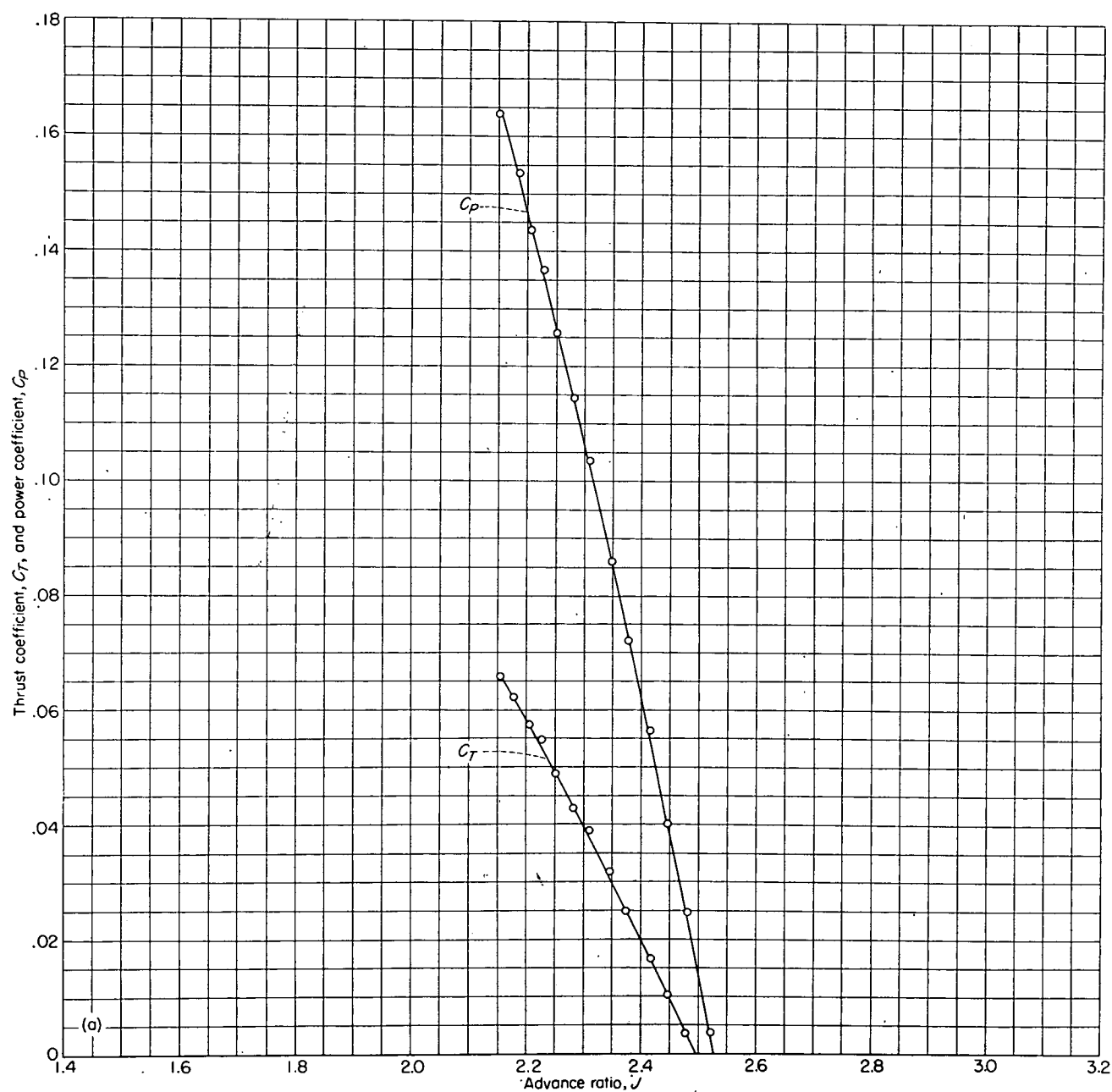
(a) Propeller thrust and power coefficients.

FIGURE 14.—Characteristics of NACA 10-(3)(062)-045A propeller at high forward speeds. Mach number of advance at maximum efficiency, 0.558; $\beta_{0.75R} = 45^\circ$.



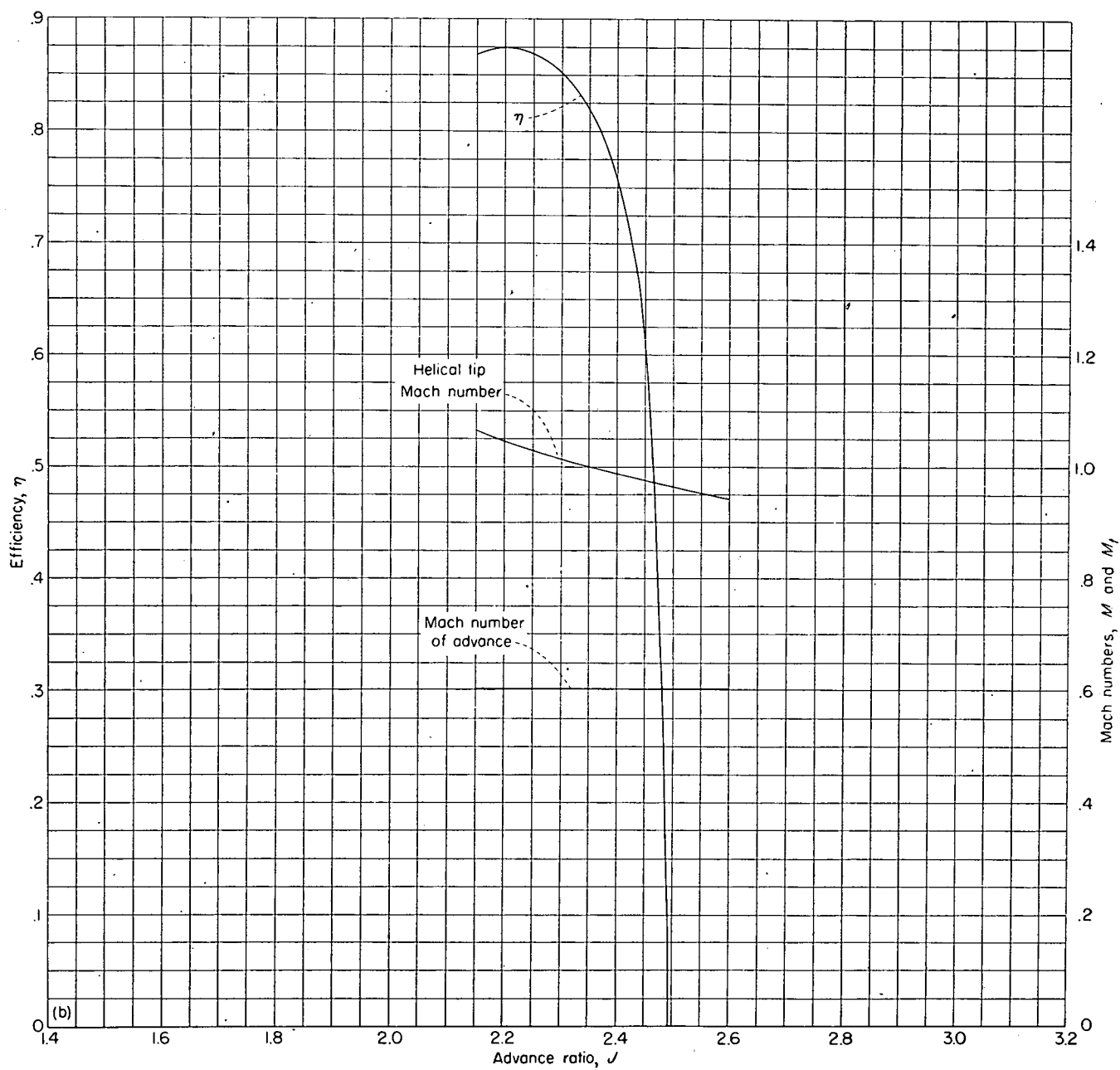
(b) Propeller efficiency.

FIGURE 14.—Concluded. Mach number of advance at maximum efficiency, 0.558.



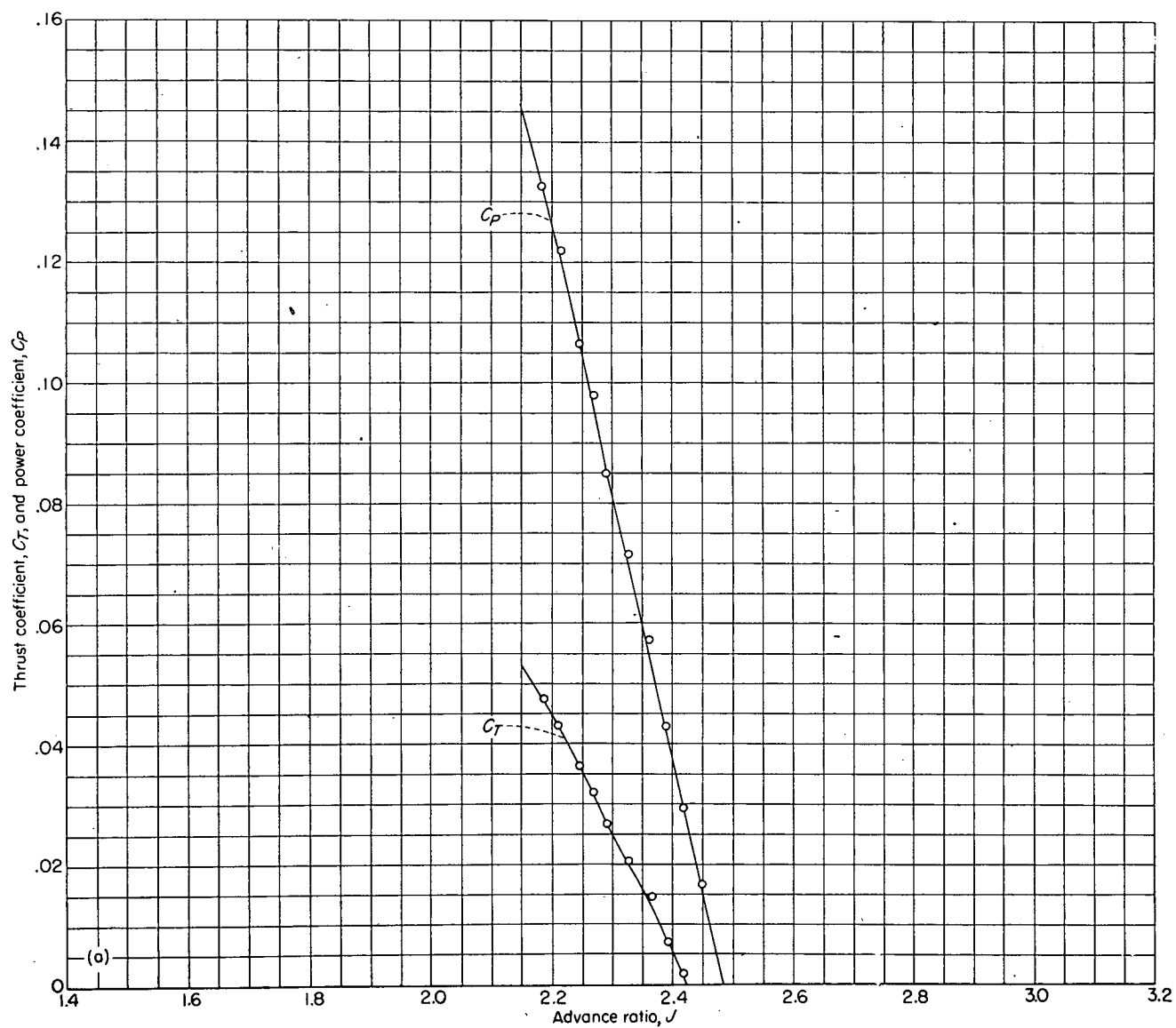
(a) Propeller thrust and power coefficients.

FIGURE 15.—Characteristics of NACA 10-(3)(062)-045A propeller at high forward speeds. Mach number of advance at maximum efficiency, 0.601; $\beta_{0.75R} = 45^\circ$.



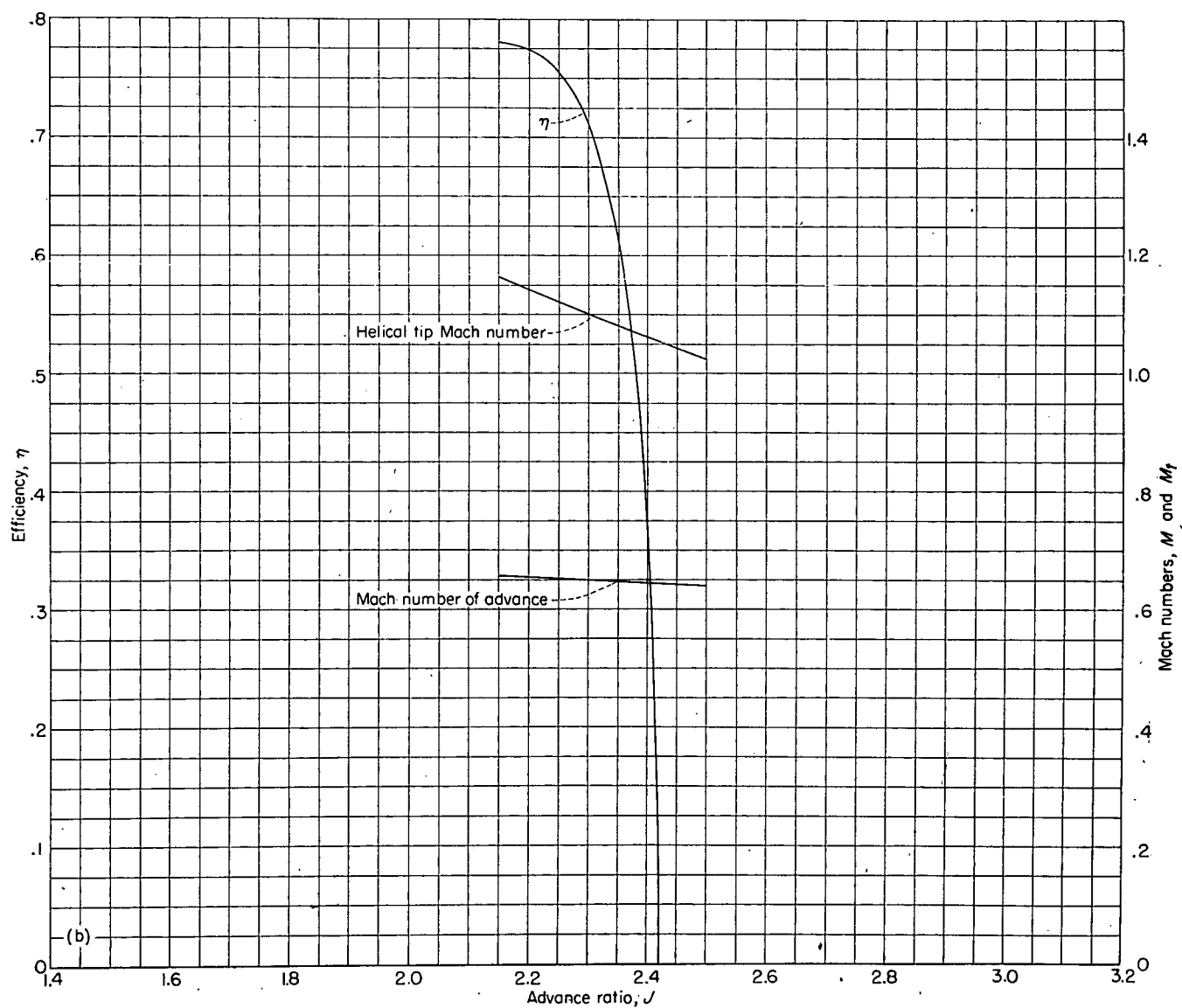
(b) Propeller efficiency.

FIGURE 15.—Concluded. Mach number of advance at maximum efficiency, 0.601.



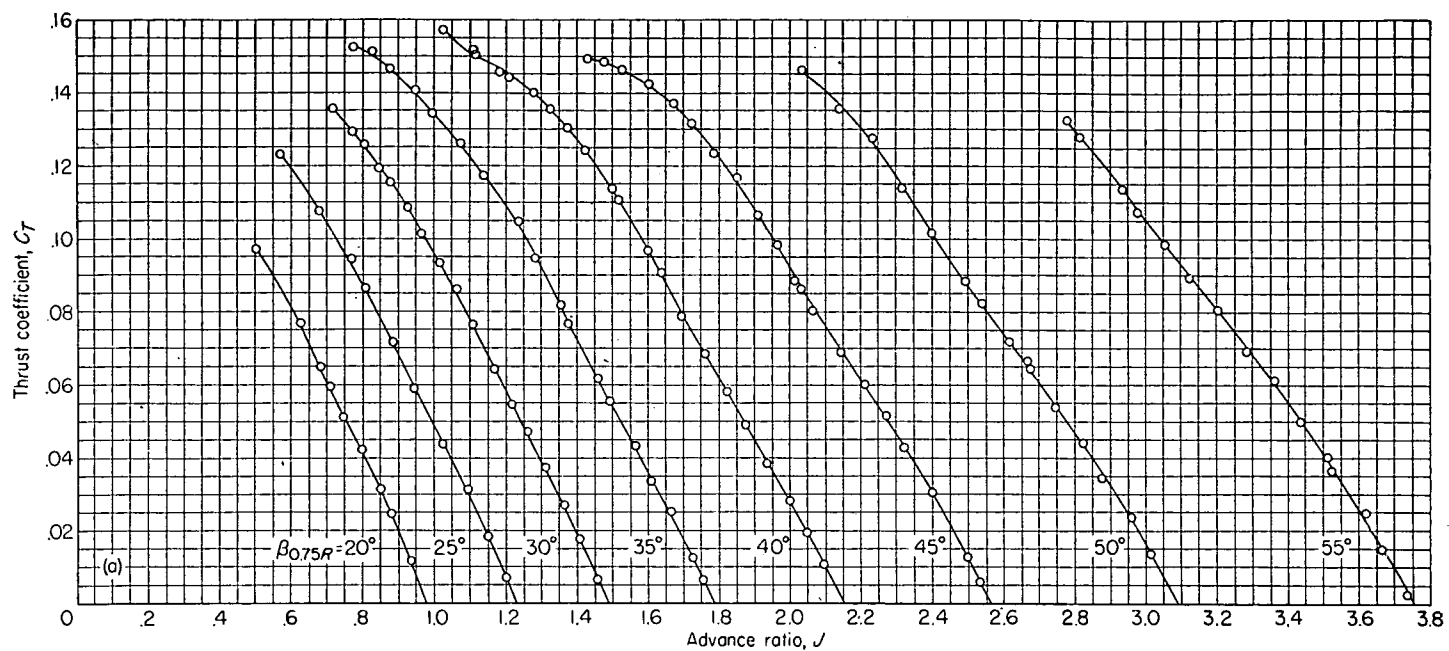
(a) Propeller thrust and power coefficients.

FIGURE 16.—Characteristics of NACA 10-(3)(062)-045A propeller at high forward speeds. . Mach number of advance at maximum efficiency, 0.657; $\beta_{0.75R} = 45^\circ$.

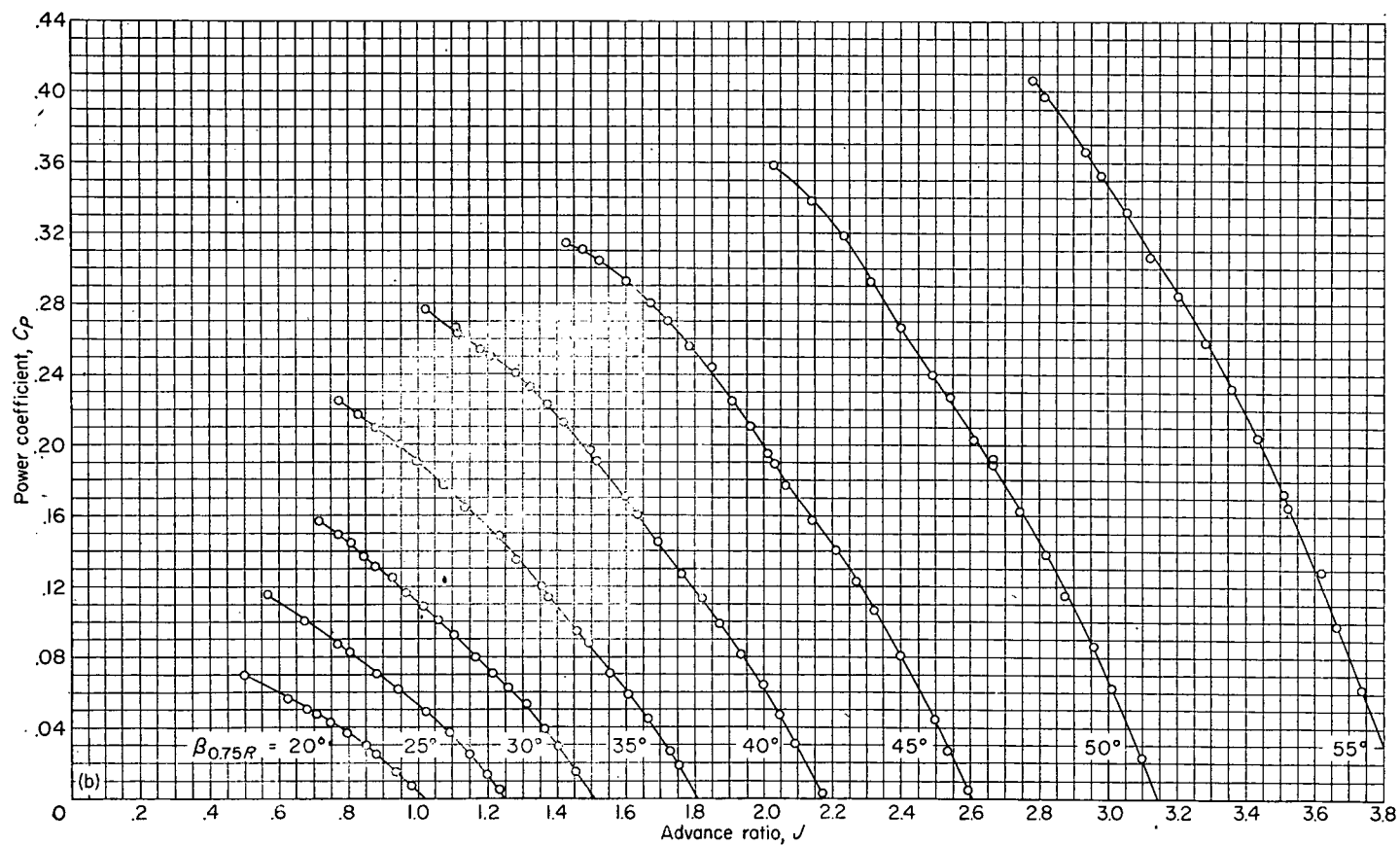


(b) Propeller efficiency.

FIGURE 16.—Concluded. Mach number of advance at maximum efficiency, 0.657.

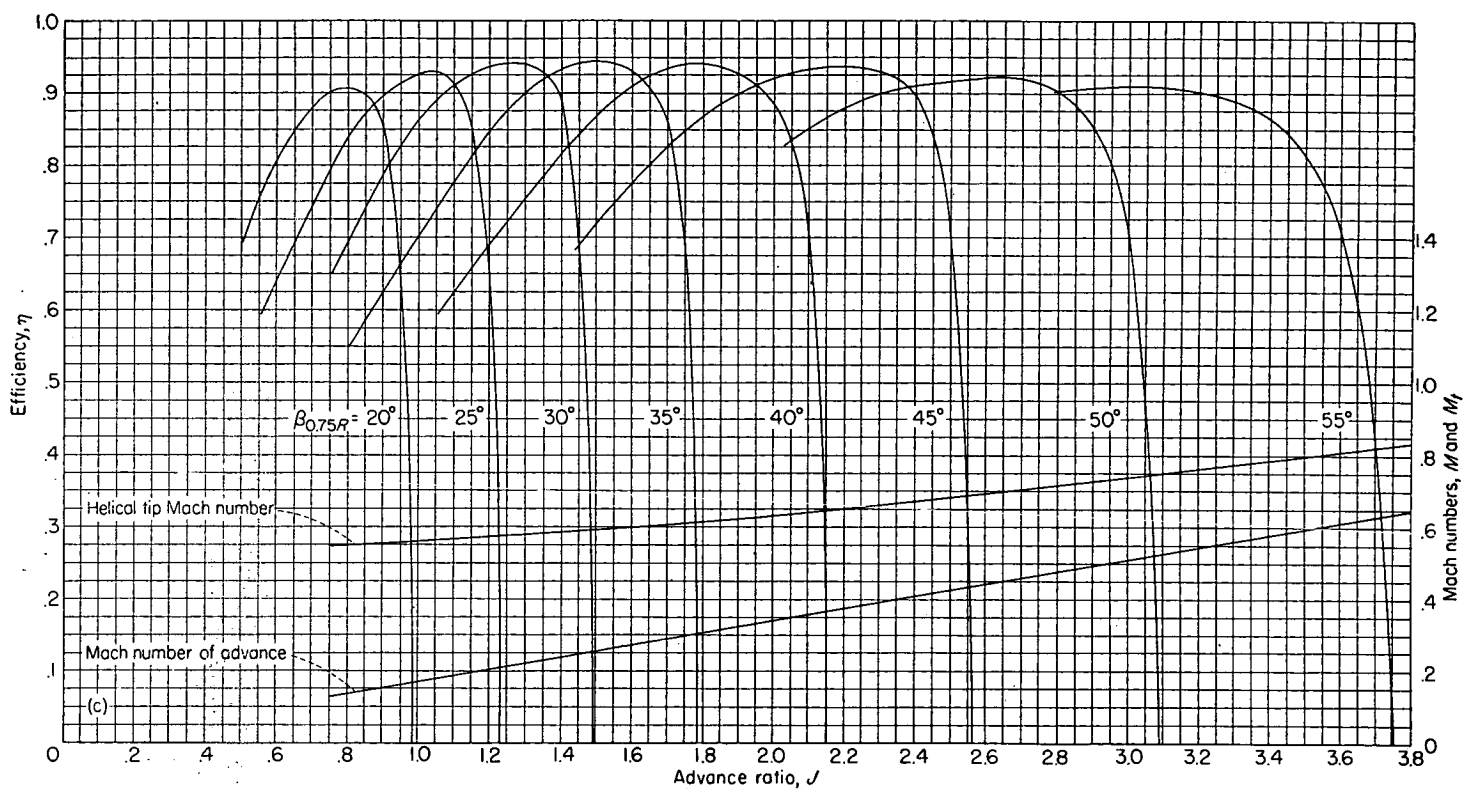


(a) Propeller thrust coefficient.



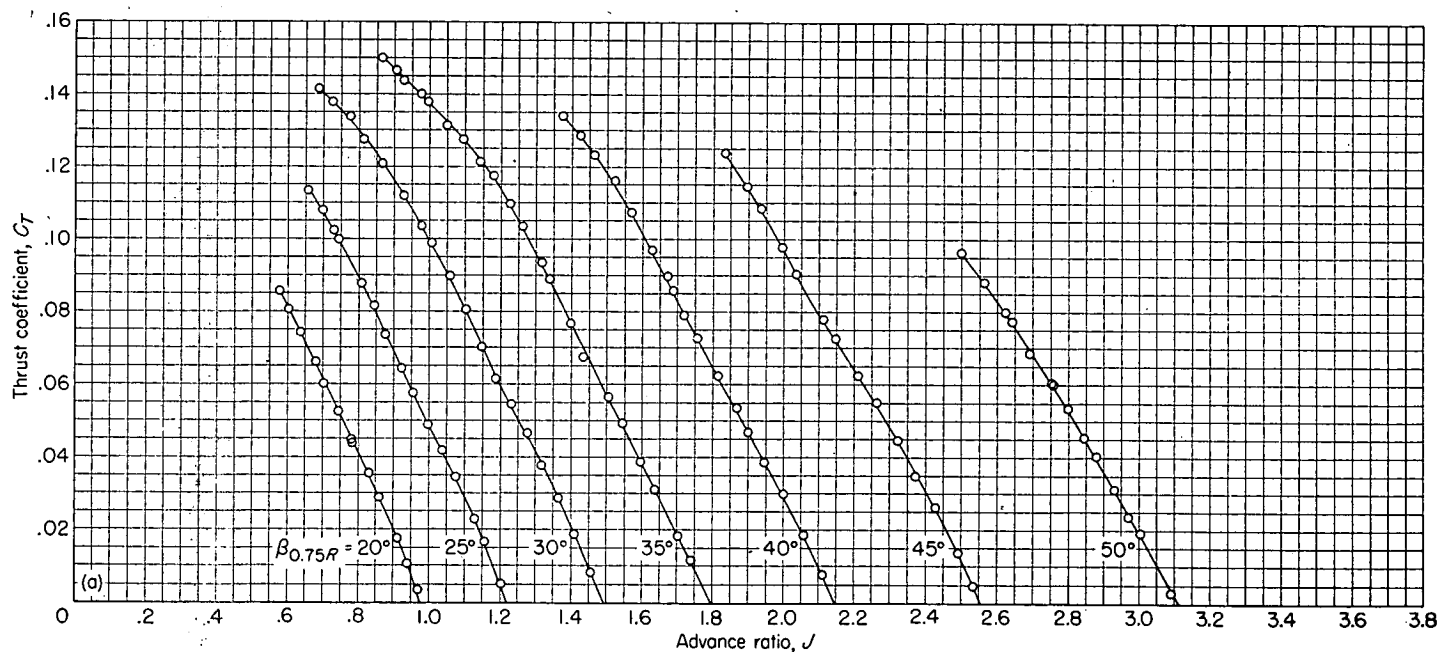
(b) Propeller power coefficient.

FIGURE 17.—Characteristics of NACA 10-(3)(05)-045 propeller. Rotational speed, 1,140 rpm.



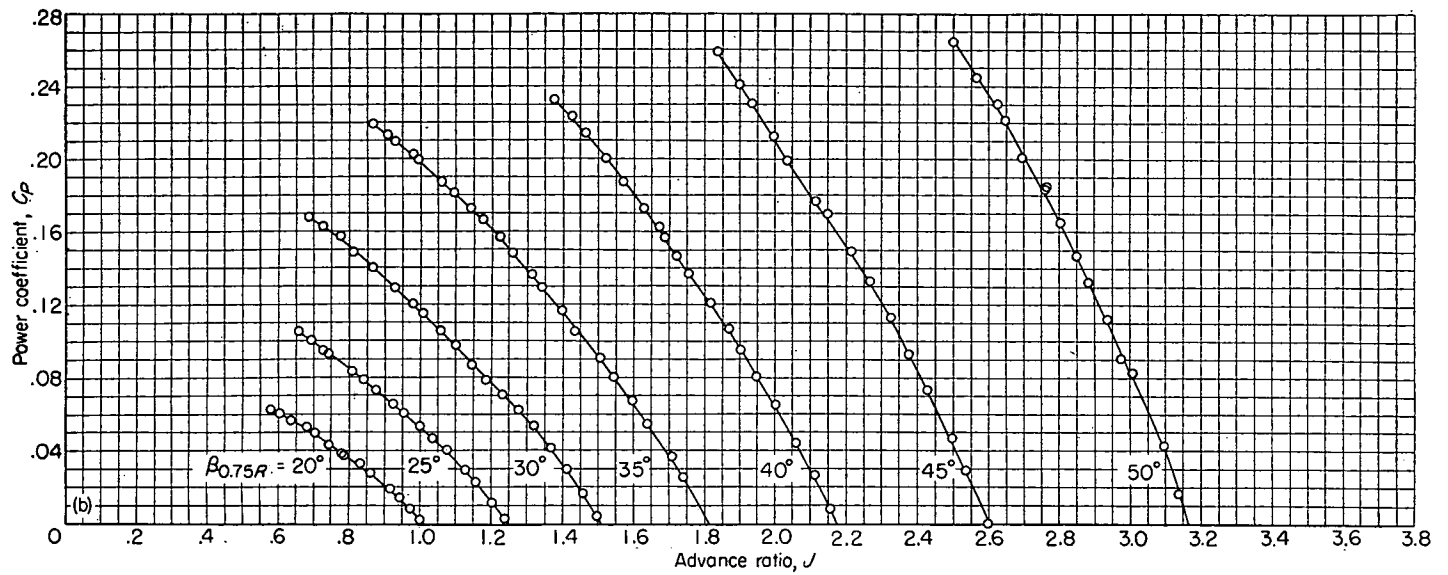
(c) Propeller efficiency.

FIGURE 17.—Concluded. Rotational speed, 1,140 rpm.

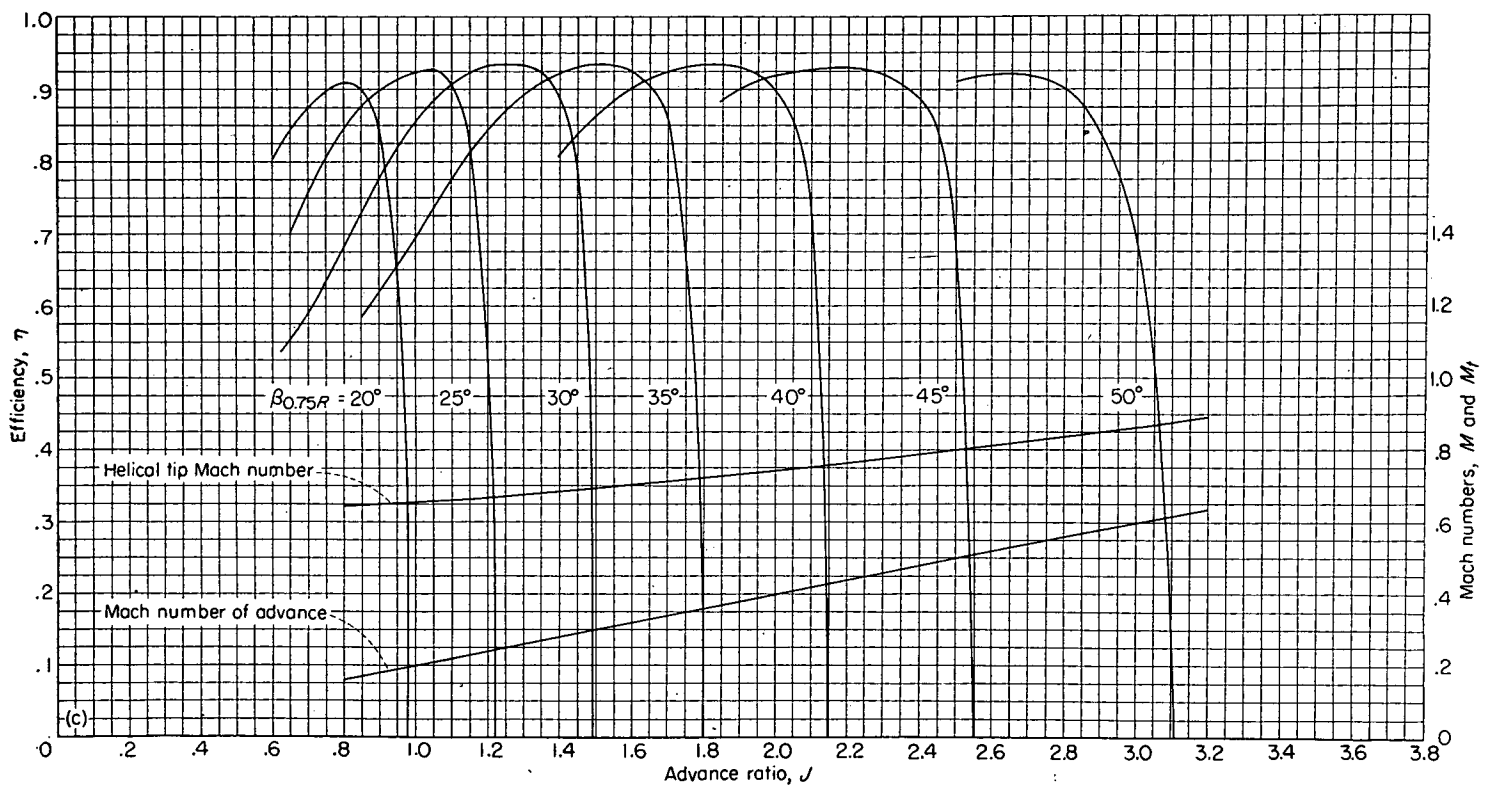


(a) Propeller thrust coefficient.

FIGURE 18.—Characteristics of NACA 10-(3)(05)-045 propeller. Rotational speed, 1,350 rpm.

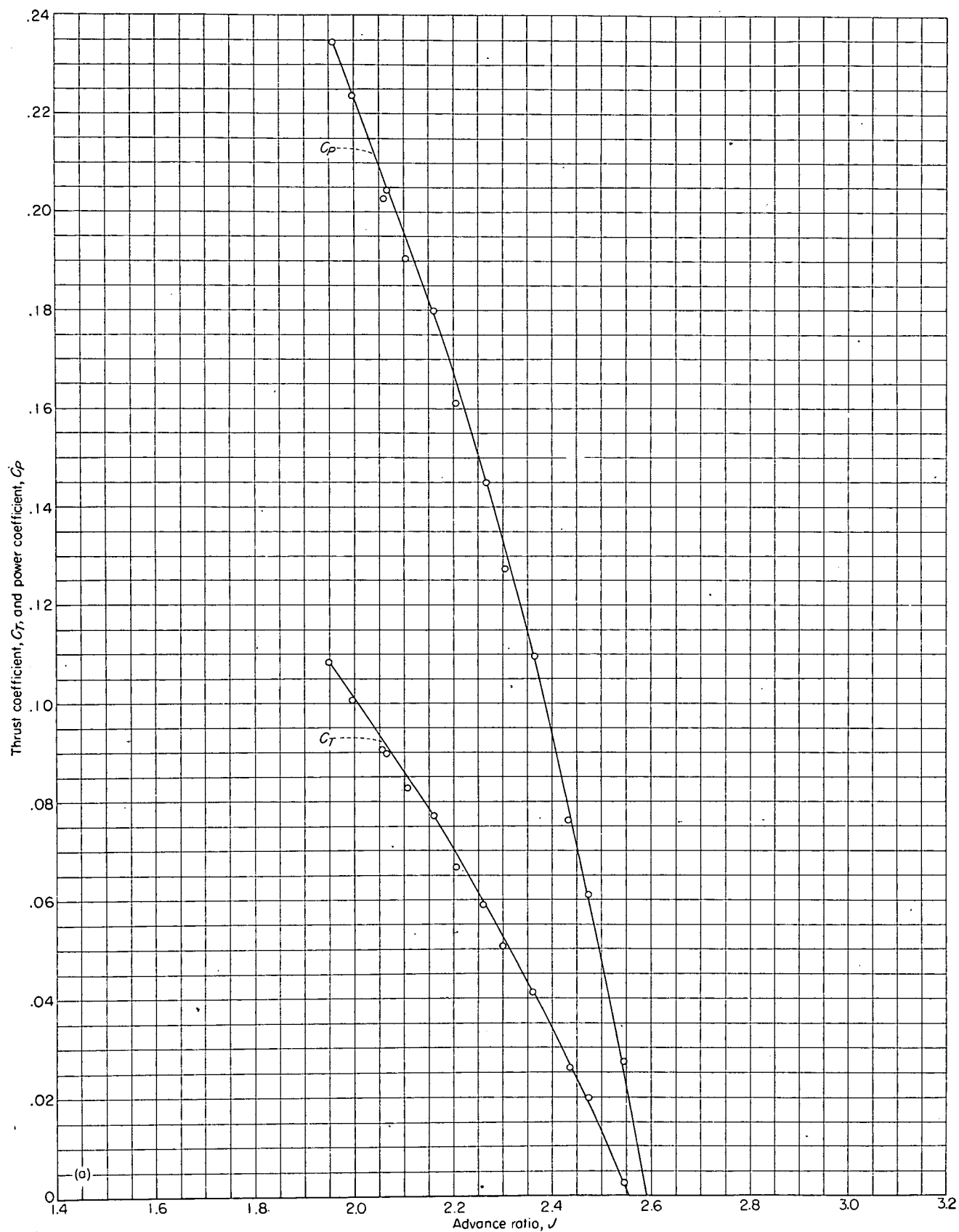


(b) Power coefficient.



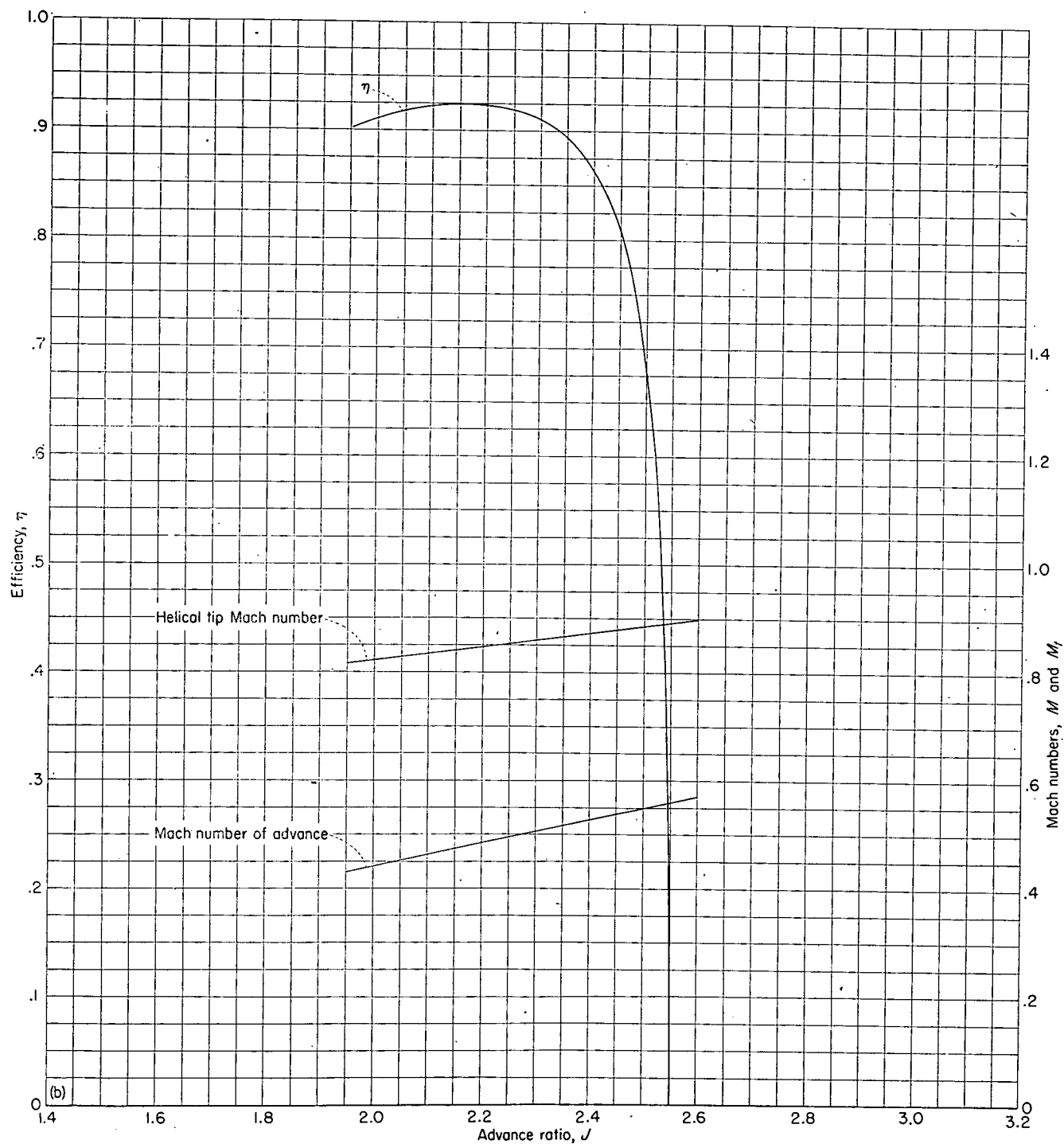
(c) Propeller efficiency.

FIGURE 18.—Concluded. Rotational speed, 1,350 rpm.



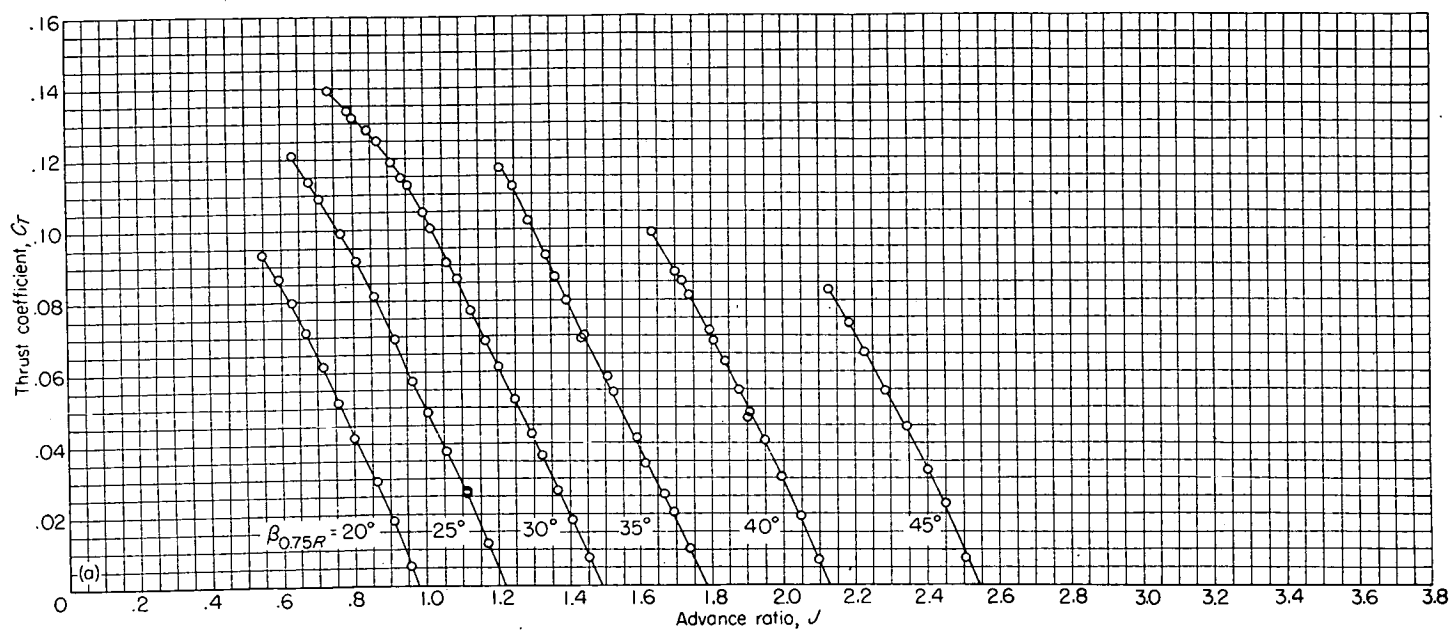
(a) Propeller thrust and power coefficients.

FIGURE 19.—Characteristics of NACA 10-(3)(05)-045 propeller. Rotational speed, 1,500 rpm; $\beta_{0.75R} = 45^\circ$.

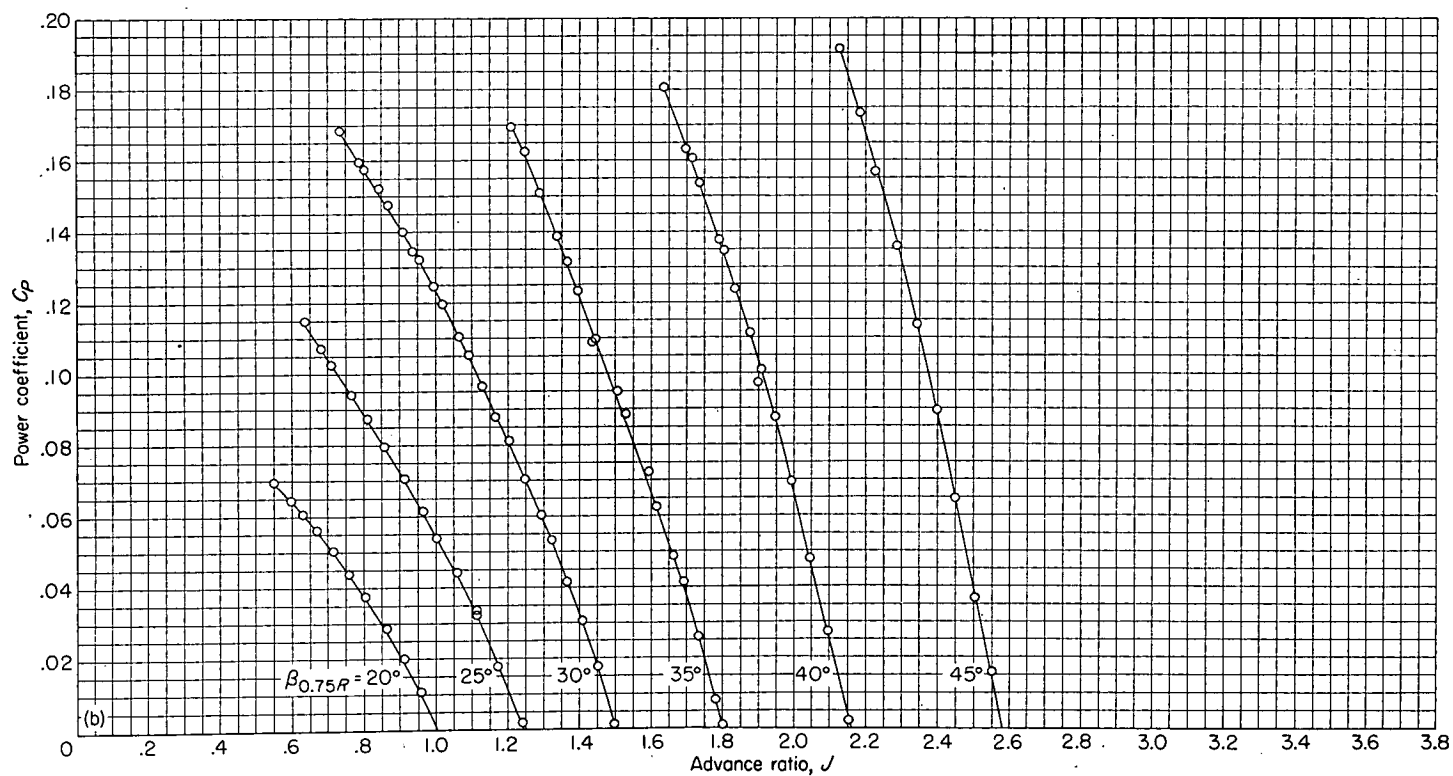


(b) Propeller efficiency.

FIGURE 19.—Concluded. Rotational speed, 1,500 rpm.

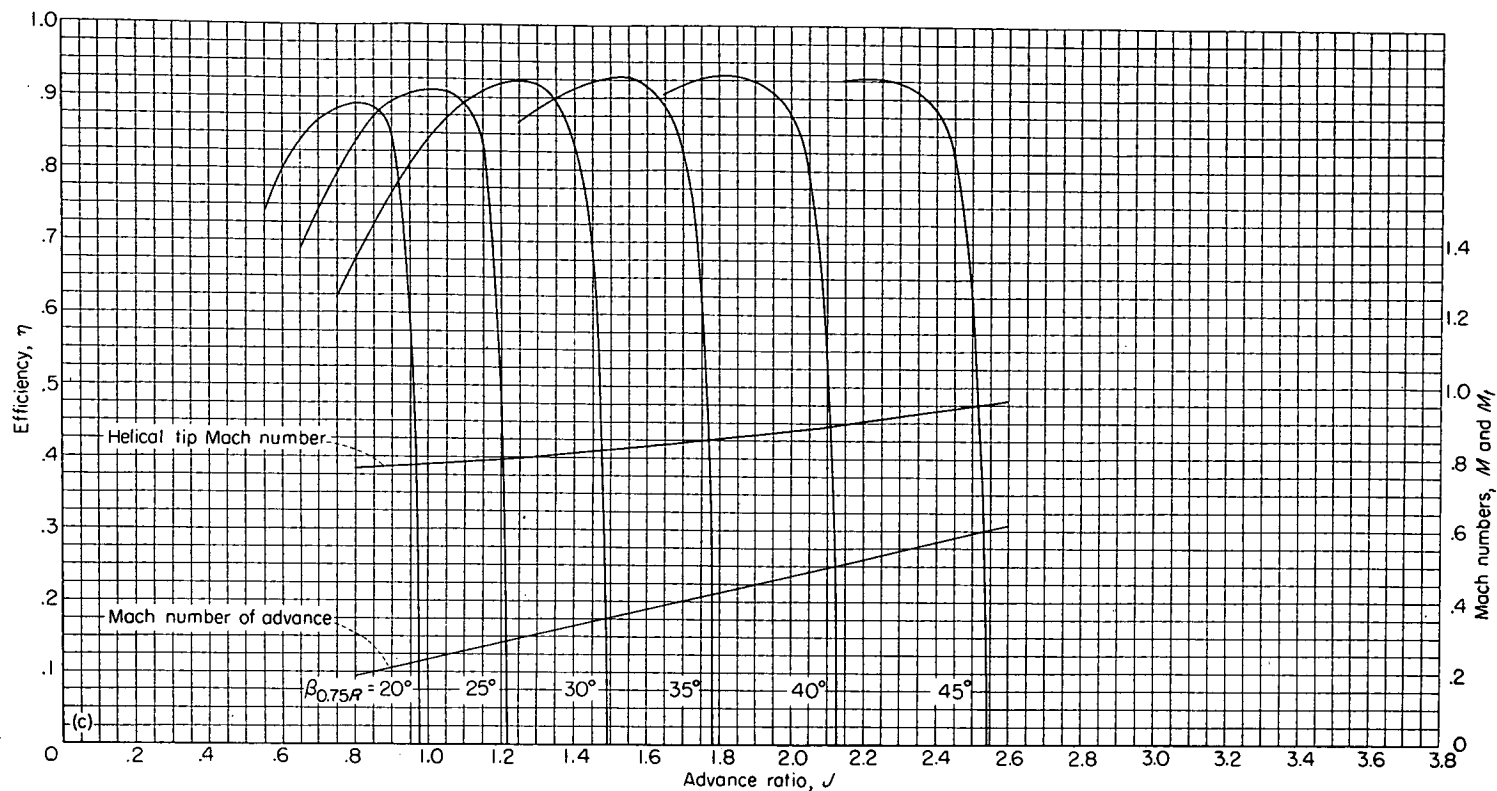


(a) Propeller thrust coefficient.



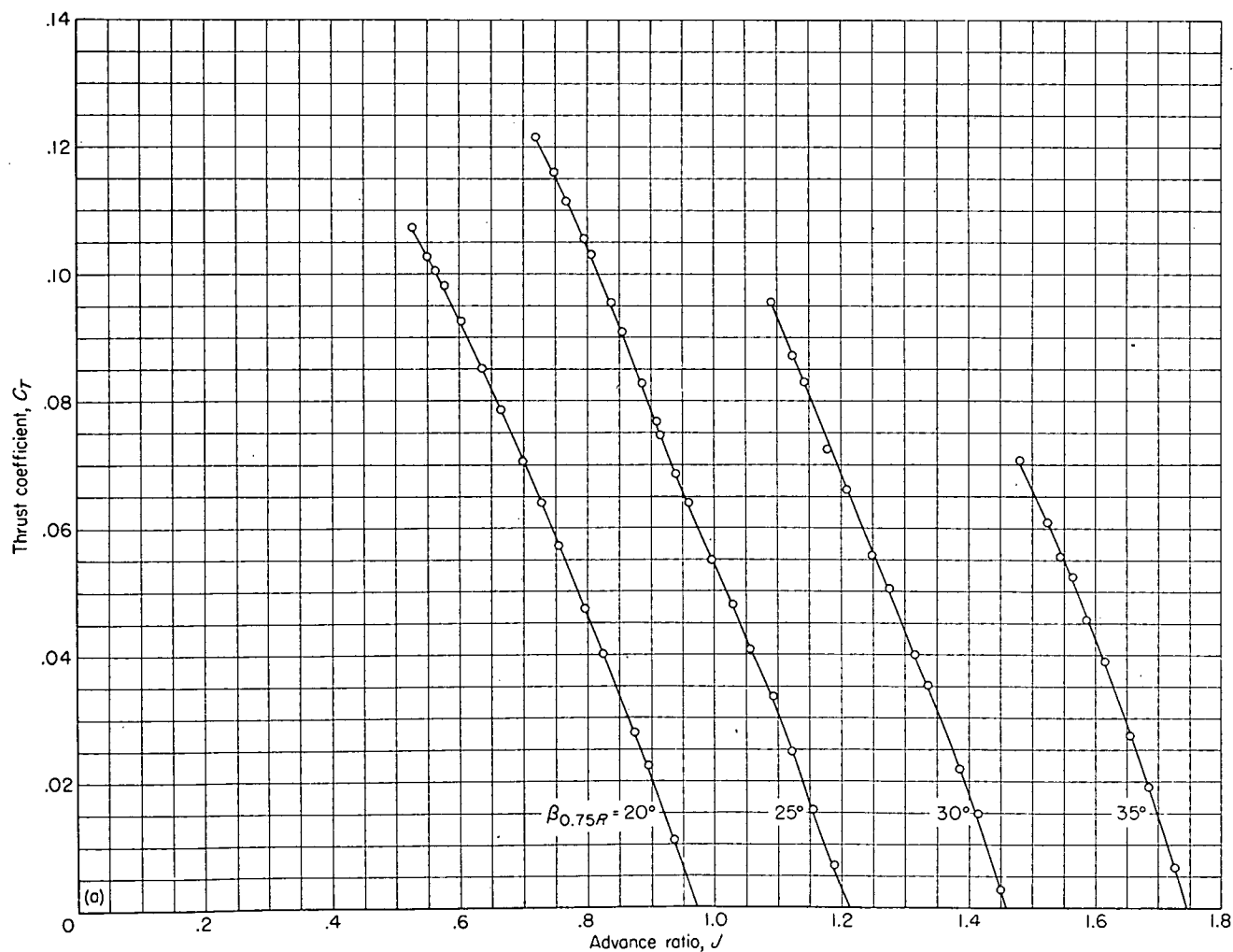
(b) Propeller power coefficient.

FIGURE 20.—Characteristics of NACA 10-(3)(05)-045 propeller. Rotational speed, 1,600 rpm.



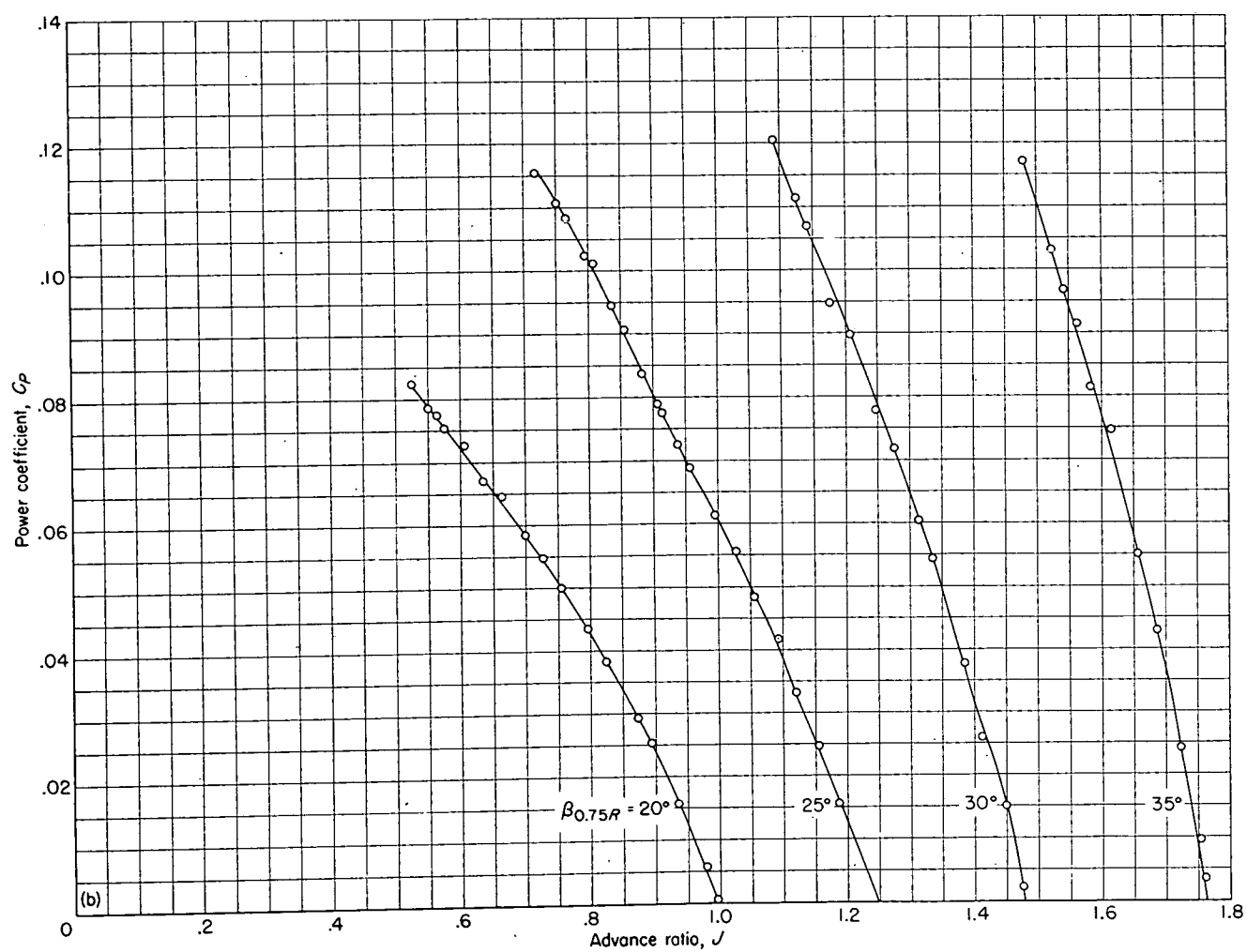
(c) Propeller efficiency.

FIGURE 20.—Concluded. Rotational speed, 1,600 rpm.



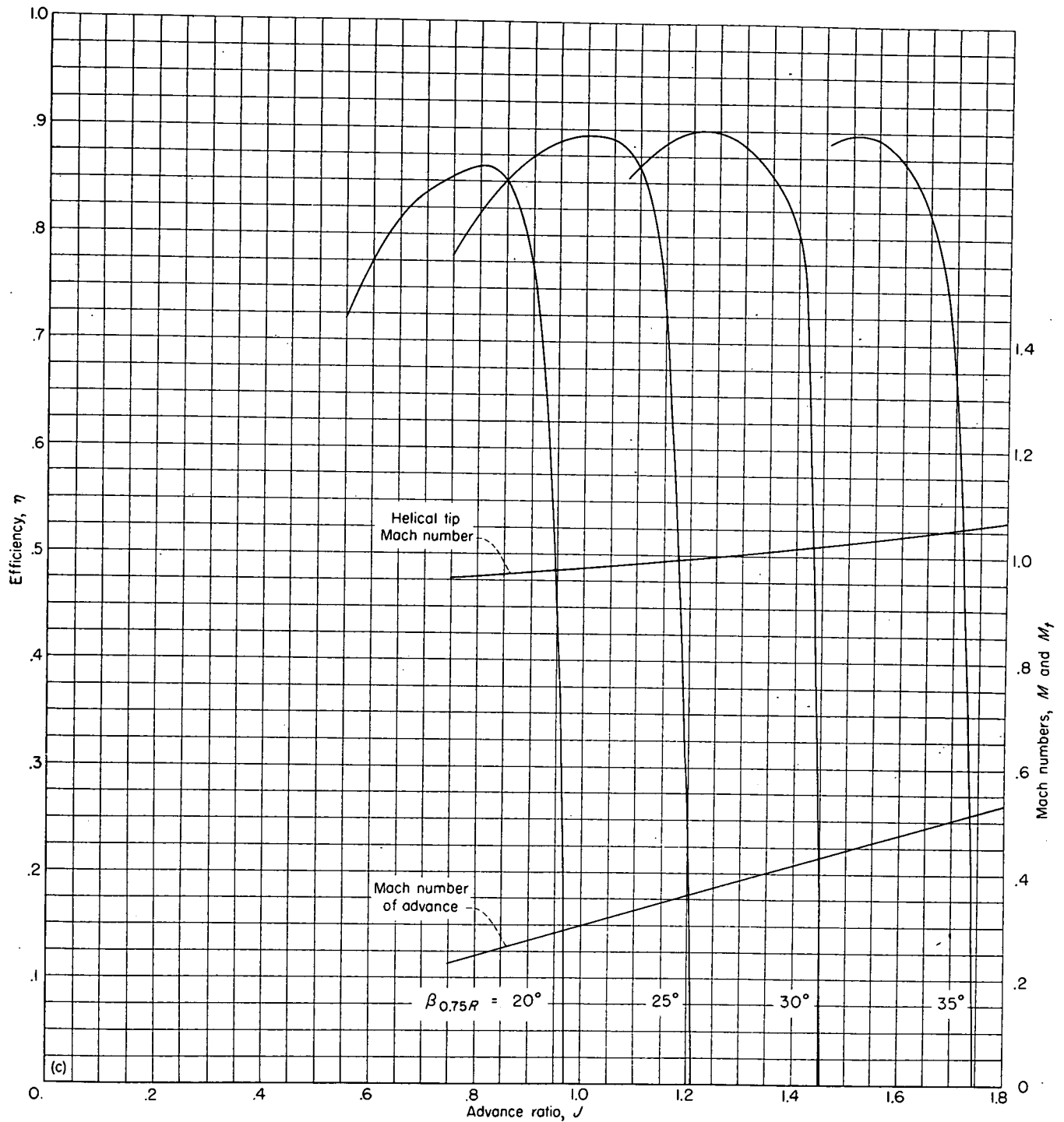
(a) Propeller thrust coefficient.

FIGURE 21.—Characteristics of NACA 10-(3)(05)-045 propeller. Rotational speed, 2,000 rpm.



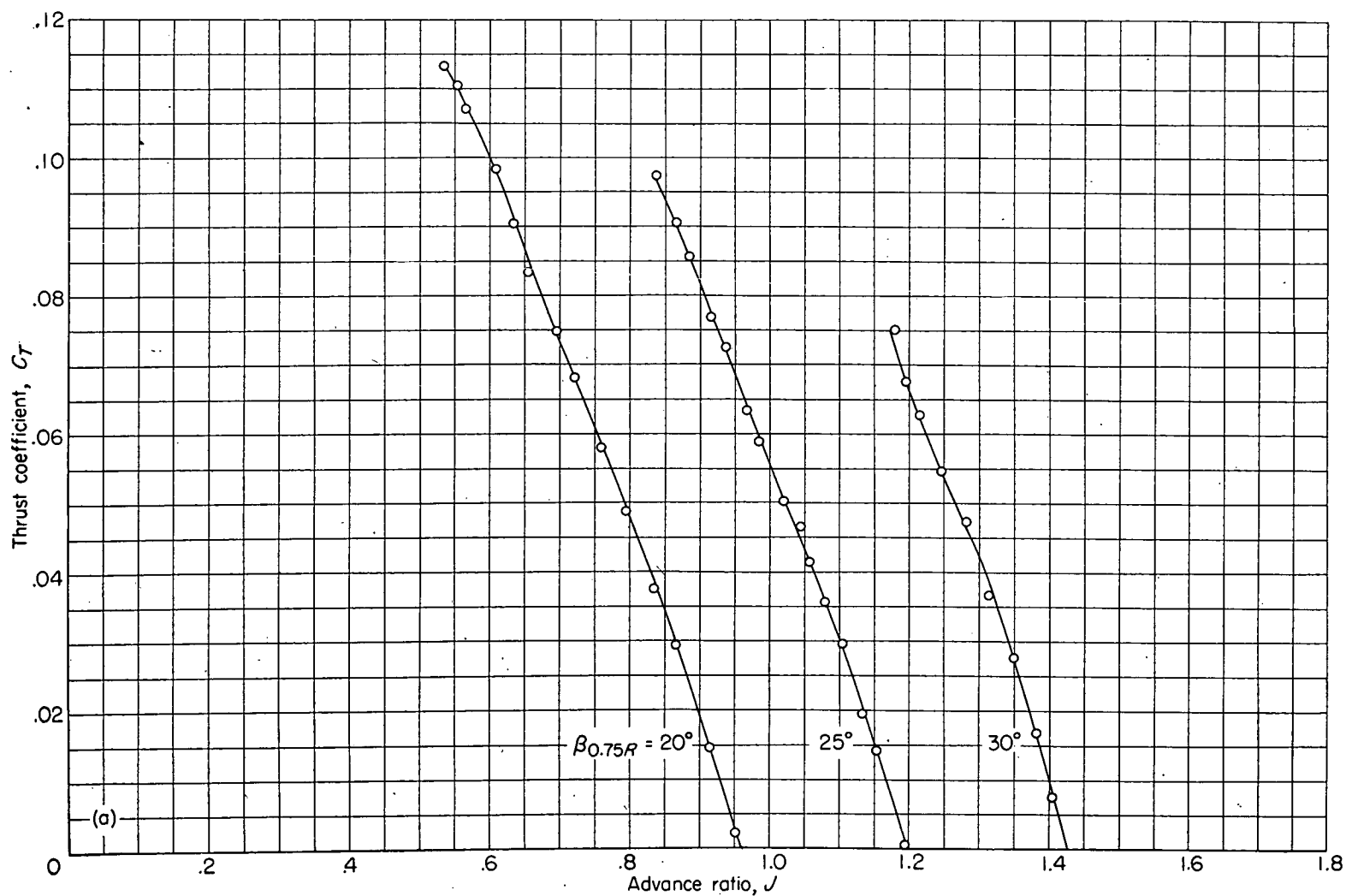
(b) Propeller power coefficient.

FIGURE 21.—Continued. Rotational speed, 2,000 rpm.



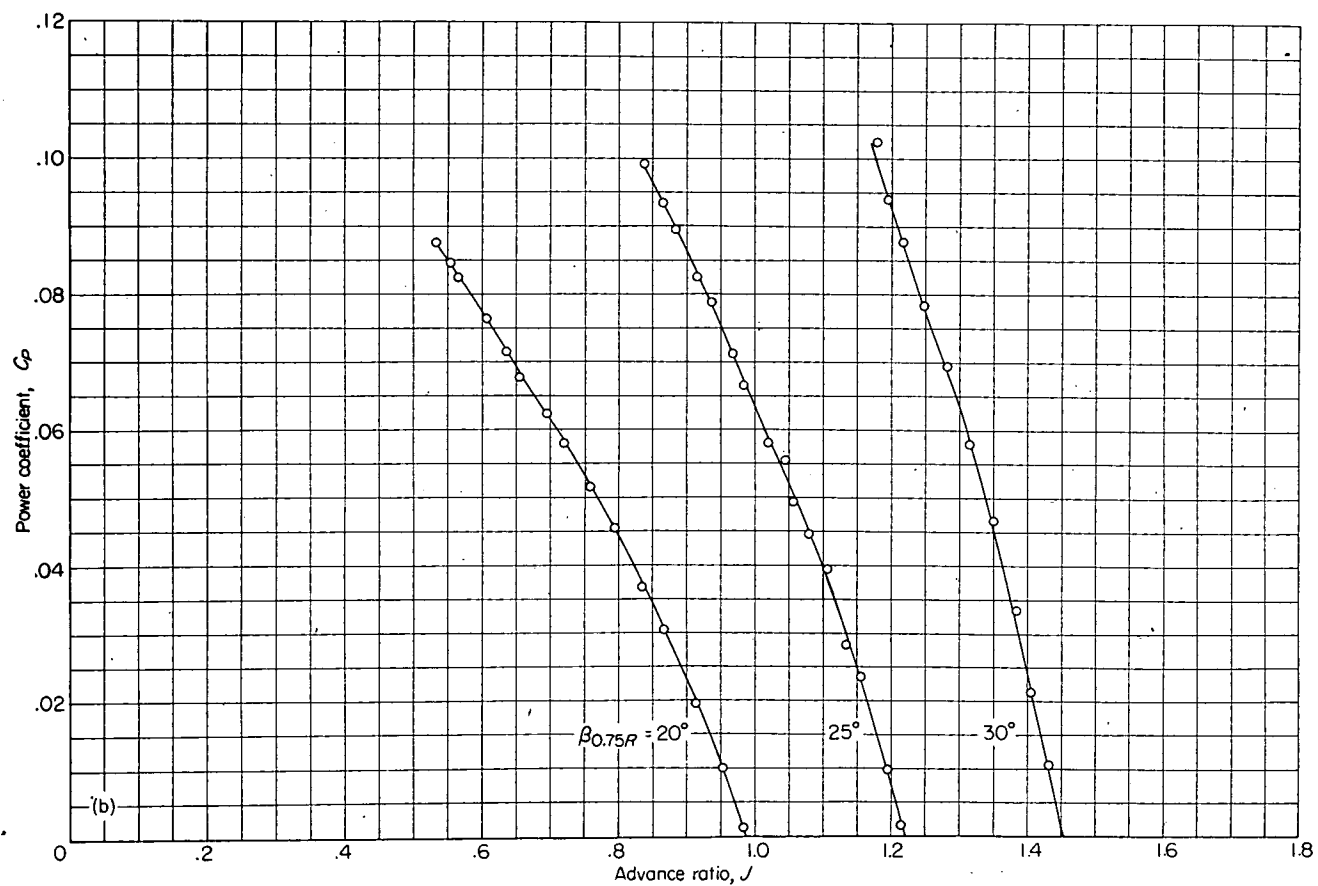
(c) Propeller efficiency.

FIGURE 21.—Concluded. Rotational speed, 2,000 rpm.



(a) Propeller thrust coefficient.

FIGURE 22.—Characteristics of NACA 10-(3)(05)-045 propeller. Rotational speed, 2,160 rpm.



(b) Propeller power coefficient.

FIGURE 22.—Continued. Rotational speed, 2,160 rpm.

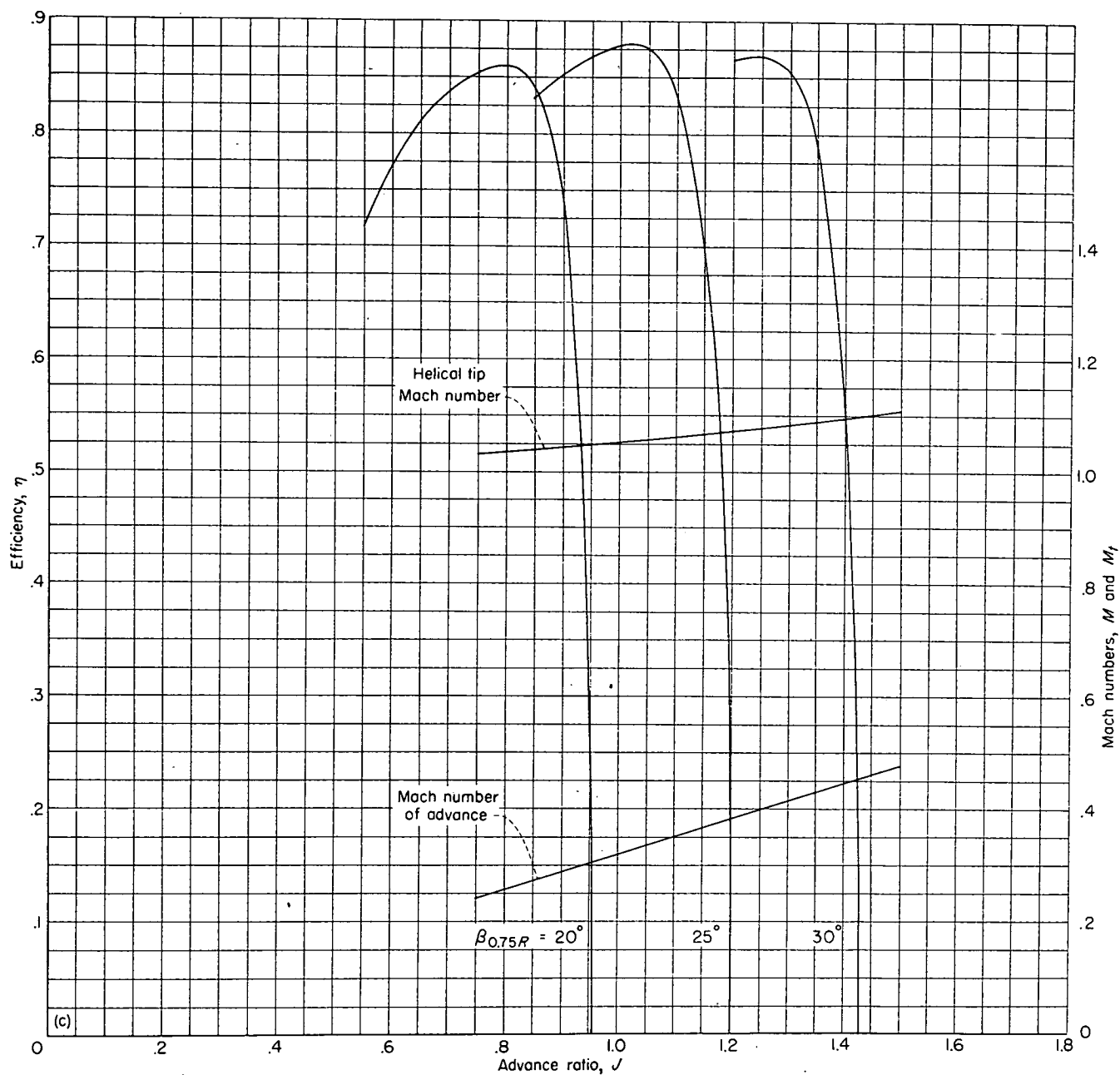
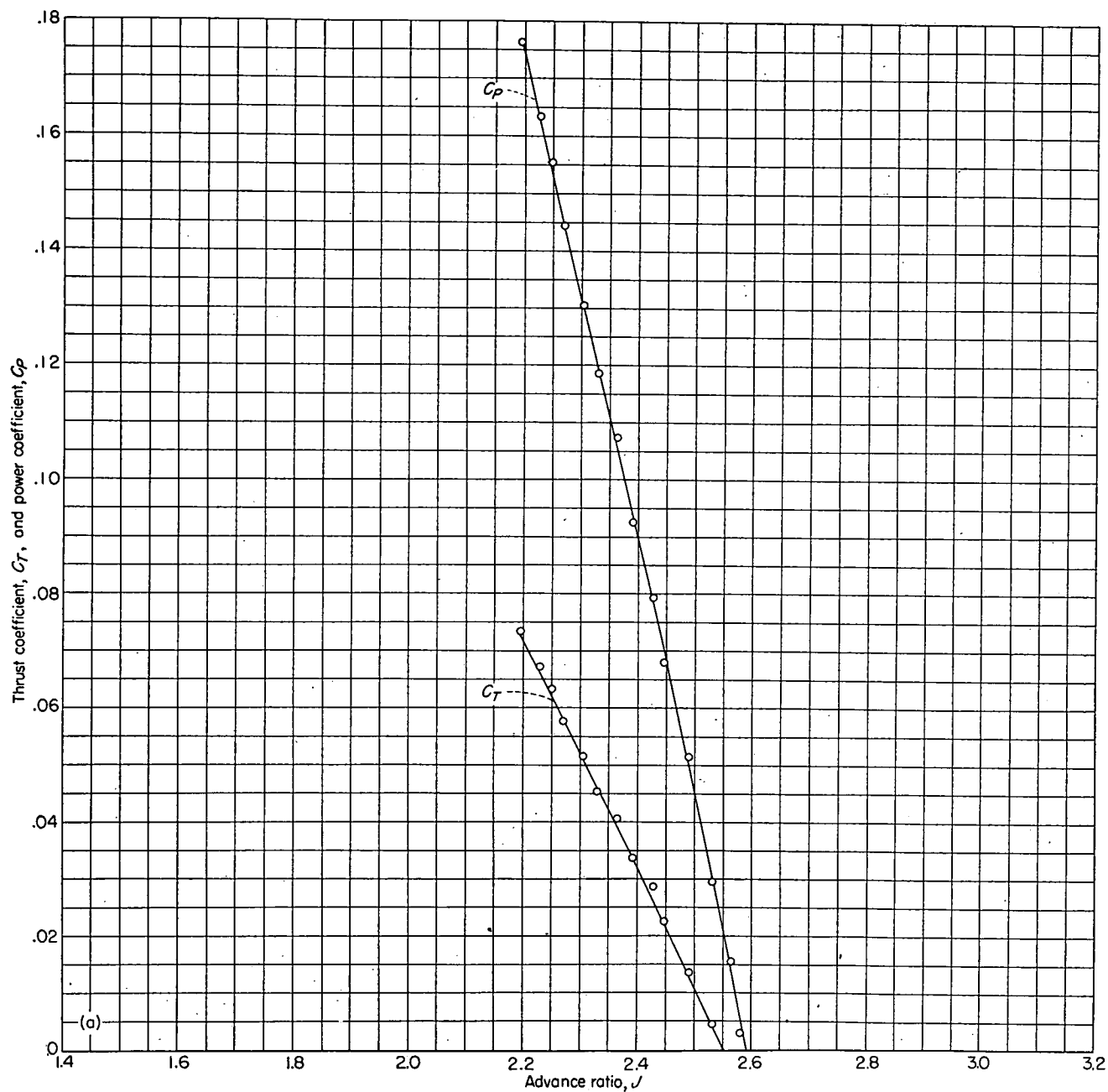
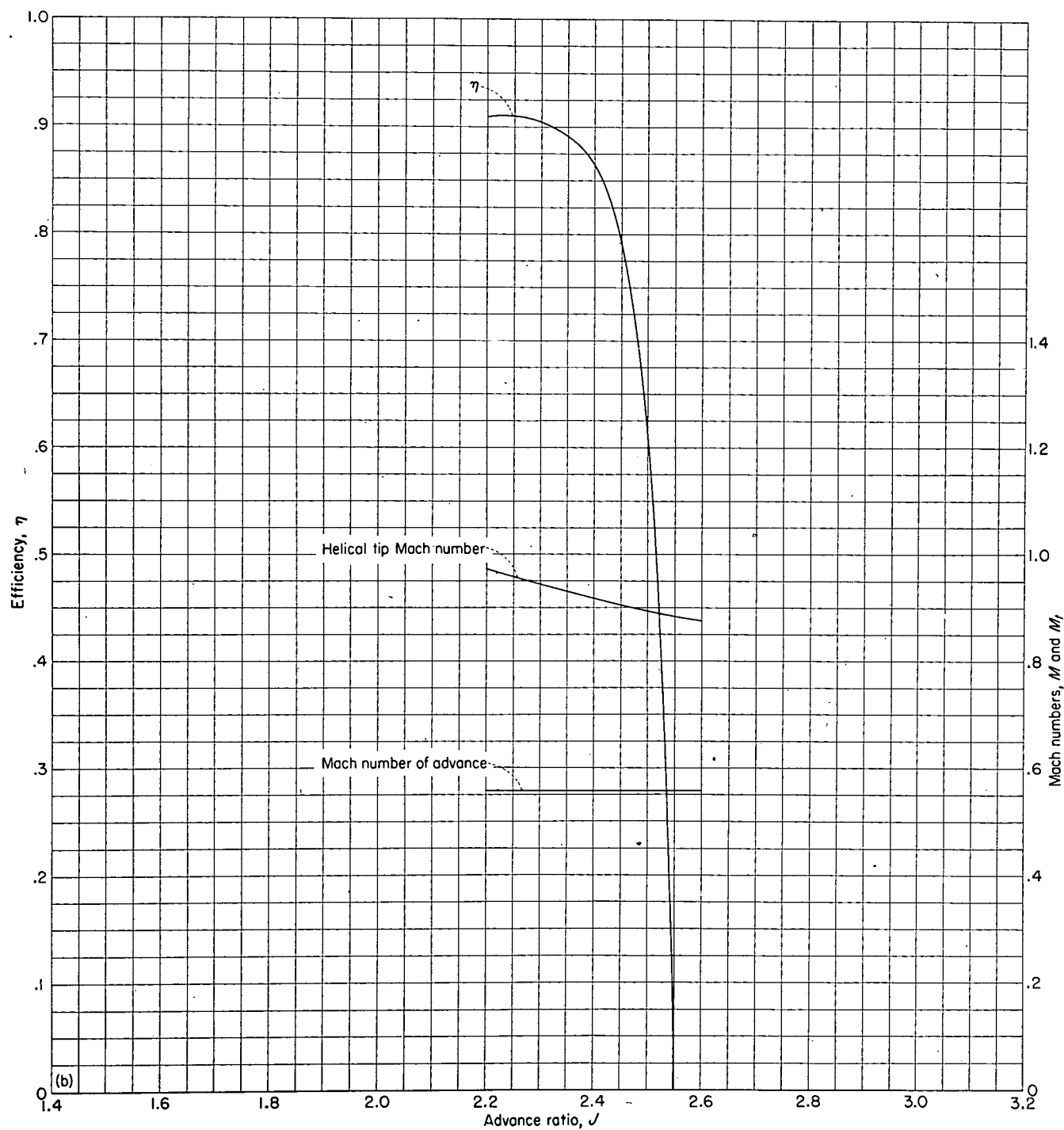


FIGURE 22.—Concluded. Rotational speed, 2,160 rpm.



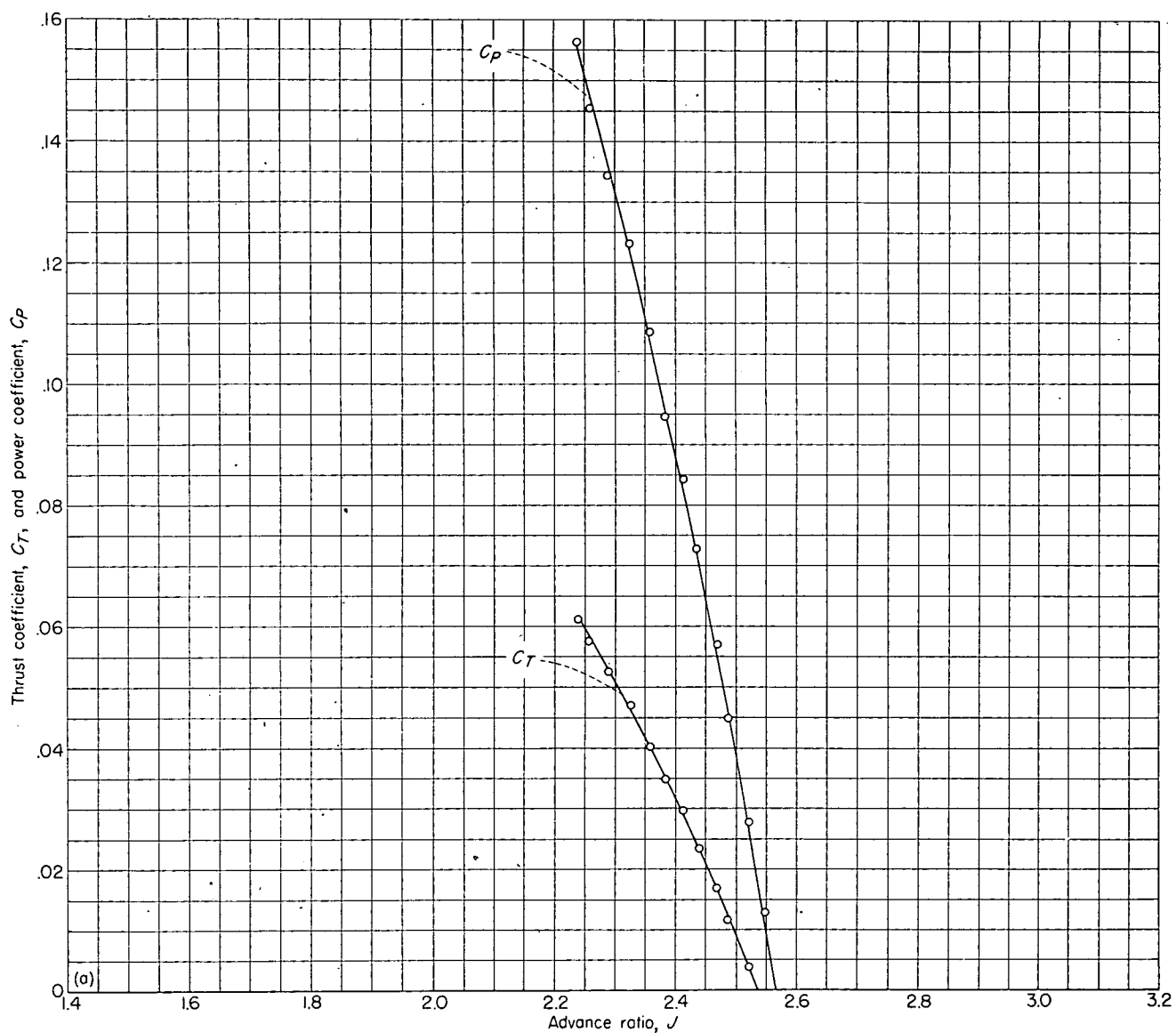
(a) Propeller thrust and power coefficients.

FIGURE 23.—Characteristics of NACA 10-(3)(05)-045 propeller at high forward speeds. Mach number of advance at maximum efficiency, 0.558; $\beta_{0.75R} = 45^\circ$.



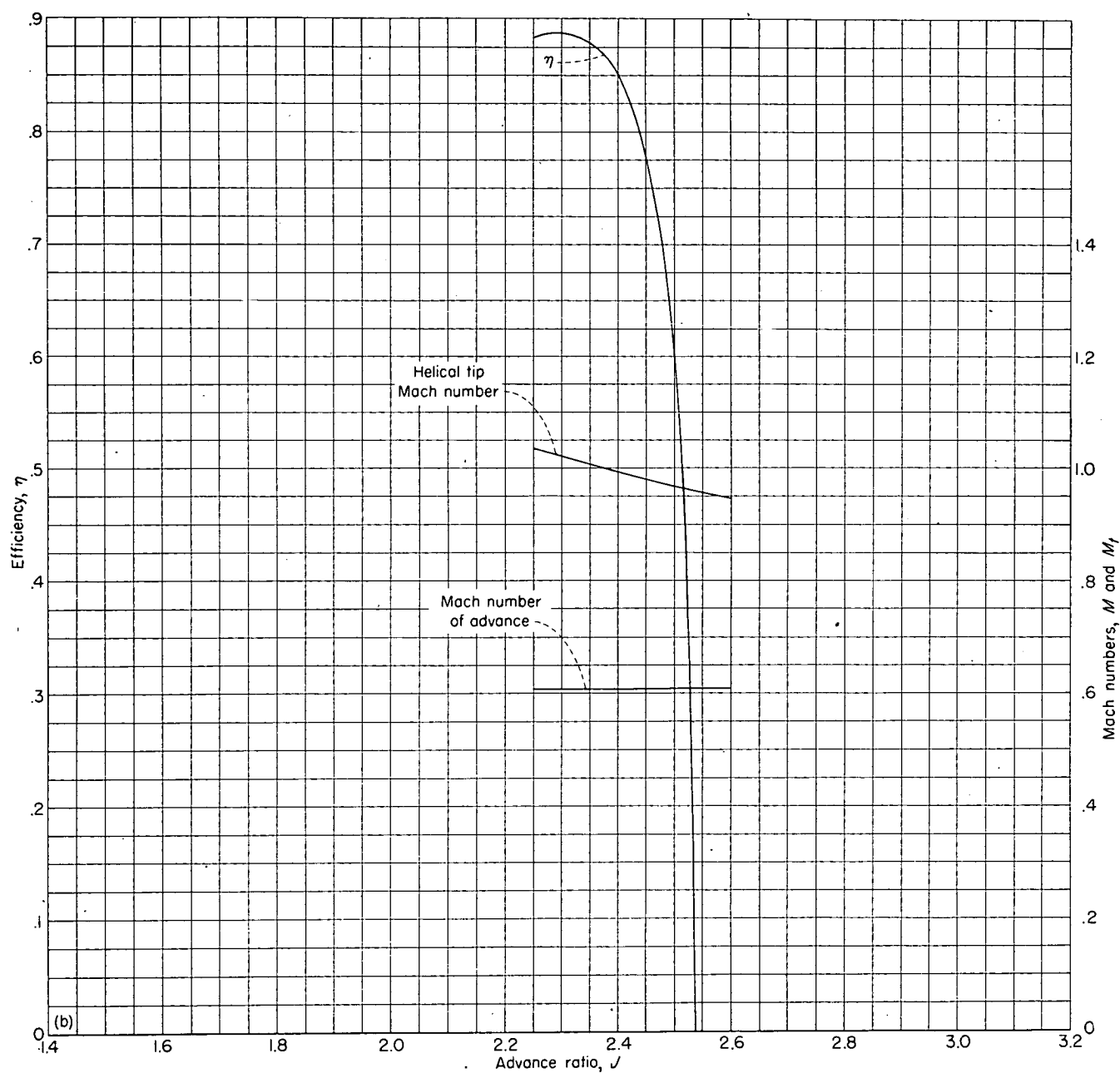
(b) Propeller efficiency.

FIGURE 23.—Concluded. Mach number of advance at maximum efficiency, 0.558.



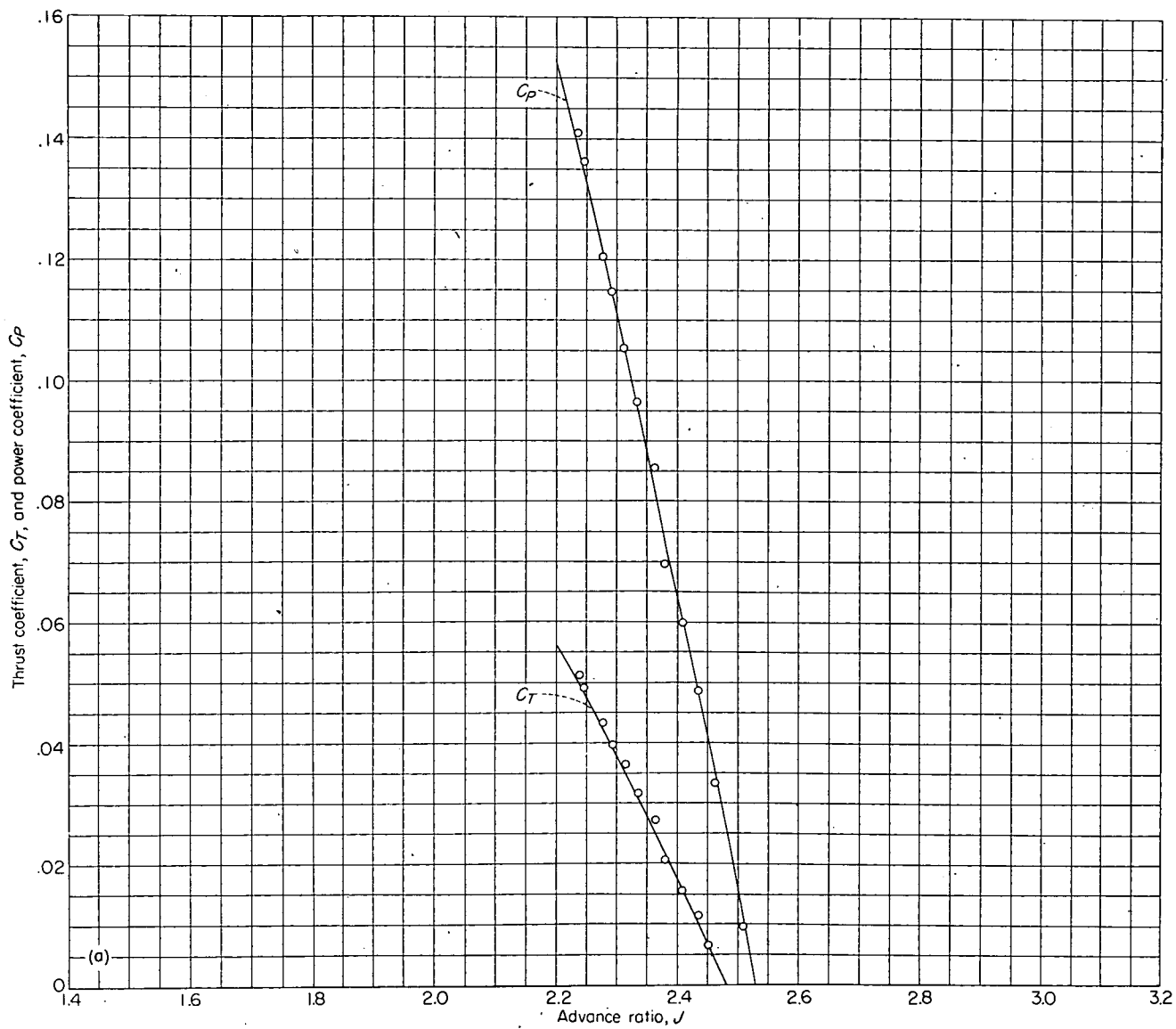
(a) Propeller thrust and power coefficients.

FIGURE 24.—Characteristics of NACA 10-(3)(05)-045 propeller at high forward speeds. Mach number of advance at maximum efficiency, 0.603; $\beta_{0.75R} = 45^\circ$.



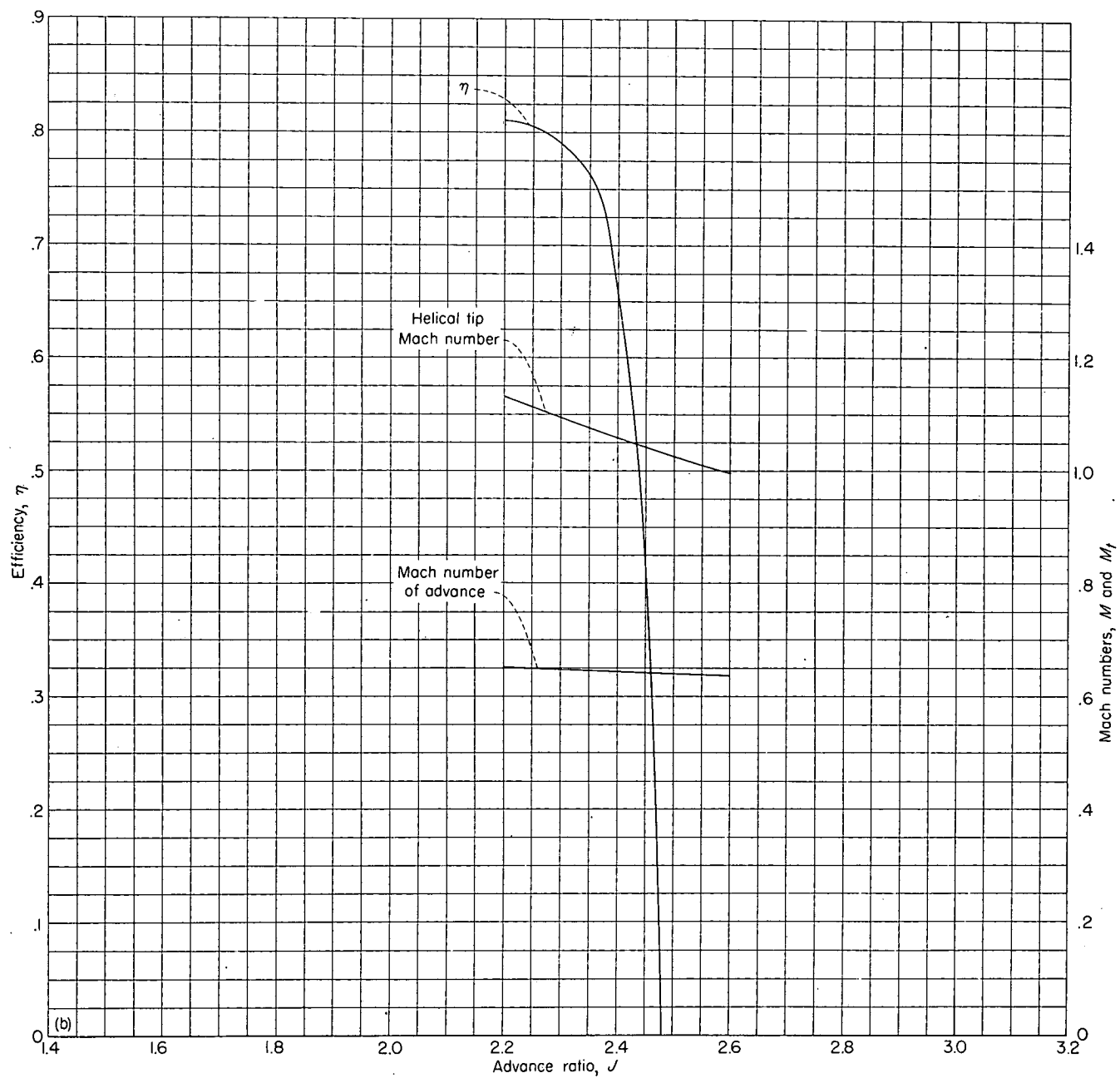
(b) Propeller efficiency.

FIGURE 24.—Concluded. Mach number of advance at maximum efficiency, 0.603.



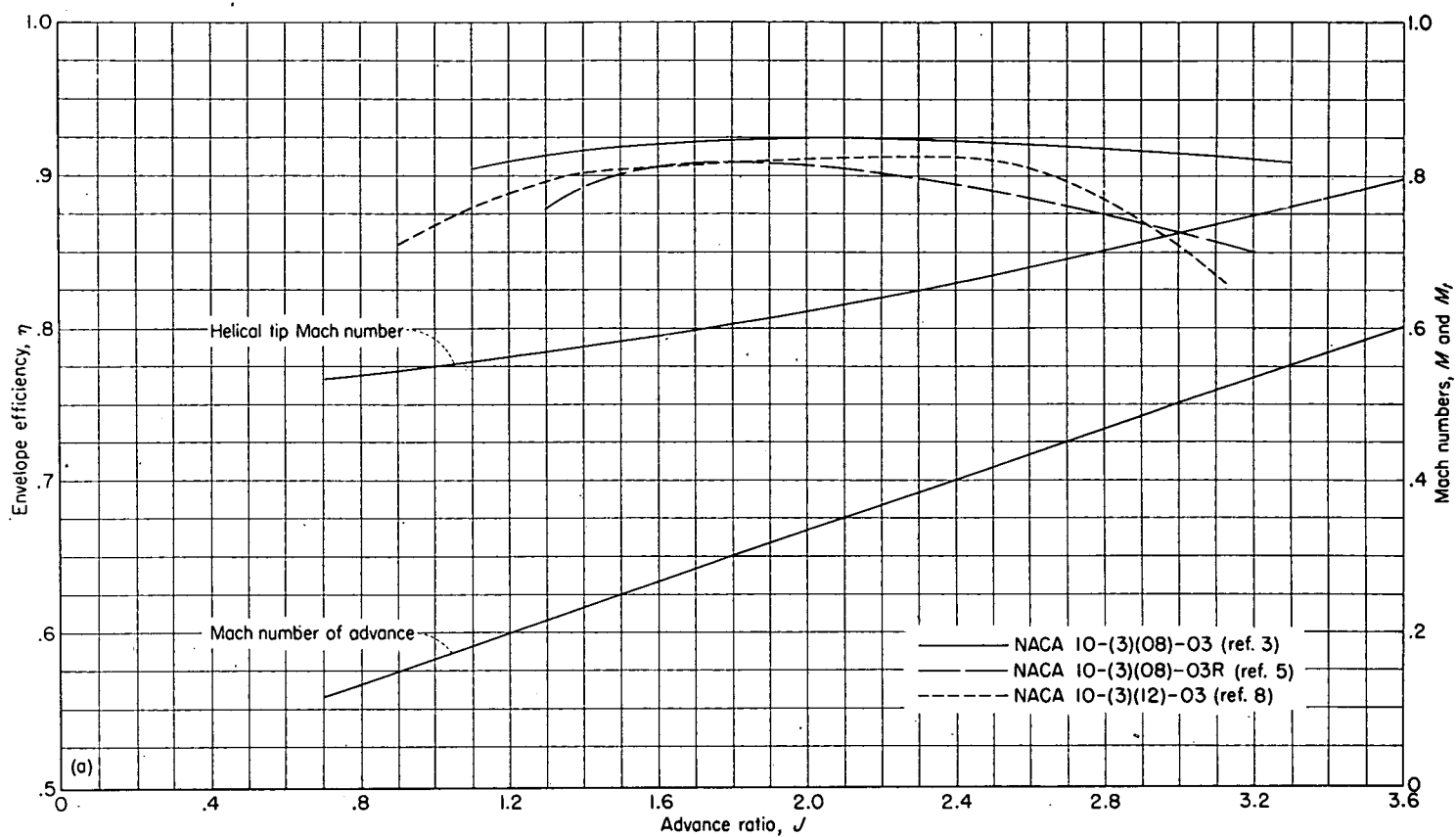
(a) Propeller thrust and power coefficients.

FIGURE 25.—Characteristics of NACA 10-(3) (05)-045 propeller at high forward speeds. Mach number of advance at maximum efficiency, 0.650; $\beta_{0.75R} = 45^\circ$.

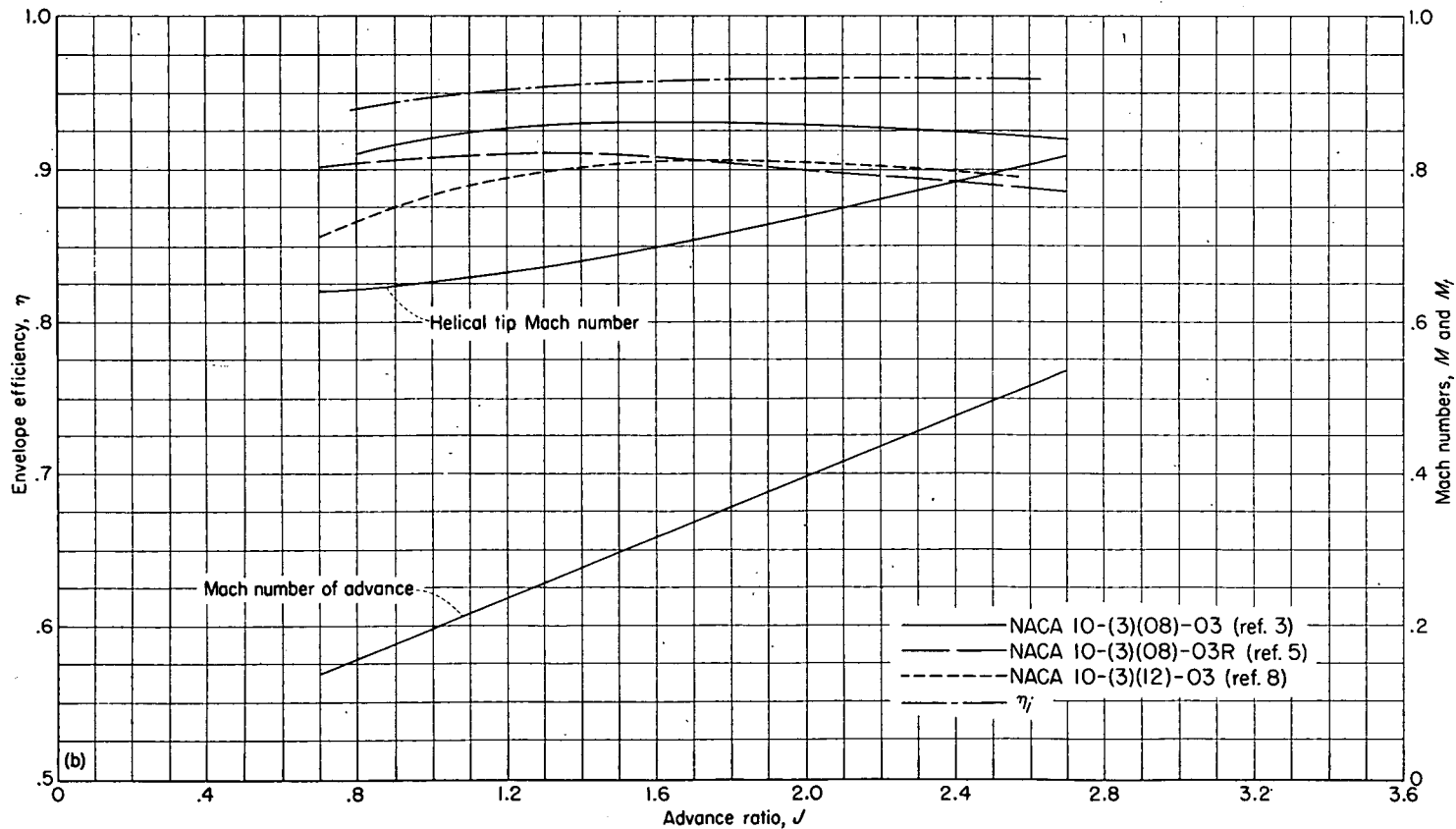


(b) Propeller efficiency.

FIGURE 25.—Concluded. Mach number of advance at maximum efficiency, 0.650.



(a) 1,140 rpm.



(b) 1,350 rpm.

FIGURE 26.—Comparison of the envelope efficiencies of NACA propellers having a solidity of 0.03 per blade at the 0.7-radius station.

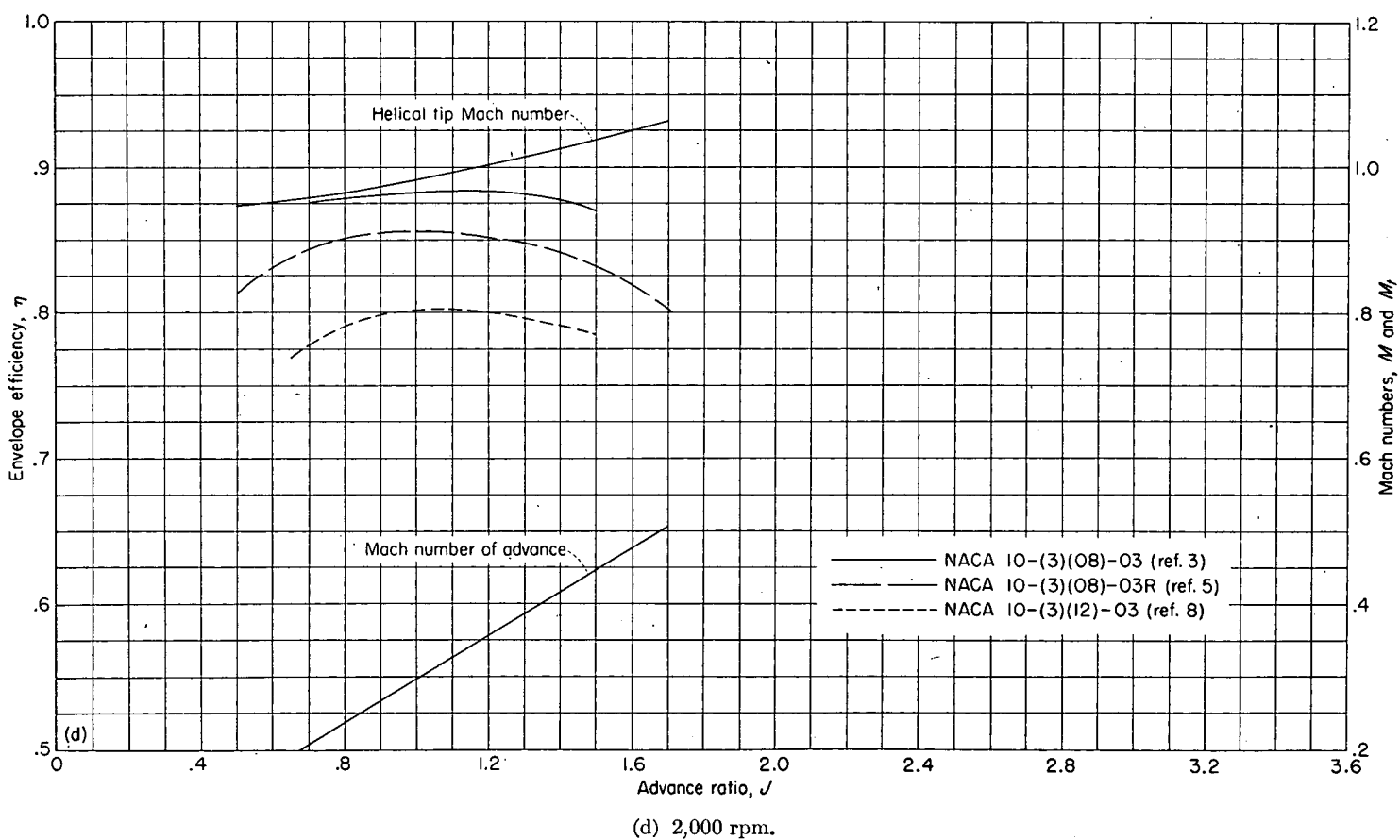
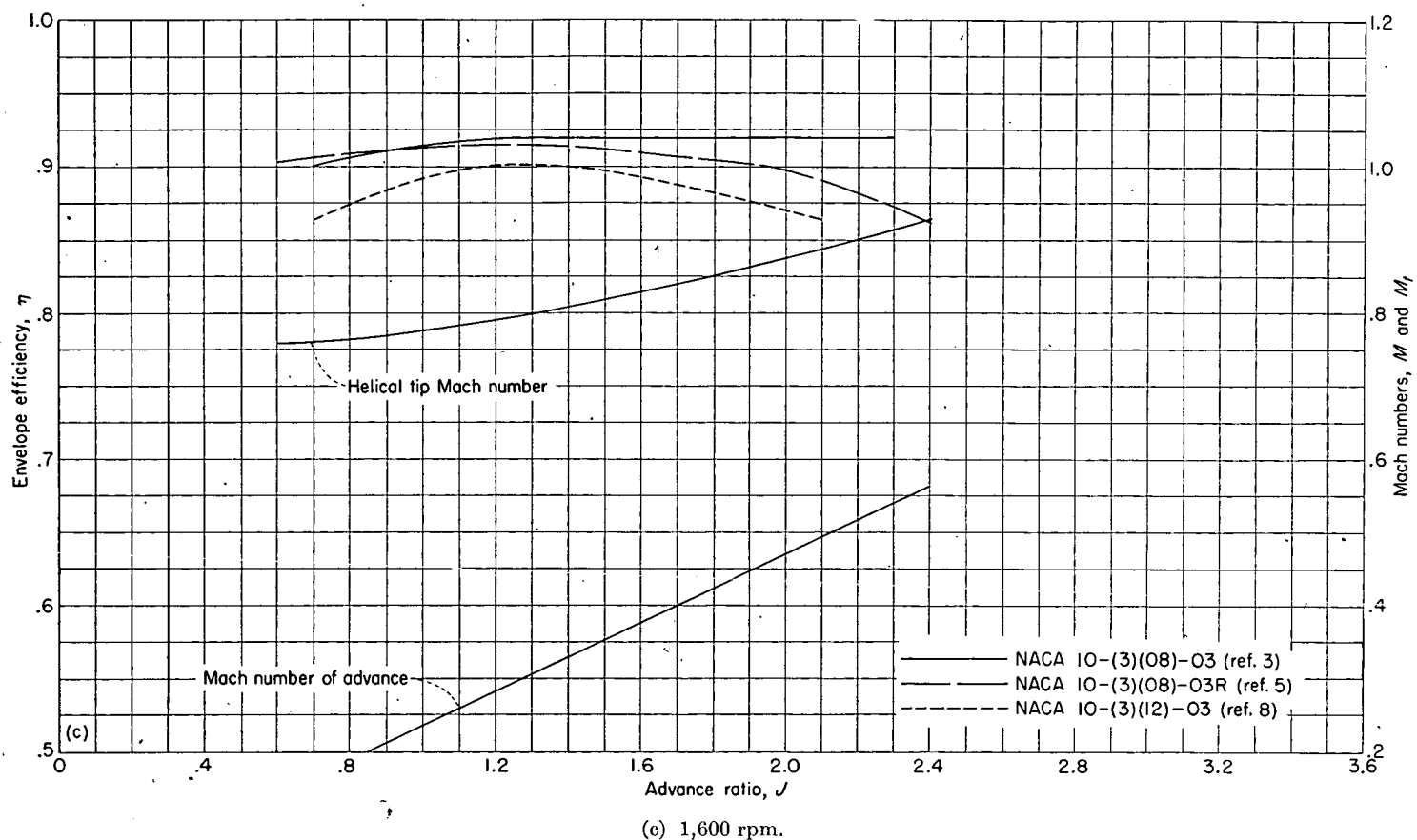


FIGURE 26.—Continued.

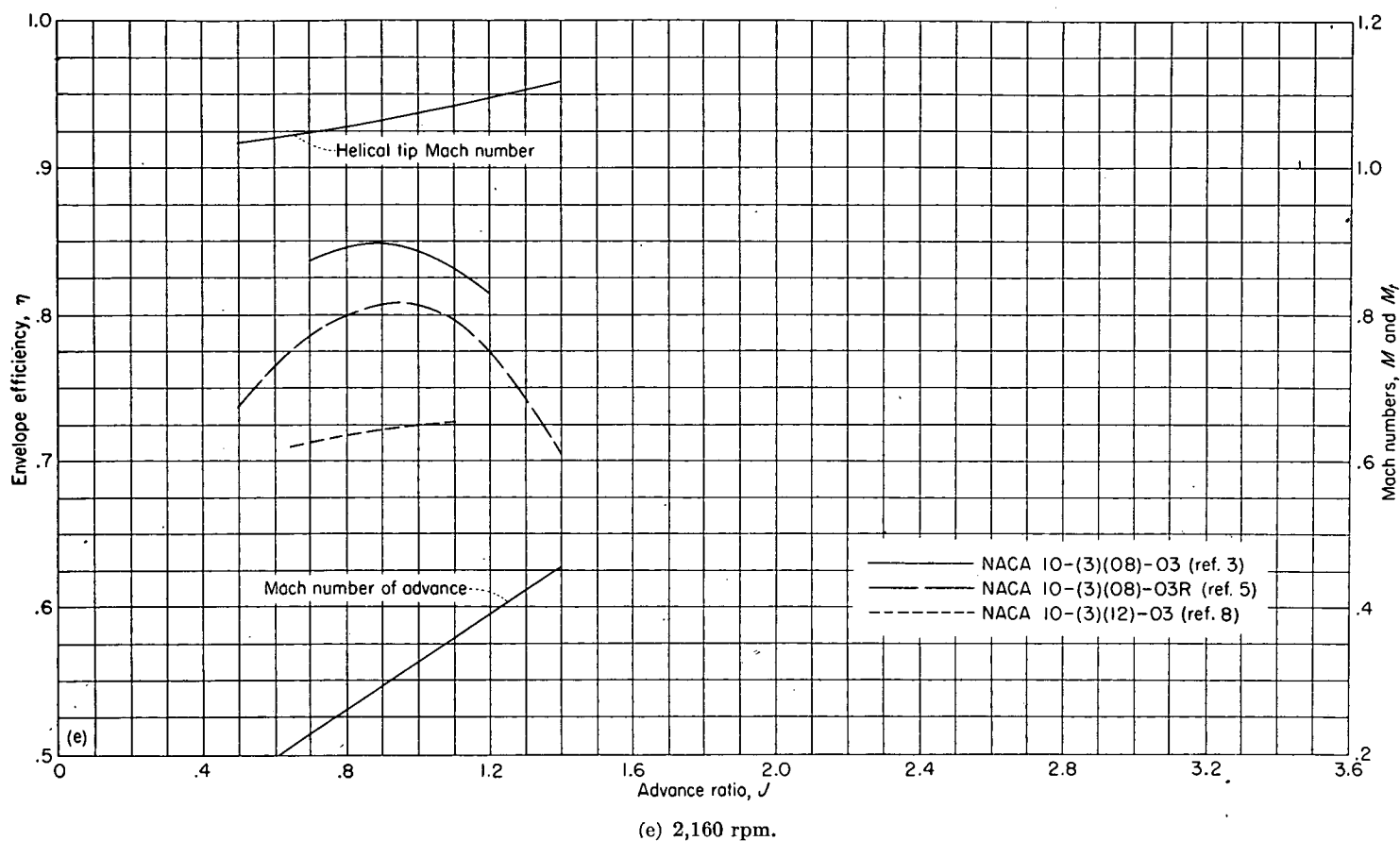


FIGURE 26.—Concluded.

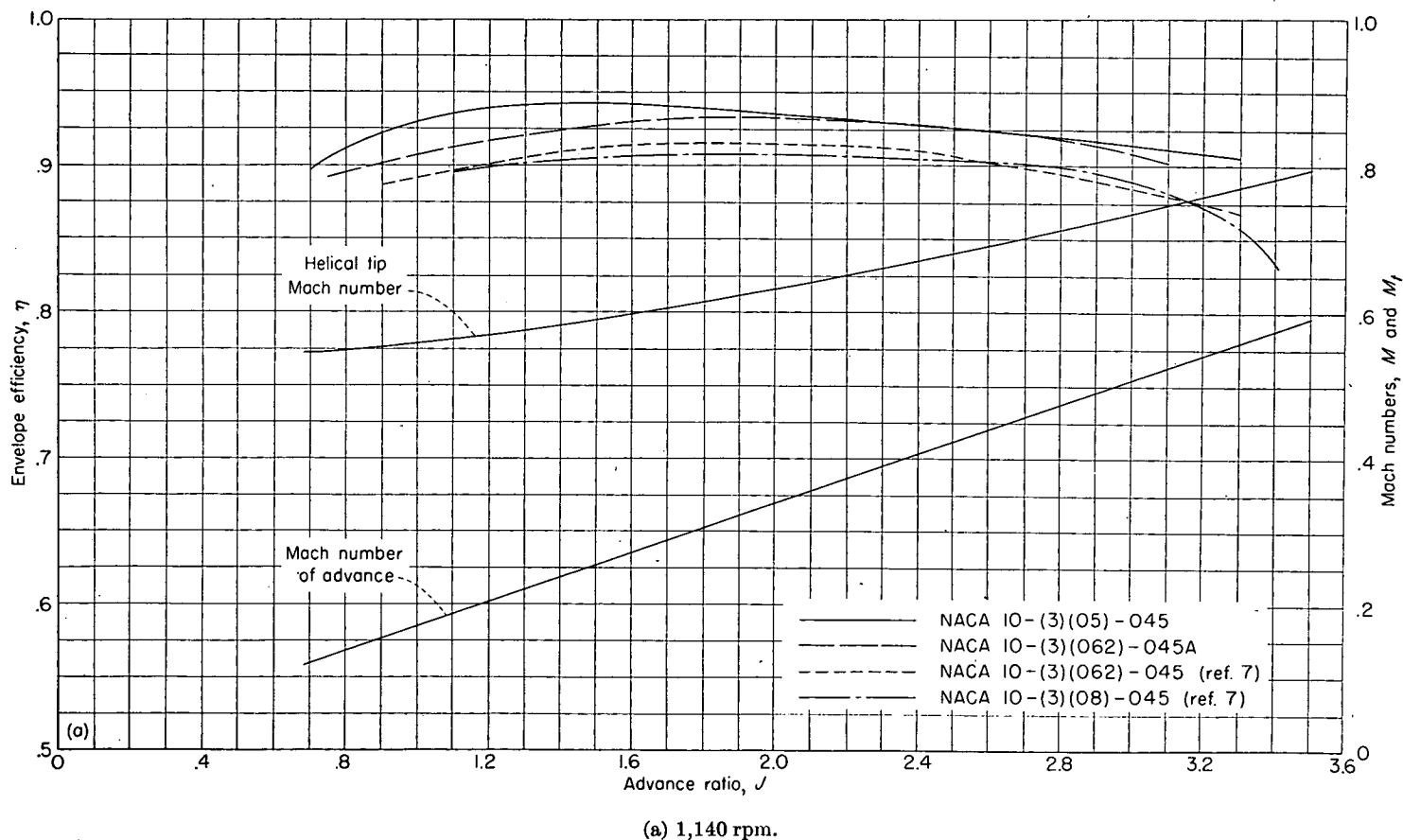
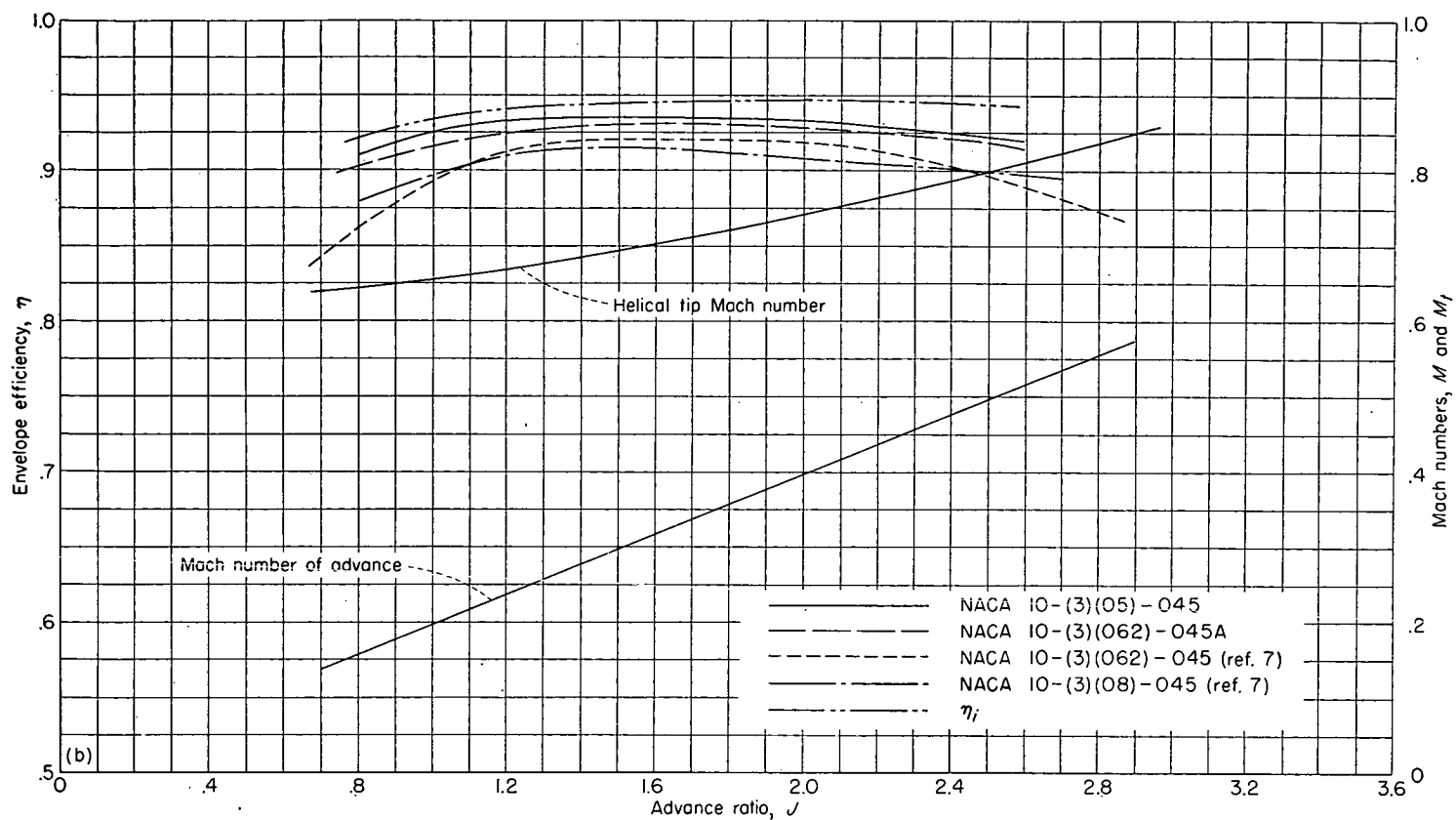
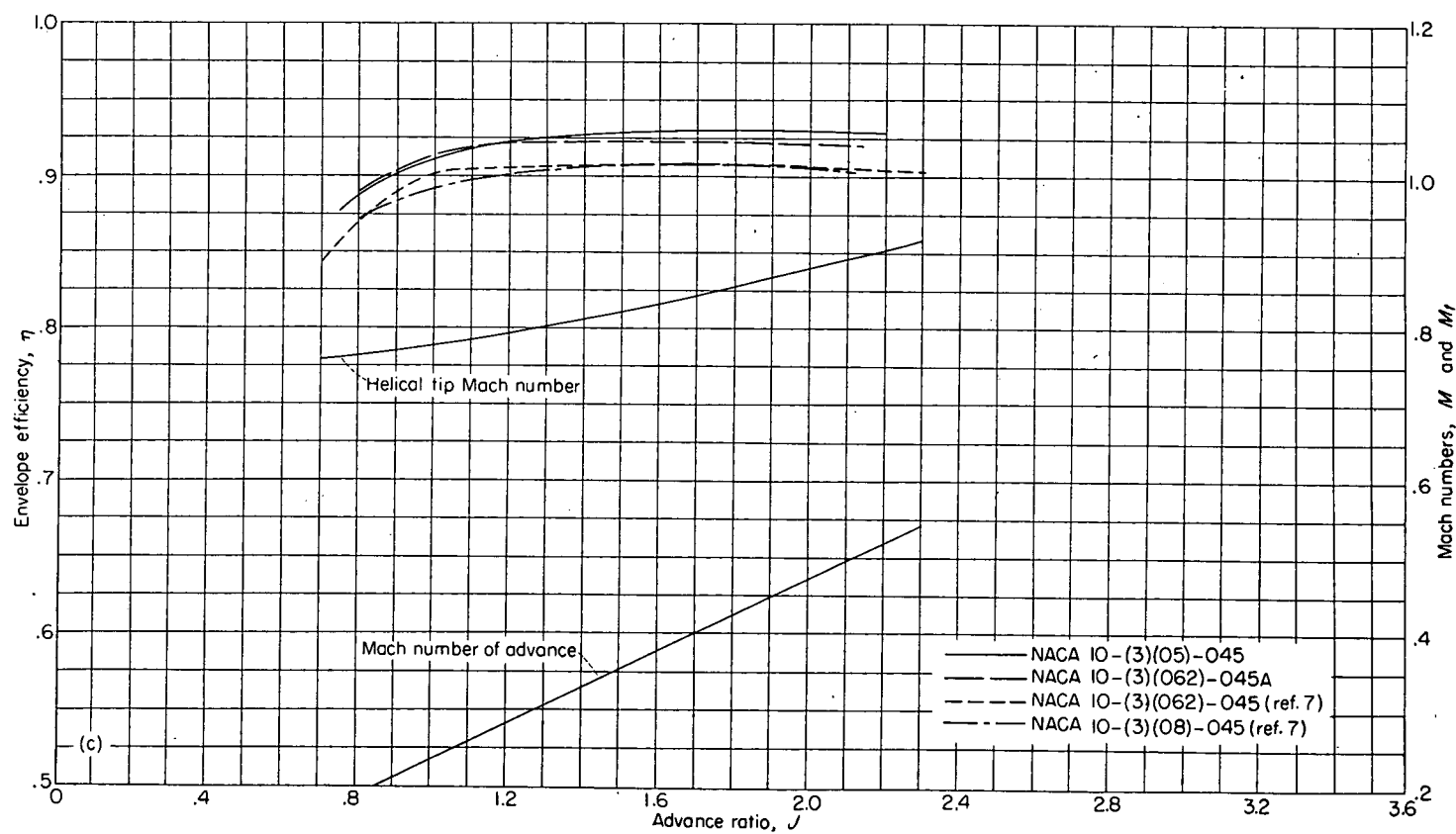


FIGURE 27.—Comparison of the envelope efficiencies of NACA propellers having a solidity of 0.045 per blade at the 0.7-radius station.



(b) 1,350 rpm.



(c) 1,600 rpm.

FIGURE 27.—Continued.

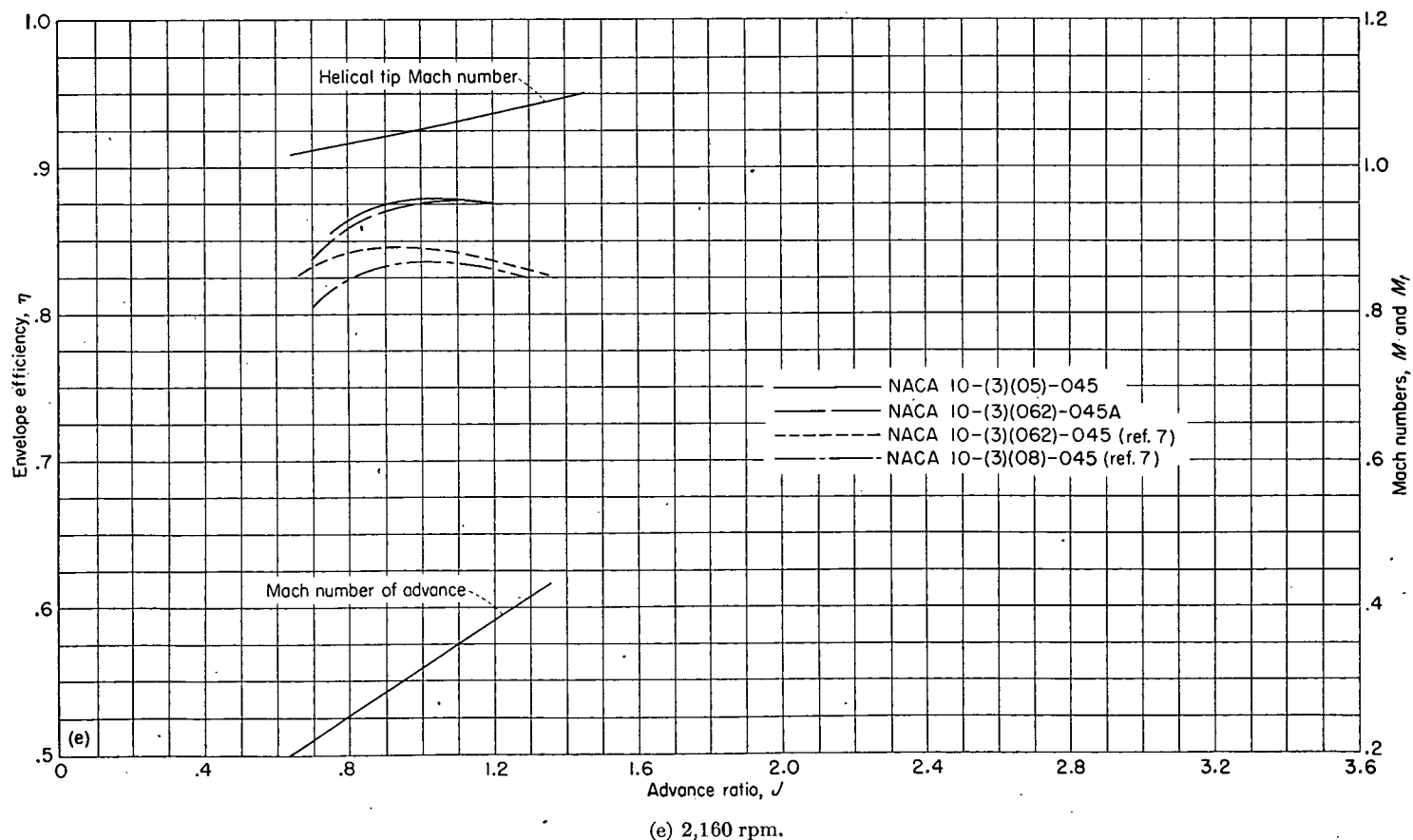
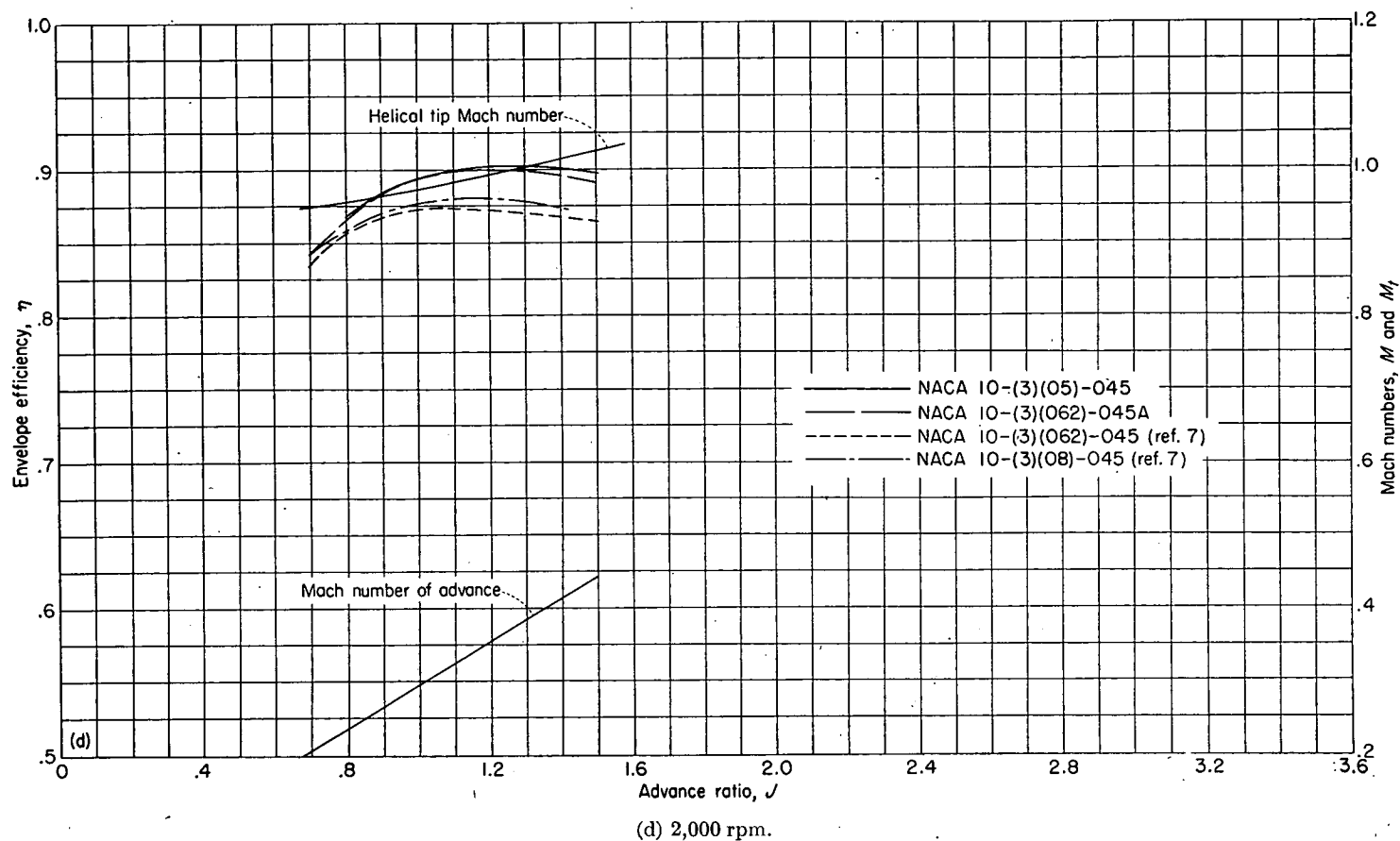
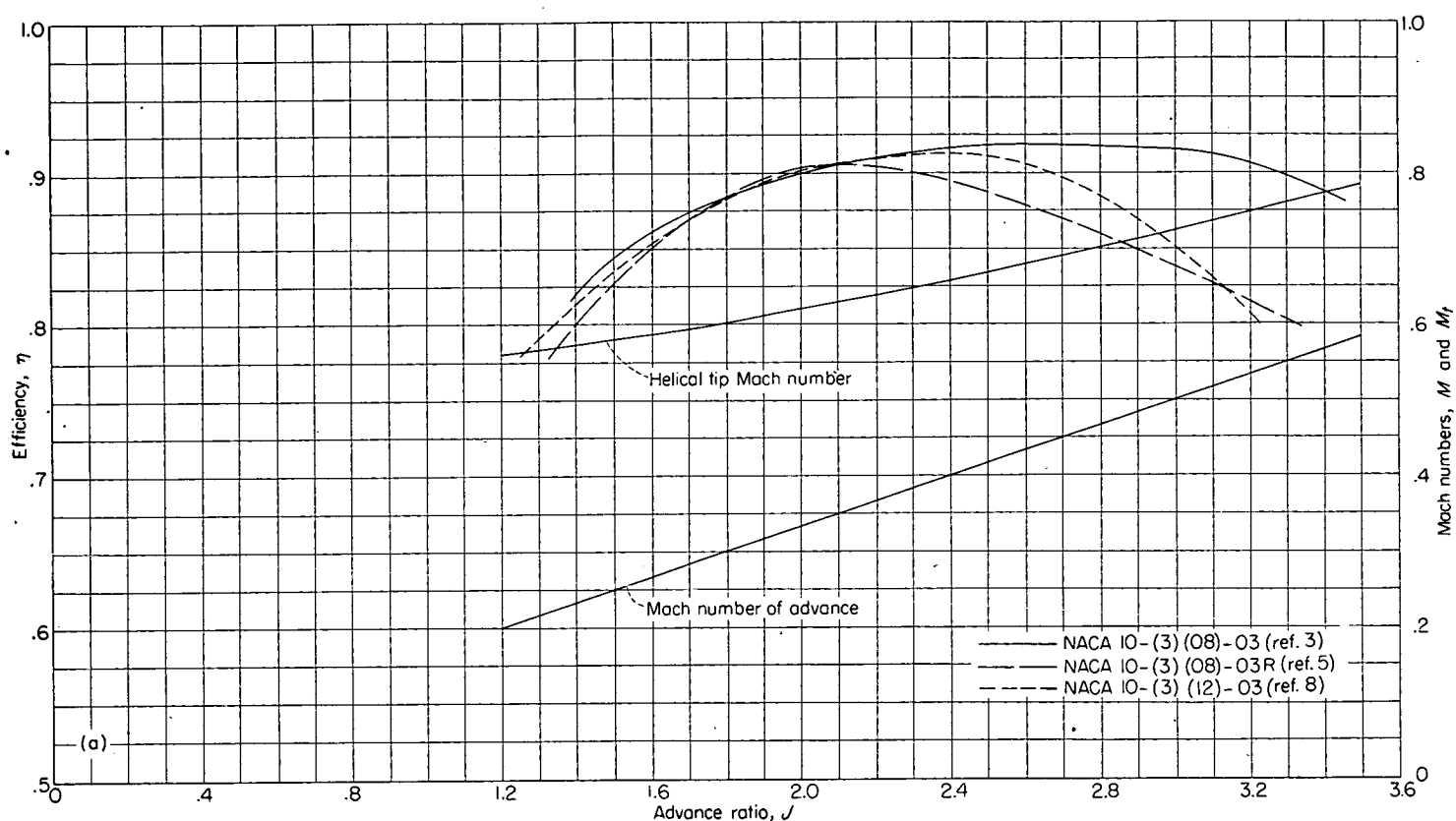
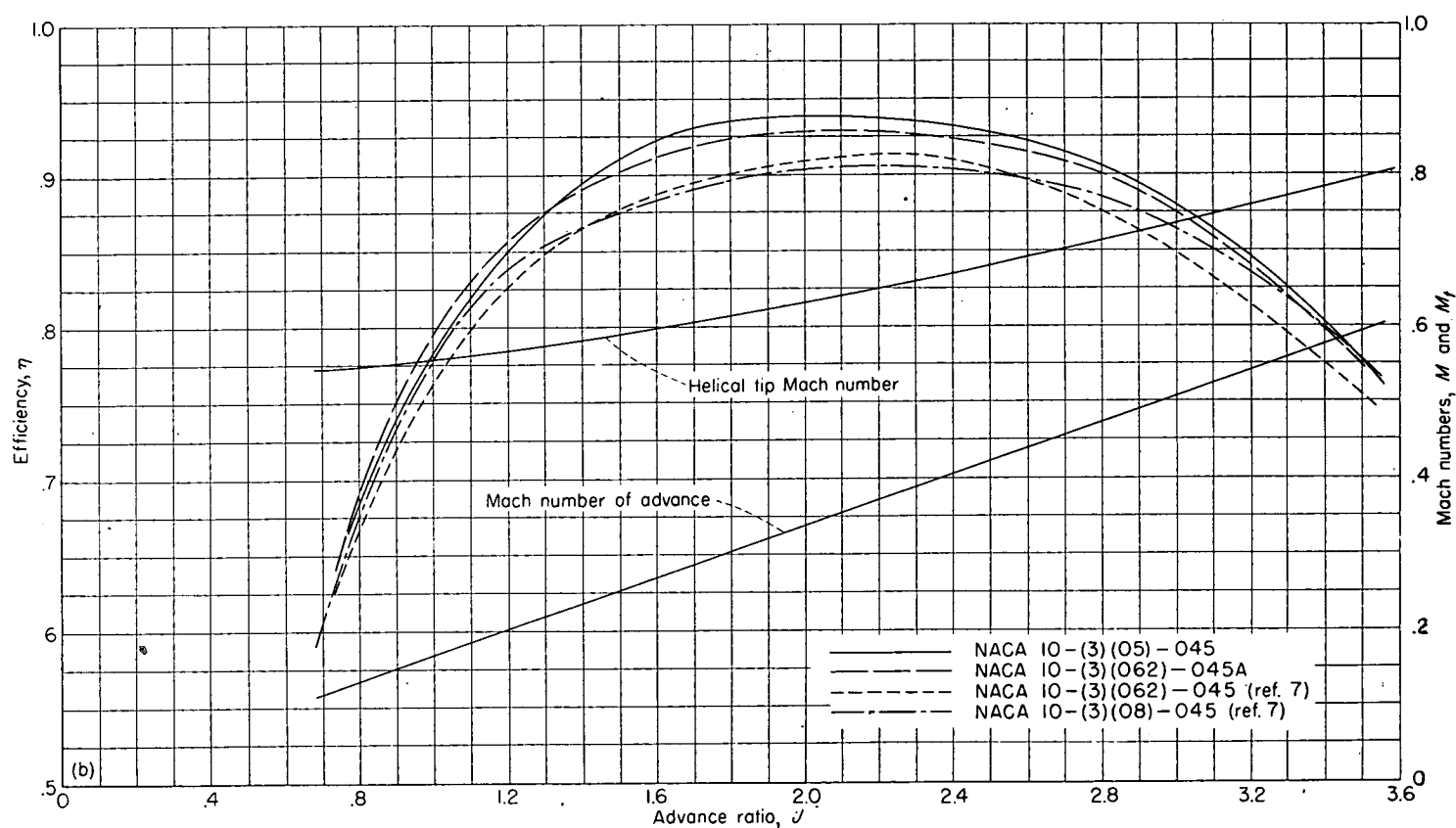


FIGURE 27.—Concluded.



(a) Solidity, 0.03 per blade at the 0.7-radius station.



(b) Solidity, 0.045 per blade at the 0.7-radius station.

FIGURE 28.—The efficiency of NACA propellers having different blade-section thicknesses when operating at a constant power coefficient of 0.15 and a rotational speed of 1,140 rpm.

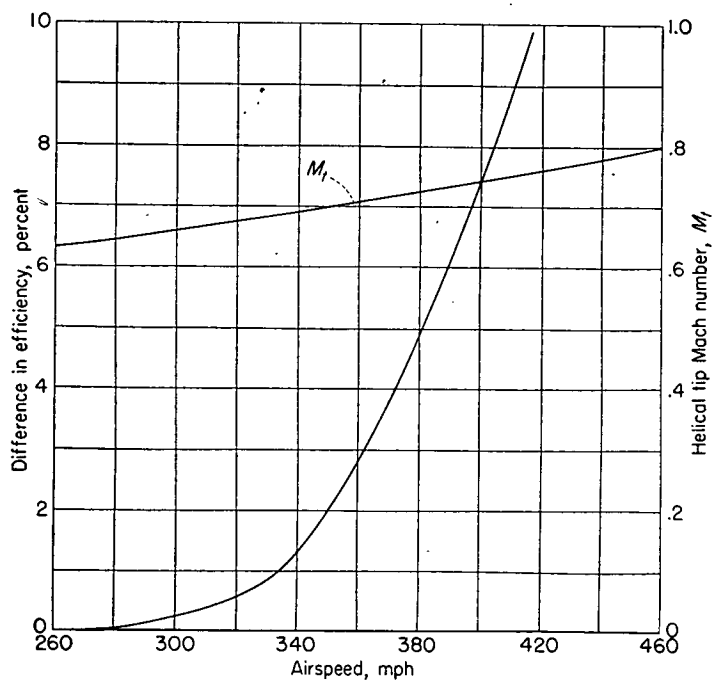


FIGURE 29.—The effect of airspeed on the difference in efficiency between the NACA 10-(3)(08)-03 and NACA 10-(3)(12)-03 two-blade propellers. Constant propeller rotational speed, 1,140 rpm; constant power coefficient, 0.15.

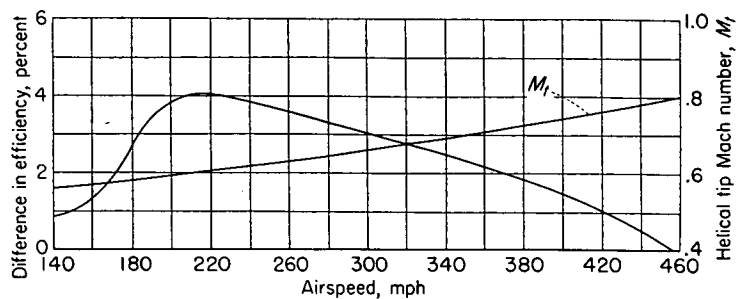
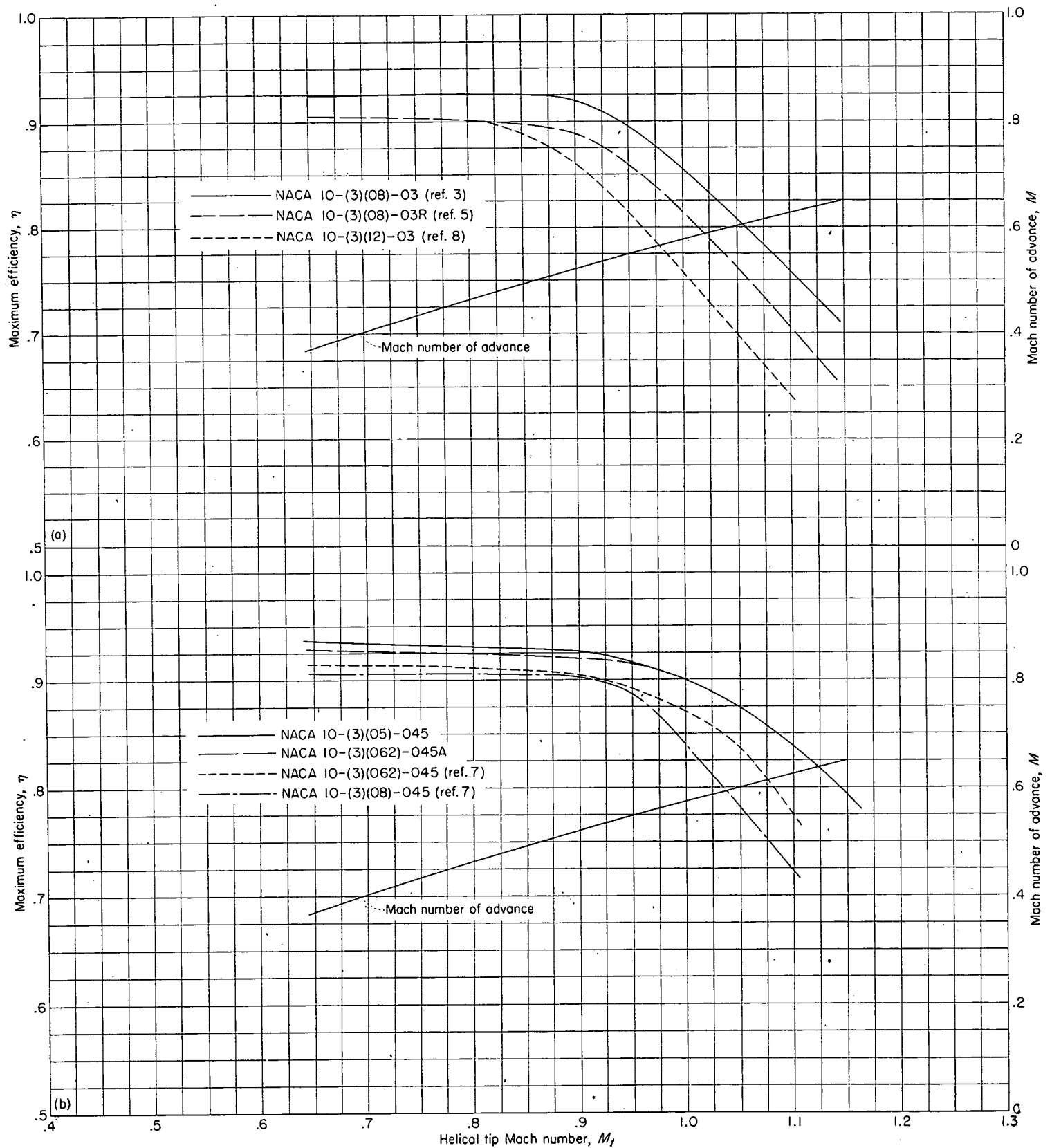


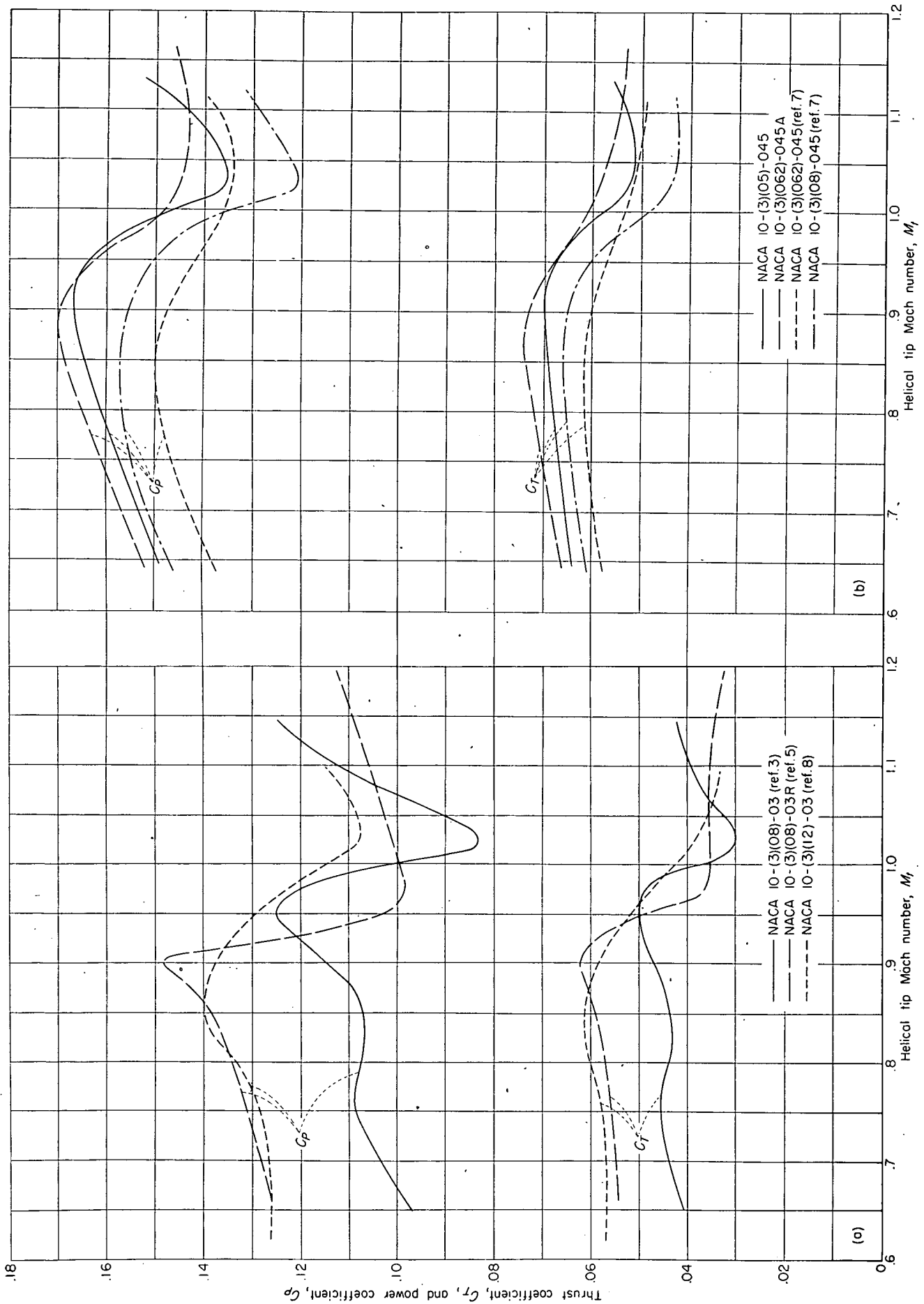
FIGURE 30.—The effect of airspeed on the difference in efficiency between the NACA 10-(3)(05)-045 and NACA 10-(3)(08)-045 two-blade propellers. Constant propeller rotational speed, 1,140 rpm; constant power coefficient, 0.15.



(a) Solidity, 0.03 per blade at the 0.7-radius station.

(b) Solidity, 0.045 per blade at the 0.7-radius station.

FIGURE 31.—The effect of compressibility on the maximum efficiency of NACA propellers having different blade-section thicknesses.
 $\beta_{0.75R} = 45^\circ$.



(a) Solidity, 0.03 per blade at the 0.7-radius station.
 (b) Solidity, 0.045 per blade at the 0.7-radius station.
 Figure 32.—The effect of compressibility on the thrust and power coefficients for maximum efficiency of NACA propellers having different blade-section thicknesses. $\beta_{0.7R} = 45^\circ$.

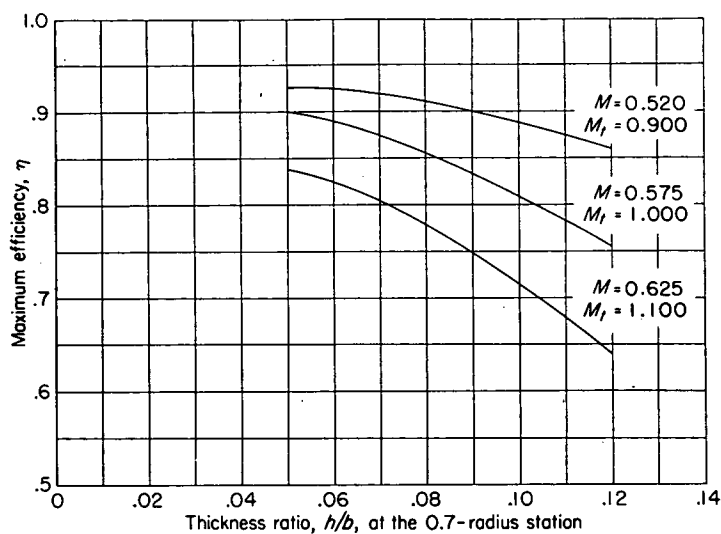


FIGURE 33.—The effect of thickness ratio and compressibility on the maximum efficiency of the NACA propellers. $\beta_{0.75R} = 45^\circ$.

TABLE I—RANGE OF BLADE ANGLES AT VARIOUS ROTATIONAL SPEEDS FOR NACA PROPELLER TESTS

Figure (a)	Rotational speed, rpm	Blade angle at 0.75-radius station, $\beta_{0.75R}$, deg							
NACA 10-(3)(062)-045A propeller									
8	1140	20	25	30	35	40	45	50	55
9	1350	20	25	30	35	40	45	50	
10	1500	---	---	---	---	---	45		
11	1600	20	25	30	35	40	45		
12	2000	20	25	30	35				
13	2160	20	25	30					
14, 15, 16	Varied						45		
NACA 10-(3)(05)-045 propeller									
17	1140	20	25	30	35	40	45	50	55
18	1350	20	25	30	35	40	45	50	
19	1500	---	---	---	---	---	45		
20	1600	20	25	30	35	40	45		
21	2000	20	25	30	35				
22	2160	20	25	30					
23, 24, 25	Varied						45		
NACA 10-(3)(062)-045 propeller (ref. 7)									
4	1140	---	25	30	35	40	45	50	55
5	1350	20	25	30	35	40	45	50	
6	1500	---	---	---	---	---	45		
7	1600	20	25	30	35	40	45		
8	2000	20	25	30	35				
9	2160	20	25	30					
10	Varied						45		
NACA 10-(3)(08)-045 propeller (ref. 7)									
11	1140	---	---	30	35	40	45	50	55
12	1350	20	25	30	35	40	45	50	
13	1500	---	---	---	---	---	45		
14	1600	20	25	30	35	40	45		
15	2000	20	25	30	35				
16	2160	20	25	30					
17	Varied						45		
NACA 10-(3)(08)-03 propeller (ref. 3)									
19	1140	---	---	30	35	40	45	50	55
20	1350	20	25	30	35	40	45	50	
21	1500	---	---	---	---	---	45		
22	1600	20	25	30	35	40	45		
23	2000	20	25	30	35				
24	2160	20	25	30					
25	Varied						45		
NACA 10-(3)(08)-03R propeller (ref. 5)									
3	1140	---	---	---	35	40	45	50	55
4	1350	20	25	30	35	40	45	50	
8	1500	---	---	---	---	---	45		
5	1600	20	25	30	35	40	45		
6	2000	20	25	30	35				
7	2160	20	25	30					
9	Varied						45		
10	Varied							50	
NACA 10-(3)(12)-03 propeller (ref. 8)									
2	1140	---	25	30	35	40	45	50	55
3	1350	20	25	30	35	40	45	50	
4	1500	---	---	---	---	---	45		
5	1600	20	25	30	35	40	45		
6	2000	20	25	30	35				
7	2160	20	25	30					
8	Varied						45		

* Figure numbers refer to figures in the references except for the NACA 10-(3)(062)-045A and NACA 10-(3)(05)-045 propellers.

REPORT 1127

ONE-DIMENSIONAL ANALYSIS OF CHOKED-FLOW TURBINES¹

By ROBERT E. ENGLISH and RICHARD H. CAVICCHI

SUMMARY

Turbines for most applications requiring high work output per stage have one or more blade rows which are choked. This analysis indicated that the area ratios and equivalent blade speed are the controlling factors in the design and operation of such turbines. For the usual class of turbine, increasing the equivalent blade speed of a given stage makes the internal flow conditions less critical. Flow conditions within choked-flow turbines are not unusually sensitive to manufacturing errors in area ratio even near the stator choking limit. Six criteria are stated that will aid in establishing from test data of multistage turbines which blade rows are choked and which are not. The variation in internal flow conditions with operating conditions for a turbine equipped with an adjustable stator is determined for one application of stator adjustment.

INTRODUCTION

Turbines for most applications requiring high work output per stage have one or more blade rows which are choked for at least a part of the operating range. Choking is a condition which exists at a high pressure ratio and during which an increase in turbine pressure ratio with constant equivalent blade speed does not affect the equivalent weight flow of the turbine. As the pressure ratio is increased above this choking value, the flow conditions upstream of the choked throat are independent of the pressure ratio and dependent upon only the turbine geometry and the equivalent blade speed.

An analysis of the flow conditions upstream of such a choked throat is important for two reasons:

(1) The basic principles controlling the operation of choked-flow turbines should be understood in order that such principles may be effectively used in design and in order that the data from tests of such turbines may be analyzed.

(2) The comparative sensitivity of factors affecting turbine efficiency to operating conditions should be known if choked-flow turbines are to be operated effectively.

A method is described for extending the analysis to include flow having variations in entropy.

This report therefore presents a one-dimensional analysis made at the NACA Lewis laboratory during 1952 of flow

within turbines having choking in one or more blade rows in order to

- (1) Indicate those factors which control turbine design and operation
- (2) Determine the effect of variation in operating conditions on internal flow conditions
- (3) Determine the effect of manufacturing errors on internal flow conditions

- (4) Provide aids for analysis of turbine test data

The following cases are analyzed:

- (1) A choked stator followed by a choked rotor
- (2) A choked rotor followed by a choked stator
- (3) A stage having a choked stator and followed by a choked stator
- (4) A turbine having a choked first stator and a choked last rotor, and followed by a choked exhaust nozzle
- (5) A nonchoked stator followed by a choked rotor
- (6) Two-stage turbine having one or more choked blade rows
- (7) Turbine equipped with adjustable inlet stator

The one-dimensional analysis is based on isentropic flow through successive throats. The results are thus qualitative, showing trends and orders of magnitude.

SYMBOLS

The following symbols are used in this report:

A	flow area, sq ft
a	sonic velocity, $\sqrt{\gamma gRT}$, ft/sec
a'_{cr}	critical velocity relative to stator, $\sqrt{\frac{2\gamma}{\gamma+1} gRT'}$, ft/sec
a''_{cr}	critical velocity relative to rotor, $\sqrt{\frac{2\gamma}{\gamma+1} gRT''}$, ft/sec
c_p	specific heat at constant pressure, Btu/(lb)(°R)
g	mass ratio, lb/slug
J	mechanical equivalent of heat, ft-lb/Btu
k, l, m, n	constants
M_r	Mach number relative to rotor, W/a
M_s	Mach number relative to stator, V/a
p	pressure, lb/sq ft absolute
R	gas constant, ft-lb/(lb)(°R)
s	entropy, Btu/(lb)(°R)
T	temperature, °R

¹Supersedes NACA TN 2810, "One-Dimensional Analysis of Choked-Flow Turbines" by Robert E. English and Richard H. Cavicchi, 1952.

U	blade velocity, ft/sec
V	absolute velocity of gas, ft/sec
W	relative velocity of gas, ft/sec
w	rate of weight flow of gas, lb/sec
$\bar{w}$	specific-mass-flow parameter, $\frac{\rho_2 V_2}{\rho_1 a_{cr,1}}$
α	flow angle of absolute velocity of gas measured from tangential direction (fig. 1), deg
β	flow angle of relative velocity of gas measured from tangential direction (fig. 1), deg
γ	ratio of specific heats
δ	ratio of pressure to NACA standard sea-level pressure, $p/2116$
θ	ratio of temperature to NACA standard sea-level temperature, $T/518.4$
ρ	density, lb/cu ft
ψ	reaction factor

Subscripts:

1, 3, 5, 7, 9	axial stations ahead of or behind blade rows
2, 4, 6, 8, 10	stations at throats
c	compressor inlet conditions
cr	critical, state at speed of sound
u	tangential component (positive in direction of blade speed U)
x	axial component

Superscripts:

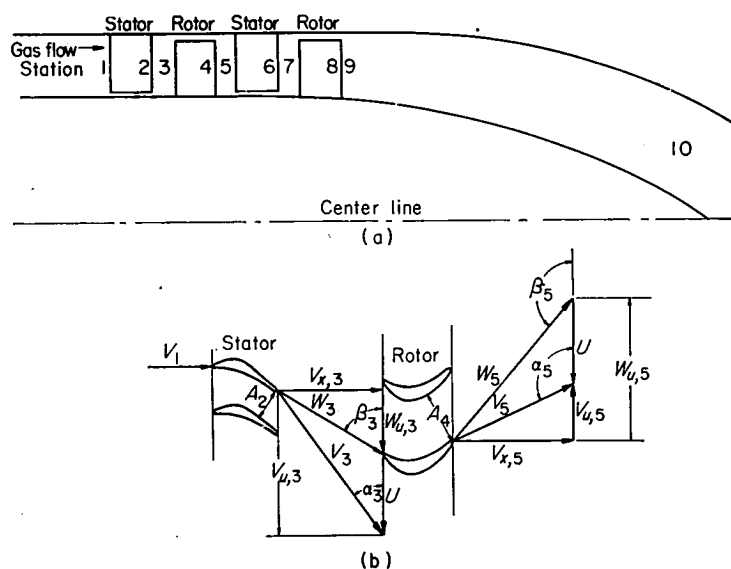
'	stagnation state relative to stator
"	stagnation state relative to rotor

GENERAL ANALYSIS

The analysis presented herein is divided into two sections, the **GENERAL ANALYSIS**, in which certain basic equations are presented, and the **ANALYSIS AND DISCUSSION**, in which various modifications of these basic equations are developed and discussed. The greater part of this investigation considers two or more choking throats; the remainder treats a turbine stage having a choked rotor but a nonchoked stator.

The basis of the analysis is an assumption of one-dimensional, isentropic flow at constant mean radius. Relative motion of the blade rows results, in general, in changes in stagnation enthalpy from one blade row to another. For such an energy extraction from a moving stream, the areas of successive choking throats differ from one another. This is in contrast with the common one-dimensional analysis of choking in successive throats (ref. 1, pp. 70-72) for which no energy is added to or extracted from the moving stream; in such a case, an assumption of constant entropy results in equal areas for the successive choking throats. Because of the assumption of one-dimensional isentropic flow at constant mean radius, the results of the investigation are qualitative.

The numerical subscripts and the axial stations within the turbine are related in figure 1. The even numbers 2, 4, 6, 8, and 10 represent stations at throats, and the odd num-



(a) Turbine cross section showing numbered stations.
(b) Typical velocity diagram for first stage.

FIGURE 1.—Schematic illustrations showing notation used.

bers 1, 3, 5, 7, and 9 represent stations ahead of or behind blade rows.

The stagnation state relative to any particular blade row is assumed to be constant from the entrance to the exit of each blade row; that is, for example,

$$\left. \begin{aligned} p'_1 &= p'_2 = p'_3 \\ T'_1 &= T'_2 = T'_3 \\ p''_3 &= p''_4 = p''_5 \\ T''_3 &= T''_4 = T''_5 \end{aligned} \right\} \quad (1)$$

Although this assumption is not exactly satisfied in the general case, typical deviations from this assumption will not change the qualitative results of the analysis. For the assumed condition of constant radius, each of the following cases illustrates the flow conditions which may exist without a change in relative stagnation temperature across the radial element of a blade row:

- (1) The flow does not shift radially across the blade row.
- (2) For a stator, the flow at the entrance is isoenergetic.
- (3) For a rotor, an isoenergetic flow at the entrance has a free-vortex distribution of tangential velocity.

If the stagnation values of both temperature and entropy are constant across a blade row, the stagnation pressure is also constant.

The condition of continuity of mass flow between two throats, for example, 2 and 4, can be stated.

$$\rho_2 A_2 V_2 = \rho_4 A_4 W_4 \quad (2)$$

Equation (2) can be modified to read

$$\frac{\rho_2 V_2}{\rho_2' a_{cr,2}} A_2 = \frac{\rho_4 W_4}{\rho_4' a_{cr,4}} A_4 \frac{\rho_4'' a_{cr,4}}{\rho_2' a_{cr,2}} \quad (3)$$

For choking at the two throats,

$$\left. \begin{aligned} \frac{V_2}{a'_{cr,2}} = \frac{W_4}{a''_{cr,4}} = 1 \\ \text{and} \\ \frac{\rho_2}{\rho_4} = \frac{\rho_4}{\rho_2} = \left(\frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \end{aligned} \right\} \quad (4)$$

For a perfect gas at constant entropy,

$$\frac{\rho_4'' a'_{cr,4}}{\rho_2' a'_{cr,2}} = \left(\frac{T_4''}{T_2'} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad (5)$$

Combining equations (1), (3), (4), and (5) yields

$$\frac{A_2}{A_4} = \left(\frac{T_3''}{T_1'} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad (6)$$

If station 4 is choked and station 2 is nonchoked, that is,

$$\frac{W_4}{a'_{cr,4}} = 1 \text{ and } \frac{V_2}{a'_{cr,2}} < 1$$

then

$$\frac{\rho_2 V_2}{\rho_2' a'_{cr,2}} < \frac{\rho_4 W_4}{\rho_4' a'_{cr,4}} = \left(\frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \quad (7)$$

For this case, equation (3) reduces to

$$\frac{\rho_2 V_2}{\rho_2' a'_{cr,2}} = \frac{A_4}{A_2} \left(\frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \left(\frac{T_3''}{T_1'} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad (8)$$

The stagnation temperature relative to a stator and relative to a rotor may be stated in terms of the velocity components in the following way:

$$c_p(T' - T) = \frac{V_x^2 + V_u^2}{2gJ} \quad (9)$$

$$c_p(T'' - T) = \frac{V_x^2 + (V_u - U)^2}{2gJ} \quad (10)$$

With the use of

$$c_p = \frac{\gamma R}{(\gamma-1)J} \quad (11)$$

and the definition of critical velocity, equations (9) and (10) can be combined to yield

$$\frac{T''}{T'} = 1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr}} \left(2 \frac{V_u}{a'_{cr}} - \frac{U}{a'_{cr}} \right) \quad (12)$$

By employing

$$V_u = W_u + U$$

equation (12) can be changed to

$$\frac{T''}{T'} = 1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr}} \left(2 \frac{W_u}{a'_{cr}} + \frac{U}{a'_{cr}} \right) \quad (13)$$

and

$$\frac{T'}{T''} = 1 + \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr}} \left(2 \frac{W_u}{a'_{cr}} + \frac{U}{a'_{cr}} \right) \quad (14)$$

The equations in this section are general in the sense that they are applicable to any or all the axial stations in the turbine. Equation (6), although derived by considering conditions at stations 2 and 4, may be employed to relate temperatures and areas at any two choking stations; for example, equation (6) may be modified to read

$$\frac{A_2}{A_{10}} = \frac{T_9'}{T_1'}^{\frac{\gamma+1}{2(\gamma-1)}} \quad (15)$$

Equations (12) to (14) are applicable to any particular axial station; for example,

$$\frac{T_3''}{T_1'} = 1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr,1}} \left(2 \frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right) \quad (16)$$

In order to show the relation among the relative-absolute stagnation-temperature ratio T''/T' , the blade speed parameter U/a'_{cr} , and the tangential velocity parameter V_u/a'_{cr} , equation (12) was used to plot figure 2. A value of 4/3 for the ratio of specific heats was employed in all the calculations for the figures presented herein. The lines of constant tangential velocity parameter V_u/a'_{cr} are parabolas passing through the point $(T''/T' = 1, U/a'_{cr} = 0)$. Along a line having a stagnation-temperature ratio T''/T' of 1,

$$\frac{U}{a'_{cr}} = 2 \frac{V_u}{a'_{cr}}$$

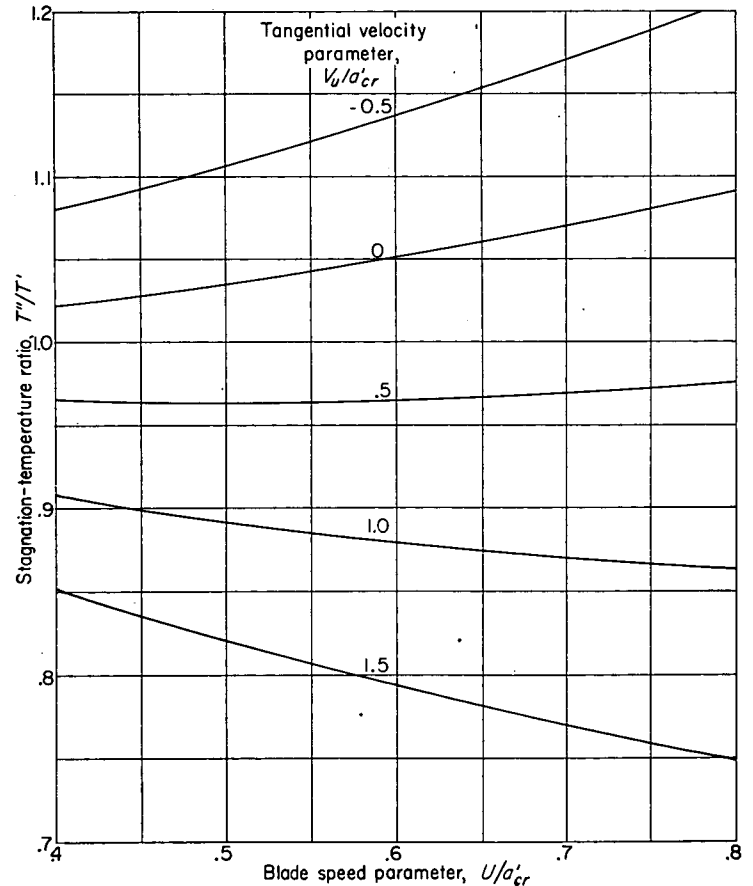


FIGURE 2.—Variation of stagnation-temperature ratio with blade speed parameter for various tangential velocity parameters (eq. (12)).

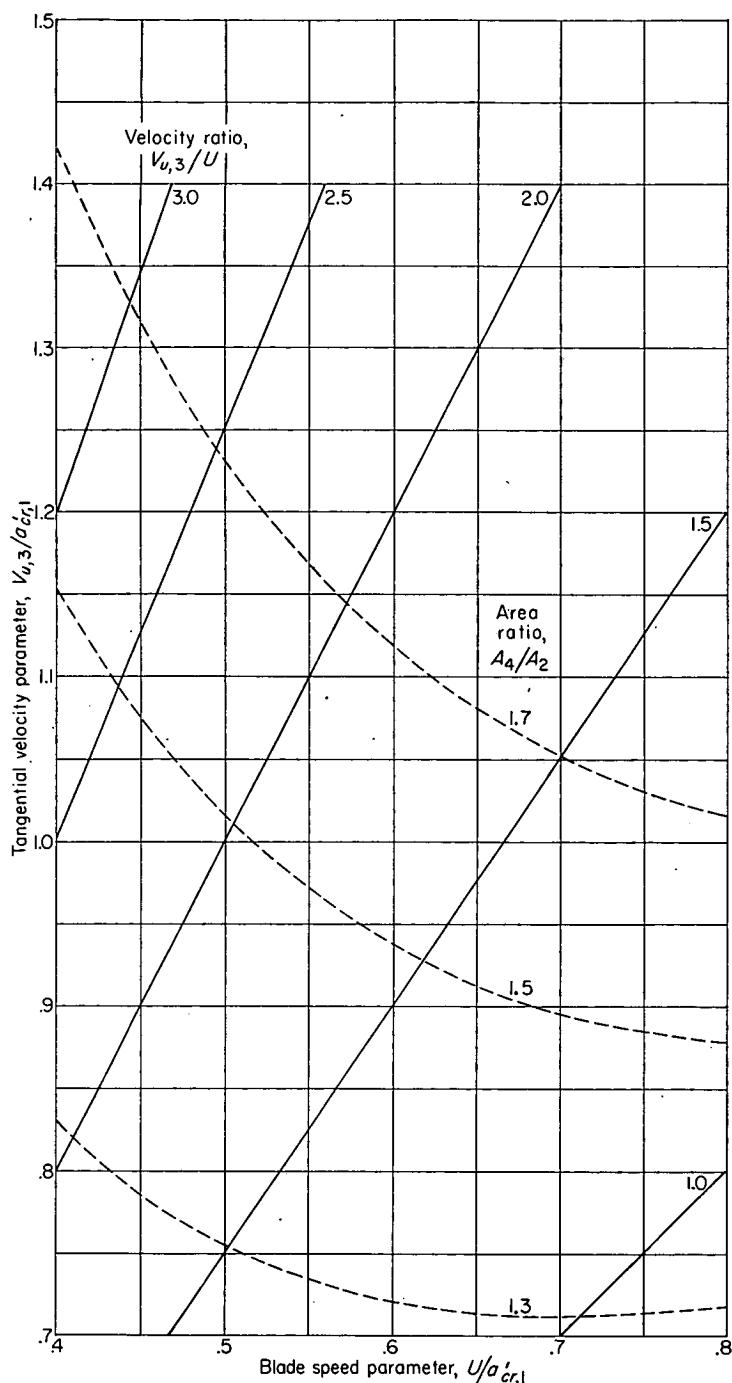


FIGURE 3.—Effect of area ratio on velocity parameters at exit of choked stator followed by choked rotor (eq. (17)).

The minimums of these parabolas occur if

$$\frac{U}{a'_{cr}} = \frac{V_u}{a'_{cr}}$$

The physical reason that the stagnation-temperature ratio T''/T' is a minimum under this condition becomes apparent upon examination of equation (10); for given values of axial velocity V_x and static temperature T , the stagnation temperature T'' is a minimum if $V_u = U$. A minimum value for the stagnation-temperature ratio is thus collateral with minimum kinetic energy relative to the rotor.

ANALYSIS AND DISCUSSION

The modifications of these basic equations, which are subsequently presented, show the effect of turbine manufacture and operation on factors influencing the turbine performance; namely, tangential velocity, entrance Mach number, turning angle, reaction, and weight flow. The following variables are postulated to be independent: area ratios, flow angles at blade-row exits, and blade speed. Blade speed is controllable during turbine operation. Area ratios and blade exit angles are established during turbine manufacture unless the exhaust nozzle or turbine stator is adjustable. In any event, the turbine geometry establishes the exit flow angle.

CHOKED STATOR FOLLOWED BY CHOKED ROTOR

Stator exit tangential velocity parameter.—Equations (6) and (12) were combined in order to determine the tangential velocity parameter $V_{u,3}/a'_{cr,1}$, a factor which considerably influences the turbine work.

$$\frac{V_{u,3}}{a'_{cr,1}} = \frac{1}{2} \left\{ \frac{U}{a'_{cr,1}} + \frac{\gamma+1}{\gamma-1} \frac{a'_{cr,1}}{U} \left[1 - \left(\frac{A_2}{A_4} \right)^{\frac{2(\gamma-1)}{\gamma+1}} \right] \right\} \quad (17)$$

Equation (17) shows that for a given value of γ the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ may be expressed in terms of the independent variables blade speed parameter $U/a'_{cr,1}$ and area ratio A_4/A_2 , as was postulated. With the turbine rotor choking, the tangential velocity parameter at the rotor entrance $V_{u,3}/a'_{cr,1}$ is independent of the over-all pressure ratio.

The lines of constant area ratio A_4/A_2 in figure 3 are a graphical representation of equation (17). The coordinates are tangential velocity parameter $V_{u,3}/a'_{cr,1}$ and blade speed parameter $U/a'_{cr,1}$; these same coordinates are used for most of the figures concerning a choked stator followed by a choked rotor. The lines of constant velocity ratio $V_{u,3}/U$ are straight lines having slopes equal to the velocity ratio $V_{u,3}/U$ and passing through the origin. The ranges of variables presented are representative of most choked-flow aircraft gas turbines.

For most of the range shown in figure 3, increasing the blade speed parameter $U/a'_{cr,1}$ along a line of constant area ratio A_4/A_2 causes a reduction in the tangential velocity parameter $V_{u,3}/a'_{cr,1}$. For a turbine being operated with a given set of inlet conditions, this means that increasing the blade speed usually reduces the tangential velocity at the stator exit; in turn, the resultant velocity at the stator exit must decrease. The range over which this trend of changing tangential velocity parameter $V_{u,3}/a'_{cr,1}$ with blade speed parameter $U/a'_{cr,1}$ is exhibited is more precisely delineated by closer examination of equation (17).

With the blade speed parameter $U/a'_{cr,1}$ and the area ratio A_4/A_2 considered to be independent, equation (17) was differentiated to yield

$$\frac{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)}{\partial \left(\frac{U}{a'_{cr,1}} \right)} = 1 - \frac{V_{u,3}}{U} \quad (18)$$

Equation (18) shows that

if $\frac{V_{u,3}}{U}$	=1	>1	<1
then $\frac{\partial(\frac{V_{u,3}}{a_{cr,1}})}{\partial(\frac{U}{a_{cr,1}})}$	=0	<0	>0

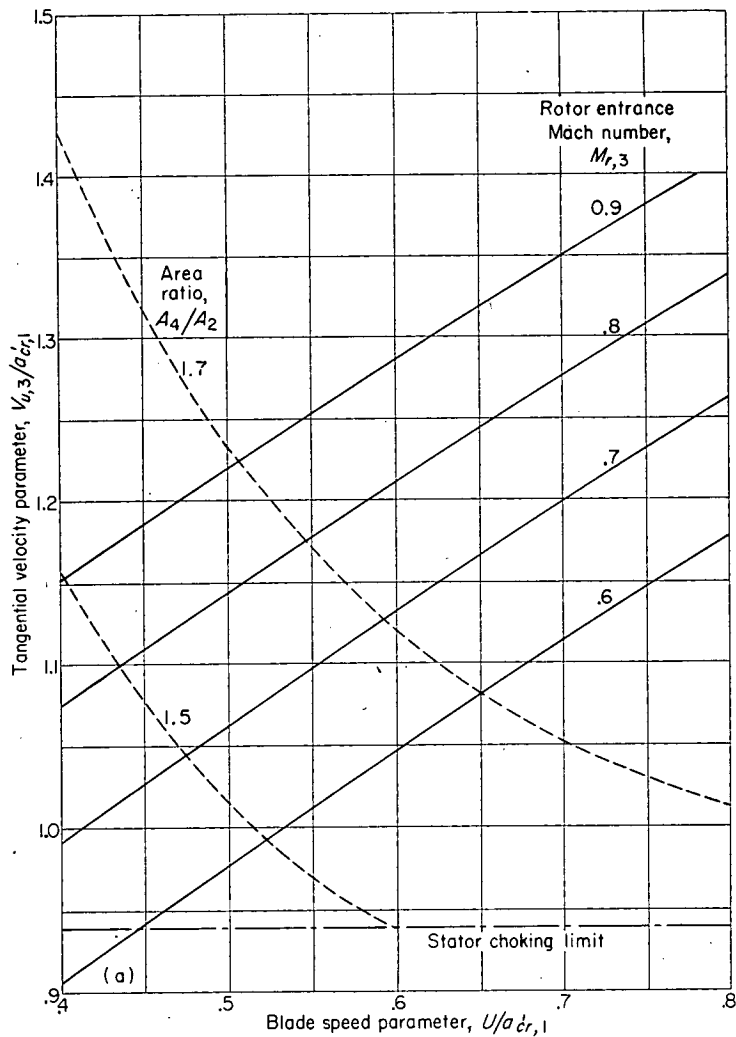
(19)

Thus, each of the lines of constant area ratio A_4/A_2 reaches a minimum where the velocity ratio $V_{u,3}/U$ is unity; this condition corresponds to axial relative flow at the rotor entrance and is typical of a turbine stage having 50 percent reaction and zero exit tangential velocity. A turbine having a velocity ratio $V_{u,3}/U$ in the neighborhood of unity therefore has essentially no change in tangential velocity parameter with blade speed parameter; whereas a turbine having a velocity ratio $V_{u,3}/U$ greater than unity experiences a de-

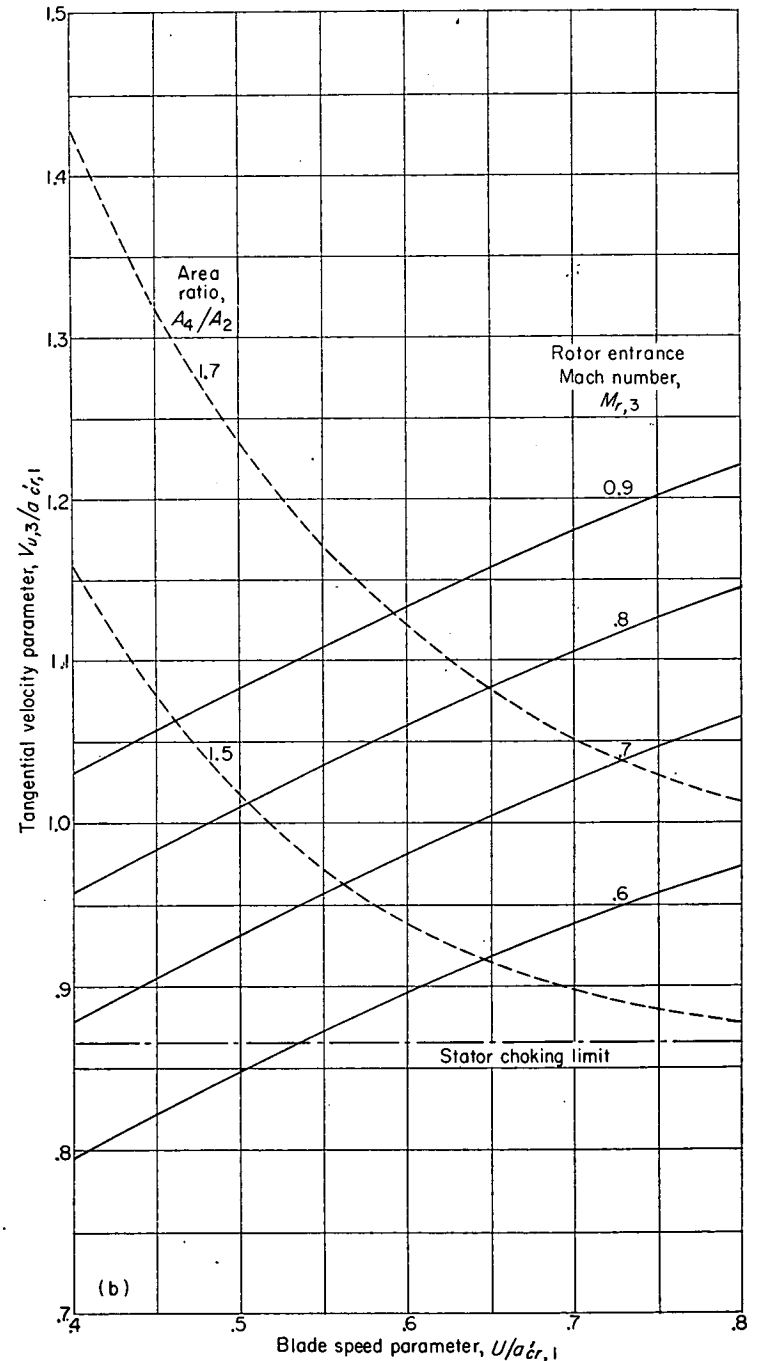
crease in tangential velocity parameter with increasing blade speed parameter.

Rotor entrance Mach number.—For either a choked or a nonchoked stator, the relative Mach number at the rotor entrance $M_{r,3}$ may be expressed

$$M_{r,3} = \sqrt{\frac{2}{\gamma+1} \frac{\left(\frac{V_{u,3}}{a_{cr,1}}\right)^2 \tan^2 \alpha_3 + \left(\frac{V_{u,3}}{a_{cr,1}} - \frac{U}{a_{cr,1}}\right)^2}{1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V_{u,3}}{a_{cr,1}}\right)^2 \sec^2 \alpha_3}} \quad (20)$$



(a) Stator exit flow angle, 20°.



(b) Stator exit flow angle, 30°.

FIGURE 4.—Relation among area ratio, rotor entrance Mach number, and velocity parameters at exit of choked stator followed by choked rotor (eqs. (17) and (20)).

For a choked stator followed by a choked rotor, equation (17) may be used to replace the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ in equation (20) by a function of area ratio A_4/A_2 and blade speed parameter $U/a'_{cr,1}$. For any given value of specific heat ratio γ , the rotor entrance Mach number $M_{r,3}$ in equation (20) can therefore be reduced to a function of three variables—area ratio A_4/A_2 , blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3 . The relation among

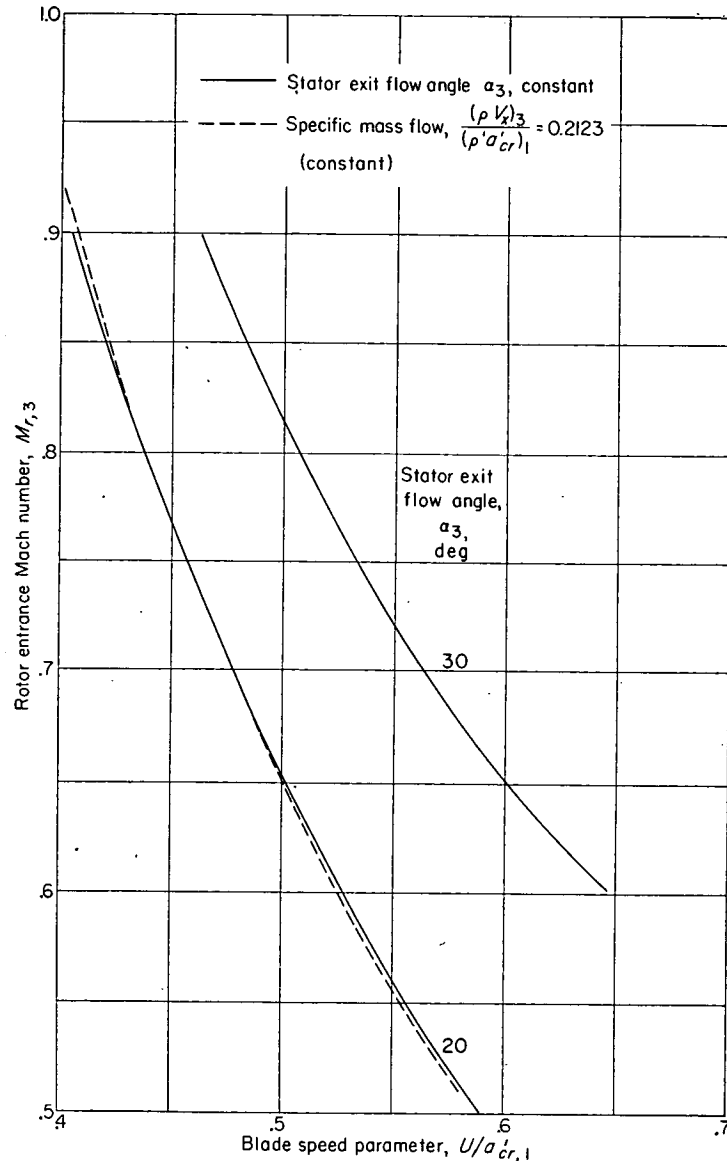


FIGURE 5.—Comparative effect of assumption of both constant stator exit flow angle and constant specific mass flow on rotor entrance Mach number at area ratio of 1.5.

$$\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)} = \frac{V_{u,3}}{U} \left[1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr,1}} \left(2 \frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right) \right] \left[\sec^2 \alpha_3 - \frac{U}{V_{u,3}} \frac{\gamma-1}{\gamma+1} \sec^2 \alpha_3 \frac{U}{a'_{cr,1}} \left(\frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right) \right] \left[\frac{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)}{\partial \left(\frac{A_4}{A_2} \right)} \right] \left(\frac{A_2}{A_4} \right) \quad (21)$$

Evaluation of the four factors on the right side of equation (21) by means of equations (17) and (20) yields the following expression for the manufacturing-error parameter:

$$\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)} = \frac{V_{u,3}}{U} \left[1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr,1}} \left(2 \frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right) \right] \left[\sec^2 \alpha_3 - \frac{U}{V_{u,3}} \frac{\gamma-1}{\gamma+1} \sec^2 \alpha_3 \frac{U}{a'_{cr,1}} \left(\frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right) \right] \left[\frac{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)}{\partial \left(\frac{A_4}{A_2} \right)} \right] \left(\frac{A_2}{A_4} \right) \quad (22)$$

the change in rotor entrance Mach number $M_{r,3}$. Equation (22) has been plotted in figure 6 in order to show the variation in manufacturing-error parameter $\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)}$ with tangential velocity parameter $V_{u,3}/a'_{cr,1}$, blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3 .

these parameters is shown in figure 4. For stator choking, the velocity at the throat (station 2) is equal to the entrance value of critical velocity $a'_{cr,1}$. At the stator choking limit, the stator exit velocity V_3 is presumed to be equal to the throat velocity V_2 with the result that at the stator choking limit

$$\frac{V_{u,3}}{V_3} = \frac{V_{u,3}}{a'_{cr,1}} = \cos \alpha_3$$

For values of tangential velocity parameter $V_{u,3}/a'_{cr,1}$ above the stator choking limit, lines of constant area ratio A_4/A_2 have been reproduced from figure 3 (eq. (17)).

When figure 4 is employed to estimate the variation in rotor entrance Mach number $M_{r,3}$ with blade speed parameter $U/a'_{cr,1}$, an error is introduced by considering the stator exit flow angle α_3 to be constant. This error arises because, as the blade speed parameter is changed, the mass flow through the stator will vary for constant stator exit flow angle α_3 , whereas for actual operation with a given stator the mass flow will remain constant and the stator exit flow angle α_3 will vary. The effect of this discrepancy on rotor entrance Mach number $M_{r,3}$ is shown in figure 5; the error is comparatively small. The presence of two lines of constant stator exit flow angle α_3 in figure 5 permits the deviation of the stator exit flow angle to be estimated from comparative distances separating the lines. Because this error introduces only a small discrepancy into the results, the stator exit flow angle α_3 was, for simplicity, assumed to be independent of the tangential velocity parameter $V_{u,3}/a'_{cr,1}$.

Effect of manufacturing errors.—Figure 4 shows that increasing the area ratio A_4/A_2 increases the rotor entrance Mach number $M_{r,3}$ for any given value of blade speed parameter $U/a'_{cr,1}$. In order to examine in greater detail the variation in rotor entrance Mach number $M_{r,3}$ with manufacturing errors in area ratio A_4/A_2 , equation (20) was differentiated with the qualification that area ratio A_4/A_2 , blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3 are independent variables. In formulating the desired relation, the following identity was employed:

$$\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)} = - \left(\frac{1}{2 M_{r,3}} \right) \left[\frac{\partial (M_{r,3}^2)}{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)} \right] \left[\frac{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)}{\partial \left(\frac{A_4}{A_2} \right)} \right] \left(\frac{A_2}{A_4} \right) \quad (21)$$

Evaluation of the four factors on the right side of equation (21) by means of equations (17) and (20) yields the following expression for the manufacturing-error parameter:

$$\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)} = \frac{V_{u,3}}{U} \left[1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr,1}} \left(2 \frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right) \right] \left[\sec^2 \alpha_3 - \frac{U}{V_{u,3}} \frac{\gamma-1}{\gamma+1} \sec^2 \alpha_3 \frac{U}{a'_{cr,1}} \left(\frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right) \right] \left[\frac{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)}{\partial \left(\frac{A_4}{A_2} \right)} \right] \left(\frac{A_2}{A_4} \right) \quad (22)$$

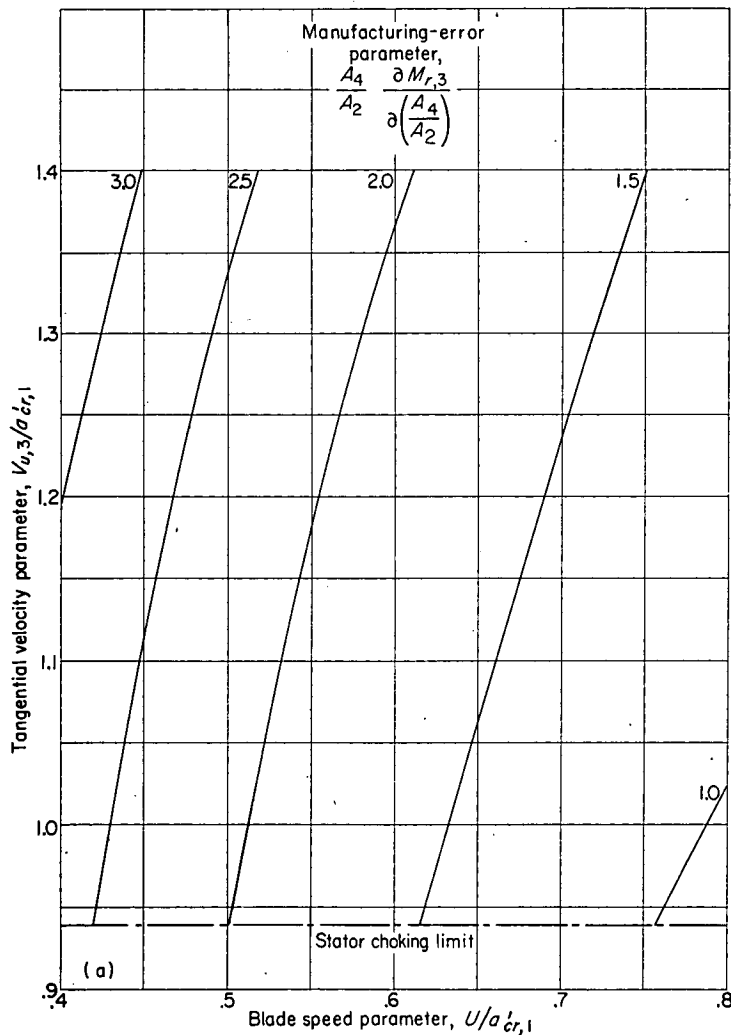
the change in rotor entrance Mach number $M_{r,3}$. Equation (22) has been plotted in figure 6 in order to show the variation in manufacturing-error parameter $\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)}$ with tangential velocity parameter $V_{u,3}/a'_{cr,1}$, blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3 .

This parameter $\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)}$ is herein termed "manufacturing-error parameter." For small changes in area ratio A_4/A_2 such as might occur during turbine manufacture, the manufacturing-error parameter $\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)}$ can be used to estimate

The following example illustrates the use of the manufacturing-error parameter in estimating the effect of errors in area ratio on rotor entrance Mach number for a stator exit flow angle α_3 of 20° , a tangential velocity parameter $V_{u,3}/a'_{cr,1}$ of 1.2, and a blade speed parameter $U/a'_{cr,1}$ of 0.69. Figure 6 shows that the manufacturing-error parameter

$\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)}$ is 1.5, and figure 4 indicates that the rotor entrance Mach number $M_{r,3}$ is 0.71. If, in the course of turbine manufacture, the rotor throat area A_4 is made 1 percent greater than the design value and the stator throat area A_2 is made 1 percent smaller than the design value, the area ratio A_4/A_2 is increased by approximately 2 percent. The increment in rotor entrance Mach number $M_{r,3}$ associated with this change in area ratio A_4/A_2 can be estimated from

$$\Delta M_{r,3} \approx \left[\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)} \right] \left[\frac{\Delta \left(\frac{A_4}{A_2}\right)}{\frac{A_4}{A_2}} \right]$$

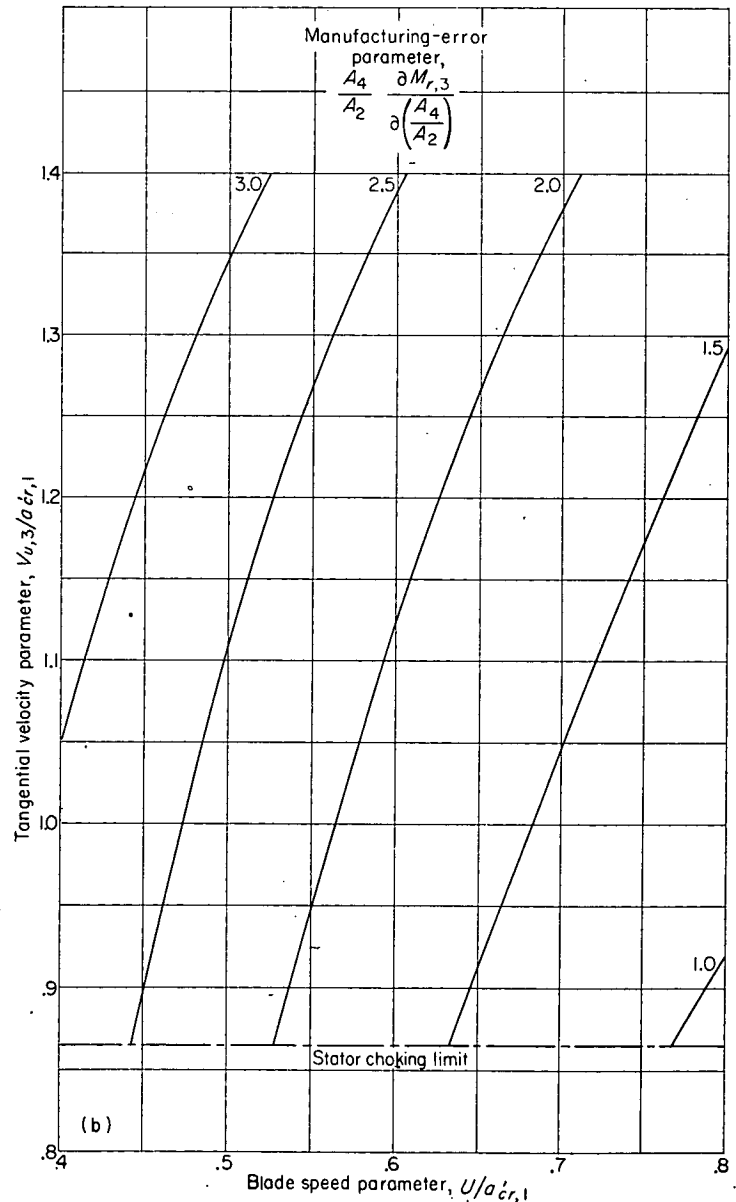


(a) Stator exit flow angle, 20° .

which, for this case, is 0.03. Thus, the rotor entrance Mach number $M_{r,3}$ will be increased to approximately 0.74 for operation of the turbine at the design value of blade speed parameter.

Effect of blade speed on rotor entrance conditions.—For most of the range presented in figure 4, increasing the blade speed parameter $U/a'_{cr,1}$ at constant area ratio A_4/A_2 corresponds to decreasing the rotor entrance Mach number $M_{r,3}$. In order to examine in greater detail the range of conditions for which this relation is true, equation (20) was differentiated with the qualification that area ratio A_4/A_2 , blade speed parameter $U/a'_{cr,1}$, and stator exit flow angle α_3 are independent variables. The result of this differentiation is

$$\frac{\partial M_{r,3}}{\partial \left(\frac{U}{a'_{cr,1}}\right)} = \frac{2}{M_{r,3}(\gamma+1)} \frac{V_{u,3}}{a'_{cr,1}} \left(1 - \frac{V_{u,3}}{U}\right) \sec^2 \alpha_3 \frac{T_3''}{\left(\frac{T_3'}{T_1}\right)^2} \quad (23)$$



(b) Stator exit flow angle, 30° .

FIGURE 6.—Effect of manufacturing errors in throat areas on rotor entrance Mach number for choked stator followed by choked rotor (eq. (22)).

where the relative-absolute temperature ratio T_3''/T_1' is stated in equation (16), and, by combination of equation (9) and the definition of critical velocity $a_{cr,1}'$,

$$\frac{T_3}{T_1'} = 1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V_{u,3}}{a_{cr,1}'} \right)^2 \sec^2 \alpha_3 \quad (24)$$

With the single exception of $\left(1 - \frac{V_{u,3}}{U}\right)$, every factor on the right side of equation (23) is positive at the entrance to any turbine rotor. Therefore,

if $\frac{V_{u,3}}{U}$	= 1	> 1	< 1
then $\frac{\partial M_{r,3}}{\partial \left(\frac{U}{a_{cr,1}'}\right)}$	= 0	< 0	> 0

(25)

A reaction factor ${}_3\psi_4$ is defined

$${}_3\psi_4 \equiv \frac{W_4^2 - W_3^2}{W_3^2} \quad (26)$$

For choking in the rotor,

$$W_4 = a_{cr,3}''$$

Equation (26) therefore reduces to

$${}_3\psi_4 = \left(\frac{a_{cr,3}''}{W_3} \right)^2 - 1 \quad (27)$$

Because

$$\frac{dM_{r,3}}{d \left(\frac{W_3}{a_{cr,3}''} \right)} > 0$$

differentiation of equation (27) and combination of this result with equation (25) yield the following:

if $\frac{V_{u,3}}{U}$	= 1	> 1	< 1
then $\frac{\partial {}_3\psi_4}{\partial \left(\frac{U}{a_{cr,1}'}\right)}$	= 0	> 0	< 0

(28)

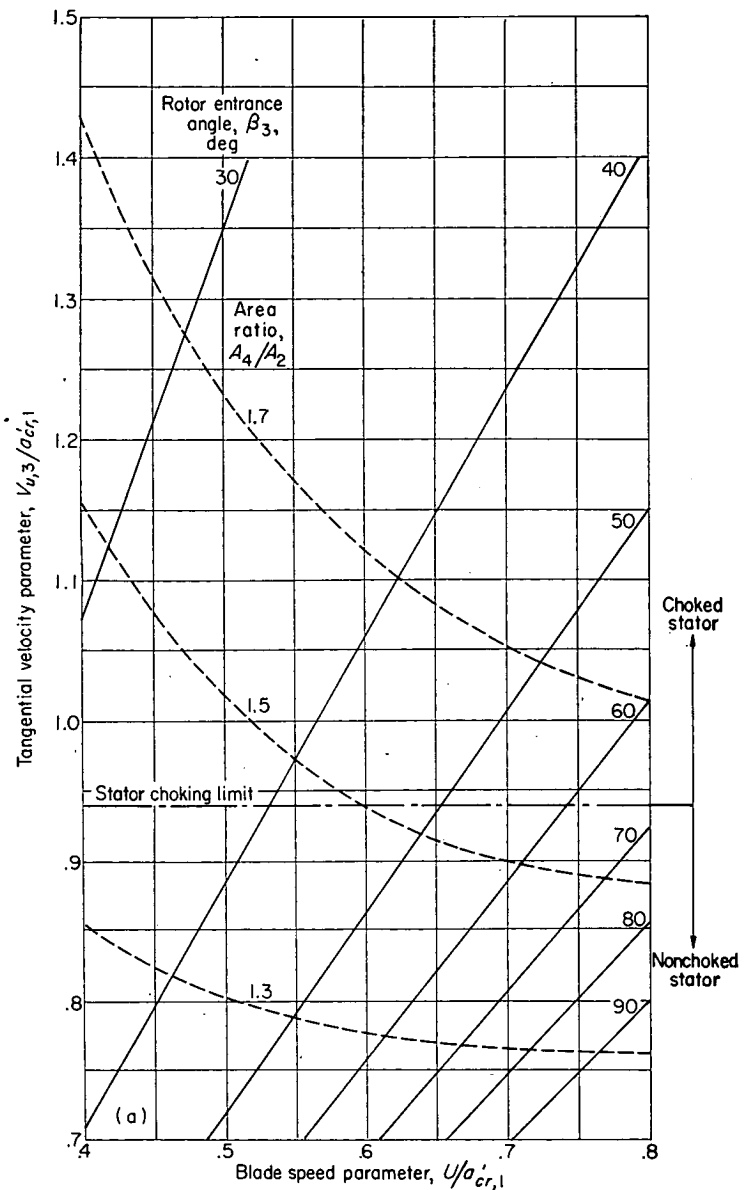
From the geometry of the vector diagram in figure 1,

$$\frac{V_{u,3}}{a_{cr,1}'} \tan \alpha_3 = \left(\frac{V_{u,3}}{a_{cr,1}'} - \frac{U}{a_{cr,1}'} \right) \tan \beta_3$$

or

$$\tan \beta_3 = \frac{\tan \alpha_3}{1 - \frac{U}{V_{u,3}}} \quad (29)$$

Equation (29) is plotted in figure 7 for two values of stator exit flow angle α_3 . The lines of constant area ratio A_4/A_2 shown in the region above the stator choking limit are



(a) Stator exit flow angle, 20° .

FIGURE 7.—Relation among area ratio, rotor entrance angle, and velocity parameters at exit of choked stator followed by choked rotor (eq. (29)). For lines of constant area ratio, equation (17) applies above stator choking limit and equation (50) applies below stator choking limit.

reproduced from figure 3, whereas the extension of these lines into the region below the stator choking limit has been accomplished by means of a relation yet to be developed (eq. (50)). Along a line of constant area ratio A_4/A_2 for a given stator exit flow angle α_3 , increasing the blade speed parameter $U/a_{cr,1}'$ increases the rotor entrance flow angle β_3 . Increased values of rotor entrance flow angle β_3 correspond to reduced angles of attack on the rotor blades and reduced amounts of turning by the rotor.

A consideration of equations (25) and (28) and figure 7 indicates the conditions for which a turbine should be designed if that turbine is to be operated over a range of blade speed parameter $U/a_{cr,1}'$. The values of rotor entrance

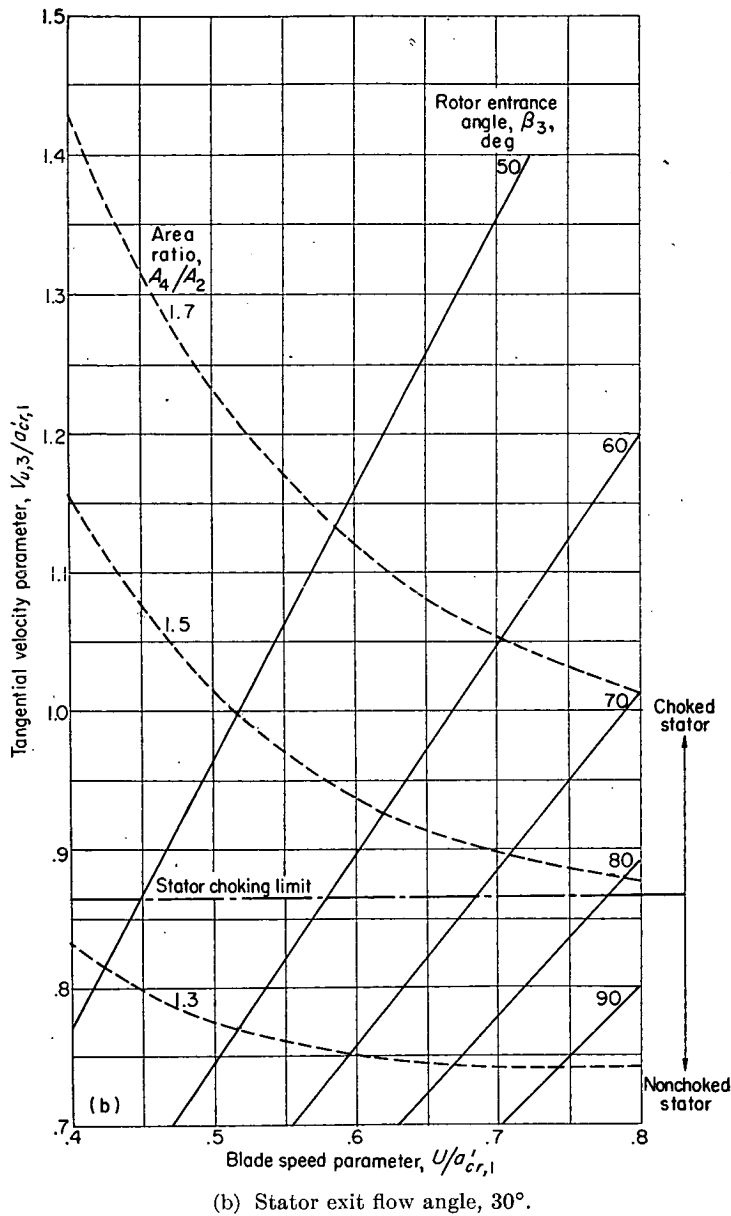


FIGURE 7.—Concluded. Relation among area ratio, rotor entrance angle, and velocity parameters at exit of choked stator followed by choked rotor (eq. (29)). For lines of constant area ratio, equation (17) applies above stator choking limit and equation (50) applies below stator choking limit.

Mach number $M_{r,3}$, reaction factor ψ_4 , and rotor entrance flow angle β_3 are factors which are commonly considered to have critical effects on turbine efficiency. For turbine stages having $\frac{V_{u,3}}{U} > 1$, small increases in blade speed parameter $U/a'_{cr,1}$ cause the magnitude of each of these factors to become more favorable. On the other hand, large increases in blade speed parameter $U/a'_{cr,1}$ will result in large negative angles of incidence and thus poor performance. Unless this is the case, the turbine should be designed for the lowest anticipated value of blade speed parameter $U/a'_{cr,1}$ in order that satisfactory rotor performance may be guaranteed for the required range of operation.

Recapitulation.—This analysis of conditions between a choked stator and a choked rotor indicates those factors which control the flow conditions between the stator and the rotor, namely, blade speed parameter $U/a'_{cr,1}$, area ratio A_4/A_2 , and stator exit flow angle α_3 . The importance of maintaining proper values of stator and rotor throat areas in turbine manufacture thus becomes apparent if the rotor entrance conditions are to be preserved. The sensitivity of rotor entrance Mach number $M_{r,3}$ to manufacturing errors in area ratio A_4/A_2 was determined; and rotor entrance Mach number $M_{r,3}$ was found to be only moderately sensitive to errors in area ratio A_4/A_2 , even in the neighborhood of the stator choking limit. The variations in rotor entrance Mach number $M_{r,3}$, rotor entrance flow angle β_3 , and reaction factor ψ_4 with blade speed parameter $U/a'_{cr,1}$ were determined. For the common range of design, for which $V_{u,3} > U$, each of these factors becomes more conservative as the blade speed parameter $U/a'_{cr,1}$ is increased.

CHOKED ROTOR FOLLOWED BY CHOKED STATOR

Stator entrance tangential velocity parameter.—The analysis of flow between a choked rotor and a choked stator is essentially the same as that for a choked stator followed by a choked rotor. The stator entrance tangential velocity parameter $V_{u,5}/a'_{cr,5}$ may be determined from a modification of equation (6) in combination with equation (12).

$$\frac{V_{u,5}}{a'_{cr,5}} = \frac{1}{2} \left\{ \frac{U}{a'_{cr,5}} - \frac{\gamma+1}{\gamma-1} \frac{a'_{cr,5}}{U} \left[\left(\frac{A_6}{A_4} \right)^{\frac{2(\gamma-1)}{\gamma+1}} - 1 \right] \right\} \quad (30)$$

With the blade speed parameter $U/a'_{cr,5}$ and the area ratio A_6/A_4 considered independent variables, equation (30) may be differentiated to yield

$$\frac{\partial \left(\frac{V_{u,5}}{a'_{cr,5}} \right)}{\partial \left(\frac{U}{a'_{cr,5}} \right)} = 1 - \frac{V_{u,5}}{U} \quad (31)$$

Because, almost without variation,

$$\frac{V_{u,5}}{U} < 1 \quad \text{then as a result} \quad \frac{\partial \left(\frac{V_{u,5}}{a'_{cr,5}} \right)}{\partial \left(\frac{U}{a'_{cr,5}} \right)} > 0 \quad (32)$$

Rotor exit tangential velocity parameter.—In the same way,

$$\frac{W_{u,5}}{a''_{cr,4}} = -\frac{1}{2} \left\{ \frac{U}{a''_{cr,4}} + \frac{\gamma+1}{\gamma-1} \frac{a''_{cr,4}}{U} \left[1 - \left(\frac{A_4}{A_6} \right)^{\frac{2(\gamma-1)}{\gamma+1}} \right] \right\} \quad (33)$$

and

$$\frac{\partial \left(\frac{W_{u,5}}{a''_{cr,4}} \right)}{\partial \left(\frac{U}{a''_{cr,4}} \right)} = -\frac{W_{u,5}}{U} - 1 \quad (34)$$

Equation (34) shows that

if $\frac{-W_{u,5}}{U}$	= 1	> 1	< 1
then $\frac{\partial \left(\frac{W_{u,5}}{a_{cr,4}''} \right)}{\partial \left(\frac{U}{a_{cr,4}''} \right)}$	= 0	> 0	< 0

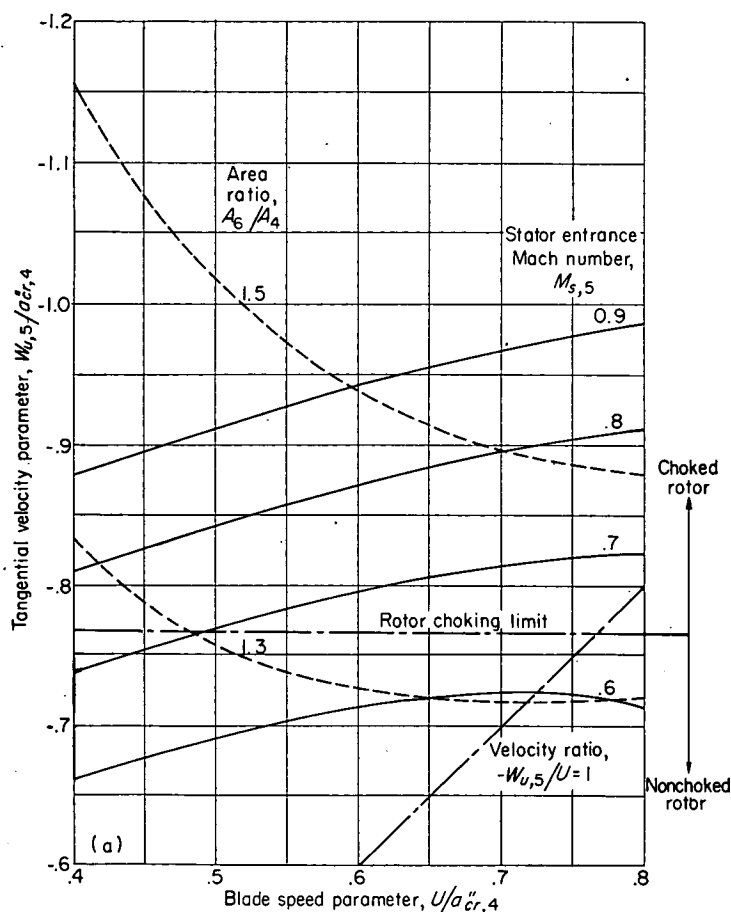
(35)

Figure 3 can be interpreted as a graphical representation of equation (33) if

$$\frac{V_{u,3}}{a_{cr,1}'} \text{ is replaced by } \frac{-W_{u,5}}{a_{cr,4}''}$$

$$\frac{U}{a_{cr,1}'} \text{ is replaced by } \frac{U}{a_{cr,4}''}$$

$$\frac{U}{a_{cr,4}''} = \pm \sqrt{\frac{\gamma+1}{2} M_{s,5}^2 \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{W_{u,5}}{a_{cr,4}''} \right)^2 \sec^2 \beta_5 \right] - \left(\frac{W_{u,5}}{a_{cr,4}''} \right)^2 \tan^2 \beta_5 - \frac{W_{u,5}}{a_{cr,4}''}} \quad (37)$$



(a) Rotor exit flow angle, 140° .

FIGURE 8.—Relation among area ratio, stator entrance Mach number, and velocity parameters at exit of choked rotor followed by choked stator (eqs. (36) and (37)). For lines of constant area ratio, equation (33) applies above rotor choking limit and equation (50) applies below rotor choking limit.

$$\frac{A_4}{A_2} \text{ is replaced by } \frac{A_6}{A_4}$$

$$\frac{V_{u,3}}{U} \text{ is replaced by } \frac{-W_{u,5}}{U}$$

Stator entrance Mach number.—The absolute Mach number at the stator entrance $M_{s,5}$ may be expressed in a form similar to equation (20)

$$M_{s,5} = \sqrt{\frac{2}{\gamma+1} \frac{\left(\frac{W_{u,5}}{a_{cr,4}''} \right)^2 \tan^2 \beta_5 + \left(\frac{W_{u,5}}{a_{cr,4}''} + \frac{U}{a_{cr,4}''} \right)^2}{1 - \frac{\gamma-1}{\gamma+1} \left(\frac{W_{u,5}}{a_{cr,4}''} \right)^2 \sec^2 \beta_5}} \quad (36)$$

Equation (36) may be revised to read

The radical in equation (37) represents the tangential velocity parameter $V_{u,5}/a_{cr,4}''$. For the conventional range of design, the negative sign should be used.

In figure 8, equation (37) is plotted for values of rotor exit flow angle β_5 of 140° and 150° . Lines of constant area ratio A_6/A_4 were plotted by using equation (33). For most of the range of these maps, the stator entrance Mach number $M_{s,5}$ decreases as the blade speed parameter $U/a_{cr,4}''$ is increased along a line of constant area ratio A_6/A_4 . For a velocity ratio $-W_{u,5}/U$ of unity, each line of constant stator entrance Mach number $M_{s,5}$ reaches a maximum and each line of constant area ratio A_6/A_4 reaches a minimum.

With the blade speed parameter $U/a_{cr,4}''$, the area ratio A_6/A_4 , and the rotor exit flow angle β_5 considered to be independent variables, equation (36) was differentiated to yield

$$\frac{\partial M_{s,5}}{\partial \left(\frac{U}{a_{cr,4}''} \right)} = \frac{2}{M_{s,5}(\gamma+1)} \frac{W_{u,5}}{a_{cr,4}''} \left(\frac{-W_{u,5}}{U} - 1 \right) \sec^2 \beta_5 \frac{T_5''}{T_5'} \quad (38)$$

The similarity between equations (23) and (38) is immediately apparent. Because for any practical turbine

$$\frac{W_{u,5}}{a_{cr,4}''} < 0$$

an equation similar to equation (25) may be written.

If $\frac{-W_{u,5}}{U}$	= 1	> 1	< 1
then $\frac{\partial M_{s,5}}{\partial \left(\frac{U}{a_{cr,4}''} \right)}$	= 0	< 0	> 0

(39)

In like manner, an equation similar to equation (28) may be written.

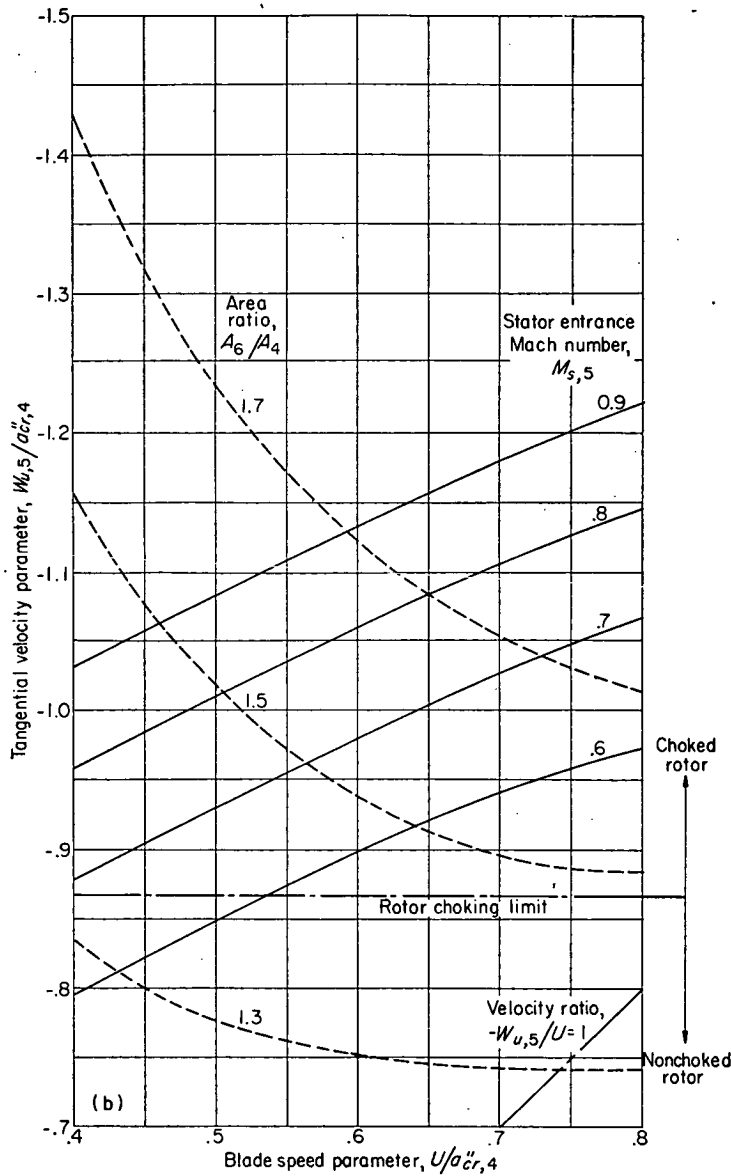
(b) Rotor exit flow angle, 150° .

FIGURE 8.—Concluded. Relation among area ratio, stator entrance Mach number, and velocity parameters at exit of choked rotor followed by choked stator (eqs. (36) and (37)). For lines of constant area ratio, equation (33) applies above rotor choking limit and equation (50) applies below rotor choking limit.

If $\frac{-W_{u,5}}{U}$	= 1	> 1	< 1
then $\frac{\partial \psi_5}{\partial \left(\frac{U}{a_{cr,4}}\right)}$	= 0	> 0	< 0

(40)

Recapitulation.—For a choked rotor followed by a choked stator the flow conditions between the rotor and the stator have essentially the same characteristics as the flow behind a choked stator which is followed by a choked rotor. The principal variables controlling the flow conditions are the blade speed parameter $U/a_{cr,4}$, the area ratio A_6/A_4 , and

the rotor exit flow angle β_5 . As for a choked stator followed by a choked rotor, the flow conditions at the entrance to a choked stator which follows a choked rotor are generally improved by increasing the blade speed parameter $U/a_{cr,4}$.

STAGE HAVING CHOKED STATOR AND FOLLOWED BY CHOKED STATOR

For choking at stations 2 and 6 (see fig. 1), equation (6) may be modified to read

$$\frac{T'_1}{T'_5} = \left(\frac{A_6}{A_2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \quad (41)$$

Equation (41) shows that for a stage having a choked stator and followed by a choked stator the stage temperature ratio T'_1/T'_5 depends upon only the area ratio A_6/A_2 and is thus independent of the blade speed parameter. If either of the stators is not choking, however, the stage temperature ratio T'_1/T'_5 is a function of blade speed parameter as well as area ratio. The significance of area ratio A_3/A_2 in determining the stage temperature ratio T'_1/T'_5 emphasizes the importance of maintaining the proper throat areas during turbine manufacture.

Differentiation of equation (41) gives

$$\frac{d\left(\frac{T'_1}{T'_5}\right)}{\frac{T'_1}{T'_5}} = \frac{2(\gamma-1)}{\gamma+1} \frac{d\left(\frac{A_6}{A_2}\right)}{\frac{A_6}{A_2}} \quad (42)$$

The fractional error in stage temperature ratio T'_1/T'_5 is thus about 0.3 the fractional error in area ratio A_6/A_2 . For a typical stage temperature ratio T'_1/T'_5 of 1.2, the fractional error in turbine equivalent work is then about 1.5 the fractional error in area ratio A_6/A_2 for a stage having a choked stator and followed by a choked stator.

TURBINE HAVING CHOKED FIRST STATOR AND CHOKED LAST ROTOR AND FOLLOWED BY CHOKED EXHAUST NOZZLE

For choking at stations 2, 8, and 10 (see fig. 1), equation (6) can be rewritten

$$\frac{T'_1}{T'_8} = \left(\frac{A_8}{A_2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \quad (43)$$

and

$$\frac{T'_1}{T'_9} = \left(\frac{A_{10}}{A_2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \quad (44)$$

Rewriting equation (12) results in

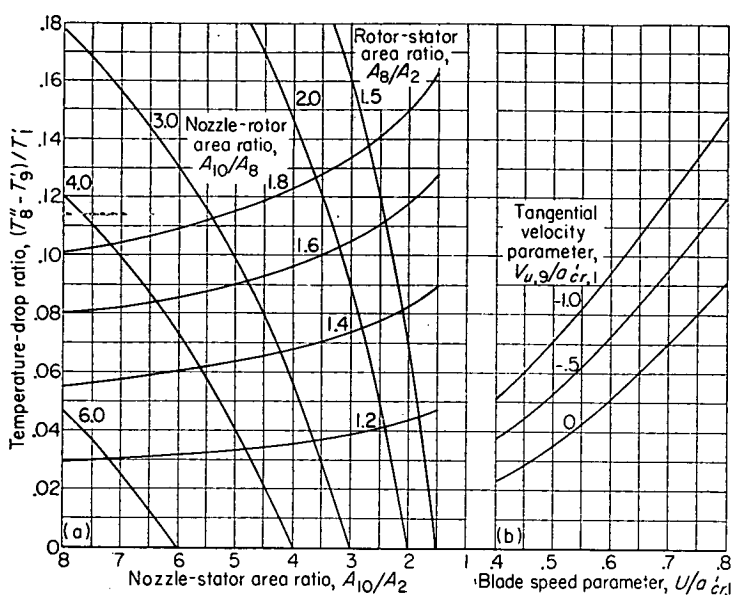
$$\frac{T'_8}{T'_9} = 1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr,9}} \left(2 \frac{V_{u,9}}{a'_{cr,9}} - \frac{U}{a'_{cr,9}} \right) \quad (45)$$

Combining equations (43), (44), and (45) with the definition of critical velocity a'_{cr} yields

$$\frac{V_{u,9}}{a'_{cr,1}} = \frac{1}{2} \left[\frac{U}{a'_{cr,1}} - \frac{\gamma+1}{\gamma-1} \frac{a'_{cr,1}}{U} \left(\frac{T'_8}{T'_1} - 1 \right) \right] \quad (46)$$

and

$$\frac{T'_8 - T'_9}{T'_1} = \left(\frac{A_2}{A_8}\right)^{\frac{2(\gamma-1)}{\gamma+1}} - \left(\frac{A_2}{A_{10}}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \quad (47)$$



(a) Relation among temperature-drop ratio, nozzle-stator area ratio, rotor-stator area ratio, and nozzle-rotor area ratio (eq. (47)).
 (b) Relation among temperature-drop ratio, blade speed parameter, and tangential velocity parameter (eq. (46)).

FIGURE 9.—Conditions for choked first stator, last rotor, and exhaust nozzle.

Equations (46) and (47) are plotted in figure 9. The blade speed parameter $U/a'_{cr,1}$ is in close agreement with equivalent blade speed $U/\sqrt{\theta_1}$ if the blade speed parameter is based on the entrance value of critical velocity $a'_{cr,1}$ rather than on $a'_{cr,9}$. The tangential velocity parameter $V_{u,3}/a'_{cr,1}$ is also based on the entrance value of critical velocity $a'_{cr,1}$ in order that the kinetic energy $V_{u,3}^2/2gJ$ may more easily be related to the turbine work.

The use of figure 9 is illustrated by the following example: Consider the operation of a turbine with a rotor-stator area ratio A_8/A_2 of 2.0, a nozzle-stator area ratio A_{10}/A_2 of 2.3, and a blade speed parameter $U/a'_{cr,1}$ of 0.472; for these conditions, the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ is 0. If the nozzle-stator area ratio A_{10}/A_2 is increased from 2.3 to 2.68, the rotor-stator area ratio A_8/A_2 and blade speed parameter $U/a'_{cr,1}$ being held constant, the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ decreases from 0 to -1.0. If the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ is to be increased to 0, the blade speed parameter $U/a'_{cr,1}$ must then be increased from 0.472 to 0.677.

In order to illustrate the effect of adjusting the throat area A_2 of a choked turbine stator on turbine exit conditions, figure 9 includes lines of constant nozzle-rotor area ratio A_{10}/A_8 . If the area of only the first turbine stator is changed, the turbine operation will be along a line of constant nozzle-rotor area ratio A_{10}/A_8 . If the areas of the first stator and the exhaust nozzle are varied simultaneously, the combined effect of such a variation on the temperature-drop ratio $(T_8'' - T_9'')/T_1'$ can be determined by changing both the rotor-stator area ratio A_8/A_2 in inverse proportion to the change in stator area and the nozzle-rotor area ratio A_{10}/A_8 in direct proportion to the change in nozzle area. If the concomitant change in blade speed parameter $U/a'_{cr,1}$ is also known from required conditions of engine operation, the effect of this

variation on the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ can be determined.

Recapitulation.—The over-all temperature ratio T_1'/T_9' across the turbine is shown to depend on only the nozzle-stator area ratio A_{10}/A_2 , and it is thus independent of the blade speed parameter $U/a'_{cr,1}$ or the blade profile design. Figure 9 permits variation in the turbine-exit conditions to be approximately estimated for changes in blade speed parameter $U/a'_{cr,1}$ and adjustment of the turbine stator and exhaust-nozzle areas.

NONCHOKED STATOR FOLLOWED BY CHOKED ROTOR

Stator exit tangential velocity parameter.—For choking in the rotor and absence of choking in the stator, equation (8) rather than equation (6) is appropriate. For subsonic flow at the stator exit, the assumptions were made that

$$\left. \begin{aligned} V_2 &= V_3 \\ \frac{d\alpha_3}{dV_{u,3}} &= 0 \end{aligned} \right\} \quad (48)$$

For a perfect gas,

$$\frac{\rho_2 V_2}{\rho_1 a'_{cr,1}} = \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V_2}{a'_{cr,1}} \right)^2 \right]^{\frac{1}{\gamma-1}} \frac{V_2}{a'_{cr,1}} \quad (49)$$

For subsonic stator exit conditions, equations (8), (12), (48), and (49) were combined to yield

$$\frac{A_2}{A_4} = \frac{\left(\frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \left\{ 1 - \frac{\gamma-1}{\gamma+1} \left(\frac{U}{a'_{cr,1}} \right) \left[2 \frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}} \right] \right\}^{\frac{\gamma+1}{2(\gamma-1)}}}{\frac{V_{u,3}}{a'_{cr,1}} \sec \alpha_3 \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V_{u,3}}{a'_{cr,1}} \right)^2 \sec^2 \alpha_3 \right]^{\frac{1}{\gamma-1}}} \quad (50)$$

which is plotted in figure 10 for two values of stator exit flow angle α_3 . At the stator choking limit,

$$\frac{V_3}{a'_{cr,1}} = \frac{V_{u,3}}{a'_{cr,1}} \sec \alpha_3 = 1$$

Under this condition equation (50) reduces to equation (17).

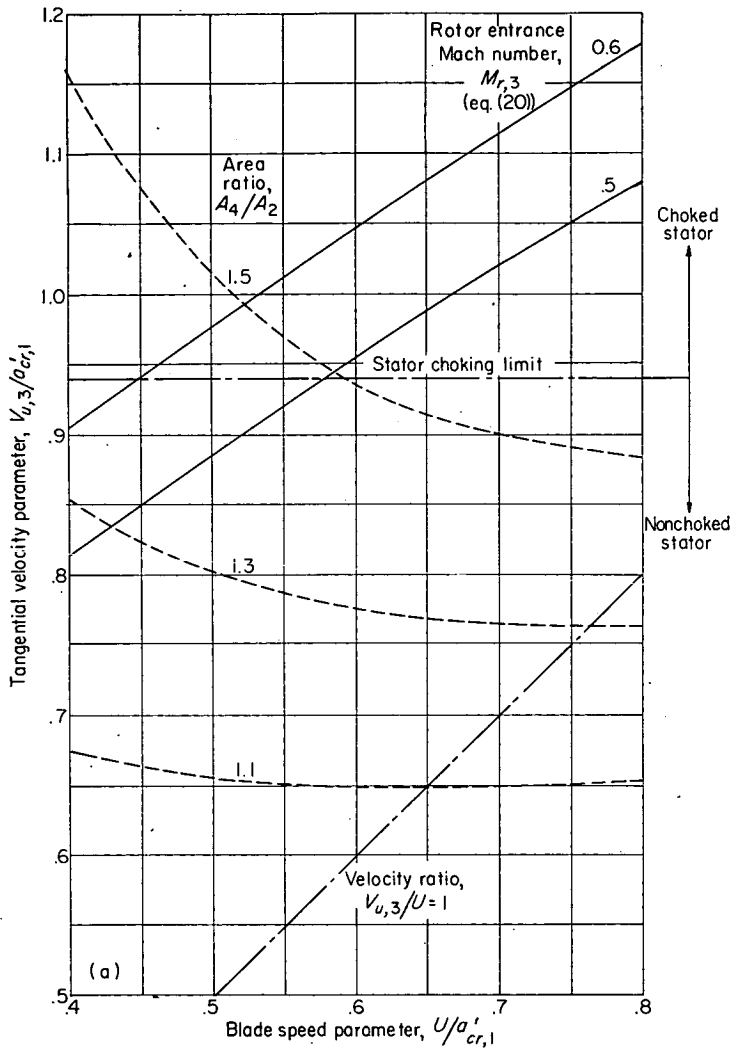
Differentiation of equation (50) and inclusion of equations (16) and (24) yield

$$\frac{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)}{\partial \left(\frac{U}{a'_{cr,1}} \right)} = \frac{1 - \frac{V_{u,3}}{U}}{1 + \frac{(a'_{cr,1})^2}{UV_{u,3}} \left[1 - \left(\frac{V_{u,3}}{a'_{cr,1}} \right)^2 \sec^2 \alpha_3 \right] \left(\frac{T_3''/T_1''}{T_3/T_1} \right)} \quad (51)$$

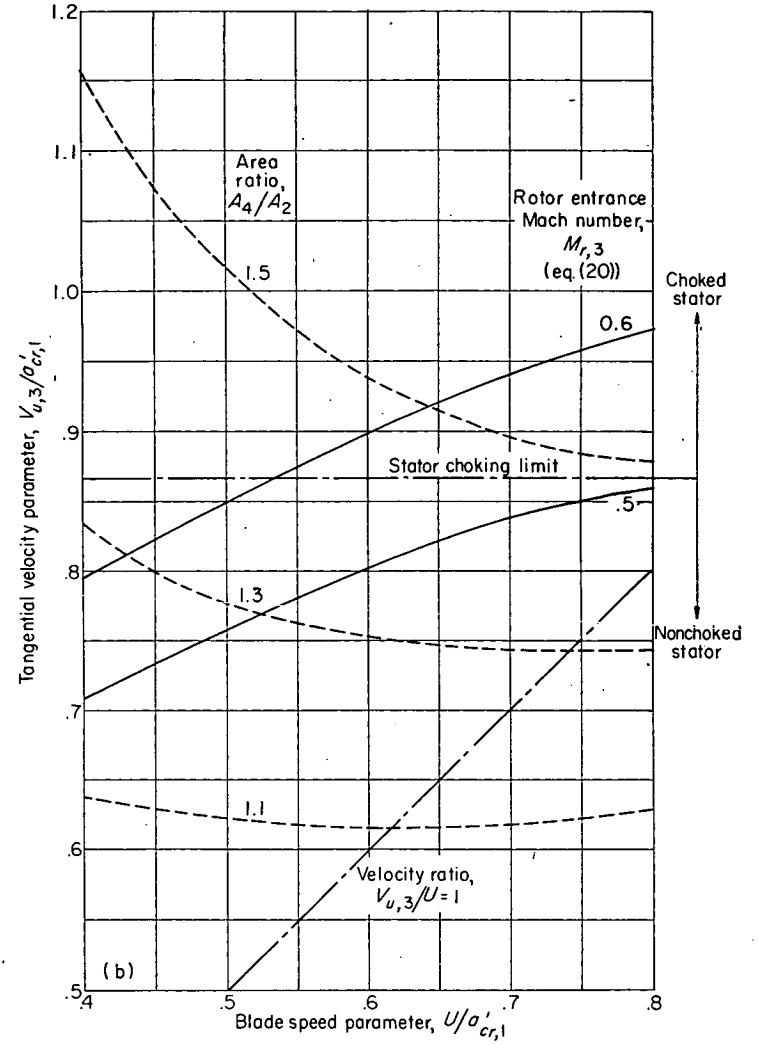
At the stator choking limit, equation (51) reduces to equation (18). Because for the subsonic case the denominator of this expression is always positive,

if	$\frac{V_{u,3}}{U}$	= 1	> 1	< 1
then	$\frac{\partial \left(\frac{V_{u,3}}{a'_{cr,1}} \right)}{\partial \left(\frac{U}{a'_{cr,1}} \right)}$	= 0	< 0	> 0

Thus, just as for a choked stator, the tangential velocity parameter $V_{u,3}/a'_{cr,1}$ is a minimum if the velocity ratio $V_{u,3}/U$ is 1.



(a) Stator exit flow angle, 20°.



(b) Stator exit flow angle, 30°.

FIGURE 10.—Relation among area ratio, rotor entrance Mach number, and velocity parameters at exit of nonchoked stator followed by choked rotor (eq. (50)). For lines of constant area ratio, equation (17) applies above stator choking limit and equation (50) applies below stator choking limit.

Effect of manufacturing errors.—In order to make a complete presentation of the effect of manufacturing errors in throat areas upon rotor entrance Mach number for a turbine stage having choking flow in the rotor, the case of a nonchoked stator followed by a choked rotor was analyzed in a manner similar to that used in the derivation of equa-

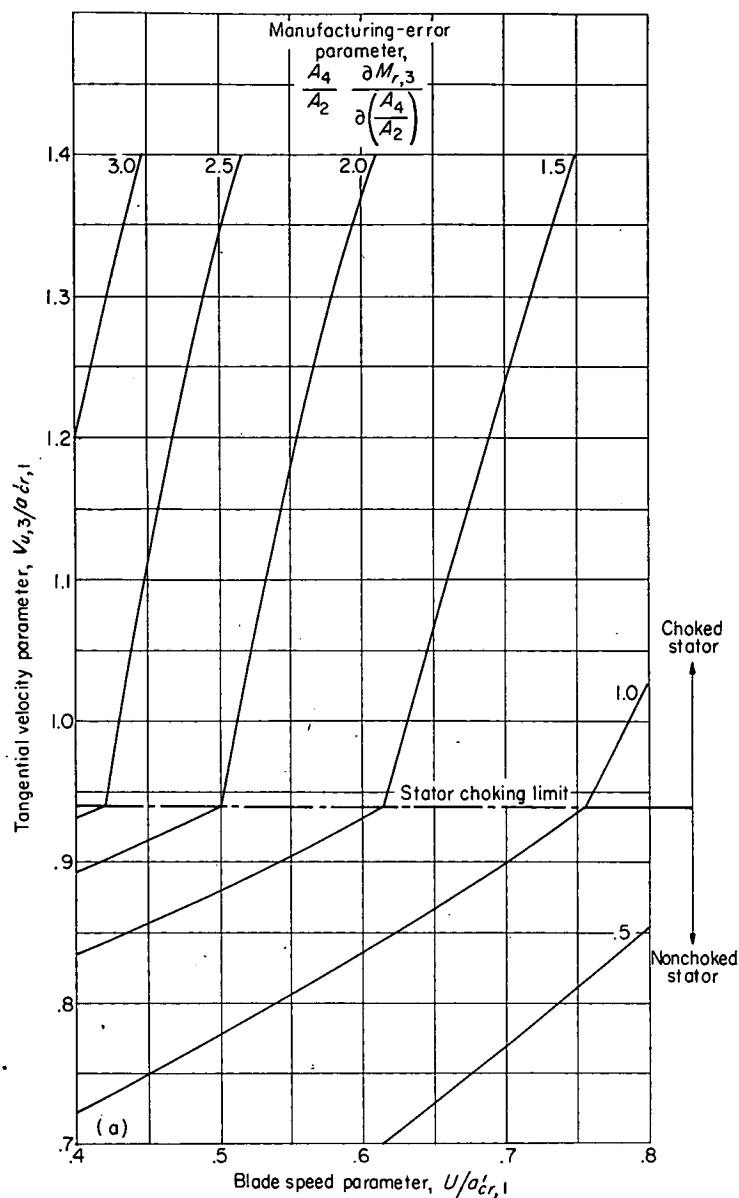
tion (22). Once again, equation (21) was used; evaluation of the four factors on the right side of this identity was made this time by means of equations (20) and (50) to yield the following expression for the manufacturing-error parameter:

$$\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)} = \frac{\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2 \left[1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr,1}} \left(2 \frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}}\right)\right]}{\sqrt{1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2} \sec^2 \alpha_3 \sqrt{\frac{\gamma+1}{2} \left[\left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2 \tan^2 \alpha_3 + \left(\frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}}\right)^2\right]}} \times$$

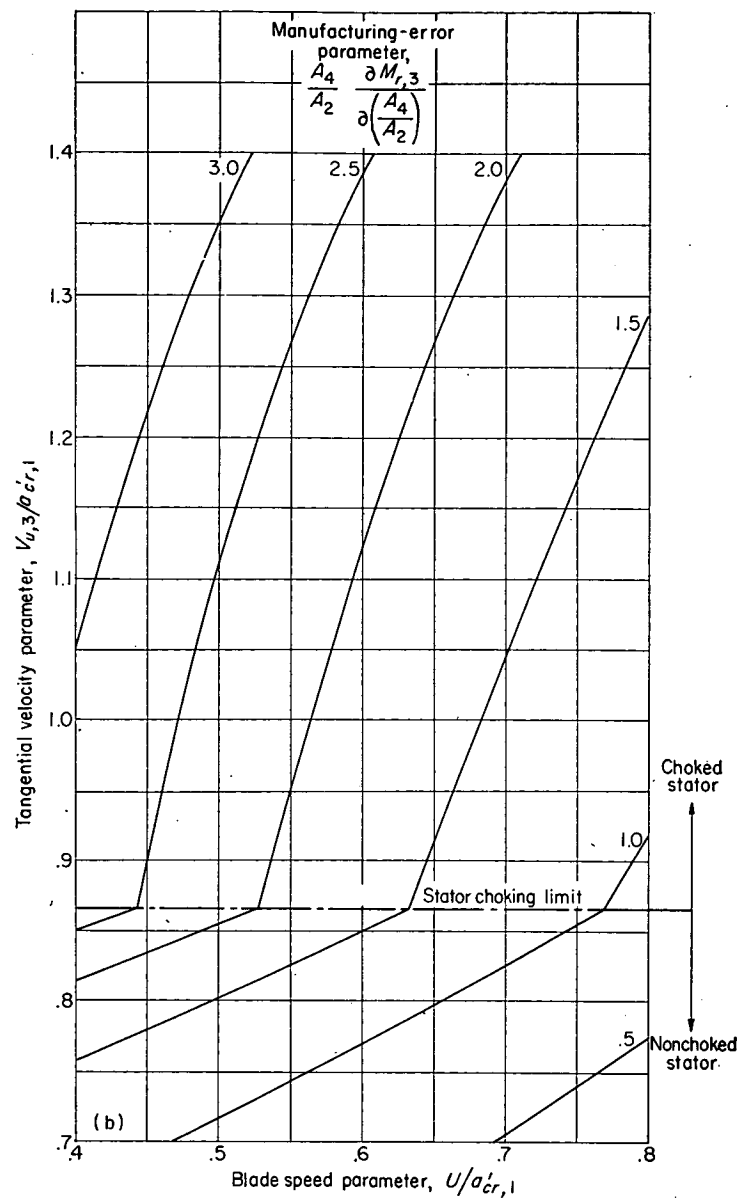
$$\frac{\left[\sec^2 \alpha_3 - \frac{U}{V_{u,3}} - \frac{\gamma-1}{\gamma+1} \sec^2 \alpha_3 \frac{U}{a'_{cr,1}} \left(\frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}}\right)\right]}{\left\{\frac{UV_{u,3}}{(a'_{cr,1})^2} \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2 \sec^2 \alpha_3\right] + \left[1 - \left(\frac{V_{u,3}}{a'_{cr,1}}\right)^2 \sec^2 \alpha_3\right] \left[1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a'_{cr,1}} \left(2 \frac{V_{u,3}}{a'_{cr,1}} - \frac{U}{a'_{cr,1}}\right)\right]\right\}} \quad (53)$$

Equation (53) was plotted in figure 11 in the region below the stator choking limit; figure 6 was reproduced in the region above the stator choking limit to complete the presen-

tation. At the stator choking limit (i.e., $V_{u,3}/a'_{cr,1} = \cos \alpha_3$), equations (22) and (53) coincide.



(a) Stator exit flow angle, 20°.



(b) Stator exit flow angle, 30°.

FIGURE 11.—Effect of manufacturing errors in throat areas on rotor entrance Mach number for stator followed by choked rotor. Equation (22) applies above stator choking limit and equation (53) applies below stator choking limit.

The rotor entrance Mach number is frequently, but erroneously, considered to be extremely sensitive to errors in area ratio A_4/A_2 as the stator exit Mach number V_3/a_3 approaches unity. Figure 11 shows that at this stator choking limit, the values of manufacturing-error parameter $\frac{A_4}{A_2} \frac{\partial M_{r,3}}{\partial \left(\frac{A_4}{A_2}\right)}$ remain finite and exhibit no marked variations

even in the immediate neighborhood of this limit. It may therefore be concluded that at the choking limit the rotor entrance Mach number is not extremely sensitive to small changes in area ratio A_4/A_2 .

Specific-mass-flow parameter.—Equations (8) and (12) were combined to determine the specific-mass-flow parameter $\bar{w}$ where

$$\bar{w} = \frac{\rho_2 V_2}{\rho_1 a_{cr,1}} = \frac{A_4}{A_2} \left(\frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \left[1 - \frac{\gamma-1}{\gamma+1} \frac{U}{a_{cr,1}} \left(2 \frac{V_{u,3}}{a_{cr,1}} - \frac{U}{a_{cr,1}} \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad (54)$$

Equation (54) was used in combination with the computations for figure 10 to plot figure 12. For values of area ratio A_4/A_2 sufficiently large to produce choking in the stator, the specific-mass-flow parameter $\bar{w}$ is, of course,

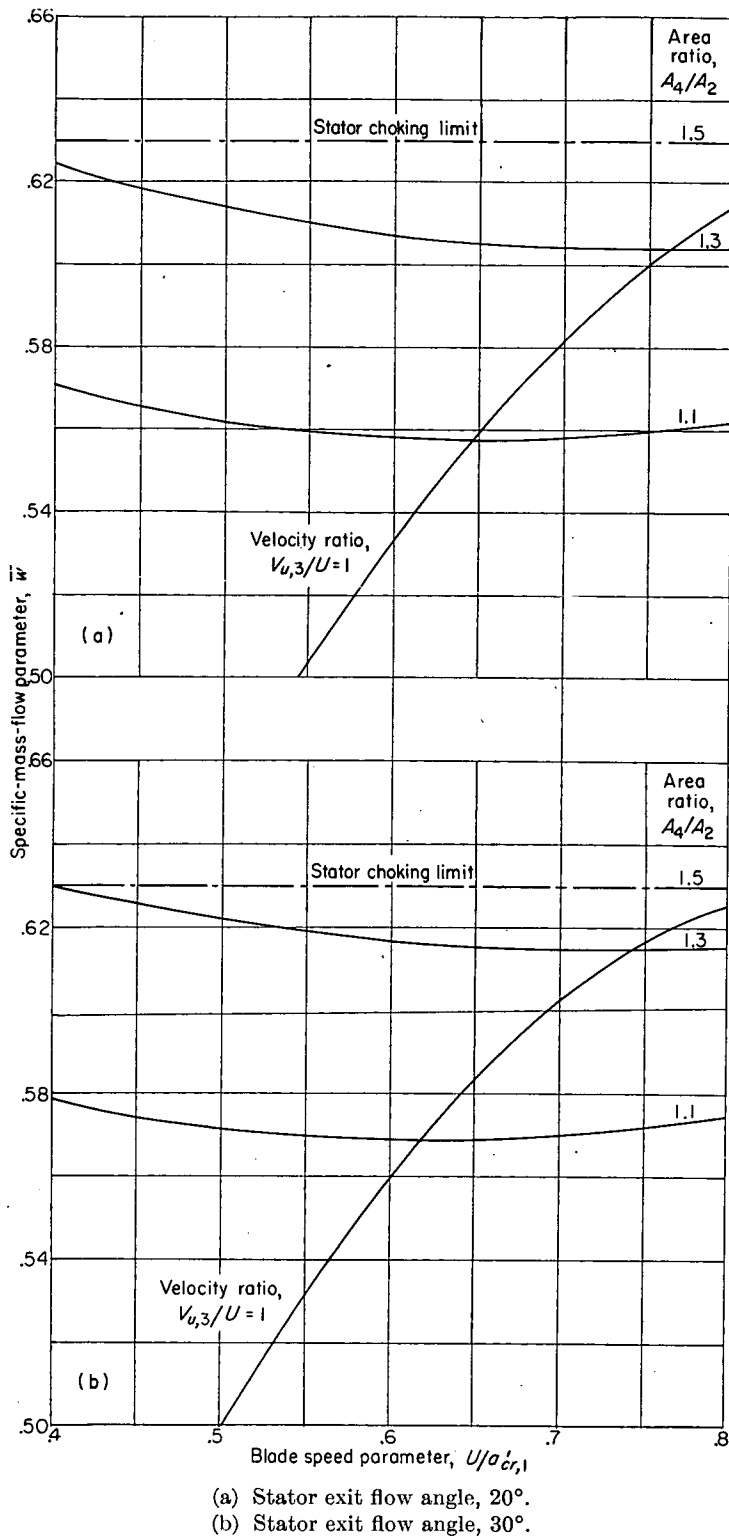


FIGURE 12.—Variation of specific-mass-flow parameter with blade speed parameter and area ratio for nonchoked stator followed by choked rotor (eqs. (50) and (54)).

independent of blade speed parameter $U/a'_{cr,1}$ or area ratio A_4/A_2 (see eq. (4)). For ordinary variations in blade speed parameter $U/a'_{cr,1}$, the variation in specific-mass-flow parameter $\bar{w}$ is comparatively small. The rate of variation in specific-mass-flow parameter $\bar{w}$ with blade speed parameter $U/a'_{cr,1}$ may be determined by differentiating equation (54) and combining this result with equation (51), that is,

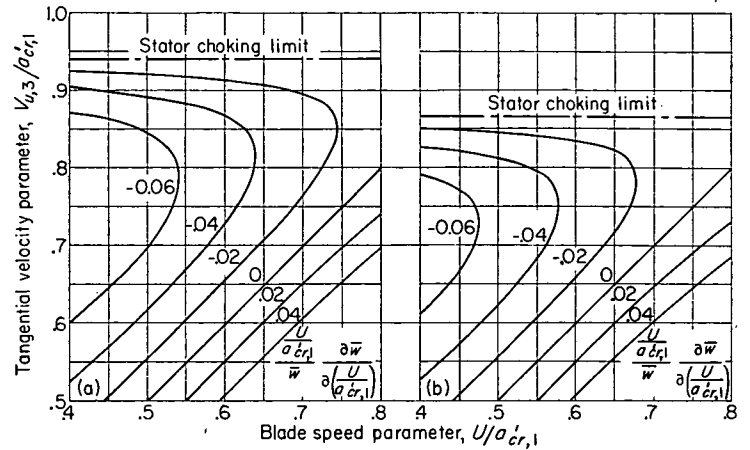


FIGURE 13.—Rate of variation of specific-mass-flow parameter with blade speed parameter for nonchoked stator followed by choked rotor (eq. (55)).

$$\frac{U}{a'_{cr,1}} \frac{\partial \bar{w}}{\partial \left(\frac{U}{a'_{cr,1}} \right)} = \frac{\left(\frac{U}{a'_{cr,1}} \right)^2 \frac{T'_1}{T''_3} \left(1 - \frac{V_{u,3}}{U} \right) \left[1 - \left(\frac{V_{u,3}}{a'_{cr,1}} \right)^2 \sec^2 \alpha_3 \right]}{\left(\frac{T_3}{T''_3} \right) \frac{U V_{u,3}}{(a'_{cr,1})^2} + 1 - \left(\frac{V_{u,3}}{a'_{cr,1}} \right)^2 \sec^2 \alpha_3} \quad (55)$$

Equation (55) is plotted in figure 13 for two values of stator exit flow angle α_3 . For the conventional range of design,

the factor $\frac{U}{a'_{cr,1}} \frac{\partial \bar{w}}{\partial \left(\frac{U}{a'_{cr,1}} \right)}$ is between 0 and -0.06. From equation (55),

if	$\frac{V_{u,3}}{U}$	= 1	> 1	< 1
then	$\frac{\left(\frac{U}{a'_{cr,1}} \right)}{\bar{w}} \frac{\partial \bar{w}}{\partial \left(\frac{U}{a'_{cr,1}} \right)}$	= 0	< 0	> 0

(56)

or at the stator choking limit

$$\left. \begin{aligned} \frac{V_{u,3}}{a'_{cr,1}} &= \cos \alpha_3 \\ \text{and thus} \quad \left(\frac{U}{a'_{cr,1}} \right) \frac{\partial \bar{w}}{\partial \left(\frac{U}{a'_{cr,1}} \right)} &= 0 \end{aligned} \right\} \quad (57)$$

Recapitulation.—Analysis of conditions between a nonchoked stator and a choked rotor indicates that those factors which control the flow conditions between the stator and the rotor are the same as those for a choked stator followed by a choked rotor, namely, blade speed parameter $U/a'_{cr,1}$, area ratio A_4/A_2 , and stator exit flow angle α_3 . The sensitivity of rotor entrance Mach number $M_{r,3}$ to manufacturing errors

in area ratio A_4/A_2 was determined; rotor entrance Mach number $M_{r,3}$ was found to be only moderately sensitive to errors in area ratio A_4/A_2 , even near the stator choking limit. The variation in specific-mass-flow parameter $\bar{w}$ with blade speed parameter $U/a'_{cr,1}$ was determined.

TEST-DATA ANALYSIS FOR A TWO-STAGE TURBINE

The analysis of turbines having choked or nonchoked stators provides an understanding of flow processes in choked-flow turbines which permits some general observations to be made for turbines of this class. In particular, criteria can be set down which will aid in establishing from test data of single-stage or multistage turbines which blade rows are choked and which are not. The following analysis considers a two-stage turbine such as that in figure 1, but such an analysis may be extended to apply to any number of turbine stages.

For a given value of ratio of specific heats γ , equivalent weight flow $w\sqrt{\theta'_1}/\delta'_1$ for any particular turbine is directly proportional to specific-mass-flow parameter $\bar{w}$. Similarly, equivalent blade speed $U/\sqrt{\theta'_1}$ is proportional to blade speed parameter $U/a'_{cr,1}$. The results of the preceding section can thus be employed to predict the manner in which the equivalent weight flow of a given type of turbine will vary with turbine operating conditions.

From the test data, equivalent weight flow at the entrance to each stage, namely, $w\sqrt{\theta'_1}/\delta'_1$ and $w\sqrt{\theta'_5}/\delta'_5$, should be plotted against either the stagnation-pressure ratio p'_1/p'_0 or the stagnation-static-pressure ratio p'_1/p_0 with lines of constant equivalent blade speed $U/\sqrt{\theta'_1}$. In the discussion which follows, it will be assumed that the stagnation-static-pressure ratio p'_1/p_0 is employed. These plots should then be examined in order to determine whether one or more of the following conditions are satisfied:

$$(1) \text{ If } \frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} > 0, \text{ the flow is not choked. If the pres-}$$

sure ratio can be made sufficiently high, at least one blade row in the turbine will choke. For all higher pressure ratios,

$$\frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} = 0.$$

$$(2) \text{ If } \frac{\partial \left(\frac{w\sqrt{\theta'_5}}{\delta'_5} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} > 0, \text{ the flow is not choked downstream of}$$

station 5, that is, in either the second stator or the second rotor. If the pressure ratio can be made sufficiently high, choking will occur at some station downstream of station 5.

$$\text{For all higher pressure ratios, } \frac{\partial \left(\frac{w\sqrt{\theta'_5}}{\delta'_5} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} = 0. \text{ Under this}$$

condition, the flow may or may not be choked upstream of station 5.

$$(3) \text{ If } \frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} = 0 \text{ but } \frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{U}{\sqrt{\theta'_1}} \right)} \neq 0, \text{ the first stator is not}$$

$$\text{choked. If, in addition, } \frac{\partial \left(\frac{w\sqrt{\theta'_5}}{\delta'_5} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} > 0, \text{ then the first rotor is}$$

the only blade row which is choked.

$$(4) \text{ If } \frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} = 0 \text{ and } \frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{U}{\sqrt{\theta'_1}} \right)} = 0 \text{ and at the mean}$$

radius $V_{u,3}/U \neq 1$, then the first stator is choked.

$$(5) \text{ If } \frac{\partial \left(\frac{w\sqrt{\theta'_5}}{\delta'_5} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} = 0 \text{ but } \frac{\partial \left(\frac{w\sqrt{\theta'_5}}{\delta'_5} \right)}{\partial \left(\frac{U}{\sqrt{\theta'_1}} \right)} \neq 0, \text{ the second stator is}$$

not choked and the second rotor is choked. The preceding blade rows may or may not be choked.

$$(6) \text{ If } \frac{\partial \left(\frac{w\sqrt{\theta'_5}}{\delta'_5} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} = 0 \text{ and } \frac{\partial \left(\frac{w\sqrt{\theta'_5}}{\delta'_5} \right)}{\partial \left(\frac{U}{\sqrt{\theta'_1}} \right)} = 0 \text{ and at the mean}$$

radius $V_{u,7}/U \neq 1$, then the second stator is choked.

Recapitulation.—These six criteria will in some cases permit discrimination between choking and nonchoking blade rows, but whether or not a particular blade row is choking

cannot always be determined. For example, if $\frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{p'_1}{p_0} \right)} = 0$

and $\frac{\partial \left(\frac{w\sqrt{\theta'_1}}{\delta'_1} \right)}{\partial \left(\frac{U}{\sqrt{\theta'_1}} \right)} = 0$ and the stator exit velocity ratio $V_{u,3}/U$

at the mean radius is nearly equal to 1, then whether or not the first stator is choking cannot be determined from this test (see eq. (56)).

Even if the first stator is not choked, variations in equivalent weight flow with blade speed may be difficult to detect. Figures 12 and 13 indicate for a nonchoked stator followed by a choked rotor the amount which the specific-mass-flow parameter $\bar{w}$, and thus equivalent weight flow $w\sqrt{\theta'_1}/\delta'_1$, will vary with blade speed parameter $U/a'_{cr,1}$ or, alternatively, equivalent blade speed $U/\sqrt{\theta'_1}$ for constant amount of loss in the turbine. Whether or not the variation in equivalent weight flow can be detected will depend on the range of blade speed for which the rotor is choked and on the precision of measurement.

Any static-pressure measurement within the turbine can also be used to indicate whether or not the flow is choked downstream of the measuring station. For example, if $\frac{\partial(p'_1/p_3)}{\partial(p'_1/p_9)} = 0$, then one of the throats downstream of station 3 is choked. If for this condition the second rotor is not choked, this nonchoking can be recognized because $\frac{\partial(p'_1/p_7)}{\partial(p'_1/p_9)} \neq 0$.

TURBINE STATOR ADJUSTMENT

An analysis is needed in order to evaluate the effects of turbine stator adjustment on flow conditions within turbines. Such an analysis, which provides a means for determining the range of usefulness of turbine stator adjustment within aerodynamic limits on internal flow conditions, is presented in the following paragraphs.

Stator adjustment provides an extension of the range of effective turbine operation. When stator adjustment is employed for this purpose, the relation among the turbine operating parameters and the stator setting will depend on the specific application of stator adjustment to be considered. The analysis presented is therefore based on only one application of turbine stator adjustment, and the results are thus useful in appraising turbine stator adjustment for only this application. This general method of analysis may, of course, be employed to study turbine stator adjustment for other applications.

Type of application.—For some applications of turbojet engines, operation of the engine at constant compressor pressure ratio and constant equivalent rotative speed may be desirable. In comparison with the take-off value of engine temperature ratio (turbine inlet temperature to compressor inlet temperature), low values of engine temperature ratio may result from reduced turbine inlet temperature or, for flight at high flight Mach numbers, from increased compressor inlet stagnation temperature. A turbojet engine is therefore considered to be operated in the following manner:

- (1) The engine equivalent rotative speed based on compressor inlet conditions is constant.
- (2) The compressor pressure ratio is constant.
- (3) The engine temperature ratio is varied.

Analysis.—Operating an engine in this way will require adjustment of both the turbine stator and the exhaust nozzle; the equivalent weight flow of the compressor will remain constant. These conditions can be expressed in equation form

$$\frac{w\sqrt{T'_c}}{p'_c} = k \quad (58)$$

$$\frac{U}{\sqrt{T'_c}} = l \quad (59)$$

$$\frac{p'_1}{p'_c} = m \quad (60)$$

where k , l , and m are constants. Combining equations (58) to (60) gives

$$\left(\frac{w\sqrt{T'_c}}{p'_c}\right)\left(\frac{U}{\sqrt{T'_c}}\right) = \frac{kl}{m}$$

or, neglecting variations in γ with engine operating conditions,

$$\left(\frac{\rho_2 V_2}{\rho'_1 a'_{cr,1}}\right)\left(\frac{A_2}{A_4}\right)\left(\frac{U}{a'_{cr,1}}\right) = n = \text{a constant} \quad (61)$$

For operation under the specified conditions, the square of the blade speed parameter $(U/a'_{cr,1})^2$ is inversely proportional to the engine temperature ratio T'_1/T'_c .

Discussion.—For choking of the turbine stator, the specific-mass-flow parameter $\rho_2 V_2 / \rho'_1 a'_{cr,1}$ is a constant. The required variation in area ratio A_2/A_4 is thus proportional to the variation in the reciprocal of blade speed parameter $U/a'_{cr,1}$.

For conditions other than those for which the turbine stator chokes, the variation in specific-mass-flow parameter $\rho_2 V_2 / \rho'_1 a'_{cr,1}$ must be taken into account. For the purposes of this analysis, the turbine rotor was assumed to be choked. A critical stator exit flow angle $\alpha_{cr,3}$ was defined as the maximum value of stator exit flow angle α_3 for which the stator is choked. Then equations (8), (12), (49), and (61) were combined to plot figure 14 for two values of critical stator exit flow angle $\alpha_{cr,3}$.

In figure 14, the lines of constant blade speed parameter $U/a'_{cr,1}$ are straight lines through the origin with slopes equal to $a'_{cr,1}/U$. In the region of stator choking, which is below the stator choking limit line in figure 14, the lines of constant area ratio A_2/A_4 are independent of critical stator exit flow angle $\alpha_{cr,3}$. A comparison of figures 4, 10, and 14 indicates that the more critical rotor entrance Mach numbers occur at high values of blade speed parameter $U/a'_{cr,1}$, which cor-

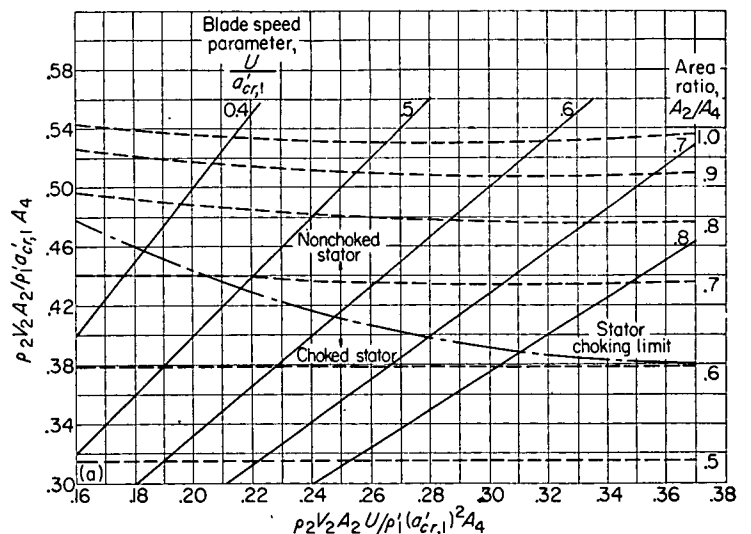
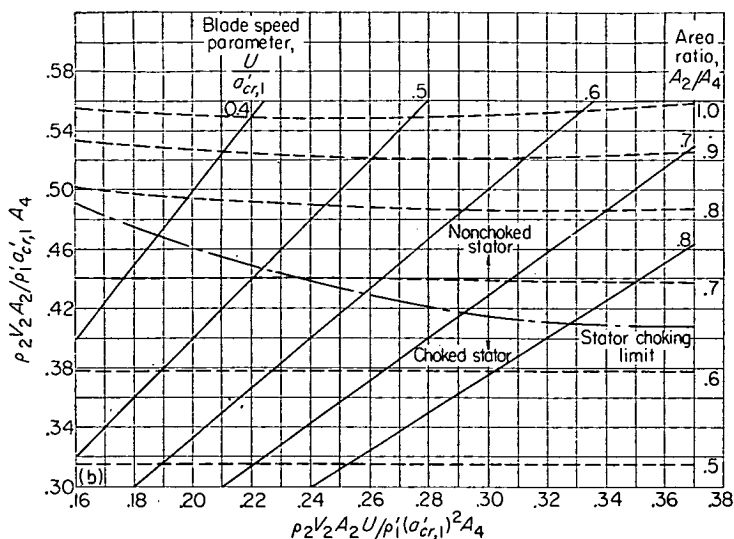


FIGURE 14.—Variation in stator exit flow conditions with stator setting for constant values of compressor equivalent blade speed and compressor pressure ratio.



(b) Critical stator exit flow angle, 25°.

FIGURE 14.—Concluded. Variation in stator exit flow conditions with stator setting for constant values of compressor equivalent blade speed and compressor pressure ratio.

respond to low engine temperature ratio. The variation in rotor entrance conditions with stator setting can be determined from plots such as those in figures 4, 10, and 14.

EFFECT OF VARIATION IN ENTROPY

Although for this analysis the flow through turbines was assumed to be isentropic, the analysis is readily adaptable to include those cases having variations in entropy s . For a perfect gas which undergoes an entropy change, equation (5) becomes

$$e^{\frac{J(s_4-s_2)}{R}} \frac{\rho_4'' a_{cr,4}''}{\rho_2'' a_{cr,2}''} = \left(\frac{T_4''}{T_2''} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad (62)$$

The area ratio A_2/A_4 in equation (6) is thereby changed to

$$\frac{A_2}{A_4} = \left(\frac{T_3''}{T_1''} \right)^{\frac{\gamma+1}{2(\gamma-1)}} e^{\frac{J(s_2-s_4)}{R}} \quad (63)$$

that is, the change in area ratio is proportional to the factor

$e^{J\Delta s/R}$, where Δs is the entropy change between the two stations being considered.

Values of $\bar{w}$ or $\rho_2 V_2 / \rho_1 a_{cr,1}'$ taken from the figures or used in the equations must be multiplied by the factor $e^{J(s_1-s_2)/R}$ to obtain the specific-mass-flow parameter. Because the stagnation temperature relative to a blade row remains constant irrespective of any increase in entropy across the blade row, no adjustment in any of the velocity parameters is necessary. Except for the adjustments in area ratio and specific-mass-flow parameter, the remainder of the analysis presented herein is unchanged.

CONCLUDING REMARKS

Turbines having choking flow were investigated by means of a one-dimensional analysis based on isentropic flow. This analysis indicated that the area ratios, equivalent blade speed, and blade-row exit flow angles are the factors which control design and operation of such turbines. For the usual class of turbine, increasing the equivalent blade speed of a given turbine makes the internal flow conditions become less critical. Flow conditions within choked-flow turbines are not unusually sensitive to manufacturing errors in area ratio even near the stator choking limit. Six criteria are stated that will aid in establishing from test data of turbines which blade rows are choked and which are not. For a turbine equipped with an adjustable stator, the variation in internal flow conditions with operating conditions is determined for one application of stator adjustment. A method is described for extending the analysis to include flow having variations in entropy.

LEWIS FLIGHT PROPULSION LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

CLEVELAND, OHIO, July 9, 1952

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REPORT 1128

CALCULATIONS ON THE FORCES AND MOMENTS FOR AN OSCILLATING WING-AILERON COMBINATION IN TWO-DIMENSIONAL POTENTIAL FLOW AT SONIC SPEED¹

By HERBERT C. NELSON and JULIAN H. BERMAN

SUMMARY

The linearized theory for compressible unsteady flow is used, as suggested in recent contributions to the subject, to obtain the velocity potential and the lift and moment for a thin, harmonically oscillating, two-dimensional wing-aileron combination moving at sonic speed. The velocity potential is derived by considering the sonic case as the limit of the linearized supersonic theory. From the velocity potential explicit expressions for the lift and moment are developed for vertical translation and pitching of the wing and rotation of the aileron. The report provides extensive tables of numerical values for the coefficients contained in the expressions for lift and moment, for various values of the reduced frequency k ($0 < k \leq 3.5$), and aileron hinge position (from 10 to 90 percent of the wing chord). The sonic results are compared and found to be consistent with previously obtained subsonic and supersonic results. Several figures are presented showing the variation of lift and moment with reduced frequency and Mach number and the influence of Mach number on some cases of bending-torsion flutter.

INTRODUCTION

Instability investigations for high-speed aircraft often require a knowledge of the air forces and moments that act on an oscillating wing moving at high speed. For subsonic and supersonic speeds the main source of theoretical information has been the solution of the linearized differential equation for compressible flow. For sonic or near-sonic speed, however, the linearized theory has been generally assumed inapplicable, since it does not allow for thickness effects, shocks, and strong disturbances. As is well known, it predicts infinite forces on a nonoscillating, thin, unswept wing moving at sonic speed.

Important differences exist, however, between the steady and unsteady cases. By a discussion of the order of magnitude of the terms of the general nonlinear differential equation for compressible flow, reference 1 shows that for unsteady two-dimensional flow at sonic speed this equation is essentially linear and in linear form leads to physically plausible results for the forces on a thin oscillating wing, provided the

frequency of oscillation is sufficiently large. A similar conclusion was reached in reference 2, where linear methods applied to a wing in two-dimensional nonstationary flow at sonic speed yielded perturbation velocities of the same order of magnitude as those obtained for subsonic or supersonic speeds. In references 3 and 4 expressions and some numerical values are given for the lift forces and moments on an oscillating two-dimensional wing moving at sonic speed. Because of the importance of the sonic problem in present-day flight considerations and because of the insight into the three-dimensional problem that the solution for two-dimensional flow will probably afford, the purpose of the present report is to develop the two-dimensional case more fully.

Consideration is thus given to the case of an oscillating wing-aileron combination in two-dimensional flow at sonic speed. The velocity potential for this case is obtained, and from the velocity potential expressions for the air forces and moments on the wing-aileron combination are developed in terms of the frequency of oscillation. Numerical tables of the coefficients contained in the expressions for lift and moment are supplied which may be used for the theoretical calculations involved in wing flutter and other instability problems for sonic speed. The tables provide a means for obtaining continuity of calculation between high-subsonic and low-supersonic results for the oscillating wing-aileron combination in two-dimensional flow.

Because of the small-disturbance assumption, the theory and subsequent results are subject to the same restrictions imposed on all small-perturbation theory, subsonic and supersonic. In addition, as the frequency approaches zero, the difficulties of the steady linearized problem are encountered; therefore the validity of the subsequent results is subject to question for the range of low frequencies. Moreover, uncertainty exists because the linear unsteady results are considered to represent disturbances from an equilibrium position that is governed by nonlinear relations, and a great amount of experimentation may be necessary to determine the region of validity for the calculations. Nevertheless, the results serve as a bridge between subsonic and supersonic theory and may be applicable for a range of high frequencies.

¹ Supersedes NACA TN 2590, "Calculations on the Forces and Moments for an Oscillating Wing-Aileron Combination in Two-Dimensional Potential Flow at Sonic Speed" by Herbert C. Nelson and Julian H. Berman, 1952.

SYMBOLS

a	velocity of sound in undisturbed medium
b	wing semichord
c_l	section lift coefficient
c_m	section moment coefficient about leading edge
$f(r_j)$	Fresnel integrals contained in equation (23)
h	vertical displacement of axis of rotation
$J_0(\lambda)$	Bessel function of zero order (first kind)
k	reduced frequency, $\omega b/V$
$k' = \frac{\omega b}{a}$	
L_i, M_i, N_i	quantities defined by equation (22); $i=1, 2, 3, 4, 5$, and 6
L'_i, M'_i, N'_i	quantities defined by equation (23), independent of wing-axis-of-rotation position
m	mass of wing per unit span
M	Mach number, V/a
M_α	aerodynamic section moment on wing about axis of rotation, positive leading edge up
M_β	aerodynamic section moment on aileron about its hinge, positive leading edge up
Δp	pressure difference
P	aerodynamic section normal force, positive downward
t	time
V	flight speed
$w(2bx, t)$	normal velocity at x at time t
x, z	nondimensional rectangular coordinates, referred to $2b$
$x' = 2bx$	
$y' = 2by$	
$z' = 2bz$	
x_0	abscissa of axis of rotation of wing section, referred to $2b$
x_1	abscissa of aileron hinge, referred to $2b$
x_α	location of center of gravity of wing-aileron system measured from elastic axis (see ref. 5)
α	angular displacement (pitch) about axis of rotation
α_h	effective angle of attack due to vertical translation, h/V
β	angular displacement of aileron, measured relative to α
θ_h	phase angle between lift due to h and bending velocity $\dot{h}$
θ_α	phase angle between lift due to α and position α
θ_{hm}	phase angle between moment due to h and bending velocity $\dot{h}$
$\theta_{\alpha m}$	phase angle between moment due to α and position α
κ	density parameter, $\frac{\pi \rho b^2}{m}$; reference 5 uses $\mu = \frac{\pi}{4} \frac{1}{\kappa}$
ξ	abscissa of point of disturbance, referred to $2b$
$\xi' = 2b\xi$	

ρ	density in main stream
τ	time variable
τ_1, τ_2	times required for transmittal of disturbance to field point
ϕ	disturbance velocity potential
ω	angular frequency of oscillation
ω_h	natural bending frequency of wing
ω_α	natural torsional frequency of wing

ANALYSIS

The theory presented herein for two-dimensional flow at sonic speed is based on the assumptions that the two wing surfaces act independently and that wake effects are absent. Thus the sonic case as treated is more akin to the supersonic than the subsonic case. The velocity potential for the oscillating two-dimensional wing moving at sonic speed is derived by allowing the Mach number M to approach unity in the velocity potential for the wing moving at supersonic speed. An alternative derivation is also given in which the potential is obtained directly from the linearized differential equation by a method of solution employing the Laplace transformation. In reference 3 Rott obtained the velocity potential by superposition of the elementary source solution of the linearized differential equation.

Velocity potential for sonic speed.—Consider first the velocity potential for a harmonically oscillating two-dimensional wing moving at supersonic speed, given in reference 5 as

$$\phi(2bx, t) = -\frac{2b}{\pi\sqrt{M^2-1}} \int_0^x \int_{\tau_1}^{\tau_2} \frac{w(2b\xi, t) e^{-i\omega\tau} d\tau d\xi}{\sqrt{(\tau-\tau_1)(\tau_2-\tau)}} \quad (1)$$

where

$$\tau_1 = \frac{2b(x-\xi)}{a(M+1)}$$

$$\tau_2 = \frac{2b(x-\xi)}{a(M-1)}$$

a is the speed of sound in the undisturbed medium, x and ξ are nondimensional coordinates referred to the wing chord $2b$, $w(2b\xi, t)$ is the prescribed local normal velocity at the wing surface, and ω is the frequency of oscillation. The integral in equation (1) represents the total effect of all the disturbances created by the wing. The time-lag functions τ_1 and τ_2 are associated with the two pulses that occur at the point x because of a disturbance created at the point ξ (see ref. 5 for more complete discussion). Another form for equation (1), also given in reference 5, is

$$\phi(2bx, t) = -\frac{2b}{\sqrt{M^2-1}} \int_0^x w(2b\xi, t) e^{-i2k' \frac{M(x-\xi)}{M^2-1}} J_0\left(2k' \frac{x-\xi}{M^2-1}\right) d\xi \quad (2)$$

where $k' = \frac{b\omega}{a}$.

As the Mach number M approaches unity, the argument of the Bessel function J_0 in equation (2) becomes infinite and the following asymptotic approximation is applicable:

$$\lim_{M \rightarrow 1} J_0 \left(2k' \frac{x-\xi}{M^2-1} \right) = \sqrt{\frac{M^2-1}{\pi k(x-\xi)}} \cos \left(2k \frac{x-\xi}{M^2-1} - \frac{\pi}{4} \right) \quad (3)$$

where on the right-hand side k' has been replaced by k since $k = \frac{b\omega}{V}$ and $k' = \lim_{M \rightarrow 1} k$. At $M=1$ the time-lag function τ_2 contained in equation (1) becomes infinite and the influence of one of the two pulses characteristic of supersonic flow becomes vanishingly small. (By considering the sonic case as a limit from the subsonic side, the wing at sonic speed cannot overtake the second pulse.)

By letting M approach unity in equation (2) and using equation (3) in the process, the sonic velocity potential is found to be

$$\phi(2bx, t) = -2b \int_0^x w(2b\xi, t) G(x-\xi) d\xi \quad (4)$$

where

$$G(x) = \frac{1}{2} \frac{e^{-tkx}}{\sqrt{i\pi kx}}$$

As a matter of possible interest an alternative derivation of equation (4) that makes use of the Laplace transformation (as was done by Stewartson in ref. 6 for supersonic flow) is presented. The linearized differential equation for two-dimensional compressible flow may be written as

$$\frac{1}{a^2} \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x'} \right)^2 \phi = \frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial z'^2} \quad (5)$$

For the harmonically oscillating wing moving at sonic speed, equation (5) becomes

$$\frac{\partial^2 \psi}{\partial z'^2} = -\frac{\omega^2}{a^2} \psi + \frac{2i\omega}{a} \frac{\partial \psi}{\partial x'} \quad (6)$$

where the disturbance velocity potential ϕ is related to ψ by

$$\phi(x', z', t) = \psi(x', z') e^{i\omega t}$$

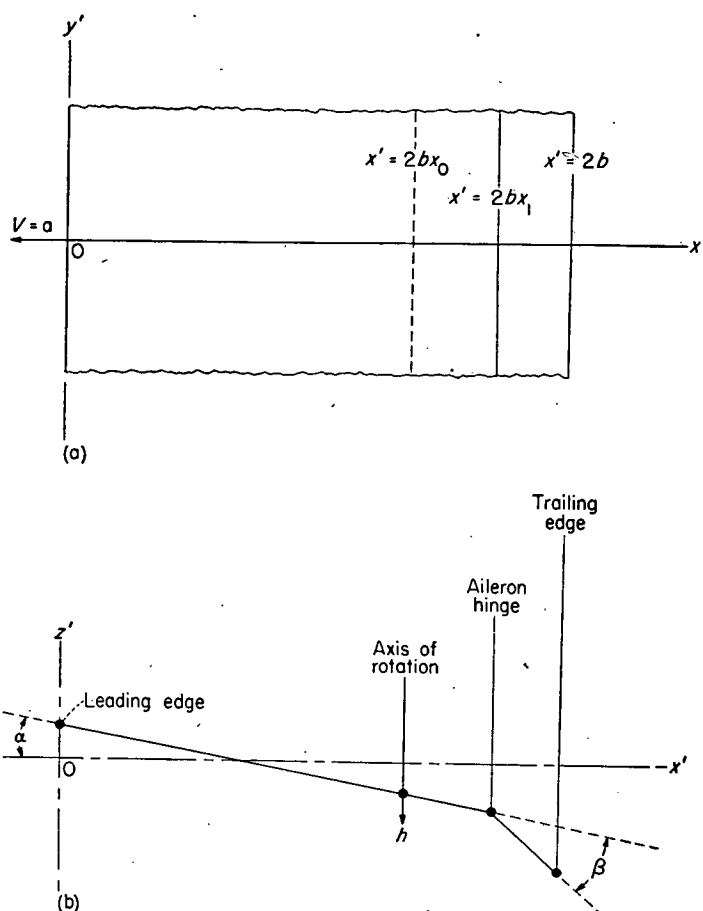
and $x' = 2bx$ and $z' = 2bz$. The mean position of the wing (given by $z' = 0$ and $x' \geq 0$) and the rectangular coordinate system being used are moving at velocity $V = a$ in the negative x' -direction, as shown in figure 1. Since this report is concerned only with the lift of a thin wing, the boundary conditions that equation (6) must satisfy are

$$\left(\frac{\partial \psi}{\partial z'} \right)_{z'=\pm 0} = w(x') \quad (x' \geq 0) \quad (7)$$

$$\psi \rightarrow 0 \text{ as } z' \rightarrow \pm \infty \quad (8)$$

$$\psi = 0 \quad (x' < 0) \quad (9)$$

In accordance with small-disturbance linearized theory the boundary conditions are expressed for the mean position of



(a) Projection of wing strip on $x'y'$ -plane.
(b) Section $y' = 0$.

FIGURE 1.—Sketch illustrating coordinate system and the degrees of freedom α , h , and β .

the wing rather than the wing itself. Equation (7) implies that the normal-velocity distribution on the wing is given; equation (8) is a condition on the behavior at infinity (the manner of approaching zero is associated with the radiation condition of Sommerfeld); equation (9) is the condition that no disturbances be propagated forward of the wing. Since the velocity potential must be continuous, equation (9) implies that

$$\psi(+0, z') = \psi(-0, z') = 0$$

Equations (6) to (9) constitute the boundary-value problem for the velocity potential ϕ .

Applying the Laplace transform

$$\bar{\psi}(s, z') = \int_0^\infty e^{-sx'} \psi(x', z') dx'$$

to equations (6) to (9) yields

$$\left(\frac{d^2 \bar{\psi}}{dz'^2} \right) = \left(\frac{2i\omega}{a} s - \frac{\omega^2}{a^2} \right) \bar{\psi} = \mu^2 \bar{\psi} \quad (10)$$

$$\left(\frac{d\bar{\psi}}{dz'}\right)_{z'=\pm\infty} = w(s) \quad (11)$$

$$\bar{\psi} \rightarrow 0 \text{ as } z' \rightarrow \pm\infty \quad (12)$$

From equations (10) to (12) the value for $\bar{\psi}$ is

$$\bar{\psi} = -\frac{z'}{|z'|} \frac{w(s)e^{-\mu z'}}{\mu} \quad (13)$$

From equation (13) the value for $\bar{\psi}$ at the upper surface of the wing ($z' = +0$) is

$$\bar{\psi} = -\frac{w(s)}{\mu} \quad (14)$$

Applying the inverse transform to equation (14) yields

$$\psi(x', +0) = -\int_0^{x'} w(\xi') G(x' - \xi') d\xi'$$

or

$$\phi(x', +0, t) = -e^{i\omega t} \int_0^{x'} w(\xi') G(x' - \xi') d\xi' \quad (15)$$

where

$$G(x') = \frac{1}{2} \frac{e^{-\frac{i\omega}{2a}x'}}{\sqrt{\pi \frac{i\omega}{2a}x'}} \\ \xi' = 2b\xi$$

Equations (15) and (4) are identical, each giving the velocity potential at the upper surface of the wing.

Application to wing-aileron combination.—For the particular case of the wing-aileron combination oscillating harmonically in vertical translation h , pitch α , and aileron rotation β (see fig. 1(b)), the normal velocity at a point x of the wing chord may be expressed as

$$w(2bx, t) = -[h + V\alpha + 2b(x - x_0)\dot{\alpha} + V\beta + 2b(x - x_1)\dot{\beta}] \quad (16)$$

where

$$h = h_0 e^{i\omega t}$$

$$\alpha = \alpha_0 e^{i\omega t}$$

$$\beta = \beta_0 e^{i\omega t}$$

h_0 , α_0 , and β_0 are complex amplitudes, and the β terms are to be interpreted as zero for $x < x_1$. Since linearized theory is being employed, the potential given in equation (4)

may be considered as the sum of five potentials, each of which is associated with one of the terms of the right-hand side of equation (16). Hence the potential may be written as

$$\phi = \phi_h + \phi_\alpha + \phi_{\dot{\alpha}} + \phi_\beta + \phi_{\dot{\beta}} \quad (17)$$

where upon substituting equation (16) into equation (4)

$$\phi_h = 2bh \int_0^x G(x - \xi) d\xi$$

$$\phi_\alpha = 2bV\alpha \int_0^x G(x - \xi) d\xi$$

$$\phi_{\dot{\alpha}} = 4b^2\dot{\alpha} \int_0^x (\xi - x_0) G(x - \xi) d\xi$$

$$\phi_\beta = 2bV\beta \int_{x_1}^x G(x - \xi) d\xi$$

$$\phi_{\dot{\beta}} = 4b^2\dot{\beta} \int_{x_1}^x (\xi - x_1) G(x - \xi) d\xi$$

Forces and moments.—The velocity potential for the upper wing surface given in equation (4) is antisymmetric with respect to the plane $z=0$, as may be noted in the boundary condition (eq. (7)). The local pressure difference, positive downward, between the upper and lower surfaces of the wing at any point x is obtained from equation (4) by means of

$$\Delta p = -2\rho \left(\frac{\partial \phi}{\partial t} + \frac{V}{2b} \frac{\partial \phi}{\partial x} \right)$$

where ρ is the density of the undisturbed medium. The force (positive downward) acting on a wing section is therefore

$$P = 2b \int_0^1 \Delta p dx \quad (18)$$

The moments (positive leading edge up) acting on the entire wing section about the axis of rotation at x_0 and on the aileron section about the hinge point x_1 are, respectively,

$$M_\alpha = 4b^2 \int_0^1 (x - x_0) \Delta p dx \quad (19)$$

$$M_\beta = 4b^2 \int_{x_1}^1 (x - x_1) \Delta p dx \quad (20)$$

Upon substituting equation (17) into equations (18), (19), and (20) and performing the indicated integrations, the results may be written as

$$\left. \begin{aligned} P &= -4\rho b V^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (L_1 + iL_2) + \alpha_0 (L_3 + iL_4) + \beta_0 (L_5 + iL_6) \right] \\ M_\alpha &= -4\rho b^2 V^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (M_1 + iM_2) + \alpha_0 (M_3 + iM_4) + \beta_0 (M_5 + iM_6) \right] \\ M_\beta &= -4\rho b^2 V^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (N_1 + iN_2) + \alpha_0 (N_3 + iN_4) + \beta_0 (N_5 + iN_6) \right] \end{aligned} \right\} \quad (21)$$

The coefficients of equations (21) can be expressed as follows with primed quantities introduced for convenience in numerical tabulation to denote terms independent of the wing-axis-of-rotation position x_0 (referred to $x_0=0$):

$$\left. \begin{aligned} L_1 + iL_2 &= L_1' + iL_2' \\ L_3 + iL_4 &= L_3' + iL_4' - 2x_0(L_1' + iL_2') \\ L_5 + iL_6 &= L_5' + iL_6' \\ M_1 + iM_2 &= M_1' + iM_2' - 2x_0(L_1' + iL_2') \\ M_3 + iM_4 &= M_3' + iM_4' - 2x_0[(M_1' + iM_2') + (L_3' + iL_4')] + 4x_0^2(L_1' + iL_2') \\ M_5 + iM_6 &= N_5' + iN_6' + 2(x_1 - x_0)(L_5' + iL_6') \\ N_1 + iN_2 &= N_1' + iN_2' + M_1' + iM_2' - 2x_1(L_1' + iL_2') \\ N_3 + iN_4 &= N_3' + iN_4' - 2x_0(N_1' + iN_2') \\ N_5 + iN_6 &= N_5' + iN_6' \end{aligned} \right\} \quad (22)$$

The primed quantities, as a result of integration by parts, can be expressed as

$$\left. \begin{aligned} L_1' + iL_2' &= -\frac{1-i}{r_0} f(r_0) + \frac{1+i}{r_0^2} \sqrt{\frac{r_0}{2\pi}} e^{-ir_0} \\ L_3' + iL_4' &= \frac{1-i}{2r_0} \left(-2 + \frac{2i}{r_0} + \frac{1}{2r_0^2} \right) f(r_0) + \frac{1+i}{2r_0^2} \sqrt{\frac{r_0}{2\pi}} e^{-ir_0} \left(2 - \frac{i}{r_0} \right) \\ L_5' + iL_6' &= (1-x_1)^3 \left[\frac{1-i}{2r_1} \left(-2 + \frac{2i}{r_1} + \frac{1}{2r_1^2} \right) f(r_1) + \frac{1+i}{2r_1^2} \sqrt{\frac{r_1}{2\pi}} e^{-ir_1} \left(2 - \frac{i}{r_1} \right) \right] \\ M_1' + iM_2' &= \frac{1-i}{2r_0} \left(-2 - \frac{1}{2r_0^2} \right) f(r_0) + \frac{1+i}{2r_0^2} \sqrt{\frac{r_0}{2\pi}} e^{-ir_0} \left(2 - \frac{i}{r_0} \right) \\ M_3' + iM_4' &= \frac{1-i}{2r_0} \left(-\frac{8}{3} + \frac{2i}{r_0} - \frac{i}{2r_0^3} \right) f(r_0) + \frac{1+i}{2r_0^2} \sqrt{\frac{r_0}{2\pi}} e^{-ir_0} \left(\frac{8}{3} - \frac{2i}{3r_0} + \frac{1}{r_0^2} \right) \\ N_1' + iN_2' &= x_1^3 \left[\frac{1-i}{2r_2} \left(-2 + \frac{1}{2r_2^2} \right) f(r_2) + \frac{1+i}{2r_2^2} \sqrt{\frac{r_2}{2\pi}} e^{-ir_2} \left(2 + \frac{i}{r_2} \right) \right] \\ N_3' + iN_4' &= x_1^4 \left[\frac{1-i}{2r_2} \left(-\frac{4}{3} + \frac{2i}{r_2} + \frac{1}{r_2^2} + \frac{i}{2r_2^3} \right) f(r_2) + \frac{1+i}{2r_2^2} \sqrt{\frac{r_2}{2\pi}} \left(\frac{4}{3} - \frac{4i}{3r_2} - \frac{1}{r_2^2} \right) e^{-ir_2} \right] + M_3' + iM_4' - 2x_1(L_3' + iL_4') \\ N_5' + iN_6' &= (1-x_1)^4 \left[\frac{1-i}{2r_1} \left(-\frac{8}{3} + \frac{2i}{r_1} - \frac{i}{2r_1^3} \right) f(r_1) + \frac{1+i}{2r_1^2} \sqrt{\frac{r_1}{2\pi}} \left(\frac{8}{3} - \frac{2i}{3r_1} + \frac{1}{r_1^2} \right) e^{-ir_1} \right] \end{aligned} \right\} \quad (23)$$

where

$$r_0 = k$$

$$r_1 = (1-x_1)k$$

$$r_2 = x_1k$$

and the quantities $f(r_j)$ are the Fresnel integrals

$$f(r_j) = \int_0^{r_j} \frac{e^{-ix}}{\sqrt{2\pi x}} dx \quad (j=0,1,2)$$

The primed quantities L_i' and M_i' ($i=1, 2, 3$, and 4), associated with wing bending torsion, are tabulated in table I

as functions of the reduced frequency k for the range $0 < k \leq 3.5$.

The primed quantities L_i' , M_i' ($i=5$ and 6), and N_i' ($i=1, 2, 3, 4, 5$, and 6), introduced by the aileron degree of freedom are tabulated in or can be obtained from table II for the same values of k and for values of the aileron hinge position x_1 ranging from 0.1 to 0.9 in increments of 0.1. In order to make the tabulated values more uniform, each of the primed quantities listed in the tables has been multiplied by the reduced frequency squared k^2 , which appears in the force and moment equations (eqs. (21)).

DISCUSSION

Lift forces and moments.—The lift forces and moments, the coefficients of which are given in table I, apply to a thin, oscillating, two-dimensional wing moving at sonic speed. A comparison of these results with the forces and moments previously obtained for the same type of wing moving at subsonic and supersonic speeds (refs. 5, 7, 8, and other papers) may be of interest.

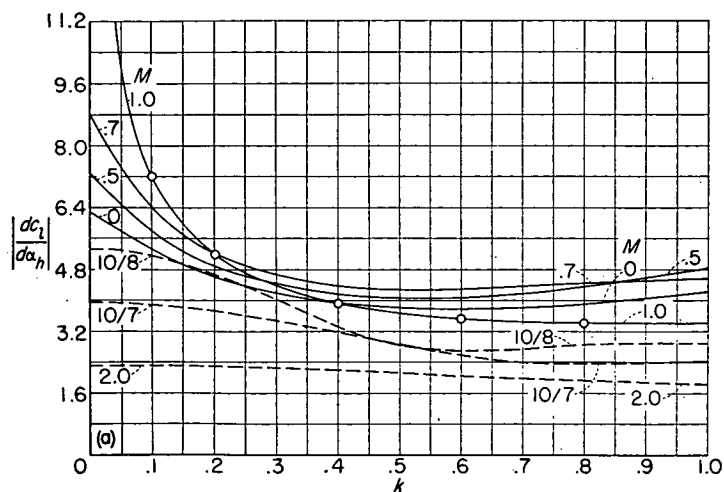
For purposes of comparison, consider the case of a wing pitching about its leading edge and translating vertically. The lift coefficient c_l and the moment coefficient about the leading edge c_m can be expressed as

$$\left. \begin{aligned} c_l &= -\frac{P}{\rho b V^2} = 4[-ik(L_1' + iL_2')\alpha_h + k^2(L_3' + iL_4')\alpha] \\ c_m &= \frac{M_\alpha}{2\rho b^2 V^2} = -2[-ik(M_1' + iM_2')\alpha_h + k^2(M_3' + iM_4')\alpha] \end{aligned} \right\} \quad (24)$$

where $\alpha_h = \frac{h}{V}$ is the angle of attack due to vertical translation and the quantities L_i' and M_i' are now dependent on M as well as k . For the nonoscillating wing in incompressible flow ($k=0, M=0$) $c_l = 2\pi\alpha$ and $c_m = -\frac{\pi}{2}\alpha$. From equation (24) the lift- and moment-curve slopes (complex derivatives) associated with vertical translation and pitching are, respectively,

$$\left. \begin{aligned} \frac{dc_l}{d\alpha_h} &= -i4k(L_1' + iL_2') \\ \frac{dc_m}{d\alpha_h} &= i2k(M_1' + iM_2') \\ \frac{dc_l}{d\alpha} &= 4k^2(L_3' + iL_4') \\ \frac{dc_m}{d\alpha} &= -2k^2(M_3' + iM_4') \end{aligned} \right\} \quad (25)$$

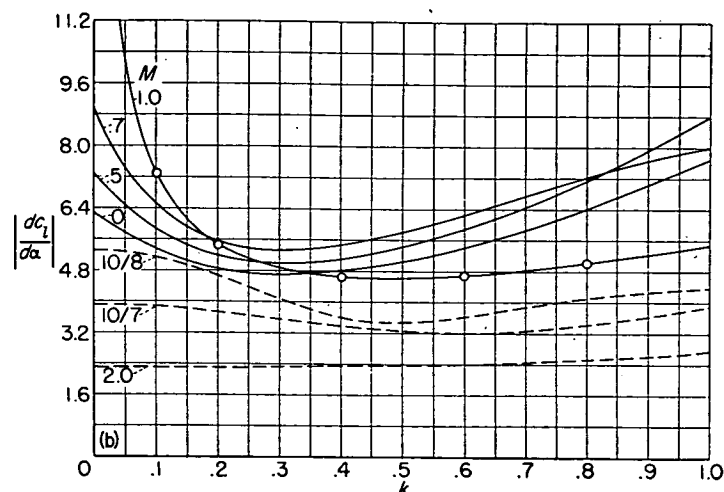
In figure 2 the magnitudes of the slopes given by equation (25) are plotted against k for several values of M , and in figure 3 the associated phase angles are plotted.



(a) Lift-curve slope associated with vertical translation of wing.

$$\left| \frac{dc_l}{d\alpha_h} \right| = 4k\sqrt{L_1'^2 + L_2'^2}$$

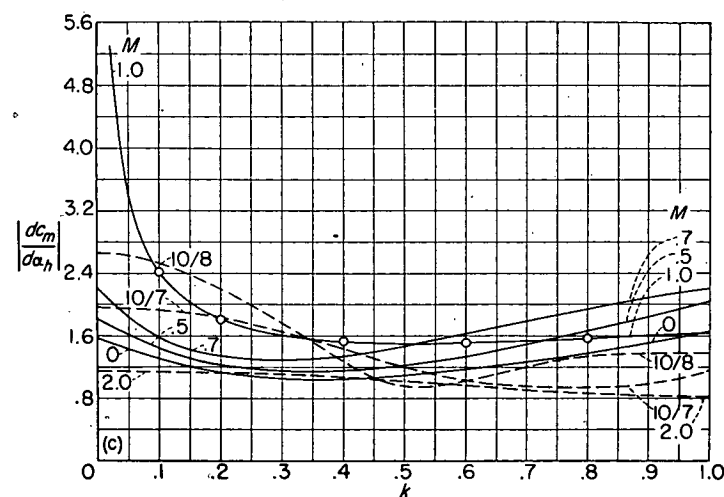
FIGURE 2.—Magnitude of lift-curve slope and moment-curve slope against reduced frequency for several Mach numbers.



(b) Lift-curve slope associated with pitching of wing.

$$\left| \frac{dc_l}{d\alpha} \right| = 4k^2\sqrt{L_3'^2 + L_4'^2}$$

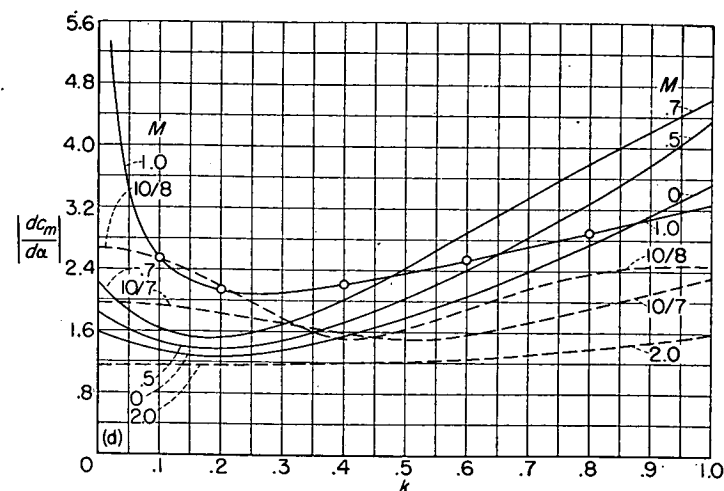
FIGURE 2.—Continued.



(c) Moment-curve slope associated with vertical translation.

$$\left| \frac{dc_m}{d\alpha_h} \right| = 2k\sqrt{M_1'^2 + M_2'^2}$$

FIGURE 2.—Continued.



(d) Moment-curve slope associated with pitching of wing.

$$\left| \frac{dc_m}{d\alpha} \right| = 2k^2\sqrt{M_3'^2 + M_4'^2}$$

FIGURE 2.—Concluded.

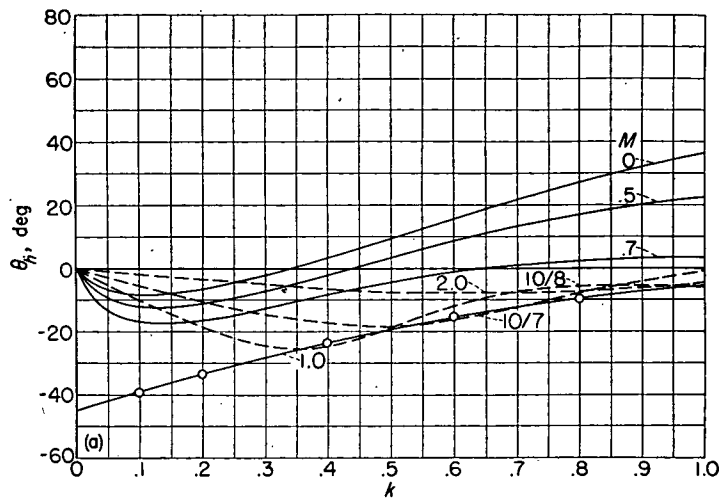

 (a) Phase angle between lift vector due to vertical translation and vertical velocity vector $\bar{h}$.

FIGURE 3.—Phase angles plotted against reduced frequency for several Mach numbers.

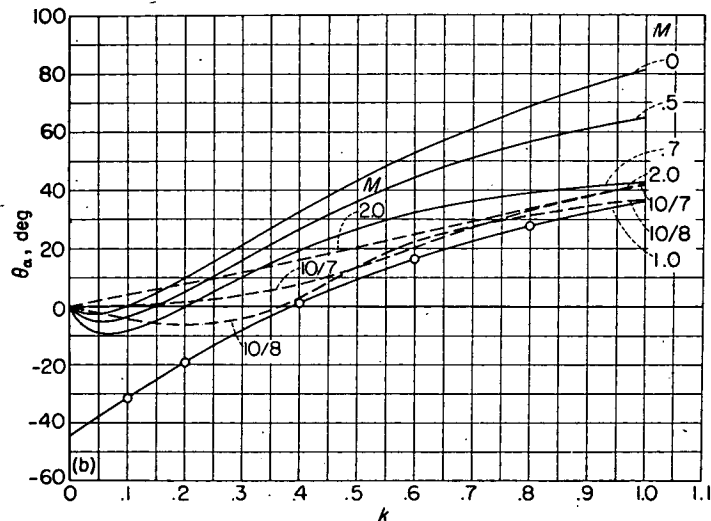

 (b) Phase angle between lift vector due to pitching and angular displacement vector α .

FIGURE 3.—Continued.

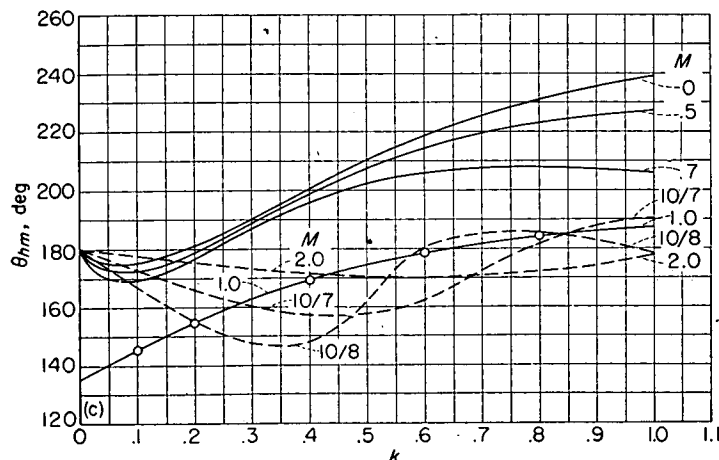

 (c) Phase angle between moment vector due to vertical translation and vertical velocity vector $\bar{h}$.

FIGURE 3.—Continued.

In figures 2 and 3 the dashed curves represent the supersonic results, the solid-line curves represent the subsonic results, and the solid-line curves with several of the computed points circled represent the sonic results.

In figure 2 the variation of slope with Mach number for the steady case (along ordinate $k=0$) is given by the Prandtl-Glauert rule for subsonic speeds and the Ackeret rule for supersonic speeds. Each of these rules predicts an infinite slope at $M=1$. In the figure, the values for the slope magnitude become excessive only for Mach numbers approaching unity and values of k approaching zero. In this neighborhood the linearized theory does not apply, and the Mach number and k range in which the theory is applicable awaits experimental or theoretical determination. In figure 3 the phase-angle curves for $M=1$ depart from those for the other Mach numbers in the low k range. At $k=0$, the phase angle for $M=1$ differs from the constant phase angle of all the other Mach numbers by 45° .

Figure 4 contains a cross plot against Mach number of figure 2 (a) for several values of k . Note that the maximum lift-curve slope occurs at $M=1$ only for small values of k .

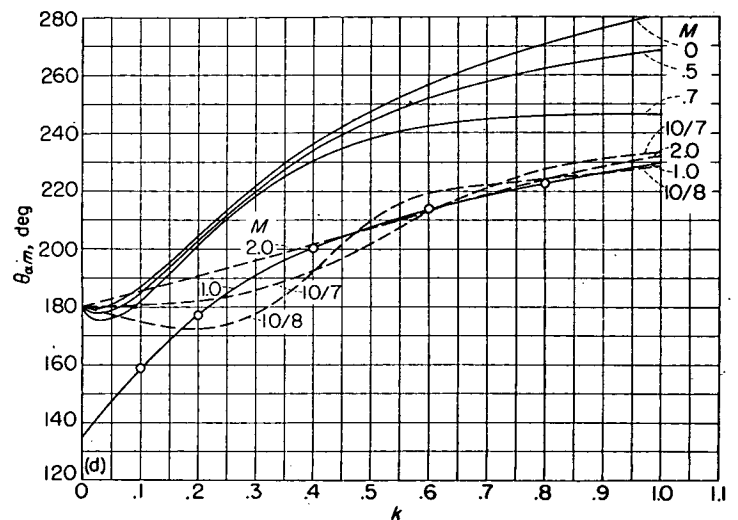
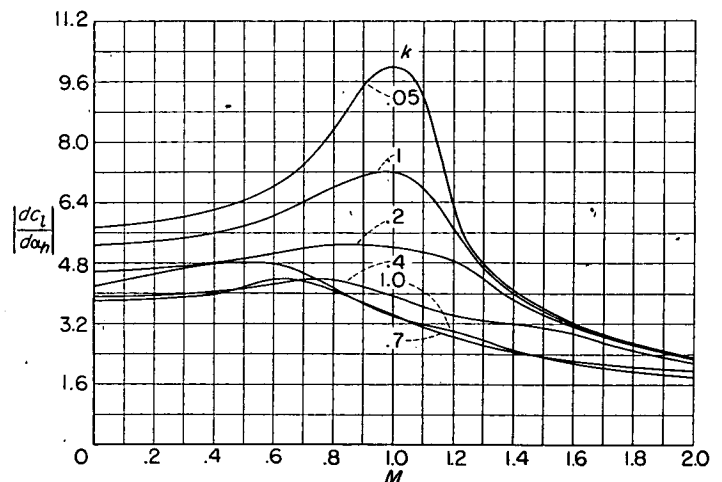

 (d) Phase angle between moment vector due to pitching and angular displacement vector α .

FIGURE 3.—Concluded.


 FIGURE 4.—Magnitude of lift-curve slope associated with vertical translation of wing against Mach number for several values of reduced frequency. $\left| \frac{dc_l}{d\alpha_h} \right| = 4k\sqrt{L_1'^2 + L_2'^2}$.

Above a k of around 0.2, as may also be noted in figure 2 (a), the maximum lift-curve slope for a particular value of k occurs at a Mach number less than 1.

Some applications to bending-torsion flutter.—In reference 5 a systematic numerical study of the bending-torsion flutter of a two-dimensional wing was made including, among other considerations, the effect of Mach number on this type of flutter. The results were presented in the form of figures. Table I of the present report is used to obtain points at $M=1$ for figures 18 and 19 of reference 5. These figures of reference 5 with the $M=1$ points added are presented as figures 5 and 6.

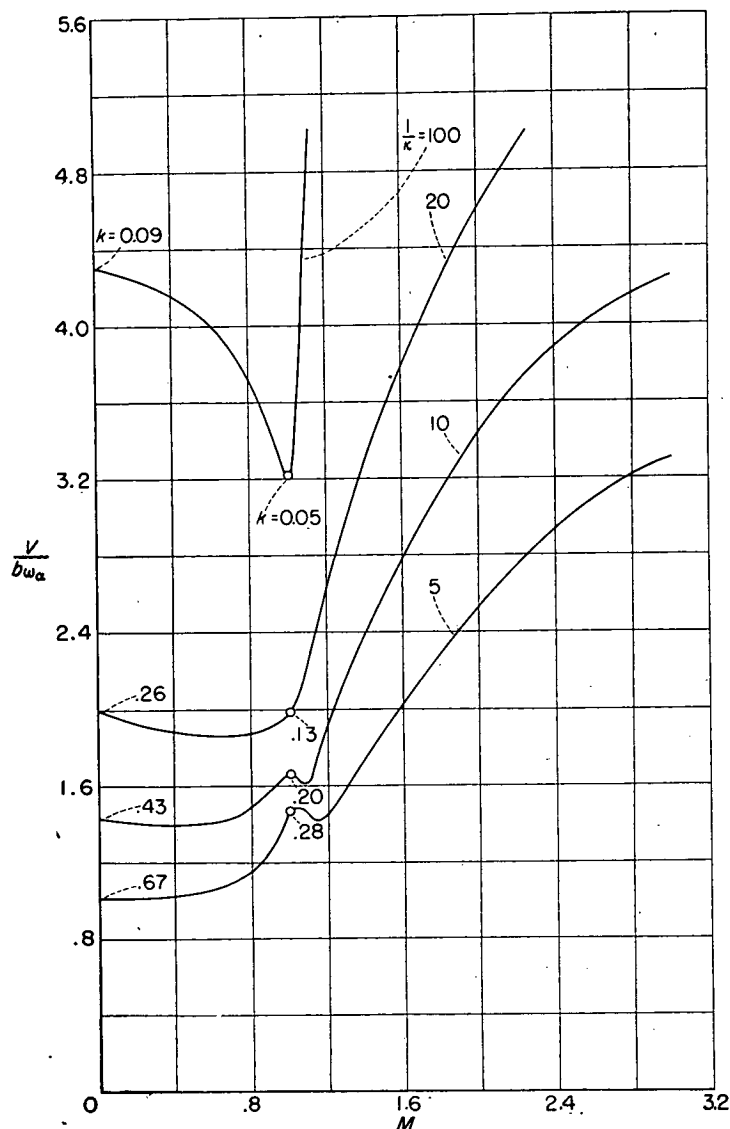


FIGURE 5.—Flutter-speed coefficient against Mach number for several values of $1/\kappa$ when $\frac{\omega_h}{\omega_a}=0$, $x_a=0.2$, and $a=0$. (Fig. 18 of ref. 5 modified to include calculated values indicated by circles.)

In figure 5 the flutter-speed coefficient $V/b\omega_a$ is plotted against Mach number M for several values of the density parameter $1/\kappa$, for wings with the center of gravity at 60 percent chord and the elastic axis at 50 percent chord. The points for Mach number 1, indicated by circles, are consistent with the results of reference 5. As a matter of possible interest some values of the reduced frequency are indicated at $M=0$ and $M=1$.

In figure 6 a plot of the flutter-speed coefficient $V/b\omega_a$ against the ratio of wing bending frequency to wing torsional frequency ω_h/ω_a is shown for several Mach numbers. The curve for $M=1$, calculated points of which are circled, is shown in relation to the curves previously given in reference 5.

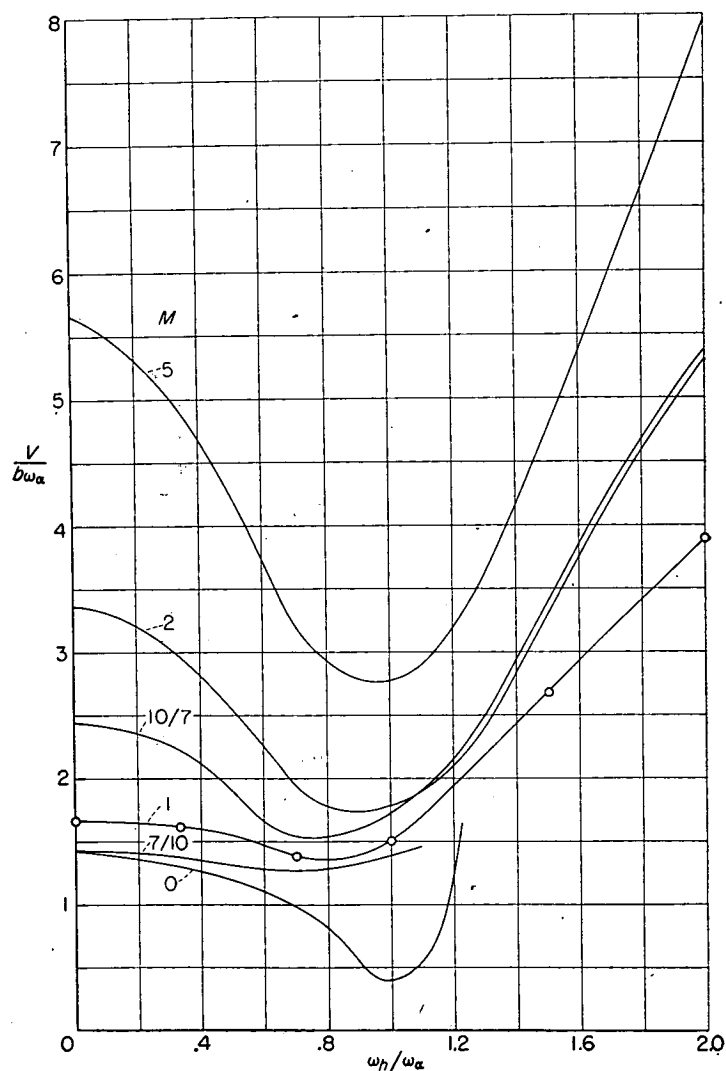


FIGURE 6.—Flutter-speed coefficient against frequency ratio for several values of M when $a=0$, $x_a=0.2$, and $\frac{1}{\kappa}=10$. (Fig. 19 of ref. 5 modified to include calculated values indicated by circles.)

CONCLUDING REMARKS

The linearized theory for compressible unsteady flow has been used to obtain the forces and moments for a thin, harmonically oscillating, two-dimensional wing-aileron combination moving at sonic speed. These forces and moments and the flutter results obtained from them were found to be consistent with similar calculations previously obtained for other Mach numbers. In assessing or applying the results for a Mach number of 1, the limitations associated with linearized theory should be kept in mind. In addition, aspect ratio considerations become increasingly important as the Mach number approaches 1 and may render the two-dimensional results inapplicable to a finite wing even for high frequencies.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., September 4, 1951.

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TABLE I.—VALUES OF FUNCTIONS FOR WING FLUTTER CALCULATIONS

k	$k^2 L_1'$	$k^2 L_2'$	$k^2 L_3'$	$k^2 L_4'$	$k^2 M_1'$	$k^2 M_2'$	$k^2 M_3'$	$k^2 M_4'$
3.5	-0.092268	3.5956	0.97979	3.4024	-0.13699	3.8153	0.92484	4.6249
3.0	-.13457	3.0365	.99531	2.8776	-.25228	3.2404	.93852	3.9240
2.5	-.14084	2.4591	1.0199	2.3562	-.31793	2.6183	.96756	3.2302
2.0	-.10268	1.8845	1.0503	1.8409	-.31343	1.9795	1.0071	2.5514
1.5	-.023461	1.3390	1.0808	1.3296	-.23499	1.3642	1.0459	1.8920
1.0	-.077100	.84912	1.1048	.80532	-.10151	.81581	1.0653	1.2439
.8	.11455	.67354	1.1134	.57971	-.042407	.62412	1.0625	.97907
.6	.14412	.51052	1.1261	.32969	.013043	.45119	1.0525	.69934
.4	.15861	.35939	1.1582	.024216	.057482	.29803	1.0398	.38091
.36	.15871	.33031	1.1709	-.049992	.064095	.26975	1.0387	.30774
.32	.15754	.30149	1.1875	-.13156	.069652	.24221	1.0392	.22918
.28	.15489	.27281	1.2101	-.22295	.073971	.21535	1.0419	.14346
.24	.15050	.24413	1.2414	-.32793	.076821	.18912	1.0488	.047825
.20	.14396	.21524	1.2862	-.45272	.077892	.16338	1.0627	-.062204
.18	.13974	.20060	1.3163	-.52575	.077634	.15065	1.0738	-.12488
.16	.13475	.18577	1.3538	-.60859	.076751	.13796	1.0890	-.19460
.14	.12887	.17064	1.4015	-.70450	.075154	.12527	1.1101	-.27367
.12	.12195	.15510	1.4635	-.81857	.072717	.11249	1.1399	-.36562
.10	.11376	.13898	1.5480	-.95913	.069272	.099537	1.1830	-.47627
.09	.10907	.13061	1.6023	-.1.0437	.067098	.092939	1.2120	-.54159
.08	.10394	.12198	1.6681	-.1.1414	.064570	.086227	1.2482	-.61613
.07	.098250	.11302	1.7495	-.1.2569	.061627	.079356	1.2941	-.70295
.06	.091915	.10364	1.8530	-.1.3969	.058198	.072266	1.3541	-.80683
.05	.084783	.093703	1.9895	-.1.5732	.054173	.064875	1.4352	-.93558
.045	.080848	.088464	2.0756	-.1.6808	.051891	.061029	1.4873	-.1.0133
.04	.076618	.083002	2.1789	-.1.8066	.049394	.057053	1.5503	-.1.1019
.035	.072039	.077263	2.3050	-.1.9566	.046044	.052913	1.6283	-.1.2103
.03	.067037	.071182	2.4632	-.2.1407	.043593	.048509	1.7271	-.1.3402
.025	.061508	.064663	2.6693	-.2.3749	.040169	.043954	1.8573	-.1.5041
.024	.060324	.063291	2.7184	-.2.4300	.039431	.042991	1.8885	-.1.5424
.023	.059116	.061898	2.7708	-.2.4885	.038673	.042013	1.9219	-.1.5830
.022	.057877	.060481	2.8269	-.2.5508	.037893	.041018	1.9576	-.1.6262
.021	.056602	.059032	2.8871	-.2.6173	.037091	.040004	1.9961	-.1.6723
.020	.055296	.057552	2.9519	-.2.6886	.036264	.038972	2.0376	-.1.7216
.019	.053948	.056038	3.0219	-.2.7653	.035411	.037919	2.0825	-.1.7746
.018	.052563	.054490	3.0978	-.2.8480	.034532	.036842	2.1314	-.1.8316
.017	.051133	.052901	3.1804	-.2.9377	.033619	.035741	2.1837	-.1.8934
.016	.049656	.051272	3.2712	-.3.0356	.032676	.034614	2.2432	-.1.9606
.015	.048130	.049595	3.3710	-.3.1428	.031698	.033455	2.3078	-.2.0341
.014	.046544	.047865	3.4814	-.3.2610	.030678	.032264	2.3794	-.2.1150
.013	.044897	.046080	3.6048	-.3.3925	.029617	.031035	2.4595	-.2.2048
.012	.043178	.044227	3.7434	-.3.5395	.028506	.029765	2.5498	-.2.3050
.011	.041382	.042303	3.9012	-.3.7059	.027344	.028448	2.6526	-.2.4183
.010	.039496	.040294	4.0823	-.3.8961	.026118	.027076	2.7710	-.2.5476

TABLE II.—VALUES OF FUNCTIONS FOR AILERON FLUTTER CALCULATIONS²

z_1 (wing chords)	$k^2 L_1'$	$k^2 L_2'$	$k^2 N_1$	$k^2 N_2$	$k^2 N_3'$	$k^2 N_4'$	$k^2 N_5'$	$k^2 N_6'$
$k=3.5$								
0.1	0.89067	2.7314	-0.09339	3.1344	0.74199	3.9401	0.75549	3.3484
.2	.80334	2.1347	-.045042	2.5061	.58009	3.2668	.60700	2.3329
.3	.71591	1.6131	-.0065088	1.9349	.43962	2.6203	.47589	1.5491
.4	.62642	1.1660	.018221	1.4282	.32026	2.0135	.35957	.96683
.5	.53285	.79307	.028928	.99321	.22116	1.4603	.25674	.55513
.6	.43448	.49071	.029296	.63394	.14121	.97433	.16833	.28195
.7	.33082	.25786	.022080	.35459	.07954	.57050	.095843	.11782
.8	.22375	.091798	.012059	.15626	.035526	.26349	.042332	.033681
.9	.11746	-.0069616	.0034955	.038640	.0089560	.068345	.010386	.0028866
$k=3.0$								
0.1	0.90819	2.3077	-0.20128	2.6676	0.75299	3.3431	0.77297	2.8401
.2	.82052	1.8021	-.14342	2.1398	.58836	2.7714	.62394	1.9796
.3	.73081	1.3604	-.092088	1.6590	.44511	2.2224	.48942	1.3160
.4	.63775	.98181	-.051800	1.2308	.32322	1.7075	.36851	.82271
.5	.54039	.66478	.023670	.86078	.22208	1.2382	.26148	.47300
.6	.43845	.40760	-.0068100	.55321	.14093	.82635	.16982	.24055
.7	.33269	.20837	.00094230	.31164	.078792	.48402	.095832	.10013
.8	.22522	.065938	.0025070	.13836	.034885	.22367	.042098	.027973
.9	.11980	-.017583	.0011246	.034469	.0087147	.058061	.010400	.0018736
$k=2.5$								
0.1	0.93119	1.8879	-0.26703	2.1572	0.77781	2.7522	0.79913	2.3397
.2	.84025	1.4727	-.20750	1.7337	.60881	2.2808	.64456	1.6329
.3	.74619	1.1096	-.15129	1.3481	.46103	1.8281	.50353	1.0874
.4	.64844	.79775	-.10298	1.0040	.33475	1.4039	.37653	.68113
.5	.54688	.53640	.064825	.70575	.22984	1.0179	.26473	.39246
.6	.44193	.32213	.036076	.45578	.14533	.67894	.17045	.19903
.7	.33476	.15607	.017119	.25831	.080881	.39763	.095456	.082000
.8	.22755	.037479	.0061219	.11543	.035620	.18376	.041826	.021904
.9	.12326	-.030013	-.0011501	.028958	.0088363	.047711	.010466	.00072869
$k=2.0$								
0.1	0.95660	1.4727	-0.27189	1.6293	0.81216	2.1748	0.82913	1.8511
.2	.86004	1.1456	-.22041	1.3098	.63808	1.8017	.66496	1.2943
.3	.76036	.85876	-.16926	1.0197	.48488	1.4432	.51552	.86344
.4	.65768	.61140	-.12269	.76104	.35314	1.1075	.38209	.51124
.5	.55240	.40268	.082900	.53616	.24290	.80216	.26633	.31098
.6	.44536	.23188	.050992	.34768	.15385	.53476	.17000	.15666
.7	.33782	.098908	.027244	.19788	.085640	.31294	.094720	.062940
.8	.23164	.0048432	.011370	.088860	.037677	.14454	.041592	.015236
.9	.12862	-.045272	-.0026384	.022415	.0093252	.037512	.010627	-.00062204
$k=1.5$								
0.1	0.97994	1.0577	-0.21153	1.1188	0.84629	1.6153	0.85440	1.3749
.2	.87691	.81520	-.17744	.89766	.66784	1.3386	.67928	.96219
.3	.77191	.60170	-.14113	.69811	.50999	1.0719	.52182	.64145
.4	.66539	.41675	-.10616	.52088	.37404	.82193	.38333	.40055
.5	.55793	.26010	.074653	.36716	.25792	.59472	.26516	.22779
.6	.45043	.13188	.047957	.23837	.16412	.39602	.16839	.11189
.7	.34394	.032740	.026874	.13591	.091712	.23148	.093816	.042001
.8	.23960	-.035165	.011820	.061184	.040478	.10679	.041603	.0074945
.9	.13761	-.065466	-.0029068	.015481	.010045	.027684	.010987	-.0023279
$k=1.0$								
0.1	0.99808	0.62511	-0.10111	0.66383	0.86349	1.0685	0.86251	0.90122
.2	.89071	.46377	-.090344	.52958	.68378	.88808	.68000	.62662
.3	.78309	.32129	-.075513	.41001	.52446	.71223	.51857	.41259
.4	.67565	.19781	-.05929	.30482	.38573	.54649	.37888	.25176
.5	.56887	.093697	-.043368	.21425	.26795	.39542	.26141	.13690
.6	.46329	.0096862	.028919	.13880	.17142	.26317	.16637	.060946
.7	.35939	-.052749	.016806	.079021	.096326	.15370	.093606	.016863
.8	.25724	-.090548	.0076650	.035540	.042741	.070828	.042507	-.0024881
.9	.15480	-.095913	-.0019552	.0089892	.010663	.018337	.011830	-.0047628
$k=0.8$								
0.1	1.0058	0.43534	-0.050987	0.50518	0.80067	0.84646	0.85802	0.70455
.2	.89824	.30614	-.049679	.40143	.68186	.70630	.67520	.48465
.3	.79104	.19227	-.043885	.30975	.52352	.56794	.51439	.31359
.4	.68454	.094067	-.03587	.22961	.38556	.43656	.37595	.18566
.5	.57912	.012108	.027084	.16099	.26827	.31627	.25995	.095226
.6	.47503	-.052625	.018542	.10405	.17194	.21067	.16625	.036670
.7	.37242	-.098381	.011024	.059124	.096806	.12310	.094394	.0043043
.8	.27076	-.12172	.0051309	.026545	.043044	.056737	.043561	-.0077843
.9	.16681	-.11414	-.0013332	.0067040	.010760	.014689	.012482	-.0061613

² Corrections to table II of NACA TN 2590 have been incorporated herein.

TABLE II.—VALUES OF FUNCTIONS FOR AILERON FLUTTER CALCULATIONS²—Continued

z_1 (wing chords)	$k^2 L_1'$	$k^2 L_2'$	$k^2 N_1$	$k^2 N_2$	$k^2 N_3'$	$k^2 N_4'$	$k^2 N_5'$	$k^2 N_6'$
$k=0.6$								
0.1	1.0191	0.22164	-0.0032162	0.36259	0.85075	0.61351	0.84920	0.49383
.2	.91274	.12542	-.010365	.28651	.67352	.51660	.66834	.33006
.3	.80716	.041375	-.012874	.22003	.51710	.41810	.50994	.20378
.4	.70250	-.029996	-.012425	.16257	.38102	.32298	.37393	.11079
.5	.59900	-.087916	-.010642	.11344	.26534	.23487	.26002	.046840
.6	.49655	-.13117	-.0078988	.073069	.17025	.15691	.16781	.0076522
.7	.39488	-.15773	-.0049936	.041386	.095990	.091897	.096638	-.011239
.8	.29274	-.16371	-.0024409	.018527	.042746	.042430	.045598	-.014625
.9	.18530	-.13969	-.00066060	.0046667	.010705	.011000	.013541	-.0080683
$k=0.4$								
0.1	1.0538	-0.044994	0.036142	0.23706	0.83622	0.35098	0.84132	0.24927
.2	.95006	-.10525	.022694	.18582	.66002	.30456	.66499	.14668
.3	.84709	-.15606	.013670	.14172	.50573	.25195	.51053	.070298
.4	.74483	-.19675	.0076867	.10395	.37218	.19792	.37757	.017218
.5	.64310	-.22637	.0038858	.072186	.25899	.14588	.26566	-.015551
.6	.54152	-.24344	.0016488	.046250	.16614	.098523	.17424	.031136
.7	.43912	-.24557	.00049747	.026066	.093675	.058224	.10259	.032906
.8	.33362	-.22829	.000045360	.011615	.041740	.027088	.049926	.024645
.9	.21789	-.18066	-.000029310	.0029130	.010457	.0070666	.015504	-.011035
$k=0.36$								
0.1	1.0671	-0.11071	0.042228	0.21401	0.83394	0.29112	0.84152	0.19224
.2	.96403	-.16298	.027934	.16742	.65750	.25654	.66627	.10323
.3	.86190	-.20630	.017950	.12745	.50339	.21455	.51262	.038111
.4	.75972	-.23993	.011013	.093337	.37023	.16994	.38017	-.0057839
.5	.65815	-.26288	.0063034	.064718	.25752	.12607	.26844	-.031221
.6	.55637	-.27363	.0022579	.041408	.16512	.085600	.17685	-.041114
.7	.45330	-.26969	.0014338	.023307	.093079	.050808	.10474	.038632
.8	.34634	-.24642	.00047434	.010373	.041458	.023727	.051359	-.027377
.9	.22775	-.19243	.000080716	.0025984	.010388	.0062091	.016112	-.011873
$k=0.32$								
0.1	1.0845	-0.18335	0.047477	0.19161	0.83258	0.22710	0.84325	0.13065
.2	.98207	-.22718	.032523	.14957	.65555	.20533	.66906	.056622
.3	.88011	-.26250	.021755	.11364	.50139	.17477	.51612	.0028801
.4	.77850	-.28856	.013986	.083076	.36845	.14025	.38401	-.031155
.5	.67690	-.30430	.0084806	.057511	.25611	.10510	.27225	-.048651
.6	.57468	-.30813	.0047159	.036740	.16430	.071940	.18023	-.052319
.7	.47057	-.29747	.0022870	.020650	.092482	.042991	.10745	-.045130
.8	.36168	-.26747	.00086655	.0091777	.041178	.020187	.053123	-.030513
.9	.23952	-.20622	.00018225	.0022962	.010314	.0053115	.016845	-.012849
$k=0.28$								
0.1	1.1078	-0.26523	0.051744	0.16984	0.83284	0.15751	0.84743	0.062999
.2	1.0058	-.30000	.036348	.13225	.65471	.14985	.67411	.0037623
.3	.90422	-.32667	.024976	.10027	.50009	.13180	.52167	-.036387
.4	.80265	-.34446	.016540	.073161	.36710	.10826	.38964	-.059668
.5	.70073	-.35225	.010372	.050555	.25495	.082571	.27754	-.068418
.6	.59769	-.34839	.0059935	.032242	.16328	.057313	.18483	-.065151
.7	.49205	-.33015	.0030403	.018093	.091940	.034632	.11094	-.052654
.8	.38056	-.29244	.0012156	.0080289	.040915	.016414	.055339	-.034189
.9	.25387	-.22272	.00027249	.0020056	.010245	.0043502	.017746	-.014009
$k=0.24$								
0.1	1.1396	-0.35089	0.054843	0.14865	0.83601	0.080237	0.85536	-0.013014
.2	1.0380	-.38474	.039262	.11545	.65583	.088468	.68269	-.055387
.3	.93643	-.40187	.027507	.087322	.50013	.084407	.53034	-.081216
.4	.83456	-.41044	.018594	.063573	.36662	.073094	.39792	-.092506
.5	.73187	-.40928	.011918	.043844	.25432	.057876	.28498	-.091405
.6	.62742	-.39663	.0070537	.027810	.16270	.041317	.19102	-.080231
.7	.51951	-.36962	.0036728	.015634	.091538	.025521	.11556	-.061597
.8	.40444	-.32286	.0015114	.0069261	.040697	.012311	.058193	-.038616
.9	.27184	-.24300	.00034985	.0017273	.010186	.0033103	.018885	-.015424
$k=0.20$								
0.1	1.1847	-0.47320	0.056532	0.12795	0.84412	-0.0082108	0.86976	-0.10115
.2	1.0830	-.48687	.041076	.099080	.66048	.018512	.69696	-.12455
.3	.98104	-.49316	.029205	.074748	.50264	.030608	.54396	-.13410
.4	.87824	-.49112	.020040	.054292	.36780	.033313	.41036	-.13162
.5	.77400	-.47956	.013047	.037358	.25474	.030037	.29575	-.11907
.6	.66724	-.45656	.0078484	.023731	.16275	.023342	.19971	-.098580
.7	.55588	-.41908	.0041572	.013266	.091452	.015313	.12187	-.072616
.8	.43576	-.36130	.0017422	.0058664	.040604	.0077280	.062012	-.044140
.9	.29519	-.26886	.00041096	.0014604	.010154	.0021518	.020376	-.017216

² Corrections to table II of NACA TN 2590 have been incorporated herein.

TABLE II.—VALUES OF FUNCTIONS FOR AILERON FLUTTER CALCULATIONS²—Continued

z_1 (wing chords)	$k^2 L_1'$	$k^2 L_2'$	$k^2 N_1$	$k^2 N_2$	$k^2 N_3'$	$k^2 N_4'$	$k^2 N_5'$	$k^2 N_6'$
$k=0.18$								
0.1	1.2147	-0.53982	0.056745	0.11774	0.85115	-0.058372	0.87830	-0.15168
.2	1.1127	-.54727	.041491	.091031	.66501	-.021025	.70742	-.16446
.3	1.0102	-.54743	.029680	.068581	.50541	.00029990	.55362	-.16484
.4	.90658	-.53940	.020491	.049747	.36942	.010968	.41893	-.15453
.5	.80115	-.52183	.013424	.034188	.25563	.014442	.30300	-.13540
.6	.69268	-.49284	.0081269	.021694	.16318	.013299	.20544	-.10951
.7	.57899	-.44919	.0043332	.012115	.091621	.0096244	.12596	-.079234
.8	.45551	-.38485	.0018283	.0053512	.040672	.0051801	.064450	-.047492
.9	.30973	-.28480	.00043445	.0013308	.010160	.0015101	.021314	-.015316
$k=0.16$								
0.1	1.2518	-0.61565	0.056466	0.10760	0.86126	-0.11399	0.89533	-0.20814
.2	1.1494	-.61627	.041518	.083052	.67174	-.064748	.72110	-.20927
.3	1.0460	-.60969	.029865	.062474	.50980	-.033137	.56604	-.19952
.4	.94113	-.59494	.020730	.045256	.37217	-.013629	.42980	-.18052
.5	.83405	-.57070	.013654	.031063	.25723	-.0026913	.31204	-.15403
.6	.72335	-.53494	.0083118	.019686	.16404	.0022897	.21250	-.12205
.7	.60667	-.48430	.0044564	.010981	.092024	.0033999	.13094	-.086886
.8	.47905	-.41244	.0018911	.0048451	.040814	.0023970	.067382	-.051395
.9	.32712	-.30356	.00045197	.0012036	.010189	.00081027	.022432	-.019606
$k=0.14$								
0.1	1.2988	-0.70384	0.055619	0.097485	0.87569	-0.17684	0.91518	-0.27248
.2	1.1954	-.69678	.041107	.075111	.68167	-.11401	.73931	-.26060
.3	1.0907	-.68261	.029714	.056413	.51650	-.070717	.58228	-.23947
.4	.98410	-.66030	.020725	.040805	.37652	-.041207	.44376	-.21062
.5	.87475	-.62843	.013716	.027971	.25990	-.021854	.32354	-.17574
.6	.76111	-.58488	.0083900	.017703	.16555	-.010000	.22136	-.13676
.7	.64057	-.52610	.0045205	.0098625	.092767	-.0035341	.13712	-.095915
.8	.50774	-.44543	.0019277	.0043467	.041105	-.00069611	.070985	-.056032
.9	.34814	-.32610	.00046303	.0010787	.010252	.000033518	.023794	-.021150
$k=0.12$								
0.1	1.3599	-0.80908	0.054111	0.087342	0.89653	-0.24965	0.94262	-0.34769
.2	1.2548	-.79327	.040180	.067170	.69636	-.17090	.76408	-.32092
.3	1.1481	-.77034	.029172	.050365	.52661	-.11399	.60402	-.28668
.4	1.0390	-.73925	.020435	.036374	.38324	-.072883	.46224	-.24638
.5	.92650	-.69846	.013583	.024898	.26414	-.043813	.33853	-.20170
.6	.80889	-.64572	.0083434	.015736	.16802	-.024051	.23279	-.15447
.7	.68325	-.57728	.0045147	.0087552	.094032	-.011444	.14501	-.10686
.8	.54369	-.48600	.0019336	.0038536	.041617	-.0042175	.075541	-.061694
.9	.37434	-.35395	.00046633	.00095528	.010364	-.00085127	.025498	-.023050
$k=0.10$								
0.1	1.4421	-0.93930	0.051808	0.077093	0.92732	-0.33689	0.98172	-0.43869
.2	1.3345	-.91314	.038635	.059172	.71843	-.23884	.79882	-.39432
.3	1.2246	-.87981	.028163	.044289	.54212	-.16551	.63413	-.34445
.4	1.1118	-.83816	.019806	.031934	.39376	-.11048	.48748	-.29046
.5	.99473	-.78657	.013214	.021824	.27091	-.069811	.35879	-.23390
.6	.87155	-.72260	.0081483	.013774	.17204	-.040637	.24805	-.17656
.7	.73896	-.64222	.0044256	.0076525	.096139	-.020758	.15544	-.12062
.8	.59038	-.53772	.0019023	.0033639	.042488	-.0083561	.081503	-.068866
.9	.40823	-.38961	.00046075	.00083249	.010573	-.0018817	.027709	-.025476
$k=0.09$								
0.1	1.4948	-1.0178	0.050302	0.071892	0.94835	-0.38822	1.0078	-0.49263
.2	1.3853	-.98569	.037586	.055123	.73365	-.27870	.82175	-.43802
.3	1.2734	-.94624	.027450	.041220	.55291	-.19566	.65382	-.37902
.4	1.1580	-.89837	.019339	.029696	.40116	-.13245	.50385	-.31694
.5	1.0379	-.84038	.012926	.020278	.27572	-.084961	.37182	-.25334
.6	.91101	-.76970	.0079844	.012788	.17494	-.050287	.25780	-.18997
.7	.77391	-.68214	.0043442	.0070997	.097670	-.026166	.16205	-.12901
.8	.61956	-.56061	.0018706	.0031187	.043128	-.010754	.085253	-.073265
.9	.42933	-.41167	.00045375	.00077151	.010721	-.0024828	.029092	-.026971
$k=0.08$								
0.1	1.5585	-1.1089	0.048516	0.066611	0.97466	-0.44665	1.0400	-0.55438
.2	1.4467	-.1.0899	.036323	.051020	.75290	-.32399	.84998	-.48821
.3	1.3320	-.1.0236	.026575	.038115	.56666	-.22987	.67789	-.41882
.4	1.2133	-.96858	.018755	.027435	.41064	-.15731	.52376	-.34755
.5	1.0894	-.90323	.012557	.018718	.28193	-.10209	.38758	-.27588
.6	.95808	-.82490	.0077690	.011795	.17869	-.061178	.26952	-.20558
.7	.81555	-.72902	.0042339	.0065434	.099667	-.032263	.16996	-.13882
.8	.65421	-.60713	.0018264	.0028271	.043974	-.013450	.098728	-.078426
.9	.45434	-.43769	.00044387	.00070995	.010923	-.0031549	.030732	-.028735

² Corrections to table II of NACA TN 2593 have been incorporated herein.

TABLE II.—VALUES OF FUNCTIONS FOR AILERON FLUTTER CALCULATIONS²—Continued

x_1 (wing chords)	$k^2 L_5'$	$k^2 L_6'$	$k^2 N_1$	$k^2 N_2$	$k^2 N_3'$	$k^2 N_4'$	$k^2 N_5'$	$k^2 N_6'$
$k=0.07$								
0.1	1.6372	-1.2166	0.046407	0.061221	1.0085	-0.51450	1.0808	-0.62653
.2	1.5222	-1.1697	.034808	.046838	.77768	-.37651	.88543	-.54704
.3	1.4039	-1.1154	.025510	.034957	.58447	-.26945	.70800	-.46564
.4	1.2812	-1.0522	.018033	.025138	.42300	-.18604	.54851	-.38366
.5	1.1525	-.97833	.012093	.017137	.29005	-.12185	.40706	-.30257
.6	1.0155	-.89087	.0074936	.010789	.18363	-.073725	.28394	-.22413
.7	.86612	-.78518	.0040903	.0059805	.10232	-.039270	.17964	-.15051
.8	.69629	-.65224	.0017668	.0026232	.045097	-.016551	.095173	-.084599
.9	.48460	-.46902	.00043011	.00064778	.011194	-.0039242	.032722	-.030851
$k=0.06$								
0.1	1.7369	-1.3476	0.043916	0.055674	1.0528	-0.59555	1.1337	-0.71312
.2	1.6178	-1.2914	.032999	.042548	.81043	-.43902	.93114	-.61787
.3	1.4947	-1.2276	.024225	.031722	.60811	-.31647	.74657	-.52218
.4	1.3665	-1.1546	.017151	.022791	.43945	-.22011	.58003	-.42743
.5	1.2316	-1.0704	.011519	.015522	.30095	-.14523	.43178	-.33504
.6	1.0874	-.97200	.0071485	.0097646	.19029	-.088553	.30216	-.24678
.7	.92934	-.85442	.0039074	.0054083	.10591	-.047545	.19182	-.16484
.8	.74869	-.70790	.0016905	.0023700	.046631	-.020200	.10199	-.092200
.9	.52222	-.50782	.00041195	.00058482	.011560	-.0048319	.035201	-.033467
$k=0.05$								
0.1	1.8681	-1.5127	0.040965	0.049908	1.1131	-0.69565	1.2047	-0.82080
.2	1.7431	-1.4452	.030833	.038098	.85515	-.51610	.99223	-.70625
.3	1.6135	-1.3697	.022671	.028375	.64058	-.37435	.79785	-.59303
.4	1.4779	-1.2845	.016076	.020365	.46220	-.26195	.62178	-.48245
.5	1.3347	-1.1875	.010812	.013857	.31605	-.17391	.46433	-.37600
.6	1.1808	-1.0754	.0067195	.0087093	.19957	-.10670	.32603	-.27545
.7	1.0113	-.94288	.0036780	.0048198	.11093	-.057655	.20770	-.18307
.8	.81660	-.77923	.0015940	.0021103	.048778	-.024646	.11084	-.10190
.9	.57076	-.55760	.00038898	.00052030	.012077	-.0059388	.038400	-.036823
$k=0.045$								
0.1	1.9509	-1.6137	0.039279	0.046915	1.1519	-0.75597	1.2502	-0.88599
.2	1.8220	-1.5394	.029589	.035790	.88412	-.56248	1.0312	-.75988
.3	1.6881	-1.4568	.021773	.026641	.66167	-.40911	.83043	-.63613
.4	1.5478	-1.3643	.015450	.019112	.48308	-.28704	.64818	-.51605
.5	1.3992	-1.2596	.010398	.012999	.32592	-.19108	.48487	-.40107
.6	1.2391	-1.1392	.0064672	.0081656	.20566	-.11754	.34101	-.29306
.7	1.0624	-.99749	.0035423	.0045168	.11423	-.063684	.21765	-.19428
.8	.85866	-.82334	.0015355	.0019771	.050194	-.027313	.11636	-.10789
.9	.60094	-.58847	.00037489	.00048738	.012420	-.0065977	.040395	-.038898
$k=0.04$								
0.1	2.0498	-1.7318	0.037426	0.043824	1.1991	-0.82584	1.3051	-0.96172
.2	1.9162	-1.6498	.028214	.033413	.91933	-.61611	1.0781	-.82232
.3	1.7771	-1.5590	.020776	.024858	.68736	-.44926	.86957	-.68640
.4	1.6310	-1.4580	.014754	.017824	.49510	-.31602	.67986	-.55525
.5	1.4759	-1.3443	.0099366	.012116	.33802	-.21088	.50939	-.43040
.6	1.3085	-1.2143	.0061837	.0076080	.21310	-.13005	.35891	-.31370
.7	1.1230	-1.0619	.0033896	.0042058	.11829	-.070624	.22949	-.20746
.8	.90867	-.87538	.0014699	.0018402	.051938	-.030368	.12293	-.11493
.9	.63667	-.62490	.00035870	.00045342	.012837	-.0073584	.042760	-.041344
$k=0.035$								
0.1	2.1705	-1.8730	0.035377	0.040614	1.2576	-0.90842	1.3728	-1.0516
.2	2.0309	-1.7818	.026693	.030946	.96307	-.67943	1.1358	-.89653
.3	1.8854	-1.6814	.019670	.023010	.71933	-.49062	.91759	-.74623
.4	1.7323	-1.5703	.013978	.016490	.51765	-.35013	.71859	-.60203
.5	1.5691	-1.4460	.0094210	.011203	.35309	-.23417	.53936	-.46546
.6	1.3926	-1.3045	.0058673	.0070813	.22245	-.14473	.38069	-.33841
.7	1.1965	-1.1393	.0032180	.0038855	.12336	-.078792	.24387	-.22326
.8	.96920	-.93804	.0013972	.0016986	.054136	-.033945	.13089	-.12341
.9	.67984	-.66883	.00034181	.00041808	.013371	-.0082444	.045619	-.044291
$k=0.03$								
0.1	2.3217	-2.0464	0.033095	0.037250	1.3320	-1.0087	1.4584	-1.1611
.2	2.1748	-1.9440	.024989	.028365	1.0188	-.75625	1.2086	-.98712
.3	2.0210	-1.8321	.018428	.021080	.78018	-.55399	.97812	-.81945
.4	1.8587	-1.7088	.013105	.015098	.54651	-.39141	.76729	-.65939
.5	1.6854	-1.5714	.0088380	.010253	.37243	-.26235	.57694	-.50852
.6	1.4974	-1.4158	.0055076	.0064317	.23442	-.16248	.40796	-.36881
.7	1.2880	-1.2350	.0030229	.0035523	.12991	-.088626	.26183	-.24274
.8	1.0445	-1.0157	.0013127	.0015530	.056942	-.038279	.14080	-.13388
.9	.73346	-.72327	.00032172	.00038174	.014072	-.0092997	.049169	-.047946

² Corrections to table II of NACA TN 2590 have been incorporated herein.

TABLE II.—VALUES OF FUNCTIONS FOR AILERON FLUTTER CALCULATIONS²—Continued

x_1 (wing chords)	$k^2 L_2'$	$k^2 L_4'$	$k^2 N_1$	$k^2 N_2$	$k^2 N_3'$	$k^2 N_4'$	$k^2 N_5'$	$k^2 N_6'$
$k=0.025$								
0.1	2.5186	-2.2672	0.030530	0.033691	1.4302	-1.1351	1.5709	-1.2995
.2	2.3615	-2.1509	.023067	.025634	1.0926	-.85288	1.3041	-1.1018
.3	2.1968	-2.0244	.017023	.019039	.81427	-.62606	1.0571	-.91231
.4	2.0226	-1.8858	.012113	.013629	.58480	-.44322	.83081	-.73225
.5	1.8360	-1.7319	.0081744	.0092506	.39814	-.29765	.62581	-.56335
.6	1.6329	-1.5584	.0050971	.0058001	.25036	-.18471	.44336	-.40761
.7	1.4061	-1.3578	.0027995	.0032017	.13863	-.10094	.28509	-.26766
.8	1.1415	-1.1152	.0012168	.0013987	.060719	-.043673	.15361	-.14728
.9	.80256	-.79325	.00029798	.00034401	.014969	-.010652	.053757	-.052627
$k=0.024$								
0.1	2.5654	-2.3191	0.029968	0.032942	1.4538	-1.1646	1.5979	-1.3319
.2	2.4060	-2.1996	.022650	.025065	1.1103	-.87540	1.3268	-1.1288
.3	2.2386	-2.0697	.016717	.018614	.81026	-.65969	1.0760	-.93410
.4	2.0614	-1.9274	.011897	.013323	.59403	-.45529	.84591	-.74943
.5	1.8717	-1.7698	.0080300	.0090426	.40435	-.30586	.63746	-.57629
.6	1.6650	-1.5921	.0050076	.0056691	.25422	-.18987	.45176	-.41676
.7	1.4341	-1.3867	.0027499	.0031298	.14072	-.10382	.29060	-.27355
.8	1.1646	-1.1388	.0011959	.0013666	.061661	-.044896	.15664	-.15046
.9	.81890	-.80980	.00029266	.00033636	.015205	-.010948	.054835	-.053743
$k=0.023$								
0.1	2.6154	-2.3744	0.029397	0.032188	1.4789	-1.1959	1.6266	-1.3663
.2	2.4533	-2.2513	.022222	.024488	1.1293	-.89930	1.3512	-1.1573
.3	2.2832	-2.1179	.016403	.018183	.84122	-.66067	1.0961	-.95728
.4	2.1030	-1.9717	.011675	.013014	.60391	-.46808	.86206	-.76758
.5	1.9098	-1.8100	.0078810	.0088317	.41099	-.31458	.64988	-.58999
.6	1.6993	-1.6279	.0049150	.0055365	.25834	-.19535	.46075	-.42646
.7	1.4640	-1.4176	.0026999	.0030559	.14299	-.10683	.29650	-.27979
.8	1.1890	-1.1638	.0011738	.0013347	.062634	-.046232	.15988	-.15382
.9	.83630	-.82741	.00028746	.00032830	.015457	-.011263	.055995	-.054921
$k=0.022$								
0.1	2.6689	-2.4331	0.028810	0.031420	1.5060	-1.2292	1.6575	-1.4028
.2	2.5040	-2.3065	.021781	.023901	1.1496	-.92473	1.3774	-1.1877
.3	2.3308	-2.1691	.016080	.017745	.85620	-.67963	1.1177	-.98189
.4	2.1474	-2.0190	.011447	.012699	.61449	-.48169	.87938	-.78694
.5	1.9506	-1.8529	.0077275	.0086171	.41810	-.32384	.66318	-.60456
.6	1.7360	-1.6661	.0048202	.0054014	.26275	-.20118	.47035	-.43681
.7	1.4959	-1.4505	.0026477	.0029813	.14539	-.11008	.30280	-.28645
.8	1.2152	-1.1905	.0011512	.0013020	.063661	-.047664	.16335	-.15741
.9	.85489	-.84618	.00028163	.00032059	.015677	-.011643	.057233	-.056173
$k=0.021$								
0.1	2.7263	-2.4959	0.028205	0.030639	1.5352	-1.2645	1.6906	-1.4419
.2	2.5584	-2.3653	.021326	.023305	1.1715	-.95181	1.4054	-1.2201
.3	2.3820	-2.2240	.015747	.017300	.87230	-.69978	1.1409	-.98082
.4	2.1949	-2.0695	.011211	.012379	.62591	-.49617	.89792	-.80765
.5	1.9942	-1.8988	.0075693	.0083984	.42579	-.33369	.67742	-.62018
.6	1.7752	-1.7069	.0047218	.0052642	.26754	-.20737	.48065	-.44788
.7	1.5300	-1.4856	.0025940	.0029055	.14801	-.11350	.30955	-.29356
.8	1.2432	-1.2191	.0011283	.0012685	.064818	-.049141	.16705	-.16125
.9	.87481	-.86630	.00027604	.00031230	.015974	-.011994	.058556	-.057515
$k=0.020$								
0.1	2.7880	-2.5632	0.027582	0.029843	1.5665	-1.3025	1.7264	-1.4836
.2	2.6169	-2.4285	.020859	.022696	1.1951	-.98076	1.4357	-1.2547
.3	2.4370	-2.2828	.015404	.016846	.88968	-.72136	1.1659	-.9363
.4	2.2461	-2.1237	.010968	.012053	.63824	-.51164	.91792	-.82980
.5	2.0411	-1.9480	.0074064	.0081768	.43408	-.34422	.69276	-.63688
.6	1.8174	-1.7508	.0046212	.0051240	.27269	-.21399	.49172	-.45972
.7	1.5667	-1.5234	.0025391	.0028277	.15082	-.11717	.31681	-.30120
.8	1.2733	-1.2498	.0011040	.0012350	.065984	-.050796	.17104	-.16537
.9	.89624	-.88788	.00027078	.00030336	.016262	-.012401	.059980	-.058960
$k=0.019$								
0.1	2.8548	-2.6357	0.026938	0.029032	1.6005	-1.3432	1.7651	-1.5285
.2	2.6801	-2.4965	.020375	.022076	1.2208	-.91018	1.4684	-1.2921
.3	2.4964	-2.3461	.015049	.016384	.90853	-.74445	1.1929	-.91066
.4	2.3013	-2.1820	.010717	.011721	.65164	-.52822	.93954	-.85362
.5	2.0918	-2.0011	.0072381	.0079503	.44309	-.35549	.70933	-.65485
.6	1.8629	-1.7980	.0045165	.0049818	.27829	-.22108	.50367	-.47251
.7	1.6063	-1.5641	.0024821	.0027487	.15390	-.12108	.32464	-.30943
.8	1.3058	-1.2828	.0010796	.0012001	.067334	-.052500	.17533	-.16981
.9	.91929	-.91116	.00026495	.00029461	.016595	-.012818	.061514	-.060518

² Corrections to table II of NACA TN 2590 have been incorporated herein.

TABLE II.—VALUES OF FUNCTIONS FOR AILERON FLUTTER CALCULATIONS²—Continued

τ_1 (wing chords)	$k^2 L_1'$	$k^2 L_2'$	$k^2 N_1$	$k^2 N_2$	$k^2 N_3'$	$k^2 N_4'$	$k^2 N_5'$	$k^2 N_6'$
$k=0.018$								
0.1	2.9271	-2.7139	0.026272	0.028204	1.6375	-1.3871	1.8071	-1.5768
.2	2.7486	-2.5699	.019874	.021444	1.2486	-1.0452	1.5038	-1.3323
.3	2.5607	-2.4144	.014681	.015914	.92904	-.76934	1.2209	-1.0993
.4	2.3612	-2.2451	.010456	.011383	.66621	-.54607	.96296	-.87937
.5	2.1467	-2.0583	.0070622	.0077206	.45289	-.36764	.72728	-.67428
.6	1.9122	-1.8490	.0044074	.0048376	.28440	-.22870	.51665	-.48629
.7	1.6492	-1.6081	.0024221	.0026691	.15725	-.12529	.33310	-.31833
.8	1.3410	-1.3186	.0010533	.0011654	.068795	-.054332	.17999	-.17462
.9	.94427	-.93636	.00025855	.00028609	.016983	-.013238	.063170	-.062205
$k=0.017$								
0.1	3.0059	-2.7987	0.025583	0.027357	1.6779	-1.4345	1.8530	-1.6292
.2	2.8232	-2.6496	.019356	.020797	1.2791	-1.0814	1.5426	-1.3759
.3	2.6308	-2.4887	.014300	.015431	.95150	-.79625	1.2541	-1.1347
.4	2.4263	-2.3135	.010187	.011037	.68216	-.56537	.98850	-.90732
.5	2.2064	-2.1206	.0068811	.0074851	.46364	-.38073	.74683	-.69539
.6	1.9658	-1.9044	.0042948	.0046893	.29111	-.23691	.53072	-.50130
.7	1.6958	-1.6559	.0023608	.0025869	.16094	-.12982	.34232	-.32802
.8	1.3792	-1.3575	.0010270	.0011292	.070418	-.056294	.18504	-.17985
.9	.97142	-.96373	.00025201	.00027727	.017388	-.013714	.064967	-.064042
$k=0.016$								
0.1	3.0922	-2.8913	0.024870	0.026488	1.7222	-1.4861	1.9033	-1.6862
.2	2.9048	-2.7364	.018820	.020134	1.3125	-1.1208	1.5850	-1.4233
.3	2.7075	-2.5695	.013906	.014938	.97610	-.82552	1.2891	-1.1733
.4	2.4976	-2.3881	.0099067	.010683	.69962	-.58637	1.0165	-.93768
.5	2.2717	-2.1884	.0066931	.0072438	.47542	-.39501	.76831	-.71834
.6	2.0244	-1.9649	.0041779	.0045379	.29842	-.24589	.54620	-.51761
.7	1.7468	-1.7081	.0022970	.0025028	.16495	-.13479	.35246	-.33851
.8	1.4210	-1.3999	.0009935	.0010925	.072133	-.058488	.19059	-.18553
.9	1.0011	-.99364	.00024528	.00026819	.017780	-.014281	.066939	-.066043
$k=0.015$								
0.1	3.1871	-2.9925	0.024129	0.025598	1.7712	-1.5426	1.9588	-1.7485
.2	2.9948	-2.8916	.018262	.019455	1.3494	-1.1638	1.6319	-1.4752
.3	2.7918	-2.6584	.013495	.014432	1.0033	-.85752	1.3277	-1.2155
.4	2.5760	-2.4701	.0090154	.010320	.71894	-.60828	1.0473	-.97099
.5	2.3436	-2.2628	.0064969	.0069973	.48843	-.41063	.79191	-.74351
.6	2.0889	-2.0312	.0040561	.0043828	.30652	-.25569	.56320	-.53548
.7	1.8028	-1.7654	.0022300	.0024172	.16937	-.14022	.36358	-.35006
.8	1.4669	-1.4465	.00097034	.0010550	.074050	-.060865	.19668	-.19177
.9	1.0337	-1.0265	.00023810	.00025902	.018238	-.014878	.069109	-.068234
$k=0.014$								
0.1	3.2924	-3.1042	0.023357	0.024682	1.8255	-1.6047	2.0203	-1.8172
.2	3.0943	-2.9367	.017680	.018757	1.3905	-1.2111	1.6838	-1.5324
.3	2.8853	-2.7563	.013068	.013912	1.0335	-.89270	1.3705	-1.2621
.4	2.6629	-2.5603	.0093118	.0099470	.74045	-.63449	1.0814	-1.0077
.5	2.4230	-2.3451	.0062926	.0067438	.50292	-.42775	.81805	-.77128
.6	2.1603	-2.1045	.0039288	.0042238	.31554	-.26642	.58200	-.55525
.7	1.8648	-1.8286	.0021603	.0023293	.17433	-.14615	.37587	-.36282
.8	1.5177	-1.4980	.00094017	.0010165	.076211	-.063451	.20341	-.19869
.9	1.0697	-1.0627	.00023065	.00024963	.018772	-.015509	.071516	-.070646
$k=0.013$								
0.1	3.4097	-3.2286	0.022555	0.023739	1.8864	-1.6735	2.0891	-1.8934
.2	3.2053	-3.0533	.017074	.018037	1.4364	-1.2635	1.7417	-1.5960
.3	2.9894	-2.8651	.012621	.013378	1.0674	-.93165	1.4181	-1.3138
.4	2.7594	-2.6607	.0089943	.0095637	.76454	-.66238	1.1195	-1.0486
.5	2.5115	-2.4365	.0060789	.0064832	.51920	-.44665	.84715	-.80219
.6	2.2398	-2.1860	.0037957	.0040601	.32570	-.27828	.60292	-.57725
.7	1.9339	-1.8991	.0020873	.0022387	.17993	-.15267	.38955	-.37704
.8	1.5743	-1.5553	.00090843	.00097697	.078678	-.066270	.21090	-.20638
.9	1.1098	-1.1031	.00022294	.00023984	.019430	-.016150	.074147	-.073381
$k=0.012$								
0.1	3.5417	-3.3676	0.021712	0.022764	1.9549	-1.7505	2.1667	-1.9786
.2	3.3301	-3.1841	.016439	.017294	1.4882	-1.3221	1.8071	-1.6669
.3	3.1065	-2.9871	.012153	.012825	1.1056	-.97518	1.4720	-1.3716
.4	2.8682	-2.7734	.0086626	.0091670	.79170	-.69356	1.1624	-1.0942
.5	2.6112	-2.5390	.0058552	.0062137	.53749	-.46784	.87996	-.83673
.6	2.3291	-2.2775	.0036566	.0038909	.33710	-.29157	.62656	-.60182
.7	2.0114	-1.9780	.0020110	.0021455	.18618	-.16003	.40496	-.39290
.8	1.6379	-1.6196	.00087519	.00093623	.081361	-.069513	.21933	-.21498
.9	1.1549	-1.1484	.00021465	.00023000	.020058	-.016976	.077175	-.076378

² Corrections to table II of NACA TN 2590 have been incorporated herein.

TABLE II.—VALUES OF FUNCTIONS FOR AILERON FLUTTER CALCULATIONS²—Concluded

z_1 (wing chords)	$k^2 L_3'$	$k^2 L_6'$	$k^2 N_1$	$k^2 N_2$	$k^2 N_3'$	$k^2 N_4'$	$k^2 N_5'$	$k^2 N_6'$
$k=0.011$								
0.1	3.6917	-3.5250	0.020830	0.021752	2.0329	-1.8373	2.2549	-2.0748
.2	3.4719	-3.3322	.015774	.016523	1.5471	-1.3881	1.8814	-1.7472
.3	3.2395	-3.1252	.011663	.012251	1.1491	-1.0243	1.5331	-1.4370
.4	2.9917	-2.9010	.0083139	.0087565	.82262	-.72876	1.2112	-1.1458
.5	2.7242	-2.6551	.0056202	.0059347	.55834	-.49176	.91728	-.87576
.6	2.4304	-2.3810	.0035105	.0037155	.35009	-.30658	.65338	-.62964
.7	2.0995	-2.0674	.0019308	.0020485	.19327	-.16835	.42248	-.41087
.8	1.7099	-1.6923	.00084042	.00089384	.084414	-.077263	.23988	-.23588
.9	1.2059	-1.1998	.00020593	.00021976	.020733	-.017948	.080575	-.079806
$k=0.010$								
0.1	3.8640	-3.7050	0.019900	0.020700	2.1229	-1.9363	2.3564	-2.1847
.2	3.6347	-3.5015	.015072	.015722	1.6152	-1.4636	1.9668	-1.8390
.3	3.3922	-3.2832	.011145	.011656	1.1993	-1.0802	1.6034	-1.5117
.4	3.1334	-3.0469	.0079462	.0083296	.85836	-.76880	1.2672	-1.2048
.5	2.8538	-2.7880	.0053724	.0056448	.58247	-.51893	.96003	-.92053
.6	2.5467	-2.4996	.0033557	.0035340	.36512	-.32364	.68415	-.66149
.7	2.2004	-2.1698	.0018459	.0019482	.20154	-.17775	.44253	-.43150
.8	1.7924	-1.7758	.00080360	.00084993	.088031	-.077263	.23988	-.23588
.9	1.2645	-1.2586	.00019691	.00020898	.021638	-.018934	.084451	-.083757

² Corrections to table II of NACA TN 2590 have been incorporated herein.

REPORT 1129

TRANSVERSE VIBRATIONS OF HOLLOW THIN-WALLED CYLINDRICAL BEAMS¹

By BERNARD BUDIANSKY and EDWIN T. KRUSZEWSKI

SUMMARY

The variational principle, differential equations, and boundary conditions considered appropriate to the analysis of transverse vibrations of hollow thin-walled cylindrical beams are shown. General solutions for the modes and frequencies of cantilever and free-free cylindrical beams of arbitrary cross section but of uniform thickness are given. The combined influence of the secondary effects of transverse shear deformation, shear lag, and longitudinal inertia is shown in the form of curves for cylinders of rectangular cross section and uniform thickness. The contribution of each of the secondary effects to the total reduction in the actual frequency is also indicated.

INTRODUCTION

The elementary theory of bending vibration is often inadequate for the accurate calculation of natural modes and frequencies of hollow, thin-walled cylindrical beams. Such secondary effects as transverse shear deformation, shear lag, and longitudinal inertia, which are not considered in the elementary theory of lateral oscillations, can have appreciable influence, particularly on the higher modes and frequencies of vibration. The effects of transverse shear deformation and of rotary (rather than longitudinal) inertia have been studied by many on the basis of the original investigations of Rayleigh (ref. 1) and Timoshenko (ref. 2). Anderson and Houbolt (ref. 3) have presented a procedure for including the effects of shear lag in the numerical calculation of modes and frequencies of box beams of rectangular cross section. However, there does not appear to exist a general solution for the vibration of hollow beams that incorporates the influence of all the secondary effects mentioned.

The purpose of the present report is threefold: First, to exhibit the variational principle, differential equations, and boundary conditions appropriate for the analysis of the uncoupled bending vibration of hollow thin-walled cylindrical beams; second, to give general solutions for cantilever and free-free cylinders of arbitrary cross section but of uniform thickness; and finally, to show quantitatively the influence

of the secondary effects by means of numerical results for hollow beams of rectangular cross section of various lengths, widths, and depths.

SYMBOLS

A	cross-sectional area
A_n	Fourier coefficient
A_s	effective shear-carrying area
B_i	parameter defined in equation (30)
C	constant
E	modulus of elasticity
G	shear modulus of elasticity
I	moment of inertia
K	geometrical parameter defined in equation (29)
L	length of cantilever beam, half-length of free-free beam
N_i	parameter defined in equation (38)
T	maximum kinetic energy
U	maximum strain energy
a	half-depth of rectangular beam
b	half-width of rectangular beam
a_{mn}, b_n	Fourier series coefficients
i, j, m, n	integers
k_B	frequency coefficient, $\omega \sqrt{\frac{\mu L^4}{EI}}$
k_s	coefficient of shear rigidity, $\frac{1}{L} \sqrt{\frac{EI}{A_s G}}$
k_{RI}	coefficient of rotary inertia, $\frac{1}{L} \sqrt{\frac{I}{A}}$
p	perimeter of cross section
s	distance along periphery of cross section (see fig. 1)
t	wall thickness
$u(x, s)$	longitudinal displacement in x -direction
$w(x)$	vertical displacement in y -direction
x	longitudinal coordinate
y	vertical coordinate
$\bar{y}$	y -coordinate of center of gravity of cross section
γ_{xs}	shear strain

¹ Supersedes NACA TN 2682, "Transverse Vibrations of Hollow Thin-Walled Cylindrical Beams" by Bernard Budiansky and Edwin T. Kruszewski, 1952.

ϵ_x	longitudinal strain
θ	inclination of normal with vertical (see fig. 1)
λ	Lagrangian multiplier
μ	mass of beam per unit length
ρ	mass density of beam
σ	longitudinal direct stress
τ	shear stress
ω	natural frequency of beam
ω_0	natural frequency of beam calculated from elementary beam theory
δ_{ij}	Kronecker delta (1 if $i=j$; 0 if $i \neq j$)
ϕ	constraining relationship

BASIC EQUATIONS

Assumptions.—The problem to be considered is that of the natural bending vibration of a thin-walled hollow cylindrical beam whose cross section is symmetrical about at least one axis (see fig. 1). The transverse vibration is supposed to take place in the direction of this axis of symmetry of the cross section so that no torsional oscillations are induced.

In the present analysis, the following simplifications are introduced:

(a) Changes in the size and shape of the cross section are neglected.

(b) Stress and strain are assumed to be uniform across the wall thickness.

(c) The small effect of circumferential stress upon longitudinal strain is neglected.

In accordance with statements (a) and (b), the distortions of the vibrating beam are completely described by the vertical displacement $w(x)$ of a cross section and the longitudinal

displacement $u(x,s)$ of each point of the median line of the beam wall.

The longitudinal and shear strains are given in terms of $u(x,s)$ and $w(x)$ as

$$\epsilon_x = \frac{\partial u}{\partial x} \quad (1)$$

and

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{dw}{dx} \sin \theta \quad (2)$$

and the corresponding stresses become

$$\sigma_x = E \frac{\partial u}{\partial x} \quad (3)$$

and

$$\tau_{xs} = G \left(\frac{\partial u}{\partial s} + \frac{dw}{dx} \sin \theta \right) \quad (4)$$

where θ is the inclination of the normal with the vertical (see fig. 1).

In elementary beam theory, where the effects of all shear distortion are neglected, the longitudinal distortion $u(x,s)$ is related to the vertical displacement $w(x)$ by

$$u(x,s) = (\bar{y} - y) \frac{dw}{dx}$$

where $\bar{y}$ is the y -coordinate of the center of gravity of the cross section. In the present report, however, $u(x,s)$ is allowed to be perfectly general, so that shear distortions (and consequently the so-called shear-lag and transverse-shear-deformation effects) are fully taken into account. Furthermore, because cross sections are not constrained to remain plane, the inertia effect associated with motion in the longitudinal direction is more properly designated as the effect of *longitudinal* inertia than the effect of *rotary* inertia.

Variational principle and geometrical boundary conditions.—The variational equation to be written is appropriate to beams whose ends are either fixed, simply supported, or free. For some such beam vibrating in a natural mode, the maximum strain energy is

$$U = \frac{1}{2} \int_0^L \oint E \left(\frac{\partial u}{\partial x} \right)^2 t ds dx + \frac{1}{2} \int_0^L \oint G \left(\frac{\partial u}{\partial s} + \frac{dw}{dx} \sin \theta \right)^2 t ds dx \quad (5)$$

where $u(x,s)$ and $w(x)$ are the amplitudes of displacement for the particular mode considered. The maximum kinetic energy is

$$T = \frac{1}{2} \int_0^L \oint \rho t \omega^2 w^2 ds dx + \frac{1}{2} \int_0^L \oint \rho t \omega^2 u^2 ds dx \quad (6)$$

where ω is the natural frequency of the mode under consideration and ρ is the mass density of the beam. The second term in equation (6) constitutes the contribution of longitudinal inertia to the kinetic energy.

A natural mode of vibration must satisfy the variational

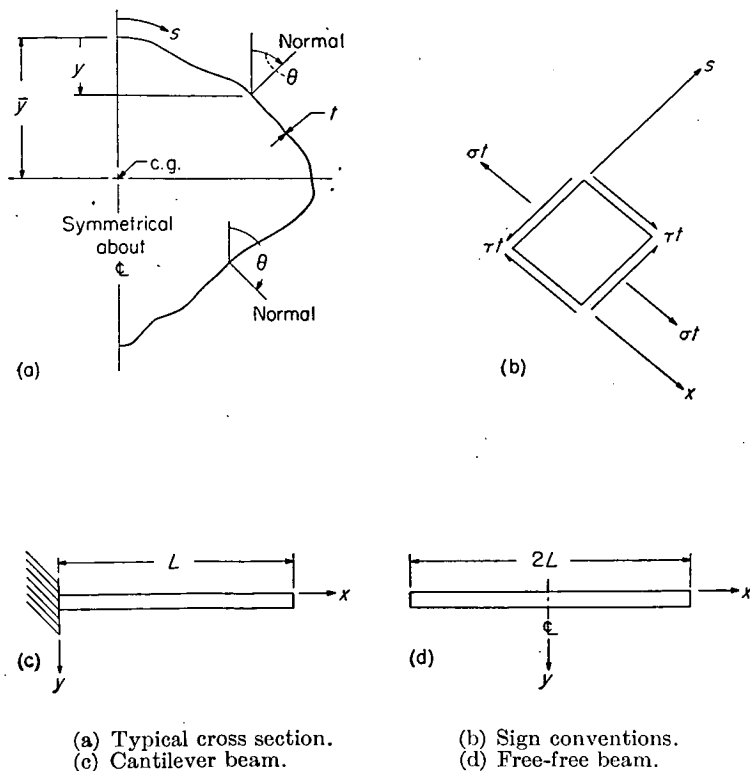


FIGURE 1.—Coordinate systems and sign conventions.

equation

$$\delta(U-T)=0 \quad (7)$$

where the variation is taken independently with respect to $u(x,s)$ and $w(x)$ and with the provision that both $u(x,s)$ and $w(x)$ must satisfy the geometrical boundary conditions of the problem; furthermore, $u(x,s)$ must be periodic in the coordinate s with a period equal to the perimeter p . The geometrical boundary conditions are $w=0$ and $u=0$ at a fixed end and only $w=0$ at a simply supported end. At a free end no geometrical boundary conditions are imposed.

Differential equations and natural boundary conditions.—Equations (5), (6), and (7) in conjunction with the usual procedure of the calculus of variations yield the following simultaneous integrodifferential equations for u and w :

$$Et \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial s} \left[Gt \left(\frac{\partial u}{\partial s} + \frac{dw}{dx} \sin \theta \right) \right] + \rho t \omega^2 u = 0 \quad (8)$$

$$\oint Gt \left(\frac{\partial^2 u}{\partial s \partial x} \sin \theta + \frac{d^2 w}{dx^2} \sin^2 \theta \right) ds + \mu \omega^2 w = 0 \quad (9)$$

where

$$\mu = \oint \rho t ds \quad (10)$$

and the boundary equations at each end of the beam are

$$\oint Et \left(\frac{\partial u}{\partial x} \right) \delta u ds = 0 \quad (11)$$

$$\oint Gt \left(\frac{\partial u}{\partial s} + \frac{dw}{dx} \sin \theta \right) \sin \theta ds \delta w = 0 \quad (12)$$

At a fixed end, both boundary equations (11) and (12) are satisfied by virtue of the fact that the geometrical boundary conditions require that both δu and δw be zero. At a simply supported end $\delta w=0$, but, since $\delta u(x,s)$ is perfectly arbitrary, the variational process forces the equality

$$Et \frac{\partial u}{\partial x} = 0 \quad (13)$$

Finally, at a free end, since there are no geometrical constraints, both δu and δw are arbitrary and hence the variational process forces, in addition to equation (13), the equality

$$\oint Gt \left(\frac{\partial u}{\partial s} + \frac{dw}{dx} \sin \theta \right) \sin \theta ds = 0 \quad (14)$$

Equations (13) and (14) constitute so-called "natural boundary conditions" because they are automatically satisfied as the result of a variational process. Equation (13) is recognized as the condition of zero longitudinal direct stress while equation (14) simply stipulates that the total vertical shear force vanish.

Thus to summarize, the appropriate boundary conditions required for the solution of equations (8) and (9) are

Fixed end:

$$w=0$$

$$u=0$$

Simply supported end:

$$w=0$$

$$Et \frac{\partial u}{\partial x} = 0$$

Free end:

$$\oint Gt \left(\frac{\partial u}{\partial s} + \frac{dw}{dx} \sin \theta \right) \sin \theta ds = 0$$

$$Et \frac{\partial u}{\partial x} = 0$$

The integrodifferential equations (8) and (9), which specify equilibrium in the longitudinal and transverse directions respectively, can, of course, be written directly without recourse to the variational principle.

GENERAL SOLUTIONS FOR CYLINDERS OF UNIFORM WALL THICKNESS

The following exact solutions for cylinders of uniform wall thickness are carried out by means of Fourier series in conjunction with the application of the variational condition (eq. (7)). This procedure, which does not require explicit consideration of the natural boundary conditions, was believed to be more expedient than a direct attack upon the simultaneous integrodifferential equations (8) and (9) and all their associated boundary conditions.

Cantilever beam.—The geometrical boundary conditions, for a cantilever beam, as previously shown, are

$$w(0)=u(0,s)=0$$

(see fig. 1). Appropriate assumptions for the displacements $w(x)$ and $u(x,s)$ are

$$w(x) = C + \sum_{n=1,3,5}^{\infty} b_n \cos \frac{n\pi x}{2L} \quad (15)$$

and

$$u(x,s) = \sum_{m=1,3,5}^{\infty} \sum_{n=0,1,2}^{\infty} a_{mn} \sin \frac{m\pi x}{2L} \cos \frac{2n\pi s}{p} \quad (16)$$

The condition $u(0,s)=0$ is satisfied by each term of equation (16); the condition

$$w(0) = C + \sum_{n=1,3,5}^{\infty} b_n = 0 \quad (17)$$

is introduced into the variational procedure by means of the Lagrangian multiplier method. The choice of the particular trigonometric functions used in the Fourier series (15) and (16) was guided by consideration of the orthogonality required for the simplification of expressions in the strain energy. The constant C is needed in the expression for $w(x)$ in order that $w(L)$ be unrestricted.

Using equations (15) and (16) in equations (5) and (6) yields

$$U - T =$$

$$\begin{aligned} & \frac{1}{2} \int_0^L \oint Et \left(\sum_{m=1,3,5} \sum_{n=0,1,2} a_{mn} \frac{m\pi}{2L} \cos \frac{m\pi x}{2L} \cos \frac{2n\pi s}{p} \right)^2 ds dx + \\ & \frac{1}{2} \int_0^L \oint Gt \left(\sum_{m=1,3,5} \sum_{n=0,1,2} -a_{mn} \frac{2n\pi}{p} \sin \frac{m\pi x}{2L} \sin \frac{2n\pi s}{p} + \right. \\ & \left. \sin \theta \sum_{n=1,3,5} -b_n \frac{n\pi}{2L} \sin \frac{n\pi x}{2L} \right)^2 ds dx - \\ & \frac{1}{2} \int_0^L \oint \omega^2 \rho t \left(\sum_{n=1,3,5} b_n \cos \frac{n\pi x}{2L} + C \right)^2 ds dx - \\ & \frac{\omega^2}{2} \int_0^L \oint \rho t \left(\sum_{m=1,3,5} \sum_{n=0,1,2} a_{mn} \sin \frac{m\pi x}{2L} \cos \frac{2n\pi s}{p} \right)^2 ds dx \quad (18) \end{aligned}$$

To make equation (18) stationary and at the same time satisfy the constraining relationship

$$\varphi = C + \sum_{n=1,3,5} b_n = 0 \quad (19)$$

it is sufficient to set

$$\delta(U - T - \lambda \varphi) = 0 \quad (20)$$

where the variation is with respect to the a 's, b 's, and C considered as independent variables; here λ is a Lagrangian multiplier. This variational process results in the following equations:

$$\begin{aligned} \frac{\partial(U - T)}{\partial b_i} - \lambda \frac{\partial \varphi}{\partial b_i} &= \sum_{n=0,1,2} Gt \frac{in\pi^2}{2} \frac{A_n}{2} a_{in} + G \left(\frac{i\pi}{2L} \right)^2 \frac{A_s L}{2} b_i - \\ & \mu \omega^2 \frac{L}{2} b_i - (-1)^{\frac{i-1}{2}} \frac{2L}{i\pi} \mu \omega^2 C - \lambda \\ &= 0 \quad (i=1, 3, 5, \dots) \quad (21) \end{aligned}$$

$$\begin{aligned} \frac{\partial(U - T)}{\partial a_{ij}} &= Et \left(\frac{i\pi}{2L} \right)^2 \frac{Lp}{4} (1 + \delta_{0j}) a_{ij} + Gt \left(\frac{2j\pi}{p} \right)^2 \frac{Lp}{4} a_{ij} + \\ & Gt \frac{ij\pi^2}{4} A_j b_i - \omega^2 \rho t \frac{Lp}{4} (1 + \delta_{0j}) a_{ij} = 0 \\ & \quad (i=1, 3, 5, \dots) \\ & \quad (j=0, 1, 2, \dots) \quad (22) \end{aligned}$$

$$\begin{aligned} \frac{\partial(U - T)}{\partial C} - \lambda \frac{\partial \varphi}{\partial C} &= -\mu \omega^2 \sum_{n=1,3,5} \frac{2L}{n\pi} (-1)^{\frac{n-1}{2}} b_n - \mu \omega^2 LC - \lambda \\ &= 0 \quad (23) \end{aligned}$$

where

$$A_n = \frac{2}{p} \oint \sin \theta \sin \frac{2n\pi s}{p} ds \quad (24)$$

$$A_s = \oint t \sin^2 \theta ds \quad (25)$$

With the use of the nondimensional parameters

$$k_B^2 = \frac{\mu L^4}{EI} \omega^2 \quad (26)$$

$$k_s^2 = \frac{EI}{A_s G L^2} \quad (27)$$

$$k_{Ri}^2 = \frac{I}{pt L^2} = \frac{I}{A L^2} \quad (28)$$

$$K^2 = \frac{16I}{A_s p^2} \quad (29)$$

and

$$B_i^2 = i^2 - k_{Ri}^2 k_B^2 \left(\frac{2}{\pi} \right)^2 \quad (30)$$

equations (21), (22), and (23) may be reduced to

$$\begin{aligned} \sum_{n=0,1,2} \frac{in\pi^2}{4k_s^2} \frac{Lt}{A_s} A_n a_{in} + \frac{1}{2} \left(\frac{i\pi}{2} \right)^2 \frac{1}{k_s^2} b_i - \frac{1}{2} k_B^2 b_i - \\ (-1)^{\frac{i-1}{2}} \frac{2}{i\pi} k_B^2 C - \frac{L^3 \lambda}{EI} = 0 \quad (i=1, 3, 5, \dots) \quad (31) \end{aligned}$$

$$\begin{aligned} (k_s^2 B_i^2 + K^2 j^2) (1 + \delta_{0j}) a_{ij} + K^2 \frac{p}{4L} A_j i j b_i = 0 \\ (i=1, 3, 5, \dots) \\ (j=0, 1, 2, \dots) \quad (32) \end{aligned}$$

$$k_B^2 \sum_{n=1,3,5} \frac{2}{n\pi} (-1)^{\frac{n-1}{2}} b_n + k_B^2 C + \frac{L^3 \lambda}{EI} = 0 \quad (33)$$

For $j=0$, equation (32) becomes

$$k_s^2 \left[i^2 - k_B^2 k_{Ri}^2 \left(\frac{2}{\pi} \right)^2 \right] a_{i0} = 0 \quad (i=1, 3, 5, \dots) \quad (34)$$

Equation (34) is not coupled to any of equations (31) to (33). A given value of a_{i0} corresponds to the amplitude of the i th mode of longitudinal oscillation, and if this value of a_{i0} is not equal to 0, then equation (34) simply gives the frequency of this longitudinal mode. Consequently those equations in equation (32) for values of $j=0$ are not associated with transverse bending and so are ignored henceforth. For the remaining values of j (that is, $j \neq 0$) equation (32) yields

$$a_{ij} = \frac{-K^2 \frac{p}{4L} A_j i j}{k_s^2 B_i^2 + K^2 j^2} b_i \quad \begin{matrix} (i=1, 3, 5, \dots) \\ (j=1, 2, 3, \dots) \end{matrix} \quad (35)$$

Substituting the expression for a_{ij} in equation (35) into equation (31) and solving for b_i gives

$$b_i = \frac{(-1)^{\frac{i-1}{2}} \frac{2}{i\pi} k_B^2 C + \frac{L^3 \lambda}{EI}}{N_i} \quad (i=1, 3, 5, \dots) \quad (36)$$

where

$$N_i = \frac{i^2 \pi^2}{8} \frac{1}{k_s^2} - \sum_{n=1,2,3}^{\infty} \frac{\frac{\pi^2}{16} \frac{K^2}{k_s^2} \frac{A}{A_s} A_n^2 (in)^2}{k_s^2 B_i^2 + K^2 n^2} - \frac{1}{2} k_B^2 \quad (37)$$

In the appendix this expression for N_i is shown to be equivalent to

$$N_i = \frac{i^2 \pi^4}{32} B_i^2 - \frac{i^2 \pi^2}{16} \frac{A}{A_s} B_i^4 k_s^2 \sum_{n=1,2,3}^{\infty} \frac{A_n^2}{K^2 n^2 (k_s^2 B_i^2 + K^2 n^2)} - \frac{1}{2} k_B^2 \quad (38)$$

Since the series in equation (38) is considerably more quickly convergent than that in equation (37), equation (38) should be used in actual numerical calculations of N_i .

Substitution of equation (36) into equation (33) and the constraining-relationship equation (19) gives the following two homogeneous equations in C and λ :

$$k_B^2 \left[1 + k_B^2 \sum_{n=1,3,5}^{\infty} \left(\frac{2}{n\pi} \right)^2 \frac{1}{N_n} \right] C + \left[1 + k_B^2 \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{2}{n\pi} \frac{1}{N_n} \right] \frac{L^3 \lambda}{EI} = 0 \quad (39a)$$

$$\left[1 + k_B^2 \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{2}{n\pi} \frac{1}{N_n} \right] C + \left(\sum_{n=1,3,5}^{\infty} \frac{1}{N_n} \right) \frac{L^3 \lambda}{EI} = 0 \quad (39b)$$

Finally the condition for a nontrivial solution for C and λ gives the frequency equation

$$\begin{vmatrix} k_B^2 \left[1 + k_B^2 \sum_{n=1,3,5}^{\infty} \left(\frac{2}{n\pi} \right)^2 \frac{1}{N_n} \right] & 1 + k_B^2 \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{2}{n\pi} \frac{1}{N_n} \\ 1 + k_B^2 \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{2}{n\pi} \frac{1}{N_n} & \sum_{n=1,3,5}^{\infty} \frac{1}{N_n} \end{vmatrix} = 0 \quad (40)$$

which the frequency parameter k_B must satisfy. Since the terms of the infinite series which appear in the frequency equation contain k_B itself, the roots of equation (40) are most conveniently found by trial. Fortunately the infinite series in equation (40) as well as the series in the definition of N_i converge rapidly so that only a few terms are needed to evaluate them with sufficient accuracy.

Once k_B has been determined for a particular mode, the corresponding mode shape can be found by letting $C=1$ and solving either of equations (39) for λ and then finally evaluating b_i and a_{ij} successively from equations (36) and (35).

Free-free beam—symmetrical modes.—If the origin of a free-free beam of length $2L$ is taken at the midspan (see fig. 1), the form of the Fourier series assumed for $w(x)$ and $u(x,s)$ when the beam is undergoing a symmetrical mode of vibration may be exactly the same as that assumed for the cantilever beam of length L (see eqs. (15) and (16)). The only difference in the ensuing calculations is that the constraining condition (19) is not introduced. Consequently, it can be readily seen that the frequency equation for the

symmetrically vibrating free-free beam is obtained from equation (39a) by setting $\lambda=0$ and is

$$k_B^2 \left[1 + k_B^2 \sum_{n=1,3,5}^{\infty} \left(\frac{2}{n\pi} \right)^2 \frac{1}{N_n} \right] = 0 \quad (41)$$

After a particular root k_B is found from equation (41), the shape of the corresponding symmetrical free-free mode may be obtained from equations (36) (with $\lambda=0$) and equations (35).

Free-free beam—antisymmetrical modes.—Consider a free-free beam of length $2L$ undergoing antisymmetrical vibrations. Explicit consideration need be given only to the right half of the beam (see fig. 1), and for this half-beam the only geometrical boundary condition that must be imposed is that $w(0)=0$. The spanwise displacement $u(0,s)$ is unrestrained by virtue of antisymmetry.

Appropriate assumptions for the displacements $w(x)$ and $u(x,s)$ are then

$$w(x) = \sum_{n=2,4,6}^{\infty} b_n \sin \frac{n\pi x}{2L} + Cx \quad (42)$$

and

$$u(x, s) = \sum_{m=0,2,4}^{\infty} \sum_{n=1,2,3}^{\infty} a_{mn} \cos \frac{m\pi x}{2L} \cos \frac{2n\pi s}{p} \quad (43)$$

The linear portion Cx of the expression for $w(x)$ is needed in order to give the beam sufficient freedom at the tip ($x=L$). The choice of the particular trigonometric function in the series expansion for $u(x, s)$ was, as in the case of the cantilever beam, guided by consideration of the orthogonality required for the simplification of the expressions in the strain energy. The zeroth term in the series for $u(x, s)$ in the s -direction was omitted because it only leads to the frequency equation for longitudinal oscillations.

Using equations (42) and (43) in equations (5) and (6) yields

$$U - T = \frac{1}{2} \int_0^L \oint Et \left(\sum_{m=0,2,4}^{\infty} \sum_{n=1,2,3}^{\infty} -a_{mn} \frac{m\pi}{2L} \sin \frac{m\pi x}{2L} \cos \frac{2n\pi s}{p} \right)^2 ds dx + \frac{1}{2} \int_0^L \oint Gt \left[\sum_{m=0,2,4}^{\infty} \sum_{n=1,2,3}^{\infty} -a_{mn} \frac{2n\pi}{p} \cos \frac{m\pi x}{2L} \sin \frac{2n\pi s}{p} + \sin \theta \left(\sum_{n=2,4,6}^{\infty} b_n \frac{n\pi}{2L} \cos \frac{n\pi x}{2L} + C \right) \right]^2 ds dx - \frac{1}{2} \int_0^L \oint \rho t \omega^2 \left(\sum_{n=2,4,6}^{\infty} b_n \sin \frac{n\pi x}{2L} + Cx \right)^2 ds dx - \frac{1}{2} \int_0^L \oint \rho t \omega^2 \left(\sum_{m=0,2,4}^{\infty} \sum_{n=1,2,3}^{\infty} a_{mn} \cos \frac{m\pi x}{2L} \cos \frac{2n\pi s}{p} \right)^2 ds dx \quad (44)$$

The variation of equation (44) with respect to the a 's, b 's, and C gives, after suitable simplification,

$$(B_i^2 k_s^2 + K^2 j^2) a_{ij} - K^2 \frac{p}{4L} A_j i j b_i = 0 \quad (i=2, 4, 6, \dots) \\ (j=1, 2, 3, \dots) \quad (45)$$

$$\left[K^2 j^2 - k_s^2 k_{RI}^2 k_B^2 \left(\frac{2}{\pi} \right)^2 \right] a_{0j} - K^2 \frac{p}{2\pi} A_j j C = 0 \\ (j=1, 2, 3, \dots) \quad (46)$$

$$\sum_{n=1,2,3}^{\infty} \frac{1}{k_s^2} \frac{Lt}{A_s} \frac{i n \pi^2}{4} A_n a_{in} - \frac{1}{2} \frac{1}{k_s^2} \left(\frac{i \pi}{2} \right)^2 b_i + \frac{1}{2} k_B^2 b_i - \\ (-1)^{i/2} \frac{2}{i \pi} k_B^2 C L = 0 \quad (i=2, 4, 6, \dots) \quad (47)$$

$$\sum_{n=1,2,3}^{\infty} \frac{1}{k_s^2} \frac{Lt}{A_s} n \pi A_n a_{0n} - \frac{CL}{k_s^2} - \\ k_B^2 \sum_{n=2,4,6}^{\infty} \frac{2}{n \pi} (-1)^{n/2} b_n + \frac{1}{3} k_B^2 C L = 0 \quad (48)$$

From equation (45)

$$a_{ij} = \frac{K^2 \frac{p}{4L} A_j i j}{B_i^2 k_s^2 + K^2 j^2} b_i \quad (i=2, 4, 6, \dots) \\ (j=1, 2, 3, \dots) \quad (49)$$

which, except for sign, is the same expression as that obtained for the cantilever and symmetrically vibrating free-free beams (eq. (35)). From equation (46)

$$a_{0j} = \frac{K^2 \frac{p}{2\pi} A_j j}{B_0^2 k_s^2 + K^2 j^2} C \quad (j=1, 2, 3, \dots) \quad (50)$$

Substitution of equation (49) into equation (47) gives

$$b_i = -(-1)^{i/2} \frac{2}{i \pi} \frac{k_B^2}{N_i} C L \quad (i=2, 4, 6, \dots) \quad (51)$$

where N_i is defined in equation (37).

Substitution of equations (50) and (51) into equation (48) and simplification gives as the frequency equation for the antisymmetrically vibrating free-free beam

$$k_B^2 \left[\sum_{n=2,4,6}^{\infty} \left(\frac{2}{n \pi} \right)^2 \frac{k_B^2}{N_n} + \frac{2}{\pi^2} \frac{A}{A_s} k_{RI}^2 \sum_{n=1,2,3}^{\infty} \frac{A_n^2}{B_0^2 k_s^2 + K^2 n^2} + \frac{1}{3} \right] = 0 \quad (52)$$

After a particular value of k_B is found from equation (52), the shape of the corresponding antisymmetrical free-free mode may be obtained by giving C the arbitrary value of unity and calculating the b 's and a 's successively from equations (51), (50), and (49).

Discussion of parameters.—The parameters entering in the frequency equations merit discussion. The unknown natural frequency is contained only in the frequency coefficient k_B , which is defined by the formula $\omega = k_B \sqrt{\frac{EI}{\mu L^4}}$ and is in common use in beam-vibration analysis. The parameters k_s and k_{RI} are identical with the shear and inertia parameters defined in reference 4, which considers the effect of only transverse shear and rotary inertia on beam vibrations. The quantity A_s which appears in the present definition of k_s is actually the effective shear-carrying area when plane sections are constrained to remain plane; that is, when shear lag is neglected. The remaining parameters appearing in the present derivation, namely, A/A_s , K , and $A_1, A_2, \dots$ are essentially shape parameters which actually depend only on the contour of the cross section; as shown in the appendix,

$$\frac{A_s}{A} = \frac{1}{2} \sum_{n=1,2,3}^{\infty} A_n^2$$

and

$$K^2 = \frac{2}{\pi^2} \frac{A}{A_s} \sum_{n=1,2,3}^{\infty} \frac{A_n^2}{n^2}$$

and the A_n 's are simply the Fourier coefficients of the function $\sin \theta$, which is dependent only on the shape of the

cross section. These shape parameters are related to shear-lag effects and their interaction with transverse shear and longitudinal inertia.

The effect of longitudinal inertia is associated with the parameter k_{RI} . If the effect of longitudinal inertia is to be neglected, it is sufficient to set k_{RI} equal to zero in the final frequency equation. If k_{RI} is equal to zero, B_i becomes independent of k_B . Appreciable simplification in a trial-and-error solution for the natural frequency then results since, with B_i independent of k_B , the infinite summation contained in N_i is also independent of k_B and need be calculated only once for any particular beam. As is shown in the following section, the effect of disregarding the influence of longitudinal inertia may often be negligible.

Without presentation of details, it may be mentioned that for the case of a circular cylinder, which has no shear lag, all the A_n 's except A_1 vanish and the frequency equations (40), (41), and (52) may be put into closed forms identical to those given in reference 4. Again, if in the general frequency equations k_s is set equal to zero, the equations may be put into closed forms equivalent to those of reference 4 where only rotary inertia is considered.

RESULTS FOR CYLINDRICAL BEAMS OF RECTANGULAR CROSS SECTION

In order to show quantitatively the effects of shear lag, transverse shear deformation, and longitudinal inertia on the natural frequencies of hollow thin-walled cylindrical

beams, numerical calculations have been performed for cylinders of rectangular cross section oscillating as free-free beams. The calculations have been limited to symmetrical modes of vibration, and consequently the frequency equation (41) is applicable. For rectangular cross sections the quantity N_i may be put into closed form as shown in the appendix, and this closed-form version of N_i was used in the calculations. A value of E/G equal to 2.65 (appropriate for aluminum alloys) was assumed.

The results of these calculations are shown in figures 2, 3, and 4. In figure 2, the ratio of the natural frequency ω to the natural frequency ω_0 obtained from elementary beam theory is shown as a function of the plan-form aspect ratio L/b for cross-sectional aspect ratios of 1.0, 3.6, and ∞ . The contribution of each of the secondary effects to the total reduction in the natural frequency for the cross-sectional aspect ratios $\frac{b}{a}=3.6$ and 1.0 can be seen in figures 3 and 4,

respectively. The cross-sectional aspect ratio of $\frac{b}{a}=\infty$ corresponds to the limiting case of a beam where the effects of transverse shear deformation and longitudinal inertia are negligible and therefore the reduction in natural frequency is due entirely to shear lag.

The dashed lines in figures 3 and 4 show the reduction in frequency due to the inclusion of the effect of only transverse shear deformation as obtained from reference 4.

The long- and short-dash lines are calculated from the frequency equation (41) with $k_{RI}=0$ and consequently

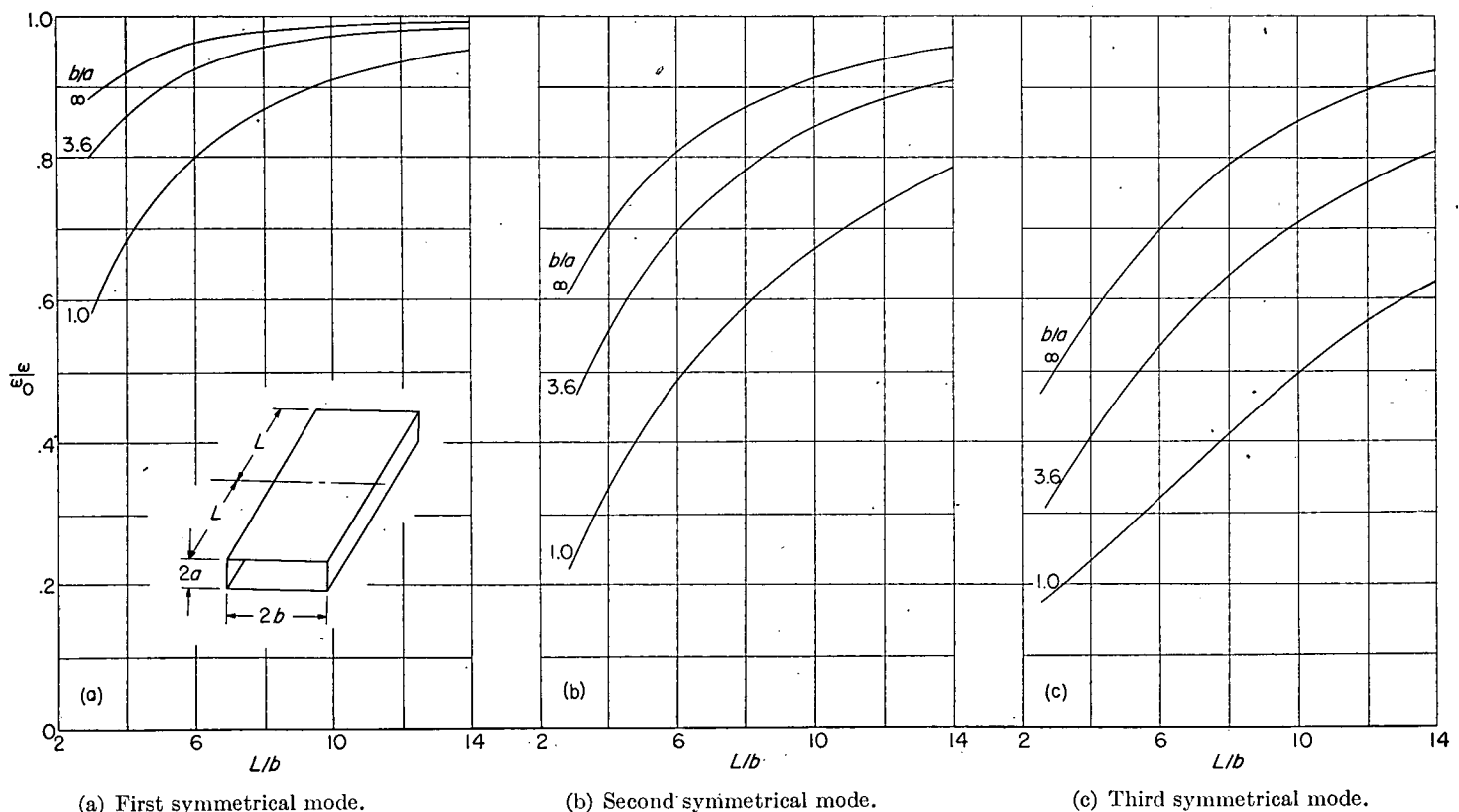


FIGURE 2.—Change in the natural frequency of a symmetrically vibrating free-free cylinder due to the inclusion of secondary effects.

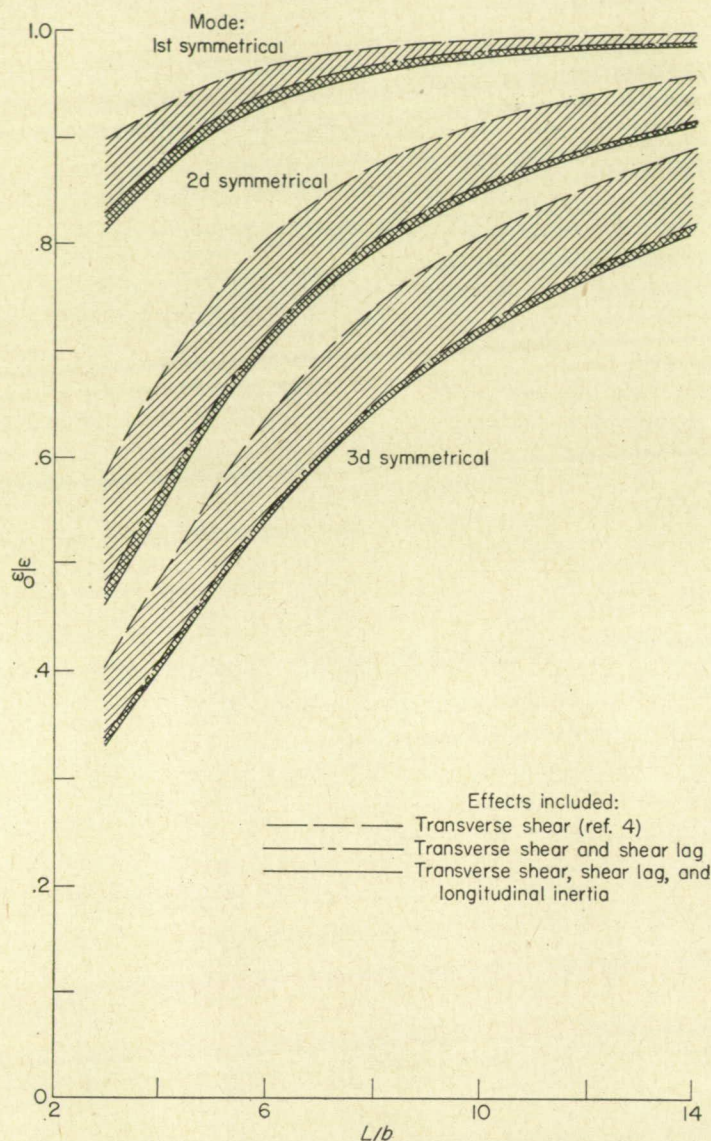


FIGURE 3.—Contribution of transverse shear deformation, shear lag, and longitudinal inertia to the reduction in natural frequency for $\frac{b}{a}=3.6$.

represent the reduction in natural frequency when both shear lag and transverse shear deformation are taken into account. Thus the hatched area between the dashed and the long- and short-dash lines may be considered as showing the additional reduction in natural frequency when the influence of shear lag is considered. Finally, the solid lines are calculated with k_{RI} taken into account, and consequently the shaded area shows the additional influence of longitudinal inertia in reducing the frequency.

Examination of figures 3 and 4 and the curves for $\frac{b}{a} = \infty$ in figure 2 shows that the influence of shear lag increases as the

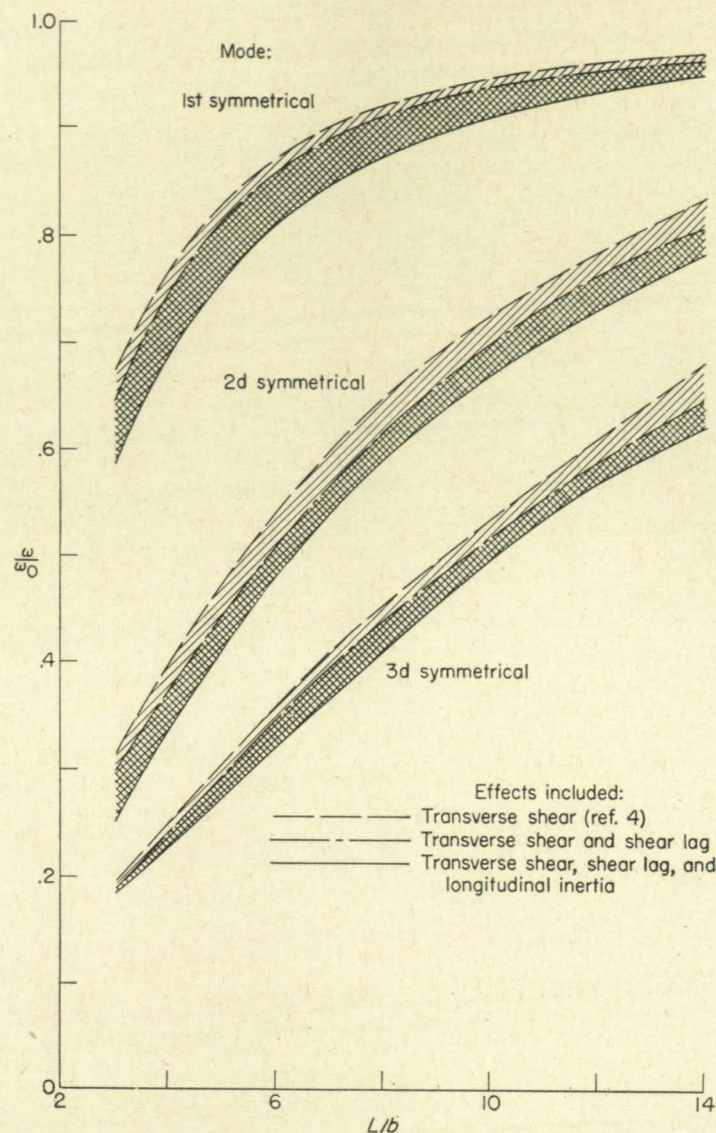


FIGURE 4.—Contribution of transverse shear deformation, shear lag, and longitudinal inertia to the reduction in natural frequency for $\frac{b}{a}=1.0$.

cross-sectional aspect ratio increases; whereas the influence of transverse shear and longitudinal inertia decreases with increasing cross-sectional aspect ratio. Indeed, it appears from the results for $\frac{b}{a}=3.6$ that for this aspect ratio the effects of longitudinal inertia may already be considered practically negligible.

A word of caution concerning the interpretation of figures 3 and 4 may be in order. Since in some cases the depth of the hatching increases with increasing L/b , it might appear, at first glance, that the shear-lag effect increases with increasing plan-form aspect ratio. However, if the additional effects

of shear lag are considered on a percentage basis with the dashed line as a base, it will be found that shear-lag effects actually reduce in percentage with increasing L/b . A similar criterion should be used in judging the influence of longitudinal inertia.

CONCLUDING REMARKS

The numerical calculations show that secondary effects have appreciable influence on the natural frequencies of rectangular box beams of uniform wall thickness. These results constitute an indication of the probable inadequacy of elementary beam theory for the vibration analysis of actual aircraft structures of the monocoque and semimonocoque

type and emphasize the need for practical calculation procedures for such structures that would take into account transverse shear deformation, shear lag, and, when necessary, longitudinal inertia. The general solutions presented for cylinders of uniform thickness, as well as the numerical results for rectangular box beams, should be useful in the assessment of the accuracy of any procedure of this kind that may be developed.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., January 21, 1952.

APPENDIX

TRANSFORMATION OF PARAMETERS

Expressions for A_s/A , I , and K^2 .—If $\sin \theta$ is expanded into a Fourier series

$$\sin \theta = \sum_{n=1,2,3}^{\infty} A_n \sin \frac{2n\pi s}{p} \quad (\text{A1})$$

the Fourier coefficients A_n are the same as those defined in equation (24); that is,

$$A_n = \frac{2}{p} \oint \sin \theta \sin \frac{2n\pi s}{p} ds \quad (\text{A2})$$

The effective shear area A_s (eq. (25)) can now be written as a function of the Fourier series expansion for $\sin \theta$ as

$$A_s = \oint t \left(\sum_{n=1,2,3}^{\infty} A_n \sin \frac{2n\pi s}{p} \right)^2 ds \quad (\text{A3})$$

With the use of the appropriate orthogonality conditions, equation (A3) becomes, after the integration is performed,

$$A_s = \frac{pt}{2} \sum_{n=1,2,3}^{\infty} A_n^2 = \frac{A}{2} \sum_{n=1,2,3}^{\infty} A_n^2$$

or

$$\frac{A_s}{A} = \frac{1}{2} \sum_{n=1,2,3}^{\infty} A_n^2 \quad (\text{A4})$$

The moment of inertia I of a cylinder is defined as (see fig. 1)

$$I = \int_0^p y^2 t ds - A \bar{y}^2 \quad (\text{A5})$$

where $\bar{y}$ is the y -distance to the center of gravity of the cross section and is given by

$$\bar{y} = \frac{\int_0^p y t ds}{pt} \quad (\text{A6})$$

But

$$y = \int_0^s \sin \theta ds \quad (\text{A7})$$

or

$$y = \sum_{n=1,2,3}^{\infty} A_n \frac{p}{2n\pi} \left(1 - \cos \frac{2n\pi s}{p} \right) \quad (\text{A8})$$

and, consequently,

$$\bar{y} = \sum_{n=1,2,3}^{\infty} A_n \frac{p}{2n\pi} \quad (\text{A9})$$

With the use of equations (A8) and (A9), the expression for I in equation (A5) becomes

$$I = \frac{Ap^2}{8\pi^2} \sum_{n=1,2,3}^{\infty} \frac{A_n^2}{n^2} \quad (\text{A10})$$

With the series expansion for I in equation (A10), the parameter K^2 , as defined in equation (29), becomes

$$K^2 = \frac{2}{\pi^2} \frac{A}{A_s} \sum_{n=1,2,3}^{\infty} \frac{A_n^2}{n^2} \quad (\text{A11})$$

Transformation of expression for N_i .—In equation (37) N_i was defined as

$$N_i = \frac{i^2 \pi^2}{8k_s^2} - \frac{i^2 \pi^2}{16k_s^2} K^2 \frac{A}{A_s} \sum_{n=1,2,3}^{\infty} \frac{n^2 A_n^2}{k_s^2 B_i^2 + K^2 n^2} - \frac{1}{2} k_B^2 \quad (\text{A12})$$

The infinite series that appears in this expression converges as A_n^2 and therefore is a relatively slowly converging series. In order to increase its rate of convergence, the following transformations are made.

By adding and subtracting A_n^2/K^2 inside the infinite summation in equation (A12) and using equation (A4), the equation simplifies to

$$N_i = \frac{i^2 \pi^2}{16} \frac{A}{A_s} B_i^2 \sum_{n=1,2,3}^{\infty} \frac{A_n^2}{k_s^2 B_i^2 + K^2 n^2} - \frac{1}{2} k_B^2 \quad (\text{A13})$$

By adding and subtracting $A_n^2/K^2 n^2$ inside the infinite summation in equation (A13) and using equation (A11), the

expression for N_i can be transformed to

$$N_i = \frac{i^2 \pi^4}{32} B_i^2 - \frac{i^2 \pi^2}{16} \frac{A}{A_s} B_i^4 k_s^2 \sum_{n=1,2,3}^{\infty} \frac{A_n^2}{K^2 n^2 (k_s^2 B_i^2 + K^2 n^2)} - \frac{1}{2} k_B^2 \quad (\text{A14})$$

The infinite series in equation (A14) converges as A_n^2/n^4 and therefore is considerably more quickly convergent than the series in equations (A12) and (A13), which converge as A_n^2 and A_n^2/n^2 , respectively.

Closed form of N_i for cylindrical beams of rectangular cross section.—For a cylindrical beam of rectangular cross section, with dimensions as shown in figure 2, it is possible to write the expression for N_i in a closed form. The parameters for such a cross section become

$$\left. \begin{aligned} A_s &= 4at \\ A &= 4(a+b)t = pt \\ A_n &= 0 \quad (n \text{ even}) \\ &= \frac{4}{n\pi} \cos \frac{2n\pi b}{p} \quad (n \text{ odd}) \end{aligned} \right\} \quad (\text{A15})$$

With equations (A15) the parameter N_i shown in equation (A12) becomes

$$N_i = \frac{i^2 \pi^2}{8k_s^2} - \frac{i^2}{4k_s^2} \frac{p}{a} \sum_{n=1,3,5}^{\infty} \frac{\cos^2 \frac{2n\pi b}{p}}{\frac{k_s^2}{K^2} B_i^2 + n^2} - \frac{1}{2} k_B^2 \quad (\text{A16})$$

or

$$N_i = \frac{i^2 \pi^2}{8k_s^2} - \frac{i^2}{8k_s^2} \frac{p}{a} \left(\sum_{n=1,3,5}^{\infty} \frac{1}{\frac{k_s^2}{K^2} B_i^2 + n^2} + \sum_{n=1,3,5}^{\infty} \frac{\cos \frac{4n\pi b}{p}}{\frac{k_s^2}{K^2} B_i^2 + n^2} \right) - \frac{1}{2} k_B^2 \quad (\text{A17})$$

Each of the infinite summations in equation (A17) can now be written in closed form as shown in reference 5, and the closed expression for N_i then becomes

$$N_i = \frac{i^2 \pi}{8k_s^2} \left\{ \pi - \frac{K}{4k_s B_i} \frac{p}{a} \left[\frac{\sinh \frac{\pi}{2} \frac{k_s}{K} B_i \left(\frac{8a}{p} - 1 \right)}{\cosh \frac{\pi}{2} \frac{k_s}{K} B_i} + \tanh \frac{\pi}{2} \frac{k_s}{K} B_i \right] \right\} - \frac{1}{2} k_B^2 \quad (\text{A18})$$

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REPORT 1130

SOME EFFECTS OF FREQUENCY ON THE CONTRIBUTION OF A VERTICAL TAIL TO THE FREE AERODYNAMIC DAMPING OF A MODEL OSCILLATING IN YAW¹

By JOHN D. BIRD, LEWIS R. FISHER, and SADIE M. HUBBARD

SUMMARY

The directional damping and stability of a fuselage—vertical-tail model oscillating freely in yaw were measured at a Mach number of 0.14 and compared with the damping and stability obtained by consideration of the effects of unsteady lift. The contribution of the vertical tail to the damping in yaw of the model agreed within the limits of experimental accuracy with predictions of the approximate finite-aspect-ratio unsteady-lift theories for the range of frequencies tested, and the two-dimensional unsteady-lift theory predicted a much greater loss in damping with reduction in frequency than was shown by experiment or by the approximate finite-aspect-ratio unsteady-lift theories. For frequencies comparable to those of lateral airplane motions, both the unsteady-lift theories and the experimental results indicated that the contribution of the vertical tail to the directional stability of the model was relatively independent of frequency.

INTRODUCTION

The advent of high-speed airplanes of high relative density has focused attention on certain problems associated with the dynamic stability of airplanes which, because of previous unimportance, have heretofore been neglected. Among these problems is the effect of the periodicity of the airplane motion on the effective values of the various stability derivatives. Reference 1 and, more recently, references 2 and 3 have indicated the possibility of sizeable effects from this problem.

An appreciable amount of theoretical work on these unsteady-lift effects exists at present; however, only a small amount of experimental substantiation of the results is available for the low-frequency range of oscillation. As a result, a program has been undertaken in the Langley stability tunnel to determine the effects of such variables as frequency and amplitude of motion on the contribution of the various airplane components to the stability derivatives of present-day airplane configurations. The work reported herein covers that phase of the investigation which considers frequency effects on the directional damping and stability of a model undergoing a freely damped oscillatory yawing motion. The effects of vertical-tail aspect ratio and compressibility as predicted by the theoretical treatments are discussed in relation to the experimental stability characteristics obtained by the free-oscillation and by the curved-flow procedures.

SYMBOLS

The data are referred to the system of stability axes and are presented in the form of standard NACA coefficients of forces and moments about the quarter-chord point of the mean aerodynamic chord of the normal wing location of the model tested. (See fig. 1.) The coefficients and symbols used herein are defined as follows:

C_L	lift coefficient, L/qS_w
C_D	drag coefficient, D/qS_w
C_Y	lateral-force coefficient, Y/qS_w
C_l	rolling-moment coefficient, $L'/qS_w b_w$
C_m	pitching-moment coefficient, $M/qS_w \bar{c}_w$
C_n	yawing-moment coefficient, $N/qS_w b_w$

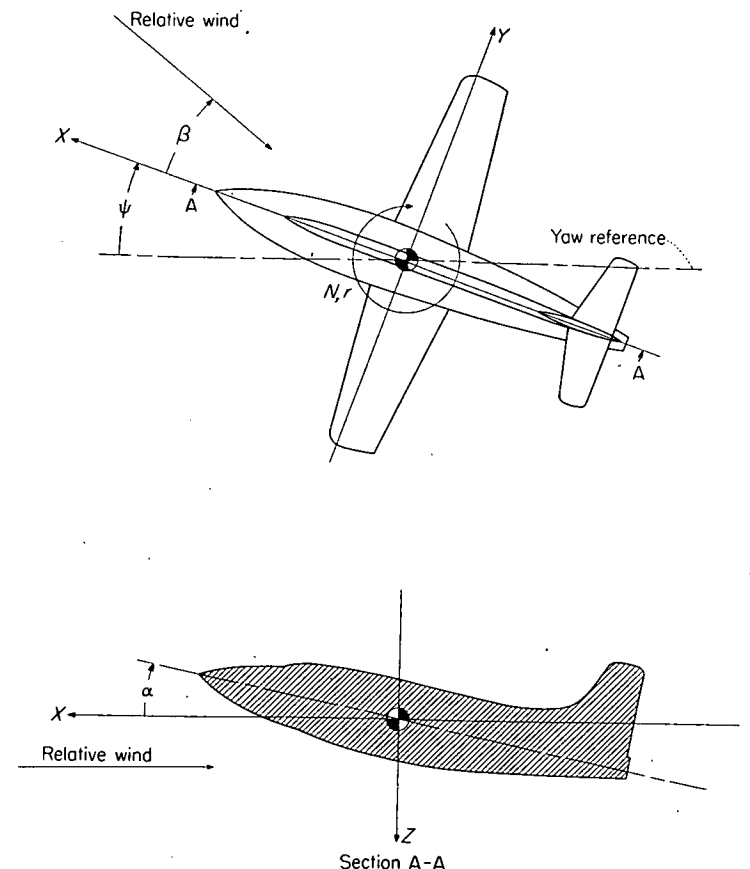


FIGURE 1.—System of stability axes. Arrows indicate positive forces, moments, and angular displacements. Yaw reference generally chosen to coincide with initial relative wind.

¹ Supersedes NACA TN 2657, "Some Effects of Frequency on the Contribution of a Vertical Tail to the Free Aerodynamic Damping of a Model Oscillating in Yaw" by John D. Bird, Lewis R. Fisher, and Sadie M. Hubbard, 1952

L	lift, lb
D	drag, lb
Y	lateral force, lb
L'	rolling moment, ft-lb
M	pitching moment, ft-lb
N	yawing moment, ft-lb
q	dynamic pressure, $\frac{1}{2} \rho V^2$, lb/sq ft
ρ	mass density of air, slugs/cu ft
V	free-stream velocity, ft/sec
S	area, sq ft
b	span, ft
c	chord, ft
c_t	mean tail chord, ft
$\bar{c}$	mean aerodynamic chord, $\frac{2}{S} \int_0^{b/2} c^2 dy$, ft
y	spanwise distance from the plane of symmetry, ft
A	aspect ratio, b^2/S
α	angle of attack, deg
β	angle of sideslip, radians
$\dot{\beta} = \frac{d\beta}{dt}$	
t	time, sec
$\frac{rb}{2V}$	yawing-velocity parameter referred to semispan of wing
ψ	angle of yaw, radians
$r, \dot{\psi}$	yawing velocity, $d\psi/dt$, radians/sec
$\ddot{\psi}$	yawing acceleration, $d^2\psi/dt^2$, radians/sec ²
ω	angular velocity, radians/sec
f	frequency, cps
k	reduced-frequency parameter referred to semichord of vertical tail, $\omega c_t/2V$
N_ψ	mechanical spring constant, ft-lb/radian
I_z	yawing moment of inertia, ft-lb-sec ²
$t_{1/2}$	time to damp to one-half amplitude, sec
m	damping constant
P	period of oscillation, sec
a	nondimensional tail length referred to semichord of vertical tail, $-\frac{l}{c_t/2}$
l	distance from origin of axes to midchord point of vertical tail, ft
$C(k) = F + iG$	
F, G	circulation functions used by Theodorsen (ref. 6)
$\bar{P} = F + iG$ $\bar{Q} = H + iJ$	finite-span functions used by Biot and Boehnlein (ref. 9)
σ	finite-span correction used by Reissner and Stevens (ref. 8)
Z, M	compressible-flow functions used in place of F and G (refs. 10 and 11)

A, B functions of k defined in text

$$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$$

$$C_{n_r} = \frac{\partial C_n}{\partial \frac{\dot{r}b^2}{4V^2}}$$

$$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \frac{\dot{\beta}b}{2V}}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

$$N_{\dot{\psi}} = \frac{\partial N}{\partial \dot{\psi}}$$

$$N_{\ddot{\psi}} = \frac{\partial N}{\partial \ddot{\psi}}$$

$$N_\beta = \frac{\partial N}{\partial \beta}$$

$$N_{\dot{\beta}} = \frac{\partial N}{\partial \dot{\beta}}$$

Subscripts:

w	wing
t	vertical tail
0	amplitude
f	due to friction

APPARATUS

MODEL AND OSCILLATING STRUT

The model used in this investigation and shown in figure 2 had separable wing, fuselage, and tail surfaces and was constructed almost entirely of balsa wood in order to minimize the mass and make the natural frequency of the model on its mounting as high as possible. The configuration is representative of that of several present-day high-speed airplanes. The aspect ratio A_t and mean chord c_t of the vertical tail were 2 and 4.9 inches, respectively.

The model was mounted at the quarter chord of the wing mean aerodynamic chord on a 1-inch-diameter rod which

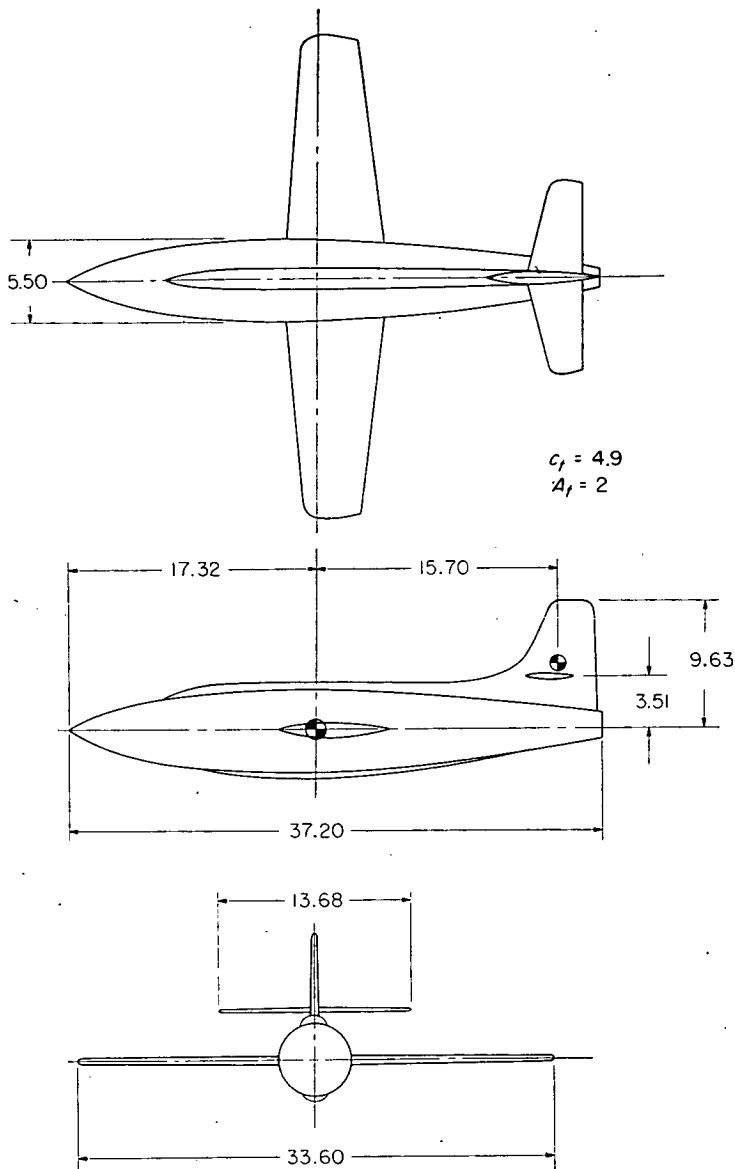


FIGURE 2.—Drawing of model tested. All dimensions are in inches.

rotated with respect to a circular hollow strut and which was connected to it by means of flexure pivots made of Swedish steel. Flexure pivots were used rather than bearings in order to minimize the friction in the system. The pivots were aligned with the center of the rod on which the model was mounted. The entire assembly, which is shown in figures 3 and 4, was fastened to the ceiling of the tunnel test section and covered by a streamlined fairing.

The inner rod or oscillating strut, which was permitted to rotate and upon which the model was mounted, extended through the ceiling of the tunnel test section and was capped by a horizontal crosspiece. The period of oscillation of the model was controlled by varying the moment of inertia

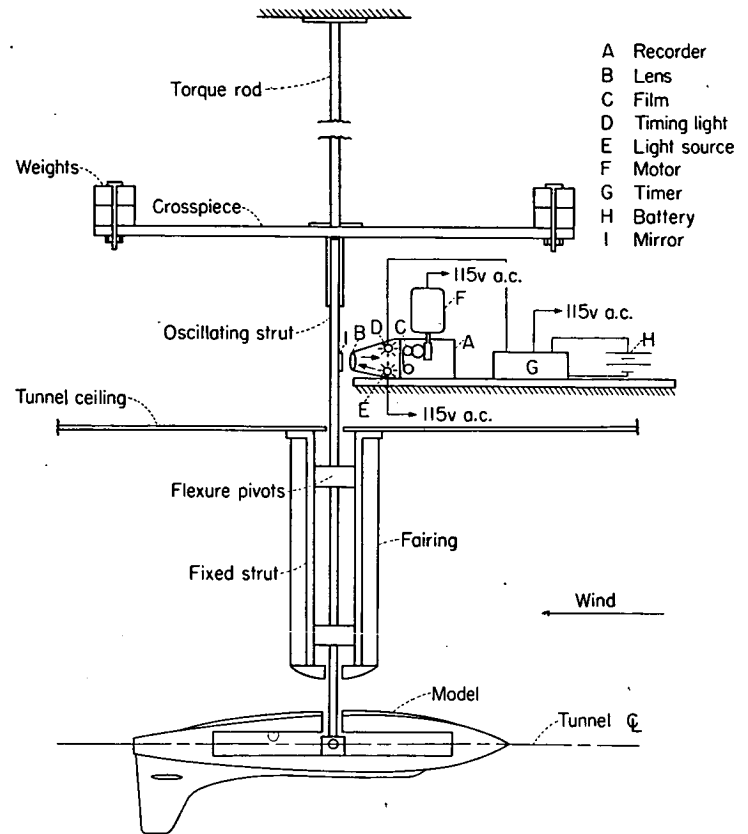


FIGURE 3.—Diagram of test equipment.

about the vertical axis of the system. This variation was accomplished by using crosspieces of different lengths and by clamping weights at the ends of the crosspieces.

RECORDING DEVICE

A continuous record of the displacement in yaw of the model was made on photographic film by means of an optical recording system. A mirror clamped to the section of the oscillating strut which extended outside the tunnel reflected a beam of light from a point light source within the recorder. The beam was focused on a roll of photographic film by means of an 8-inch-focal-length double-concave lens. The film roll was driven at constant speed by gearing it to an electric motor. A 10-cycle-per-second neon timing light incorporated in the recorder exposed timing lines on the film in order that time as well as model displacement could be measured accurately on the film record.

TESTS

OSCILLATION TESTS

The free-damping tests were conducted with the model in an inverted position, in order to minimize strut interference effects, and at zero angle of attack with respect to the fuselage reference line. The tests were conducted on the

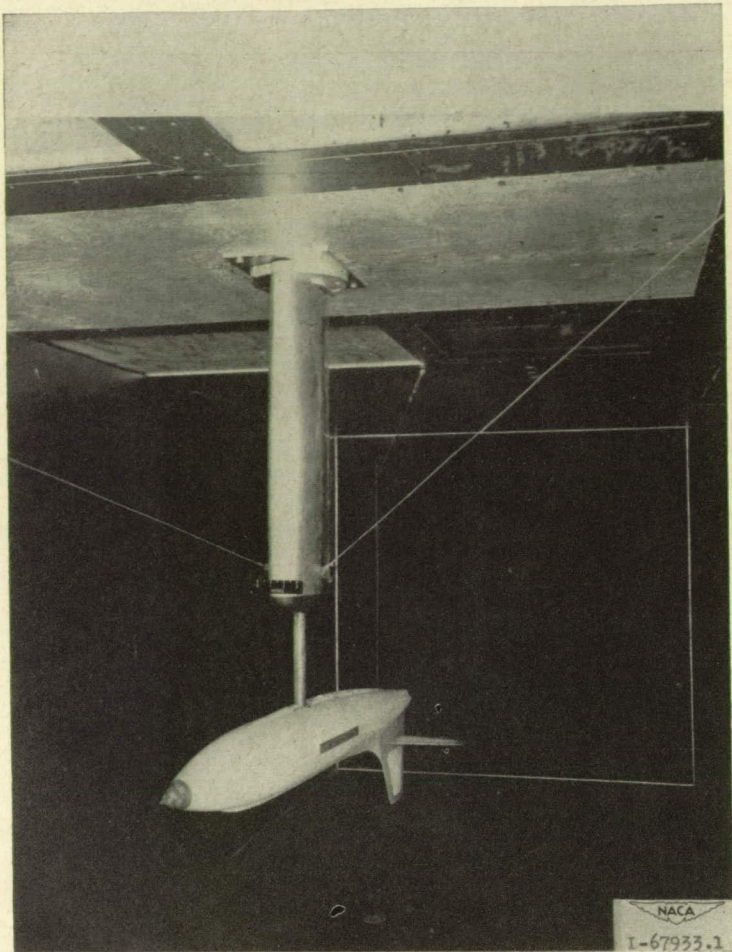


FIGURE 4.—Test model mounted on free-oscillation strut in Langley stability tunnel.

combination of fuselage and tail surface and on the fuselage alone because preliminary tests had shown that the presence of the wing had no apparent effect on the damping at zero angle of attack. The range of moments of inertia and frequencies covered are shown together with the test results in table I. Results of runs made with wind on and off are given in order to illustrate the magnitude of the tares applied.

When tests were made of the fuselage alone, a torque rod was employed to provide a stable restoring moment to the fuselage which was equal to the aerodynamic moment provided by the vertical tail when attached. The torque rod was clamped between the extended oscillating strut and a building structural member as shown in figure 3. The spring constant of the system was 6.8 foot-pounds per radian for the flexure pivots alone and 32.8 foot-pounds per radian for the combination of the flexure pivots and the torque rod.

FORCE TESTS

The steady-state stability characteristics of the model were determined by means of wind-tunnel force tests through an angle-of-attack range from 0° to 8° for the fuselage alone, the fuselage and tail surfaces, and the complete configuration. The model was mounted on a single-strut support and all forces and moments were measured by a six-component balance system. The steady-state sideslipping derivatives were determined through a sideslip range of ±5°, and the yawing derivatives were determined at values of the yawing-velocity parameter $rb/2V$ of 0, -0.0291, -0.0616, and -0.0810 by standard Langley stability-tunnel curved-flow testing procedures.

All tests were conducted in the 6- by 6-foot test section of the Langley stability tunnel at a dynamic pressure of 24.9 pounds per square foot, a Mach number of about 0.14, and a Reynolds number of 442,000 based on the wing mean aerodynamic chord. Standard jet-boundary corrections have been applied to the angle-of-attack, the drag-coefficient and the rolling-moment-coefficient data obtained from the force tests. In addition, the steady-state yawing data have been corrected for the effects of the lateral static-pressure gradient peculiar to the curved-flow testing procedure. Jet-boundary corrections were not applied to the oscillation tests because of their insignificance for the test conditions involved. See reference 4.

REDUCTION OF OSCILLATION TEST DATA

From the continuous film record taken of the motion of the model after an initial displacement, the amplitudes of the

TABLE I
EXPERIMENTAL RESULTS

Configuration	$P, \text{ sec}$		k	$t_{1/2}, \text{ sec}$		$I_z, \text{ ft-lb-sec}^2$	$-(C_{n_r} - C_{n_{\dot{\beta}}})$			$C_{n_{\beta}} + C_{n_r} k^2 \left(\frac{b_w}{c_t}\right)^2$
	Wind off	Wind on		Wind off	Wind on		Total	Friction	Aero-dynamic	
Fuselage and tail	13. 21	7. 15	0. 0012	190. 08	85. 80	32. 89	0. 610	0. 275	0. 335	0. 204
	4. 08	2. 19	. 0040	49. 70	11. 47	3. 09	. 429	. 099	. 330	. 204
	3. 96	2. 25	. 0039	54. 65	10. 35	3. 09	. 475	. 090	. 385	. 190
	3. 96	2. 19	. 0040	54. 65	11. 39	3. 09	. 432	. 090	. 342	. 204
	. 75	. 77	. 0114	34. 36	1. 72	. 38	. 351	. 018	. 333	. 213
	. 22	. 30	. 0293	3. 40	. 28	. 06	. 340	. 028	. 312	. 203
Fuselage alone	. 29	. 27	. 0321	13. 80	3. 21	. 06	. 030	. 007	. 023	-. 058
	. 75	. 74	. 0119	34. 36	13. 77	. 38	. 044	. 018	. 026	-. 015

successive cycles were measured and plotted to a logarithmic scale against time. Inasmuch as the damping is of a logarithmic nature, the resulting plot is a straightline from which may be read the time for the motion to damp to one-half amplitude ($t_{1/2}$).

For a system having a single degree of freedom in yaw such that for the stability-axis system the angle of sideslip β is the negative of the angle of yaw ψ , the equation of motion can be expressed as

$$(I_Z - N_{\ddot{\psi}})\ddot{\psi} - (N_{\dot{\psi}} - N_{\beta} + N_{\dot{\psi}_f})\dot{\psi} - (N_{\psi} - N_{\beta})\psi = 0 \quad (1)$$

where $N_{\dot{\psi}_f}$ and N_{ψ} refer to the mechanical friction of the system and the flexure-pivot spring constant, respectively. The solution of equation (1) can be written as

$$\psi = e^{-mt}(A \sin 2\pi ft + B \cos 2\pi ft) \quad (2)$$

where

$$2\pi f = \sqrt{-\frac{N_{\psi} - N_{\beta}}{I_Z - N_{\ddot{\psi}}} - \left[\frac{N_{\dot{\psi}} - N_{\beta} + N_{\dot{\psi}_f}}{2(I_Z - N_{\ddot{\psi}})} \right]^2}$$

$$m = -\frac{N_{\dot{\psi}} - N_{\beta} + N_{\dot{\psi}_f}}{2(I_Z - N_{\ddot{\psi}})}$$

The damping constant may be expressed in terms of the time to damp to one-half amplitude as

$$m = \frac{0.693}{t_{1/2}}$$

The directional stability and damping of the system may then be expressed in nondimensional form in a manner similar to that of reference 5 as

$$C_{n_{\beta}} + C_{n_r} k^2 \left(\frac{b_w}{c_t} \right)^2 = \frac{1}{q S_w b_w} [I_Z (2\pi f)^2 + N_{\psi}] \quad (3)$$

$$C_{n_r} - C_{n_{\beta}} = \frac{-2.772 V}{q S_w b_w^2} \left\{ I_Z \left[\frac{1}{t_{1/2}} - \frac{1}{(t_{1/2})_{wind\ off}} \right] - \frac{N_{\ddot{\psi}}}{t_{1/2}} \right\} \quad (4)$$

The terms C_{n_r} and $C_{n_{\beta}}$ are necessarily in combination with $C_{n_{\beta}}$ and C_{n_r} , respectively, because of the nature of the motion being considered. For the frequencies employed in these tests, the factor $C_{n_r} k^2 \left(\frac{b_w}{c_t} \right)^2$ should be small in comparison with $C_{n_{\beta}}$.

In determining $(t_{1/2})_{wind\ off}$, which corrects for the damping due to friction in the system, the model was oscillated at zero wind velocity with an equivalent mass in place of the vertical tail and with a torque rod of sufficient strength to produce the proper frequency inserted in the system. The influence of the $\frac{N_{\ddot{\psi}}}{t_{1/2}}$ term is very small compared with that of the inertia term in equation (4) and was neglected in these tests.

The damping derivatives were obtained from data similar to the sample records of model motion and plots of logarithm of amplitude against time given as figure 5. A small residual yawing motion of the model attributable to turbulence in the tunnel airstream necessitated determining the time to damp

to one-half amplitude by the slope of the amplitude envelope at the largest angles of yaw in order to minimize the error in determining the damping derivatives. This source is believed to have resulted in no more than a 5-percent uncertainty in the results.

ANALYTICAL CONSIDERATIONS

Various unsteady-lift theories were employed to calculate the contributions of the vertical tail to the directional stability and damping of the model for comparison with the experimental results. These theories included the incompressible cases covered by Theodorsen (ref. 6) for two dimensions and by Jones (ref. 7), Reissner and Stevens (ref. 8), and Biot and Boehnlein (ref. 9) with various degrees of approximation for finite span. The two-dimensional work including the subsonic effects of compressibility as given by Possio and Frazer and Skan is also employed. A short summary indicating procedures for obtaining the necessary derivatives from these theories is included here for completeness.

TWO-DIMENSIONAL DERIVATIVES

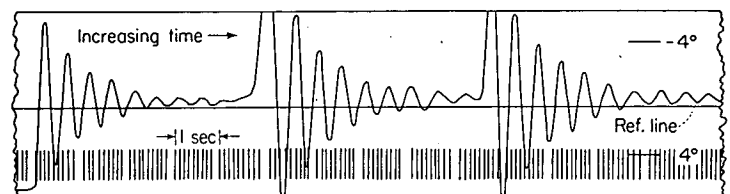
The work of Theodorsen on the derivation of the expressions for the unsteady lift and moment of a two-dimensional surface undergoing sinusoidal oscillations in an incompressible fluid is given in reference 6. From this source, the moment per unit span about the airplane center of gravity of a vertical tail mounted on an airplane which is performing sideslipping oscillations can be shown to be

$$N(\beta) = -\pi q c_t^2 \left[\frac{a c_t \beta}{4V} + \left(a + \frac{1}{2} \right) C(k) \beta \right] \quad (5)$$

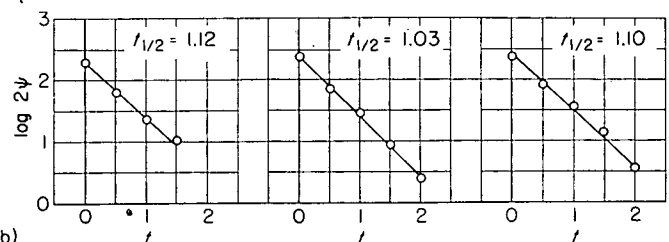
A similar expression for an airplane performing yawing oscillations is

$$N(\psi) = -\frac{\pi}{2} q c_t^3 \left\{ \left[\frac{1}{4} - C(k) \left(\frac{1}{4} - a^2 \right) \right] \frac{\dot{\psi}}{V} + \frac{c_t}{4} \left(\frac{1}{8} + a^2 \right) \frac{\ddot{\psi}}{V^2} \right\} \quad (6)$$

An appendix to this report gives a discussion of the conversion of these results from reference 6.



(a)



(b)

(a) Film record of yawing oscillation.

(b) Logarithmic plots of amplitude variation with time.

FIGURE 5.—Representative film record and resulting logarithmic plots.

The Theodorsen function $C(k)$ is a complex circulation function of the reduced frequency and in terms of its real and imaginary parts is written

$$C(k) = F + iG$$

The functions F and G may be computed from expressions given in reference 6 in terms of Bessel functions of the first and second kinds of orders zero and one and of argument k . In stability practice equations (5) and (6) are usually interpreted in terms of derivatives as follows:

$$\left. \begin{aligned} N(\beta) &= C_{n\beta} q S_w b_w \beta + C_{n\dot{\beta}} \frac{q S_w b_w^2}{2V} \dot{\beta} \\ N(\psi) &= C_{n\psi} \frac{q S_w b_w^2}{2V} \dot{\psi} + C_{n\ddot{\psi}} \frac{q S_w b_w^3}{4V^2} \ddot{\psi} \end{aligned} \right\} \quad (7)$$

If the sinusoidal motions

$$\beta = \beta_0 e^{i\omega t}$$

$$\psi = \psi_0 e^{i\omega t}$$

are employed for a substitution into equations (5), (6), and (7), the real and imaginary terms may be equated and the derivatives $C_{n\beta}$, $C_{n\dot{\beta}}$, $C_{n\psi}$, and $C_{n\ddot{\psi}}$ determined. The results of such an analysis would be as follows:

$$\left. \begin{aligned} C_{n\beta} &= -\frac{S_t}{S_w} \frac{c_t}{b_w} \pi \left(a + \frac{1}{2} \right) F \\ C_{n\dot{\beta}} &= -\frac{S_t}{S_w} \left(\frac{c_t}{b_w} \right)^2 \pi \left[\frac{a}{2} + \left(a + \frac{1}{2} \right) \frac{G}{k} \right] \\ C_{n\psi} &= \frac{S_t}{S_w} \left(\frac{c_t}{b_w} \right)^2 \pi \left[-\frac{1}{4} + F \left(\frac{1}{4} - a^2 \right) \right] \\ C_{n\ddot{\psi}} &= \frac{S_t}{S_w} \left(\frac{c_t}{b_w} \right)^3 \pi \left[-\frac{1}{2} \left(\frac{1}{8} + a^2 \right) + \frac{G}{k} \left(\frac{1}{4} - a^2 \right) \right] \end{aligned} \right\} \quad (8)$$

The velocity terms contain only the F term of the Theodorsen function, whereas the acceleration terms contain the G term divided by k . The corresponding side-force derivatives are derived in the appendix. For the free-oscillation tests conducted for this report, the motion is such that the angle of sideslip is the negative of the angle of yaw. The damping derivative measured therefore is the difference between the derivatives $C_{n\psi}$ and $C_{n\dot{\beta}}$ determined from equation (8). This fact was illustrated previously with respect to the reduction of the oscillation test data. In a like manner, the directional stability parameter measured can be shown to be a combination of the $C_{n\beta}$ and $C_{n\dot{\beta}}$ terms, specifically

$$C_{n\beta} + C_{n\dot{\beta}} k^2 \left(\frac{b_w}{c_t} \right)^2$$

Combination of the results of equations (8) to obtain the factors calculated for comparison with the oscillation tests gives

$$\left. \begin{aligned} C_{n\psi} - C_{n\dot{\beta}} &= -\frac{\pi}{2} \left(\frac{c_t}{b_w} \right)^2 \frac{S_t}{S_w} \frac{B}{k} \\ C_{n\beta} + C_{n\dot{\beta}} k^2 \left(\frac{b_w}{c_t} \right)^2 &= -\frac{\pi}{2} \frac{c_t}{b_w} \frac{S_t}{S_w} A \end{aligned} \right\} \quad (9)$$

where

$$B = -\left(a - \frac{1}{2} \right) k - \left(a + \frac{1}{2} \right) 2G + \left(a^2 - \frac{1}{4} \right) 2kF$$

$$A = \left(a^2 + \frac{1}{8} \right) k^2 + \left(a + \frac{1}{2} \right) 2F + \left(a^2 - \frac{1}{4} \right) 2kG$$

FINITE-SPAN DERIVATIVES

The aerodynamic-span effect is considered in reference 7 for wings of aspect ratios of 6 and 3 by correcting the aerodynamic inertia and the angle of attack of the infinite-span surface. An approximation employed in this analysis concerning motions of long wave length may make the results subject to question for values of k near zero. The reference presents expressions for calculating the finite-span values of the Theodorsen function $C(k)$. The use of these F and G circulation functions in equations (8) and (9) results in values of the various derivatives considered for aspect ratios of 6 and 3.

The aerodynamic-span effect is considered for wings of arbitrary aspect ratio in reference 8. The three-dimensional effect of the finite span may be obtained by adding a correction term σ to the basic Theodorsen two-dimensional function. The finite-span function $C(k) + \sigma$ is employed in a rather lengthy and tedious computation in order to determine the components of the unsteady lift for each of five points along the span of the oscillating surface. The various stability derivatives may then be determined by a graphical integration of the in-phase and out-of-phase components. An approximation made in this analysis is of such a nature as to make the method primarily suited to the higher aspect ratios. A practical lower limit of applicability is not known.

A one-point approximation to the span effect for the calculation of the unsteady lift may be employed rather than the five-point procedure. Comparative calculations made for the present investigation have shown the one-point approximation to yield results for low aspect ratios which are essentially the same as those calculated by the longer procedure.

Biot and Boehnlein (ref. 9) considered the aerodynamic-span effect on unsteady lift by an approach that yields a closed form for a one-point approximation to the span effect. The finite-span circulation functions used are

$$\bar{P} = F + iG$$

$$\bar{Q} = H + iJ$$

For infinite aspect ratio

$$\bar{P} = \bar{Q} = C(k)$$

where $C(k)$ is the two-dimensional Theodorsen function. The functions $\bar{P}$ and $\bar{Q}$ may be calculated for the argument k by methods given in reference 9, and equations (8) and (9) are again used to calculate the various stability derivatives. The factors A and B which appear in equations (9) must, however, be written in terms of $\bar{P}$ and $\bar{Q}$ as follows:

$$A = (2a - 1)akG + \left(a - \frac{1}{2} \right) kJ + \left(a^2 + \frac{1}{8} \right) k^2 + 2aF + H$$

$$B = (2a - 1)akF + \left(a - \frac{1}{2}\right)kH - \left(a - \frac{1}{2}\right)k - 2aG - J$$

EFFECT OF COMPRESSIBILITY

Subsonic compressibility effects on the unsteady damping in yaw and directional stability of a two-dimensional oscillating lifting surface can be considered through the use of reference 10 which is an extension of the calculations of Possio (ref. 11). The infinite-span compressible-flow values for the stability derivatives can be obtained by the use of equations (8) and (9) for Mach numbers 0, 0.5, and 0.7. In the notation of Frazer and Skan (ref. 10), however,

$$A = 4M_3 - \frac{1}{2}\left(\frac{1}{2} + a\right)\left(M_1 + Z_3 - \frac{a}{2}Z_1\right) + M_1$$

$$B = 4M_4 - \frac{1}{2}\left(\frac{1}{2} + a\right)\left(M_2 + Z_4 - \frac{a}{2}Z_2\right) + M_2$$

The correspondence between the notation of Theodorsen and that of Frazer and Skan is shown in reference 12. The values of the Z and M functions are tabulated by Frazer and Skan for Mach numbers 0, 0.5, and 0.7 and a significant range of reduced frequencies. Reissner has recently extended his work on oscillating wings of finite span to include, to the same degree of approximation as for the incompressible case, the subsonic effects of compressibility. (See ref. 13.)

RESULTS AND DISCUSSION

PRESENTATION OF RESULTS

The experimental values of the longitudinal, directional, and yawing characteristics of the complete model, the model with wing removed, and the fuselage alone as determined by force tests are given in figure 6. The experimental values of the damping in yaw of the fuselage and tail assembly (designated FT) and the fuselage alone (designated F) obtained by the freely damped oscillation technique for several frequencies of oscillation are shown in figures 7 and 8. The frequency and aspect-ratio effects on the damping in yaw contributed by the vertical tail as calculated from the unsteady-lift theories discussed previously are also shown in these figures. Figure 9 gives a comparison of the relative importance of C_{n_r} and $C_{n_{\dot{\beta}}}$ on the damping in yaw of the model. Figures 10 to 12 are similar figures summarizing frequency and aspect-ratio effects on directional stability. Figures 13 and 14 give the computed effects of compressibility on the vertical-tail contribution to the damping in yaw and directional stability of the test model.

DISCUSSION

The variation of the experimentally determined damping in yaw of the test model with frequency is in reasonably good agreement for the range covered by the tests with the various approximate theoretical treatments which consider the effect of finite span. (See figs. 7 and 8.) The contribution of the fuselage to the damping in yaw is small for the range of test frequencies. A two-dimensional approximation to the damping as is frequently employed for flutter work appears to be remarkably good for $k > 0.1$ and for aspect

ratios of 6 and greater (fig. 8). A large effect of aspect ratio, however, is shown for aspect ratios below 6 for the entire frequency range and for all aspect ratios for frequencies less than $k = 0.1$. The two-dimensional result indicates a loss in damping at low frequencies much greater than the finite-span result and, in fact, shows a reversal in sign of the damping at sufficiently low frequencies. A number of papers dealing with unsteady lift have noted this characteristic of the two-dimensional aerodynamic forces. (See refs. 1 to 3 and 14.)

An examination of the values of the derivatives C_{n_r} and $C_{n_{\dot{\beta}}}$ as computed from reference 6 for infinite aspect ratio and from reference 7 for aspect ratios of 3 and 6 (fig. 9) indicates that the derivative $C_{n_{\dot{\beta}}}$ is critically dependent on reduced frequency for the low range of reduced frequencies and infinite aspect ratio. The derivative C_{n_r} is only slightly dependent on the reduced frequency for infinite aspect ratio and even less so for the lower aspect ratios. The dependence of $C_{n_{\dot{\beta}}}$ on reduced frequency is responsible for the decrease in damping shown for the infinite-aspect-ratio case as the frequency is reduced (figs. 7 and 8).

The aspect-ratio-2 results shown in figures 7 and 10 were obtained from the theory of Jones (ref. 7) for aspect ratio 3 by use of a correction factor which was the ratio of the lifting-line-theory lift-curve slopes for aspect ratios of 2 and 3. This correction is only approximate and was made in order to place the results on a comparable basis. The variation of $C_{n_r} - C_{n_{\dot{\beta}}}$ with reduced frequency, shown in figure 7, indicates no appreciable effect of frequency to zero frequency. In fact, the result of Jones for aspect ratio 3 reduces to a finite value at $k = 0$ given by

$$C_{n_r} - C_{n_{\dot{\beta}}} = -\frac{\pi}{2} \left(\frac{c_t}{b_w}\right)^2 \frac{S_t}{S_w} (1.2a^2 - 0.370a + 0.515)$$

which is not the case for the two-dimensional result that approaches infinity as k approaches zero. This result is not expected to be very accurate close to $k = 0$, however, because of the nature of the assumptions made in its derivation as mentioned previously.

When the finite-span effects are considered, it appears that these effects are especially significant with regard to the damping in yaw of this model for aspect ratios smaller than 6. (See fig. 8.) The theory also indicates a significant decrease in the damping at low frequencies when the aspect ratio was increased from 20 to ∞ .

The experimental values of damping in yaw obtained by free oscillation agree closely with the values of C_{n_r} obtained by the standard curved-flow testing procedure used in the Langley stability tunnel. The curved-flow technique results in the $k = 0$ value of C_{n_r} . No unsteady effects, however, are obtained by this technique.

The unsteady-lift theories indicate that, at frequencies of oscillation from $k = 0$ to $k = 0.1$, the frequency has little effect on the directional stability contributed by the vertical tail but that high frequencies of oscillation can cause the directional stability contributed by the vertical tail to decrease greatly and eventually to change sign (figs. 10 and 11).

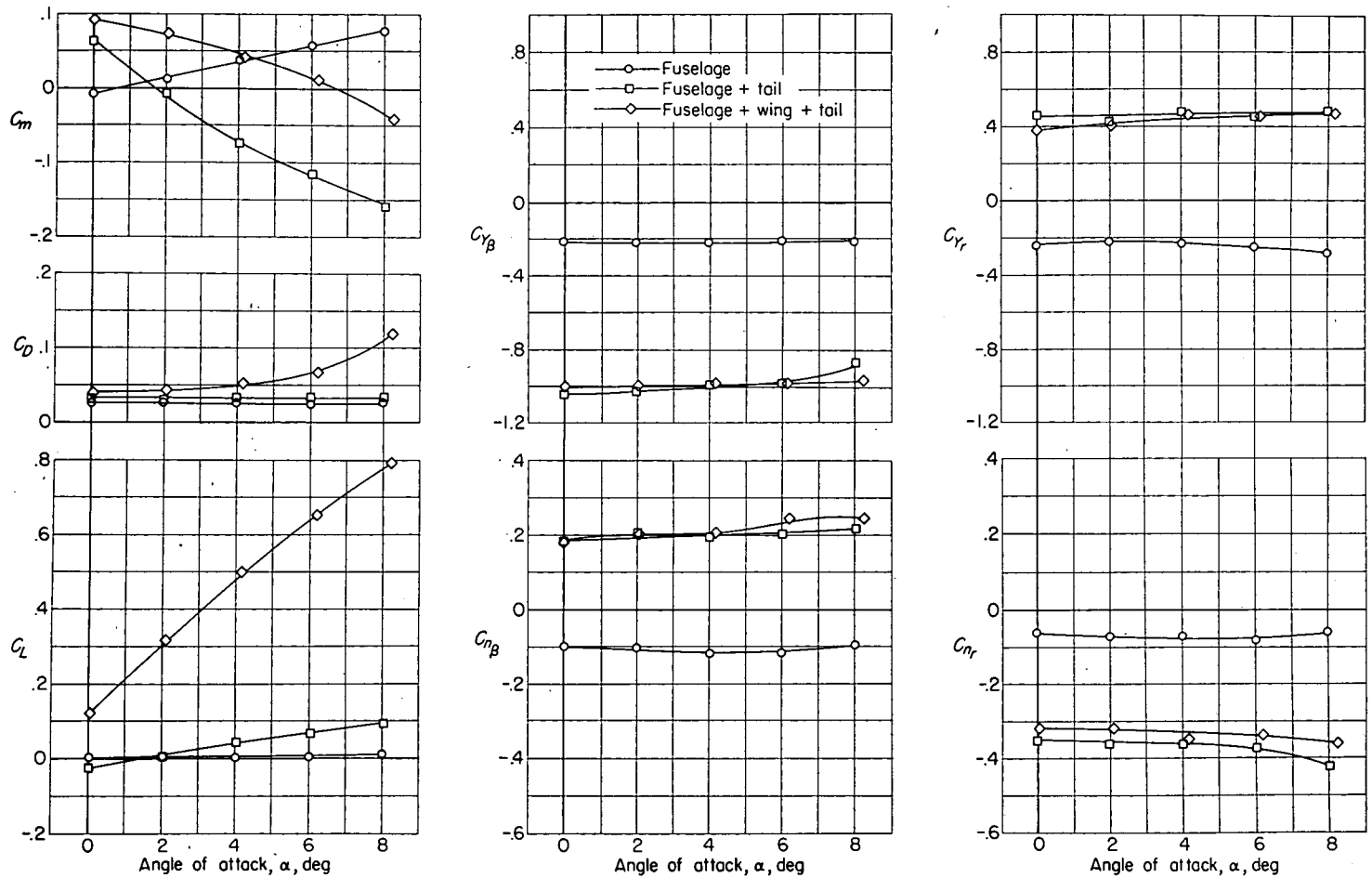


FIGURE 6.—The steady-state stability characteristics of the model tested.

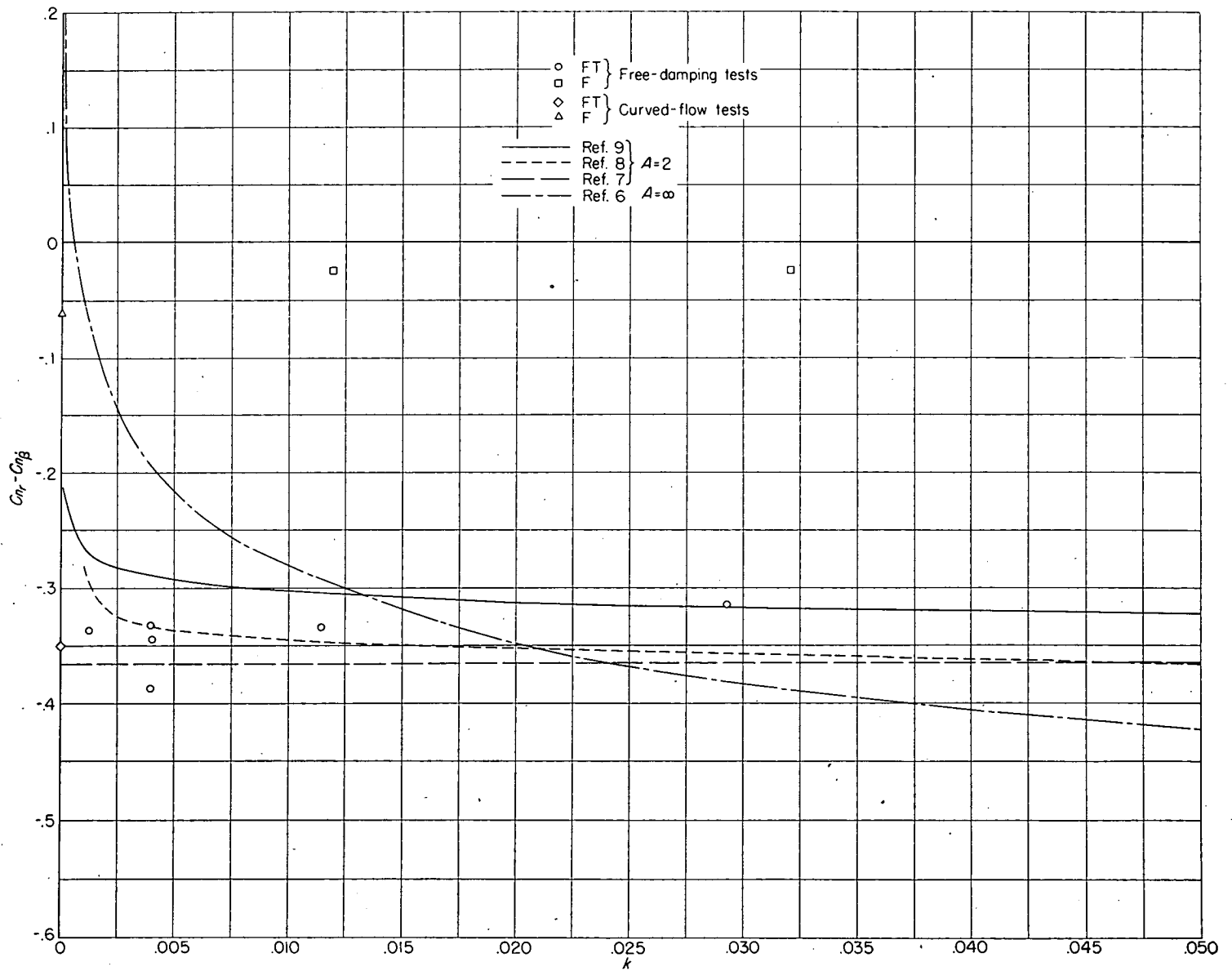


FIGURE 7.—The effect of frequency on the damping in yaw for a low range of the reduced frequency. Comparison of several methods for calculating frequency effect on the damping in yaw.

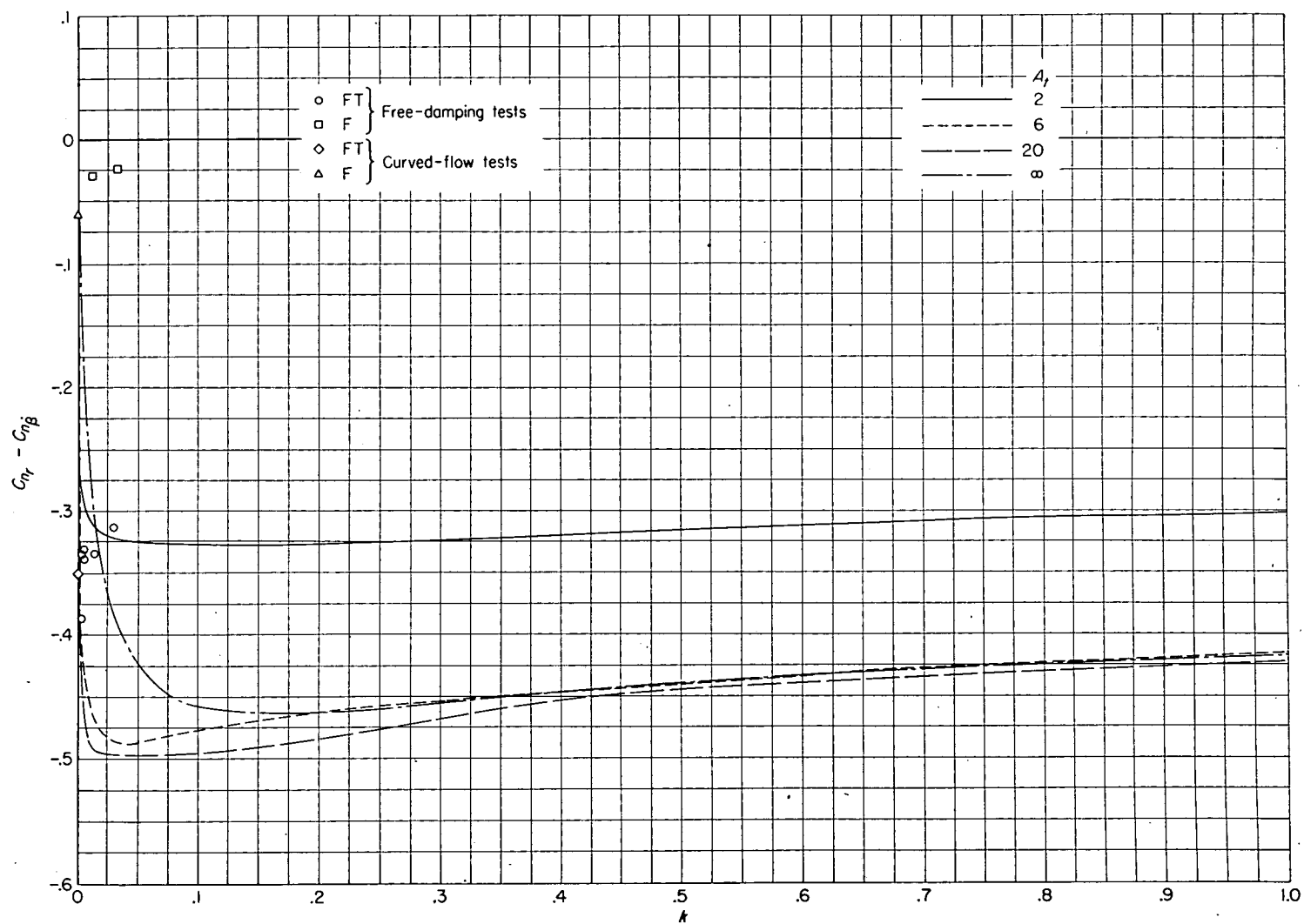


FIGURE 8.—The effect of aspect ratio on the damping in yaw as calculated by the method of Biot and Boehnlein for a range of reduced frequencies.

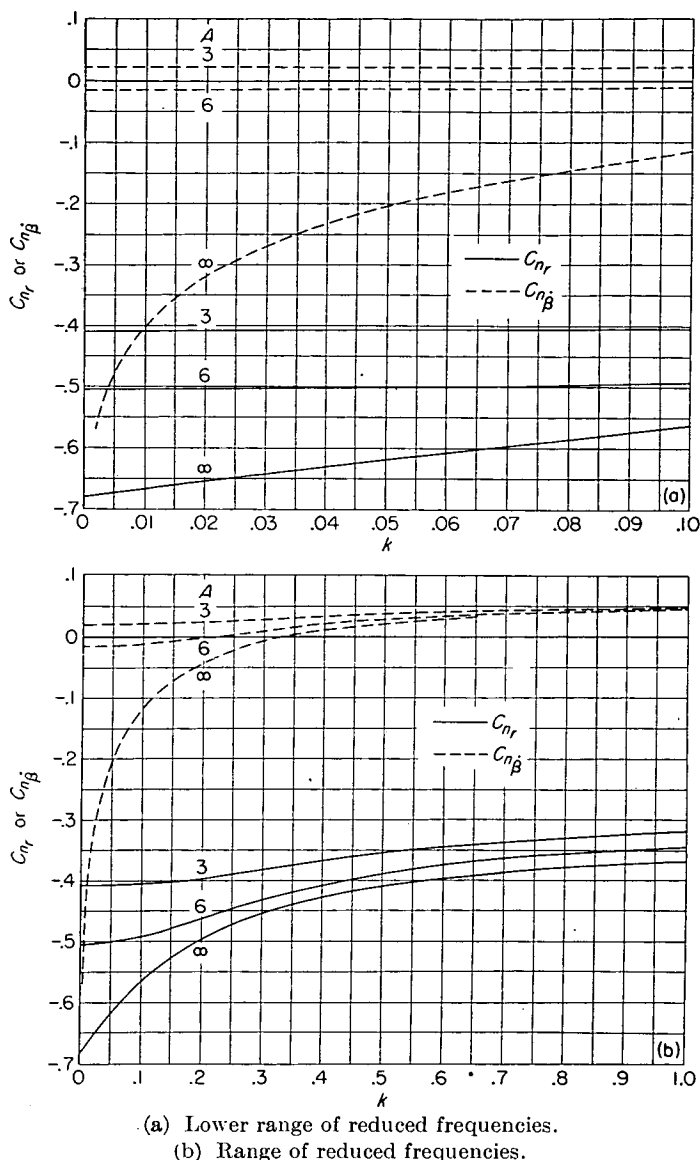


FIGURE 9.—The effect of reduced frequency on the stability derivatives C_{n_r} and $C_{n_{\beta}}$.

Changes in aspect ratio, for low aspect ratios, have a large effect on the directional stability; for high aspect ratios, changes in aspect ratio are less significant. The experimental values of directional stability obtained from the free-damping tests remained constant for the range of frequencies investigated, although the magnitudes are somewhat smaller than those predicted for the tail contribution by the calculation methods for aspect ratio 2. This result is not unexpected because all the finite-span unsteady-lift theories employed maintain some resemblance to the Prandtl lifting-line theory which, of course, yields values of the lift-curve slope for low aspect ratios which are unrealistically high. The values of directional stability obtained by both the free-oscillation and the static-test procedures were about the same magnitude.

Figure 12 gives a comparison of the contribution of a vertical tail to the stability derivatives $C_{n_{\beta}}$ and C_{n_r} as

computed from references 6 and 7. These derivatives were combined in the proportion

$$C_{n_{\beta}} + C_{n_r} k^2 \left(\frac{b_w}{c_t} \right)^2$$

for comparison with the results of the oscillation tests. Although C_{n_r} is of appreciable magnitude itself, the factor $C_{n_r} k^2 \left(\frac{b_w}{c_t} \right)^2$ is very small relative to $C_{n_{\beta}}$ for the range of frequencies around $k=0.01$ and thus should not greatly affect the directional stability of the usual airplane.

The computed two-dimensional Mach number effects on the damping in yaw and directional stability of the tail surface of the test model are shown in figures 13 and 14. These results were included to illustrate the powerful effect of compressibility on the damping contributed by the tail surface. A complete coverage of one-degree-of-freedom flutter by use of these aerodynamic forces which may be obtained from references 10 to 12 is given in reference 14. Compressibility effects decrease the tail-surface damping at low frequencies and increase the damping at higher frequencies. The directional stability increases at all frequencies with increase in Mach number with the largest increases occurring in the high-frequency range of oscillations. Reference 15 which presents the two-dimensional oscillating aerodynamic forces at sonic speed indicates larger deviations from the results for incompressible flow than for the subsonic results given here.

CONCLUSIONS

The following conclusions were drawn from the results of an investigation made at a Mach number of 0.14 of the effects of frequency and aspect ratio on the directional damping and stability of a fuselage and vertical tail oscillating freely in yaw:

1. The contribution of the vertical tail to the damping in yaw of the model agreed within the limits of experimental accuracy with predictions of the approximate finite-aspect-ratio unsteady-lift theories for the range of frequencies tested.
2. Two-dimensional unsteady-lift theory predicted a much greater loss in damping with reduction in frequency than was shown by experiment or by the approximate finite-aspect-ratio unsteady-lift theories.
3. Both the unsteady-lift theories and the experimental results indicated that the contribution of the vertical tail to the directional stability of the model was relatively independent of frequency for frequencies comparable to those of lateral airplane motions.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., November 26, 1951.

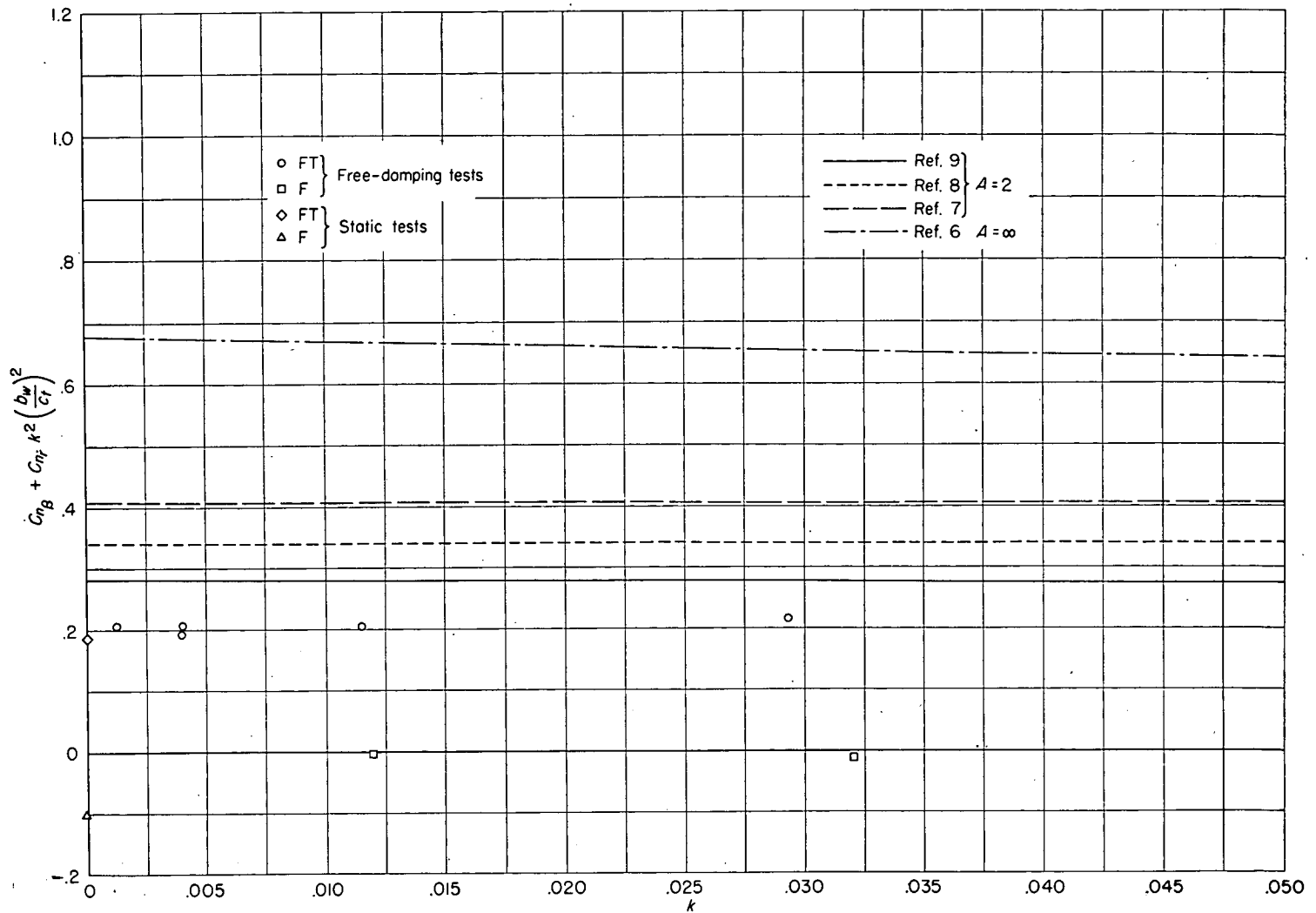


FIGURE 10.—The effect of frequency on the directional stability for a low range of the reduced frequency. Comparison of several methods for calculating frequency effect on the directional stability.

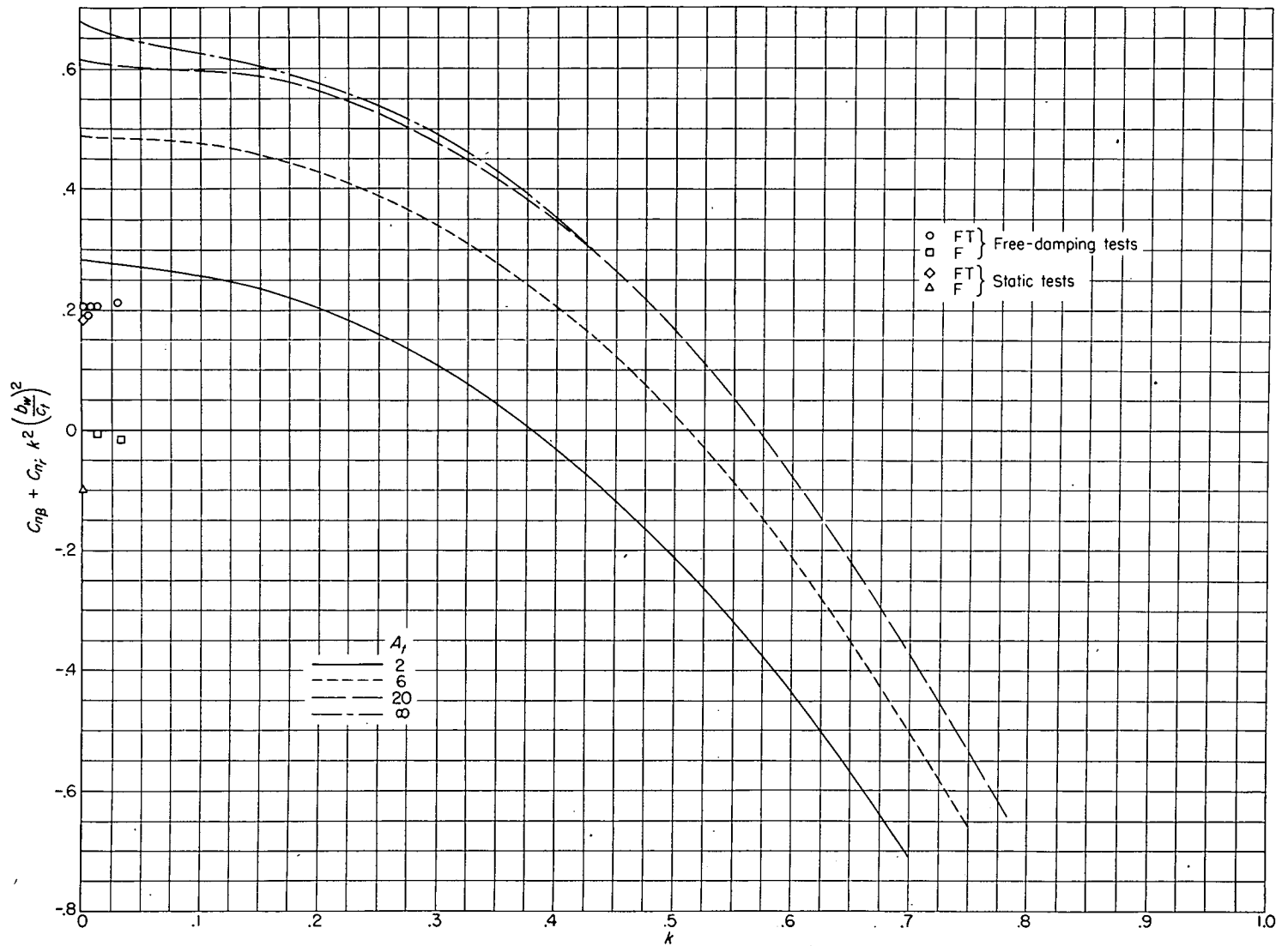
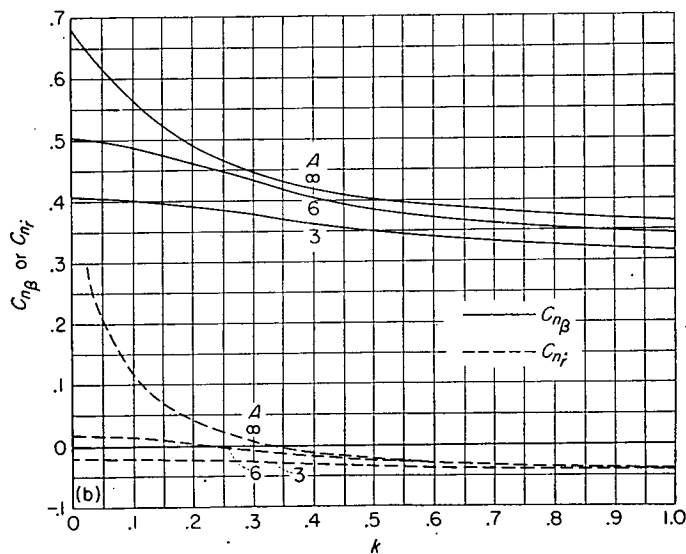
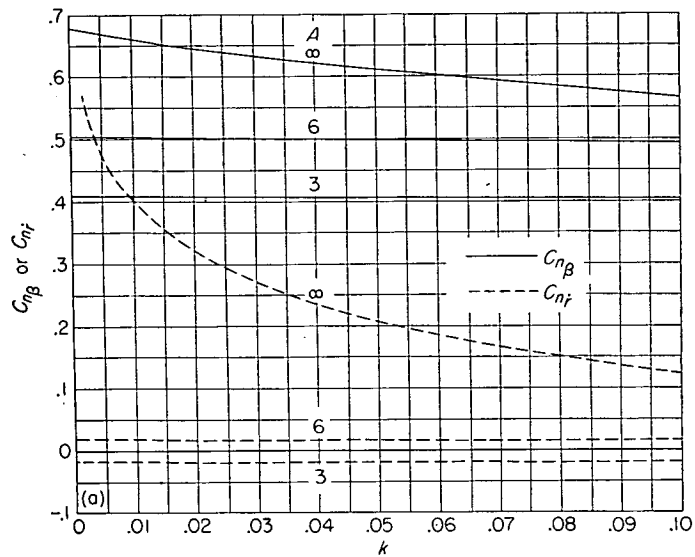
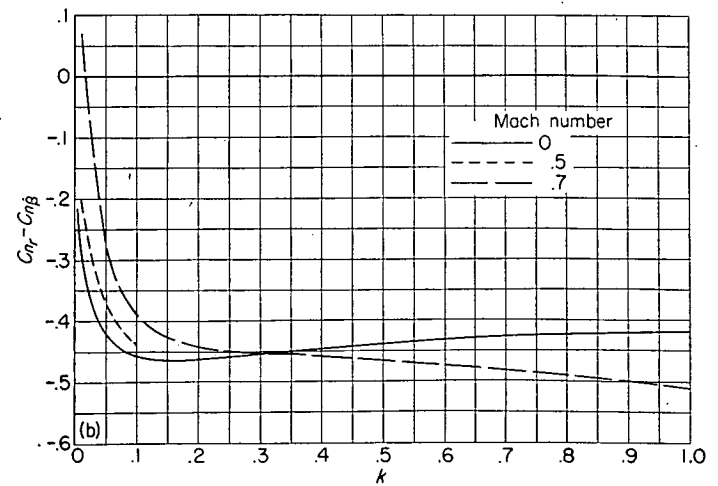
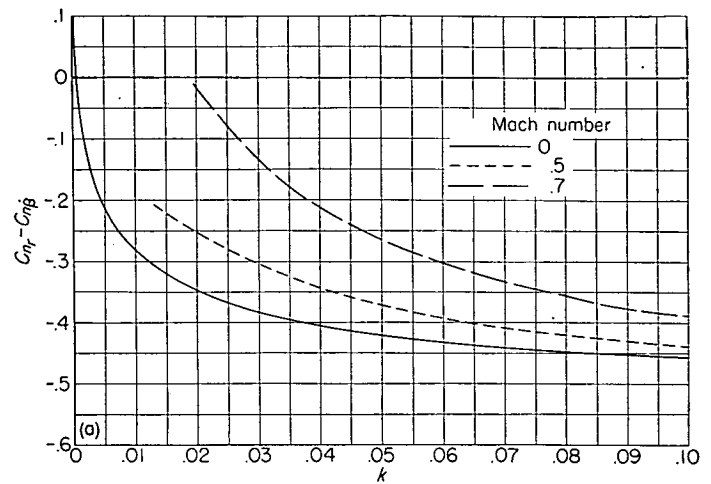


FIGURE 11.—The effect of aspect ratio on the directional stability as calculated by the method of Biot and Boehnlein for a range of reduced frequencies.



(a) Lower range of reduced frequencies.

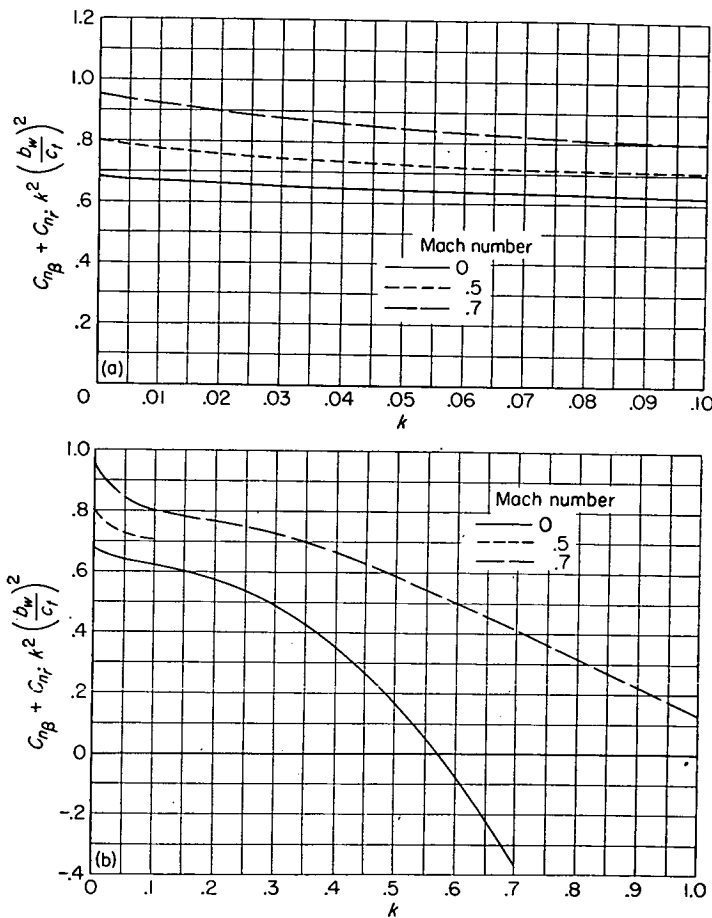
(b) Range of reduced frequencies.

FIGURE 12.—The effect of reduced frequency on the stability derivatives $C_{n\beta}$ and $C_{n\dot{\gamma}}$.

(a) Lower range of reduced frequencies.

(b) Range of reduced frequencies.

FIGURE 13.—The effect of compressibility on the damping in yaw.



(a) Lower range of reduced frequencies.

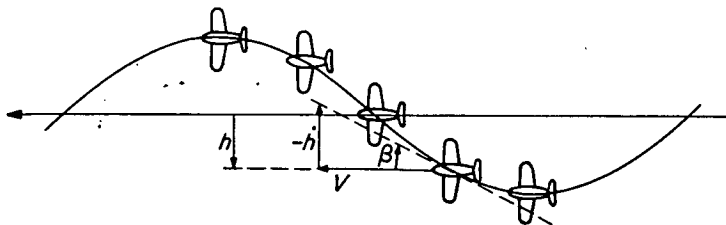
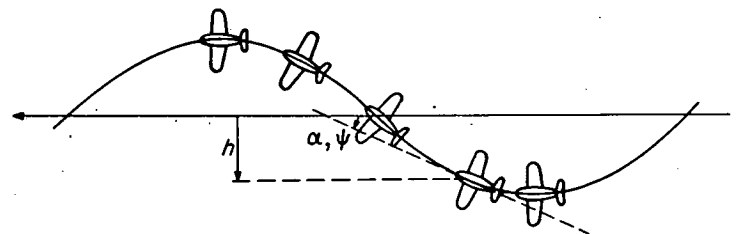
(b) Range of reduced frequencies.

FIGURE 14.—The effect of compressibility on the directional stability.

APPENDIX

CONVERSION OF UNSTEADY FORCES AND MOMENTS FROM FLUTTER TO STABILITY AXES

The conversion of the contribution of the vertical tail to the yawing moment of an airplane in terms of unsteady-lift functions and of the Theodorsen flutter axes to the stability axes which are fixed in the body is included here as a matter of general interest. Reference 16 gives considerable attention to such axis conversions. The following sketches indicate the paths and attitudes of airplanes performing sinusoidal sideslipping and yawing oscillations within the limits of the stability-axis system:

Sideslipping, $\beta = \beta_0 e^{i\omega t}$ Yawing, $\psi = \psi_0 e^{i\omega t}$

The Theodorsen coordinates h and α and the stability-axis coordinates β and ψ are shown with respect to the paths followed by the airplanes in sideslipping and yawing flight

In sideslipping flight

$$\beta = -\frac{1}{V} h$$

$$\alpha = 0$$

and in yawing flight

$$\psi = -\frac{1}{V} \dot{h} = \alpha$$

From reference 6 the contribution of the vertical tail to the yawing moment of an airplane in terms of the flutter coordinates is

$$N(\alpha, h) = -\frac{\rho c_t^2}{4} \left[\pi \left(\frac{1}{2} - a \right) \frac{V c_t}{2} \dot{\alpha} + \frac{\pi c_t^2}{4} \left(\frac{1}{8} + a^2 \right) \ddot{\alpha} - \frac{a \pi c_t}{2} \ddot{h} \right] + \rho V \frac{c_t^2}{2} \pi \left(a + \frac{1}{2} \right) C(k) \left[V \alpha + \dot{h} + \frac{c_t}{2} \left(\frac{1}{2} - a \right) \dot{\alpha} \right]$$

If this moment is for a sideslipping motion or a yawing motion in the sense given by stability axes fixed in the body and if the following terms are substituted for the sideslipping case

$$\begin{aligned} \alpha &= 0 \\ \dot{\alpha} &= 0 \\ \ddot{\alpha} &= 0 \\ \dot{h} &= -V\beta \\ \ddot{h} &= -V\dot{\beta} \end{aligned}$$

and for the yawing case

$$\begin{aligned} \dot{h} &= -V\psi \\ \ddot{h} &= -V\dot{\psi} \\ \alpha &= \psi \\ \dot{\alpha} &= \dot{\psi} \\ \ddot{\alpha} &= \ddot{\psi} \end{aligned}$$

the following results are obtained for the sideslipping case

$$N(\beta) = -\frac{\rho c_t^2}{4} \left(\frac{a \pi c_t V}{2} \dot{\beta} \right) + \frac{\rho V c_t^2 \pi}{2} \left(a + \frac{1}{2} \right) C(k) (-V\beta)$$

and for the yawing case

$$N(\psi) = -\frac{\rho c_t^2}{4} \left[\pi \left(\frac{1}{2} - a \right) \frac{V c_t}{2} \dot{\psi} + \frac{\pi c_t^2}{4} \left(\frac{1}{8} + a^2 \right) \ddot{\psi} + \frac{a \pi c_t V}{2} \dot{\psi} \right] + \frac{\rho V c_t^2}{2} \pi \left(a + \frac{1}{2} \right) C(k) \left[\frac{c_t}{2} \left(\frac{1}{2} - a \right) \dot{\psi} \right]$$

which reduce for the sideslipping case to

$$N(\beta) = -\pi q c_t^2 \left[\frac{a c_t \beta}{4 V} + \left(a + \frac{1}{2} \right) C(k) \beta \right]$$

and for the yawing case to

$$N(\psi) = -\frac{\pi}{2} q c_t^3 \left\{ \left[\frac{1}{4} - C(k) \left(\frac{1}{4} - a^2 \right) \right] \frac{\dot{\psi}}{V} + \frac{c_t}{4} \left(\frac{1}{8} + a^2 \right) \frac{\ddot{\psi}}{V^2} \right\}$$

Similar expressions may be obtained for the side force in pure

sideslipping and yawing flight. Reference 6 gives the side force of the vertical tail in terms of the flutter axes as

$$Y(\alpha, h) = \frac{\rho c_t^2}{4} \left(V \pi \dot{\alpha} + \pi \ddot{h} - \frac{\pi c_t a \ddot{\alpha}}{2} \right) + \pi \rho V c_t C(k) \left[V \alpha + \dot{h} + \frac{c_t}{2} \left(\frac{1}{2} - a \right) \dot{\alpha} \right]$$

which, when the following terms are substituted for the sideslipping case

$$\begin{aligned} \dot{h} &= -V\beta \\ \ddot{h} &= -V\dot{\beta} \\ \alpha &= 0 \\ \dot{\alpha} &= 0 \\ \ddot{\alpha} &= 0 \end{aligned}$$

and for the yawing case

$$\begin{aligned} \dot{h} &= -V\psi \\ \ddot{h} &= -V\dot{\psi} \\ \alpha &= \psi \\ \dot{\alpha} &= \dot{\psi} \\ \ddot{\alpha} &= \ddot{\psi} \end{aligned}$$

becomes for pure sideslipping and yawing flight

$$Y(\beta) = -q c_t \frac{\pi}{2} \left[c_t \left(\frac{\dot{\beta}}{V} \right) + 4 C(k) \beta \right]$$

and

$$Y(\psi) = q c_t^2 \frac{\pi}{2} \left[2 C(k) \left(\frac{1}{2} - a \right) \frac{\dot{\psi}}{V} - \frac{c_t}{2} a \frac{\ddot{\psi}}{V^2} \right]$$

Substituting the motions, in the manner done in the text, for the yawing moment,

$$\begin{aligned} \beta &= \beta_0 e^{i\omega t} \\ \psi &= \psi_0 e^{i\omega t} \end{aligned}$$

into these expressions and into the expressions in terms of stability derivatives which are

$$Y(\beta) = C_{Y_\beta} \beta q S_w + C_{Y_\beta} \beta \frac{q S_w b_w}{2 V}$$

$$Y(\psi) = C_{Y_r} \dot{\psi} \frac{q S_w b_w}{2 V} + C_{Y_r} \ddot{\psi} \frac{q S_w b_w^2}{4 V^2}$$

gives the following relationships for the tail contribution to the side-force derivatives when the real and imaginary parts of these equations are equated:

$$C_{Y_\beta} = -2\pi \frac{S_t}{S_w} F$$

$$C_{Y_\beta} = -\pi \frac{S_t}{S_w} \frac{c_t}{b_w} \left(1 + 2 \frac{G}{k} \right)$$

$$C_{Y_r} = 2 \frac{S_t}{S_w} \frac{c_t}{b_w} \pi F \left(\frac{1}{2} - a \right)$$

$$C_{Y_r} = -\frac{S_t}{S_w} \left(\frac{c_t}{b_w} \right)^2 \pi \left[a - 2 \frac{G}{k} \left(\frac{1}{2} - a \right) \right]$$

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REPORT 1131

DEFLECTION AND STRESS ANALYSIS OF THIN SOLID WINGS OF ARBITRARY PLAN FORM WITH PARTICULAR REFERENCE TO DELTA WINGS¹

By MANUEL STEIN, J. EDWARD ANDERSON, and JOHN M. HEDGEPEETH

SUMMARY

The structural analysis of arbitrary solid cantilever wings by small-deflection thin-plate theory is reduced to the solution of linear ordinary differential equations by the assumption that the chordwise deflections at any spanwise station may be expressed in the form of a power series in which the coefficients are functions of the spanwise coordinate. If the series is limited to the first two and three terms (that is, if linear and parabolic chordwise deflections, respectively, are assumed), the differential equations for the coefficients are solved exactly for uniformly loaded solid delta wings of constant thickness and of symmetrical double-wedge airfoil section with constant thickness ratio. For cases for which exact solutions to the differential equations cannot be obtained, a numerical procedure is derived. Experimental deflection and stress data for constant-thickness delta-plate specimens of 45° and 60° sweep are presented and are found to compare favorably with the present theory.

INTRODUCTION

One of the present trends in the development of high-speed airplanes and missiles is toward the use of thin low-aspect-ratio wings. The structural analysis of these wings often cannot be based on beam theory since the structural deformations may vary considerably from those of a beam and, indeed, may more closely approach those of a plate. In cases where the wing construction is solid or nearly solid the use of plate theory in the analysis is particularly valid, and it is this type of wing which is considered in the present report.

Exact solutions to the partial-differential equation of plate theory are not readily obtained, especially for plates of arbitrary shape and loading; however, a number of approximate solutions to specific problems on cantilever plates have appeared in the literature (see, for example, refs. 1 to 7). Of the approaches used in these references, only the one in references 6 and 7 is readily applicable to plates of arbitrary plan form, thickness distribution, and load distribution; thus it is the most useful one for the analysis of actual wings.

In reference 6 the cantilever-plate problem is simplified by the assumption that the deformations of the plate in the chordwise direction (parallel to the root) are linear. By minimizing the potential energy of the plate, the partial-differential equation of plate theory is replaced by two

simultaneous ordinary differential equations for the spanwise variations of the bending deflection and twist. In reference 7 the same ordinary differential equations are obtained in a different manner. Refinement of the analysis by inclusion of the effect of parabolic, cubic, or higher-order chordwise camber terms is indicated in reference 6, and as the order of refinement is increased a corresponding increase in the number of ordinary differential equations is obtained.

In the present report, which is an extension of reference 6, a general set of ordinary differential equations is presented which may be used to obtain any desired degree of approximation to the deflection of the plate. These equations are solved exactly for several cases of delta plates under uniform load first by considering linear chordwise deformation only and second by including the effect of parabolic chordwise camber. Comparisons are drawn between the stresses and deflections computed from the equations of each approximation and also with some experimental results.

The differential equations presented contain coefficients that depend on the plan form and stiffness distribution of the plate and on the loading. In this report, the plates considered in detail have coefficients such that the differential equations can be solved exactly; however, in cases for which exact solutions cannot be obtained a numerical procedure must be used. One such procedure is derived and its accuracy is demonstrated.

SYMBOLS

l	length of plate measured perpendicular to root
c	root chord of plate
p	lateral load per unit area, positive in z -direction
t	local thickness of plate
t_{av}	average thickness of plate
D	local flexural stiffness, $\frac{Et^3}{12(1-\mu^2)}$
$\bar{D}$	flexural stiffness based on average thickness, $\frac{Et_{av}^3}{12(1-\mu^2)}$
E	modulus of elasticity of material
μ	Poisson's ratio
w	deflection of plate, positive in z -direction
x, y, z	coordinates defined in figure 1

¹ Supersedes NACA TN 2621, "Deflection and Stress Analysis of Thin Solid Wings of Arbitrary Plan Form With Particular Reference to Delta Wings" by Manuel Stein, J. Edward Anderson, and John M. Hedgepeth, 1952.

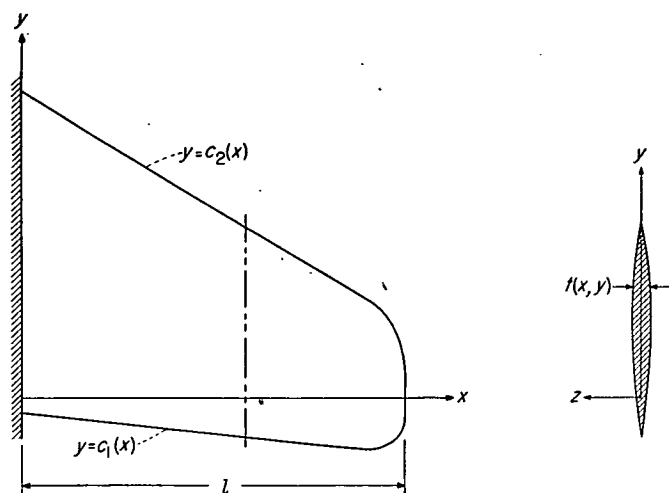


FIGURE 1.—Coordinate system used in the present analysis for a cantilever plate of arbitrary shape with arbitrary thickness variation.

φ_n	function of x , coefficient in power series for deflection $w = \sum_{n=0}^N \varphi_n(x)y^n$
$c_1(x), c_2(x)$	functions defining plan form (see fig. 1)
x_1	variable obtained by transformation $x_1 = 1 - \frac{x}{l}$
σ_x, σ_y	normal stresses
τ_{xy}	shear stress
σ	maximum principal stress
λ	aspect-ratio parameter, $\frac{l}{c} \sqrt{\frac{3}{2}(1-\mu)}$

RESULTS

The derivation of the general set of ordinary differential equations is given in appendix A. The general procedure outlined in reference 6 is followed; that is, the deflection of the plate w is expanded into a power series in y the chordwise coordinate with coefficients which are functions of x the spanwise coordinate (see fig. 1)

$$w = \varphi_0(x) + \varphi_1(x)y + \varphi_2(x)y^2 + \dots + \varphi_N(x)y^N \quad (1)$$

Equation (1) is substituted into the expression for the potential energy of the plate-load combination which is in turn minimized by the calculus of variations with respect to each of the coefficients φ_n . The results of the variational procedure appear as $N+1$ simultaneous differential equations with the coefficients φ_n as unknowns.

By taking a sufficient number of terms in the expansion of w , the resulting differential equations can be used to obtain any desired degree of accuracy in the solution for the deflections of any given cantilever plate subjected to an arbitrary lateral load. Of most interest, perhaps, are the particular cases for $N=1$ and $N=2$, which are obtained from the general set of equations and are simplified in appendix A. The case for $N=1$ (also derived in refs. 6

and 7) includes linear chordwise deflections, and the case for $N=2$ takes into account parabolic chordwise curvature. Although for most practical problems the solution by the parabolic theory should be adequate, cases might exist in which cubic, quartic, or even higher-order chordwise terms should be included, depending on the convergence of the series for the configuration considered.

The particular equations for $N=1$ and $N=2$ are used to determine the deflections and stresses of the following cantilever plates subjected to uniform lateral load:

- (1) A 45° delta plate of uniform thickness
- (2) A 60° delta plate of uniform thickness
- (3) A 45° delta plate of symmetrical double-wedge airfoil section with constant thickness ratio

Fortunately, for these configurations, the solution can be carried out exactly by both the linear and parabolic theories, and the details of these exact solutions are included in appendix B. In general, however, exact solutions cannot be obtained and some numerical method must be used. One such method, based on replacing derivatives by their first-order-approximation difference forms, is derived in appendix C.

A summary of the results for the three particular problems is shown in figures 2 to 11. Deflections obtained by the

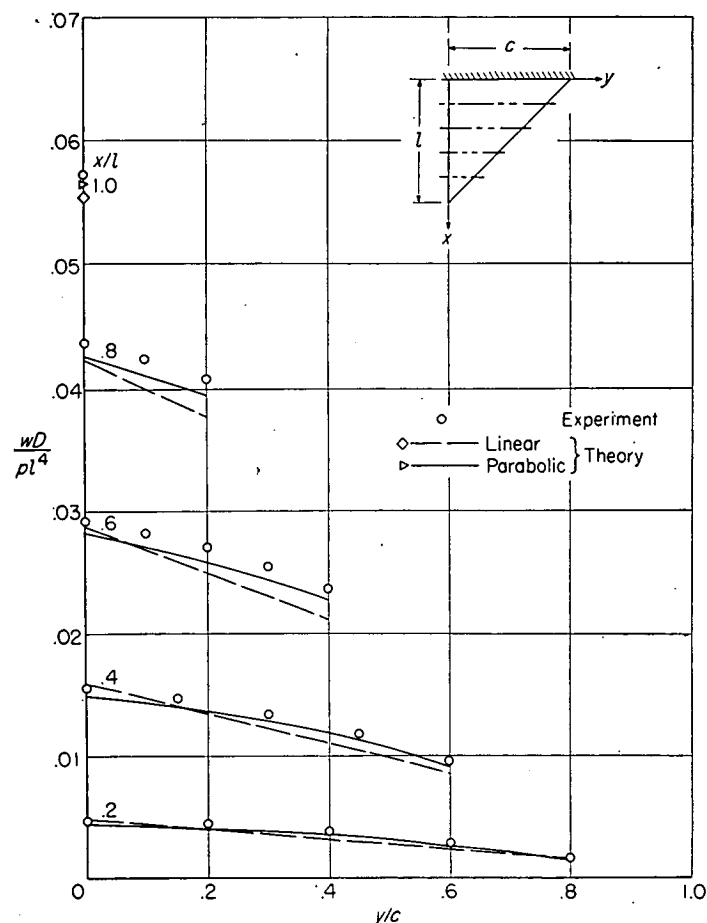


FIGURE 2.—Deflections of a 45° delta plate of uniform thickness under uniform load.

linear theory and the parabolic theory for the three configurations are compared in figures 2, 3, and 4. Stresses obtained by the linear theory and the parabolic theory for the three configurations are compared in figures 5, 6, and 7. Where available, experimental deflections and stresses are also shown in these figures. The details of the procedure used to obtain the experimental deflections of the 45° and 60° uniform-thickness plates and the experimental stresses in the 45° uniform-thickness plate are contained in appendix D; whereas the experimental root stresses for the 60° uniform-thickness plate were obtained from reference 8. Figures 8 to 11 present the comparison between deflections and stresses computed from the exact solutions of the differential equations and those computed from the numerical solutions of the same equations.

DISCUSSION

The results shown in figures 2 and 3 indicate that, with regard to deflections, either the linear theory or the parabolic theory is adequate for the case of a constant-thickness delta plate subjected to a uniform load, the comparison being somewhat better for the 60° plate than for the 45° plate. If accurate slopes in the chordwise direction (angle of attack)

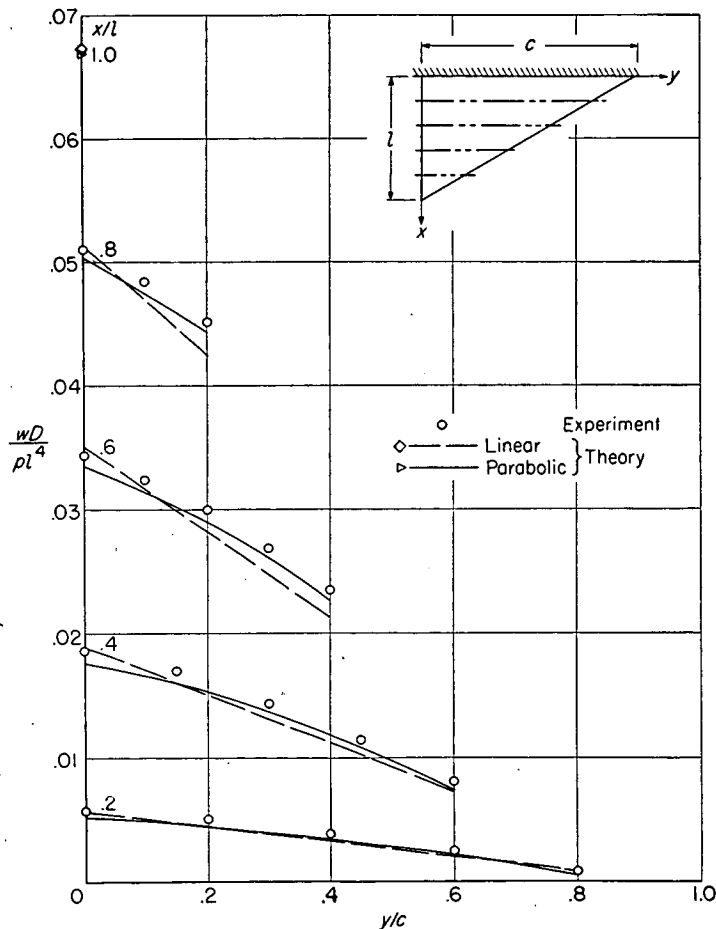


FIGURE 3.—Deflections of a 60° delta plate of uniform thickness under uniform load.

are desired, however, the parabolic theory must be used because the error in the angle of attack as computed by the linear theory is as much as 30 percent (see figs. 2 and 3). The appreciable anticlastic curvature, evidenced by the experimental results of figures 2 and 3, may be important aerodynamically and is, of course, not taken into account by the linear theory.

The apparent convergence of the aforementioned series in the case of the double-wedge-section plate (see fig. 4) implies that the linear theory is adequate for this case. The lack of chordwise curvature in the result obtained by the parabolic theory is attributable to the fact that the natural tendency of the plate to have anticlastic curvature is canceled by the opposite tendency of the thin edges to bend down under the load. Unfortunately, no experimental results are available for this configuration.

In figure 4 the plate stiffness $\bar{D}$ in the nondimensional parameter $w\bar{D}/pl^4$ is the local value of D at a point where the thickness is equal to the average thickness of the plate as a whole. Thus the results of figure 4 are comparable with the results of figure 2 on an equal-weight basis. It can be seen that the deflections of the double-wedge-section, constant-thickness-ratio plate are everywhere less than

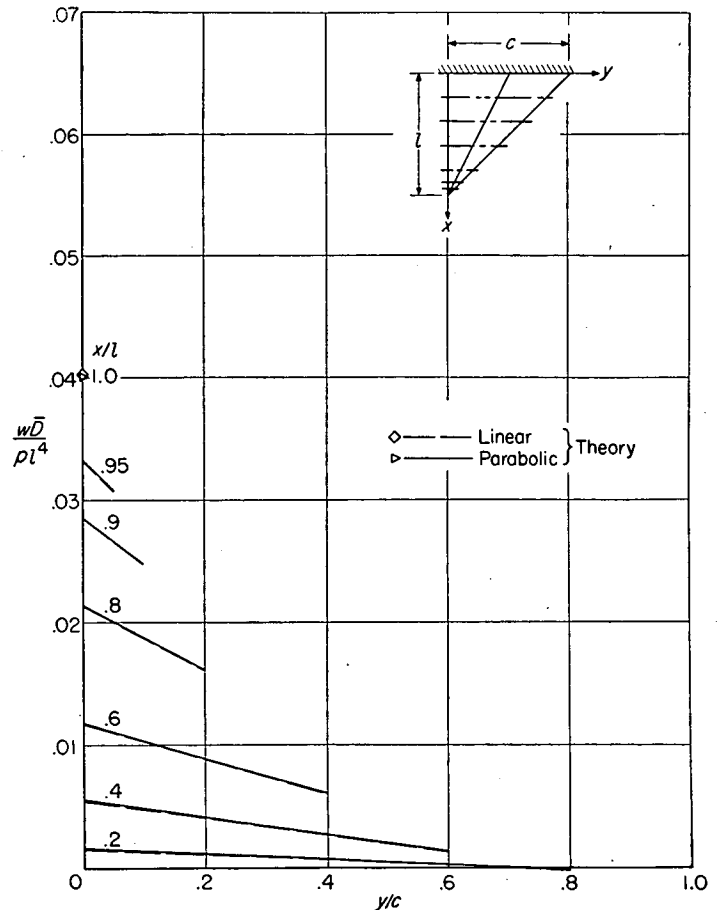


FIGURE 4.—Deflections of a 45° delta plate of symmetrical double-wedge section and constant thickness ratio under uniform load, $\bar{D} = \frac{Et_a^3}{12(1-\mu^2)}$.

those of the uniform-thickness plate although they increase rapidly near the tip. This curling-up or singularity in slope at the tip is a result of using a small-deflection theory and probably would not be so marked in an actual case.

The stress results for the 45° and 60° uniform-thickness delta plates indicate that both the linear and the parabolic theories are adequate and that the parabolic theory is better than the linear theory only near the root. It should be noted that, although the maximum principal stress over a large part of the 45° plate is plotted in figure 5, only the stresses normal to the root along the line $\frac{x}{l}=0.0087$ of the 60° plate are plotted in figure 6 since only these stresses are given in reference 8. (The maximum principal stress and the stress normal to the root are theoretically equal at the root since the root shear stress is zero.)

Experimental data are lacking for the double-wedge-section delta plate and, therefore, only theoretical stresses are shown in figure 7. As in the case of deflections, the results obtained from the linear theory and those obtained from the parabolic theory are almost coincident, the difference being greatest near the root. Figure 7 has also been plotted so that the results are directly comparable with those for the 45°

uniform-thickness plate in figure 5 on an equal-weight basis. As can be expected, the double-wedge-section, constant-thickness-ratio plate is a better design structurally; the stresses in the double-wedge-section plate are everywhere smaller and are almost constant in the spanwise direction.

The theoretical results in figures 2 to 7 have been obtained from exact solutions of the differential equations of the linear and parabolic theories. In order to test the reliability of the numerical method derived in appendix C, the differential equations were also solved numerically. The results shown in figures 8 and 9 indicate that the agreement is good between the numerical solution in which five equal intervals were used and the exact solution of the differential equations for the case of the 45° uniform-thickness plate. The same good agreement can be expected in other cases where the thickness and load distributions are not too erratic and where the plate stiffness does not go to zero at the tip—that is, when no singularities appear at the tip.

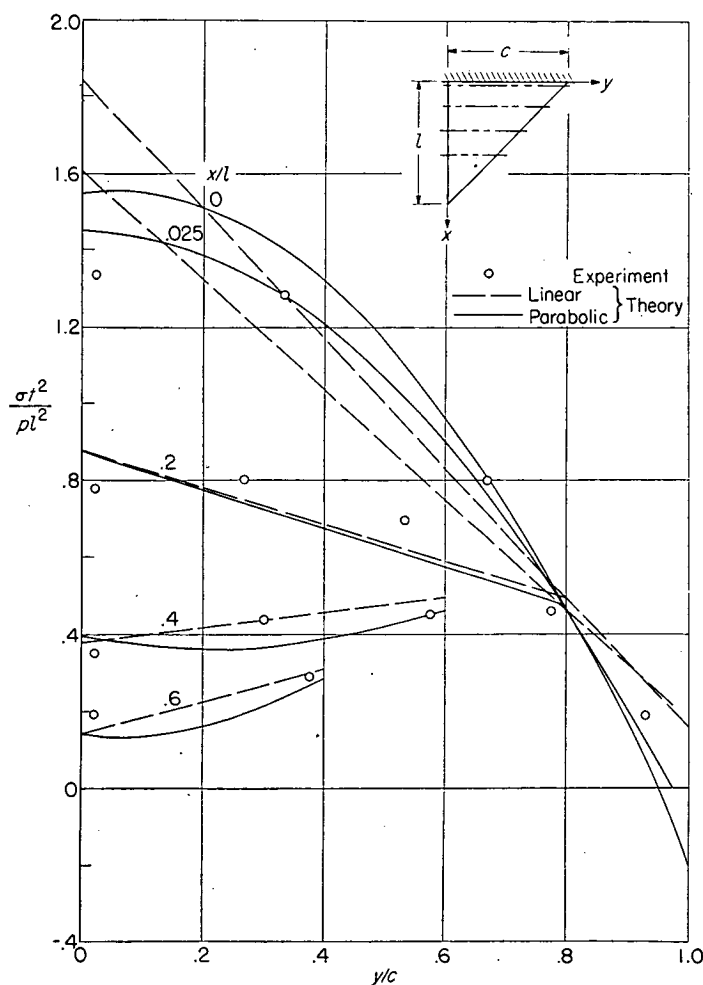


FIGURE 5.—Maximum principal stresses in a 45° delta plate of uniform thickness under uniform load.

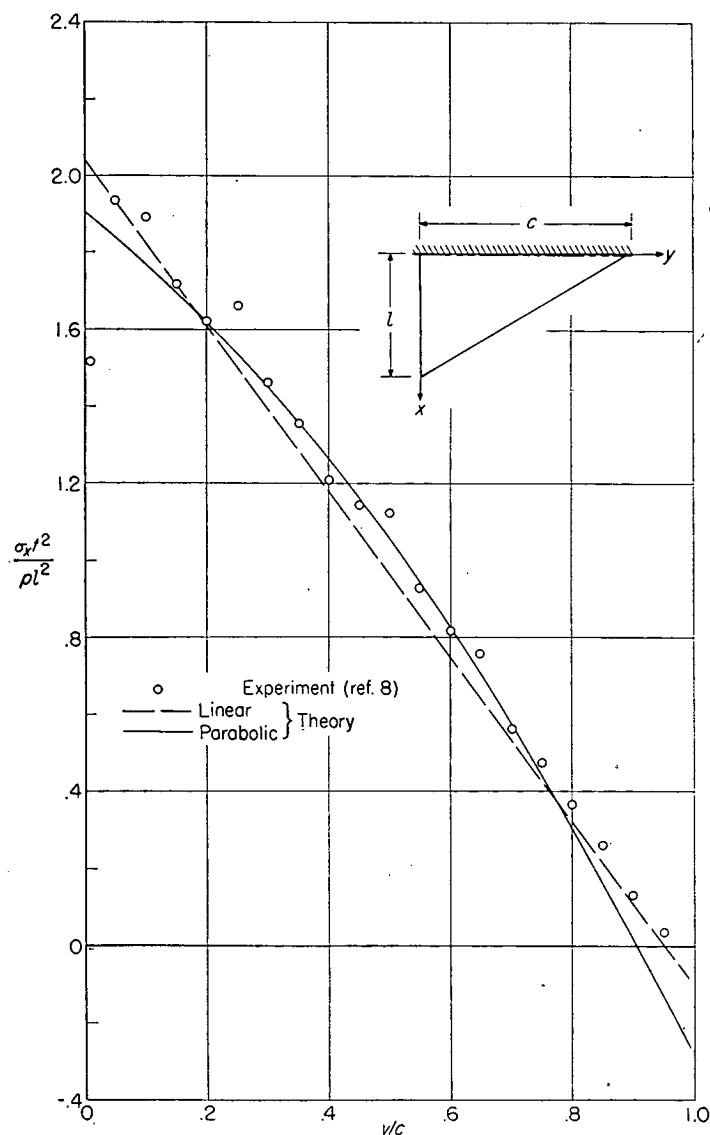


FIGURE 6.—Normal-stress distribution near the root (at $\frac{x}{l}=0.0087$) of a 60° delta plate of uniform thickness under uniform load.

Since the efficacy of the numerical method depends on how well parabolic arcs fit the various functions between stations, serious error can result from blind application. An example of the seriousness of these errors and of the manner in which they can be remedied is shown in figures 10 and 11. In these figures a comparison is made between exact and numerical results obtained on the 45° double-wedge-section, constant-thickness-ratio plate. As can be expected, the five-station numerical solution fails to follow the exact solution in the neighborhood of the singularity at the tip. Since the region of trouble is localized at the tip, a reasonable remedy would be to decrease the spacing of the station points near the tip. This decrease in spacing may be accomplished either by using a greater number of equally spaced stations or by using unequally spaced stations crowded near the tip. The increase in accuracy obtained by increasing the number of equally spaced station points to ten is shown in figures 10 and 11.

CONCLUDING REMARKS

The general method presented herein for finding deflections and stresses of solid or nearly solid wings is, in principle,

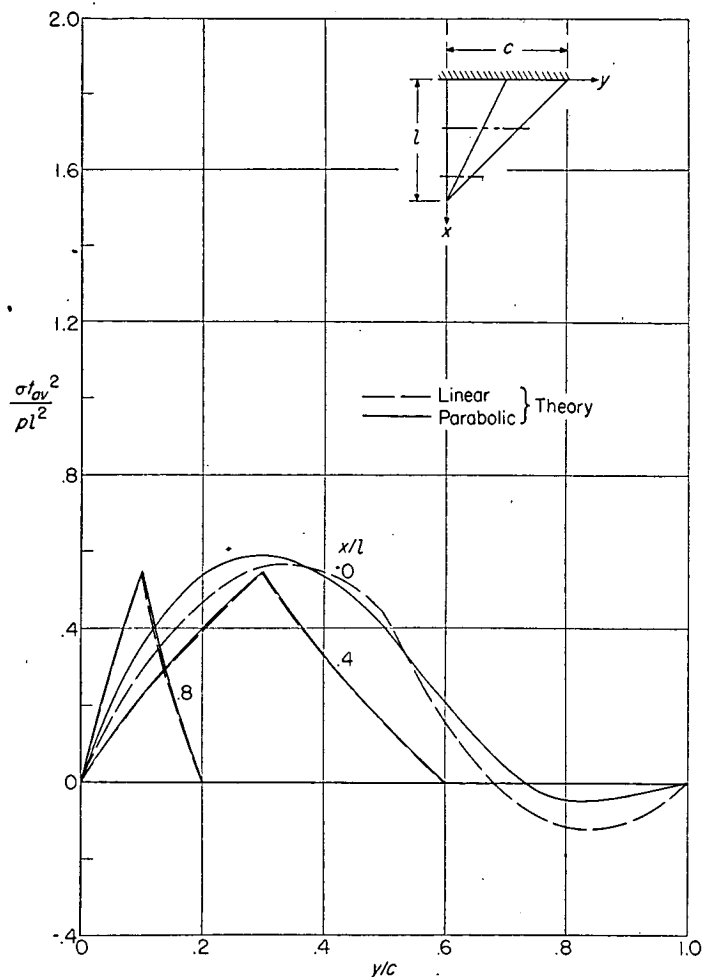


FIGURE 7.—Maximum principal stress in a 45° delta plate of symmetrical double-wedge airfoil section and constant thickness ratio under uniform load.

capable of yielding arbitrarily accurate results for any configuration. It is seen that, for the examples considered, only the first two or three terms in the series expansion need be considered to obtain adequate accuracy. In addition, for most practical plate-like wings with clamped roots the first two or three terms will probably be adequate, although problems may exist wherein more terms are needed.

The numerical procedure, derived for application in cases where exact solutions cannot be obtained, gives good agreement when compared with exact solutions if enough stations are taken along the span. The necessary number of stations is dependent on the type of thickness and loading distribution considered, five equally spaced stations being enough for the uniform-thickness delta wing subjected to uniform loading and ten being necessary for the double-wedge-section, constant-thickness-ratio delta wing subjected to uniform loading.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., November 30, 1951.

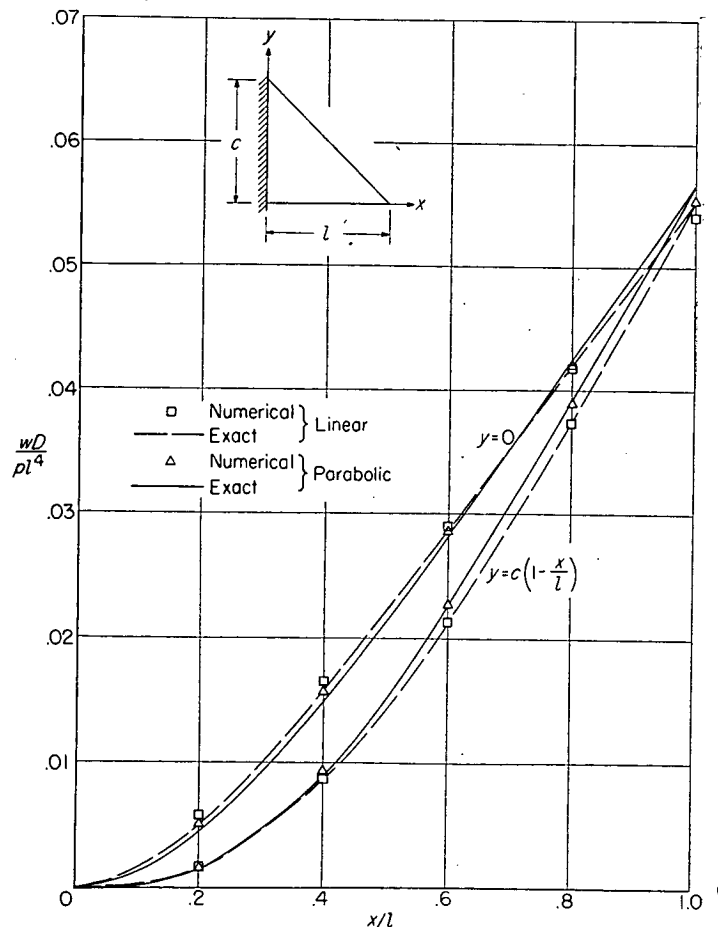


FIGURE 8.—Numerical and exact solutions of the differential equations for the deflections of the free edges of a 45° delta plate of uniform thickness under uniform load.

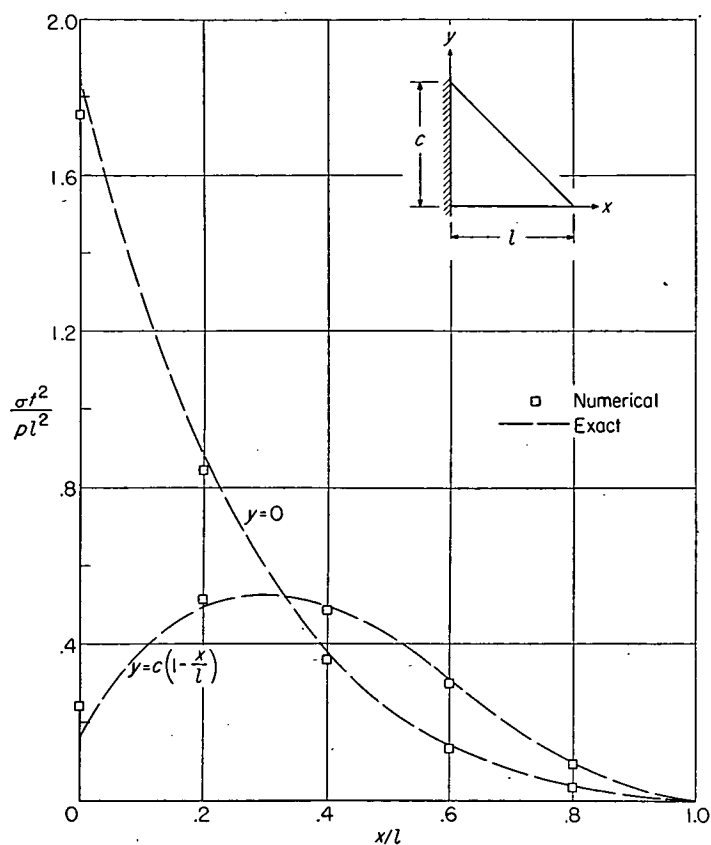


FIGURE 9.—Numerical and exact solutions of the differential equations (obtained by assuming linear chordwise deflections) for the maximum principal stresses along the free edges of a 45° delta plate of uniform thickness under uniform load.

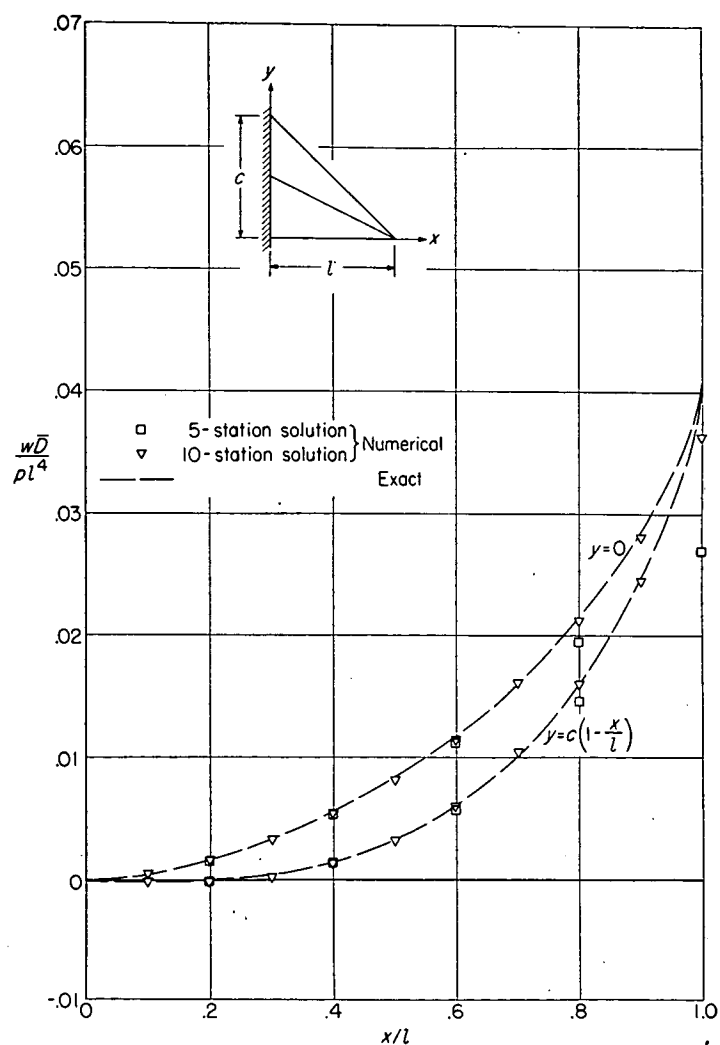


FIGURE 10.—Numerical and exact solutions of the differential equations (obtained by assuming linear chordwise deflections) for the deflections along the free edges of a 45° delta plate of symmetrical double-wedge airfoil section and constant thickness ratio under uniform load.

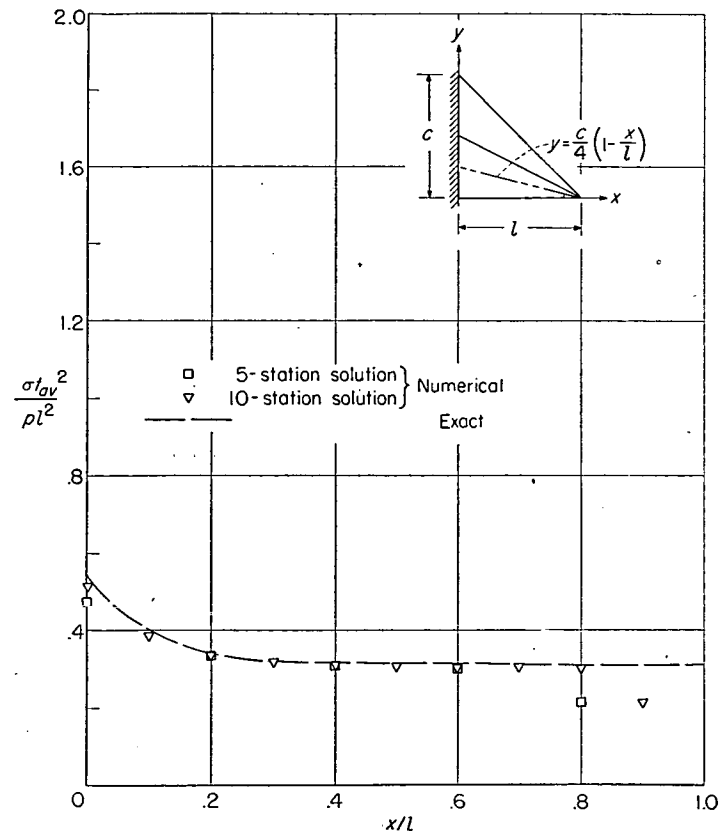


FIGURE 11.—Numerical and exact solutions of the differential equations (obtained by assuming linear chordwise deflections) for the maximum principal stress along the line $y = \frac{c}{4}(1 - \frac{x}{l})$ of a 45° delta plate of symmetrical double-wedge airfoil section and constant thickness ratio under uniform load.

APPENDIX A

DERIVATION OF DIFFERENTIAL EQUATIONS

The structure considered herein is a thin, elastic, isotropic, cantilever plate of arbitrary plan form and slowly varying thickness subjected to distributed lateral load (see fig. 1). By assuming that the deflection of the plate can be represented by a power series in the chordwise coordinate and by applying the minimum-potential-energy principle, a set of ordinary differential equations in the spanwise coordinate is obtained from which the coefficients of the power series may be determined. First the general set of equations is derived; then the particular equations for the cases of linear chordwise deflections and parabolic chordwise deflections are deduced from the general set and simplified.

General equations.—The potential energy of the system under consideration is

$$\text{Potential energy} = \int_0^l \int_{c_1(x)}^{c_2(x)} \left\{ \frac{D(x,y)}{2} \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\mu) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] - p(x,y)w \right\} dy dx \quad (A1)$$

in which

$$D(x,y) = \frac{E[t(x,y)]^3}{12(1-\mu^2)}$$

and $p(x,y)$ is the distributed lateral load.

The assumption is made that the deflection w can be represented by the power series

$$w = \sum_{n=0}^N \varphi_n(x) y^n \quad (A2)$$

Substitution of this expression for w into equation (A1) gives

$$\begin{aligned} \text{Potential energy} = \int_0^l dx \left\{ \frac{1}{2} \sum_{m=0}^N \sum_{n=0}^N [a_{m+n+1} \varphi_m'' \varphi_n'' + mn(m-1)(n-1) a_{m+n-3} \varphi_m \varphi_n + \right. \\ \left. 2\mu n(n-1) a_{m+n-1} \varphi_m'' \varphi_n + 2(1-\mu) mn a_{m+n-1} \varphi_m' \varphi_n'] - \sum_{n=0}^N p_{n+1} \varphi_n \right\} \end{aligned} \quad (A3)$$

in which

$$\left. \begin{aligned} a_r &= \int_{c_1(x)}^{c_2(x)} D(x, y) y^{r-1} dy & (r=1, 2, \dots, 2N+1) \\ p_r &= \int_{c_1(x)}^{c_2(x)} p(x, y) y^{r-1} dy & (r=1, 2, \dots, N+1) \end{aligned} \right\} \quad (A4)$$

and the primes denote differentiation with respect to x .

Minimization of the potential energy by means of the calculus of variations gives

$$\delta(\text{Potential energy}) = 0$$

$$\begin{aligned} = \int_0^l dx \left\{ \frac{1}{2} \sum_{m=0}^N \sum_{n=0}^N [a_{m+n+1} (\varphi_m'' \delta \varphi_n'' + \varphi_n'' \delta \varphi_m'') + mn(m-1)(n-1) a_{m+n-3} (\varphi_m \delta \varphi_n + \varphi_n \delta \varphi_m) + \right. \\ \left. 2\mu n(n-1) a_{m+n-1} (\varphi_m'' \delta \varphi_n + \varphi_n \delta \varphi_m'') + 2(1-\mu) mn a_{m+n-1} (\varphi_m' \delta \varphi_n' + \varphi_n' \delta \varphi_m')] - \sum_{n=0}^N p_{n+1} \delta \varphi_n \right\} \end{aligned}$$

Integrating by parts and collecting terms results in

$$\begin{aligned} 0 = \int_0^l dx \sum_{n=0}^N \delta \varphi_n \left\{ \sum_{m=0}^N [(a_{m+n+1} \varphi_m'')' + \mu m(m-1)(a_{m+n-1} \varphi_m)' - 2(1-\mu) mn(a_{m+n-1} \varphi_m')' + \mu n(n-1) a_{m+n-1} \varphi_m'' + \right. \\ \left. mn(m-1)(n-1) a_{m+n-3} \varphi_m] - p_{n+1} \right\} + \left\{ \sum_{n=0}^N \delta \varphi_n' \sum_{m=0}^N [a_{m+n+1} \varphi_m'' + \mu m(m-1) a_{m+n-1} \varphi_m] \right\}_0^l - \left\{ \sum_{n=0}^N \delta \varphi_n \sum_{m=0}^N [(a_{m+n+1} \varphi_m'')' + \right. \\ \left. \mu m(m-1)(a_{m+n-1} \varphi_m)' - 2(1-\mu) mn a_{m+n-1} \varphi_m'] \right\}_0^l \end{aligned} \quad (A5)$$

Everywhere in the region of the plate, except at the boundary $x=0$, the variation of w is arbitrary. At $x=0$ the cantilever boundary conditions

$$w = \frac{\partial w}{\partial x} = 0$$

yield

$$\varphi_n(0) = \varphi_n'(0) = 0 \quad (n=0, 1, \dots, N) \quad (A6)$$

and therefore the variation in these quantities must also be zero at $x=0$.

Equation (A5) is then satisfied if, in addition to equation (A6),

$$\sum_{m=0}^N [(a_{m+n+1} \varphi_m'')' + \mu m(m-1)(a_{m+n-1} \varphi_m)' - 2(1-\mu) mn(a_{m+n-1} \varphi_m')' + \mu n(n-1) a_{m+n-1} \varphi_m'' + mn(m-1)(n-1) a_{m+n-3} \varphi_m] = p_{n+1} \quad (n=0, 1, \dots, N) \quad (A7)$$

$$\sum_{m=0}^N [a_{m+n+1} \varphi_m'' + \mu m(m-1) a_{m+n-1} \varphi_m]_{x=l} = 0 \quad (n=0, 1, \dots, N) \quad (A8)$$

and

$$\sum_{m=0}^N [(a_{m+n+1} \varphi_m'')' + \mu m(m-1)(a_{m+n-1} \varphi_m)' - 2(1-\mu) mn a_{m+n-1} \varphi_m']_{x=l} = 0 \quad (n=0, 1, \dots, N) \quad (A9)$$

Equations (A7) form a set of $N+1$ simultaneous ordinary differential equations for the functions $\varphi_n(x)$. The functions φ_n are completely determined by these differential equations and the boundary conditions (A6), (A8), and (A9).

Particular case of linear chordwise deflections.—If $N=1$, the deflection function becomes

$$w = \varphi_0 + y\varphi_1 \quad (\text{A10})$$

a linear function in the chordwise direction, where φ_0 is the bending deflection and φ_1 is the twist. Equations (A7) become

$$(a_1\varphi_0'')'' + (a_2\varphi_1'')'' = p_1 \quad (\text{A11})$$

$$(a_2\varphi_0'')'' + (a_3\varphi_1'')'' - 2(1-\mu)(a_1\varphi_1')' = p_2 \quad (\text{A12})$$

The root boundary conditions, given by equation (A6), become

$$\varphi_0(0) = \varphi_0'(0) = \varphi_1(0) = \varphi_1'(0) = 0 \quad (\text{A13})$$

The tip boundary conditions, given by equations (A8) and (A9), become

$$(a_1\varphi_0'' + a_2\varphi_1'')_{x=l} = 0 \quad (\text{A14})$$

$$(a_2\varphi_0'' + a_3\varphi_1'')_{x=l} = 0 \quad (\text{A15})$$

$$[(a_1\varphi_0'')' + (a_2\varphi_1'')]_{x=l} = 0 \quad (\text{A16})$$

$$[(a_2\varphi_0'')' + (a_3\varphi_1'')] - 2(1-\mu)a_1\varphi_1'_{x=l} = 0 \quad (\text{A17})$$

Equations (A11) to (A17) are the differential equations and corresponding boundary conditions presented in reference 6 (if only distributed load is considered) where the symbols W and θ are used instead of φ_0 and φ_1 , respectively.

If equation (A11) is integrated twice and the boundary conditions (A14) and (A16) are used,

$$\varphi_0'' = -\frac{a_2}{a_1}\varphi_1'' + \frac{1}{a_1}\int_x^l \int_x^l p_1 dx^2 \quad (\text{A18})$$

Substitution of φ_0'' into equations (A12), (A15), and (A17) gives

$$(b_1\varphi_1'')'' - 2(1-\mu)(a_1\varphi_1')' = p_2 - \left(\frac{a_2}{a_1}\int_x^l \int_x^l p_1 dx^2\right)'' \quad (\text{A19})$$

$$(b_1\varphi_1'')_{x=l} = 0 \quad (\text{A20})$$

$$[(b_1\varphi_1'')' - 2(1-\mu)a_1\varphi_1']_{x=l} = 0 \quad (\text{A21})$$

in which

$$b_1 = a_3 - \frac{a_2^2}{a_1}$$

If equation (A19) is integrated once and the boundary condition (A21) is used,

$$(b_1\varphi_1'')' - 2(1-\mu)a_1\varphi_1' = -\int_x^l p_2 dx - \left(\frac{a_2}{a_1}\int_x^l \int_x^l p_1 dx^2\right)' \quad (\text{A22})$$

The differential equation (A22) is a second-order differential equation in φ_1' . The twist φ_1 and then the bending deflection φ_0 are obtained by solving equations (A22) and (A18), respectively, by applying the boundary conditions (A13) and (A20).

Particular case of parabolic chordwise deflections.—The effect of parabolic chordwise camber may be included by

letting $N=2$ in the general power series (eq. (A2)). If $N=2$, the deflection function becomes

$$w = \varphi_0 + y\varphi_1 + y^2\varphi_2$$

Here φ_2 represents the spanwise distribution of parabolic chordwise camber. For this case the differential equations (A7) become

$$(a_1\varphi_0'')'' + (a_2\varphi_1'')'' + (a_3\varphi_2'')'' + 2\mu(a_1\varphi_2)'' = p_1 \quad (\text{A23})$$

$$(a_2\varphi_0'')'' + (a_3\varphi_1'')'' + (a_4\varphi_2'')'' + 2\mu(a_2\varphi_2)'' -$$

$$2(1-\mu)[(a_1\varphi_1')' + 2(a_2\varphi_2')'] = p_2 \quad (\text{A24})$$

$$(a_3\varphi_0'')'' + (a_4\varphi_1'')'' + (a_5\varphi_2'')'' + 2\mu[a_1\varphi_0'' + a_2\varphi_1'' + a_3\varphi_2'' + (a_3\varphi_2)'] - 4(1-\mu)[(a_2\varphi_1')' + 2(a_3\varphi_2')'] + 4a_1\varphi_2 = p_3 \quad (\text{A25})$$

with the boundary conditions

$$\varphi_0(0) = \varphi_0'(0) = \varphi_1(0) = \varphi_1'(0) = \varphi_2(0) = \varphi_2'(0) = 0 \quad (\text{A26})$$

$$(a_1\varphi_0'' + a_2\varphi_1'' + a_3\varphi_2'' + 2\mu a_1\varphi_2)_{x=l} = 0 \quad (\text{A27})$$

$$(a_2\varphi_0'' + a_3\varphi_1'' + a_4\varphi_2'' + 2\mu a_2\varphi_2)_{x=l} = 0 \quad (\text{A28})$$

$$(a_3\varphi_0'' + a_4\varphi_1'' + a_5\varphi_2'' + 2\mu a_3\varphi_2)_{x=l} = 0 \quad (\text{A29})$$

$$[(a_1\varphi_0'')' + (a_2\varphi_1'')' + (a_3\varphi_2'')' + 2\mu(a_1\varphi_2)']_{x=l} = 0 \quad (\text{A30})$$

$$[(a_2\varphi_0'')' + (a_3\varphi_1'')' + (a_4\varphi_2'')' + 2\mu(a_2\varphi_2)'] -$$

$$2(1-\mu)(a_1\varphi_1' + 2a_2\varphi_2')_{x=l} = 0 \quad (\text{A31})$$

$$[(a_3\varphi_0'')' + (a_4\varphi_1'')' + (a_5\varphi_2'')' + 2\mu(a_3\varphi_2)'] -$$

$$4(1-\mu)(a_2\varphi_1' + 2a_3\varphi_2')_{x=l} = 0 \quad (\text{A32})$$

If equation (A23) is integrated twice and the boundary conditions (A27) and (A30) are used,

$$\varphi_0'' = -\frac{a_2}{a_1}\varphi_1'' - \frac{a_3}{a_1}\varphi_2'' - 2\mu\varphi_2 + \frac{1}{a_1}\int_x^l \int_x^l p_1 dx^2 \quad (\text{A33})$$

Substitution of φ_0'' into the remaining differential equations and boundary conditions results in

$$(b_1\varphi_1'')'' + (b_2\varphi_2'')'' - 2(1-\mu)[(a_1\varphi_1')' + 2(a_2\varphi_2')'] = p_2 - \left(\frac{a_2}{a_1}\int_x^l \int_x^l p_1 dx^2\right)'' \quad (\text{A34})$$

$$(b_2\varphi_1'')'' + (b_3\varphi_2'')'' - 4(1-\mu)[(a_2\varphi_1')' + 2(a_3\varphi_2')'] + 4(1-\mu^2)a_1\varphi_2 = p_3 - 2\mu\int_x^l \int_x^l p_1 dx^2 - \left(\frac{a_3}{a_1}\int_x^l \int_x^l p_1 dx^2\right)'' \quad (\text{A35})$$

$$(b_1\varphi_1'' + b_2\varphi_2'')_{x=l} = 0 \quad (\text{A36})$$

$$[(b_1\varphi_1'')' + (b_2\varphi_2'')] - 2(1-\mu)(a_1\varphi_1' + 2a_2\varphi_2')_{x=l} = 0 \quad (\text{A37})$$

$$(b_2\varphi_1'' + b_3\varphi_2'')_{x=l} = 0 \quad (\text{A38})$$

$$[(b_2\varphi_1'')' + (b_3\varphi_2'')] - 4(1-\mu)(a_2\varphi_1' + 2a_3\varphi_2')_{x=l} = 0 \quad (\text{A39})$$

$$\varphi_1(0) = \varphi_1'(0) = \varphi_2(0) = \varphi_2'(0) = 0 \quad (\text{A40})$$

in which

$$b_1 = a_3 - \frac{a_2^2}{a_1}$$

$$b_2 = a_4 - \frac{a_2 a_3}{a_1}$$

$$b_3 = a_5 - \frac{a_3^2}{a_1}$$

If equation (A34) is integrated and the boundary condition (A37) is used,

$$(b_1 \varphi_1'')' + (b_2 \varphi_2'')' - 2(1-\mu)(a_1 \varphi_1' + 2a_2 \varphi_2') \\ = - \int_x^l p_2 dx - \left(\frac{a_2}{a_1} \int_x^l \int_x^l p_1 dx^2 \right)' \quad (\text{A41})$$

Thus φ_1 and φ_2 are obtained by solving equations (A35) and (A41) with the boundary conditions (A36), (A38), (A39), and (A40). Subsequently, φ_0 can be obtained by solving equation (A33) with the boundary conditions $\varphi_0(0) = \varphi_0'(0) = 0$.

Stresses.—After the approximate deflection of the plate is determined from equations (A18) and (A22) or from equations (A33), (A35), and (A41), the extreme-fiber stresses may be calculated from the well-known equations of thin-plate theory, which are (see, for example, ref. 9):

$$\sigma_x = \frac{6D}{t^2} \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma_y = \frac{6D}{t^2} \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right)$$

$$\tau_{xy} = \frac{6(1-\mu)D}{t^2} \frac{\partial^2 w}{\partial x \partial y}$$

The maximum principal stress σ at any point in the plate can be determined from

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

APPENDIX B

EXACT SOLUTIONS OF DIFFERENTIAL EQUATIONS FOR SOME SPECIFIC DELTA-PLATE PROBLEMS

The differential equations of appendix A for linear and parabolic chordwise deflections are solved exactly for uniformly loaded delta plates of constant thickness and of symmetrical double-wedge airfoil section with constant thickness ratio. The equations for deflections obtained by the linear theory are presented in terms of the aspect-ratio parameter λ for both kinds of delta plates. The equations for deflections obtained by the parabolic theory are presented for $\frac{l}{c}=1$ and $\frac{\sqrt{3}}{3}$ with $\mu=\frac{1}{3}$ for the constant-thickness delta plate and for $\frac{l}{c}=1$, also with $\mu=\frac{1}{3}$, for the delta plate of symmetrical double-wedge airfoil section with constant thickness ratio.

If the x -axis is passed through the edge perpendicular to the root and the substitution $x_1=1-\frac{x}{l}$ is made, the differential equations are clearly of the homogeneous type for which the solutions are of the form x_1^γ , where γ is a constant. For the configurations considered, the functions that define the plan form (see fig. 1) are then $c_1(x)=0$ and $c_2(x)=cx_1$, where c is the root chord. In all the equations of this appendix the primes denote differentiation with respect to the new independent variable x_1 .

DELTA PLATE OF UNIFORM THICKNESS UNDER UNIFORM LOAD

Since the stiffness D is a constant for uniform-thickness plates, the coefficients in the differential equations (see eq. (A4)) become

$$a_n = \frac{Dc^n}{n} x_1^n \quad (\text{B1a})$$

$$b_1 = a_3 - \frac{a_2^2}{a_1} = \frac{Dc^3}{12} x_1^3 \quad (\text{B1b})$$

$$b_2 = a_4 - \frac{a_2 a_3}{a_1} = \frac{Dc^4}{12} x_1^4 \quad (\text{B1c})$$

$$b_3 = a_5 - \frac{a_3^2}{a_1} = \frac{4Dc^5}{45} x_1^5 \quad (\text{B1d})$$

$$p_n = \frac{pc^n}{n} x_1^n \quad (\text{B1e})$$

Solution for linear chordwise deflections.—If the coefficients given by equations (B1) are substituted into equations (A22) and (A18) and the independent variable is changed to $x_1=1-\frac{x}{l}$, the following equations for linear

chordwise deflections result:

$$(x_1^3 \varphi_1'')' - 16\lambda^2 x_1 \varphi_1' = -2 \frac{pl^4}{Dc} x_1^3 \quad (\text{B2})$$

$$\varphi_0'' = -\frac{c}{2} x_1 \varphi_1'' + \frac{pl^4}{6D} x_1^2 \quad (\text{B3})$$

where

$$\lambda = \frac{l}{c} \sqrt{\frac{3}{2} (1-\mu)}$$

The boundary conditions to be used with these equations are obtained from equations (A13) and (A20) and are

$$\varphi_0(1) = \varphi_0'(1) = \varphi_1(1) = \varphi_1'(1) = 0 \quad (\text{B4})$$

$$(x_1^3 \varphi_1'')_{x_1=0} = 0 \quad (\text{B5})$$

The general solution of equation (B2) is

$$\varphi_1' = A_1 x_1^{\gamma-1} + A_2 x_1^{-\gamma-1} - \frac{x_1^2}{4(1-2\lambda^2)} \frac{pl^4}{Dc} \quad (\text{B6})$$

where

$$\gamma = \sqrt{1+16\lambda^2}$$

and A_1 and A_2 are arbitrary constants. Since λ^2 is inherently positive, the boundary condition (B5) requires that $A_2=0$. One integration of equation (B6) and the application of the conditions $\varphi_1(1) = \varphi_1'(1) = 0$ yields

$$\varphi_1 = \frac{1}{4(1-2\lambda^2)} \frac{pl^4}{Dc} \left(\frac{x_1^\gamma - 1}{\gamma} - \frac{x_1^3 - 1}{3} \right) \quad (\text{B7})$$

If equation (B3) is solved for φ_0 with the conditions $\varphi_0(1) = \varphi_0'(1) = 0$, the result is

$$\begin{aligned} \varphi_0 = \frac{pl^4}{8D} \frac{1}{1-2\lambda^2} \left[\frac{2}{9} (5-4\lambda^2) \left(1-x_1 - \frac{1-x_1^4}{4} \right) - \right. \\ \left. \frac{\gamma-1}{\gamma} \left(1-x_1 - \frac{1-x_1^{\gamma+1}}{\gamma+1} \right) \right] \quad (\text{B8}) \end{aligned}$$

Substitution of equations (B7) and (B8) into the equation

$$w = \varphi_0 + \gamma \varphi_1$$

gives the expression for the deflection w of the plate under the assumption of linear chordwise deflections.

Solution for parabolic chordwise deflections.—If the coefficients given by equations (B1) are substituted into equations (A41), (A35), and (A33) and the independent variable is again changed to $x_1=1-\frac{x}{l}$, the following equations for

parabolic chordwise deflections result:

$$(x_1^3 \varphi_1'')' + (x_1^4 c \varphi_2'')' - 16\lambda^2 (x_1 \varphi_1' + x_1^2 c \varphi_2') = -2 \frac{pl^4}{Dc} x_1^3 \quad (B9)$$

$$(x_1^4 \varphi_1'')'' + \frac{16}{15} (x_1^5 c \varphi_2'')'' - 16\lambda^2 \left[(x_1^2 \varphi_1')' + \frac{4}{3} (x_1^3 c \varphi_2')' \right] + \frac{64}{3} \lambda^4 \frac{1+\mu}{1-\mu} x_1 c \varphi_2 = -\frac{4}{3} \left(7 + \frac{2\mu\lambda^2}{1-\mu} \right) \frac{pl^4}{Dc} x_1^3 \quad (B10)$$

$$\varphi_0'' = -\frac{c}{2} x_1 \varphi_1'' - \frac{c^2}{3} x_1^2 \varphi_2'' - 2\mu l^2 \varphi_2 + \frac{pl^4}{D} \frac{x_1^2}{6} \quad (B11)$$

The boundary conditions to be used with these equations are

$$\varphi_0(1) = \varphi_0'(1) = \varphi_1(1) = \varphi_1'(1) = \varphi_2(1) = \varphi_2'(1) = 0 \quad (B12)$$

$$(x_1^3 \varphi_1'' + x_1^4 c \varphi_2'')_{x_1=0} = 0 \quad (B13)$$

$$\left(x_1^4 \varphi_1'' + \frac{16}{15} x_1^5 c \varphi_2'' \right)_{x_1=0} = 0 \quad (B14)$$

$$\left[(x_1^4 \varphi_1'')' + \frac{16}{15} (x_1^5 c \varphi_2'')' - 16\lambda^2 \left(x_1^2 \varphi_1' + \frac{4}{3} x_1^3 c \varphi_2' \right) \right]_{x_1=0} = 0 \quad (B15)$$

The homogeneous solutions of the simultaneous equations (B9) and (B10) are of the form

$$\varphi_1' = A x_1^{\gamma-1}$$

$$\varphi_2 = B x_1^{\gamma-1}$$

Substitution of these expressions into the homogeneous parts of equations (B9) and (B10) leads to the following characteristic equation from which λ may be determined:

$$\gamma^6 - 6(1+16\lambda^2)\gamma^4 + \left[320 \left(4 + \frac{1+\mu}{1-\mu} \right) \lambda^4 + 480\lambda^2 + 9 \right] \gamma^2 - 4 \left[1280 \frac{1+\mu}{1-\mu} \lambda^6 + 80 \left(4 + \frac{1+\mu}{1-\mu} \right) \lambda^4 + 96\lambda^2 + 1 \right] = 0 \quad (B16)$$

and gives the following relationship between A and B :

$$A = -(\gamma-1) \left[\frac{(\gamma-2)(\gamma+1) - 16\lambda^2}{\gamma^2 - 1 - 16\lambda^2} \right] cB$$

The particular solutions for uniform loading are given by

$$\varphi_1' = A_p x_1^2$$

$$\varphi_2 = B_p x_1^2$$

where

$$A_p = \frac{1}{2} \frac{4 \frac{3\mu+1}{1-\mu} \lambda^4 - 2 \frac{2-\mu}{1-\mu} \lambda^2 + 1}{8 \frac{1+\mu}{1-\mu} (2\lambda^2-1)\lambda^4 - (8\lambda^2-1)(4\lambda^2-1)} \frac{pl^4}{Dc}$$

$$B_p = -\frac{1}{4} \frac{\frac{4\mu\lambda^2}{1-\mu} (2\lambda^2-1) + 1 + 4\lambda^2}{8 \frac{1+\mu}{1-\mu} (2\lambda^2-1)\lambda^4 - (8\lambda^2-1)(4\lambda^2-1)} \frac{pl^4}{Dc^2}$$

The general solution is the sum of the homogeneous solutions and the particular integral

$$\varphi_1' = \sum_{n=1}^6 A_n x_1^{\gamma_n-1} + A_p x_1^2$$

$$\varphi_2 = \sum_{n=1}^6 B_n x_1^{\gamma_n-1} + B_p x_1^2 \quad (B17)$$

where the values γ_n are the six roots of the characteristic equation (B16) and the coefficients A_n and B_n are the coefficients corresponding to each of these roots. After integration φ_1 becomes

$$\varphi_1 = \sum_{n=1}^6 A_n \frac{x_1^{\gamma_n}}{\gamma_n} + A_p \frac{x_1^3}{3} + A_q \quad (B18)$$

The general solution for φ_0 from equation (B11) is found to be

$$\varphi_0 = \sum_{n=1}^6 C_n x_1^{\gamma_n+1} + C_p x_1^4 + C_q x_1 + C_r \quad (B19)$$

where, for $n=1, 2, \dots, 6$,

$$C_n = -\frac{c}{\gamma_n(\gamma_n+1)} \left\{ \frac{\gamma_n-1}{2} A_n + \frac{c}{3} \left[(\gamma_n-1)(\gamma_n-2) + \frac{4\mu\lambda^2}{1-\mu} \right] B_n \right\}$$

and

$$C_p = -\frac{c}{12} \left[A_p + \frac{2c}{3} \left(1 + \frac{2\mu\lambda^2}{1-\mu} \right) B_p - \frac{pl^4}{6Dc} \right]$$

The coefficients A_1 to A_6 , A_q , C_q , and C_r must be determined by the boundary conditions (B12) to (B15).

A complete set of coefficients is given in the following table for delta plates with Poisson's ratio μ equal to $1/3$ and with $\lambda = \frac{l}{c} = 1$ and $\frac{\sqrt{3}}{3}$. Deflection curves plotted from these results are shown in figures 2 and 3 in which the 45° plate corresponds to $\frac{l}{c} = 1$ and the 60° plate corresponds to $\frac{l}{c} = \frac{\sqrt{3}}{3}$.

m	γ_m		$A_m \frac{Dc}{pl^4}$		$B_m \frac{Dc^2}{pl^4}$		$C_m \frac{D}{pl^4}$	
	$\lambda=1$	$\lambda=\frac{\sqrt{3}}{3}$	$\lambda=1$	$\lambda=\frac{\sqrt{3}}{3}$	$\lambda=1$	$\lambda=\frac{\sqrt{3}}{3}$	$\lambda=1$	$\lambda=\frac{\sqrt{3}}{3}$
1	2.7034	1.5671	0.7378	0.09632	-0.3133	-0.1022	-0.02931	-0.003223
2	4.9437	3.6347	.02411	.3707	-.03039	-.4313	.003074	.01347
3	8.3816	4.7258	.03827	-.1766	-.006293	.07379	.000486	.002317
4	-2.7034	-1.5671	0	0	0	0	0	0
5	-4.9437	-3.6347	0	0	0	0	0	0
6	-8.3816	-4.7258	0	0	0	0	0	0
p	-----	-----	-.8000	-.2903	.3500	.4597	.04167	.004032
q	-----	-----	-.01557	-.02924	-----	-----	-.07152	-.08354
r	-----	-----	-----	-----	-----	-----	.05668	.06692

Substitution of equations (B17), (B18), and (B19) into the equation

$$w = \varphi_0 + y\varphi_1 + y^2\varphi_2$$

gives the expression for the deflection w of the plate under the assumption of parabolic chordwise deflection.

DELTA PLATE OF SYMMETRICAL DOUBLE-WEDGE AIRFOIL SECTION WITH CONSTANT THICKNESS RATIO UNDER UNIFORM LOAD

For a delta plate of symmetrical double-wedge airfoil section with a constant thickness ratio the thickness is a function of x and y and is given by the following equations:

$$t = 6t_{av} \frac{y}{c} \quad \left(0 \leq y \leq \frac{cx_1}{2}\right)$$

$$t = 6t_{av} \left(x_1 - \frac{y}{c}\right) \quad \left(\frac{cx_1}{2} \leq y \leq cx_1\right)$$

where t_{av} is the average thickness. From these expressions for the thickness the stiffness can be found and the coefficients in the differential equations become

$$\left. \begin{aligned} a_1 &= \frac{27\bar{D}c}{4} x_1^4 \\ a_2 &= \frac{27\bar{D}c^2}{8} x_1^5 \\ a_3 &= \frac{9\bar{D}c^3}{5} x_1^6 \\ a_4 &= \frac{81\bar{D}c^4}{80} x_1^7 \\ a_5 &= \frac{2673\bar{D}c^5}{4480} x_1^8 \\ b_1 &= \frac{9\bar{D}c^3}{80} x_1^6 \\ b_2 &= \frac{9\bar{D}c^4}{80} x_1^7 \\ b_3 &= \frac{2613\bar{D}c^5}{22400} x_1^8 \\ p_n &= \frac{pc^n}{n} x_1^n \end{aligned} \right\} \quad (B20)$$

Solution for linear chordwise deflections.—By use of the coefficients given by equations (B20), equations (A22) and

(A18) for linear chordwise deflections may be solved for φ_1 and φ_0 . The steps in the solution are the same in form as those for the uniform-thickness plate and the resulting equations are

$$\varphi_1 = \frac{40}{27 \left(\gamma^2 - \frac{9}{4}\right)} \left(\frac{1 - x_1^{\gamma - \frac{3}{2}}}{\gamma - \frac{3}{2}} + \log_e x_1 \right) \frac{pl^4}{\bar{D}c} \quad (B21)$$

and

$$\varphi_0 = \frac{1}{27} \left[\frac{20}{\gamma^2 - \frac{9}{4}} \frac{\gamma - \frac{5}{2}}{\gamma - \frac{3}{2}} \left(\frac{x_1^{\gamma - \frac{1}{2}} - 1}{\gamma - \frac{1}{2}} + 1 - x_1 \right) + \left(\frac{20}{\gamma^2 - \frac{9}{4}} + \frac{2}{3} \right) (x_1 \log_e x_1 + 1 - x_1) \right] \frac{pl^4}{\bar{D}} \quad (B22)$$

where

$$\gamma = \sqrt{\left(\frac{5}{2}\right)^2 + 80\lambda^2}$$

Solution for parabolic chordwise deflections.—By use of the coefficients given by equations (B20), equations (A41), (A35), and (A33) for parabolic chordwise deflections may be solved for φ_1 , φ_2 , and φ_0 . The steps in the solution are again the same in form as those for the uniform-thickness plate and the resulting general expressions for φ_1 , φ_2 , and φ_0 are

$$\varphi_1 = \sum_{n=1}^6 A_n \frac{x_1^{\gamma_n - \frac{3}{2}} - 1}{\gamma_n - \frac{3}{2}} + A_p \log_e x_1 \quad (B23)$$

$$\varphi_2 = \sum_{n=1}^6 B_n x_1^{\gamma_n - \frac{5}{2}} + B_p \frac{1}{x_1} \quad (B24)$$

$$\varphi_0 = \sum_{n=1}^6 C_n x_1^{\gamma_n - \frac{1}{2}} + C_p x_1 \log_e x_1 + C_q x_1 + C_r \quad (B25)$$

where the exponents γ_n are the roots of the characteristic equation

$$\left(\gamma^2 - \frac{25}{4} - 80\lambda^2\right) \left[\frac{871}{840} \left(\gamma^2 - \frac{25}{4}\right) \left(\gamma^2 - \frac{49}{4}\right) - \frac{256}{3} \lambda^2 \left(\gamma^2 - \frac{25}{4}\right) + \frac{320}{3} \frac{1+\mu}{1-\mu} \lambda^4 \right] - \gamma^2 \left(\gamma^2 - \frac{25}{4} - 80\lambda^2\right)^2 + \left[\frac{7}{2} \left(\gamma^2 - \frac{25}{4}\right) - 200\lambda^2 \right]^2 = 0 \quad (B26)$$

For $n=1, 2, \dots, 6$, A_n , B_n , and C_n are related by

$$B_n = - \frac{\gamma^2 - \frac{25}{4} - 80\lambda^2}{\left(\gamma - \frac{5}{2}\right) \left[\left(\gamma - \frac{7}{2}\right) \left(\gamma + \frac{5}{2}\right) - 80\lambda^2 \right]} \frac{A_n}{c}$$

$$C_n = - \frac{c}{\gamma_n - \frac{3}{2}} \left\{ \frac{\gamma_n - \frac{5}{2}}{2} A_n + \frac{4}{15} c \left[\left(\gamma_n - \frac{5}{2}\right) \left(\gamma_n - \frac{7}{2}\right) + \frac{5\mu\lambda^2}{1-\mu} \right] B_n \right\}$$

For uniform load the coefficients in the particular integrals of equations (B23), (B24), and (B25) are

$$A_p = \frac{10}{27} \frac{pl^4}{Dc} \frac{\frac{871}{7} + 1024\lambda^2 + 320 \frac{1+\mu}{1-\mu} \lambda^4 - 16 \left(5 + \frac{2\mu}{1-\mu} \lambda^2\right) (10\lambda^2 + 1)}{(20\lambda^2 + 1) \left(\frac{871}{7} + 1024\lambda^2 + 320 \frac{1+\mu}{1-\mu} \lambda^4\right) - 120(16\lambda^2 + 1)(10\lambda^2 + 1)}$$

$$B_p = \frac{40}{27} \frac{pl^4}{Dc^2} \frac{15(16\lambda^2 + 1) - 2(20\lambda^2 + 1) \left(5 + \frac{2\mu\lambda^2}{1-\mu}\right)}{(20\lambda^2 + 1) \left(\frac{871}{7} + 1024\lambda^2 + 320 \frac{1+\mu}{1-\mu} \lambda^4\right) - 120(16\lambda^2 + 1)(10\lambda^2 + 1)}$$

$$C_p = \frac{c}{2} A_p - \frac{8}{15} c^2 \left(1 + \frac{5}{2} \frac{\mu\lambda^2}{1-\mu}\right) B_p + \frac{2}{81} \frac{pl^4}{D}$$

The coefficients A_1 to A_6 , A_q , C_q , and C_r are again determined by the boundary conditions (A26), (A36), (A38), and (A39) in which the coefficients given by equations (B20) are substituted.

For Poisson's ratio μ equal to $\frac{1}{3}$ and $\lambda = \frac{l}{c} = 1$, the solution of the characteristic equation (B26) leads to two real values and two pairs of complex conjugate values for γ . The identity

$$x_1^{a \pm ib} = x_1^a \cos(b \log_e x_1) \pm ix_1^a \sin(b \log_e x_1)$$

was therefore used to transform the terms involving the complex conjugate values into real form. If $\frac{l}{c} = 1$ and $\mu = \frac{1}{3}$, the solution is

$$\varphi_2 = \frac{pl^4}{Dc^2} \left[0.004070x_1^{3.947} - 0.004363x_1^{8.075} \cos(2.825 \log_e x_1) + 0.006893x_1^{8.075} \sin(2.825 \log_e x_1) + 0.000294 \frac{1}{x_1} \right]$$

$$\varphi_1 = \frac{pl^4}{Dc} \left[-0.003896x_1^{4.947} + 0.002134x_1^{9.075} \cos(2.825 \log_e x_1) - 0.006381x_1^{9.075} \sin(2.825 \log_e x_1) + 0.01794 \log_e x_1 + 0.001763 \right]$$

$$\varphi_0 = \frac{pl^4}{D} \left[0.0007715x_1^{5.947} - 0.0000708x_1^{10.075} \cos(2.825 \log_e x_1) + 0.001234x_1^{10.075} \sin(2.825 \log_e x_1) + 0.03331x_1 \log_e x_1 - 0.04096x_1 + 0.04026 \right]$$

APPENDIX C

NUMERICAL PROCEDURE FOR SOLVING DIFFERENTIAL EQUATIONS

In cases where the equations of the present theory cannot be solved exactly a numerical method must be used. In this appendix, equations (A19) and equations (A34) and (A35) are set up in difference form for numerical solution. Initially the assumption is made that the functions involved in the differential equations are continuous and nonsingular. In this case, first and second derivatives can be expressed by the standard difference forms

$$\left(\frac{d^2y}{dx^2}\right)_n = \frac{y_{n+1} - 2y_n + y_{n-1}}{\epsilon^2}$$

$$\left(\frac{dy}{dx}\right)_n = \frac{y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}}{\epsilon}$$

where ϵ is the distance between equally spaced station points.

In the following development five equally spaced spanwise stations are used; however, the extension to a different number of stations can be readily made.

First, consider equation (A19) resulting from the linear theory

$$(b_1\varphi_1'')'' - 2(1-\mu)(a_1\varphi_1')' = p_2 - \left(\frac{a_2}{a_1} \int_x^t \int_x^t p_1(x) dx^2\right)'' = q_1$$

Because of the nature of the tip boundary conditions for this equation, it can be conveniently put in the form

$$T' = q_1 \quad (C1)$$

where

$$T = (b_1\varphi_1'')' - 2(1-\mu)a_1\varphi_1'$$

In finding the difference equation equivalent to equation (C1), the quantity $(b_1\varphi_1'')$ is found in matrix form; from this expression is subtracted the matrix equivalent of $2(1-\mu)a_1\varphi_1'$; the resulting expression for T is multiplied by a differentiating matrix; and the product is equated to the right-hand side.

The quantity $(b_1\varphi_1'')$ at the half-stations can be expressed in matrix form as follows:

$$\begin{bmatrix} \varphi_1''_0 \\ \varphi_1''_1 \\ \varphi_1''_2 \\ \varphi_1''_3 \\ \varphi_1''_4 \end{bmatrix} = \frac{1}{\epsilon^2} \begin{bmatrix} 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{1-1} \\ \varphi_{10} \\ \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix} \quad (C2)$$

where the second subscript denotes the station point, the subscript at the root station being 0 and at the tip 5. The root boundary conditions are now applied; namely,

$$\varphi_1(0) = 0 = \varphi_{10}$$

$$\varphi_1'(0) = 0 = \frac{\varphi_{11} - \varphi_{1-1}}{2\epsilon}$$

Thus, after the values of $\varphi_{10} = 0$ and $\varphi_{1-1} = \varphi_{11}$ are substituted, equation (C2) becomes

$$\begin{bmatrix} \varphi_1''_0 \\ \varphi_1''_1 \\ \varphi_1''_2 \\ \varphi_1''_3 \\ \varphi_1''_4 \end{bmatrix} = \frac{1}{\epsilon^2} \begin{bmatrix} 2 & & & & \\ -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} (b_1\varphi_1'')_0 \\ (b_1\varphi_1'')_1 \\ (b_1\varphi_1'')_2 \\ (b_1\varphi_1'')_3 \\ (b_1\varphi_1'')_4 \end{bmatrix} = \frac{1}{\epsilon^2} \begin{bmatrix} b_{10} & & & & \\ & b_{11} & & & \\ & & b_{12} & & \\ & & & b_{13} & \\ & & & & b_{14} \end{bmatrix} \begin{bmatrix} 2 \\ -2 & 1 \\ 1 & -2 & 1 \\ & 1 & -2 & 1 \\ & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix} \quad (C3)$$

One of the tip boundary conditions is

$$(b_1\varphi_1'')_{x=l} = 0 = (b_1\varphi_1'')_5$$

Thus,

$$\begin{bmatrix} (b_1\varphi_1'')'_{1/2} \\ (b_1\varphi_1'')'_{3/2} \\ (b_1\varphi_1'')'_{5/2} \\ (b_1\varphi_1'')'_{7/2} \\ (b_1\varphi_1'')'_{9/2} \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} (b_1\varphi_1'')_0 \\ (b_1\varphi_1'')_1 \\ (b_1\varphi_1'')_2 \\ (b_1\varphi_1'')_3 \\ (b_1\varphi_1'')_4 \end{bmatrix} \quad (C4)$$

The matrix equivalent for the second term of T is

$$2(1-\mu) \begin{bmatrix} (a_1\varphi_1')_{1/2} \\ (a_1\varphi_1')_{3/2} \\ (a_1\varphi_1')_{5/2} \\ (a_1\varphi_1')_{7/2} \\ (a_1\varphi_1')_{9/2} \end{bmatrix} = \frac{2(1-\mu)}{\epsilon} \begin{bmatrix} a_{1,1/2} & & & & \\ & a_{1,3/2} & & & \\ & & a_{1,5/2} & & \\ & & & a_{1,7/2} & \\ & & & & a_{1,9/2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 & 1 \\ & -1 & 1 \\ & & -1 & 1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix}$$

Therefore T becomes

$$\begin{bmatrix} T_{1/2} \\ T_{3/2} \\ T_{5/2} \\ T_{7/2} \\ T_{9/2} \end{bmatrix} = \frac{1}{\epsilon^3} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} b_{10} \\ b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \end{bmatrix} \begin{bmatrix} 2 \\ -2 & 1 \\ 1 & -2 & 1 \\ & 1 & -2 & 1 \\ & & 1 & -2 & 1 \end{bmatrix} - \frac{2(1-\mu)}{\epsilon} \begin{bmatrix} a_{1,1/2} & & & & \\ & a_{1,3/2} & & & \\ & & a_{1,5/2} & & \\ & & & a_{1,7/2} & \\ & & & & a_{1,9/2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 & 1 \\ & -1 & 1 \\ & & -1 & 1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix} \quad (C5)$$

The right-hand side of equation (C1) can now be equated to the derivative of equation (C5); thus,

$$\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} T_{1/2} \\ T_{3/2} \\ T_{5/2} \\ T_{7/2} \\ T_{9/2} \end{bmatrix}$$

In order to obtain q_{15} , the boundary condition

$$T=0$$

at $x=l$ must be used. In other words, T goes from $T_{9/2}$ at station $4\frac{1}{2}$ to 0 at station 5. A straight line drawn between

these two points would have the slope $-\frac{2T_{9/2}}{\epsilon}$. The value of q_{15} is considered to be this slope; therefore,

$$\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \\ q_{15}/2 \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} T_{1/2} \\ T_{3/2} \\ T_{5/2} \\ T_{7/2} \\ T_{9/2} \end{bmatrix}$$

or

$$\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \\ q_{15}/2 \end{bmatrix} = \left\{ \frac{1}{\epsilon^4} \begin{bmatrix} 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} b_{10} \\ b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \end{bmatrix} \begin{bmatrix} 2 \\ -2 & 1 \\ 1 & -2 & 1 \\ & 1 & -2 & 1 \\ & & 1 & -2 & 1 \end{bmatrix} - \right. \\ \left. \frac{2(1-\mu)}{\epsilon^2} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} a_{1,1/2} \\ a_{1,3/2} \\ a_{1,5/2} \\ a_{1,7/2} \\ a_{1,9/2} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix} \right\} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix}$$

If the matrix multiplication is carried out, the difference equivalent of equation (C1) finally becomes

$$\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \\ q_{15}/2 \end{bmatrix} = \left\{ \frac{1}{\epsilon^4} [C_1] - \frac{2(1-\mu)}{\epsilon^2} [D_1] \right\} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix} \quad (C6)$$

where

$$[C_1] = \begin{bmatrix} 2b_{10} + 4b_{11} + b_{12} & -2b_{11} - 2b_{12} & b_{12} & & \\ -2b_{11} - 2b_{12} & b_{11} + 4b_{12} + b_{13} & -2b_{12} - 2b_{13} & b_{13} & \\ b_{12} & -2b_{12} - 2b_{13} & b_{12} + 4b_{13} + b_{14} & -2b_{13} - 2b_{14} & b_{14} \\ & b_{13} & -2b_{13} - 2b_{14} & b_{13} + 4b_{14} & -2b_{14} \\ & & b_{14} & -2b_{14} & b_{14} \end{bmatrix}$$

$$[D_1] = \begin{bmatrix} -a_{1,1/2} - a_{1,3/2} & a_{1,3/2} & & & \\ a_{1,3/2} & -a_{1,3/2} - a_{1,5/2} & a_{1,5/2} & & \\ & a_{1,5/2} & -a_{1,5/2} - a_{1,7/2} & a_{1,7/2} & \\ & & a_{1,7/2} & -a_{1,7/2} - a_{1,9/2} & a_{1,9/2} \\ & & & a_{1,9/2} & -a_{1,9/2} \end{bmatrix}$$

In order to determine φ_0 from φ_1 , use must be made of equation (A18)

$$\varphi_0'' = \frac{1}{a_1} \int_x^t \int_x^t p_1 dx^2 - \frac{a_2}{a_1} \varphi_1''$$

or, by use of the boundary condition $\varphi_0(0) = \varphi_0'(0) = 0$,

$$\varphi_0 = \int_0^x \int_0^x \frac{1}{a_1} \int_x^t \int_x^t p_1 dx^4 - \int_0^x \int_0^x \frac{a_2}{a_1} \varphi_1'' dx^2 \quad (C7)$$

In matrix form equation (C7) becomes

$$\begin{aligned}
 \begin{bmatrix} \varphi_{01} \\ \varphi_{02} \\ \varphi_{03} \\ \varphi_{04} \\ \varphi_{05} \end{bmatrix} = \epsilon^4 \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & 1/2 & 1 & & \\ & 1/2 & 1 & 1 & \\ & 1/2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/a_{10} & & & & \\ & 1/a_{11} & & & \\ & & 1/a_{12} & & \\ & & & 1/a_{13} & \\ & & & & 1/a_{14} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{15}/2 \end{bmatrix} - \\
 \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & 1/2 & 1 & & \\ & 1/2 & 1 & 1 & \\ & 1/2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (a_2/a_1)_0 & & & & \\ & (a_2/a_1)_1 & & & \\ & & (a_2/a_1)_2 & & \\ & & & (a_2/a_1)_3 & \\ & & & & (a_2/a_1)_4 \end{bmatrix} \begin{bmatrix} 2 & & & & \\ -2 & 1 & & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix} \quad (C8)
 \end{aligned}$$

Thus, if the values of q_1 (which can be determined numerically or analytically according to preference and feasibility) are known, the values of φ_1 can be found by solving equation (C6) and the values of φ_0 in turn by means of equation (C8).

The foregoing development applies to the case where only linear chordwise deformations are allowed. A similar procedure is followed in expressing the differential equations pertaining to the parabolic theory in difference form; only the results are shown herein.

The matrix equivalent to equations (A34) and (A35) is

$$\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \\ q_{15}/2 \\ q_{21} \\ q_{22} \\ q_{23} \\ q_{24} \\ q_{25}/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\epsilon^4} [C_1] - \frac{2(1-\mu)}{\epsilon^2} [D_1] & \frac{1}{\epsilon^4} [C_2] - \frac{4(1-\mu)}{\epsilon^2} [D_2] \\ \frac{1}{\epsilon^4} [C_2] - \frac{4(1-\mu)}{\epsilon^2} [D_2] & \frac{1}{\epsilon^4} [C_3] - \frac{8(1-\mu)}{\epsilon^2} [D_3] + 4(1-\mu^2)[E] \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \\ \varphi_{21} \\ \varphi_{22} \\ \varphi_{23} \\ \varphi_{24} \\ \varphi_{25} \end{bmatrix} \quad (C9)$$

where

$$\begin{aligned}
 [C_n] &= \begin{bmatrix} 2b_{n0} + 4b_{n1} + b_{n2} & -2b_{n1} - 2b_{n2} & b_{n2} & & \\ -2b_{n1} - 2b_{n2} & b_{n1} + 4b_{n2} + b_{n3} & -2b_{n2} - 2b_{n3} & b_{n3} & \\ b_{n2} & -2b_{n2} - 2b_{n3} & b_{n2} + 4b_{n3} + b_{n4} & -2b_{n3} - 2b_{n4} & b_{n4} \\ & b_{n3} & -2b_{n3} - 2b_{n4} & b_{n3} + 4b_{n4} & -2b_{n4} \\ & & b_{n4} & -2b_{n4} & b_{n4} \end{bmatrix} \\
 [D_n] &= \begin{bmatrix} -a_{n,1/2} - a_{n,3/2} & a_{n,3/2} & & & \\ a_{n,3/2} & -a_{n,3/2} - a_{n,5/2} & a_{n,5/2} & & \\ & a_{n,5/2} & -a_{n,5/2} - a_{n,7/2} & a_{n,7/2} & \\ & & a_{n,7/2} & -a_{n,7/2} - a_{n,9/2} & a_{n,9/2} \\ & & & a_{n,9/2} & -a_{n,9/2} \end{bmatrix}
 \end{aligned}$$

$$[E] = \begin{bmatrix} a_{11} & & & & \\ & a_{12} & & & \\ & & a_{13} & & \\ & & & a_{14} & \\ & & & & a_{15} \end{bmatrix}$$

and q_1 and q_2 are the right-hand sides of equations (A34) and (A35), respectively; that is,

$$q_1 = p_2 - \left(\frac{a_2}{a_1} \int_x^l \int_x^l p_1 dx^2 \right)''$$

$$q_2 = p_3 - 2\mu \int_x^l \int_x^l p_1 dx^2 - \left(\frac{a_3}{a_1} \int_x^l \int_x^l p_1 dx^2 \right)''$$

With φ_1 and φ_2 known, φ_0 can be obtained by use of equation (A33)

$$\begin{bmatrix} \varphi_{01} \\ \varphi_{02} \\ \varphi_{03} \\ \varphi_{04} \\ \varphi_{05} \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & 1/2 \end{bmatrix} \left\{ \epsilon^4 \begin{bmatrix} 1/a_{10} & & & & \\ & 1/a_{11} & & & \\ & & 1/a_{12} & & \\ & & & 1/a_{13} & \\ & & & & 1/a_{14} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{15}/2 \end{bmatrix} \right\} - \begin{bmatrix} (a_2/a_1)_0 \\ & (a_2/a_1)_1 \\ & & (a_2/a_1)_2 \\ & & & (a_2/a_1)_3 \\ & & & & (a_2/a_1)_4 \end{bmatrix} \begin{bmatrix} 2 & & & & \\ -2 & 1 & & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix} - \begin{bmatrix} (a_3/a_1)_0 \\ & (a_3/a_1)_1 \\ & & (a_3/a_1)_2 \\ & & & (a_3/a_1)_3 \\ & & & & (a_3/a_1)_4 \end{bmatrix} \begin{bmatrix} 2 & & & & \\ -2 & 1 & & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{21} \\ \varphi_{22} \\ \varphi_{23} \\ \varphi_{24} \\ \varphi_{25} \end{bmatrix} - 2\mu \epsilon^2 \begin{bmatrix} \varphi_{21} \\ \varphi_{22} \\ \varphi_{23} \\ \varphi_{24} \\ \varphi_{25} \end{bmatrix} \quad (C10)$$

It should be noted that, as can be expected, the matrix equations (C6) and (C8) are merely special cases of equations (C9) and (C10), respectively. In addition, the square matrices in equations (C6) and (C9) are symmetric, a result that is consistent with the fact that the differential equations under consideration are self-adjoint.

In the beginning of this appendix the assumption was made that the functions involved in the differential equations are continuous and nonsingular. The difference solution, however, may be adequate for some cases in which this assumption is not strictly correct. For instance, the deflections of a plate with a discontinuous stiffness distribution could conceivably be not very different from the deflections of a plate

with a continuous stiffness distribution closely approximating the discontinuous distribution except in the neighborhood of the discontinuity. The results yielded by the difference solution in this case would be those associated with the continuous stiffness distribution. The number of stations may have to be increased, however, in order to minimize the inaccuracy introduced by the discontinuity or, in other cases, by a singularity. The case of the symmetrical double-wedge airfoil section, constant-thickness-ratio delta plate, discussed in the body of this report, is an example of a treatment of a singularity. In this case, although the solution is singular, adequate accuracy is obtained by the difference solution if ten equal intervals are used.

APPENDIX D

DEFLECTION AND STRESS EXPERIMENTS ON SOME TRIANGULAR CANTILEVER PLATES

Test specimens.—The specimens tested were: (1) a 45° right-triangular plate clamped along one leg and (2) a 60° right-triangular plate clamped along the longer leg. Each specimen, cut from 24S-T4 aluminum-alloy sheet of 0.250-inch thickness, had a length perpendicular to the clamped edge of 30 inches.

Method of testing.—Figure 12, a photograph of the test setup, shows the methods of clamping, loading, and measurement of deflections. A 1,000,000-pound clamping load (held constant during the test) was applied to the root area of each specimen and a uniform load of 0.204 psi was applied by 2-inch washers giving a tip deflection in each case of approximately $\frac{3}{4}$ inch.

The deflections were measured by dial gages placed at the points indicated in figures 2 and 3.

Stresses were obtained from the 45° specimen only. On this specimen, 13 resistance-wire rosette strain gages were placed at the points indicated in figure 5. The plate was loaded with 2-inch washers in four increments of 0.0847 psi per increment and the maximum tip deflection was 1.13 inches. Readings of all the strain gages were recorded at each increment of loading.

Analysis and discussion of data.—The deflection w was plotted in figures 2 and 3 in terms of the nondimensional

parameter wD/pl^4 , in which the elastic constants were taken as $E=10.6 \times 10^6$ psi and $\mu=\frac{1}{3}$. It was found that the dial-gage forces reduced the tip deflection of the plate by approximately 2 percent; however, since this error is of the same order of magnitude as that in the material properties and from other sources, no corrections are made in the results presented.

The readings of each of the 39 individual strain gages were plotted against load, and the slope of each of the resulting linear curves was taken as the average strain per unit load of the individual gage. The principal stresses were then calculated and plotted in figure 5 in terms of the nondimensional parameter $\sigma t^2/pl^2$.

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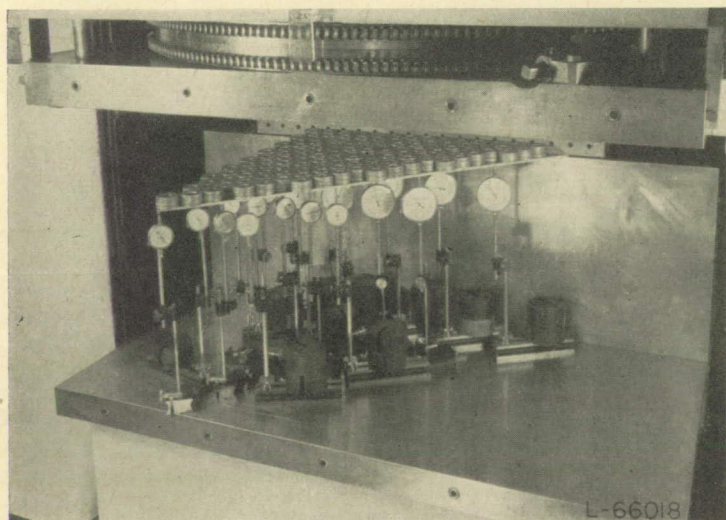


FIGURE 12.—Deflection test setup of the 45° delta plate under uniform load.

REPORT 1132

LAMINAR BOUNDARY LAYER ON CONE IN SUPERSONIC FLOW AT LARGE ANGLE OF ATTACK ¹

By FRANKLIN K. MOORE

SUMMARY

The laminar boundary-layer flow about a circular cone at large angles of attack to a supersonic stream has been analyzed in the plane of symmetry by a method applicable in general to the flow about conical bodies.

At the bottom of the cone, velocity profiles were obtained showing the expected tendency of the boundary layer to become thinner on the under side of the cone as the angle of attack is increased.

At the top of the cone, the analysis failed to yield unique solutions, except for small angles of attack. Beyond a certain critical angle of attack, boundary-layer flow does not exist in the plane of symmetry, thus indicating separation. This critical angle is presented as a function of Mach number and cone vertex angle.

INTRODUCTION

The supersonic aerodynamics of pointed bodies has considerable current interest in connection with the design of aircraft and missile fuselages. An important feature of the flow about such bodies is the behavior of the boundary layer and, in particular, the flow separation which may occur along the low-pressure side of the body due to angle of attack. The present report will consider the development of the laminar boundary layer on the surface of a right circular cone at an angle of attack to a supersonic stream (see fig. 1). The conical configuration may be considered an idealization of the nose portion of a supersonic aircraft fuselage.

Outside a thin boundary layer on a cone, the nonviscous supersonic flow (upon which the boundary layer itself depends) is "conical" in the sense that physical quantities

(such as velocity and pressure) are constant along any ray proceeding from the cone apex. The description of this outer flow, contained in references 1 to 3 and elucidated in reference 4, is considered adequate for the purposes of this report, but subject to restrictions which will be discussed subsequently.

In figure 2 is shown qualitatively the circumferential pressure distribution on the cone surface predicted for various angles of attack (see ref. 4). These pressure distributions depend only on the character of the nonviscous flow beyond the boundary layer, on the assumption that the boundary layer is extremely thin. When the angle of attack is very small, the pressure decreases monotonically from the bottom of the cone around to the top. For larger angles of attack there appears a region near the top of the cone wherein the pressure gradient reverses and the pressure increases toward the top. As the angle of attack is further increased, this region becomes greater in extent.

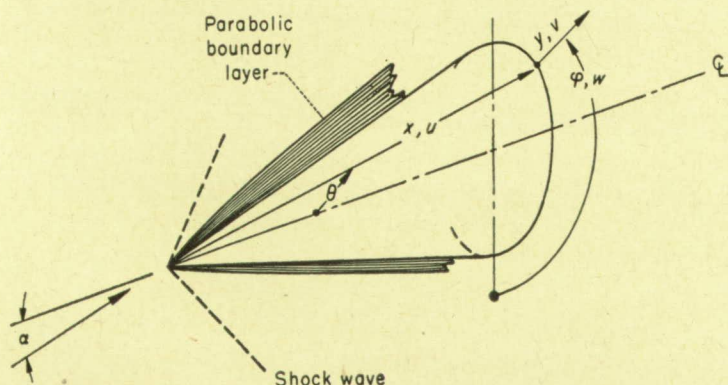


FIGURE 1.—Cone at angle of attack to supersonic stream.

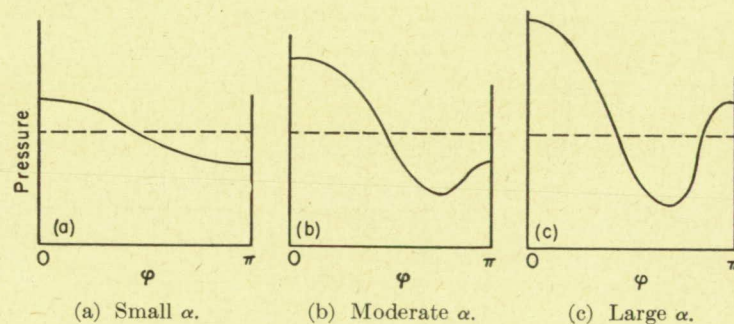


FIGURE 2.—Pressure distribution around cone for various ranges of angle of attack.

As a consequence of the conical nature of the nonviscous flow, it is shown in references 5 and 6 that the laminar boundary layer has parabolic similarity along generators of the cone; that is, velocity, pressure, and density inside the boundary layer are constant along any parabola (see fig. 1) drawn in any one meridional plane (plane passing through the body axis). Of course, circumferential variation of these quantities is to be expected when the cone is at angle of attack.

In reference 7 the effect of angle of attack on the laminar boundary layer is analyzed, in the limit of very small angle of attack, with the result that the boundary layer tends to be thicker on the top of the cone than on the bottom (fig. 3(a)).

¹ Supersedes NACA TN 2844, "Laminar Boundary Layer on Cone in Supersonic Flow at Large Angle of Attack" by Franklin K. Moore, 1952.

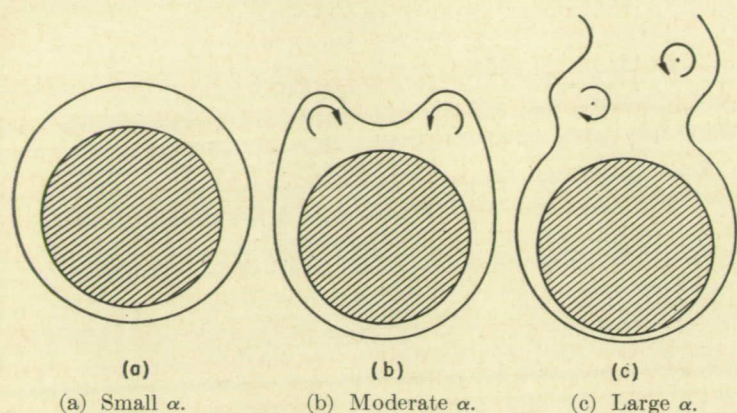
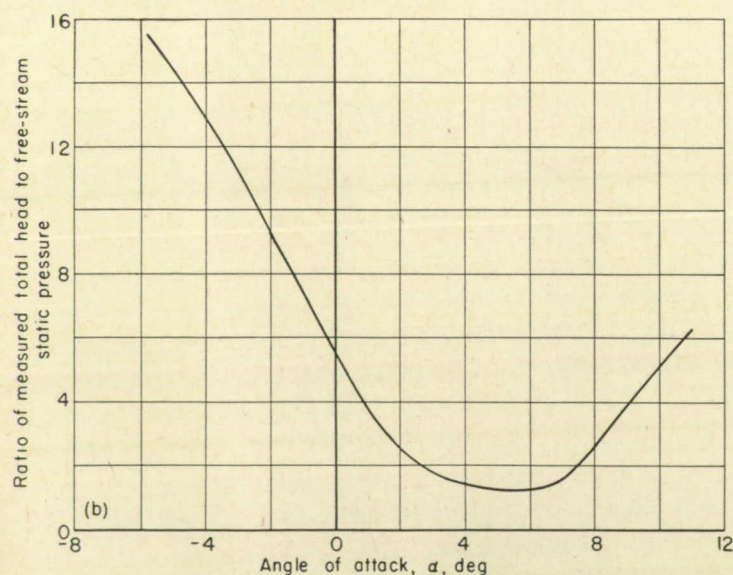
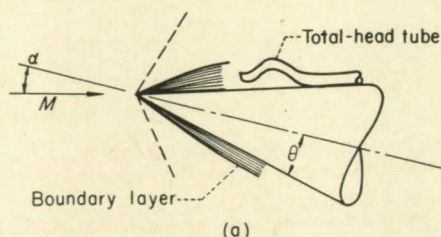


FIGURE 3.—Cross-sectional views of boundary layer on cone at various angles of attack.

This is to be expected since the fluid near the base of the boundary layer has low inertia, is therefore inclined to follow the direction of the circumferential pressure gradient more closely than is the outer flow, and thus tends to drain away from beneath the cone and accumulate near the top. No separation is encountered because, for small angle of attack, the pressure gradient is always favorable (fig. 2(a)).

For larger angles of attack, when the pressure gradient reverses direction near the top of the cone (fig. 2(b)), experiment indicates the formation of boundary-layer "lobes"



(a) Test configuration.

(b) Variation of total head with angle of attack.

FIGURE 4.—Variation with angle of attack of total head measured in boundary layer of cone. M , 3.095; θ , 7.5° .

(fig. 3(b)). When the angle of attack is further increased, the lobe pattern finally breaks away from the body to form a vortex street (fig. 3(c)).

Recently (ref. 8), at the Lewis laboratory a brief experiment was carried out in which a total-head probe was placed near the surface at the top of a cone and pointed toward the cone apex (fig. 4(a)). The cone was mounted in a supersonic wind tunnel, and the probe was used to measure the total head in the boundary layer at a fixed height above the surface as the angle of attack was varied by rotating the cone in the meridional plane containing the probe. Figure 4(b) shows the result of this test. The decrease in indicated total head as the angle of attack was increased from negative to positive values may be interpreted to mean an increase in boundary-layer thickness at the top of the cone. Beyond a certain angle of attack, this tendency reverses, and the boundary layer apparently becomes thinner as the angle of attack is increased. This is a possible indication of the tendency to form lobes, as illustrated in figure 3(b).

In the present report of research conducted during the summer of 1952 at the NACA Lewis laboratory, the laminar boundary layer in the meridional plane of symmetry of the flow is analyzed for large angles of attack in order to provide velocity profiles on the bottom of the cone and to provide a certain degree of insight into the question of separation on the top.

SOLUTION OF BOUNDARY-LAYER EQUATIONS IN PLANE OF SYMMETRY

BOUNDARY-LAYER EQUATIONS IN PLANE OF SYMMETRY

In reference 7, it is shown that the dimensionless laminar-boundary-layer equations for supersonic flow over a circular cone are

$$\left[f + \frac{1}{3\theta} \frac{p'(\varphi)}{p} g + \frac{2}{3\theta} g_\varphi \right] f_{\lambda\lambda} - \frac{2}{3\theta} g_\lambda f_{\lambda\varphi} + \frac{2}{3} (g_\lambda)^2 + 2f_{\lambda\lambda\lambda} = 0 \quad (1a)$$

$$\left[f + \frac{1}{3\theta} \frac{p'(\varphi)}{p} g + \frac{2}{3\theta} g_\varphi \right] g_{\lambda\lambda} - \frac{2}{3\theta} g_\lambda g_{\lambda\varphi} - \frac{2}{3} g_\lambda f_\lambda - \frac{2}{3\theta} \frac{p'(\varphi)}{p} + 2g_{\lambda\lambda\lambda} = 0 \quad (1b)$$

$$T + (f_\lambda)^2 + (g_\lambda)^2 = T_1 + u_1^2 + w_1^2 \quad (1c)$$

$$p = \frac{\gamma-1}{2\gamma} \rho T \quad (1d)$$

Equations (1a) and (1b) are momentum equations, equation (1c) is an energy balance, and equation (1d) is the equation of state. A complete list of symbols is provided in appendix A. The functions $f(\lambda, \varphi)$ and $g(\lambda, \varphi)$ are related to the two-component vector potential discussed in reference 6 and are defined according to the relations

$$\left. \begin{aligned} u &= f_\lambda \\ w &= g_\lambda \end{aligned} \right\} \quad (2)$$

in a manner such as to satisfy the continuity equation identically.

The coordinate λ has been formed as follows:

$$\lambda \equiv \sqrt{3} \left[\left(\frac{p}{p_r} \right)^{-1/2} \int_0^y \rho dy \right] x^{-1/2} \quad (3)$$

The coordinate φ is the angle between the vertical plane of symmetry of the flow and any meridional plane of the body (fig. 1). Equations (1) and (3) imply that parabolic similarity of the Blasius type exists in meridional planes. As pointed out in references 5 and 6, this conclusion of parabolic similarity applies for the boundary layer on any smooth conically symmetric body in supersonic flow (for example, a cone of elliptic cross section).

In reference 7, all quantities are made dimensionless by referring them to the properties of the nonviscous flow at the outer edge of the boundary layer when the cone is at zero angle of attack. In the present report it will be convenient to use a different reference condition (subscript r) which will be defined subsequently. The following quantities on the left are to be identified with the dimensionless groups on the right:

$$\left. \begin{aligned} u, w &\sim u/u_r, w/u_r \\ \rho &\sim \rho/\rho_r \\ T &\sim 2c_p T/u_r^2 \\ p &\sim p/\rho_r u_r^2 \\ x, y &\sim \rho_r u_r x/C\mu_r, \rho_r u_r y/C\mu_r \end{aligned} \right\} \quad (4)$$

where the constant C arises from the assumption of the temperature-viscosity relation of Chapman and Rubesin (ref. 9):

$$\frac{\mu}{\mu_r} = C \frac{T}{T_r} \quad (5a)$$

with C being defined as follows, in order to match equation (5a) to the Sutherland formula at the cone surface (denoted by subscript w):

$$C \equiv \left(\frac{T_w}{T_r} \right)^{\frac{1}{2}} \frac{1 + S/T_r}{T_w/T_r + S/T_r} \quad (5b)$$

The quantity S may be taken as equal to $(216^\circ \text{R})2c_p/u_r^2$.

The following additional physical assumptions are embodied in equations (1):

- A thin boundary layer across which the static pressure is constant
- Prandtl number of 1 and constant ratio of specific heats γ
- No heat transfer through the surface

From equation (1c), since the case of Prandtl number of 1 and no heat transfer is considered, T_w in equation (5b) may be taken equal to the dimensionless stream stagnation temperature.

The boundary conditions on the functions $f(\lambda, \varphi)$ and $g(\lambda, \varphi)$ are: At the outer edge of the boundary layer, the u and w

velocity components should take on the corresponding nonviscous values

$$f_\lambda(\infty, \varphi) = u_1(\varphi) \quad (6a)$$

$$g_\lambda(\infty, \varphi) = w_1(\varphi) \quad (6b)$$

At the cone surface, the u and w velocities should vanish

$$f_\lambda(0, \varphi) = g_\lambda(0, \varphi) = 0 \quad (6c)$$

and the normal velocity v should vanish. It is shown in reference 6 that this last requirement is met if

$$f(0, \varphi) = g(0, \varphi) = 0 \quad (6d)$$

Equations (1), involving two independent variables, would be quite difficult to solve in general. However, a certain amount of information can be obtained by restricting consideration to the plane of symmetry, thus yielding a tractable set of ordinary equations involving λ as the only independent variable.

In the plane of symmetry ($\varphi=0, \pi$), $w=g_\lambda=0$. Because $u=f_\lambda$ is even about the plane of symmetry and may be expected to be regular there, $f_{\lambda\varphi}=0$. The pressure and the density are also even, and therefore $p'(\varphi)$ vanishes at $\varphi=0, \pi$. Thus, in the plane of symmetry, equation (1a) reduces to the following equation:

$$\left(f + \frac{2}{3\theta} g_\varphi \right) f_{\lambda\lambda} + 2f_{\lambda\lambda} = 0 \quad (7a)$$

Every term in equation (1b) vanishes at the plane of symmetry; and, therefore, in order to obtain a meaningful equation, it is necessary first to differentiate equation (1b) with respect to φ and then drop terms which vanish at $\varphi=0, \pi$. This procedure yields the following result:

$$\left(f + \frac{2}{3\theta} g_\varphi \right) g_{\lambda\varphi} - \frac{2}{3\theta} (g_{\lambda\varphi})^2 - \frac{2}{3} g_{\lambda\varphi} f_\lambda - \frac{2}{3\theta} \frac{p''(\varphi)}{\rho} + 2g_{\lambda\lambda\varphi} = 0 \quad (7b)$$

Equation (1c) becomes

$$T + (f_\lambda)^2 = T_1 + u_1^2 \quad (7c)$$

Equations (7) may be considered a set of ordinary differential equations for the functions $f(\lambda, 0)$ and $g_\varphi(\lambda, 0)$, or $f(\lambda, \pi)$ and $g_\varphi(\lambda, \pi)$, depending on whether the solution is required at the bottom or the top of the cone. According to equations (2), the result $f(\lambda, 0$ or $\pi)$ may be differentiated with respect to λ to give the profile of meridional velocity u in the plane of symmetry. The form assumed by the circumferential velocity profile w as $\varphi \rightarrow 0$ or π is given by $g_{\varphi\lambda}(\lambda, 0$ or $\pi)$ in the sense that, at a small angular distance $d\varphi$ away from the plane of symmetry, $w \approx g_{\varphi\lambda} d\varphi$.

The boundary conditions (eqs. (6)) become, in the plane of symmetry,

$$\left. \begin{aligned} f_\lambda(\infty, 0 \text{ or } \pi) &= u_1(0 \text{ or } \pi) \\ g_{\varphi\lambda}(\infty, 0 \text{ or } \pi) &= w_{1\varphi}(0 \text{ or } \pi) \\ f_\lambda(0, 0 \text{ or } \pi) &= g_{\varphi\lambda}(0, 0 \text{ or } \pi) = f(0, 0 \text{ or } \pi) = g_\varphi(0, 0 \text{ or } \pi) = 0 \end{aligned} \right\} \quad (8)$$

In view of the first of equations (8) and of equation (7c), it is convenient to specify the reference condition (subscript r)

to be that existing at the outer edge of the boundary layer, for the particular angle of attack under consideration, evaluated at either $\varphi=0$ or $\varphi=\pi$, depending on whether the analysis pertains to the bottom or top of the cone.

Because the pressure is assumed constant across the boundary layer, equations (1d) and (7c) and the assumption of constant pressure across the boundary layer ($p=p_1$) provide that

$$\frac{1}{\rho} = \frac{T}{T_1} = 1 + \frac{1}{T_1} [1 - (f')^2] \quad (9)$$

From conventions (4), $T_1 = \left(\frac{2}{\gamma - 1} \right) / M_1^2$. For convenience, the following definitions are made:

$$g_\varphi(\lambda, 0 \text{ or } \pi) \equiv \frac{3\theta}{2} k \psi(\lambda) \quad (10a)$$

$$k \equiv \frac{2}{3\theta} w_{1\varphi}(0 \text{ or } \pi) \quad (10b)$$

Equation (9) and definitions (10) are introduced into equations (7a) and (7b), and a value of $p''(0 \text{ or } \pi)$ is assigned consistent with the nonviscous equations at the outer edge of the boundary layer ($p''(0 \text{ or } \pi)$ may conveniently be obtained from equations (1b) and (6) by setting $g_{\varphi\lambda\lambda} = g_{\varphi\lambda\lambda\lambda} = 0$ when $\lambda = \infty$).

The following pair of simultaneous ordinary differential equations then results:

$$(f + k\psi)f'' + 2f''' = 0 \quad (11a)$$

$$(f + k\psi)\psi'' + 2\psi''' - k(\psi')^2 - \frac{2}{3}\psi f' + \left(k + \frac{2}{3}\right) \left\{ 1 + \frac{1}{T_1} [1 - (f')^2] \right\} = 0 \quad (11b)$$

and boundary conditions (8) become

$$f'(\infty) = \psi'(\infty) = 1 \quad (12a)$$

$$f'(0) = \psi'(0) = 0 \quad (12b)$$

$$f(0) = \psi(0) = 0 \quad (12c)$$

Two parameters appear: k , which depends essentially on angle of attack, and T_1 , which is essentially dependent on Mach number. If the angle of attack (and hence k) is zero or nearly zero, equations (11) become precisely those considered in reference 7 and may be solved quite readily, since equation (11b) becomes linear and the solution of equation (11a) is well known as the Blasius function. When k differs substantially from zero (moderate or large angle of attack), equations (11) are both nonlinear and the solutions are interdependent. For any particular case, when only the stream Mach number, cone vertex angle, and angle of attack are specified, the parameters k and T_1 must be obtained by recourse to a theory of the outer nonviscous flow.

OUTER NONVISCIOUS FLOW

In references 1 to 3, the results of a theory of nonviscous supersonic flow about circular cones at angle of attack are

tabulated. The case of zero angle of attack (ref. 1) is solved exactly in the sense that no assumption of small vertex angle is made. The equations are then expanded in powers of angle of attack with the use of the zero-angle-of-attack solution as the first approximation. Terms linear in angle of attack are presented in reference 2, and terms proportional to the square of angle of attack are presented in reference 3. Reference 4 clarifies the application of the theory to the computation of flow conditions at the cone surface.

There are two objections to the use of this theory in the present application:

(1) Neglect of terms in the expansion beyond that involving the square of angle of attack may lead to an insufficiently accurate representation of the flow at the large angles of attack which are of interest. Unfortunately, no non-viscous theory is available that treats the effect of angle of attack with greater precision. In reference 4, a comparison of the theoretical and experimental pressure distributions is presented for a cone of semivertex angle of 10° , at a Mach number of 2, and of an angle of attack of 12.2° . The agreement shown is very good, especially since the angle of attack is sufficiently large that the pressure distribution is of the type shown in figure 2(b).

(2) In reference 10, Ferri points out that the method of expansion used in references 2 and 3 is improper near the cone surface and leads to an erroneous form of the entropy distribution around the cone. Therefore, the theory cannot be applied if the vertex angle, the angle of attack, or the Mach number is so large that the flow may not be considered essentially isentropic. In reference 7, an argument is presented to the effect that in the limit of infinitesimal angle of attack the presence of a boundary layer ensures that the error in entropy distribution is of no consequence even for large cone vertex angles. That argument in no way applies to the present analysis because the angles of attack considered are not infinitesimal. For the purposes of this report, the use of references 2 and 3 in their present form is justified only in cases for which isentropic flow may be assumed.

According to reference 4, the velocity components at the cone surface are, using the notation of references 1 to 3 for quantities tabulated therein,

$$\frac{w_1(\varphi)}{\bar{u}} = 1 + \alpha \frac{x}{\bar{u}} \cos \varphi + \alpha^2 \left[\frac{u_0}{\bar{u}} + \frac{1}{2} \frac{x}{\bar{u}} \cot \theta + \frac{1}{2} + \left(\frac{u_2}{\bar{u}} - \frac{1}{2} \frac{x}{\bar{u}} \cot \theta + \frac{1}{2} \right) \cos 2\varphi \right] + \dots \quad (13)$$

$$\frac{w_1(\varphi)}{\bar{u}} = \alpha \frac{z}{\bar{u}} \sin \varphi + \alpha^2 \left[\frac{w_2}{\bar{u}} - \csc \theta - \frac{z}{\bar{u}} \cot \theta \right] \sin 2\varphi + \dots \quad (14)$$

The pressure and density are

$$\frac{p_1(\varphi)}{\bar{p}} = 1 + \alpha \frac{\eta}{\bar{p}} \cos \varphi + \alpha^2 \left[\frac{p_0}{\bar{p}} + \frac{\gamma}{2} \bar{M}^2 + \frac{1}{2} \frac{\eta}{\bar{p}} \cot \theta + \left(\frac{p_2}{\bar{p}} + \frac{\gamma}{2} \bar{M}^2 - \frac{1}{2} \frac{\eta}{\bar{p}} \cot \theta \right) \cos 2\varphi \right] + \dots \quad (15)$$

$$\frac{\rho_1(\varphi)}{\rho} = 1 + \alpha \frac{\xi}{\rho} \cos \varphi + \alpha^2 \left[\frac{\rho_0}{\rho} + \frac{1}{2} \bar{M}^2 + \frac{1}{2} \frac{\xi}{\rho} \cot \Theta + \left(\frac{\rho_2}{\rho} + \frac{1}{2} \bar{M}^2 - \frac{1}{2} \frac{\xi}{\rho} \cot \Theta \right) \cos 2\varphi \right] + \dots \quad (16)$$

The barred quantities are those pertaining to the case of zero angle of attack. From equations (10b), (13), and (14),

$$k = \frac{2}{3\theta} \frac{\pm \alpha \frac{z}{u} + 2\alpha^2 \left(\frac{w_2}{u} - \frac{1}{\theta} - \frac{\sqrt{1-\theta^2}}{\theta} \frac{z}{u} \right)}{1 \pm \alpha \frac{x}{u} + \alpha^2 \left(1 + \frac{u_0}{u} + \frac{u_2}{u} \right)} + \dots$$

$$\frac{1}{T_1} = \frac{1}{T} \frac{\bar{M}^2}{\bar{M}^2} = \frac{\bar{u}^2}{1-\bar{u}^2} \left(\frac{u_1}{u} \right)^2 \frac{\bar{p}}{p_1} \frac{\rho_1}{\rho} = \frac{\bar{u}^2}{1-\bar{u}^2} \frac{\left[1 \pm \alpha \frac{x}{u} + \alpha^2 \left(1 + \frac{u_0}{u} + \frac{u_2}{u} \right) \right]^2 \left[1 \pm \alpha \frac{\eta}{p} + \alpha^2 \left(\frac{\rho_0}{\rho} + \frac{\rho_2}{\rho} + \frac{2}{\gamma-1} \frac{\bar{u}^2}{1-\bar{u}^2} \right) \right]}{1 \pm \alpha \frac{\eta}{p} + \alpha^2 \left(\frac{p_0}{p} + \frac{p_2}{p} + \frac{2\gamma}{\gamma-1} \frac{\bar{u}^2}{1-\bar{u}^2} \right)} + \dots$$

or, approximately,

$$\frac{1}{T_1} = \frac{\bar{u}^2}{1-\bar{u}^2} \left\{ 1 \pm \alpha \left(2 \frac{x}{u} + \frac{\xi}{\rho} - \frac{\eta}{p} \right) + \alpha^2 \left[2 \left(1 + \frac{u_0}{u} + \frac{u_2}{u} \right) + \frac{\rho_0}{\rho} + \frac{\rho_2}{\rho} - \frac{p_0}{p} - \frac{p_2}{p} - 2 \frac{\bar{u}^2}{1-\bar{u}^2} + 2 \frac{x}{u} \left(\frac{\xi}{\rho} - \frac{\eta}{p} \right) + \frac{x^2}{\bar{u}^2} + \frac{\eta^2}{p^2} - \frac{\eta}{p} \frac{\xi}{\rho} \right] \right\} + \dots \quad (18)$$

In figure 5 are shown k and $1/T_1$ as functions of α for a cone of semivertex angle of 7.5° and a stream Mach number of 3.1. From the tabulations of reference 2, it may be inferred that under these conditions the isentropic assumption leads to errors of less than 1 percent in quantities proportional to the angle of attack.

SOLUTION OF EQUATIONS AT $\varphi=0$

Equations (11) have been solved, subject to boundary conditions (12), at $\varphi=0$, and various angles of attack for a cone of semivertex angle of 7.5° and a stream Mach number of 3.1 for which the values of k and $1/T_1$ are given in figure 5. The computations were carried out by Dr. Lynn Albers of the Lewis laboratory and are described in appendix B. The resulting boundary-layer profiles of meridional velocity u and gradient of circumferential velocity $\partial w/\partial \varphi$ are shown in figure 6. The curves for $\alpha=0$ are obtained from reference 7. The profiles show clearly that, as the angle of attack is increased, the boundary layer becomes thinner on the bottom of the cone, and the shear stress at the wall increases.

Skin friction.—The meridional and circumferential components of the viscous shear stress at the bottom of the cone surface may be written in coefficient form as follows:

$$[C_{f_x}]_{\varphi=0} = \frac{1}{\frac{1}{2} \rho_1 u_1^2} \left(\mu \frac{\partial u}{\partial y} \right)_{y=0}$$

$$[C_{f_\varphi}]_{\varphi=0} = 0$$

$$\left[\frac{\partial C_{f_\varphi}}{\partial \varphi} \right]_{\varphi=0} = \frac{1}{\frac{1}{2} \rho_1 u_1^2} \left[\mu \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial \varphi} \right) \right]_{y=0}$$

where the quantities on the right are in dimensional form. Application of equations (1d), (2), (3), (4), (5), and (10a) yields

$$\sqrt{\frac{R_x}{3C}} [C_{f_x}]_{\varphi=0} = 2f''(0) \quad (19a)$$

or, approximately,

$$k = \frac{2}{3} \left[\pm \alpha \frac{z}{u} + 2\alpha^2 \left(\frac{w_2}{u} - \frac{1}{\theta} - \frac{\sqrt{1-\theta^2}}{\theta} \frac{z}{u} - \frac{1}{2\theta} \frac{xz}{u^2} \right) \right] + \dots \quad (17)$$

The plus or minus sign refers to $\varphi=0$ or π , respectively. From conventions (4), equations (13), (15), and (16), and the result of reference 1 that $1/\bar{T} = \bar{u}^2/(1-\bar{u}^2) = \frac{\gamma-1}{2} \bar{M}^2$,

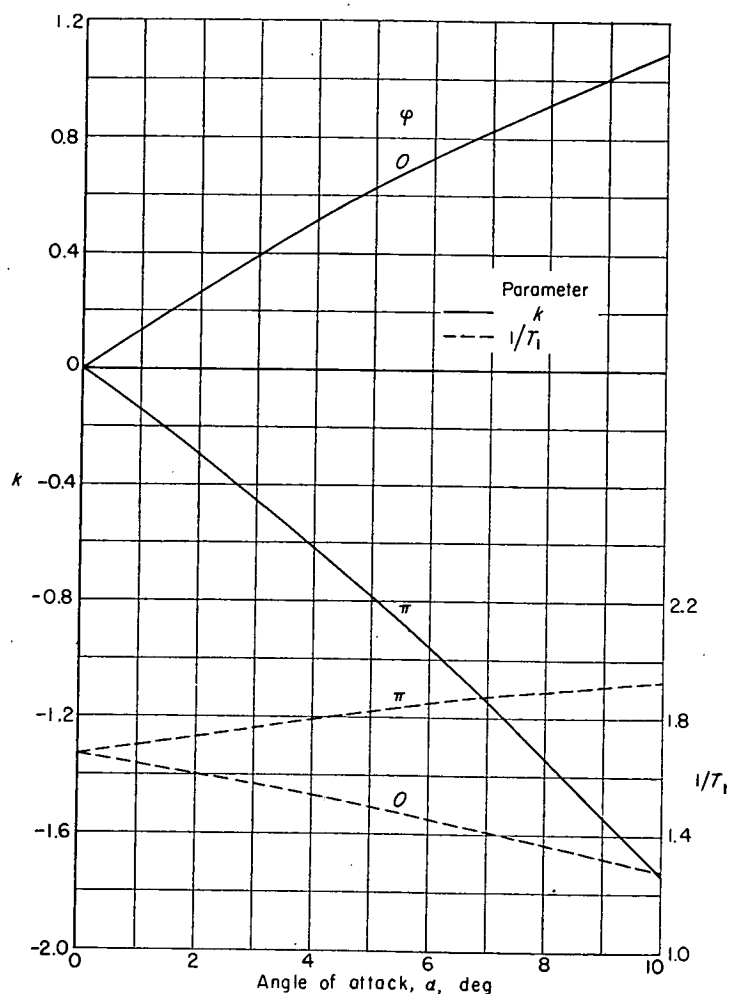
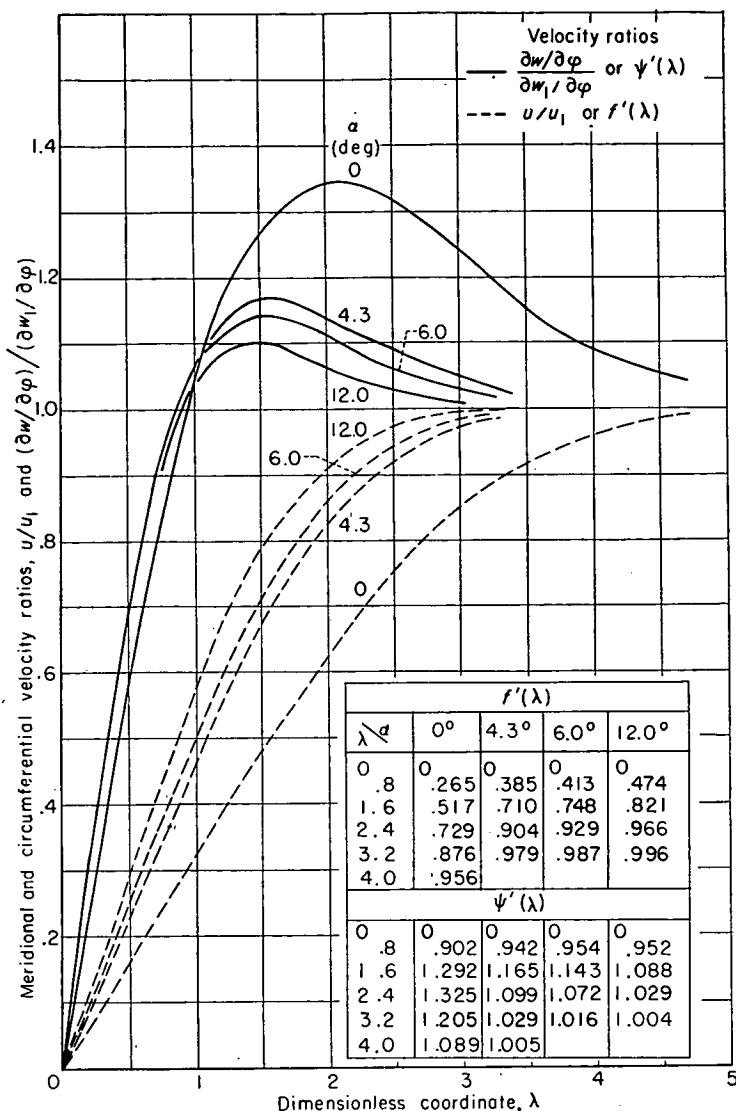


FIGURE 5.—Variation of parameters k and $1/T_1$ with angle of attack. M , 3.1; Θ , 7.5° .

$$[C_{f_\varphi}]_{\varphi=0} = 0 \quad (19b)$$

$$\sqrt{\frac{R_x}{3C}} \left[\frac{\partial C_{f_\varphi}}{\partial \varphi} \right]_{\varphi=0} = 3\theta k \psi''(0) \quad (19c)$$

FIGURE 6.—Velocity profiles at $\varphi=0$. M , 3.1; Θ , 7.5° .

where

$$R_x \equiv \frac{\rho_1 u_1 x}{\mu_1}$$

Variation of these skin-friction coefficients with angle of attack is shown in figure 7 for a particular case.

Displacement thickness.—In reference 11 it is shown that the displacement thickness Δ for a cone at angle of attack is the solution of the equation

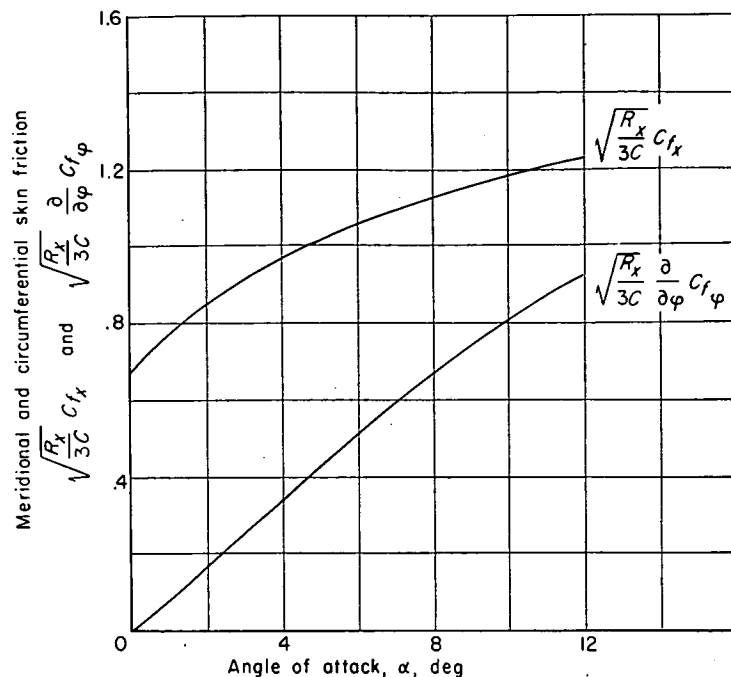
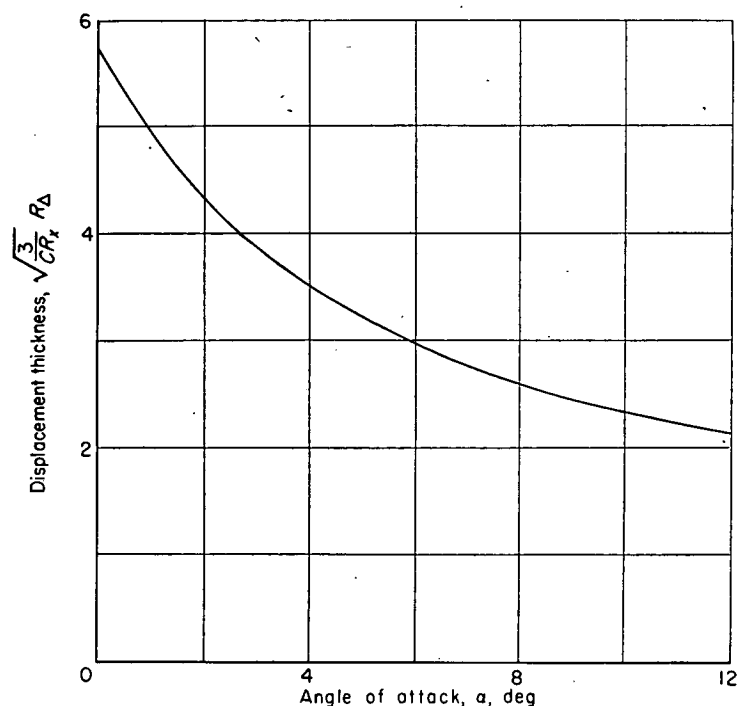
$$\frac{3}{2} \theta \rho_1 u_1 (\Delta - \delta_x) + \frac{\partial}{\partial \varphi} [\rho_1 w_1 (\Delta - \delta_\varphi)] = 0 \quad (20)$$

where

$$\left. \begin{aligned} \delta_x &\equiv \int_0^\infty \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy \\ \delta_\varphi &\equiv \int_0^\infty \left(1 - \frac{\rho w}{\rho_1 w_1}\right) dy \end{aligned} \right\} \quad (21)$$

At $\varphi=0$, $w_1=0$ and equation (20) may be solved directly

$$\Delta = \frac{\delta_x + k \delta_\varphi}{1 + k} \quad (22)$$

FIGURE 7.—Meridional and circumferential skin-friction coefficients at $\varphi=0$. M , 3.1; Θ , 7.5° .FIGURE 8.—Displacement thickness at $\varphi=0$. M , 3.1; Θ , 7.5°

where k is defined in equation (10b). From equations (5) and (9), with δ_x defined in terms of a Reynolds number,

$$R_{\delta_x} \equiv \frac{\rho_1 u_1 \delta_x}{\mu_1}$$

$$R_{\delta_x} = \sqrt{\frac{C R_x}{3}} \int_0^\infty \left[1 - f' + \frac{1}{T_1} (1 - f'^2) \right] d\lambda \quad (23a)$$

Using equations (10) and applying l'Hospital's rule to evaluate at $\varphi=0$ the limit of the ratio w/w_1 appearing in equation (21) yield

$$R_{\delta\varphi} = \sqrt{\frac{CR_x}{3}} \int_0^\infty \left[1 - \psi' + \frac{1}{T_1} (1 - f'^2) \right] d\lambda \quad (23b)$$

Therefore,

$$\sqrt{\frac{3}{CR_x}} R_{\Delta} = \int_0^\infty \left\{ \frac{k}{1+k} (f' - \psi') + (1 - f') \left[1 + \frac{1}{T_1} (1 + f') \right] \right\} d\lambda \quad (23c)$$

Figure 8 shows the variation of displacement thickness with angle of attack for a particular case and again illustrates the expected progressive shift of the boundary layer from bottom to top as the angle of attack is decreased.

LIMITATIONS OF METHOD AT $\varphi=\pi$

Except for quite small angles of attack, equations (11) cannot be solved at the top of the cone ($\varphi=\pi$). Over part of the range of angle of attack, the solutions are indeterminate; and, beyond a certain angle of attack, the solutions do not exist at all. These properties of equations (11) will be demonstrated and discussed in the following paragraphs.

Asymptotic forms of equations.—The difficulties just mentioned may best be inferred from the asymptotic forms of equations (11) at large λ . From equation (12a) it is clear that, for large λ , f and ψ may be written as follows:

$$f = \lambda + F(\lambda)$$

$$\psi = \lambda + \Psi(\lambda)$$

where $F'(\infty) = \Psi'(\infty) = 0$. Substitution into equations (11) yields the asymptotic forms for large λ

$$(1+k)\lambda F'' + 2F''' = 0 \quad (24a)$$

$$(1+k)\lambda \Psi'' + 2\Psi''' - 2\left(k + \frac{1}{3}\right)\Psi' = \frac{2}{3}F' \left[1 + \frac{3}{T_1} \left(k + \frac{2}{3}\right) \right] \quad (24b)$$

Indeterminate solutions.—Consideration will now be given to the problem of obtaining the complementary solution of equation (24b). Defining a new dependent variable

$$G \equiv e^{\frac{1+k}{8}\lambda^2} \Psi'$$

yields the equation

$$G'' - \left[\frac{5}{4}k + \frac{7}{12} + \left(\frac{1+k}{4} \right)^2 \lambda^2 \right] G = 0 \quad (25)$$

This is essentially Weber's equation (ref. 12, paragraph 16.5), and the asymptotic solutions are

$$e^{-\frac{1+k}{8}\lambda^2} \left(\sqrt{\frac{1+k}{2}} \lambda \right)^{-3\frac{k+5/9}{1+k}}$$

$$e^{\frac{1+k}{8}\lambda^2} \left(\sqrt{\frac{1+k}{2}} \lambda \right)^{2\frac{k+1/3}{1+k}}$$

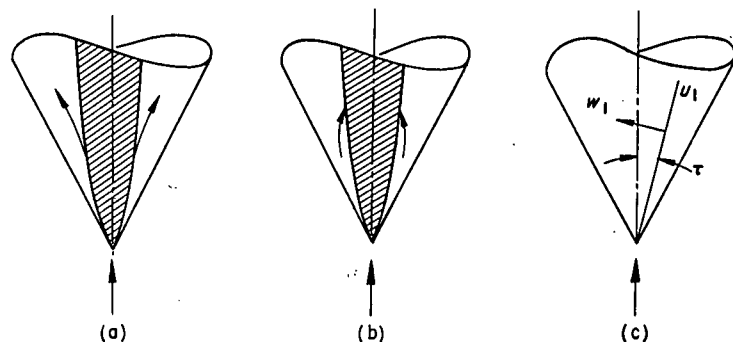
Thus, the asymptotic complementary solutions of equation (24b) are

$$\Psi' = e^{-\frac{1+k}{8}\lambda^2} \left(\sqrt{\frac{1+k}{2}} \lambda \right)^{-3\frac{k+5/9}{1+k}} \quad (26a)$$

$$\Psi' = \left(\sqrt{\frac{1+k}{2}} \lambda \right)^{2\frac{k+1/3}{1+k}} \quad (26b)$$

Because it is required that $\Psi'(\infty)=0$, solution (26b) is rejected when $k > -1/3$. If $k < -1/3$, both solutions may be retained, and an additional undetermined constant appears. When $k > 0$ (at the bottom of the cone), the complete solution of equations (11) exists and is unique, and solution (26b) is to be rejected in forming the asymptotic solution. Therefore, if solution (26b) and the associated constant must be retained when $k < -1/3$, it is clear that the complete solution for $k < -1/3$ cannot be unique. This indeterminacy has been verified by numerical integration of equations (11), as described in appendix B.

This indeterminacy arises because essential information has been lost by specializing the equations to apply only in the plane of symmetry. When the equations are so written, it is implied that the boundary-layer development in the plane of symmetry is affected only by conditions in that plane. The lateral region of influence of points on the body in the plane of symmetry grows parabolically (the shaded regions in sketches (a) and (b)), according to the law of molecular diffusion, when, as in the present instance, there is no pressure gradient in the stream direction. Fluid enters the boundary layer from the outer stream. If the fluid then moves laterally (because of the angle of attack) out of the region of influence of the plane of symmetry, as shown in sketch (a), the flow is uniquely determined by outer stream conditions in the plane of symmetry. Clearly, this is the case when $\varphi=\pi$ and α is small, and when $\varphi=0$ and α has any positive value. When $\varphi=\pi$, except for small angles of attack, the lateral motion of the fluid is inward relative to the region of influence, as shown in sketch (b). This fluid then brings into the region of influence of the plane of symmetry information concerning boundary-layer development as it proceeds around the cone from the bottom. Consequently, outer stream conditions at $\varphi=\pi$ may not uniquely determine the boundary-layer characteristics at $\varphi=\pi$, and indeterminate solutions of equations (11) may be anticipated.



The condition $k = -\frac{1}{3}$ specifies an angle of attack such that, at $\varphi = \pi$, the outer flow streamlines are just tangent to the parabolic region of influence. This statement may be proved as follows: From definition (10b), if $k = -\frac{1}{3}$, then $w_1/u_1 = -\theta/2$. Near $\varphi = \pi$ (see sketch (c)), $w_1 = w_1 \varphi d\varphi = w_1 \varphi \frac{\tau}{\theta}$. Thus, when $k = -\frac{1}{3}$, $w_1/u_1 = \tau/2$. From this last equation and the geometrical properties of parabolas, it may be inferred that the streamlines are parabolas with focus at the cone apex. Therefore, the situation shown in sketch (b) applies if $k < -\frac{1}{3}$ and equations (11) have no unique solution.

It is noteworthy that the mathematical nature of the indeterminacy so far described is quite similar to that which characterizes solutions of the Falkner-Skan equation for the plane boundary layer when the outer flow velocity is proportional to x^{-m} (x being the streamwise coordinate and m a positive number). Hartree (ref. 13) has treated this problem in detail and finds an "extra" asymptotic solution similar to equation (26b), decreasing with distance away from the surface according to a power law. This extra solution is rejected by Hartree on the ground that an unrealistically thick boundary layer would be predicted if the solution were retained.

A thorough study of the asymptotic nature of equations (1) might lead to a similar conclusion in the present problem. However, in the present problem, the mathematical difficulty goes deeper; when $k > -\frac{1}{3}$, numerical integration failed to give unique results, even though solution (26b) is certainly to be rejected. Thus, the condition $k \geq -\frac{1}{3}$ is not sufficient, but rather only necessary for uniqueness.

Inspection of the profiles shown in figure 6 for $\alpha = k = 0$ indicates that, for $0 > k \geq -\frac{1}{3}$, the streamlines within the boundary layer may be expected to incline more sharply toward the plane $\varphi = \pi$ than do the streamlines near the edge of the boundary layer and thus may bring information from beneath the cone even though the outer ones do not. Therefore, the necessary condition for uniqueness would be (see eqs. (2) and (10a)):

$$\left(\frac{w_\varphi}{u}\right)_{\max} = \frac{3}{2} \theta k \left(\frac{\psi'(\lambda)}{f'(\lambda)}\right)_{\max} > -\frac{\theta}{2}$$

or,

$$k > -\frac{1}{3} \left/ \left(\frac{\psi'}{f'}\right)_{\max} \right. \quad (27a)$$

Figure 6 indicates that perhaps the maximum value of ψ'/f' is to be found at $\lambda = 0$, in which case criterion (27a) would become

$$k > -\frac{1}{3} \frac{f''(0)}{f'''(0)} \quad (27b)$$

Nonexistence.—The solution of equation (24a) is

$$F'' = e^{-\frac{1+k}{4} \lambda^2}$$

The requirement that $F''(\infty) = 0$ is met only if $k > -1$. If $k \leq -1$, no solution of equation (24a) exists which satisfies

the boundary conditions, and, therefore, the Prandtl boundary-layer equations fail to describe the flow. This was first pointed out by Hayes in reference 5. The Prandtl equations differ from the exact equations essentially in that a thin boundary layer is assumed. Thus, if $k \leq -1$, the boundary layer cannot be regarded as thin. It may be noted that equation (22) implies that, as $k \rightarrow -1$, the displacement thickness approaches infinity.

Any boundary layer grows by the entrainment of fluid at its outer edge. That is, fluid particles acquire vorticity by entry into the boundary layer. The reverse process cannot occur—fluid particles cannot leave the boundary layer, thus losing their vorticity. In the case under consideration, it will be shown that when $k < -1$, the streamlines at the outer edge of the boundary layer would proceed outward relative to the boundary layer, if the boundary layer were to retain parabolic similarity. Because such a situation is physically impossible, the Prandtl equations fail to yield a solution.

From reference 11, the normal velocity at the outer edge of the boundary layer at $\varphi = \pi$ is

$$v_1 \cong u_1 \frac{\partial \Delta}{\partial x} + (h - \Delta) \left(\frac{\partial v_1}{\partial y} \right)_{y=0} \quad (28)$$

where h is a somewhat arbitrary definition of the outer edge of the boundary layer and $\left(\frac{\partial v_1}{\partial y} \right)_{y=0}$ is obtained from analysis of the outer nonviscous flow. The equation of continuity, for the outer flow, evaluated at the surface of the cone in the plane of symmetry, may be written

$$\frac{u_1}{x} + \frac{1}{\theta x} \frac{\partial w_1}{\partial \varphi} + \left(\frac{\partial v_1}{\partial y} \right)_{y=0} = 0 \quad (29)$$

With equations (28) and (29) combined, the flow inclination at the outer edge of the boundary layer is

$$\frac{v_1}{u_1} = \frac{\partial \Delta}{\partial x} - \frac{h - \Delta}{x} \left(1 + \frac{1}{\theta u_1} \frac{\partial w_1}{\partial \varphi} \right) \quad (30)$$

The Prandtl (thin) boundary layer may be expected to exist only if

$$\frac{\partial h}{\partial x} > \frac{v_1}{u_1}$$

or, with equation (30) introduced, if

$$\frac{1}{\theta u_1} \frac{\partial w_1}{\partial \varphi} > - \left[1 + \frac{x}{h - \Delta} \frac{\partial}{\partial x} (h - \Delta) \right] \quad (31)$$

If parabolic similarity is assumed (h and Δ each proportional to $\sqrt{x}$), inequality (31) becomes

$$\frac{1}{\theta u_1} \frac{\partial w_1}{\partial \varphi} > -\frac{3}{2}$$

or, from equation (10b),

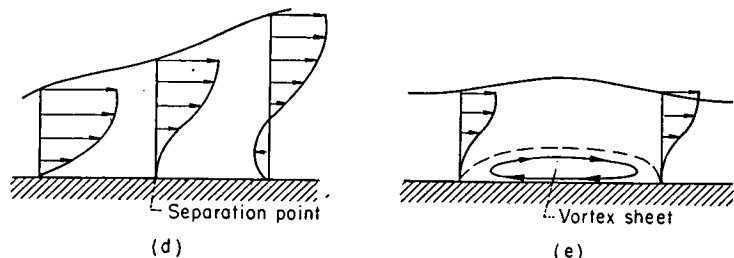
$$k > -1$$

SEPARATION

The critical condition $k = -1$ might be expected to be of physical as well as analytical significance because some sort of catastrophic thickening of the boundary layer is implied. In particular, this critical condition may reasonably be supposed to be connected with the phenomenon of laminar separation. In order to explore this possibility, it is first necessary to describe qualitatively what is meant by separation in three-dimensional boundary-layer flow. Difficulty has been encountered in establishing a satisfactory qualitative criterion for three-dimensional separation (see, for example, ref. 5). Therefore, in the subsequent paragraphs, the general problem of three-dimensional separation will be discussed, and then the particular case of the cone at angle of attack will be considered.

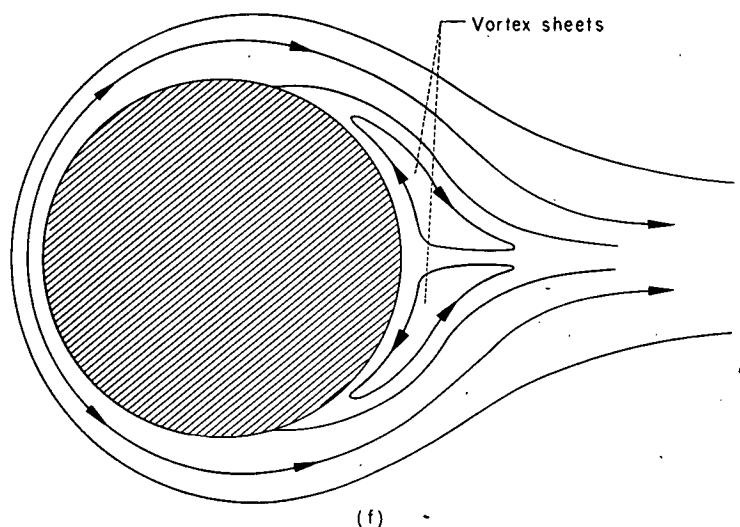
GENERAL CONSIDERATIONS

Plane flow.—In plane flow, separation is customarily identified by the appearance of reverse flow (sketch (d)). In order to generalize this concept to three-dimensional flow, it is necessary to consider the separated region as a whole. In plane flow, the separation point of sketch (d) might be regarded as the forward boundary of a vortex sheet embedded, or encapsulated, within a region bounded by the body and a stream surface meeting the body (sketch (e)). Sketch (e) shows separation followed by reattachment. Of course, the sort of separation of greatest engineering importance occurs when such an embedded vortex sheet rolls up to form a large concentrated vortex, or is shed as a vortex street, with the consequence that the outer flow is greatly disturbed and a large pressure effect (form drag) occurs.



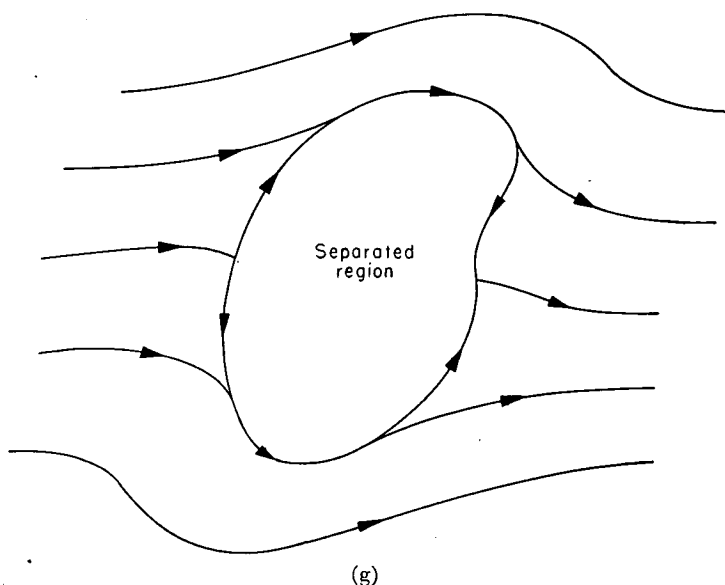
If a boundary-layer solution of the type shown in sketch (e) were obtained, the vortex sheet would be completely embedded in a thin boundary layer, and would presumably tend to remain flat against the body (leaving out of account possible effects of laminar instability). However, if the theoretically predicted vortex sheet extends downstream into a region where the thin boundary-layer equations do not apply (that is, where the solution "blows up" predicting an infinitely thick boundary layer), then in that region the vortex sheet would not be constrained to lie flat and the rolling-up process would occur.

The foregoing discussion seems to provide the proper description of what occurs behind a bluff body: A complete solution of the thin laminar boundary-layer equations for the flow over a cylinder would probably yield a streamline



pattern of the form shown in sketch (f). The boundary layer would be predicted to gain mass flow by entrainment as it proceeds around the body, until it reaches the vicinity of the rear stagnation point. There, the mass flow contained in the boundary layer must finally leave the body and proceed downstream. The boundary layer therefore cannot remain theoretically thin, but rather must approach infinite thickness in violation of the Prandtl assumptions. In this region, then, the aft boundaries of the pair of vortex sheets are free to roll up into concentrated vortices, thus distorting the outer flow in such a way that the rolling-up process engulfs most of the region which would otherwise be occupied by a flat vortex sheet. The leading edge of the sheet, however, is still constrained to lie flat against the body.

Three-dimensional flow.—The foregoing description of plane separation may be generalized to three-dimensional flow as follows: A separated region on a three-dimensional body consists of a vortex sheet embedded between the body surface and a stream surface attached to the body in a closed curve, as shown in sketch (g), which is a view of the body from above. The arrows indicate possible directions of



resultant shear stress at the surface and outside the separated region. The situation shown in the sketch would correspond to separation and reattachment in plane flow. If somewhere within or at the boundary of the separated region the theoretical boundary-layer solution would a priori be expected to blow up, then the vortex system within the separated region is free to roll up into a more or less vigorous system of vortices.

Thus, thin-boundary-layer theory can be used to obtain the following information concerning laminar separation:

(1) The solution may establish the existence of a vortex sheet which is embedded in a flat bubble on the surface and which could adhere to the surface and remain part of a thin boundary layer, provided that the Prandtl equations are valid everywhere in the separated region.

(2) The solution might predict the boundary layer to go to infinite thickness somewhere in the separated region. If this happens, then the separated region is free to roll up, thus providing a vigorous wake (which, of course, is not amenable to boundary-layer theory).

For flow about a plane body, the boundary-layer solution is not needed for predicting the breakdown of the Prandtl assumptions. Physical considerations suffice to establish where (at stagnation point of outer flow) and when (always) the breakdown occurs. In three-dimensional flow this is not always so clear, and at least certain features of the solution are required to be known. In order that the boundary-layer equations be applicable, the solution must be such that the boundary layer entrains fluid (that is, flow streamlines enter, but do not leave, the boundary layer at its outer edge). In a Cartesian system, where $h(x, z)$ is the outer edge of the boundary layer, this requirement may be written as

$$u_1 h_x + w_1 h_z > (v_1)_{y=h} \quad (32)$$

Equations (32) and (28) may be combined with the equation of continuity in the form

$$\rho_1 \left(\frac{\partial v_1}{\partial y} \right)_{y=0} = -[(\rho_1 u_1)_x + (\rho_1 w_1)_z]$$

to yield

$$\frac{\partial}{\partial x} [(h-\Delta) \rho_1 u_1] + \frac{\partial}{\partial z} [(h-\Delta) \rho_1 w_1] > 0$$

or, in vector notation,

$$\text{div} [(h-\Delta) \rho_1 \mathbf{q}_1] > 0 \quad (33)$$

where Δ is the displacement thickness, and $\mathbf{q}_1$ is the velocity vector in the outer flow evaluated at the body surface.

In many cases, circumstances may be found for which inequality (33) cannot be satisfied. For example, for plane incompressible flow about a cylinder and, as is customary, with u and x defined parallel to the surface, inequality (33) may be written

$$\frac{\frac{d}{dx} (h-\Delta)}{h-\Delta} > \frac{-\frac{du_1}{dx}}{u_1}$$

As the rear stagnation point of the outer flow is approached, u_1 tends to zero, while $-du_1/dx$ remains finite and positive. Therefore, since $h-\Delta$ by definition must be greater than zero, $d(h-\Delta)/dx$ must approach infinity in clear violation of the boundary-layer assumptions.

SEPARATION ON CONE AT ANGLE OF ATTACK

In the previous discussion, it was concluded that separation involving a strong vortex pattern occurs if a tentative boundary-layer solution predicts an embedded vortex sheet coupled with a local breakdown of the assumption of a thin boundary layer. In the case of the cone, inequality (33) may be used to predict the circumstances under which the boundary layer may not be regarded as thin: When the fact of parabolic similarity is introduced ($h-\Delta$ proportional to $-\sqrt{x}$), inequality (33) becomes

$$\frac{\frac{\partial}{\partial \varphi} (h-\Delta)}{h-\Delta} > \frac{-w_{1\varphi} - \frac{3}{2} \theta u_1}{w_1} \quad (34)$$

This inequality indicates infinite boundary-layer thickness only when $w_1=0$ and, even then, only if $w_{1\varphi}$ is negative (which is true at the top of the cone, $\varphi=\pi$) and larger in magnitude than $\frac{3}{2}\theta u_1$. This is true only for angles of attack larger than that for which $k=-1$ (by eq. (10b)). When the angle of attack is smaller than this critical value, the right member of inequality (34) is always negative, and $\partial(h-\Delta)/\partial\varphi$ may be considered to vanish by symmetry at $\varphi=\pi$ without violating inequality (34).

The foregoing result may be explained on physical grounds as follows: As the boundary layer proceeds around the cone, it entrains fluid which it then conveys toward the top (symmetrically, from both sides of the cone). In the plane cylinder case, the fluid similarly conveyed must finally erupt from the boundary layer when the stagnation point is reached. However, on the cone, the boundary layer grows parabolically along generators; and, hence, if the crossflow is not too strong (small angle of attack), the fluid brought to the top may simply become part of the growing boundary layer. For larger angles of attack, the boundary layer cannot grow at a rate sufficient to absorb the additional fluid, and eruption occurs with the consequent breakdown of the thin-boundary-layer assumptions.

Accordingly, it is proposed that when the angle of attack is less than that for which $k=-1$, a thin boundary layer may cover the cone (fig. 3(a)). For larger angles of attack, any vortex sheet present will roll up to form attached lobes (fig. 3(b)); for still larger angles of attack, a vortex street is produced.

Thus, when $k < -1$ (angle of attack greater than that for which $k=1$), strong viscous cross forces (viscous lift) on the cone may be expected. These forces are discussed by Allen and Perkins in reference 14. Of course, a weakness of the present analysis is that no indication is given as to the strength of the rolled-up vortex system because, when $k=-1$, the presence of an embedded vortex sheet over the top part of the cone has not been established. It seems likely that such a vortex sheet does exist because a rather strong adverse

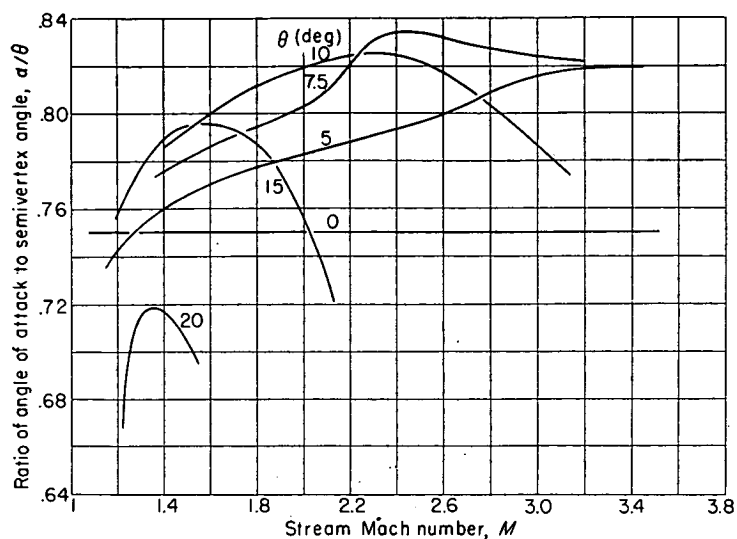


FIGURE 9.—Minimum angle of attack for which separation appears at $\varphi = \pi$.

pressure gradient (fig. 2(b)) always exists when $k = -1$. In fact, it may be shown (most conveniently by evaluating eq. (7a) at $\lambda = \infty$) that an adverse pressure gradient exists when $k < -2/3$.

Equations (13), (14), and (10b) give the critical angle of attack corresponding to $k = -1$. This angle is presented as a function of stream Mach number and vertex angle. Figure 9 shows the results of such a calculation. The critical angle of attack is given as a ratio of angle of attack to semivertex angle for convenience. The results suggest that, in general, separation involving lobes occurs later (in terms of relative angle of attack α/θ) for the smaller vertex angles, particularly at higher Mach numbers. Figure 9 indicates the possibility of rather profound qualitative differences in the flows at high Mach number about cones of different vertex angles.

The foregoing interpretation of the critical condition $k = -1$ is supported by the experimental result shown in figure 4. From figure 5, $k = -1$ when $\alpha = 6.2^\circ$, under the conditions of the test. Figure 4(b) shows the measured total head rising as the angle of attack is increased beyond 6.2° . Possibly this effect is caused by the induced field of the symmetrical pair of vortex lobes sweeping away the thick boundary layer between, thus reestablishing a thin boundary layer at the top of the cone.

It may be of interest to note that if similarity also holds for the turbulent boundary layer on a cone, and the similarity law is nearly linear (rather than parabolic as in the laminar case), separation would first appear at a higher angle of attack than in laminar flow. In fact, equations (10b) and (31) or (33) would yield the criterion $k = -4/3$.

For the boundary layer produced by supersonic flow over any smooth conically symmetric body in supersonic flow (such as a cone of elliptic cross section), inequality (33) and the condition of parabolic boundary-layer similarity may be used to find a criterion equivalent to $k = -1$ for the maximum angle of attack consistent with a thin boundary layer.

CONCLUSIONS

The laminar boundary-layer flow about a circular cone at large angles of attack to a supersonic stream has been analyzed in the plane of symmetry with the following results:

1. At the bottom of the cone, profiles of meridional velocity and of the gradient of circumferential velocity were determined and showed the expected tendency of the boundary layer to become thinner on the underside of the cone as the angle of attack is increased.

2. At the top of the cone, except for very small angles of attack, the analysis (which is restricted to the plane of symmetry) failed for the following reasons:

- (a) For angles of attack greater than some rather small value, the boundary layer brings information from beneath the cone into the vicinity of the plane of symmetry at the top. Therefore, the analysis, which deals only with the plane of symmetry, yielded indeterminate solutions.

- (b) For angles of attack greater than some angle (roughly of the order of the cone semivertex angle), no boundary-layer solution is possible. The characteristics of the outer flow and the known parabolic similarity of the boundary layer would together imply that, beyond this critical angle, there would be a component of flow leaving the boundary layer. This is physically impossible, since a boundary layer always entrains fluid. Thus, beyond the critical angle of attack, no solution can exist for equations which presume a thin boundary layer.

For three-dimensional flow it is proposed that a separated region be regarded as a vortex sheet embedded in the boundary layer, remaining flat against the body if the assumption of a thin boundary layer is valid throughout the region. If, however, the boundary-layer assumptions break down anywhere in the separated region, it is inferred that the vortex sheet may roll up to form strong vortices which may either remain attached or be shed as a vortex street.

On the cone, therefore, the critical angle of attack beyond which no boundary-layer solution is possible at the top of the cone represents the maximum angle of attack for which the boundary layer is everywhere thin or, alternatively, the minimum angle of attack for which major disruption of the flow may be expected because of the formation of strong vortex lobes. Beyond this angle of attack, strong viscous cross forces may be anticipated.

A similar criterion could easily be obtained for the boundary layer on any smooth conically symmetric body in supersonic flow.

The assumption of a suitable similarity law suffices to establish a similar criterion if the boundary layer on a conical body is turbulent.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, September 15, 1952

APPENDIX A

SYMBOLS

The following symbols are used in this report:	
C	constant appearing in temperature-viscosity relation (eq. (5a))
C_{f_x}	component of skin-friction coefficient in x -direction
C_{f_ϕ}	component of skin-friction coefficient in ϕ -direction
c_p	specific heat at constant pressure
$F(\lambda)$	function appearing in asymptotic representation of f (eqs. (24))
$f(\lambda, \phi)$	function related to meridional velocity u by eq. (2)
$g(\lambda, \phi)$	function related to circumferential velocity w by eq. (2)
h	height of outer edge of boundary layer
k	related to circumferential gradient of circumferential velocity in plane of symmetry (eq. (10b))
M	Mach number
p	static pressure
$\underline{q}_1$	velocity vector at outer edge of boundary layer
R_x	Reynolds number, $\rho_1 u_1 x / \mu_1$
$R_\Delta, R_{\delta_x}, R_{\delta_\phi}$	Reynolds numbers, $\rho_1 u_1 \Delta / \mu_1, \rho_1 u_1 \delta_x / \mu_1, \rho_1 u_1 \delta_\phi / \mu_1$, respectively
T	absolute static temperature
u	meridional component of velocity
v	component of velocity normal to surface
w	circumferential velocity component
x	coordinate along generators of cone
y	coordinate normal to surface
α	angle of attack (positive as shown in fig. 1)
γ	ratio of specific heats
Δ	displacement surface height
δ_x	mass-flow defect associated with meridional velocity profile (eq. (21))
δ_ϕ	mass-flow defect associated with circumferential profile (eq. (21))
Θ	semivertex angle of cone
θ	sine of semivertex angle of cone
λ	dimensionless variable (eq. (3))
μ	coefficient of viscosity
ρ	density
ϕ	angular coordinate around cone
$\Psi(\lambda)$	function appearing in asymptotic representation of ψ (eqs. (24))
$\psi(\lambda)$	function related to circumferential velocity w in plane of symmetry by equation (10a)
Subscripts:	
max	maximum
r	reference condition, nonviscous flow at surface, at $\phi=0$ or $\phi=\pi$, whichever is appropriate
1	evaluation at outer edge of boundary layer (alternatively, nonviscous flow at surface)
	Subscript notation for partial differentiation has been used
Superscripts:	
'	Primes denote ordinary differentiation with respect to λ or ϕ
—	Bar over quantity indicates evaluation of nonviscous flow at surface when cone is at zero angle of attack

APPENDIX B

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

By LYNN ALBERS

The two simultaneous nonlinear ordinary differential equations (11a) and (11b) together with boundary conditions (12) constitute a two-point boundary-value problem. The method of numerical solution used applies directly only to problems for which all boundary conditions are specified at a single initial point (the origin, in the present case). Each numerical integration was therefore performed starting with boundary conditions (12a) and (12b) and a tentative specification of $f''(0)$ and $\psi''(0)$. In each case, such integration was carried out for a sufficient variety of conditions $f''(0)$ and $\psi''(0)$ so that the correct set of initial conditions yielding the proper behavior at $\lambda = \infty$ (boundary condition (12a)) could be inferred to the desired degree of accuracy.

Integration was performed according to the following basic scheme: With the value of $f'''(\lambda)$ and $\psi'''(\lambda)$ given at five closely spaced values of λ , fourth-degree polynomials may be

passed through the two sets of values of f''' and ψ''' . Then, if $f, f', f'', \psi, \psi',$ and ψ'' are known at the fifth point, the polynomial representations of f''' and ψ''' may be integrated to yield $f, f', f'', \psi, \psi',$ and ψ'' at the next (sixth) point. These quantities may then be substituted into differential equations (11) to yield f''' and ψ''' at the sixth point. In this way, the solution may be extended one step at a time, in each step by use of the solution at the five previous points. In order to begin this procedure, the solution must first be found at five points starting at the origin and must be subject to boundary conditions (12b) and (12c) and the tentative selection of $f''(0)$ and $\psi''(0)$.

This preliminary calculation was done in the following manner: $f'''(0)$ was calculated directly from equations (11) and (12) and was used as an initial estimate of f''' at the next four points. Given $f(0), f'(0)$, and $f''(0)$, the values of $f, f',$

and f'' were computed at the second point by integrating a fourth-degree polynomial passed through the five values of f''' . In a similar manner, ψ , ψ' , and ψ'' at the second point were found. Direct substitution into equations (11) then yields improved estimates for f''' and ψ''' at the second point and thus an improved polynomial representation of these functions which may be used to obtain values of f , f' , f'' , ψ , ψ' , and ψ'' at the second point, and so forth, until improved values have been obtained at the fifth point. This procedure was repeated in an iterative manner until convergence was obtained at each of the five initial points.

All calculations were performed on the IBM Card Programmed Electronic Calculator. Results are considered correct to four significant figures.

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REPORT 1133

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REPORT 1133

MECHANISM OF START AND DEVELOPMENT OF AIRCRAFT CRASH FIRES¹

By I. IRVING PINKEL, G. MERRITT PRESTON, and GERARD J. PESMAN

SUMMARY

Full-scale aircraft crashes, devised to give large fuel spillage and a high incidence of fire, were made to investigate the mechanism of the start and development of aircraft crash fires. The results are discussed herein. This investigation revealed the characteristics of the ignition sources, the manner in which the combustibles spread, the mechanism of the union of the combustibles and ignition sources, and the pertinent factors governing the development of a crash fire as observed in this program.

INTRODUCTION

Recent aeromedical research has shown that the magnitude of deceleration human beings can withstand without serious injuries varies inversely with the time for which the deceleration is applied. The fact that in many airplane crashes high decelerations often exist for only extremely short periods of time indicates that worthwhile gains in crash survival might be realized if the fire that often accompanies crash were avoided. Acting on the recommendation of the NACA Committee on Operating Problems and the Subcommittee on Aircraft Fire Prevention, the NACA Lewis laboratory has engaged in a study of the airplane crash-fire problem. This

study of the manner in which crash fires start and develop is intended to serve as factual background on which features of airplane design can be based in order to reduce the likelihood of fire following crash and to improve the chances for escape or rescue should fire occur. Although this study will ultimately include aircraft powered with various types of turbine engines as well as reciprocating engines, this report considers only the work completed on aircraft with reciprocating engines. While the initiation of crash fires and the subsequent development of these fires are related events, the factors of interest in each of these events are quite different, and they are therefore treated separately in this report.

The current crash-fire research program is one of several studies made in the last 30 years. In general, the results of earlier work have been verified in this more comprehensive investigation. Of particular interest is the full-scale crash-fire study made from 1924-28 by the U. S. Army Air Corps, in which single-engine fighter aircraft powered by Hispano Swiza engines were employed. Notable contributions to the field of aircraft fires have been made by W. G. Glendinning and his associates in England.

The program on crash fire considered herein was conducted with modern aircraft and instrumentation on a scale sufficient to permit an appreciation of important factors not possible heretofore.

The current crash-fire study began in 1949 with a review of past crash accidents, civil and military. The investigation of crash-fire accident records (ref. 1), however, failed to reveal well-defined mechanisms for crash fire, primarily because the pertinent physical factors acting to initiate the fire are often concealed from view or are too short-lived to be reported accurately by eyewitnesses. Fire damage also obscured the true nature of the disruption suffered by the airplane that relates to the fuel spillage and generation of ignition sources. The need for conducting full-scale crashes under conditions permitting careful observation of the successive events in the crash was apparent from this accident study. Acquisition of service-worn twin-engine cargo aircraft from the U. S. Air Force for full-scale crash research made possible the analysis of the mechanism of crash fire discussed herein. Photographs of the low-wing C-46 and the high-wing C-82 airplanes used are shown in figures 1 and 2, respectively. A few of the features of each airplane, to which later reference will be made, are indicated in the figures. Of the 17 full-scale crashes conducted so far in



FIGURE 1.—C-46 airplane used in aircraft crash-fire program.

¹ Supersedes NACA RM E52F06, "Mechanism of Start and Development of Aircraft Crash Fires" by I. Irving Pinkel, G. Merritt Preston, and Gerard J. Pesman.



FIGURE 2.—C-82 airplane used in aircraft crash-fire program.

this program, four were C-46 airplanes and 13 were C-82 airplanes.

Since it is desirable to study the mechanism of crash fires under circumstances that approximate real crash conditions, a barrier was designed to impose the gross damage to the airplane similar to that which may occur in unsuccessful take-offs and landings, in which severe engine damage and major fuel spillage occurs. Airplane crashes at flight speed into obstructions such as buildings and mountainsides usually involve a degree of airplane disintegration so severe that design measures or equipment arranged to reduce the likelihood of fire are rendered impotent. For this reason, attention in this study is focused on crashes that occur at take-off and landing speeds, where the likelihood of personnel survival of the impact is high and design safety features and crash-fire protection equipment that may be employed have a reasonable chance to serve their function.

The results of the current work are limited to those features of airplane crash fires that have been investigated in this program. While an attempt has been made to include in this study as many factors involved in crash fires as are revealed by past experience and current results, undoubtedly there remain areas that are not covered in this work.

A synopsis of this report is included at the end for those who are not interested in a detailed discussion of the subject material.

METHOD OF CONDUCTING STUDY

A complete discussion of the crash technique employed in these studies is given in reference 2. It is useful, however, to repeat some of the salient features of this technique. A crash site, shown schematically in figure 3, was arranged to permit the airplane to accelerate from rest under its own power and, constrained by a single guide rail, to arrive at a crash barrier at approximately take-off speed (80 to 105 mph). The crash barrier is shown in figure 4 (a), and

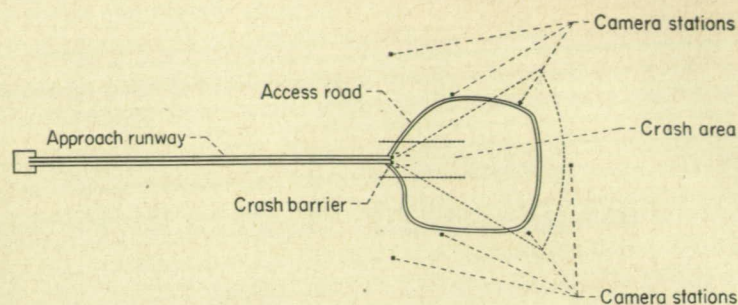


FIGURE 3.—Schematic drawing of test site for aircraft crash-fire program.

schematic views of the airplane as it passes through the barrier are shown in figures 4 (b) to (d). (The height of the abutment was adjusted to permit approximately 18 in. of the propeller tip to hit the barrier (fig. 4 (b)).) The disrupted engine and nacelle installations resulting from the propeller impact with the barrier generate a variety of ignition sources and fuel spillages, characteristic of this type airplane accident. Because the chances of obtaining a fire are higher when the damaged engine and its associated ignition sources stay with the airplane carrying the fuel than when the engine is dropped at the crash barrier, an attempt was made to restrict the barrier height to provide extensive engine damage short of engine break-out. The abutments in the path of the two main landing wheels rip out the landing gear (fig. 4 (c)). As the airplane moves through the barrier, the leading edges of the wings are cut by inclined poles (fig. 4 (d)) fitted with steel pins installed in comb-tooth fashion, which slice open the wing fuel tanks on both sides of the airplane. The airplane then slides to rest on the ground beyond the barrier. By these crash-barrier arrangements, it was hoped to impose damage sufficiently diverse so that a large percentage of actual crashes could be considered to be made up of all or several of these damaged components variously combined. By careful instrumentation of the crash airplane and camera coverage of the crash site with standard and greater-than-normal speed cameras, an appreciation was obtained of the way a variety of factors act to initiate a fire.

Instrumentation and data-recording equipment were carried on the airplane to obtain the following information at appropriate times during the crash and ensuing fire:

- (1) Fire location throughout nacelles, wings, and fuselage
- (2) Personnel compartment temperatures, ambient and radiant
- (3) Distribution of combustible mixtures in wings and nacelles
- (4) Timing and location of electrical short circuits
- (5) Timing of fuel-line ruptures
- (6) Acceleration of separate components of airplane in crash

These data were converted to electric signals that could be read on panel-located meters and indicating lights carried in a fireproof box in the airplane. The signals were photo-

graphed 10 times per second through the crash impact and ensuing fire.

All airplanes carried a total of approximately 1050 gallons of fuel in the outboard wing tanks equally distributed between the wings. In almost every crash, the fuel temperature was between 70° and 80° F. Fuel preheating was employed when required by the weather. Gasoline and low-volatility fuel (8 mm Hg Reid vapor pressure) were employed. Unless otherwise stated, the fuel employed in a given instance was aviation-grade gasoline. Inspection data of the two fuels are shown in table I. In some cases the fuel was dyed red for photographic purposes. In these cases a notation has been made on the figure.

In order to appreciate fully the significance of some of the factors in the mechanism of crash fires observed in the full-scale crash studies, a parallel set of ground studies was conducted. These ground studies helped to define the circumstances under which a particular factor or combination of factors could initiate a fire and indicated how these factors were influenced by variables such as wind, fuel volatility, state of motion of the airplane, and arrangement of the airplane components in their normal and crash configurations.

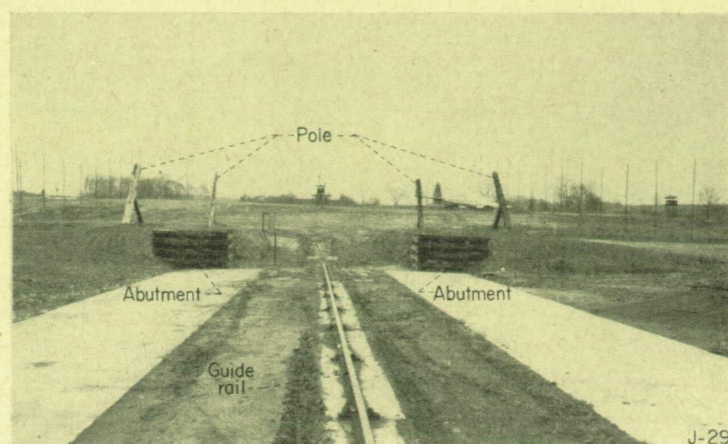
MECHANISM OF CRASH FIRE

A consideration of the results obtained in the first few airplane crashes conducted in this program showed that a detailed approach to the problem is required if erroneous and contradictory impressions are to be avoided. Factors that were believed to be of secondary importance or that were ignored entirely in a superficial approach were revealed by this study to control the methods by which fire occurs.

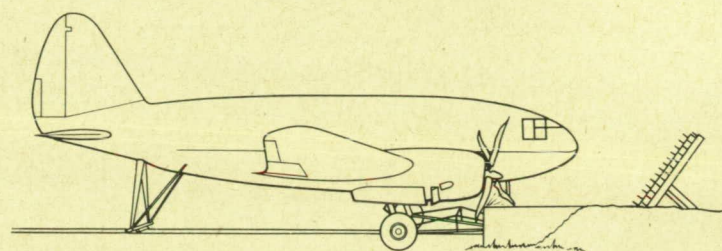
Inquiry into the manner in which the crash fire occurs centers on the answers to two principal questions: How and when do ignition sources appear in the airplane crash, and How does the fuel come into contact with the ignition sources? A proper consideration of the mechanism of crash fire thus requires that the subjects of ignition sources and fuel spillage be treated concurrently. In the organization of this discussion, however, it is useful first to consider briefly the factors controlling fuel ignition and the origin of ignition sources and then to consider them again concurrently with a discussion of the fuel spillage. A series of crash experiments with aircraft modified to reveal effects not readily apparent with the aircraft in their normal configuration is discussed separately for purposes of clarity.

FUEL IGNITION

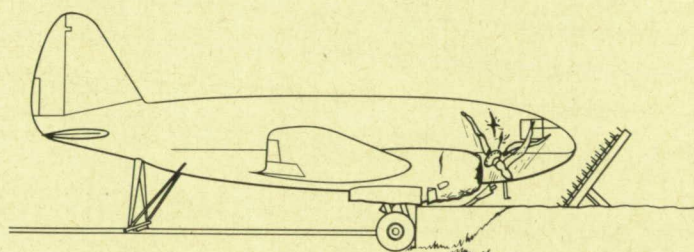
In general, the range of the variables influencing fuel ignition in an airplane crash is within the range of existing combustion experience, and this information is of significance in crash fires. The scientific literature contains several summaries of the factors controlling hydrocarbon fuel ignition. Reference 3 is an example of a recent report on the subject. Unfortunately, the space in which combustible



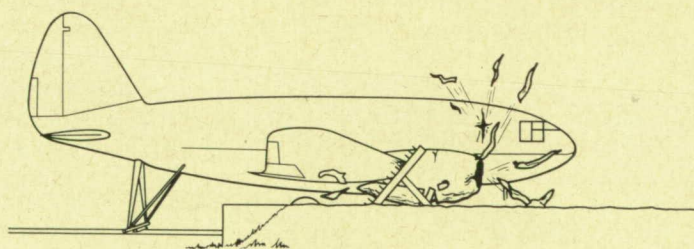
(a)



(b)



(c)

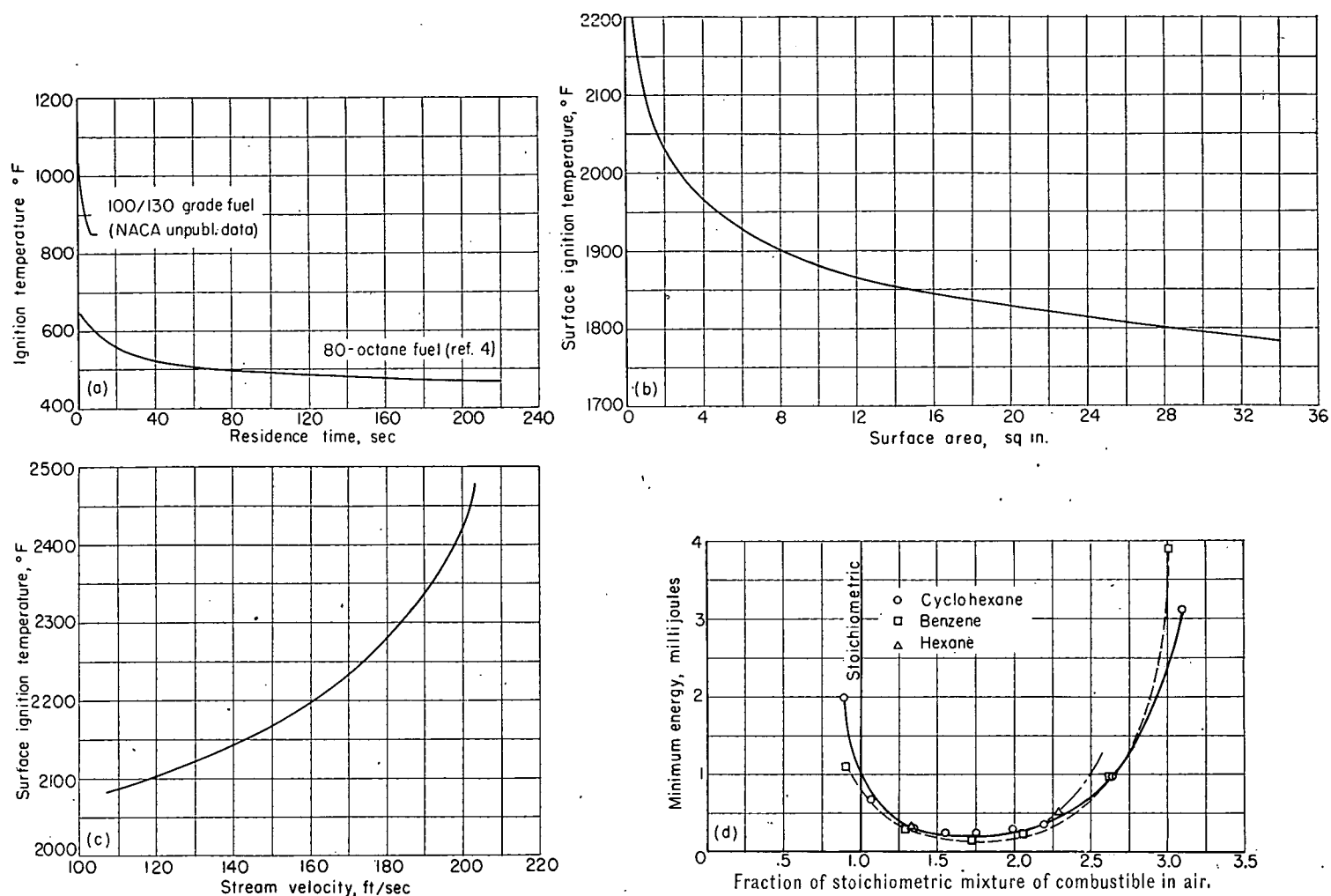


(d)

- (a) Approach runway and crash barrier.
- (b) Impact of propeller with abutment.
- (c) Impact of landing gear with abutment.
- (d) Impact of wing with pole barrier.

FIGURE 4.—Crash barrier and successive views of airplane as it passes through barrier.

mixtures are formed and the ignition sources appear in a crash is too large for a measurement of the factors influencing fuel ignition to be taken during the crash. The range of values of many of these factors is fixed, however, by the fact that the crash fire occurs at ground level, usually below 10,000 feet. This fact defines the range of air pressure, temperature, and velocity that exists in a crash. In view of



- (a) Effect of residence time on ignition temperature of aviation gasoline.
 (b) Effect of surface area on surface ignition temperature of 7 percent quiescent mixture of natural gas and air. Ignition surface, nickel. (Data from ref. 3.)
 (c) Variation of surface ignition with gas-stream velocity for a stoichiometric mixture of pentane and air at atmospheric pressure and temperature of 155° F. Ignition by 1/4-inch heated stainless-steel rods. (Data from ref. 3.)
 (d) Variation of minimum ignition energy of combustible-air mixtures with fraction stoichiometric mixture.

FIGURE 5.—Variation of spontaneous-ignition temperature with residence time, surface area, and air velocity.

these limitations, the principal function served by a background in fuel ignition is in helping to define the circumstances, sometimes quantitatively and often qualitatively, that must have existed in a crash to give the observed results.

While much concerning fuel ignition is well-known and need not be repeated here, the subject of the spontaneous-ignition temperature of hydrocarbons presents certain subtleties that have been the source of much confusion and merit consideration. Chief among these subtleties is the fact that there is no single assignable minimum ignition temperature for hydrocarbon fuels. Experiments on the so-called spontaneous-ignition temperature, conducted in apparatus in which combustible concentrations of hydrocarbons are contained in a uniformly heated cavity of known temperature, provide an ignition temperature-time curve, such as that shown in figure 5 (a), taken from reference 4 and from unpublished NACA data. On this figure appears a curve for 100/130 grade aviation fuel. These data were obtained in a

standard A.S.T.M. spontaneous-ignition-temperature apparatus. The other curve on this figure is for aviation gasoline in use at the time of the experiments (1930) which was probably 80-octane fuel. These data were obtained in an enclosed steel tube 12 inches in diameter and 12 inches long.

From these data it may be seen that, as the residence time of the fuel at elevated temperature increases, the minimum temperature at which ignition may occur reduces. For a brief residence time of 2 seconds for 100/130 grade gasoline, the ignition temperature is 1020° F. This ignition temperature reduces to 850° F for a residence time of 6 seconds. By comparing these data to those of reference 4, it can be seen that this ignition temperature may go even lower if the fuel remains in contact with the heated surface for a longer time; 6 seconds is the longest residence time for which data are available at this time. Variations in the chemical composition and fuel-air ratio will shift the position of this curve on the time-temperature coordinates without altering the essential

fact that the ignition temperature declines with increasing residence time. It is important to appreciate that the residence time refers to the time a given fuel molecule stays at the prescribed temperature. Minimum spontaneous-ignition temperatures for lubricating oil and for hydraulic fluid were obtained in references 5 and 6, respectively. A minimum spontaneous-ignition temperature for lubricating oil of 770° F was obtained with a residence time of 6 seconds. The minimum spontaneous-ignition temperature for hydraulic fluid was 437° F at a residence time of 140 seconds. When the fuel-air mixture is not contained in a uniformly heated cavity but contacts a hot surface maintained at temperatures appreciably above that of the neighborhood, the fuel contact time with the hot surface is governed by the area of the surface and the magnitude of natural-convection currents associated with the hot surface or the local forced-air circulation rate produced by wind or airplane motion. Figure 5 (b) illustrates the marked dependence on surface size of the minimum surface temperature required for ignition, and figure 5 (c) shows how the air motion adjacent to the heated surface raises the surface temperature required for ignition. It is evident from these data that the surface temperature required to ignite a mixture of fuel and air which is in motion past a hot engine exhaust-disposal system is higher than that required when the exhaust-disposal system is at rest in the same atmosphere. Combustible mixtures resident within stationary engine cylinders where prolonged residence time is possible would have spontaneous-ignition temperatures that can be as much as 100° F below the surface temperatures required of the exhaust-disposal system for fuel ignition except in a sheltered zone, according to the data of figure 5 (a).

In contrast with the time delay associated with ignition by hot surfaces, flames and electric sparks provide almost instantaneous ignition. The energy required in an electric spark to ignite the constituents of gasoline decreases from 0.9 to 0.1 millijoule as the fuel-air ratio changes from stoichiometric to 1.8 times stoichiometric (fig. 5 (d)).

Because of the interaction of all these variables in establishing an ignition temperature, a preliminary study was made with an operating engine to determine the ignition temperature of gasoline and lubricating oil on the hot exhaust-collector ring. The minimum ignition temperature obtained in these studies was 950° F for aviation gasoline and 760° F for lubricating oil. Ignition of hydraulic fluid was obtained at 600° F; however, no attempt was made to obtain a lower ignition temperature.

In the subsequent sections of this report, these data on fuel ignition will be helpful in interpreting some of the observations made in the crash-fire studies.

IGNITION SOURCES

The ignition sources revealed by the full-scale airplane crash studies can be classified in the following broad categories:

- (1) Hot surfaces
- (2) Friction and chemical sparks from abraded airplane metals
- (3) Engine-exhaust flames
- (4) Engine induction-system flames
- (5) Electric arcs, and electrically heated wiring and lamp filaments
- (6) Flames from chemical agents
- (7) Electrostatic sparks

HOT SURFACES AND FRICTION AND CHEMICAL SPARKS

Almost all hot metal surfaces present in a crash previous to the start of fire are carried by the airplane before crash impact. These surfaces include the exhaust-disposal system, exhaust-gas heat exchangers or combustion heaters, and the high-temperature spots of the engine cylinder interior. In a crash at take-off, the highest temperatures at local areas of the exhaust-disposal system exceed 1200° F because of the high engine power level employed. The lowest temperature of the exhaust system at which gasoline ignition was obtained on the external surfaces was 950° F. In a normal landing, the exhaust-disposal system has temperatures as high as 840° F in local areas. These data were obtained with a C-82 airplane. While this temperature is too low for gasoline to ignite readily, it is high enough for the ignition of lubricating oil that may, in turn, ignite the fuel. This gasoline ignition temperature of 950° F is in keeping with the data shown in figures 5 (a) and (b), in view of the large heated surface area presented and the low residence contact time between fuel and surface permitted by the convective air movement around the exhaust-system components. If high engine power is employed to correct a faulty landing approach, high exhaust-system temperatures characteristic of take-off may occur upon landing as well. Ignition sources in this category are likely to remain fixed in position with respect to the rest of the airplane in the take-off and landing type of crash considered in this study. Loss of an engine from the nacelle, however, will free some of these hot surfaces to move into different zones around the airplane.

Probability of ignition on hot surfaces is high, because they are present at the moment of crash and remain at dangerous temperatures for several minutes thereafter. The temperature history of the exhaust-disposal system, for example, taken during crash (fig. 6), following operation at take-off power, indicates that it takes 30 seconds for the hottest portions of the exhaust-collector ring to cool to 950° F, the lowest temperature at which gasoline will ignite readily on the external surfaces of the exhaust system, and 84 seconds to cool to 760° F, the lowest temperature at which lubricating oil was observed to ignite readily.

The hot surfaces of the engine cylinder interior also can ignite a fuel-air charge. Since the situation within the cylinder approximates the circumstances under which the spontaneous-ignition temperature of gasoline was measured in a cavity bounded by hot walls, the data of figure 5 (a)

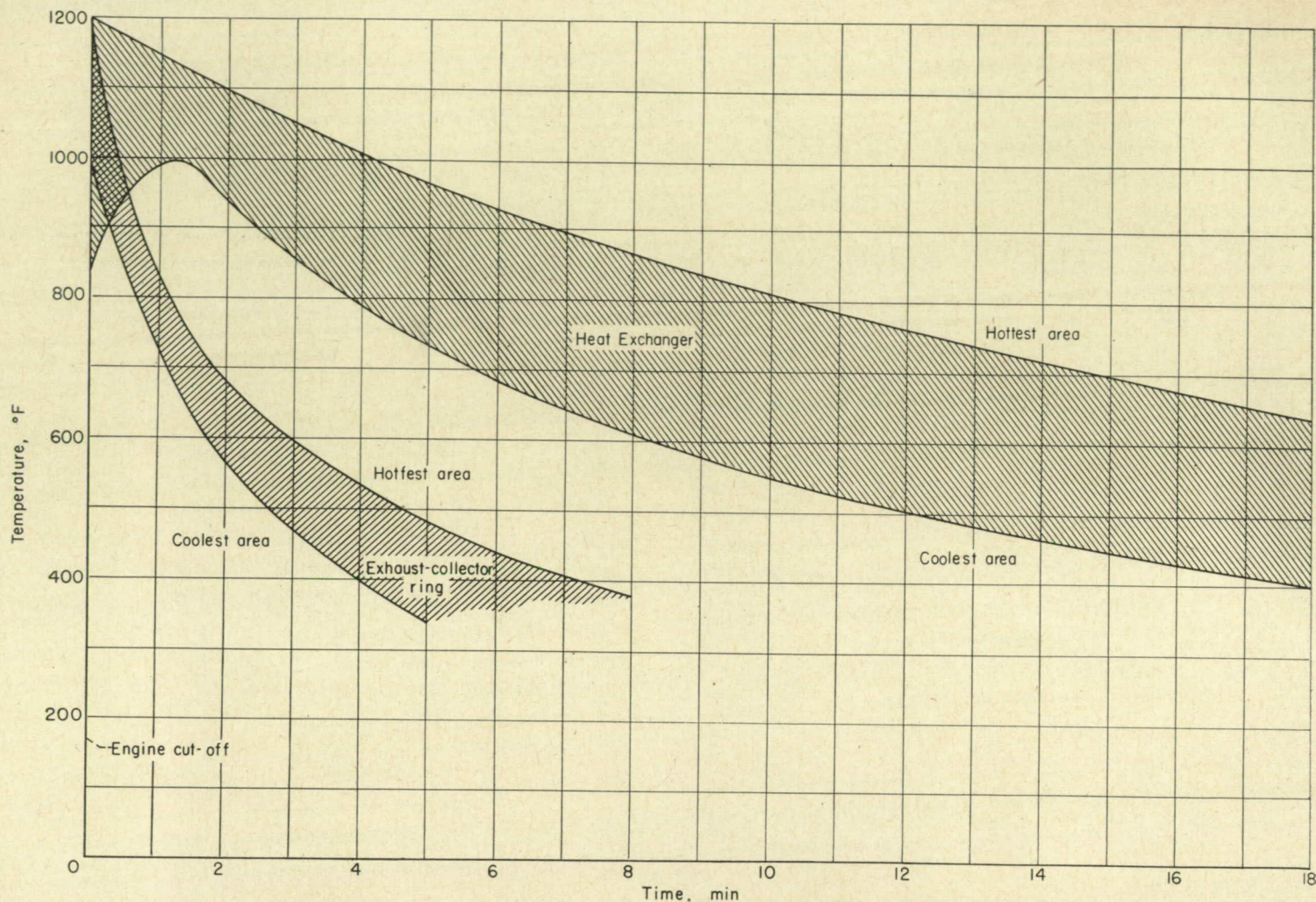


FIGURE 6.—Temperature-time history of exhaust system during crash test.

apply. These data show that the ignition of gasoline is possible if the effective temperature of a portion of the cylinder contents remains above 850°F for the 8 seconds involved. Such conditions are probable around the spark plugs in the cylinder head and the exhaust-valve assembly. Following operation at full engine power, elements of the exhaust-valve port and assembly may be hot enough to ignite the fuel-air charge almost immediately upon contact. At times, however, the cylinder charge that is ingested just as the engine rotation stops ignites after many seconds residence time in the cylinder. In one of the instances noted in this study, the resulting flash appeared as a backfire 3.7 seconds after the engine rotation ceased. The appearance of this flame at the engine inlet is shown in figure 7 (a). Another example noted in this study is shown in figure 7 (b), in which case the engine ignition system was cut 2 seconds before impact while the engine was developing full power. At 2.2 seconds after impact, the backfire shown in figure 7 (b) occurred.

The hot surfaces of friction sparks and the parent metal surface from which the sparking particles are abraded represent possible ignition sources that can appear only while the crashed airplane is in motion, by virtue of the mechanism

by which friction heat and abrasion are developed. Sparks of sufficient size and temperature to ignite gasoline have been obtained from steel airplane parts bearing on concrete paving with contact pressures in excess of 100 pounds per square inch. Ground studies under simulated crash circumstances showed the ignition hazard also associated with the abrasion of magnesium on concrete paving or stony ground. The abraded magnesium particles ignite in the air and provide ignition sources whose temperatures are considerably greater than those of abraded steel particles. The high temperatures of the latter are obtained primarily from friction heat. It is useful to designate sparks that owe their elevated temperatures to high oxidation rates as chemical sparks. Chemical sparks of sufficient size and temperature will ignite aviation gasoline with moderate loads between the materials in grinding contact. Friction sparks that would ignite aviation gasoline must be generated with high bearing force per unit area.

Wheel brakes heated to temperatures high enough to ignite hydraulic fluid and possibly gasoline by heavy application of the brakes have been reported elsewhere and must be included among the hot-surface ignition sources, even though

they were not encountered in these studies. Circumstances under which the brakes plus the hydraulic fluid might constitute an ignition source were observed in a crash in which the wheel and strut stripped from the airplane at the crash barrier followed the airplane in its skid along the ground. The hydraulic fluid, contained under air pressure in the landing-gear strut, issued as a fan-like spray from a longitudinal crack in the strut. If these circumstances were to exist in a crash in which heavy braking was employed previous to or during the crash, experience with ignition of hydraulic fluids would indicate a high probability of fire, first of the hydraulic fluid and then of the gasoline that may flow to the wheel from the crashed airplane.

EXHAUST FLAMES

Plumes of exhaust flames appear in the crash when fuel from the disrupted engine passes into the exhaust-disposal system without completing its combustion in the normal manner in the cylinder. Such a condition may result from failure of a spark plug to ignite the fuel, failure of the exhaust valve to contain the burning cylinder charge, or excessive enrichment of a cylinder charge, which burns as a torch in the air at the exhaust-pipe exit. If the engine ignition is cut off when the engine is drawing fuel, a series of flames often appears at the tail pipe (fig. 7 (c)). If the ignition is cut off just prior to a crash, the exhaust flames may continue to appear following crash impact. Several momentary flashes of exhaust flames are likely to occur immediately after impact of the engine propeller blades with the ground. These flashes may continue at irregular intervals as long as the engine drive shaft is rotating and fuel is drawn into the engine. Because an airplane crash does not always stop the engine completely, exhaust flames can appear for several minutes after crash impact. The appearance of one of the longest exhaust flames observed in these studies is shown within the circle at the engine exhaust in figure 8. In no case did an exhaust flame extend 16 inches beyond the exhaust exit and last for more than 0.2 second in the crashes studied. In the three instances in which engines were torn free from their mounts in the crash, no exhaust flames were observed during and after engine separation. Loss of the carburetor at the beginning of engine separation may be the reason for this effect.

ENGINE INDUCTION-SYSTEM FLAMES

Engine induction-system flames appeared less frequently than exhaust flames in the airplane crashes conducted so far. Backfire of an engine cylinder charge out the inlet port and consequent ignition of the induction-system fuel-air mixture is a principal mode of development of induction-system flames. Because a cylinder charge may become ignited several minutes after the engine has stopped rotating, the induction-system flame can appear at any time from the moment of the crash to several minutes thereafter. If the engine induction system is intact in the crash, the flame appears

at the entrance to the engine inlet scoop. Wherever the engine inlet system is broken, the flame appears as well.

ELECTRIC ARCS AND ELECTRICALLY HEATED WIRING AND FILAMENTS

Disruption of the extensive airplane electrical system in a crash may provide ignition sources at widely scattered locations. Wires that are completely severed may provide electric arcs between the high potential wire terminal and the grounded airplane structure of sufficient intensity to ignite fuel or other combustibles. Wires may become incandescent by short circuits resulting from abrasion of wire insulation or collapse of the metal housing of a junction box onto the terminal post located within the box. Incandescent light filaments are normally present during flight at night. Because of the relatively large diameters of the filaments of the landing light, temperature high enough for gasoline ignition may exist for at least 0.75 to 1.5 seconds after the light bulb is smashed and the filament broken (refs. 7 and 8). Momentary electric sparks produced by interrupting a current-carrying circuit are more likely to have sufficient energy to ignite fuel if the circuit contains coils such as the electromagnets employed in relays and valves.

About 0.15 millijoule is the minimum energy for spark ignition of a slightly richer than stoichiometric mixture of hydrocarbons and air under approximately standard conditions of pressure and temperature (ref. 9). This minimum energy is affected by size, material, and spacing of contacts and the rate and manner in which the energy is delivered. With an optimum contact separation of 0.65 inch, the effect of contact size is negligible. As the contact spacing is decreased, the minimum energy for ignition and the effect of contact size increase. With a needle contact, the minimum energy required for ignition is doubled when the contact spacing is decreased from 0.65 to 0.2 inch. With a contact size of 0.19 inch, the energy required with a contact spacing of 0.2 inch is five times that required at 0.65 inch. Greater energies are required with smaller separations because of the quenching action of the contact surfaces (ref. 10). Reference 11 indicates that, when the rate of delivery of energy is exponential, 90 percent of the energy must be delivered in less than 50 microseconds to provide sufficient current for ignition. In crash fires in which disruption of low-voltage circuits is the biggest problem, arcing potentials are obtained from inductances containing considerable resistance that will absorb energy; since the minimum value given represents the energy delivered to the combustibles, the total circuit energy will be larger by that amount absorbed by the resistance. Because storage batteries are part of the airplane low-voltage circuit, electrical ignition sources may persist for several minutes after crash. Very often, however, electric arcs rapidly burn away the metal forming the arc electrodes, and the arc persists for not more than 0.6 second (ref. 12). A restrike of the arc is possible as the airplane continues to deform in the crash or if the wind deflects the airplane structure adjacent to an electrical failure and new electrical contact is made momentarily.

FLAMES FROM CHEMICAL AGENTS

Combustible chemical agents commonly carried on the airplane, in addition to the fuel, include the petroleum-oil-base hydraulic fluid, lubricating oil, and alcohol for icing protection. During flight, the ignition of any of these agents can produce disastrous fire even if the fuel never becomes involved. On the ground, however, a fire is seldom serious with respect to passenger survival until the fuel is ignited. For this reason, it is convenient in a discussion of crash fire to classify these agents when aflame as ignition sources in the same sense that plumes of burning fuel issuing from the engine exhaust are considered ignition sources.

Because all these chemical agents will ignite at lower temperatures than gasoline and some may require less energy in the electric spark for ignition, the presence of these agents improves the likelihood of the appearance of flaming materials in a crash. Once ignited, these agents burn for a long time and extend the time over which ignition of the fuel may occur. Being liquid, they can flow by gravity, or be distributed explosively (as in the case of hydraulic fluid contained at high pressure), or vaporize and move by convection to form a conducting path for fire from a fixed ignition source to the fuel.

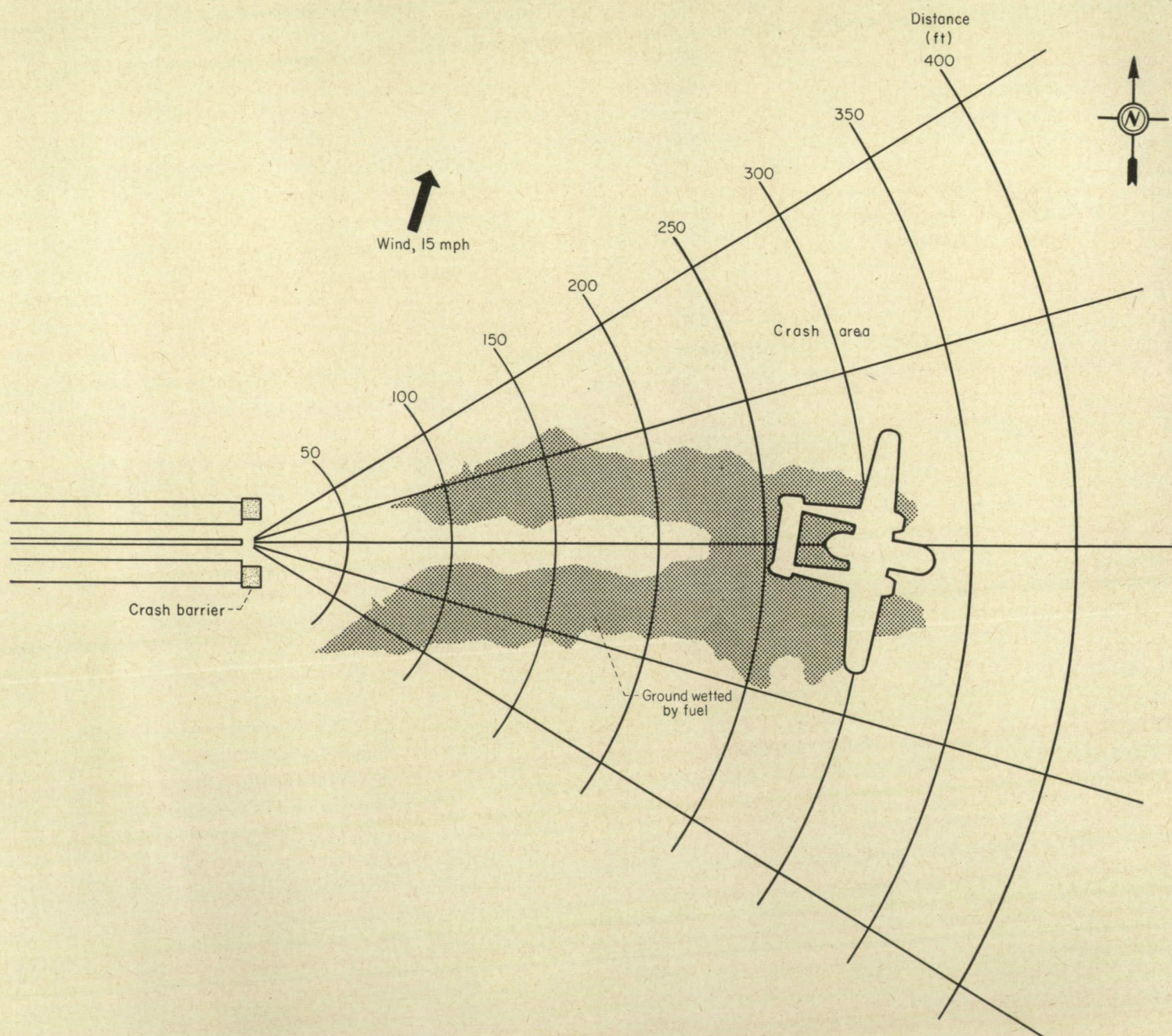


FIGURE 10.—Characteristic ground-wetting pattern of fuel deposited in wake of airplane.



(a) Backfire from engine induction system 7.7 seconds after initial impact and 3.7 seconds after end of engine rotation.



(b) Flaming out of induction system inlet 2.2 seconds after impact with barrier. Ignition system cut 2.0 seconds before impact.



(c) Exhaust flame resulting from cutting of engine ignition system during fullpower operation.

Figure 7.—Flaming out of engine induction system and exhaust flames from engine exhaust.



Fire from
previous
exhaust flame

Figure 8.—Largest exhaust flame from engine exhaust system observed during crash program; 3.3 seconds after initial impact.



(a) Front view; 0.04 second after impact with first pole barrier. Airplane speed, 125 feet per second.



(b) Side view; 0.37 second after impact with first pole barrier. Airplane speed, 125 feet per second.



(c) Front view; 1.5 seconds after impact with first pole barrier. Airplane speed, 85 feet per second.



(d) Side view; 1.9 seconds after impact with first pole barrier. Airplane speed, 70 feet per second.



(e) Front view; 3.0 seconds after impact with first pole barrier. Airplane speed, 30 feet per second.



(f) Side view; 3.4 seconds after impact with first pole barrier. Airplane speed, 20 feet per second.



(g) Front view; 4.4 seconds after impact with first pole barrier. Airplane speed, 8 feet per second.



(h) Side view; 4.8 seconds after impact with first pole barrier. Airplane speed, 0.

Figure 9.—Development of fuel mist spillage from ruptured wing tanks. (Fuel dyed red.)

ELECTROSTATIC SPARKS

Electrostatic charge can be accumulated on airplane parts torn from the airplane in the crash as these parts move above ground through the dust and fuel suspended in the airplane wake. As the torn airplane part approaches the ground, an electrical discharge may occur of sufficient intensity to ignite the fuel spread in the airplane wake. Because the bulk of the airplane structure sliding along the ground usually makes good electrical contact with the ground, significant differences from ground potential on this structure are unlikely.

In general, the ignition sources observed in the crashes conducted so far are those expected to apply in airplane crashes on the basis of past experience with normal aircraft operations and by analogy with circumstances in other technical fields that are similar to those obtained in a crash. The full-scale crash studies indicate how the ignition sources arise, the time in the crash that they are likely to appear, and the circumstances under which they will start a fire.

FUEL SPILLAGE

In a crash, fuel is spilled in liquid form from broken fuel lines and tanks, as premixed fuel vapor and air from the damaged engine induction system, and as fuel mist around the airplane when the spillage appears on the outside of the airplane while it is in motion. In the last case, the pressure and viscous forces of the air on the fuel rip it to mist that moves with the air around the airplane. In the crash arrangements employed in this study, liquid and mist spillage occurred in every crash, and carbureted fuel spillage from the engine induction system in only a few cases. These latter instances, however, were sufficient to reveal how such spillage initiates fire.

FUEL MIST

Because the poles located at the crash barrier ripped open the fuel tanks and the adjacent wing skin while the airplane was moving at take-off speed through the crash barrier, the first fuel to appear was in mist form completely suspended in the air. At this time, the fuel mist (dyed red) had the appearance shown in figures 9 (a) and (b). At the existing high relative speed between fuel issuing from the tanks and the air streaming by, a significant percentage of the fuel droplets had a sufficiently small size to be suspended in the air for many seconds. These small fuel droplets could be observed making up the less dense cloud rising above the wing in the airplane wake (figs. 9 (c) and (d)). The large fuel droplets in the dense cloud remained suspended in the highly turbulent air adjacent to a jet of fuel issuing from the wing-tank rupture. As they moved to the rear, some of these large droplets were intercepted by the fuselage and tail-assembly surfaces and the remainder rained to the ground (figs. 9 (e) and (f)). As the airplane slowed, the average droplet size of the fuel particles increased until the fuel appeared to pour in a solid stream from the wing-tank rupture when the airplane came to rest (figs. 9 (g) and (h)).

A characteristic ground-wetting pattern provided by the fuel deposited in the wake of the airplane is shown in figure 10. The fuel trail deposited at the barrier is composed of two separate bands, one for each wing, which tend to broaden and finally coalesce in the neighborhood of the airplane rest point. When a tailwind blows, the ground-wetting pattern extends forward of the wing leading edge. In general, the inertia of the fuel carries it forward of the airplane rest point.

Time duration of mist ignition hazard.—Fuel mists in ignitable concentrations seldom persist around the airplane for more than 17 seconds after the airplane comes to rest. The larger mist droplets rain to the ground, and the air-borne fuel droplets are swept from the area by the wind. In passing from the area of the crash, the fuel mist may be blown over the engine nacelles, and, during the first few seconds of this period, the probability of mist ignition is significant. The hazardous period is short, because the last portion of the fuel mist to pass over the airplane is diluted below the ignitable limit. Even on calm days, the air dragged by the crashed airplane in its slide along the ground sweeps over the airplane when it comes to rest, and brings with it some of the fuel mist suspended in the airplane wake. Fuel-mist dispersal times taken from motion pictures of the crash are plotted in figure 11 as a function of the wind speed. The dispersal time decreases inversely with the wind speed in the expected manner. Because of the large error inherent in making a visual estimate of the persistence time of the fuel mist in the neighborhood of the airplane, indicated by the scatter of the data of figure 11, it was not possible to evaluate the effect of fuel volatility on this persistence time. Fuel

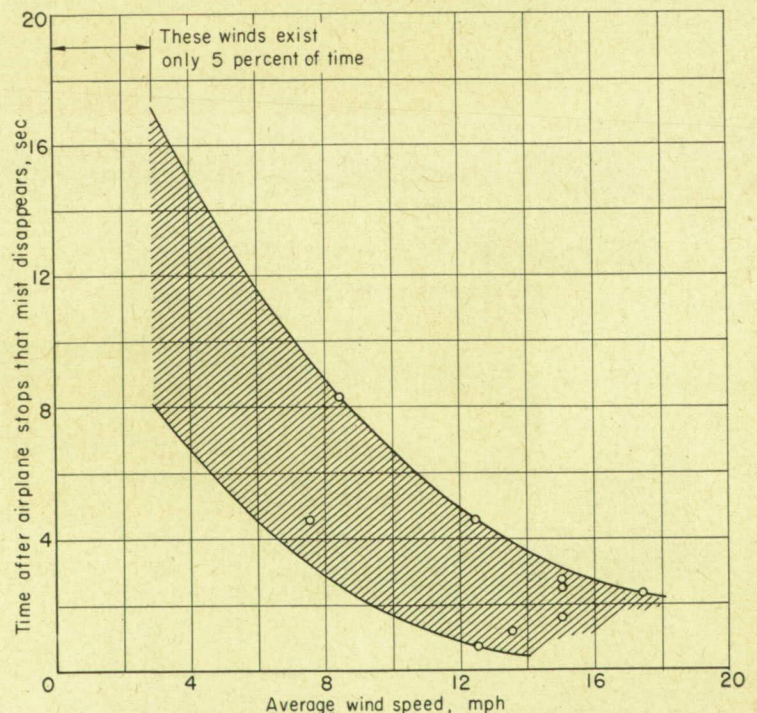


FIGURE 11.—Effect of wind velocity on time after airplane stops that fuel mist remains around crashed airplane.

vapors associated with the mist will move with the wind approximately as the smaller mist droplets and may have a persistence time a few seconds greater than the bulk mist.

Dispersion and ignition of fuel mists.—A first clue to the importance of the dynamics of the dispersion of the fuel mist in the ignition of the fuel was obtained early in the full-scale crash-fire studies when it was observed that this mist can propagate in a spanwise direction from the point of spillage on the wing and reach ignition sources located in and around the nacelle. This spanwise fuel propagation represents a displacement of the fuel droplets approximately perpendicular to the direction of the relative wind over the moving crashed airplane. It was also observed that, when the point of fuel spillage was located spanwise from the ignition source, fuel ignition occurred after the airplane slowed appreciably from its high speed at the crash barrier. A typical instance of this effect is illustrated in figure 12, in which is shown a series of pictures taken in sequence as the crashed airplane slid from the crash barrier to its rest point. Each photograph of the series shows an exhaust flame issuing from the engine tail pipe located 5 feet spanwise from the point of fuel spillage on the wing. Ignition of the fuel mist at the engine tail pipe occurred, however, when the airplane slowed from its initial speed of 137 feet per second at the crash barrier to 10 feet per second, the speed at which the photograph of figure 12 (d) was taken. (The light that appears on top of the pilot's compartment (fig. 12 (c)) is a timing light that is part of the airplane crash instrumentation.)

A study of the spanwise fuel spread conducted with taxiing airplanes and simulated fuel spillage (fig. 13) showed the following mechanism of fuel dispersion: When the leading edge of a fuel tank is breached on a decelerating airplane, the momentum of the fuel in the tank provides a forward surge of the fuel as a solid stream from the tank opening.

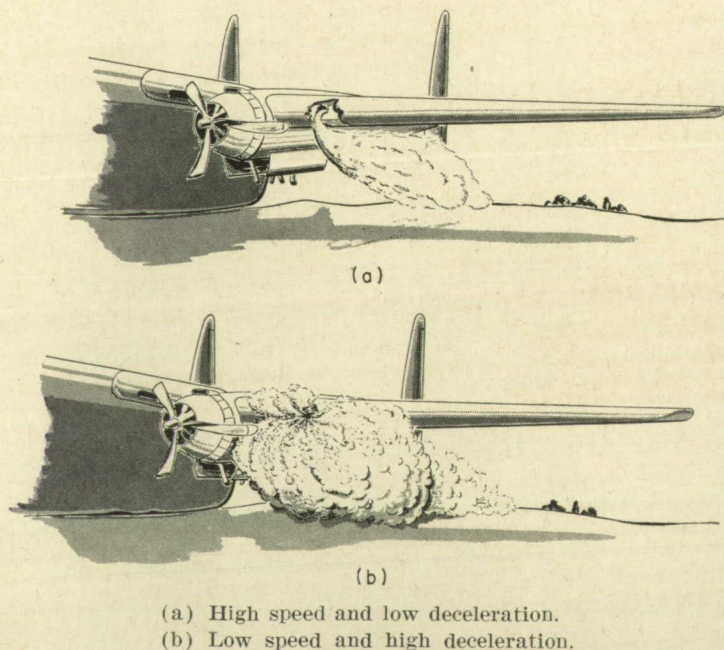


FIGURE 14.—Schematic diagram showing effect of deceleration and airplane speed on spanwise propagation of fuel mist emerging from ruptured fuel tank.

Impact with the air spreads the stream to give a spanwise velocity component to the fuel particles somewhat as would occur if the solid stream of fuel were to splash against a wall set normal to the original fuel-jet direction. The forward velocity of the fuel particles is reduced when the spanwise velocity component is acquired, and the advancing airplane intercepts the spreading fuel mist. If the airplane moves slowly, the fuel has an appreciable time to spread spanwise before interception and can extend to the nacelle. Likewise, high decelerations will produce high-velocity fuel jets that extend well ahead of the airplane and acquire high spanwise velocities. The combination of reduced airplane speed and high deceleration represents the critical condition of airplane motion with respect to fuel ignition by a source removed from the zone of fuel spillage.

A fuel-mist pattern obtained when the airplane deceleration is low and the speed is high would have the small apex angle and consequent low spanwise extension at the leading edge of the wing shown in figure 14 (a). In contrast, the mist pattern associated with high deceleration and reduced airplane speed (fig. 14 (b)) shows a wide apex angle and appreciable spanwise spread along the wing leading edge.

Wetting patterns (fig. 15) obtained with the simulated fuel spillage during the airplane taxiing studies show these effects clearly for four combinations of airplane speed and deceleration. The fuel was replaced by dyed water that issued from the wing at the location indicated in the figures. In figure 15 (a), the wetting patterns on the underside of the wing obtained with a fuel-jet velocity corresponding to a sustained deceleration of approximately $2.5g$ are less extensive spanwise than those obtained in the case of the $6.4g$ deceleration shown in figure 15 (b). In both figures, the wetting pattern is broader for the lower airplane speed. In the high-deceleration case, the fuel mist had a forward extension sufficient to wet the propeller blades and cowl inlet. Because the fuel droplets in the mist are air-borne, a relative wind having a spanwise component from the wing tip to the

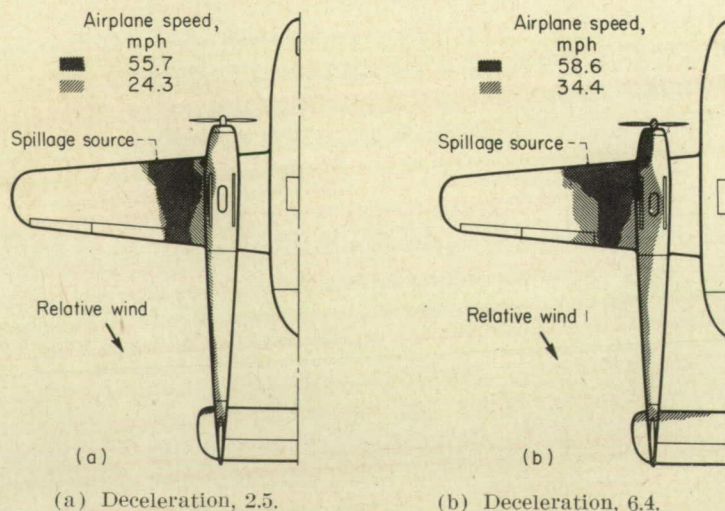
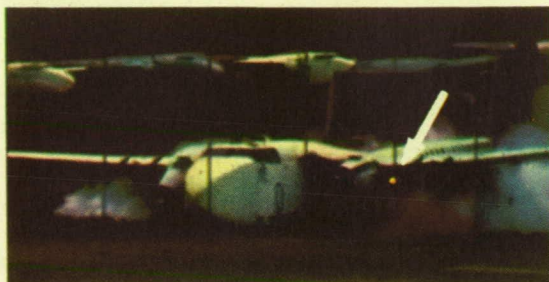


FIGURE 15.—Wetting pattern obtained with simulated fuel spillage during airplane taxiing tests.



(a) First exhaust flame and spread of fuel mist; 1.1 seconds after initial impact. Airplane speed, approximately 105 feet per second.



(b) Second exhaust flame and spread of fuel mist; 2.0 seconds after initial impact. Airplane speed, approximately 70 feet per second.



(c) Third exhaust flame and spread of fuel mist; 2.2 seconds after initial impact. Airplane speed, approximately 60 feet per second.



(d) Ignition of fuel mist; 4.1 seconds after initial impact. Airplane speed, approximately 10 feet per second.

Figure 12.—Ignition of fuel mist by exhaust flames from exhaust outlet.



Figure 13.—Test setup conducted with taxiing airplanes for simulating spanwise spread of fuel mist during aircraft crashes.



(a) Front view.



(b) Side view.

Figure 16.—Example of extreme forward and spanwise fuel mist development obtained during crash involving high airplane decelerations. Airplane speed, approximately 40 feet per second. (Fuel dyed red.)



(a) Fuel mist; 0.5 second after initial impact. Airplane speed, approximately 95 feet per second.



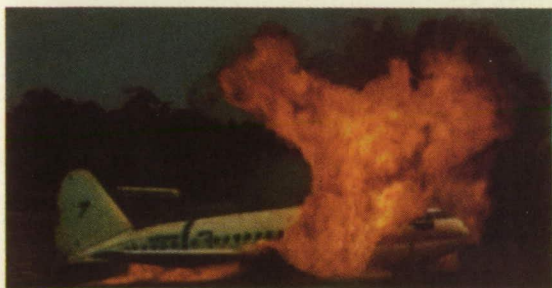
(b) Oil vapors emerging from nacelles; 3.38 seconds after initial impact. Airplane speed, approximately 3 feet per second.



(c) First oil fire outside of nacelle; 3.42 seconds after initial impact. Airplane speed, 2 feet per second.



(d) Ignition of fuel mist by oil fire; 3.5 seconds after initial impact. Airplane speed, 0.



(e) Spread of fuel mist fire; 5.3 seconds after initial impact.



(f) Spread of fuel mist fire; 6.6 seconds after initial impact.

Figure 17.—Role of oil system in aircraft crash fires.

nacelle, as in the case of figure 15, will shift the fuel mist toward the nacelle and increase the likelihood of contact with an ignitor. If the relative wind were directly from the front, the wetting pattern would be approximately symmetrical about a chordwise line through the fuel-spillage point.

An example of an extreme forward development of the fuel mist obtained during a crash involving high airplane decelerations is shown in figure 16. The fuel mist extended

The volume of fuel mist generated with a given rate of fuel spillage is greater for a high-winged airplane like the C-82 than for a low-winged airplane like the C-46. This difference results from the fact that the fuel issuing from the tank of a low-winged airplane, the wings of which are close to the ground in a crash, is intercepted by the ground before appreciable atomization of the fuel occurs (fig. 17 (a)). The fuel sweeps forward of the wing leading edge, fans out on the ground, and attains a significant spanwise extension in liquid form (fig. 17 (b)). Slightly above this liquid spillage is an associated fuel mist generated by the splashing fuel in the relative wind.

The ignition of the fuel distributed in this manner by an oil fire in the nacelle forward of the wing leading edge is illustrated in figure 17. The nacelle installation of the airplane used in this crash (fig. 18 (a)) shows an oil cooler located at the bottom of the nacelle directly behind the exhaust-collector ring. Shortly after the airplane passed through the poles at the crash barrier, the fuel spilling from the wing tanks had the pattern shown in figure 17 (a), which is associated with the low airplane deceleration and high airplane speed sketched in figure 14 (a). When the airplane nacelles struck the ground, the oil coolers were broken away from the oil lines and oil poured on the exhaust-collector ring (fig. 18 (b)). The airplane decelerations resulting from the friction and plowing of the airplane structure along the ground provided a fuel-mist pattern whose apex led the wing leading edge, as shown in figure 17 (b), in contrast to the fuel-spillage pattern at the crash barrier, which was well behind the wing leading edge (fig. 17 (a)). At 1.9 seconds after impact at the barrier, the airplane instrumentation indicated a small oil fire at the base of the exhaust-collector ring (fig. 18 (b)). Visible in the photograph taken shortly after this time (fig. 17 (b)) are dense clouds of condensed oil vapor issuing from the nacelle. The spilling fuel can be seen pouring forward of the leading edge of the wing in contrast to its earlier position under the wing at higher airplane speeds, as shown in figure 17 (a). In figure 17 (b) fuel mist and condensed oil vapor merge to form a continuous combustible atmosphere at the reduced airplane speed of 3 feet per second. As the airplane slowed, the fuel pattern increased its forward extension (fig. 17 (c)) until, just before the airplane came to rest, the fuel extended to the nacelle. Propagation of the oil fire through the dense condensed oil-vapor cloud issuing from the nacelle provided the fuel ignition close to the nacelle shown in figure 17 (d). The fuel mist and condensed oil vapor suspended in the air around the airplane were made visible to the camera by the flame that traveled through them (figs. 17 (e) and (f)).

Significance of fuel volatility in crashes involving fuel-mist ignition.—In crashes in which ignition of dense gasoline mist by a potent ignition source is involved, the full-scale crash studies indicated that the substitution of fuel of low volatility for gasoline does not prevent a fire. In figure 19 (a) is shown the ignition of the dense low-volatility fuel mist by an ex-

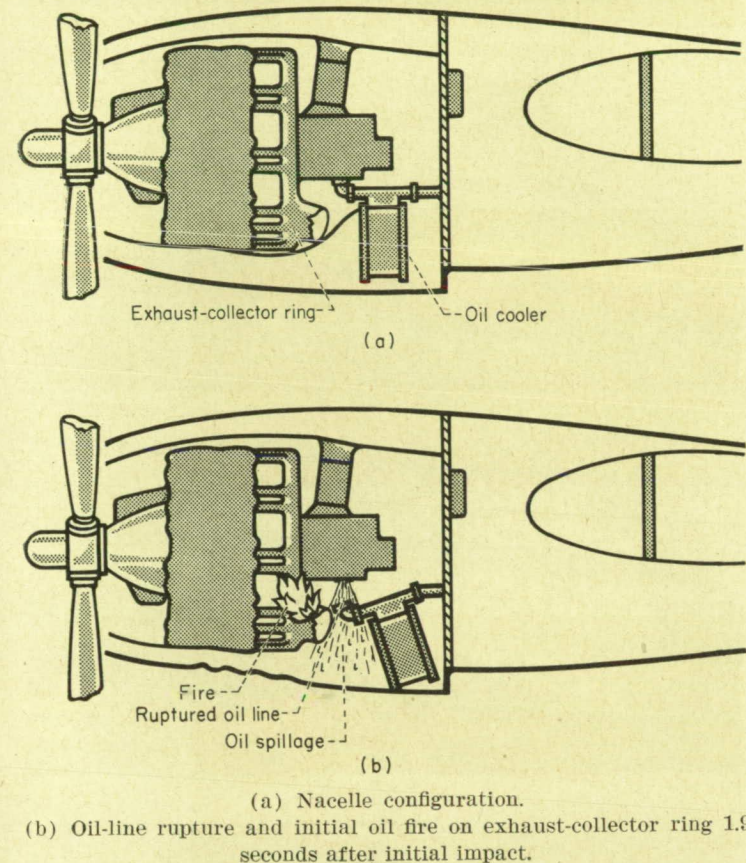


FIGURE 18.—Schematic drawing of C-46 airplane nacelle showing oil-cooler location and oil-line rupture in crashes.

28 feet ahead of the wing and reached a height of 12 feet above the top of the wing. Most of the wing span was covered by the fuel mist.

In order to establish the difference in velocity between the airplane and the tank fuel required for the fuel to project ahead of the airplane, the airplane deceleration must be sustained for a time that varies inversely with the magnitude of deceleration. In the case shown in figure 16, 0.3 second after the fuselage nose struck the ground, which represents the onset of high deceleration, fuel appeared at the leading edge of the wing with an initial velocity of approximately 30 feet per second with respect to the airplane wing. One second later the fuel achieved an extension of 28 feet ahead of the wing. In the high-deceleration phase following impact of the nose, the wing reached a momentary peak deceleration of 20g.

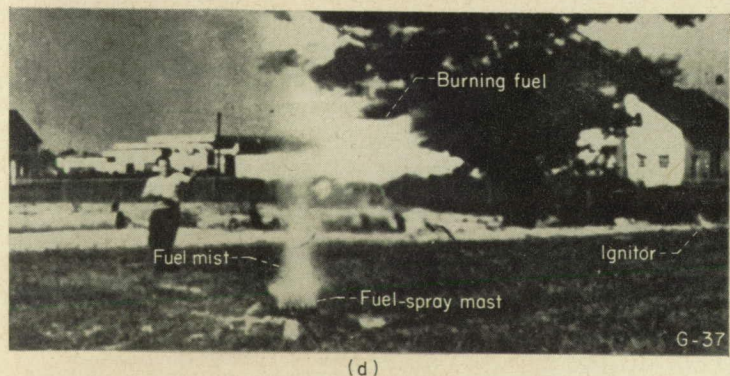
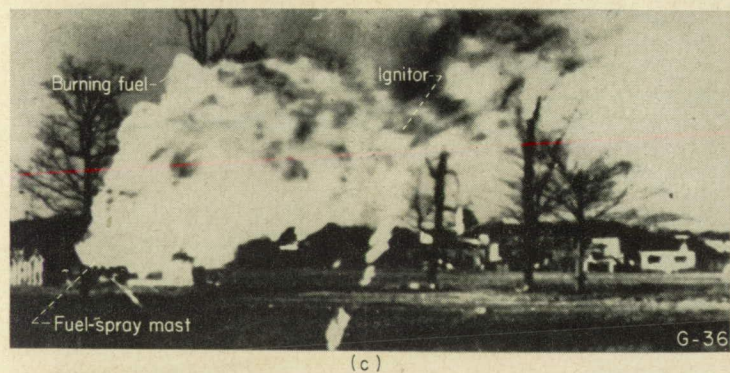
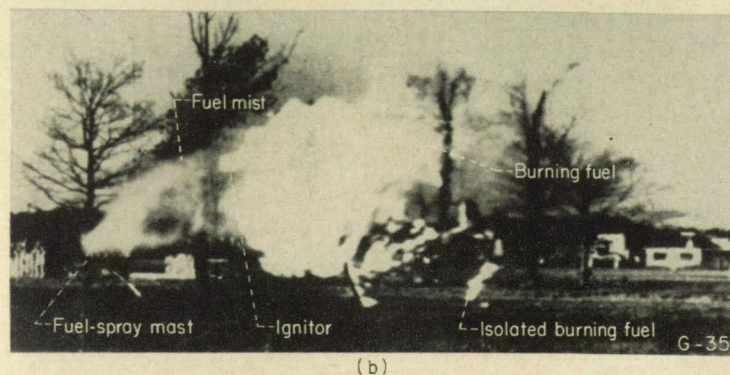
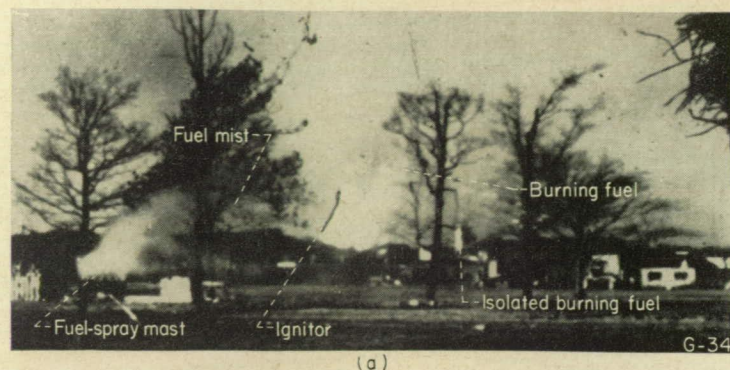
haust flame. In accordance with the mechanism of fuel-mist dispersal previously described, this ignition took place 2.0 seconds after the airplane crashed at the barrier and slowed to 98 feet per second from its crash speed of 150 feet per second.

A similar fire from fuel-mist ignition by exhaust flames occurred on the right side of the same airplane, as shown in figure 19 (b). The fire burning on the left side of the airplane (fig. 19 (a)), set 1.5 seconds earlier, is also visible. A second crash in which the low-volatility fuel was employed provided a fuel-mist ignition at reduced airplane speed under similar circumstances on the right side of the airplane (fig. 19 (c)). Exhaust flames did not appear on the right engine nacelle in this crash, and fuel ignition did not take place.

Ground studies of flame propagation through fuel mist.—In order to obtain an appreciation of the circumstances under which fuels of low volatility will provide a margin of safety over gasoline when dispersed as a mist, ground studies on the ignition and propagation of flames through mists of fuel with Reid vapor pressures ranging from 0.1 to 20 pounds per square inch were conducted with the multiple fuel-nozzle rig shown in operation in figure 20 (a). Since the details are difficult to differentiate in figure 20 as reproduced in color, black and white copies of the figure in which the details are indicated are also shown. The plume of fuel mist issuing upward from the nozzle rig was laid in a horizontal direction by the wind. When an ignitor, a burning kerosene-soaked rope, moved through the fuel mist toward the nozzle rig, the first evidence of ignition of the mist was indicated by short tongues of flames extending into the mist downwind of the ignitor (fig. 20 (a)). For fuels having a vapor pressure equal to or less than gasoline, this first appearance of mist ignition occurred in portions of the mist that appeared quite dense, the air temperature being approximately 68° F. As the ignitor was brought into the denser mist closer to the nozzle rig, the flames propagated as a continuous sheet downwind from the ignitor through the mist (fig. 20 (b)). In further displacement of the ignitor toward the nozzle rig, a point was reached at which the flame propagated upwind (fig. 20 (c)).

When isopentane, having a Reid vapor pressure of 20 pounds per square inch, was employed as the fuel, the mist evaporated a short distance downwind of the nozzle rig and provided vapor plumes that were visible because of the associated optical refraction effects. Ignition of the fuel took place in the totally evaporated plume. In daylight the flame front moving upwind through fuel vapor was visible only as a colorless circular wave. Visible flame first appeared when the colorless circular wave propagated to the tails of the fuel mist (fig. 20 (d)). The distance between the ignitor and the flame shown in the figure represents the displacement of the colorless wave through the fuel vapor. The same effects probably would be obtained with gasoline on very hot days.

Data on the maximum downwind distance from the nozzle



(a) First ignition of mist of aviation gasoline.
(b) Propagation of flame downwind in mist of aviation gasoline.
(c) Propagation of flame upwind in mist of aviation gasoline.
(d) Ignition of isopentane.

Black-and-white prints of figure 20.

rig that an ignitor must be placed for upwind propagation of the flame through the fuel mist obtained in these studies are plotted in figure 22. These data cover the range of Reid vapor pressures from 0.1 to 20 pounds per square inch, for



(a) Ignition left nacelle, test 7; 2.0 seconds after initial impact. Airplane speed, approximately 98 feet per second.



(b) Ignition right nacelle, test 7; 3.5 seconds after initial impact. Airplane speed, 50 feet per second



(c) Ignition left nacelle, test 8; 1.9 seconds after initial impact. Airplane speed, 74 feet per second.

Figure 19.—Ignition of fuel mist by exhaust flames from exhaust outlet. Low-volatility fuel.



Figure 21.—Small danger distance of aviation gasoline in liquid state in open air.



(a) First ignition of mist of aviation gasoline.



(b) Propagation of flame downwind in mist of aviation gasoline.



(c) Propagation of flame upwind in mist of aviation gasoline.



(d) Ignition of isopentane.

Figure 20.—Ground studies of ignition of fuel mists.



(a) Impact of poles with landing lights; 0.25 second after initial impact. Airplane speed, 143 feet per second.



(b) Ignition on right side of airplane; 0.60 second after initial impact. Airplane speed, 135 feet per second.



(c) Simultaneous ignition of both sides of airplane; 0.60 second after initial impact. Airplane speed, 135 feet per second.



(d) Rapid development of fire; 1.8 seconds after initial impact. Airplane speed 96 feet per second.

Figure 24.—Ignition of fuel by landing lights.

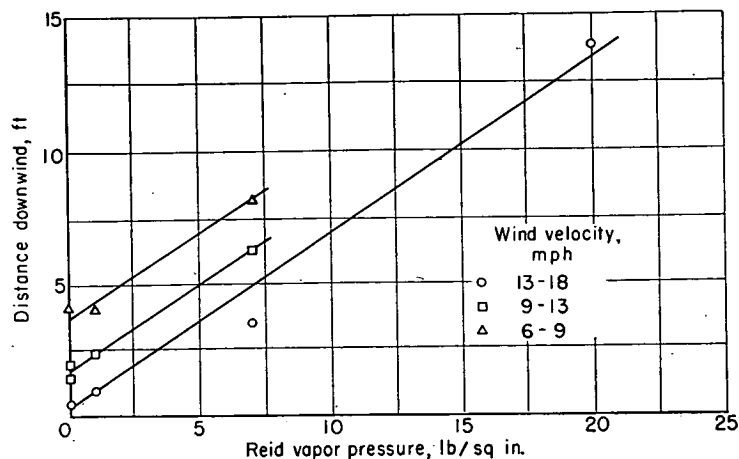


FIGURE 22.—Maximum distance downwind from nozzle rig that flame will propagate upwind as a function of fuel volatility and wind velocity. Ambient-air temperature, 68° F.

an air temperature of 68° F, with a fuel-flow rate of 178 gallons per hour through 17 hollow-cone spray nozzles each having a nominal rating of 10.5 gallons per hour. The data of figure 22 show that, in a given crash involving ignition of fuel mist, the maximum downwind distance the ignitor may be from the point of fuel spillage for the flame to propagate upwind to the airplane through the mist varies directly with fuel vapor pressure and inversely with wind speed. The maximum downwind distance from the fuel source at which upwind flame propagation will occur for gasoline mists (7 lb/sq in. Reid vapor pressure) decreases from 8 to 5 feet when the wind speed is increased from 8 to 18 miles per hour. At a wind speed of approximately 16 miles per hour, a change from gasoline to isopentane (20 lb/sq in. Reid vapor pressure) raises this maximum flame propagation distance from 5 to 14 feet. Because the distances given on figure 22 are a function of the rate of flow of the fuel generating the mist, magnitudes shown are of limited significance. The relative magnitudes given are important, however. These data are consistent with those obtained by the Texas Company in a comparable study (ref. 13), in which the fuel mists were generated by dropping fuel-filled bottles on a concrete platform exposed to a known wind. The presence of the crashed airplane and its debris can modify these results materially by providing wind-protected zones having lower wind speeds than those prevailing generally. The likelihood of upwind flame propagation through the fuel mist along these wind-protected zones is increased over that which would exist in an unobstructed wind.

These data show some advantage for fuels of low volatility under the special circumstances when the ignition source lies downwind of the fuel source producing the mist and upwind flame propagation through the mist is required to spread the fire. In the absence of statistics on the probability of the appearance of these circumstances in a crash, it is difficult to evaluate the margin of safety provided by fuels of low volatility when large fuel mists are generated.

LIQUID-FUEL SPILLAGE

The probability of obtaining combustible concentrations of fuel vapor and air from fuel spilled as liquid depends to a large degree on the local air ventilation around this spillage. Because of the low ventilation rate in the enclosed cavities of the airplane, such as the wings, large volumes of combustible concentrations of fuel vapor can accumulate from relatively little spillage. In zones such as the nacelle and landing-wheel well of a crashed airplane at rest, moderate ventilation rates exist. Relatively large fuel spillage, with evaporation taking place from extensive wetted surfaces, is required for combustible concentration to be realized in these zones. Vapors from liquid-fuel spillage on the ground exposed to the wind are subject to a high rate of air dilution, and combustible concentrations of vapor appear close to the liquid fuel only.

External fuel spillage.—Of the three types of liquid spillage just considered, only the ignition hazard associated with spillage on the ground exposed to the wind has been investigated in some detail at this time. Gasoline spilled in open air as liquid on warm ground or paved runways loses its more volatile constituents quite readily; the heat of evaporation is provided in large part by conduction from the unevaporated fuel and the fuel-wetted surfaces. Following the loss of the most volatile constituents and the associated temperature drop of the wetted surfaces, the fuel evaporation rate declines rapidly. The heat of vaporization is now provided primarily by convective heat transfer between the ambient air and the cool fuel. Exploration of the atmosphere by a combustible-vapor detector downwind of a pool of aviation gasoline arranged in pans measuring 16 feet long by 4 feet wide with the long dimension in the wind direction showed that the maximum horizontal downwind distance from the fuel at which ignition was possible was approximately 2 feet when the fuel was freshly exposed to a 3-mile-per-hour wind on a 70° F day. This danger distance declined to less than 6 inches after the fuel was exposed for several minutes. The downwind-air strata in which combustible fuel concentrations existed seldom attained a height of 6 inches above the fuel level of the ground-supported pans. This horizontal and vertical hazard distance decreased markedly with increasing wind velocities. In a 9-mile-per-hour wind a flame must be placed within 2 inches of the surface of the fuel at the downwind lip of the fuel pans in order to ignite the fuel. A photograph of the ignition of the fuel under these circumstances is given in figure 21, in which is shown the proximity of a cable-supported piece of burning waste to the downwind edge of the pool of gasoline required for gasoline ignition on a 70° F day. The pool of gasoline used in these studies is comparable in dimensions to those observed around the nacelles of airplanes crashed in this research. These results on the marked reduction in ignition danger distance with increasing wind velocity are consistent with those obtained elsewhere with prevaporized fuel released to the wind through single pipes.

The small danger distances around liquid gasoline spill-

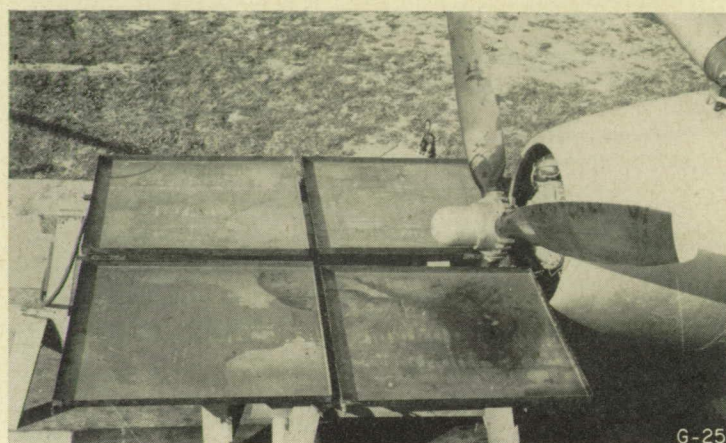
ages in open air are due, in part, to the fact that gasoline has a heat of vaporization of approximately 140 Btu per pound, whereas the air which supplies this heat of vaporization by convective heat transfer has a specific heat of approximately 0.24 Btu per pound per °F. The same air flow that provides the heat of vaporization also serves to dilute the evolved fuel

vapors. Because of the large ratio of heat of vaporization of gasoline to the specific heat of air, a reduction in temperature of about 15° F is required of the air moving over the fuel surface to transfer enough heat to the fuel to evolve the mass of vapor necessary to bring the resulting air-fuel mix to the lower combustible limit. Only the air layer moving within a few inches of the fuel surface will undergo such a temperature drop in the short transit distance over the fuel. The combustible concentrations of vapors will be within this air layer of small vertical extent. Air dilution by mixing with adjacent air flow reduces the vapor concentration in this vapor-bearing layer shortly downwind of the pool of gasoline.

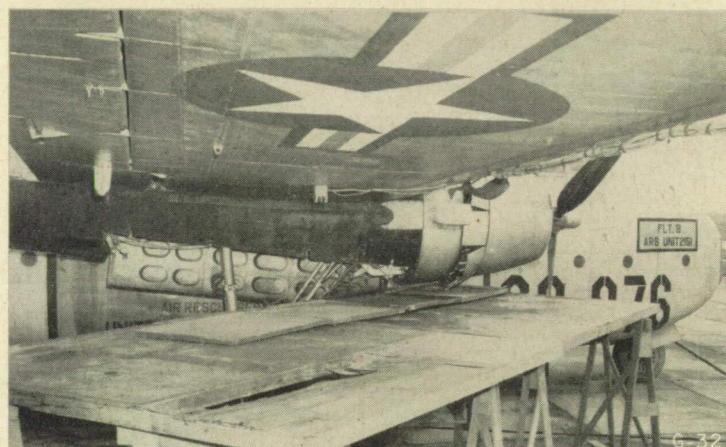
The short ignition hazard distance obtained in this work for liquid pools of gasoline exposed to unobstructed winds above 3 miles per hour, which exist more than 95 percent of the time in most of the United States, indicates a small likelihood of ignition of vapors convected by the air from these pools to an ignition source. Even when the gasoline pools were arranged at the engine cowl lips and cowl flaps as shown in figures 23 (a) and (b), respectively, wind-borne vapors from the gasoline in combustible concentration did not extend into the nacelle for a sufficient distance to be considered hazardous.

When the gasoline is spilled in tall grass and similar vegetation, the gasoline-wetted leaf surfaces increase the surface area from which fuel vaporization occurs. Protection of the vapors from air dilution by mixing is also provided. The ignition hazard distance, vertical and horizontal, is considerably longer than that which results from gasoline spillage on bare ground or pavement.

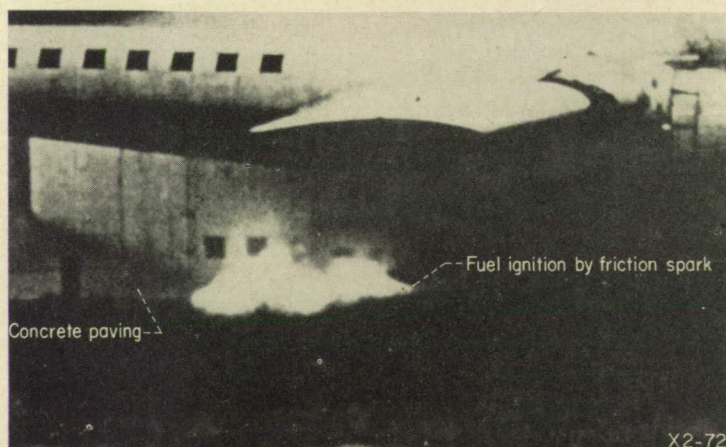
In zones around the crashed airplane that are well protected from the wind, vapor accumulation is possible. In the absence of significant heat transfer by forced convection from the wind, heat flow by conduction through the ground and the metal structure of the airplane and by radiation from the surroundings governs the fuel vaporization rate. Because vapor accumulation is possible, zones of combustible concentrations can develop with time, the magnitude of which is governed by air temperature, fuel volatility, the geometry of the airplane wreckage, and its orientation to the wind. Likewise, when spillage occurs on the ground in still air, the fuel vapors form as a layer adjacent to the ground by virtue of the high density of fuel vapor with respect to air. This fuel-vapor layer flows by gravity and may acquire considerable horizontal extent compared with the dimensions of the liquid pool from which the vapors are generated. Ignition of this vapor requires an ignition source placed close to the ground. Burning oil vapors or droplets dripping from the engine exhaust system, broken elements of the hot exhaust system falling to the ground, or sparks generated by the abrasion of magnesium and steel airplane parts on stony ground or concrete paving may provide the ignition in this instance. Ignition of fuel by friction sparks on a paved concrete slide path specially constructed for this study is shown in figure 23 (c). A portion of a steel propeller blade



(a)



(b)



(c)

(a) Pans of gasoline upwind of cowl inlet.

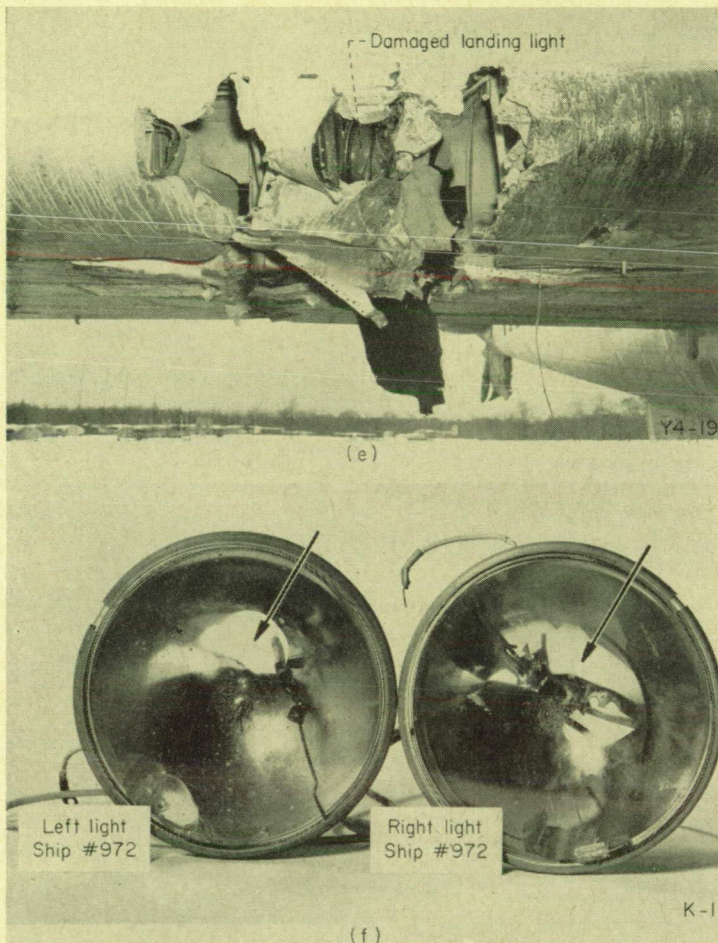
(b) Pans of gasoline upwind of cowl outlet.

(c) Ignition of gasoline by friction spark from steel propeller blade bearing on concrete paving.

FIGURE 23.—Studies of ignition hazard from liquid gasoline spilled adjacent to nacelle.

mounted on the fuselage to bear with pressures in excess of 100 pounds per square inch on the concrete paving provided the sparks that produced the ignition of the gasoline at the fuselage-ground contact line (fig. 23 (c)). Most ignition sources in airplane crashes involving moderate structural damage will lie above this fuel-vapor layer, however.

Internal fuel spillage.—In a crash, the fuel spilled within the wings of airplanes of conventional configurations is exposed to ignition sources belonging primarily to the electrical system. Typical components carried on, or within,



(e) Landing-light damage.

(f) Holes left in landing-light reflectors by filaments pulled into wing during crash.

FIGURE 24.—Concluded. Ignition of fuel by landing lights.

the wing requiring electrical wiring include wing-tip lights, landing lights, fuel pumps, and fuel-system solenoid valves. Ignition of fuel spilled in the wing by the electrical system was observed in one crash in which the poles at the crash barrier were set to smash the operating landing lights located in the leading edge of the wing and to drive them into the wing where the fuel tanks were also breached (fig. 24 (a)). The landing light on the left wing was struck squarely by the pole at the barrier to damage the light in a manner equivalent to that shown in figure 24 (e). The pole struck close to the landing light on the right wing.

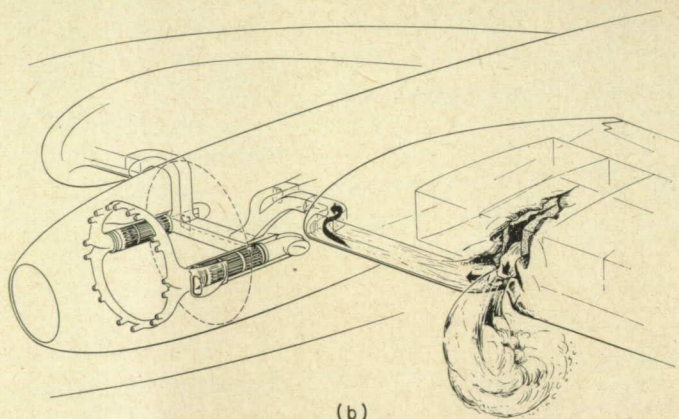
When such close strikes are obtained, the landing-light mount is seriously distorted and the filaments are pulled into the wing by the heavy electric cable serving these lights. The holes made in the landing-light reflector by the withdrawing filaments are shown in figure 24 (f). The exposed hot lamp filament ignited the fuel spilled within the wing within 0.35 second after impact with the pole barrier. The resulting fire as it first appeared issuing from the wing is shown in figure 24 (b). Because of the proximity of the ignition source to the fuel spillage, ignition occurred immediately on exposure of the fuel at an airplane speed of approximately 92 miles per hour. The flame in figure 24 (b) appeared on the outside of the wing before the airplane had moved its own length from the barrier. A front view of the airplane (fig. 24 (c)), taken at the same time, shows a similar fire on the left airplane wing produced under the same circumstances. Propagation of the fire into the fuel mist associated with the fuel spillage from the moving airplane provided the high rate of fire development indicated in figure 24 (d), which shows the airplane fire 1.8 seconds after ignition. The flame-holding action of the airplane elements, such as the damaged wings containing burning fuel, allows high airspeeds in the combustion zones without flame blow-out, as is consistent with jet-engine combustion experience where similar circumstances exist.

When the fuel spilled within the wing forms a continuous wetted path to an outside ignitor, the resulting fire moves along the path to the fuel source. Channels for the distribution of the fuel within the wing may develop in the crash or be a part of the normal airplane configuration. The hot-air duct, for example, lying along the leading edge of the wings for protection against icing, may serve as a fuel distribution channel directing fuel spilled in a crash to a combustion heater or exhaust-gas heat exchanger that normally provides the hot air for icing protection. An illustration of this mode of fuel conduction to an ignition source was provided in the crash depicted in figure 25. The passage of the pole at the crash barrier through the leading edge of the wing in figure 25 (a) bent the skin toward the interior of the wing. Part of the fuel surging forward out of the wing rent was deflected into the leading-edge hot-air duct by the scoops formed by the deformed wing skin, as indicated schematically in figure 25 (b). Because of the wing dihedral, the fuel flowed by gravity toward the heat exchanger located on the engine exhaust tail pipe slightly forward and below the wing, which supplies hot air for the icing-protection system. The fuel flowed through the clearance between the duct wall and the hot-air-flow control flap in the nearly closed position and onto the heat exchanger. Ignition occurred at the heat exchanger, and the flame propagated back to the wing to produce the wing explosion shown in figure 25 (c).

The wing also serves as a channel for conducting wing-spilled fuel to adjacent airplane components. In the aircraft of the general configuration of the C-82, these adjacent components are usually the wheel well and the engine nacelle. A photograph of the distribution of fuel throughout the



(a)



(b)

(a) Rupture of wing produced by pole barriers.

(b) Schematic diagram of hot-air anti-icing system showing path of liquid fuel flowing from wing tanks to heat exchangers.

FIGURE 25.—Mechanism of ignition of liquid fuel flowing through hot-air anti-icing duct.

wheel well of a crashed airplane is shown in figure 26 (a). This red-dyed fuel, released directly from damaged wing tanks, flowed to the wheel well through the internal wing structure. Because the wheel well contains elements of the electrical system, ignition of fuel from this source is probable when the fuel system is disrupted in a crash. Also of interest is the fuel that coursed down the landing-gear strut (fig. 26 (b)). This fuel, in conjunction with overheated wheel brakes, poses the possibility of fire initiation not observable in this crash study because brake application was not employed.

Wetting conduction.—In addition to the trough-flow of liquid fuel through internal channels of the airplane, fuel in rivulets and sheets does flow by gravity along the under side of airplane surfaces inclined to the horizontal. This type of flow is called "wetting conduction" to distinguish it from the other forms of fuel flow.

Wetting conduction of fuel occurs, for example, when some of the fuel spilled within the wing seeps through riveted seams of metal plates forming the skin and clings to the under

side of the wing. While some of this fuel drips to the ground, an appreciable portion wets and adheres to the under side of the wing and flows by gravity. If the wing slopes from the point of fuel-tank spillage toward the airplane nacelle because of the wing dihedral or the attitude imposed by the crash, an appreciable fuel flow is directed toward the nacelle, where many of the ignition sources are located. In figure 27 is shown a typical wetting-conduction trail marked by dye contained in the fuel carried by an airplane that did not burn in crash. The continuous fuel-flow path from the area around the breach in the wing to the wheel-well doors is evident. The dye trail left by the fuel flowing along the airplane skin directly above and behind the exhaust tail pipe is obscured by the dark paint on this portion of the airplane. The likelihood of ignition of this fuel by an exhaust flame is evident. Such ignition was not observed in the limited number of crashes conducted in this program, perhaps because every appearance of exhaust flames that occurred several seconds after crash involved ignition of the fuel mist.

Spreading of the fuel by wetting conduction proceeds at a relatively slow rate. In crashes in which fuel mists do not appear but fuel tank rupture does occur by inertia loading of the fuel on the tank walls during the crash deceleration, wetting conduction may well represent the mechanism by which the fuel reaches the ignition source at the nacelle. Under such circumstances, the ignition that may occur will probably take place several seconds after the airplane comes to rest, the delay involved representing the time for the relatively slow movement of the fuel by wetting conduction. While the airplane is in motion, likewise, the air flow around the wing will impose a chordwise motion on the fuel and direct it to the relatively safe zones at the wing trailing edge if the airplane is moving nose foremost. When the airplane comes to rest, a tailwind would promote the forward movement of the fuel toward the nacelle.

The same process of wetting conduction takes place within the nacelle and the wheel well by fuel lost from the ruptured fuel lines or other fuel-system components, or by fuel flowing through the airplane channels from spillage at remote locations. Fuel flowing by gravity into the nacelle, from a source of small size, can achieve appreciable spread by the combined process of wetting and dripping from one structural or engine component to another. In this way, the likelihood of contact with an ignition source is enhanced. Suitable photographs of this form of fuel spread in the nacelle are unavailable, but the process involved is evident from the spotty, yet widely distributed, fuel wetting shown in the wheel well of a crashed airplane in figure 26 (a).

When wetting conduction or fuel flow through structural channels is responsible for prolonged contact between an ignitor and the fuel in liquid form, so that appreciable quantities of vapors are generated, the use of fuels of low volatility would not materially reduce the likelihood of fire. But, when vaporization of the fuel across an air gap is required for the fuel to reach an ignitor, low-volatility fuels provide a real advantage.



(c) Explosion of wing following ignition of gasoline flowing through anti-icing duct; 13 seconds after initial impact; 0.6 second after ignition.

Figure 25.—Concluded. Mechanism of ignition of liquid fuel flowing through hot-air anti-icing duct.



(a) Wheel well.



(b) Landing gear strut.

Figure 26.—Distribution of fuel throughout wheel well and on landing gear strut from wing-tank spilled fuel. (Fuel dyed red.)



Figure 27.—Wetting conduction trail of fuel on under side of airplane wing. (Fuel dyed red.)



(a) Fire 2.8 seconds after initial impact. Airplane speed, 42 feet per second.



(b) Fire 7.4 seconds after initial impact. Airplane speed, 0.



(c) Fire 7.7 seconds after initial impact.

Figure 29.—Extension of fire from nacelle to fuel spilled from wing tanks.



(a) Ignition of induction system vapor-air mixture; 0.13 second after initial impact. Airplane speed, 115 feet per second.



(b) Ignition of engine fuel from broken main fuel line in nacelle; 0.29 second after initial impact. Airplane speed, 111 feet per second.



(c) Ignition of fuel mist from fuel tanks spillage; 0.71 second after initial impact. Airplane speed, 102 feet per second.



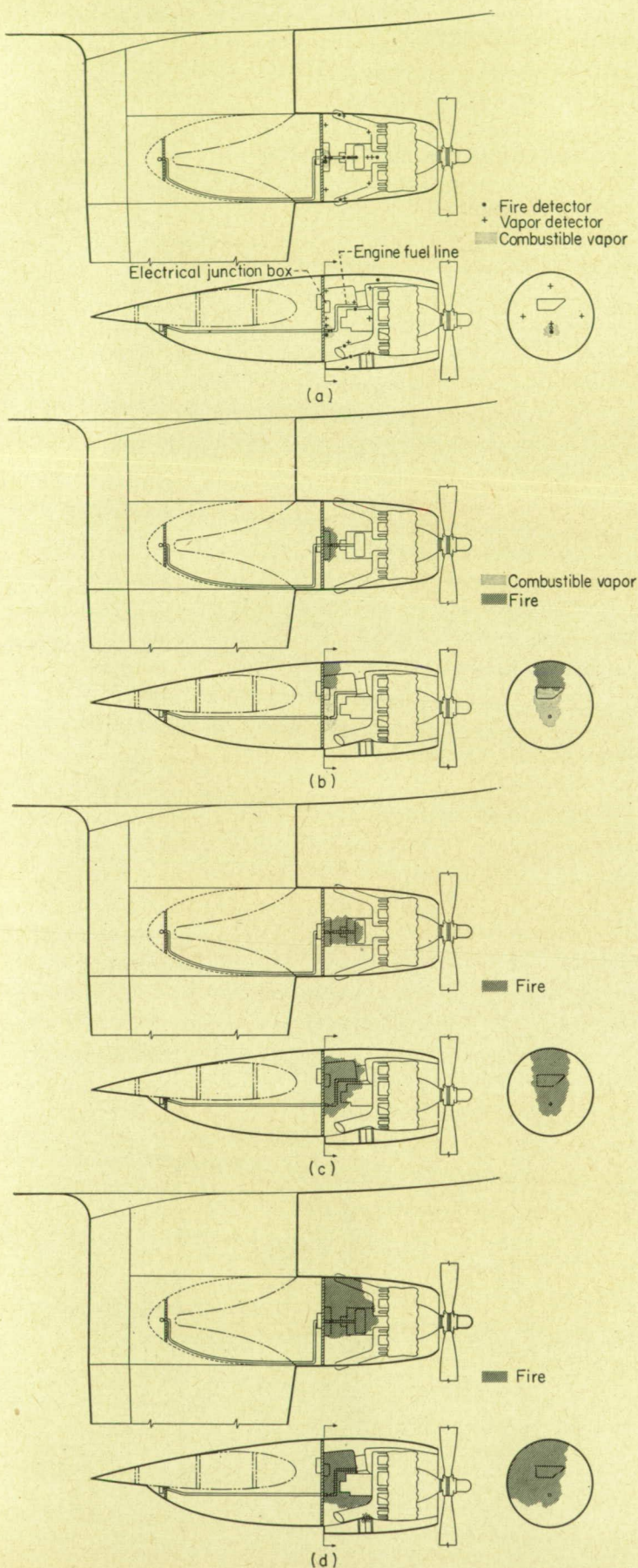
(d) Spread of fire in fuel mist; 1.2 seconds after initial impact. Airplane speed, 89 feet per second.

Figure 30.—Spread of fire resulting from ignition of vapor-air mixture in induction-system.

Nacelle fuel spillage.—When liquid-fuel spillage occurs within the nacelle, there is a high probability of ignition by a large variety of ignition sources associated with the engine and the complex electrical installation in the engine accessory section. As an example of the ignition of fuel spilled as liquid within the nacelle, it is instructive to follow the sequence of events leading to the ignition of this fuel by the accessory-section electrical system in one crash as revealed by the crash instrumentation. The crash instrumentation indicated rupture of the engine fuel line in the accessory section upon crash impact with the fuel booster pump at the wing tanks operating at normal speed. Vapor detectors indicated combustible concentrations of fuel at the fuel-line bulkhead fitting on the nacelle fire wall 0.25 second after crash impact. Location of this and other vapor detectors and neighboring fire-detection thermocouples is indicated in figure 28 (a). At this time, 0.25 second after crash impact, combustible vapors did not appear at any other location in the nacelle. Crash instrumentation indicated an electric current surge in the generator-starter circuits starting 0.50 second after crash impact, which reached a peak value of 30 amperes at 1 second after crash. The first fire-detection thermocouples to register were located immediately above the electrical junction box located on the nacelle fire wall (fig. 28 (b)), which is part of the generator-starter system. This junction box, therefore, is taken to be the location of a short-circuit arc that provided the fuel ignition. Successive indications of the fire detectors showed that the fire developed forward uniformly from the fire wall to the exhaust-collector ring, and that the exhaust system was not involved in the first fuel ignition (figs. 28 (c) and (d)). Extension of the fire from the nacelle to fuel spilling from the tanks as the airplane slowed to rest is shown in the succession of photographs in figure 29.

FUEL-VAPOR SPILLAGE

In a crash, spillage of fuel vapor premixed with air in combustible proportions may take place from the damaged engine induction system. When supercharging is employed, release of the compressed fuel-air mixture is rapid. Because of the proximity of the hot exhaust-disposal system and the electrical equipment in the accessory section of the engine nacelle, ignition of this fuel-air mixture will most likely occur within a few seconds after the engine induction system is damaged. If concurrent damage to the nacelle permits a high ventilation rate through the engine accessory section, ignition of the fuel-air mixture must occur imme-



- (a) Location of vapor and fire detectors in nacelle, and combustible vapor detected 0.25 second after initial impact.
- (b) Combustible vapor and fire detected in nacelle 1 second after initial impact.
- (c) Fire in nacelle 2 seconds after initial impact.
- (d) Fire in nacelle 3 seconds after initial impact.

FIGURE 28.—Mechanism of ignition by electrical system in accessory section of airplane nacelle.

diately after spillage if it is to take place at all, because the mixture will be rapidly diluted with air below the combustible limit. Since the quantity of fuel contained within the engine induction system at any time is only enough to produce a fire equivalent to a severe backfire, ignition of this fuel is of little consequence. If, however, other spillage of combustibles occurs previous to this ignition, then these combustibles may be inflamed by the flash fire of the engine induction-system fuel.

In the crash in which this mechanism of fuel ignition was observed, the airplane involved was equipped with steel-bladed propellers. Impact of the propeller with the abutment of the crash barrier produced a rotational displacement of the engine sufficient to rip open the engine induction system and the cylinder exhaust-stack connection to the exhaust-collector ring. The vapor-air mixture released from the engine induction system was ignited by plumes of exhaust flame issuing from the open exhaust stack before the wing fuel tanks were breached by the barrier poles (fig. 30 (a)). At the same time, the crash instrumentation indicated a parting of the engine fuel line in the nacelle produced by the engine deflection. Fuel poured from this line with the fuel booster pump at the tanks operating at normal speed. The fire of the engine induction-system fuel-air mixture ignited the fuel spillage from the broken engine fuel line to produce the fire shown 0.29 second after crash impact (fig. 30 (b)). The air flow around the engine spread the flames into the nacelle wake. The fuel mist issuing from the wing-tank rupture imposed by the poles at the crash barrier was ignited behind the wing, where the spreading fuel mist intercepted the flame extending rearward from the nacelle, as shown in figure 30 (c). At this time the fire had not propagated to the fuel mist under the wing. The resulting wing fire 1.2 seconds after impact is shown in figure 30 (d).

The engine induction-system fuel can be ignited from within the engine by the ordinary backfire mechanism. If the engine induction system is ruptured before the backfire occurs, the resulting flash may appear within the nacelle cowl and play the same role in setting the crash fire as the induction-system fuel played in the fire just described. While no airplane fires occurred by this backfire ignition of the induction-system fuel in the crashes conducted so far, the distinct possibility of setting fires in this way is shown by the under-cowl induction-system fire shown in figure 33. The disarranged engine cowl in its displacement from its normal position severed the engine cowl-inlet connection to the carburetor and provided an opening through which the engine induction-system fire appeared within the nacelle, 3.5 seconds after crash. In this instance, the fuel involved was the low-volatility (8 mm Hg Reid vapor pressure) fuel. The circumstance just described occurred in the crash discussed previously in conjunction with figure 19.

The engine induction-system fuel-air mixture that is spilled when the complete engine assembly is torn free of the nacelle in the crash was never observed to ignite in the three instances in which this spillage occurred in the full-scale crashes.

Two of these engines employed low-volatility fuel (8 mm Hg Reid vapor pressure) and one used gasoline. The brief contact between the released fuel-air mixture and the hot exhaust-system metal (permitted by the tumbling engine and the rapid fuel dilution rate with the surrounding air) is probably responsible for this result.

CRASH-FIRE EXPERIMENTS WITH MODIFIED AIRCRAFT

After the same ignition sources were observed repeatedly in the crashes conducted with the crash arrangements accepted as standard for the first phase of the full-scale crash study, modifications in procedure were employed in an effort to discover other ignition sources and to observe how they initiate fire. These modifications involved alterations to the airplane and the crash barrier. The quantity of fuel carried by the airplane remained unchanged at 1050 gallons. Aviation-grade gasoline was used throughout this phase of the program.

IGNITION-SOURCE INERTING

Since the fires observed in the first phase of the full-scale crash-fire program were set by ignition sources that functioned shortly after crash impact, it appeared probable that these early fires were masking other fire-setting mechanisms that would appear later in the crash. In an effort to prevent these early fires, therefore, the nacelle installation shown schematically in figure 31 was employed to reduce the likelihood of fire by ignition sources revealed in previous airplane crashes.

The ignition-suppression installation consisted of four main components all actuated as soon as possible after crash impact. In order to prevent the generation of flames issuing from the engine inlet, tail pipe, and other elements of a crash-disrupted exhaust-disposal system, a solenoid-operated fuel shut-off valve was placed between the carburetor and the fuel injector on the supercharger impeller. In addition, a second element consisting of a 3-pound charge of fire-extinguishing agent was arranged to discharge into the

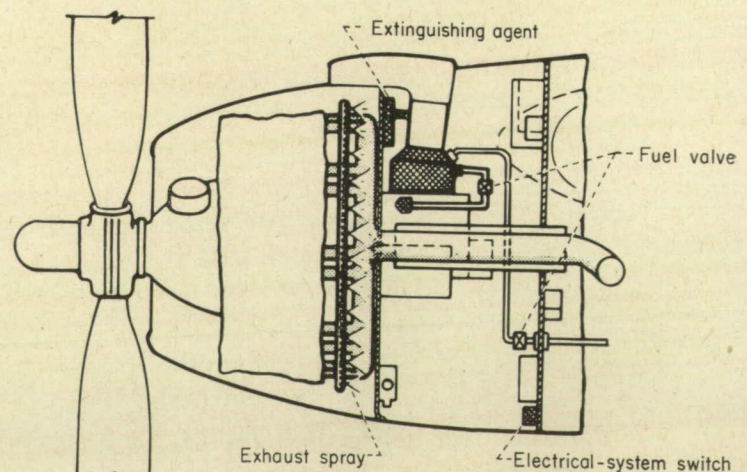


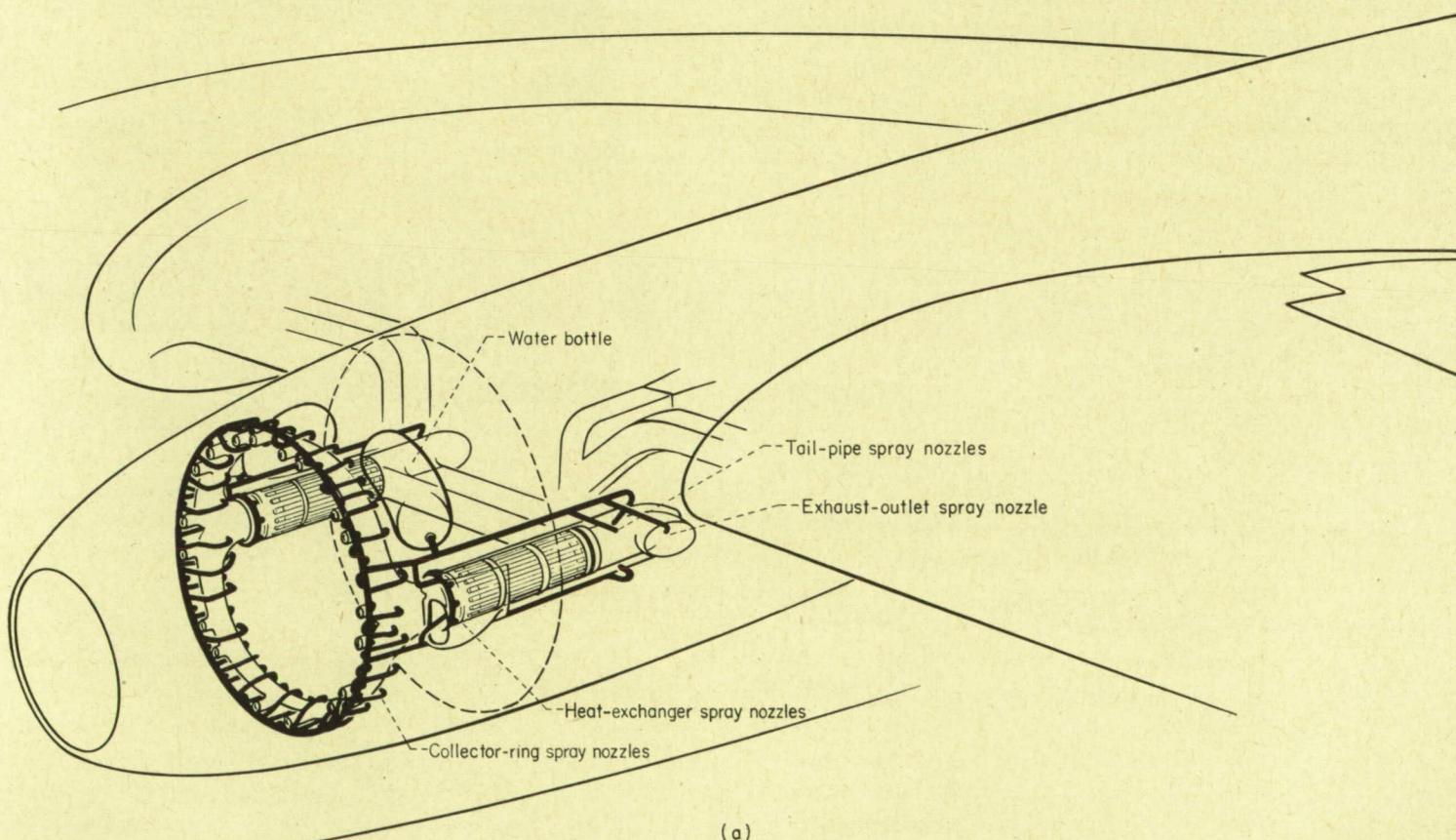
FIGURE 31.—Schematic diagram of ignition-source inerting system.

supercharger-impeller case with sufficient velocity to provide reasonably uniform mixing with the engine induction-system fuel. A $\frac{1}{2}$ -pound-per-second discharge rate was employed, giving a total discharge time of 6 seconds. This fire-extinguishing agent served to inert the engine induction-system fuel flowing through the engine that is present before the fuel shut-off valve closes completely. It was desirable to inject the extinguishing agent at the impeller housing rather than at the carburetor, in order to reduce the time required to transport the extinguishing agent from the injection station to the engine cylinders. Experience with injection of the extinguishing agent into the air flowing through the carburetor showed that the transport time involved was great enough for undesirable exhaust flames to appear momentarily at the exhaust tail pipe when the engine rotational speed was reduced abruptly by propeller impact at the barrier.

Electric ignition sources were avoided by cut-off switches on the battery and generator circuits. The engine ignition was allowed to remain operative on the thesis that, should combustible mixtures of fuel and air be ingested by the engine because of faulty operation of the fuel shut-off or fire-extinguishing-agent system, then normal combustion could take place in the cylinder. Otherwise, the fuel may pass into the exhaust system, there to ignite and produce an exhaust flame. Because the landing lights remain hot enough to ignite gasoline for at least 0.75 to 1.5 seconds after the elec-

tric current is turned off, interruption of the electric current by the cut-off switches did not represent a satisfactory control of this ignition source. For this reason, the crashes under this phase of the program were conducted with the landing lights inoperative.

The fourth element of the ignition-suppression installation was a system of water sprays distributed to give a simultaneous uniform coverage of the hot metal of the exhaust-disposal system and the associated tail-pipe heat exchanger. The water spray served the twofold purpose of rapidly cooling the hot metal to safe temperatures and providing, meanwhile, a protective blanket of steam generated on the hot surfaces that inerts the adjacent atmosphere. Water was the fluid selected for this purpose because of its high heat of vaporization and low molecular weight. Relatively small weights of water provided effective cooling and generated large volumes of inerting steam in the high-temperature zones adjacent to the hot metal. A consideration of the heat capacity of the hot metal of the exhaust-disposal system and the heat exchanger and the temperature drop of 250°F required to reduce the metal temperature from the maximum temperature of 1200°F at the moment of crash to the lowest exhaust-system surface temperature at which gasoline will readily ignite (950°F) yields a water requirement per nacelle of approximately $\frac{1}{2}$ gallon for the C-82 airplane. In order to provide a safety factor, 4 to 6 gallons of water were employed for each nacelle.

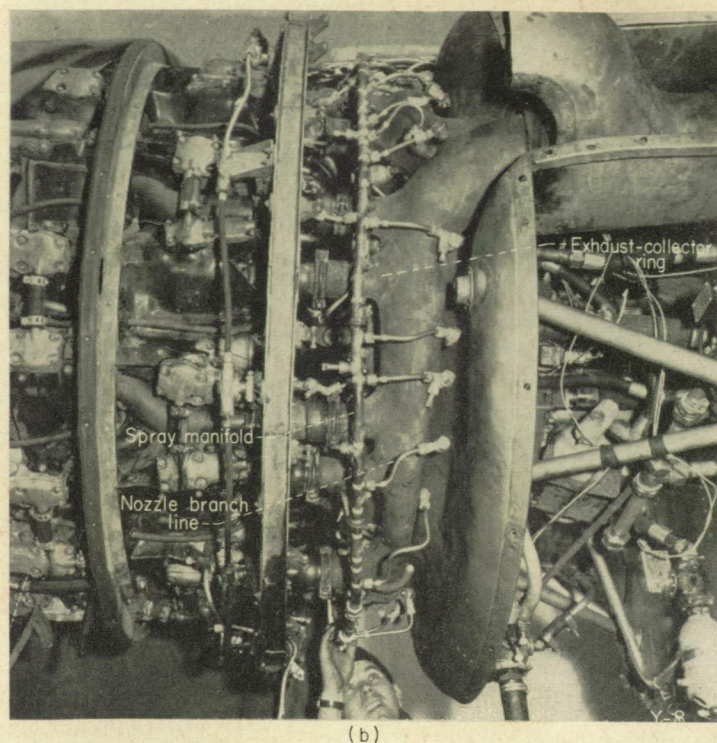


(a) Schematic diagram of nacelle installation.

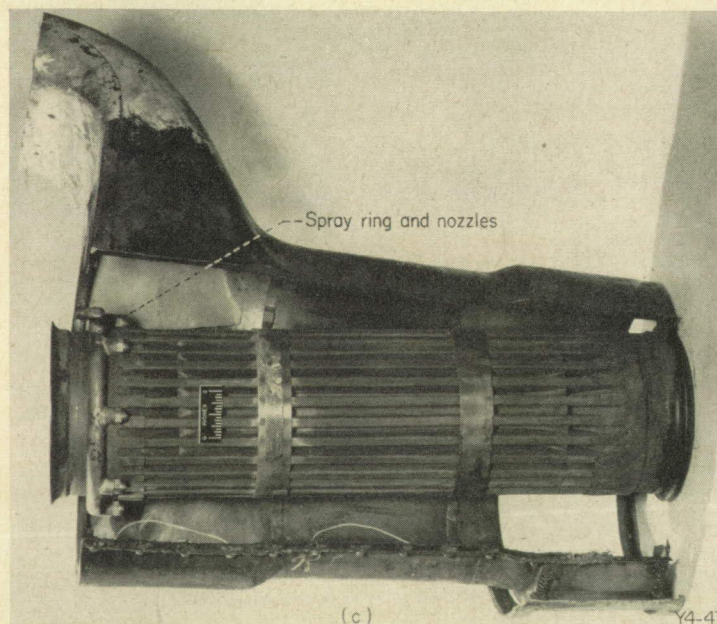
FIGURE 32.—Water-spray system of ignition-source inerting system.

In view of the 760° F minimum ignition temperature of lubricating oil measured on the hot exhaust-collector ring in these studies, some additional consideration of the temperature to which the collector-ring metal is to be cooled is in order. A review of the sequence of events in a crash with the airplane equipped with water sprays leads to the following assessment of the cooling specification: Immediately following crash impact and in the subsequent motion of the airplane's slide to rest, contact between lubricating oil and any part of the hot exhaust-collector ring is quite likely. During this time, water sprays are in operation, a portion of the oil that may be in contact with the exhaust system evaporates along with the water, and the remainder drips from the exhaust system with some of the water. When the airplane comes to rest with engines and oil pumping stopped, further contact between the exhaust system and oil occurs with the oil flowing by gravity. Under gravity flow the contact that does occur will most likely be confined to the lower octant of the exhaust-disposal system. Water sprayed on the exhaust system also drains by gravity to the lower portions of the exhaust-system elements and provides greater cooling in these zones. In general, all elements of the exhaust system considered individually experience greater cooling on the lower portions for this reason. If the water sprays are maintained for about 10 seconds after the airplane comes to rest, those surfaces likely to be wetted by lubricating oil flowing by gravity are cooled to safe temperatures. The quantity of water carried in this system was more than adequate to meet the modest added water requirements for the lower portions of the exhaust system. In the installation employed here, the water-spray nozzles were sized to give a total spray time of 20 to 30 seconds or a flow for 15 seconds or more after the airplane comes to rest. On the basis of observations made in this study, departures from the mode of oil contact with the exhaust system described are considered to have a low probability of occurrence unless a large degree of distortion and displacement of the exhaust system occurs in the crash.

The essential elements of the water-spray system are shown schematically in figure 32 (a). The water, contained under nitrogen pressure of approximately 400 pounds per square inch, was distributed to all hot metal elements of the engine exhaust system through the $\frac{3}{4}$ -inch-diameter manifold shown in figure 32 (b). The spray nozzles and branch lines terminating in spray nozzles are required to provide uniform water coverage of the hot metal parts. In order to provide rapid cooling of the hot metal, water was also sprayed into the interior of the exhaust system at six equally spaced locations, one of which is visible in figure 32 (b) at the junction of the tail pipe with the collector ring. The equivalent water-spray installations for the exhaust-gas heat exchanger are shown in figure 32 (c). The necessity for providing a protective blanket of steam around the external surfaces of the hot metal, where contact with combustibles may occur, while it cools to safe temperatures, fixes the external appli-



(b)



(c)

(b) Water-spray system for exhaust-collector ring.

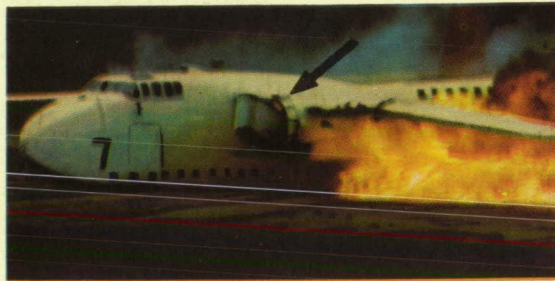
(c) Water sprays for exhaust heat exchanger.

FIGURE 32.—Concluded. Water-spray system of ignition-source inerting system.

cation of water as the primary requirement for this method of inerting. This external application is wasteful, however, because of the run-off of water in liquid form. By a proper split between internal and external utilization of water, some economy was obtained. Internal application provides the added advantage that the water is totally contained within the exhaust system regardless of twisting and displacement. Because water application had to begin as soon as possible after crash impact, while the airspeed through the



(a) Engine cowl disruption at barrier; 1.0 second after initial impact. Airplane speed, 130 feet per second.

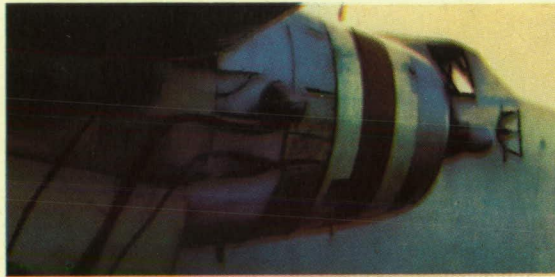


(b) Induction-system fuel ignition 3.5 seconds after initial impact. Airplane speed, 50 feet per second.

Figure 33.—Delayed ignition of induction-system fuel.

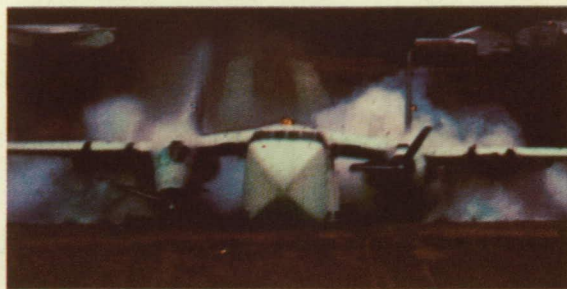


(a) Oil fire before water sprays turned on.



(b) 1.0 second after water sprays turned on oil fire.

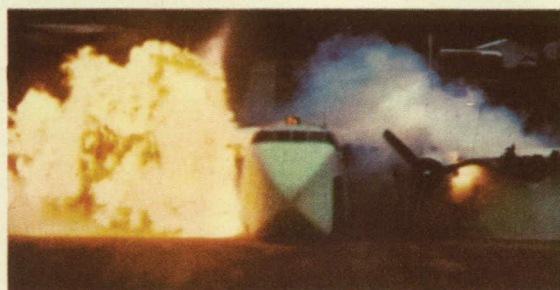
Figure 34.—Tests of water sprays of ignition-source inerting system.



(a) Deformation of right engine nacelle 2.2 seconds after initial impact. Airplane speed, 60 feet per second.



(d) First indication of fire on right nacelle 3.8 seconds after initial impact. Airplane speed, 15 feet per second.



(e) Fire development 4.1 seconds after initial impact; 0.3 second after ignition. Airplane speed, 11 feet per second.

Figure 38.—Mechanism of ignition by hot metal of exhaust collector ring.



(a) Bromochloromethane burning on outside of heated stack.



(b) Bromochloromethane burning on inside of heated stack.

Figure 39.—Bromochloromethane burning after ignition by section of exhaust stack heated to 1500° F.

nacelle is high, the spray nozzles were placed close to the hot metal to minimize water loss by air entrainment. The reduced surface coverage per nozzle occasioned by this proximity of the nozzles to the hot surfaces was compensated somewhat by arranging for the water to strike the surface at a near glancing incidence so that it may fan out along the surface. The water flow rate was adjusted to correspond roughly to the rate at which the water is evaporated initially. Faster water flow rates increase the water waste by liquid run-off.

Before any of the aircraft were committed to this part of the program, the elements constituting the complete inerting system were checked for their functional effectiveness and for the time following crash impact that they can be expected to operate. Fuel and electrical shut-off systems required development to improve reliability and actuation speed. Extinguishing-agent injection systems were engineered for uniformity of distribution within the engine inlet manifold. The effectiveness of the water-spray system was evaluated in an operating engine nacelle by spraying oil on the incandescent exhaust-collector ring and, with the oil continuing to flow, observing the action of the water spray in quenching the resulting fire. The appearance of the oil fire before the water spray was turned on is shown in figure 34 (a). One second after the water spray was applied, the fire was extinguished as shown in figure 34 (b). Because the water spray was effective in putting out an existing fire, it was considered safe to assume that it would prevent ignition in the first place. With a water-spray system that provided effective coverage of the hot metal, ignition of gaso-

line or oil did not occur when they were applied after the water-spray system was in operation.

Tests with gasoline sprayed directly onto the exhaust heat exchanger showed that the water spray did give protection against ignition when proper water distribution was provided with the nozzle arrangement shown in figure 32 (c). As additional insurance, however, against ignition of fuel by conduction through the icing-protection system hot-air duct to the exhaust-gas heat exchanger, a blind flange was inserted into the duct.

When the water spray was employed, the hottest areas of the exhaust-collector ring cooled in 12 seconds from an initial temperature of 1250° to 760° F, the lowest temperature at which lubricating oil ignites on the external surfaces of the exhaust system (fig. 35). The exhaust-gas heat exchanger cooled to temperatures safe for gasoline in 30 seconds. Contact between the hot areas of the heat exchanger and the lubricating oil is highly unlikely, and rapid cooling below 950° F is not required. The relatively stagnant atmosphere of steam within the heat exchanger continues to protect this zone until the heat exchanger cools by radiation and conduction to other portions of the airplane structure.

In the crash, the inerting system was actuated by a switch carried on the guide-rail slipper fastened to the nose-wheel strut, a photograph of which is shown in figure 36. A plywood target (fig. 37) was placed across the rail at the crash barrier to operate the switch after the propellers struck the barrier abutments. To provide a margin of safety against switch failure, a second switch connected in parallel was carried on the side of the airplane. The second plywood target shown on the angle iron support in figure 37 served this auxiliary switch.

The time delay between operation of the inerting-system actuating switch and the inerting component was 0.06 second for the extinguishing-agent discharge, 0.34 second for the fuel shut-off valve, 0.10 second for the electrical-system cut-off switch, and 0.19 second for full water-spray flow

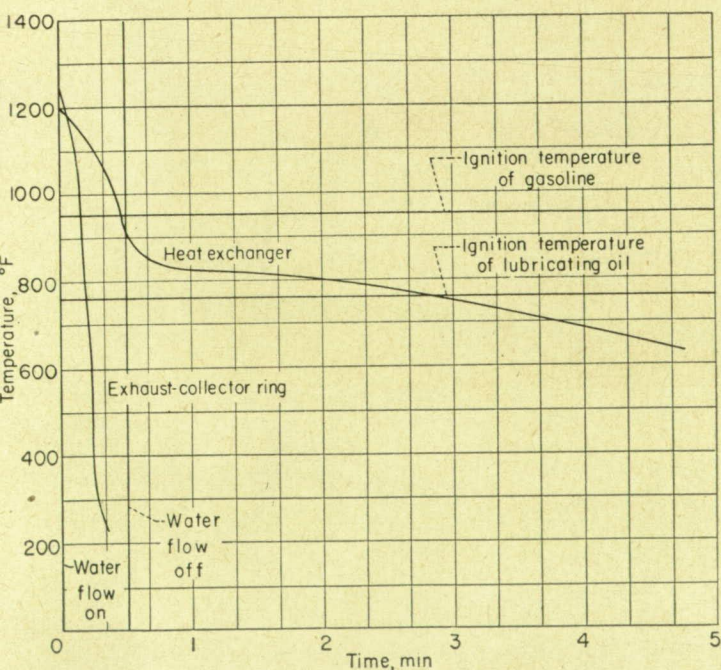


FIGURE 35.—Temperature-time history of hottest portions of exhaust system during crash test. Heat exchanger and exhaust-collector ring water-cooled and inerted.

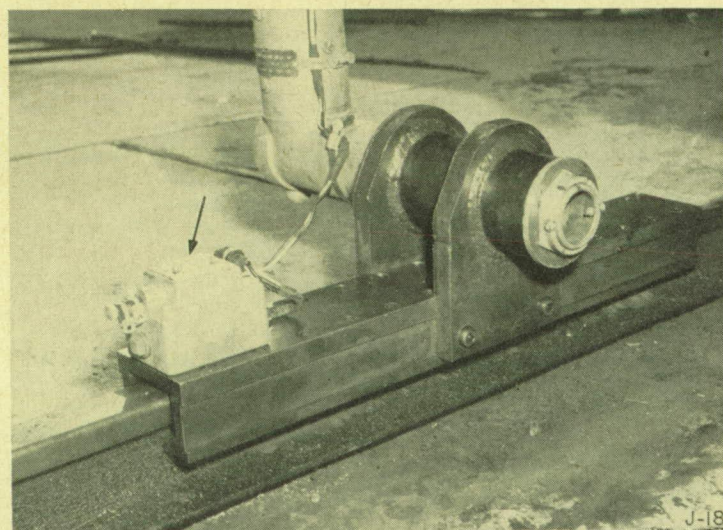


FIGURE 36.—Inerting-system switch.



FIGURE 37.—Inerting-system-switch target.

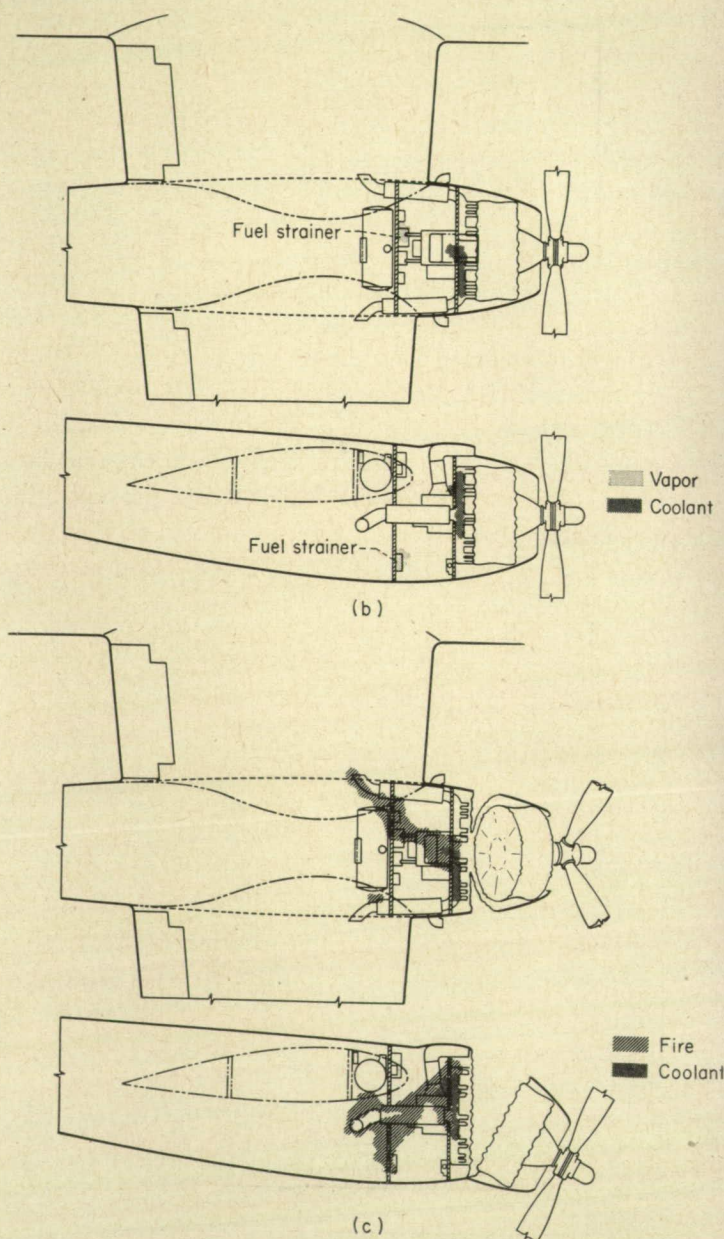
rate. These time delays are listed in table II. A comparison can be made between these time delays and the earliest times following crash that fires were obtained with the various classes of ignition sources the inerting-system components were designed to cover (table II). This comparison shows that in every case except one, the inerting-system components came into operation before the corresponding ignition sources were observed to produce fire. The exception was the full closing of the fuel shut-off valve in 0.34 second with ignition of the contents of the engine induction system occurring within 0.1 second after crash impact. The fire-extinguishing agent attained full flow discharge in 0.06 second, however, and thus provided protection until the fuel valve closed.

In the first crash conducted with airplanes equipped with the inerting system and the standard crash barrier, fire occurred on both sides of the airplane. The circumstances under which these fires took place merit consideration in some detail. In this crash, the water-spray distribution manifold was supported from convenient mounts available on the engine. In the C-82 airplane employed in this phase of the study, the exhaust-collector ring is supported from the fire wall. Upon impact at the crash barrier, the right engine separated from its upper mount and dropped to the attitude shown in figure 38 (a). As a result, the water-spray manifold was carried away from the exhaust-collector ring affixed to the fire wall. The crash instrumentation showed that the fuel shut-off valve and electrical-system switch of the inerting system functioned properly. The fire-extinguishing-agent bottle was stripped from the engine with the carburetor assembly to which it was mounted.

Three-quarters of a second after crash impact, the instrumentation showed the situation existing in the nacelle illustrated diagrammatically in figure 38 (b), which depicts the pertinent nacelle accessory-section layout and location of the fire and vapor detectors. At this time, the only combustible mixture indicated was located close to the damaged fuel strainer fastened to the face of the fire wall at

the base of the nacelle. Apparently most of the fuel loss from the strainer ran down and out of the nacelle and left only a local zone of combustible mixture. The fire-detection thermocouples mounted close to the exhaust-collector ring were normally warmed by radiations from the collector ring. These thermocouples indicated a marked drop in the temperature of the exhaust-collector ring from the 10 to 11 o'clock positions, signifying that by chance some of the water spray was directed at this section of the ring and provided local protection.

At 2.2 seconds after impact, the instrumentation indicated no change in the situation within the nacelle. A photograph



(b) Schematic diagram of nacelle showing location of combustible vapor and coolant in nacelle 0.75 second after initial impact.

(c) Schematic diagram of nacelle showing fire indicated by detectors in nacelle 4.2 seconds after initial impact.

FIGURE 38.—Mechanism of ignition by hot metal of exhaust-collector ring.

of the airplane at this time (fig. 38 (a)) moving at approximately 60 feet per second shows the marked forward development of the fuel mist close to the ground and the vertical inclination of the engine and cowl. Some of the water spraying uselessly from the damaged water manifold can be seen projecting vertically from the nacelle. At 3.8 seconds after crash, the first evidence of fire appeared between the 12 and 1 o'clock positions on the exhaust-collector ring. The initial flame that appeared at this position did not show in the printed figure. In figure 38 (d) is shown the flame after it propagated to the 11 o'clock position.

At 4.2 seconds, fire-detector thermocouples first registered over the broad zone indicated in figure 38 (c). Failure of the fire-detection thermocouple to register at the 1 o'clock position on the exhaust-collector ring where the fire first appeared before 4.2 seconds is attributed to wetting by the

water spray. Instrumentation that recorded the actuation times of the inerting-system components showed that they functioned properly. Therefore, the unprotected exhaust-collector ring is the most probable ignition source, as is consistent with the visual evidence. Migration of the fuel mist to the inboard side of the nacelle was required in order for ignition to occur, because the top of the outboard segment of the exhaust-collector ring was under water-spray protection. That the fuel mist did extend to the inboard side of the nacelle is evidenced by the flame showing around the nacelle on the inboard side in the photograph of figure 38 (e), taken immediately after ignition.

On the left side of this airplane, the engine remained in place but fire was ignited by exhaust flames that were described under the section on the mechanism of fuel-mist development shown in figure 12. The exhaust flames that

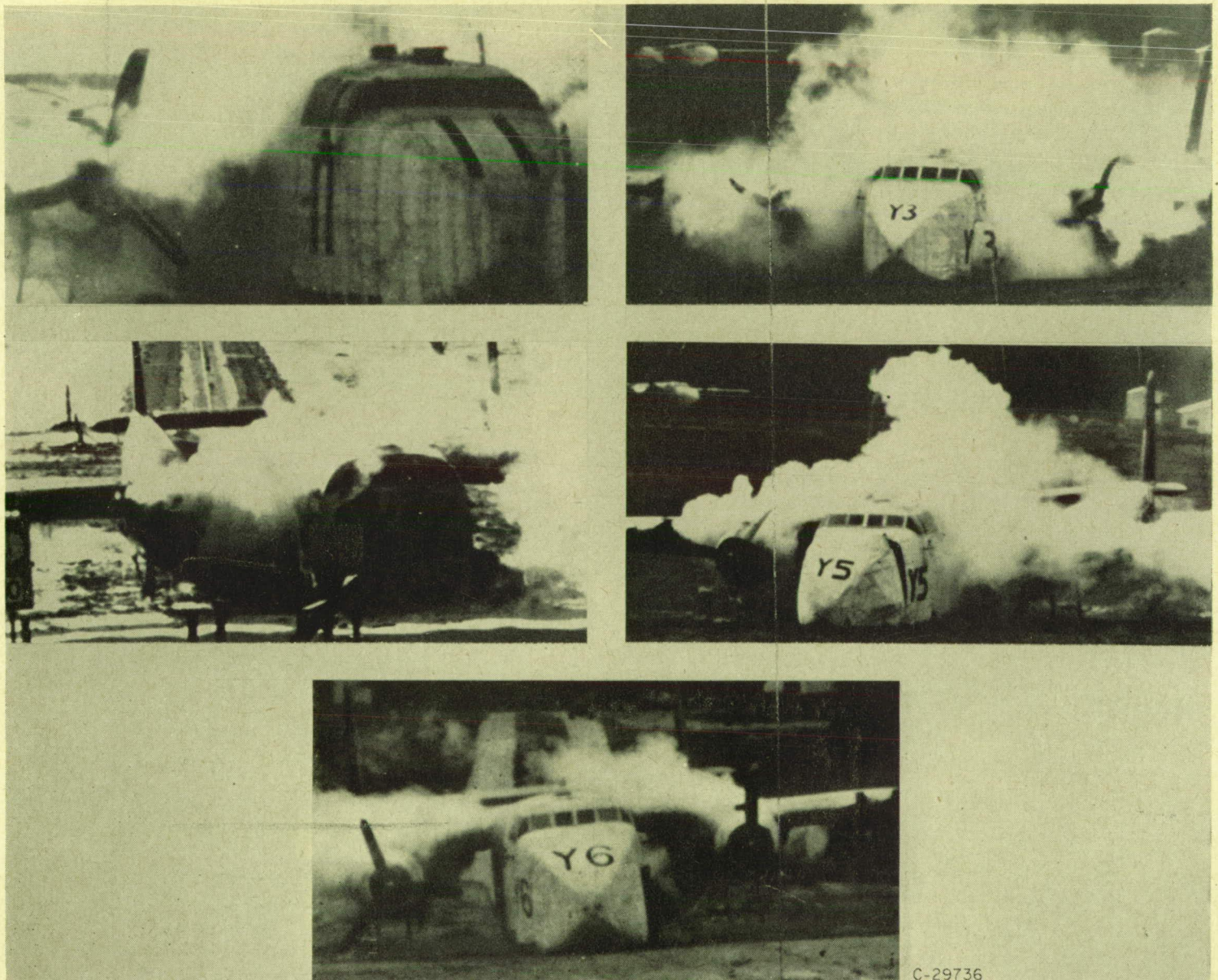


FIGURE 40.—Appearance of airplanes following crash without fire.

were observed were provided by the burning of the fire-extinguishing agent employed in the engine induction system. The agent in this case was bromochloromethane (CH_2BrCl), known as CB. This material decomposes thermally at the temperatures of the engine exhaust-system metal, and some of the decomposition fragments released will burn in air. Photographs of this agent burning on the surface of a heated section of exhaust stack are shown in figure 39. In normal application during fire extinguishment, enough CB is employed to provide an inerting atmosphere around the decomposition products, and their ignition does not occur. In a crash, unfortunately, there can be no control of the quantity of extinguishing agent passing through the engine, because the displacement of the pistons meters the extinguishing agent throughput according to the engine rotational speed and the current throttle setting of the damaged engine, neither of which can be specified in a crash. Although high concentrations of CB were provided at the engine inlet of the crashed airplane, the quantity passed through the engine was small enough to allow a sufficient residence time for the CB in the high-temperature environment of the exhaust-disposal system to decompose thermally. Upon contact with the air at the tail-pipe exit, the decomposition products ignited to provide the series of exhaust flames shown in figure 12.

Halocarbons involving bromine and fluorine, in which complete substitution of hydrogen is obtained, would represent satisfactory fire-extinguishing agents for engine inlet inerting, because their decomposition products do not burn at engine metal temperature. Compounds in this class include trifluorobromomethane CBrF_3 and difluorodibromomethane CBr_2F_2 , which have recently become available in restricted quantities.

After the first crash in this series, the water-spray system was mounted to remain with the exhaust-disposal system should the engine be displaced, and carbon dioxide was employed at the engine inlet because it does not decompose appreciably at engine exhaust temperatures. In the next six crashes, one ignition occurred by the movement of fuel through the hot-air duct to the exhaust-gas heat exchanger, as was described in the discussion of liquid-fuel spillage. This result called attention to the need for more careful distribution of the water spray in the heat exchanger and to the desirability of a safety gate in the hot-air duct. After coming to rest, the other five unburned crashed airplanes carrying the inerting system appeared as shown in figure 40. The only visible evidence of the presence of the inerting system was the volume of water vapor issuing from the nacelle. On humid days, the condensed water vapor persisted in the atmosphere long enough to have the appearance shown in some of the photographs.



FIGURE 42.—Crash area for high-contact-angle crash.



(a) 0.3 second after initial impact.



(b) 1 second after initial impact.



(c) 1.7 seconds after initial impact.



(d) 2.3 seconds after initial impact.



(e) 3.0 seconds after initial impact.



(f) 3.7 seconds after initial impact.



(g) 4.7 seconds after initial impact.

Figure 41.—Ground-loop crash. (Fuel dyed red.)



Figure 43.—Damage to forestructure of airplane produced by impact with ground 0.72 second after impact with barrier in high-contact-angle crash. Airplane speed, 100 feet per second.



(a) Wheel strut in air before ignition 1.2 seconds after initial impact. Airplane speed, 75 feet per second.



(b) Ignition of fuel mist 2.4 seconds after initial impact. Airplane speed, 24 feet per second.



(c) Spread of fire 2.9 seconds after initial impact. Airplane speed, 15 feet per second.



(d) Spread of fire 3.9 seconds after initial impact. Airplane speed, 0.

Figure 44.—Ignition of fuel mist by electrostatic charge on landing gear in high-contact-angle crash. (Fuel dyed red.)

GROUND LOOP

After five crashes in which no fires were obtained and no new ignition sources were observed, modifications to the crash barrier were employed in an effort to reveal new mechanisms of crash-fire initiation. In one crash, a ground loop was imposed on the crashed airplane by allowing the right wheel and left wing to pass through the crash barrier without damage. (The abutment on the right side of the airplane and the poles on the left side were removed.) The airplane, with known ignition sources inerted as before, ground-looped as desired, taking on the successive attitudes shown in figure 41. In executing the ground loop, the airplane exposed the long axis of the nacelle and fuselage to the fuel mist generated at the leading edge of the right wing and increased the interception of the fuel by these airplane members over that obtained when ground looping does not occur. Displacement of the fuel mist toward the nacelle on the right is promoted by the large spanwise component of the relative wind in the direction of the nacelle accompanying the yawed attitude of the airplane in the ground-loop maneuver. No new ignition sources were revealed, however, and fire did not occur.

HIGH-CONTACT-ANGLE CRASH

In order to determine whether or not ignition sources are created by the mechanical rending of the aircraft structure in a severe impact, another crash was conducted in which a 16° contact angle between the airplane and the ground occurred. The ground beyond the crash barrier, on which the airplane comes to rest, was sloped and hollowed to give the desired angle of contact. A photograph of the crash area arranged for this crash is shown in figure 42. The crash barrier was maintained in its standard configuration with two abutments and poles for the wing tanks on both sides of the airplane. The known ignition sources were inerted in the usual manner. The marked damage to the forestructure of the airplane upon impact with the ground is evident in figure 43, taken 0.72 second after impact with the barrier. Extensive coverage of the airplane with fuel occurred as in figure 16, which shows photographs of the fuel-mist development experienced in this crash. Because of the time lag associated with the movement of fuel out of the tanks, the time of the maximum forward surge of the fuel follows the period of peak airplane deceleration by 1.1 seconds.

The fact that no ignition of the fuel occurred around the deforming structure indicates the absence of ignition sources of sufficient duration to ignite the fuel in this instance. Ignition sources covered by the inerting system were inactive in spite of the severe nacelle damage sustained. As the airplane slowed to rest, however, the left wheel strut tumbling through the air behind the airplane (fig. 44 (a)) ignited the fuel in the airplane's wake when the strut approached the ground approximately 60 feet behind the left wing (fig. 44 (b)). The fire moved through the fuel mist and liquid fuel on the ground (fig. 44 (c)) toward the airplane, where ignition of the fuel in the wings exploded

both wings (fig. 44 (d)). The flame-propagation rate through the fuel mist around the airplane was determined as 75 feet per second with a 30-foot-per-second tail wind, or a net speed of 45 feet per second (fig. 45). This high rate of flame propagation includes the velocity imposed on the flame front by the expanding burning mass of fuel and air. The large column of fuel mist suspended in the air is evident from the volume of fire shown in figure 44 (d).

From inspection of the terrain in the crash area and inspection of the wheel-strut assembly, it was concluded that the ignition source was neither friction sparks nor compression ignition of the hydraulic fluid in the wheel strut. However, investigation of the electrostatic charge collected on the wheel-strut assembly by friction with the dust and the fuel mist in the airplane wake showed that the ignition in this



FIGURE 45.—Contour of fire propagation through fuel mist after ignition in high-contact-angle crash.

case was most likely produced by electrostatic discharge from the strut to the ground. Studies were made with the same wheel and strut, electrically insulated from the supports, as shown in figure 46. Air was blown over the wheel and strut assembly by the blower shown in the figure, and dust was metered into the air stream. A dust flow rate of 400 grams per second with an airspeed of 45 miles per hour past the strut generated 3900 volts, sufficient to produce sparks capable of fuel ignition in less than 0.75 second (fig. 47). In the crash, the wheel and strut traveled through the air at 43 miles per hour for 1 second, bounced on the rubber tire, and then traveled through the air for another 1-second period at approximately the same speed.

In summary, of all the ignition sources observed during this investigation, 41 percent would have been eliminated by induction-system inerting and fuel shut-off; 41 percent by inerting and cooling of the hot exhaust-system metal; and 14 percent by de-energizing the electrical system. However, if only part of the inerting-system components were used, these percentages would change materially, because the combustible would move to the next available ignition

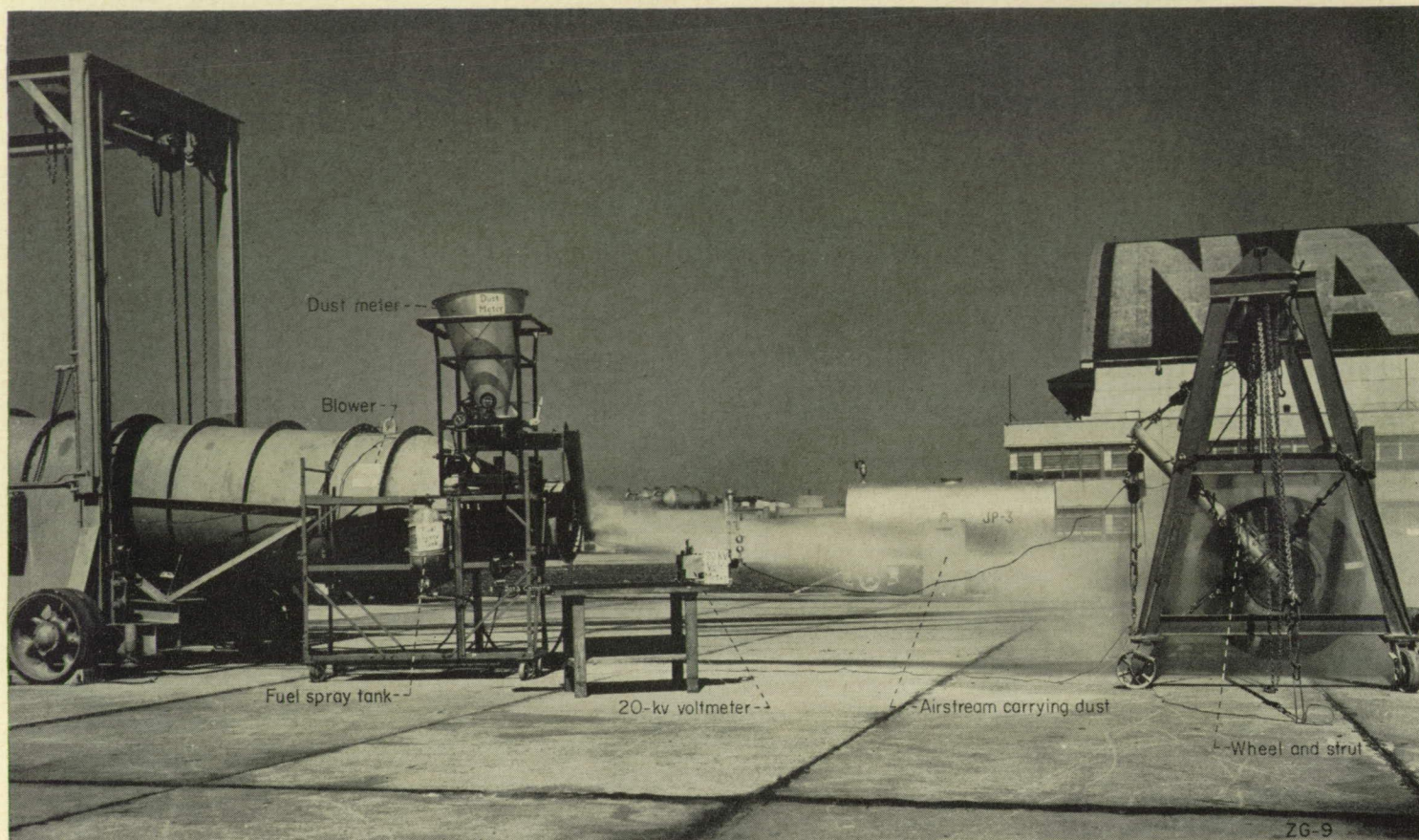
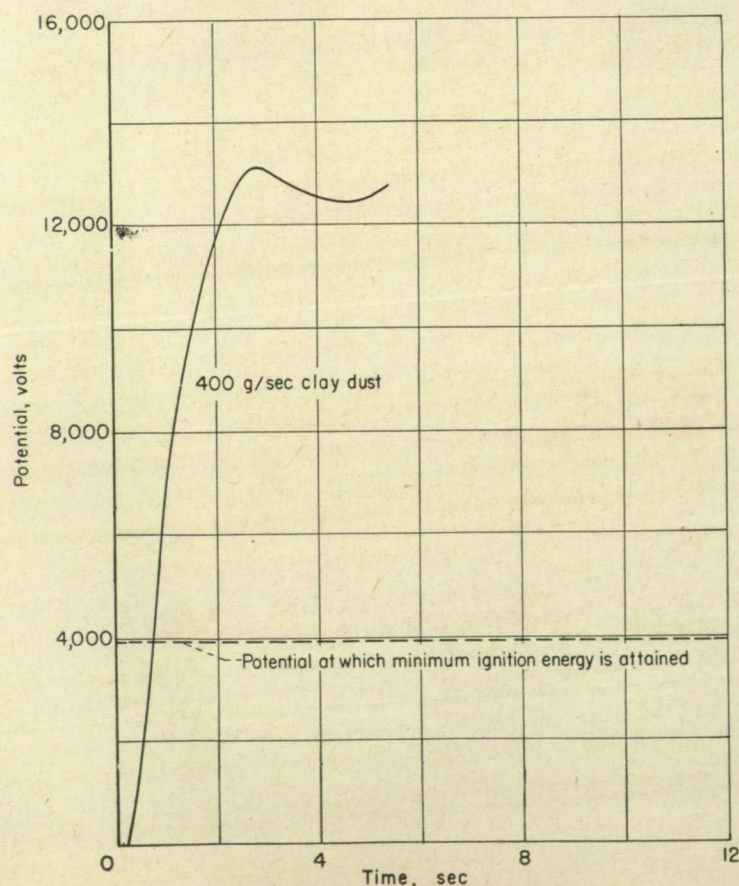


FIGURE 46.—Test setup to determine rate of accumulation of electrostatic charge on a landing-gear strut.



source. This fact is demonstrated in the case of the crash illustrated in figure 38. In this case, the electrical system was de-energized, the induction system was inerted, and the fuel to the engine was shut off. However, the exhaust cooling system pulled away from the exhaust system and did not function. Under these circumstances, the fuel mist spread until it contacted the hot exhaust system and ignited. Thus, caution must be used in evaluating the effectiveness of the inerting system when only portions of the system are used. Undoubtedly, in some crash circumstances benefits could be derived by the use of only parts of the inerting system.

FIRE DEVELOPMENT

The progress of the fire following ignition of the fuel has some aspects that are common to all crash fires and others that depend on the distribution of the fuel spillage preceding ignition, the wind magnitude and direction, the slope of the ground on which the airplane comes to rest, and the location of the opening in the fuel system from which the fuel issues. A complete description of the progress of crash fire as it is influenced by all of these factors and the con-

FIGURE 47.—Voltage produced on insulated landing gear by blowing clay dust at 45 mph. Leakage resistance, 0.5×10^{12} ohms; capacitance, 200×10^{-12} farad.



Figure 50.—Burning fuel pouring from rupture in wing fuel tank.



Figure 51.—Explosion of wing due to ignition of fuel vapors in wing and wing tanks.

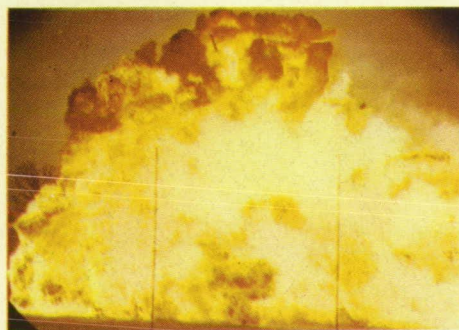


Figure 52.—Build-up of fire resulting from wing explosion. Photograph taken 0.7 second after photograph of figure 51.

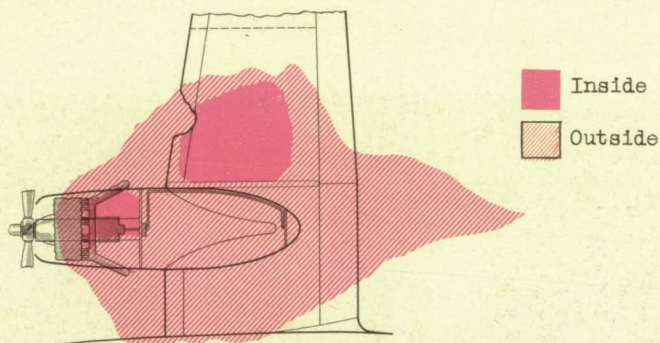


Figure 53.—Schematic diagram of fire burning inside wing and nacelle as compared with fire outside airplane.



(a) 0.9 second after ignition.



(b) 4.0 seconds after ignition.

Figure 54.—Flame movement through fuel on ground behind airplane.



(a) Fuel mist fire at its maximum; 3.0 seconds after ignition.



(b) Fuel mist fire with rising column of smoke and burning core of fuel mist; 5.3 seconds after ignition.



(c) Fire from fuel evaporating from wetted surface of airplane and ground; 10.0 seconds after ignition.



(d) Fire produced by fuel that soaked into ground vaporizing and burning in column continuous with burning fuel from tanks; 19.0 seconds after ignition.

Figure 55.—Rapid burning of fuel mist.



Figure 56.—Magnesium portions of engine burning during crash.



FIGURE 57.—Flames from burning fuel in wing tanks licking sides of fuselage.



(a) Fire before explosion.



(b) Build-up of fire due to spread of flaming hydraulic fluid.

FIGURE 58.—Fire produced by exploding hydraulic-landing-gear strut.

figuration the airplane assumes in the crash cannot be attempted here. A discussion of some of the main events observed to occur in the crash fires studied will be given, however, in which some of the factors influencing the development of the fire are considered.

In a crash, the fuel system may be broken open by impact of the airplane with obstacles such as trees, posts, boulders, and so forth, that rip holes in tanks; by hydraulic forces generated by the surging fuel contained in the tanks of the decelerating airplane; by flying debris released in the crash, particularly propeller blades, that may cut fuel lines (fig. 48) and strainers, or pierce fuel tanks (fig. 49); and by the relative displacement of airplane components as, for example, the parting of an engine from its mounts and consequent snapping of the carburetor fuel line. Fuel lines, likewise, may be burned through by oil, hydraulic fluid, or alcohol fires. Since there is usually a continuous path of fuel in liquid, mist, or vapor form from the point of spillage to an

ignition source, the fire moves from the ignition source back to the fuel-system rupture from which the fuel issues. This step in the propagation of fuel fire is common to all crashes. The fire burns at the opening from which the fuel issues and ignites the fuel as it leaves. The column of fire shown in figure 50, for example, is ignited fuel pouring from the wing. The fire at the fuel-system leak enlarges the opening and tends, thereby, to increase the fuel efflux rate.

When wing fuel tanks are ruptured, combustible concentrations of fuel vapor often accumulate in the voids around the tanks. Under these circumstances, a wing explosion occurs when the fire reaches the wing. A picture of such a wing explosion is shown in figure 51. The wing skin is generally stripped off to the wing tips as shown in the figure. A marked acceleration in the development of the fire accompanies such explosions because of the broad distribution of the wing-tank fuel that results. A typical example of such accelerated fire development followed the wing explosion of figure 51. In 0.7 second following the explosion, the fire appeared as shown in figure 52.

Fire often burns within the wing without explosion. In this case, the burning rate is governed by the air flowing into the wing. Because of the limitation of available air, the internal wing fire is often of modest proportions and tends to locate near the vents through which the air enters. The extent of one internal wing fire, determined by a grid of thermocouples located in the wing, is shown in figure 53 as the solid area. The limited size of the fire is in contrast to the extent of the fire around the wing shown as the cross-hatched area of the same figure.

In the first few seconds following ignition, the fire propagates also through all the fuel spilled previous to ignition. The fuel suspended in the air as mist ignites and the fire spreads with lineal flame-propagation speeds up to approximately 70 feet per second. This high flame-propagation speed is provided in part by the rapid expansion of burning atmosphere of fuel mist and air. The flame speed through the fuel on the ground along the skid path of the airplane and on the wetted airplane surfaces is somewhat reduced over that obtained through fuel mist, particularly if the propagation is against the wind. Two photographs of flame movement through the fuel on the ground behind the airplane 0.9 and 4.0 seconds after ignition are shown in figures 54 (a) and (b), respectively. The flame speed in this case is approximately 13 feet per second. The fire-spread along the fuel-wetted surfaces of the airplane wings, fuselage, and tailbooms occurs in a manner similar to the ground fire-spread.

If ignition occurs while appreciable fuel is suspended in the air as mist, the heat released from its combustion often represents most of the total heat release in the early phase of the fire. Because the fuel in the mist is dispersed through a large volume of air, each droplet is surrounded by a sufficient quantity of air to burn a significant portion of the droplet. The fuel mist is consumed rapidly, therefore, and

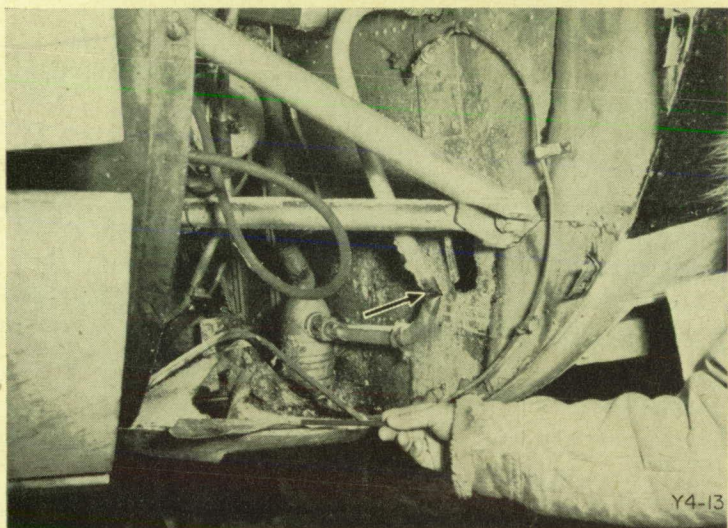


FIGURE 48.—Fuel line cut by flying propeller tip.

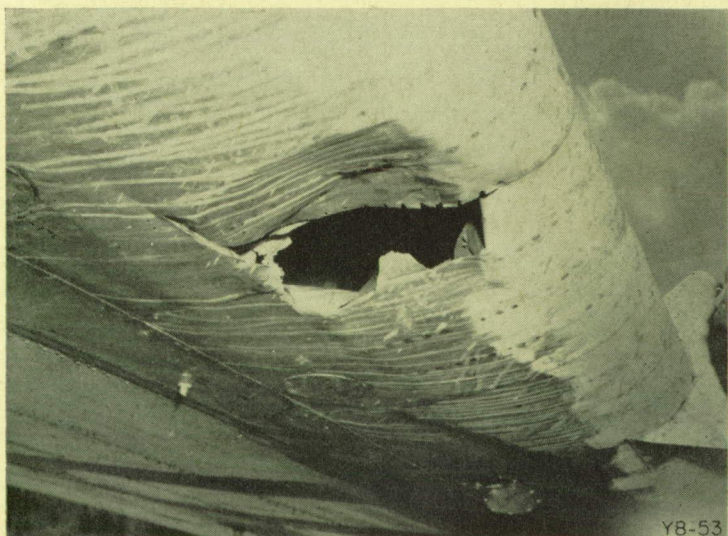


FIGURE 49.—Wing fuel tank ruptured by flying propeller tip.

seldom persists for more than 20 seconds. The fuel-mist fire at its early development, shown in figure 55 (a), 3.0 seconds after ignition, appears as a solid wall of flame extending 45 feet behind and 65 feet above the airplane. The heated fuel mist rises as it burns. The heat radiated to the airplane from burning fuel mist lessens rapidly as the fuel mist rises and acquires at the same time an envelope of opaque black smoke characteristic of petroleum fuels (such as gasoline) burning in the open air. In this phase the mist fire has the appearance shown in figure 55 (b), 5.3 seconds after ignition. The burning core of mist, visible in the figure, is the last to burn out and leaves the fuel evaporating from the wetted surface of the airplane and the ground to continue the fire as shown in the photograph of figure 55 (c).

Of the fuel spilled on the ground, a large portion soaks rapidly into dry unfrozen soil. When the crash takes place on sloping ground, as in the studies conducted here, the pools of fuel that collect are quite shallow, and only during the first few minutes after the crash are there appreciable quantities of liquid fuel exposed in the airplane slide path to provide combustible vapors in large quantities. For a few seconds after the fire starts, these vapors rise and burn to form a column continuous with the burning fuel-mist mass. The lower portion of the towering column of smoke and fire shown in figure 55 (c) is provided by this vaporized fuel. Some of the fuel that soaked into the ground released vapors at a sufficient rate in the hot environment of the fire to maintain small fires that burn close to the ground for several minutes after the fire starts (fig. 55 (d)).

Following the initial rapid consumption of the fuel that is spilled on the ground and suspended in the air while the crashed airplane is in motion, the fire shrinks to the zones adjacent to the airplane with a marked reduction in the heat radiated to the surroundings. Close approach to the burning airplane can now be made by normally clothed personnel. The air temperature at ground level close to the fire differs little from the normal air temperature, since the air heated by the fire rises overhead with the smoke and flames. Radiation from the flames and hot metal of the airplane structure may be intense, but can be tolerated by unprotected skin for several minutes. At this stage, the fire on the outside of the airplane is fed by fuel pouring from the damaged tanks and seeping through the seams on the lower surfaces of the wings; by oil from the fire-damaged or mechanically damaged lubricating system; by hydraulic fluids from the brake and wheel-strut actuating system; and by the aluminum skin and structure, the fabric skin, and the magnesium engine parts. The oil fires lie close to the engine nacelle where the oil distribution system and tanks are located. Once ignited, the magnesium engine parts continue to burn with the characteristic blue-white flame even after the surrounding fuel or oil fire subsides (fig. 56). The aluminum parts, particularly the skin, ignite early in the fire. The aluminum burns only in the high-temperature environment provided by adjacent fire. On the ground close to the fire, rivulets of molten aluminum form and flow by gravity to cooler zones and solidify.

After the flash fire that spread through the fuel spilled previous to ignition burns out and the fire is localized in the immediate area of the airplane, the relatively slow propagation of the fire provided by ignited fuel pouring to the ground and through the wing structure becomes apparent. This burning fuel, running by gravity, spreads the fire to areas not wetted by fuel in the initial fuel spillage while the airplane was in motion. If the ground slopes downward from the wings to the tail, this burning fuel will flow on the ground along the outside of the passenger compartment of the fuselage. Contact between the fuselage skin on the ground and this fuel may occur, depending on the direction taken by small ground grooves and sinuous channels. Such grooves and channels are formed by vegetation and soil erosion usually present on the ground or are plowed by the dangling airplane parts, such as propeller blades or broken wheel struts. The distance this ground-flowing fuel will extend from the fuel-tank source in these grooves and channels depends largely on the fuel flow rate and the rate of ground absorption of the fuel. The fuel fire spread in this way is maintained for many minutes, fed continuously by the fuel pouring from the tanks. Burn-through of the fuselage skin is almost a certainty if contact between fuel stream and fuselage occurs. Loss of the skin by fire burning from fuel streams that extend parallel to the fuselage without contact can occur if the local wind bends the flames to lick against the fuselage skin in the manner shown in figure 57. If the ground slopes downward toward the nose of the airplane, the fuel flows away from the passenger section of the fuselage and fire destruction of this member does not occur by the mechanism just described.

The airplane fire reaches several secondary peak intensities before it finally burns out. Barring explosions, the first of these peaks usually occurs when the fire burning at the fuel tanks enlarges the openings from which the fuel issues or opens undamaged tanks and increases the rate of fuel spillage. Such enlargement of the fuel-tank openings or burn-through of otherwise intact tanks, below the fuel level, will occur sooner with bladder-type tanks than with metal tanks which have good thermal conductivity. The cooling provided by the fuel in the tank retards burning of the tank walls. Wing explosions are usually followed by a marked resurgence of the fire because of the associated fuel scattering (figs. 51 and 52). Such explosions may occur when the fire first propagates to the wing from the point of ignition or after the wing fire has been in progress. Wing explosions that occur after the wing fire has developed are usually less violent than the explosions that sometimes accompany the first ignition of fuel within the wing. Exploding elements of the hydraulic system, such as wheel struts softened in the heat of the fire, will spread flaming hydraulic fluid with spectacular effect (fig. 58).

When the fire burns through the fuselage skin and the combustibles within are made available, the fire development achieves another peak. Exploding hydraulic-system accumulators add the hydraulic fluid to this fire and promote



(a) Smoke and hydraulic oil vapors issuing from fuselage nose.



(b) Resulting fire after burn through of skin of fuselage nose.

Figure 59.—Development of hydraulic fluid fire within fuselage.

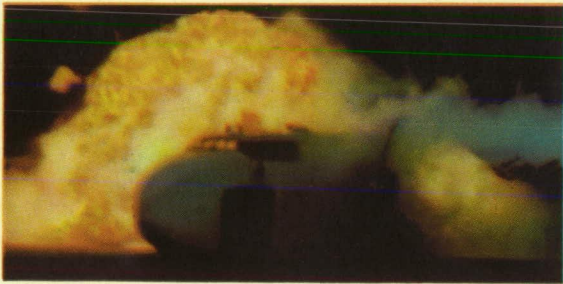


Figure 60.—Cross-over of fire from one side of airplane to other through canopy of fuel mist.

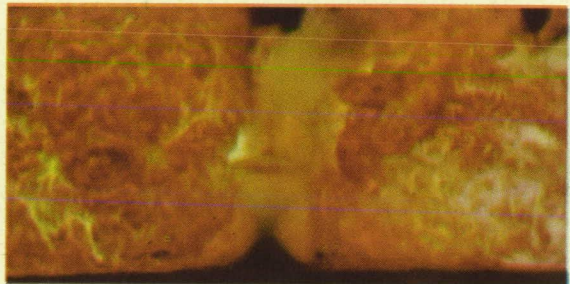


Figure 61.—Fire produced by explosion of both wings; 1.0 second after ignition.



Figure 62.—Burning fabric on tail surfaces ignited by fuel-mist flash fire.



(a) Fire burning on ground on right side under empennage of airplane 76 seconds after ignition of left side of airplane.



(b) Progress of fire burning on ground on right side 84 seconds after ignition of left side of airplane.



(d) Ignition of fuel in right wing tanks 107 seconds after ignition of left side of airplane.



(c) Ignition of fuel pouring from right wing tanks 98 seconds after ignition of left side of airplane.

Figure 66.—Cross-over of fire from left to right side of airplane through fuel trail left on ground by airplane.



FIGURE 63.—Structural elements melted in a typical crash fire.

its spread inside the fuselage. The fire within the fuselage burns slowly at first if the fuselage is essentially intact and limited combustion air is available. At this time, large volumes of smoke, representing condensed and partially burned hydraulic-oil vapors and other volatile materials, are often observed issuing from the fuselage, as shown in figure 59 (a). As the fuselage skin is consumed in the fire, access for additional air develops, with a corresponding further fire development. When a large hole burns through the skin at the top of the fuselage, the air flow is further promoted by chimney action. The fuselage fire attains its peak at this time. A view of the fuselage fire (fig. 59 (b)) shows the combustible smoke that issued earlier from the fuselage (fig. 59 (a)) burning with a bright flame. The appearance of flame at the door frame shows the fuselage being consumed by fire from within.

The last fires to burn out are often those from fuel that runs under the airplane fuselage where it burns slowly with restricted air supply, oil around the oil tank and nacelle, magnesium engine parts, and rubber tires of the landing wheels.

Progressive collapse of the main airplane structure not damaged in the crash impact occurs in the fire as main supporting elements are burned, melted, or softened by the heat. Structural elements melted in the fire show the sharp-edged and pointed stalactites of fused aluminum illustrated in figure 63. Typical of structural collapse by fire is the failure of the tailboom shown in figure 64.

At times, with fuel spillage on both sides of the crashed airplane, fuel ignition occurs on one side only. Cross-over of the fire from one side to the other appears to be possible through the canopy of fuel mist that often forms above and slightly behind the crashed airplane, although it has not been observed in the crashes conducted so far. The conditions necessary for a cross-over of fire through the fuel mist are shown in figure 60; however, in this case cross-over did not occur, because of the rapid development of the fire on the



FIGURE 64.—Structural collapse due to melting and burning of skin and structural members.

left side of the airplane. A map of the progress of this fire during the first second following ignition as the flame propagates from the right nacelle towards the left side of the airplane is shown in figure 65. The family of curves in the figure represent the outline of the fire in 0.1-second intervals, starting at the right nacelle and later at the left engine tail pipe. These data show an average vertical rate of propagation of the fire of 60 feet per second and a uniform horizontal rate of 25 feet per second over a 0.6-second period. Explosion of both wings occurred just as this flame pattern was established and produced the fire shown in figure 61.

Cross-over of fire did take place, however, along the liquid-fuel path left in the trail of the crashed airplane and on the airplane itself. In one crash of an airplane carrying fuel of low volatility (8 mm Hg Reid vapor pressure), fire occurred on the left side, as shown in figure 19 (c). The fire, as it appeared on the right side as the airplane came to rest, is shown in figure 62. Only the fabric of the empennage control surfaces, ignited by the fuel-mist flash fire, is burning on this side. Fire traveled to the right side of the airplane along the rivulet of fuel running from the right-wing tanks downhill to the fire burning in the skid path of the airplane on the left side 60 feet behind the airplane empennage. The fire that moved along this fuel rivulet on the ground is shown in the succession of photographs in figure 66. The right wing became involved in the fire approximately 107 seconds after the crash. That propagation of the fire forward through the thin mist of low-volatility fuel suspended in the air on the right side of the airplane did not occur is probably related to the relative advantage that low-volatility fuel confers under these circumstances, as shown by the data on maximum distance from the mist source for upwind flame propagation (fig. 22).

The history of the crash fires obtained with fuel of low volatility was identical to that involving gasoline. The large heat release from the burning fuel mist raised the

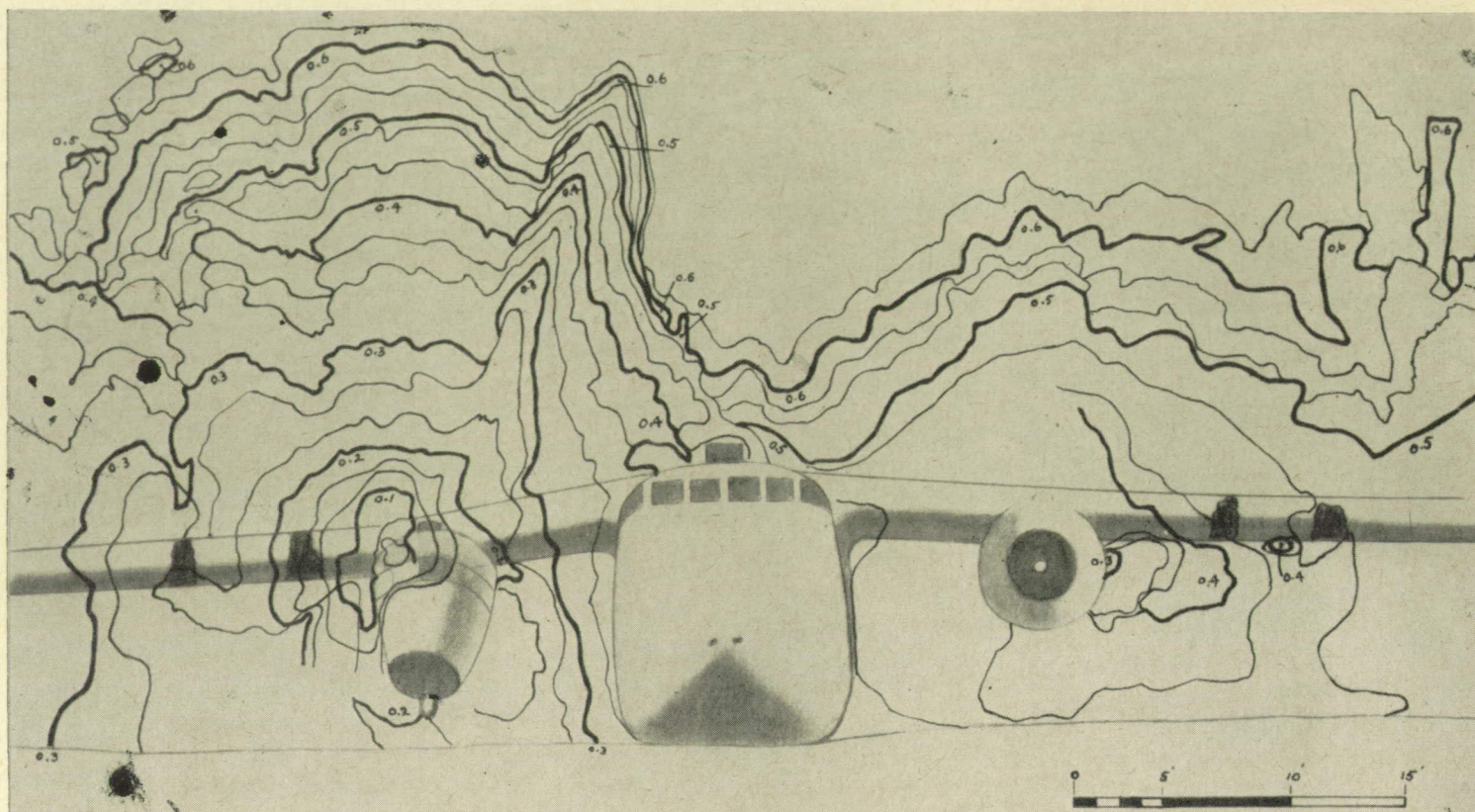


FIGURE 65.—Map of progress of fire following ignition as it appears to camera viewing airplane head-on. (Numbers refer to time after ignition in seconds.)

temperature in the airplane zone, and differences in fuel volatility measured at normal atmospheric temperature were no longer significant in this high-temperature environment. In crash fires with fuels of low volatility in which large masses of burning fuel mist are not involved, some advantage in the rate of fire spread may be gained in the early phase of the fire. Once the fire propagates back to the source of the fuel and ignites the fuel as it is released from the fuel system, the rate of fire development depends primarily on the fuel-spillage rate, with fuel volatility being of secondary importance.

The course of the crash fire described is related to those fires involving fuel spilled in the manner imposed in these full-scale studies. Except for the early fire through the fuel mist and fuel on the ground in the skid path of the crashed airplane, however, the development of the fire described is similar in main events to transport crash fires recently observed in which little fuel spillage preceded ignition.

CONCLUDING REMARKS

It is evident from the manner in which these full-scale crash studies were conducted that no evaluation can be made of the relative frequency with which the various ignition sources will initiate fires in crashes that occur in civil and military operations. Such information must come from airplane operational experience, with great doubt always pres-

ent because of the inaccuracy of observations on the initiation of the crash fire. In view of the diversity of ignition sources revealed by this crash study with a given airplane and fixed experimental crash conditions, it is difficult to make such a comparative evaluation even if the obstacles and nature of the terrain involved in the airplane crash are specified. If a detailed description of the airplane damage could be specified, however, along with the order in which the damage was imposed, the state of the airplane motions, the wind, and the contour of the crash site, some approximation of the relative likelihood of fire initiation by a particular ignition source could be made on the basis of the information obtained from this crash study.

The same observations apply to an evaluation of the margin of crash-fire safety that the use of fuels of low volatility will provide as compared with the use of gasoline in the airplane. These crash-fire studies demonstrated only some of the circumstances under which fuel volatility does not matter in fire initiation and indicated the possibility of other conditions for ignition that would not be influenced by fuel volatility. The relative frequency with which these circumstances appear in actual crashes cannot be estimated on the basis of this work with a sufficient order of accuracy to obtain a meaningful estimate of the value of fuels of low volatility in reducing the crash-fire hazard. Some little significance may be attached to the fact that, of four airplanes crashed with low-volatility fuel, two did not burn; whereas, fires

were obtained with almost all (six out of seven) airplanes crashed with gasoline when ignition-source inerting was not employed. However, in one of the crashes with low-volatility fuel in which no fire occurred, both engines were ripped from their mounts at the crash barrier and did not spend a sufficient time in the fuel mist behind the airplane to provide an ignition on the exposed hot exhaust-collector rings. The low-volatility fuel has a spontaneous-ignition temperature equal to, or slightly less than, aviation-grade gasoline. A surface hot enough to ignite gasoline in a given contact time could certainly ignite the low-volatility fuel in the same time. It is believed that fuel volatility was not the factor determining whether or not fire did occur in this case.

In the second of the crashes with low-volatility fuel in which no fire occurred, the engine ignition was cut while the airplane was in its full-power taxi run 2 seconds before reaching the crash barrier. Little nacelle damage occurred, because the propellers struck the barrier at reduced speed and no engine power. A small oil fire appeared in the left nacelle from lubricating oil burning on the exhaust-collector ring. No other combustible spillage occurred within the nacelle. The nacelle cowl was intact. The quantity of oil involved in the fire was quite small, and the fire did not appear outside the nacelle except for a brief flash $11\frac{1}{3}$ seconds after crash on the inboard side of the left nacelle. At this time, the airplane was moving too fast for the fuel to extend from the wing-tank rupture to the inboard side of the nacelles. Because it is difficult to determine from the motion pictures whether or not the fuel ever did extend to the nacelle and because the fuel does not have sufficient vapor pressure to be detected by the combustible-vapor detectors, it cannot be stated with certainty that no fire would have occurred if gasoline had been used in this crash. Some advantage from the use of low-volatility fuel may be indicated in this case. The results obtained in this crash study indicate that the use of low-volatility fuel does not in itself represent a wholly acceptable solution to the crash-fire problem.

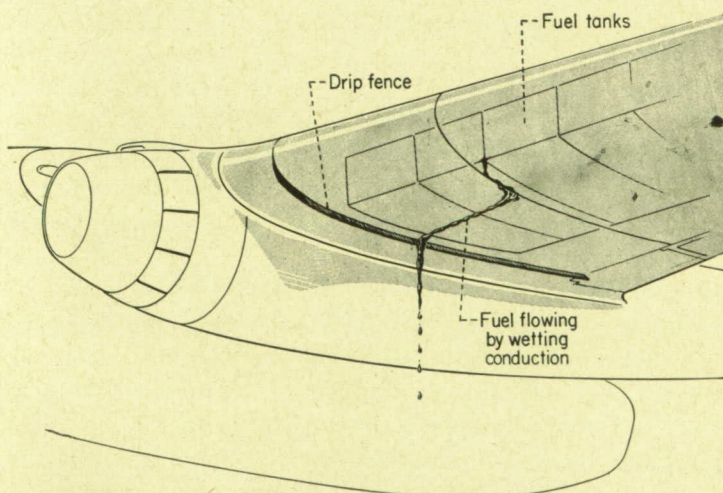


FIGURE 67.—Drip fence to prevent spanwise flow of fuel from fuel tanks to nacelle by wetting conduction on lower surface of wing.

The modes of fuel distribution to ignition sources in a crash revealed in this study indicate that, with regard to fuel spillage in liquid form, some crash-fire safety can be provided in airplane design. Any feature of wing design promotes crash-fire safety that effectively impedes the flow of wing-spilled fuel toward the engine nacelle and wheel well and provides for the drainage of this fuel at some relatively safe location at the trailing edge. Fuel running by wetting conduction likewise is effectively intercepted by any pronounced chordwise ridge or crease in the lower wing skin that serves as a drip fence (fig. 67). This drip fence is particularly desirable for intercepting fuel flowing by wetting conduction to the engine exhaust (see fig. 27). It is unfortunate with respect to crash-fire safety that such discontinuities in the wing surface do not appear in normal wing construction. Wing tanks currently under development, which do not burst under the hydraulic loading experienced in a crash, will obviously reduce the probability of crash fire by fuel spillage within the wing.

In crashes in which the wing tanks are pierced by posts or equivalent objects, however, fuel spillage will occur regardless of the material or construction of the fuel tank and the method of its support. Because the fuel mist generated when this type of fuel-system damage is imposed represents such a serious fire hazard, any measure that impedes the flow of the fuel through the tank breach lessens the fire danger. The fuel-flow impeding action desired should be equivalent to that which would be obtained if the fuel tanks were divided into small interconnected cells. In regard to the hazard of fuel-mist ignition, the greater the distance spanwise, aft, and down the fuel is stored with respect to the engine, the lower the likelihood of contact between fuel mist and ignition sources at the nacelle. The total storage of fuel in pod tanks suspended below the wing at a station 10 feet or more spanwise from the engine nacelle would lessen the likelihood of fuel-mist spread to the nacelle in a crash. Flow of the liquid fuel through the wing structure or by wetting conduction is avoided with this type of fuel storage.

The role that electrostatic charge accumulated on parts of a crash airplane plays in setting a fire, as revealed in this study, may explain the origin of some of the fires started in the wake of crashed airplanes on airport runways. Reports of such fires attributed them to friction sparks. Electrostatic discharge from debris trailing the airplane may represent another ignition means for these fires. Currently available paints that give some protection against electrostatic sparks may provide effective control of this fire hazard when applied to those lower members of the airplane likely to be separated in a crash. Because little electrostatic charge is accumulated by blowing dust when the relative humidity is above 70 percent, fire from this source is more likely to occur on clear dry days than on humid days. Crash landings on damp ground or ground well covered by green vegetation where little dust is raised around the skidding airplane should be relatively safe from ignition by dust-generated

electrostatic discharge. When sufficient time is available, light wetting down of an airfield site where a difficult landing is to be made is a good precautionary measure. Friction sparks of sufficient size and temperature to ignite fuel can be obtained from steel or magnesium airplane parts abraded under high contact pressure with stony ground or concrete paving. It is desirable in regard to crash fire that, whenever possible, these metals be in areas of the airplane not likely to scrape along the ground in a crash. The use of non-sparking materials for nuts, bolts, pitot tubes, drains, etc., that may be in ground contact in a crash would be advantageous.

These studies show how readily any combustibles spilled within the nacelle are ignited; therefore, it is desirable that the components of the fuel, lubricating, and hydraulic systems be located high in the nacelle where crash damage to these components is least likely. Oil coolers and fuel strainers often have vulnerable positions in the nacelle. Tubing containing combustibles should be arranged to accommodate the nacelle distortions produced in crash.

The action a pilot may take just previous to crash to lessen the likelihood of fire depends on the circumstances of the pending crash. Certainly, the less fuel carried by the airplane into the crash, the smaller the probability of fire. Experience gained in this crash-fire study indicates that in the event that a crash is likely, the engine fuel valve should be closed rather than the engine ignition turned off. There is no assurance that fuel flowing through an engine will not ignite if the engine ignition system is turned off. Hot cylinder-valve components or the exhaust-disposal system can provide the ignition that may result in a flame from the engine induction-system inlet or exhaust-system tail pipe. Evidence that such ignition does occur appeared in the crash previously described in which the engine switch was turned off 2 seconds before crash impact. Immediately after the ignition switches were turned off, a sequence of exhaust flames appeared, one of which is shown in figure 7 (c). At 2.2 seconds after impact the flames shown in figure 7 (b) were observed at the engine induction-system inlet. It appears more desirable, rather, to close the engine fuel valve before crash with the engine ignition system operative. By these arrangements, the fuel that dribbles into the engine burns in the normal way in the engine cylinder with less chance for producing undesirable exhaust or engine inlet flames. Several revolutions of the engine usually suffice to exhaust the fuel in the engine induction system, and, from then on, the engine ingests only air. If time permits after the available engine fuel is exhausted, engine ignition shut-off may provide an extra margin of safety. Modern aircraft ignition systems are so well protected by the engine parts that damage to the system while the engine is rotating and driving the magnetos does not appear very probable. When fuel shut-off cannot be achieved prior to crash, then turning off the engine ignition may be desirable in order that the propeller impact with the ground take place at reduced engine speed and power. In this way extensive de-

struction of the engine nacelle may be prevented, and a corresponding reduction in the number of exposed ignition sources may be realized. The likelihood of broken fuel and oil lines within the nacelle is also lessened.

The approach to a hazardous landing should be made with only those portions of the electric circuit energized that are required for airplane operation. In order to maintain the exhaust system below the ignition temperature of fuel and oil, the approach should be made at as low an engine power as possible consistent with other safety considerations and with the cowl flaps open wide. The hazard of fuel ignition by electrostatic or friction sparks is lessened if the landing can be made along a well-developed grass-covered strip parallel and adjacent to a paved runway. Greater damage to the airplane structure is sometimes obtained by landing on the turf with wheels up than on the runway because of the plowing and scooping of the ground by the deformed fuselage and nacelles. Midwing or high-wing airplanes with a solid fuselage belly structure unbroken by hatches of bomb-bays could land on turf without significant plowing. Freedom from friction and electrostatic sparks would be gained for these airplanes without increasing the likelihood of severe fuel spillage produced by the structural deformation that sometimes accompanies ground plowing. Airport crash-accident equipment can move to the airplane along the adjacent paved runway. The flight fire system should be discharged into the nacelle a few seconds after touchdown to provide a small additional margin of safety. Less damage to the nacelle will occur if the propellers are left in the unfeathered rather than in the feathered position.

In view of the effective protection provided by the water spray on the hot components of the exhaust system, the question naturally arises concerning the value of employing the fire-extinguishing agents, carried for flight fires, to reduce the likelihood of fire in a crash. Preliminary estimates indicate that the quantities of extinguishing agent carried in flight fire systems can provide a significant measure of protection in a crash, when properly employed. As in the case of the water spray, the fire-extinguishing agent should be applied directly to the hot metal parts by means of a distribution system comparable with that used in the water-spray system. In this way, cooling of the exhaust system by evaporation and decomposition of the agent is promoted, and effective inerting of the adjacent atmosphere is accomplished. The quantity of agent from the flight fire system would be adequate to inert the exhaust system for the time period that is most hazardous if a distribution system could be designed that would confine the agent to a layer around the collector ring approximately $\frac{1}{8}$ inch thick. Such a distribution system would probably have the general appearance of that employed by the water system. The extinguishing agent should be metered to give a continuous flow for 25 seconds from the moment of crash impact. The 25-second interval covers the hazardous period when the airplane is in motion following crash and fuel is spread by the methods discussed, and the brief period after the airplane comes to rest when the fuel mist is being cleared by the wind. Un-

less the fire-extinguishing agent has a sufficient heat of vaporization to cool the hot surfaces to safe temperatures, however, a residual ignition hazard exists with the fuel flowing through the airplane structure and by wetting conduction and accumulated fuel vapors. A modest amount of water, carried separately, discharged with the extinguishing agent through the same distribution system may provide the necessary additional cooling. Further study is required to establish effective combinations of extinguishing agent and water, or other liquids, that may be employed. Highly water-soluble materials, like salts and ammonia, or engine heat may be employed to keep the water in liquid form at low atmospheric temperatures. Because the experimental ignition-source inerting system proved so effective in preventing crash fires in this investigation, its further study and development for special airplane applications is desirable.

LEWIS FLIGHT PROPULSION LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

CLEVELAND, OHIO, August 1, 1952

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TABLE I.—FUEL ANALYSES

	Aviation gasoline, 100/130 grade	Low-volatility fuel
Distillation, °F		
Initial boiling point-----	104	295
Percentage evaporated:		
5	135	330
10	149	335
20	168	340
30	185	343
40	201	345
50	214	348
60	223	350
70	232	354
80	241	357
90	257	364
95	282	369
Final boiling point-----	330	378
Residue, percent-----	0.6	1.0
Loss, percent-----	0.9	0
Reid vapor pressure, lb/sq in.-----	6.1	0.1
Specific gravity at 60° F/60° F-----	0.700	0.781
Refractive index-----	1.3934	1.4374

TABLE II.—FUNCTIONING TIME OF INERTING SYSTEM

Ignition source	Time after crash impact ignitions observed, sec	Inerting-system compo- nents protecting igni- tion source	Function time of inerting- system component, sec
Hot surfaces:			
Exhaust system-----	1.3 .7 1.9 3.8	Water spray-----	0.19
Heat exchanger-----	12.1	Water spray-----	.19
Exhaust flame:			
Torching-----	4.1 1.3 2.0 3.5 1.9	CO ₂ and water and fuel shut-off	0.34
Exhaust gases-----	.1	CO ₂ and water and fuel shut-off	.34
Electrical system:			
Arcs-----	1.0	Electric shut-off---	0.10
Filaments-----	.6 .6	None-----	----
Induction-system flames.	2.2 3.5 7.7	CO ₂ -----	0.06
Chemical agents-----	4.4 3.5 3.5	All-----	----
Electrostatic sparks----	2.4	None-----	----

SYNOPSIS

A review of the literature and accident records on aircraft crash fires indicated the need for experimental work in this field with modern fuels and aircraft powered with reciprocating and turbine engines. Because human survival of crash impact is most likely in landing and take-off accidents, the study of the fire that often follows such accidents was assumed to be the most fruitful area of research in this field.

The information obtained in this study on the mechanism of the crash fire is intended to serve as a factual background on which improvement in airplane crashworthiness may be based. The crash-fire problem was studied by conducting full-scale crashes with twin-engine cargo aircraft provided by the U. S. Air Force (figs. 1 and 2). The aircraft were fully instrumented to record significant data in the crash and fire that follows regarding fuel spillage, combustible-vapor distribution in areas adjacent to potential ignition sources, locations and timing of electrical-system ignition sources, fire incidence and progression, temperatures and toxic-gas concentrations in personnel compartments, and the decelerations the several main components of the airplane undergo in the crash. High-speed motion pictures of the crash were obtained that revealed many of the details of the mechanism of crash fire. Several of the factors observed to be significant in the full-scale crash phase of this work were studied in detail under simulated crash circumstances. Only the work done on aircraft with reciprocating engines is sufficiently complete at this time to be reported herein.

The arrangements for conducting the full-scale crash study are shown in figure 4 (a). The monorail in the foreground guided the airplane, which was accelerated from rest under its own power, to the obstructions shown in the middle of the figure. Airplane speed upon impact with these obstructions ranged from 80 to 105 miles per hour. The abutments in the path of the landing wheels stripped the landing gear from the airplane. The height of the abutments was adjusted to permit the striking airplane propellers to produce extensive nacelle damage short of breaking out the engine. In this way the generation of ignition sources by the engine was promoted. Poles arranged to rip into the wing leading edges and wing fuel tanks about 5 feet outboard of the nacelles produced fuel spillage that was distributed extensively around the crashed airplane. The progressive damage to the airplane at the crash barrier is illustrated schematically in figures 4 (b) to (d). Each airplane carried 1050 gallons of gasoline or low-volatility fuel described in table I. The fuel was usually dyed red to improve its photography in color.

Inquiry into the mechanism of the crash fire centers on the answers to two questions: How and when do ignition sources appear in the airplane crash, and How does the fuel come into contact with the ignition sources? The ignition sources revealed in this study are those that are familiar from experience in normal airplane operations and other technical fields. These ignition sources are listed in table II (col-

umn 1) along with the time of their appearance after crash impact (column 2).

In these crashes the fuel was observed to spill in liquid form from broken fuel lines and tanks, as premixed fuel vapor and air from the damaged engine induction systems, and as fuel mist around the airplane when the spillage appears on the outside of the airplane while it is in motion. In the last case, the pressure and viscous forces of the air on the fuel rip it to mist that moves with air around the airplane. In the crash arrangements employed in this study, liquid and mist spillage occurred in every crash, and carbureted fuel spillage from the engine induction system in only a few cases. These latter instances, however, were sufficient to reveal how such spillage initiates a fire.

The fuel that spills to the open air through the breach in the leading edge of the wing can attain appreciable spanwise spread as mist forward of the leading edge of the wing and reach ignition sources located around the nacelle. As the airplane decelerates in the crash, the fuel opposite the breach in the tank has a speed that is greater than the existing airplane speed. The fuel surges forward through the breach in the tank and is atomized by the air. The air forces that atomize the fuel to mist also impart a spanwise velocity component to the fuel droplets. As this spanwise velocity component is acquired, the forward velocity declines. If the airplane is moving rapidly, it moves by the spreading pattern of fuel mist before the mist has an appreciable time to spread spanwise. If the airplane moves slowly, the fuel mist attains significant spanwise extension around the forward portions of the airplane, and contact with ignition sources at the nacelle is likely.

The combination of reduced airplane speed and high airplane deceleration represents the condition of airplane motion most hazardous with respect to fuel-mist ignition at the nacelle. A comparison of the small spanwise distribution of the fuel obtained when the airplane speed is high and the deceleration is low (fig. 17 (a)) with the large fuel-mist spanwise development forward of the wing leading edge obtained at low airplane speed and high deceleration (fig. 16) shows these effects clearly.

Because a few seconds are required for the airplane to slow from its high speed at crash impact, contact between the fuel mist and ignition sources of the nacelle occurs several seconds after impact. An example of this time delay required for fuel-mist contact is given in figure 12, which shows a series of exhaust flames issuing from the engine tail pipe during the airplane slide from the crash barrier to near rest, when ignition of the gasoline mist occurred at the engine tail pipe (fig. 12 (d)). Ignition of the gasoline mist by the hot exhaust-collector ring, involving a similar time delay, is shown in figure 38 (d).

The substitution of low-volatility fuel for gasoline does not materially reduce the likelihood of ignition when contact between the fuel mist and a potent ignition source occurs. The ignition of the mist of low-volatility fuel at the engine tail pipe is shown in figure 19. The large fuel-mist fire on the

left side of the airplane (fig. 19 (b)) followed the ignition $1\frac{1}{2}$ seconds earlier in figure 19 (a). These results indicate that fuels of low volatility will ignite and burn readily when dispersed as mist, and their adoption for reciprocating engines would not alone represent an acceptable solution to the crash-fire problem.

Because the fuel mists are air-borne, they persist in the air around the crashed airplane for a time that varies inversely with the wind speed (fig. 11). Fuel mists in ignitable concentrations seldom persist around the airplane for more than 17 seconds after the airplane comes to rest. If a tail wind sweeps the mist from the crash area, the likelihood of ignition is momentarily increased as the dense portions of the fuel mist are swept over the nacelle by the wind. Because of the large error inherent in making a visual estimate of the persistence time of the fuel mist in the neighborhood of the airplane, it was not possible to evaluate the effect of fuel volatility on this persistence time. Fuel vapors associated with the mist will move with the wind approximately as the smaller mist droplets and may have a persistence time a few seconds greater than the bulk mist.

Transition from fuel-mist spillage to spillage in liquid form in the open air takes place as the airplane slows to rest. Because of the high rate of air dilution provided by normal air movement in the open, combustible concentrations of gasoline vapors coming from the liquid gasoline on the ground appear only close to the fuel spillage in strata that lie close to the ground. Where protection from wind is obtained by vegetation, ground channels or ditches, or components of the crashed airplane, a marked increase exists in ignition hazard distance from the liquid spillage. In general, however, the likelihood of ignition of vapors from liquid pools of gasoline in the open air does not appear significant unless the ignition source lies within a few inches of the ground. Except for burning droplets of oil dripping from the nacelle, broken elements of the exhaust-disposal system falling to the ground, or friction sparks, the ignition sources can be expected to lie above this combustible layer of vapor.

Gasoline spilled within the enclosed cavities of the airplane, where low air-ventilation rates exist, can generate large volumes of combustible concentrations of vapor that can move through the channels provided by the airplane structure. Fuel in both liquid and vapor form can flow through these channels by gravity and achieve a considerable displacement away from the fuel-spillage point. Hot-gas ducts of the icing-protection systems or cabin air-conditioning, for example, lead to heat exchangers that are often at temperatures above the fuel-ignition temperature. In figure 25 (c) is shown the fire that follows the movement of wing-spilled fuel through the icing-system hot-air distribution duct that runs along the wing leading edge to the engine exhaust-gas heat exchanger on the engine tail pipe below the wing (fig. 25 (b)). Propagation of the fire into the wing along the path of fuel produced the wing explosion shown in figure 25 (c). Because of the time required for fuel to

move to the ignition source, this fire occurred 12 seconds after crash impact.

Ignition sources within the wing belong to the wing electrical system. Damage to this electrical system during crash impact can produce ignition sources close to the wing fuel spillage, and fire follows almost immediately. In figure 24 (c) is shown the fire produced by operating landing lights that were smashed and driven into the wings by the poles at the crash barrier. The wing fire appeared 0.60 second after crash impact.

Distribution of fuel in liquid form from the spillage point to remote areas can take place on the lower surfaces of inclined elements of the airplane structure. The liquid fuel wets and clings to these surfaces, such as the lower wing skin, and flows by gravity. This so-called "wetting conduction" of the fuel is shown in figure 27 on the wing and nacelle and in figure 26 on the elements of the wheel well and landing-gear strut. The red-dye trails left by the fuel show the wetted path of the fuel. Because of the wing dihedral, the fuel tends to move to the nacelle where many ignition sources are located, if the wing position is not altered from normal in the crash.

The fuel that is spilled as carbureted mixture of fuel and air from the damaged engine induction system is generally released in zones adjacent to the hot exhaust-disposal system and the electrical elements in the engine accessory sections. A variety of ignition sources are available, therefore, to ignite this fuel. The quantity of fuel in the engine induction system is too small to produce a serious fire unless the flash fire of this fuel extends to other fuel spillage. In figure 30 is shown the propagation of the engine induction-system fuel fire out of the nacelle to the fuel spillage from the wing.

In an effort to reveal mechanisms of crash-fire initiation that may have been obscured by the early fires set by the known ignition sources, an ignition-source inerting system was installed in nacelles of several aircraft crashed in this program. A schematic view of the inerting-system arrangements is given in figure 31. The four-component inerting system includes a fuel shut-off valve at the carburetor and fire wall that brings the engine to rest, a 3-pound charge of fire-extinguishing agent injected into the engine inlet to inert the engine interior, an electrical system cut-off switch on the battery and generator circuits to prevent the development of hot wires and electric arcs, and a water-spray system arranged to distribute water to all parts of the hot metal of the engine exhaust-disposal system. The steam generated on the hot metal surfaces of the exhaust system provided protection from fuel or oil ignition while the metal cooled to temperatures below the ignition point of fuel and oil. All components of the system were arranged to be actuated as soon after crash as possible.

The only new mechanisms of crash fire revealed in this phase of the work with the ignition-source inerting system installed in the airplane nacelles is shown in the crash illustrated in figure 44. The landing wheel and strut, which were stripped from the airplane at the crash barrier, followed in

the wake of the airplane sliding along the ground. In its passage through the soil dust and fuel mist in the wake of the airplane, the wheel and strut accumulated electrostatic charge that ignited the fuel behind the airplane when the wheel strut approached the ground and discharged (fig. 44 (b)). The fuel fire propagated through the fuel in the wake to the crashed airplane.

Five of the airplanes that were equipped with the inerting system and did not burn are shown in the photographs of figure 40, taken when the airplanes came to rest after crash. A sixth airplane, likewise equipped, which was subjected to a ground loop (fig. 41), did not burn either.

This report contains a section devoted to the description of the progress of an aircraft fire. Because the section on the progress of aircraft fires cannot be given in brief form without misinterpretation, its inclusion in this synopsis is not attempted.

The results of this work indicate that significant reductions in crash-fire hazard can be realized by any design measure that increases the forward and spanwise distance and the elevation of the engine with respect to the fuel storage. This trend in airplane component arrangement will reduce the likelihood of contact between fuel in mist form with the many ignition sources at the nacelle. Devices or design features that act to intercept spilled fuel flowing within the

channels provided by the airplane structure are also valuable. Provisions for drainage of the intercepted fuel to spillage points in the open air away from the nacelles would enhance the effectiveness of these arrangements. Location of landing lights away from chordwise positions in front of the fuel storage is indicated as well. Preliminary data suggest the value of employing paints that have a reduced tendency to accumulate electrostatic charge by friction with dust and fuel on the parts of the airplane likely to be detached in a crash and trail the airplane.

In an approach to an indicated crash landing, the pilot should de-energize all of the electrical system not required for landing. Engine operation that provides the coolest exhaust-disposal-system temperature should be practiced. Just before touchdown, the fuel flow to the engines should be cut off to allow the engine to purge itself with air before crash impact.

In view of the effectiveness of the experimental ignition-source inerting system in preventing crash fires experienced in this study, the desirability of further study of this system for special airplane applications is indicated. A combined system for both crash and flight fires may be particularly attractive, because protection for crash and flight fires may prove to be possible without serious increases in weight over the flight fire system alone.

PHOTOGRAPHIC INVESTIGATION OF COMBUSTION IN A TWO-DIMENSIONAL TRANSPARENT ROCKET ENGINE¹

By DONALD R. BELLMAN, JACK C. HUMPHREY, and THEODORE MALE

SUMMARY

Motion pictures at camera speeds up to 3000 frames per second were taken of the combustion of liquid oxygen and a hydrocarbon fuel in a transparent-sided rocket engine. This 100-pound-thrust engine consisted basically of metal contour and injection plates clamped between two plastic sheets. This design provided an essentially two-dimensional engine with a view of the entire combustion chamber. Various injectors, parallel jets and impinging jets, were used in order to study varied types of combustion.

Oxygen-hydrocarbon propellant combinations provided sufficient combustion luminosity to use direct photographic methods. An increase in the number of holes of the parallel-jet injectors tended to increase the uniformity of combustion, but all the injectors showed nonuniformity of combustion. Turbulence projections increased the apparent mixing and circulation of propellants. Variation of ignition delay, apparently from improper mixing, caused starting explosions, midrun explosions, and other short-duration transient phenomena. Both low- and high-frequency oscillations were recorded during the runs, and some of the oscillations corresponded to the resonant frequencies of the chamber. Photographic measurements of gas velocities provided data from which chamber combustion temperatures could be calculated. Patterns in the plastic windows provided additional information regarding gas-flow paths and qualitative indications of temperature variations at the walls.

INTRODUCTION

Rocket combustion involves a large heat release per unit volume, and the achievement of efficient, steady combustion with a minimum heat loss depends upon carefully designed injector and combustion-chamber configurations. Such designs are best devised on the basis of a knowledge of injector characteristics and combustion gas-flow patterns during transient and steady-state operations within the actual rocket engine. High-speed, motion-picture photography should provide a comprehensive view of flow patterns and time-space data that would be difficult to obtain by other methods. In 1947, a technique was originated at the NACA Lewis laboratory to investigate this supposition in a photographic study of rocket combustion. The technique involves the use of a two-dimensional engine with transparent plastic windows and has three principal advantages: The combustion chamber may be fabricated in uniform thickness; large windows allow the entire chamber to be photographed; and the low-melting material used as windows cannot become luminous and obscure the combustion pattern.

This technique subsequently stimulated photographic research along similar lines (ref. 1) at other laboratories, as well as at the Lewis laboratory. Slit photography, in which a continuous strip camera is used, has also yielded valuable

information about rocket combustion processes (refs. 2 and 3).

The summary presented herein of the photographic-technique research conducted at the Lewis laboratory during 1947-50 includes: (1) a description of the photographic technique; (2) an evaluation of the combustion photographs resulting from the use of three types of injector configuration; and (3) an analysis of some combustion irregularities, such as irregular starts, combustion oscillations; and explosions. Combustion patterns of eight injectors were observed in a 100-pound-thrust, two-dimensional engine in which liquid oxygen and hydrocarbons were used as propellants. Motion pictures of the combustion were taken at camera speeds up to 3000 frames per second. The most important photographs or photographic sequences illustrating the observed phenomena are included in this report. Results of this study are most favorably observed, however, by projection of the film.

APPARATUS AND PROCEDURE

Engines.—Two essentially two-dimensional engines were used in the investigation; the engines differed only in that the first had a transparent-sided nozzle and the second a copper nozzle (see fig. 1). The copper nozzle was used because very rapid erosion of the window in the nozzle of the first engine lowered the chamber pressure and changed the flow in the nozzle.

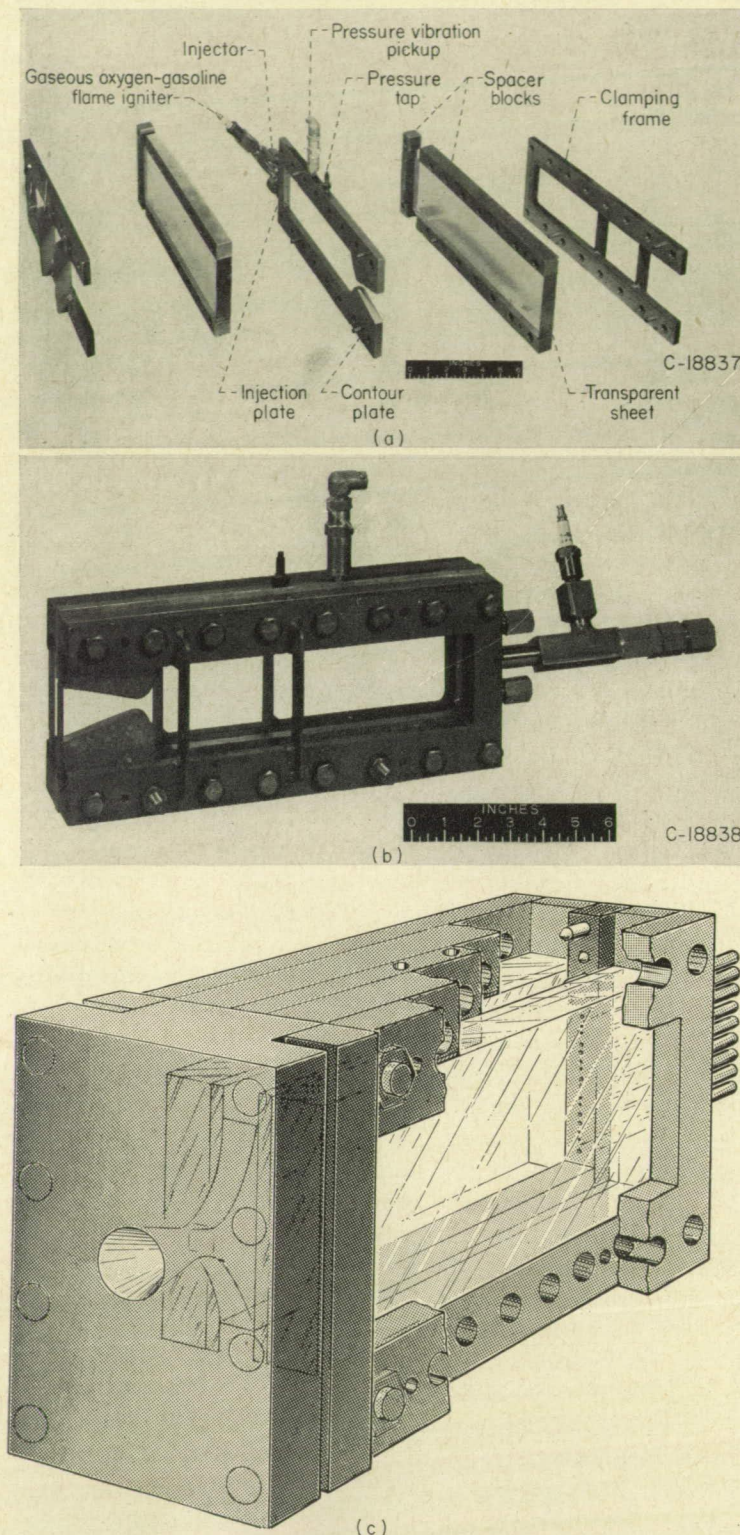
Both engines consisted of contour plates and an injection plate with sheets of transparent material held to both sides of each contour plate by means of suitable spacer blocks and clamping frames. The copper contour plates were chromium plated on the inner surface to reduce erosion by the combustion gases. The transparent sides were polymethylmethacrylate, a transparent plastic that has excellent resistance to thermal shock. The low melting point of the plastic prevented the walls from becoming luminous and thereby obscuring the combustion pattern. The low melting point of the plastic, however, limited the useful duration of a run to a few seconds.

Each side of the rocket consisted of a replaceable inner sheet of ¼-inch-thick plastic backed by a ¾-inch-thick supporting sheet of plastic to withstand the combustion pressures. The transparent sheets were sealed to the contour plates with asbestos sheet packing.

The combustion chamber including the nozzle had a uniform thickness of ½ inch and was designed to produce 100 pounds of thrust with a combustion-chamber pressure of 300 pounds per square inch absolute. The characteristic length (combustion-chamber volume divided by throat area) was about 60 inches. The rocket engine was mounted at a downward angle of 45° to prevent the accumulation of propellants in the chamber in the event of an ignition failure.

¹ Supersedes NACA RM E8F01, "Photographic Study of Combustion in a Rocket Engine. Injection," by Donald R. Bellman and Jack C. Humphrey, 1948.

I—Variation in Combustion of Liquid Oxygen and Gasoline with Seven Methods of Propellant



(a) View of unassembled engine showing transparent nozzle contour plate equipped with single impinging jets.
 (b) View of assembled engine showing transparent nozzle contour plate equipped with single impinging jets.
 (c) Cutaway view of engine with solid-copper nozzle and equipped with 16-parallel-jet injector.

FIGURE 1.—Transparent-sided rocket engine.

A combustion pressure tap and a pressure vibration pickup were installed through one of the contour plates at about the midpoint of the combustion chamber.

The eight injectors are diagrammatically shown in figures 2 and 3. These may be classed as impinging jets, impinging

jets with turbulence projections, and parallel jets (shower-head). The propellants were ignited either by a gaseous oxygen-alcohol flame, which in turn was ignited by a spark plug, or by a gunpowder squib which was fired into the chamber. For runs 1 to 12, the igniter was located at the center of the injector; for runs 13 to 15, the igniter was mounted perpendicular to the motor axis and $2\frac{1}{4}$ inches downstream of the injector.

Propellant systems.—A schematic diagram of the propellant system used with the first engine in runs 1 to 12 is shown in figure 4. The gasoline tank II was pressurized to 20 pounds per square inch by nitrogen from regulator FF in order to minimize cavitation in the inlet to the pumps. From the tank, the gasoline flowed through rotameter O to two, four-cylinder, positive-displacement pumps DD, which were coupled to a single motor EE. The eight pumping strokes were uniformly staggered to produce a minimum flow variation. From the pump, the gasoline passed through a series of hydraulic resistances BB to eliminate pulsations. The gasoline then flowed either through propellant control valve Z and into combustion chamber V or through relief valve CC and back into the supply tank. During a run, a pressure of about 400 pounds per square inch was required to send the fuel into the combustion chamber, and the relief valve was set at a pressure of 900 pounds per square inch so that no fuel was bypassed when propellant control valve Z was open.

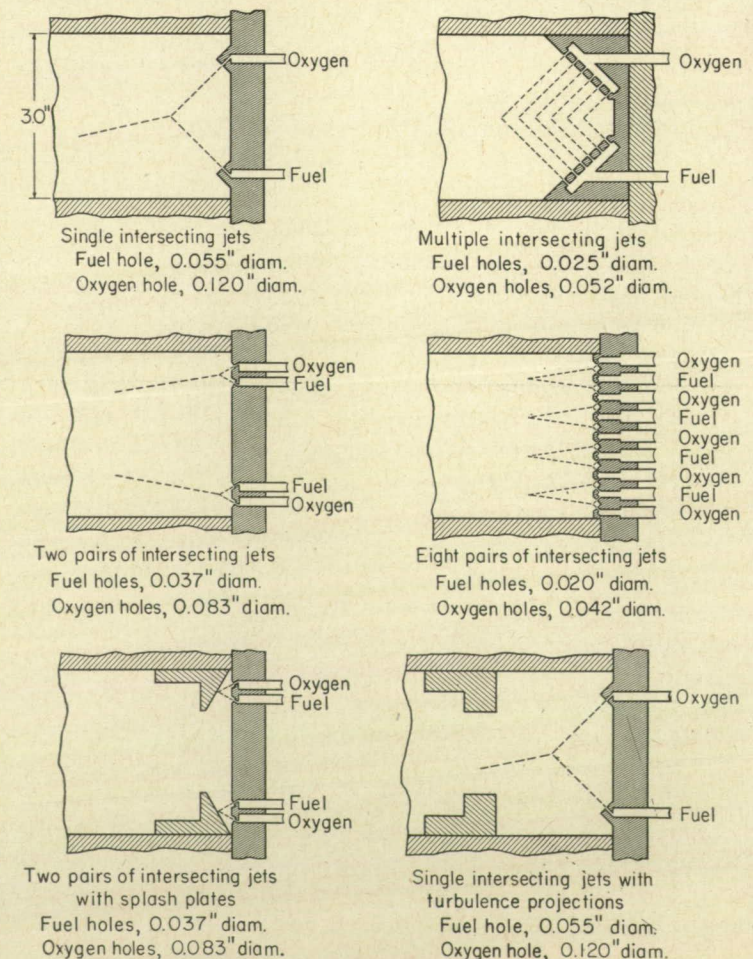


FIGURE 2.—Schematic views of impinging-jet injectors used with two-dimensional, transparent-sided rocket engine.

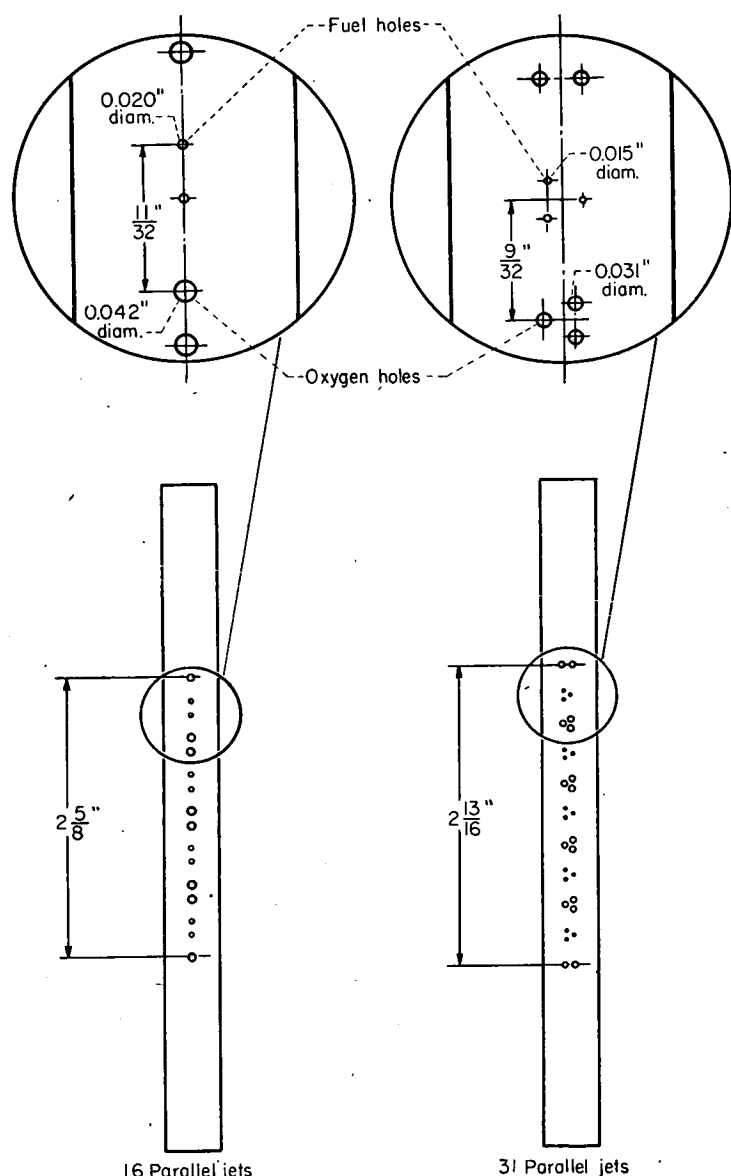
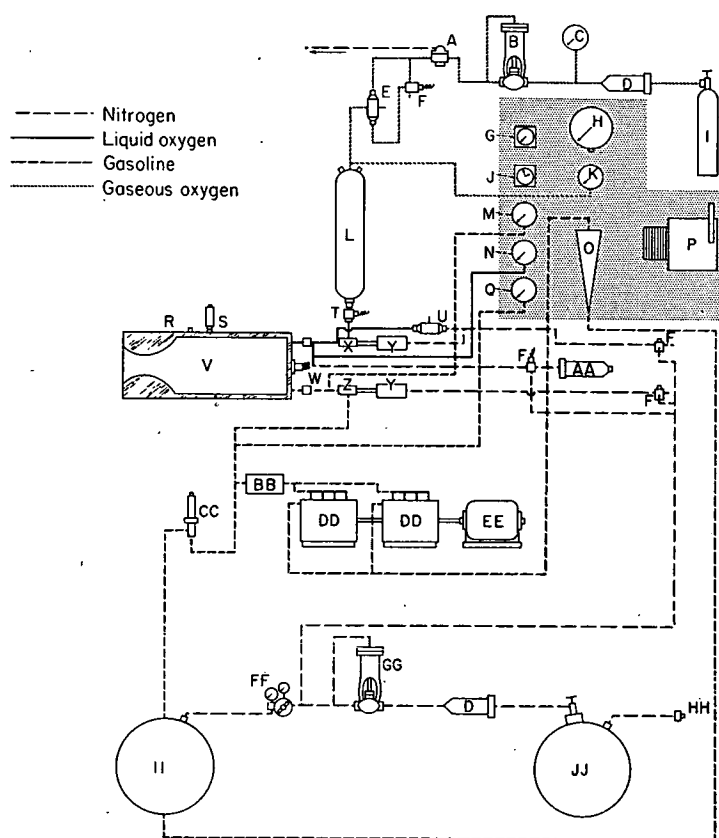


FIGURE 3.—Front view of parallel-jet injectors used with two-dimensional, transparent-sided rocket engine.

The liquid oxygen was stored in a 4-gallon, stainless-steel tank L located close to the combustion chamber. The tank was pressurized to 400 pounds per square inch by means of gaseous oxygen from pressure-reducing valves A and B. From the tank, the liquid oxygen flowed through a vane-type flowmeter T and propellant control valve X to combustion chamber V.

In order to stop the combustion quickly after each run, flushing the liquid-oxygen line with nitrogen was necessary. This flushing was accomplished by pressurizing tank AA with nitrogen during the run by means of three-way solenoid valve F. At the conclusion of each run, the three-way solenoid valve sent the contents of tank AA into the liquid-oxygen line close to the injector, which forced any oxygen remaining in the line out through valve U.

In the portion of this investigation with the engine of figure 1 (c) (runs 13 to 15), a modified system was used. The fuel flow was maintained by means of a gear pump driven at a constant predetermined speed; the flow rate was measured with a rotameter in the pump suction line. Before each



- A Secondary oxygen pressure regulator
- B Primary oxygen pressure regulator
- C Oxygen-supply pressure gage
- D Filter
- E Three-way air-operated valve
- F Three-way solenoid valve
- G Liquid-oxygen flowmeter dial
- H Combustion-chamber pressure gage
- I Gaseous-oxygen supply tank
- J Clock
- K Liquid-oxygen-tank pressure gage
- L Liquid-oxygen tank
- M Gasoline-nozzle pressure gage
- N Liquid-oxygen-nozzle pressure gage
- O Rotameter
- P Data camera
- Q Gasoline-pump pressure gage
- R Pressure tap
- S Pressure vibration pickup
- T Liquid-oxygen flowmeter element
- U Two-way air-operated valve
- V Combustion chamber
- W Squib
- X Propellant control valve—liquid oxygen
- Y Actuating air cylinder
- Z Propellant control valve—gasoline
- AA Small nitrogen pressure tank
- BB Hydraulic resistance
- CC Relief valve
- DD Gasoline pumps
- EE Electric motor
- FF Secondary nitrogen pressure regulator
- GG Primary nitrogen pressure regulator
- HH Nitrogen filling connection
- II Gasoline tank
- JJ Nitrogen supply tank

Shaded area shows instruments recorded by data camera.

FIGURE 4.—Diagrammatic sketch of propellant and control system for transparent-sided rocket engine.

run, full fuel flow was established and circulated back to the supply tank through a high-pressure relief valve.

The principal difference between this system and the gasoline-oxygen system involved the use of a positive-displacement pump that maintained a constant liquid-oxygen flow over a wide variation of injection pressure. The pump consisted of a cylinder having sufficient volume to hold all the liquid oxygen required for about 5 seconds of operation at the 100-pound-thrust level. A standard hydraulic piston actuated a second piston to force the oxygen from the cylinder, which was kept in a large Dewar flask containing liquid nitrogen. A commercial variable-delivery hydraulic pump maintained the hydraulic pressure to the actuator. Liquid-oxygen flow was determined by measuring the time displacement of the calibrated cylinder.

Instrumentation.—Thrust and chamber pressure of the engines were measured in addition to flow rates. Thrust was determined by using a bar spring equipped with strain gages connected in a bridge circuit to a self-balancing recording potentiometer. Chamber pressure was measured by photographing a Bourdon tube gage three times per second; the rate of change of pressure was measured by an engine knock indicator of the magnetostriction type, which was connected to an oscilloscope, and recorded by a moving-film camera. Other auxiliary measurements made are shown in figure 4.

The camera for studying combustion patterns was a 16-millimeter commercial motion-picture camera having a maximum speed of 3000 frames per second; an electronic-wave generator with a constant frequency of 1000 cycles per second marked the film every 1/1000 second.

OPERATING CONDITIONS AND PROCEDURE

Commercial liquid oxygen and 62-octane unleaded gasoline (runs 1 to 12) or oxygen and heptane (runs 13 to 15) were used. For a combustion-chamber pressure of 300 pounds per square inch absolute, the theoretical performance of oxygen with these fuels is as follows:

	Gasoline	Heptane
Maximum specific impulse for equilibrium expansion, lb-sec/lb.	261	263
Oxidant-fuel weight ratio for maximum specific impulse	2.4	2.4
Combustion temperature for maximum specific impulse, °R	6270	6180

Before a run, the flows were preset in an attempt to obtain an oxidant-fuel weight ratio of 2.4. All the firing operations, including shutoff, were automatically performed upon closing of a single switch. The following schedule was used: Liquid-oxygen flow was started and a 15-second delay allowed the oxygen lines and injector to cool and reach the proper flow rates; at 15 seconds, the high-speed camera was started and the squib ignited; at 16 seconds, the fuel was admitted and ignition took place. As soon as all the film in the camera was exposed, the propellant flow was stopped and the liquid-oxygen line was flushed with nitrogen.

Since the camera film length allowed an exposure time of less than 3 seconds, the run was scheduled to last about 1½ seconds. After each run, the inner sheets of the transparent sides and the squib were replaced.

GENERAL COMBUSTION PHENOMENA

The combustion-phenomena discussion is subsequently broken down into the various injector configurations, plastic-window patterns, performance, and temperature calculations. In a later section, selected photographs show irregular combustion phenomena of starting characteristics, explosions, and oscillations.

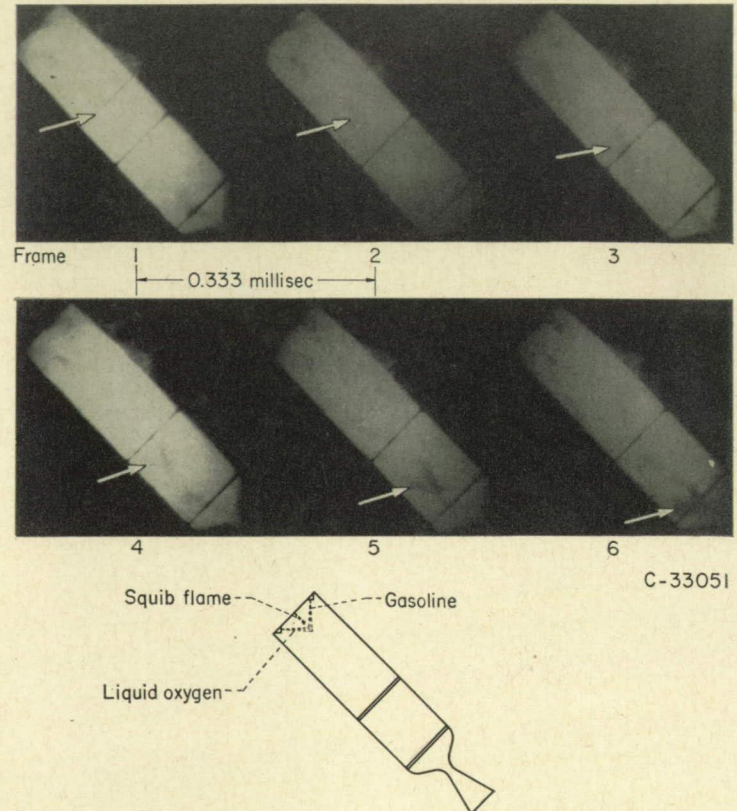
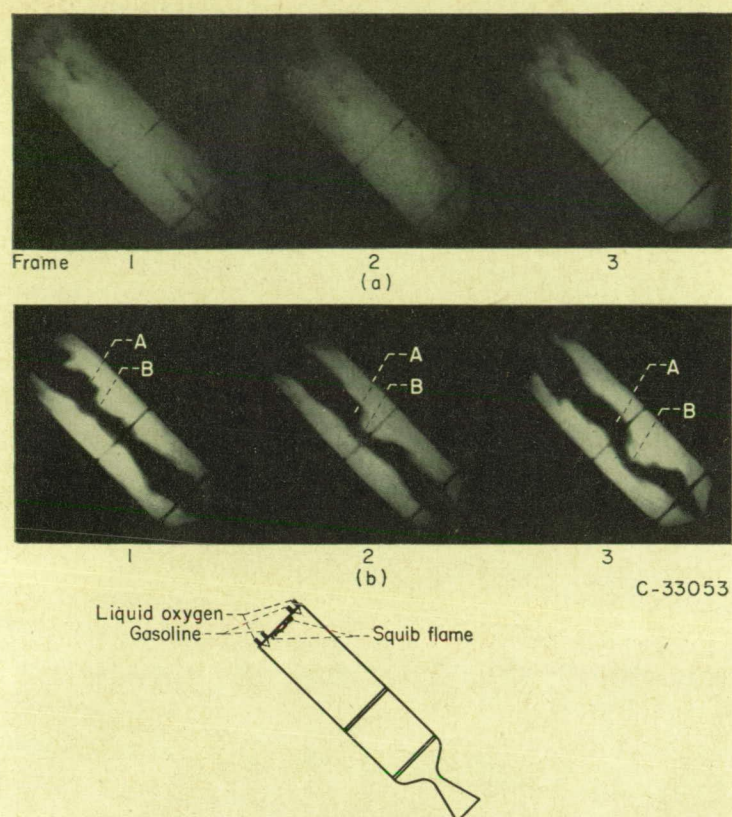


FIGURE 5.—Combustion of liquid oxygen and gasoline in transparent-sided rocket engine with single intersecting jets, and with motion of nonluminous pocket (arrows) shown. Run 1; camera speed, 3000 frames per second; thrust, 75 pounds.

COMBUSTION PATTERNS WITH VARIOUS INJECTORS

Intersecting jets.—Combustion with four combinations of intersecting-jet injectors is shown in figures 5 to 9. All the figures show some nonuniform light emission, and pockets of nonluminous gases appeared randomly through all the photographs. Some of the pockets retained their identity as they flowed through the chamber, as indicated by the arrows of figure 5. With two widely spaced pairs of intersecting jets, the resulting photographs often show, as in figure 6 (b), a dark region in the vicinity of the fuel jets and a center core of nonluminous gas that moved through the chamber as an undulating core. It might appear at first inspection that lateral movement existed to cause the undulation. However, closer inspection of the original photographs shows that dark areas arose in the injector fuel area and the luminous lobes emerged from the side areas; but the dark area, A in figure 6 (b), and the luminous areas, B in figure 6 (b), passed through the chamber almost undeviated from the longitudinal path originating near the injector. The wavy core was apparently made up of pockets of non-reacting propellant mixtures that were displaced laterally and were so close together that they were joined.



(a) Run 11; camera speed, 2380 frames per second; thrust, 73 pounds.
 (b) Run 12; camera speed, 2780 frames per second; thrust, 65 pounds.

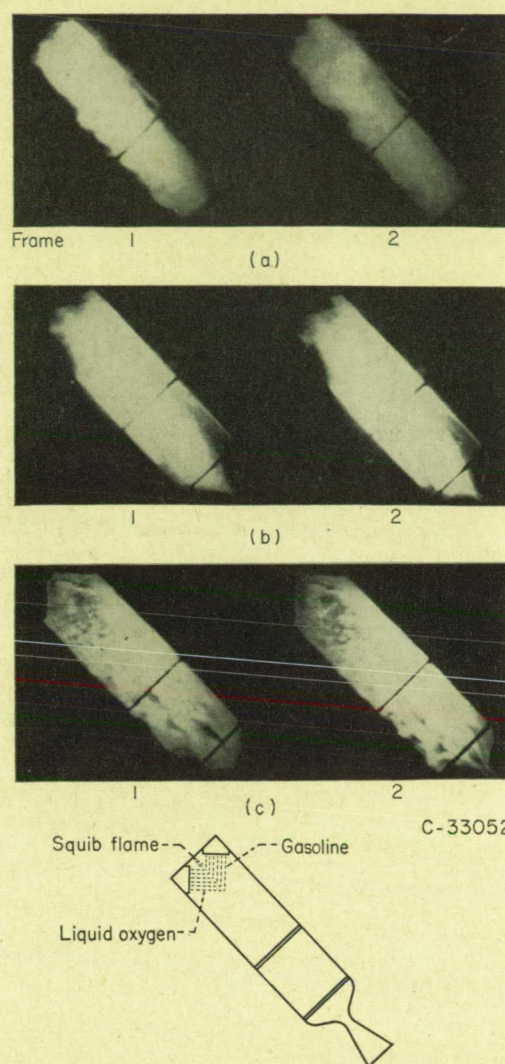
FIGURE 6.—Combustion of liquid oxygen and gasoline in transparent-sided rocket engine with two sets of single intersecting jets.

The luminous gas outside the engine (figs. 5 and 7) was a reflection of combustion light by external oxygen vapor. Figure 8 shows two pairs of intersecting jets with projections mounted as splash plates; areas of nonluminous gas existed in the injector area, and two dark streaks often appeared to stream from the tips of the splash plates.

Intersecting jets with turbulence projections.—Photographs of intersecting jets with the turbulence projections mounted about $3\frac{3}{4}$ inches downstream of the injector faces are shown in figures 10 and 11. The projections created additional turbulence and circulation of propellants and tended to form pockets of nonluminous gases downstream of the projections.

Parallel jets.—Combustion with 16 and 31 parallel jets is shown in figures 12 and 13, respectively. Both of these injectors showed a striated pattern of smooth stream flow with varying intensity of luminosity across the stream lines. An increase in the subdivision of the propellant streams tended to increase the uniformity of luminosity.

A major problem of rocket combustion, poor propellant mixing, is illustrated by the 16-parallel-jet injector (fig. 14). The injector was located at the left of the photograph in figure 14 (a) and propellant flow was from left to right and out of the nozzle not shown in the photograph. From the midchamber area to the nozzle, five bright zones A of illumination or apparent combustion are separated by four dark zones B of noncombustion. About halfway between the midchamber area and the injector, zones of intense brightness C alternate with zones of lesser brightness D.

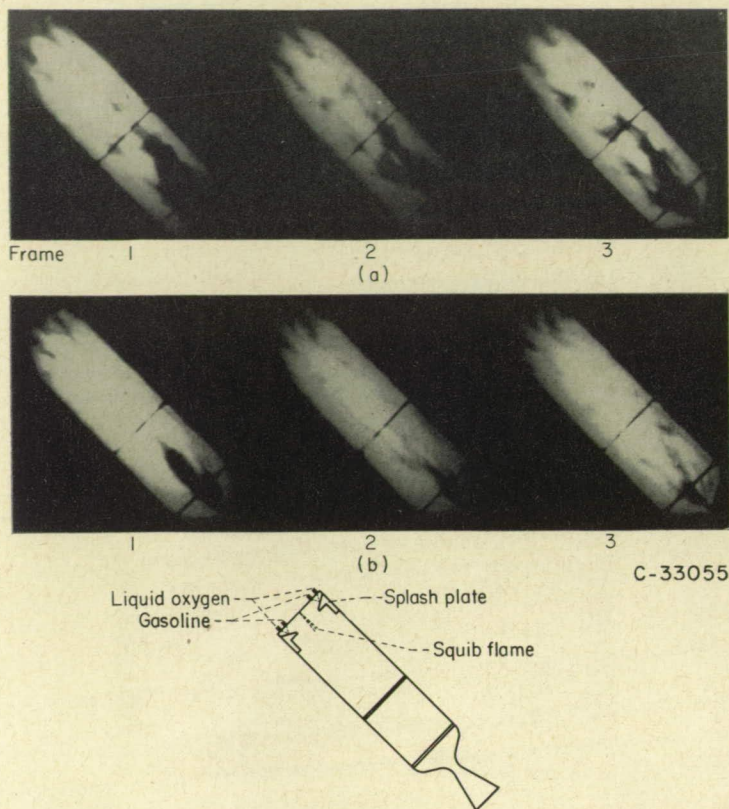


(a) Run 3; camera speed, 2835 frames per second; thrust, 112 pounds.
 (b) Run 4; camera speed, 2810 frames per second; thrust, 84 pounds.
 (c) Run 5; camera speed, 2675 frames per second; thrust, 91 pounds.

FIGURE 7.—Combustion of liquid oxygen and gasoline in transparent-sided rocket engine with multiple intersecting jets.

The over-all bright area near the injector may be explained as follows: With most liquid sprays, the friction of gases causes small fragments of liquid to peel off the liquid spray and form a mist. It is reasonable to assume that some mist mixing would occur as a result of interaction of the propellant pairs as well as interaction between the two different propellants.

Primary mixing of the propellants would occur as the streams individually expand and finally touch. Run 14 (fig. 12), when examined under motion-picture projection, shows practically no longitudinal turbulence except near the injector and very little lateral turbulence. A possible schematic spray configuration is shown in figure 14 (b), which illustrates how the propellant streams can mix at some station downstream and react. This mixing and reacting cause illumination and exposure of the photographic film as shown by figure 14 (a). The gray regions represent the unburned propellants, and the white regions represent the combustion where interaction takes place. The schematic illustration is based upon the actual injector in which there were five oxidant zones and four fuel zones resulting in eight



(a) Run 9; camera speed, 2860 frames per second; thrust, approximately 100 pounds.

(b) Run 10; camera speed, 2940 frames per second; thrust, 115 pounds.

FIGURE 8.—Combustion of liquid oxygen and gasoline in transparent-sided rocket engine with two pairs of intersecting jets with splash plates.

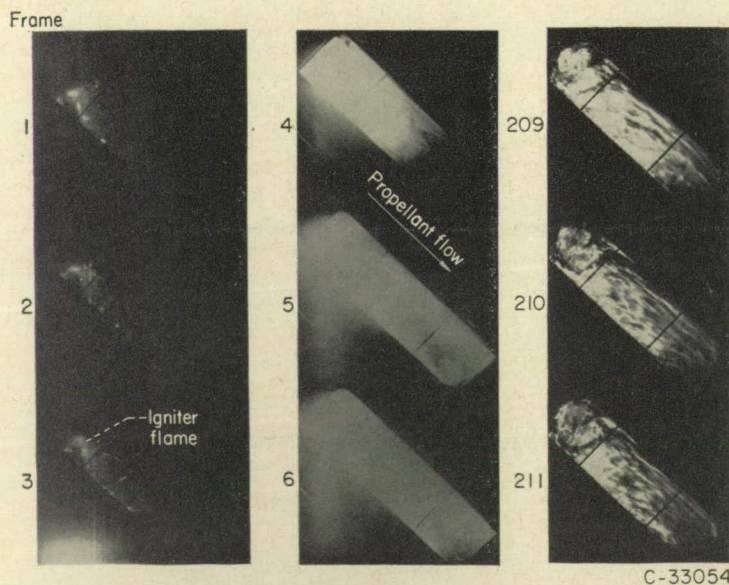


FIGURE 9.—Start of combustion and steady-state burning of liquid oxygen and gasoline in transparent-sided rocket engine with multiple intersecting jets. Run 13; camera speed at frame 4, 857 frames per second; camera speed at frame 210, 1091 frames per second.

mixing zones. Figure 14 (a) shows eight zones of high intensity, and the high-intensity zones can actually be perceived in the original photographs as close as 1 inch from the injector. The mist and earlier turbulent mixing could cause the apparent blurred brightness near the injector,

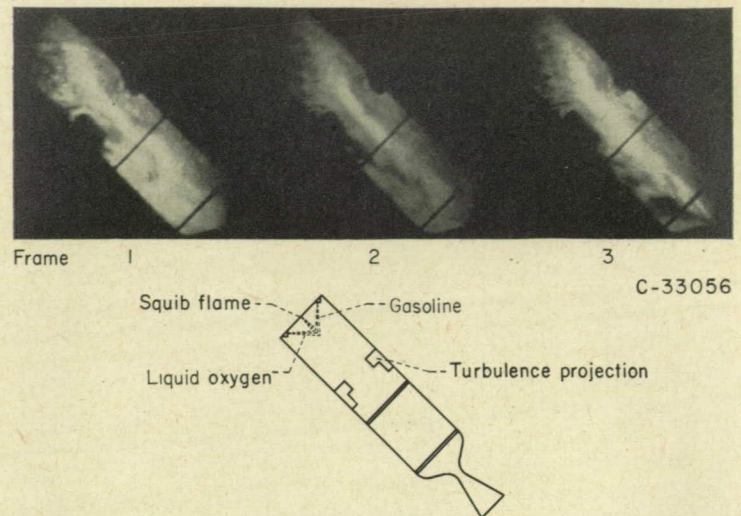
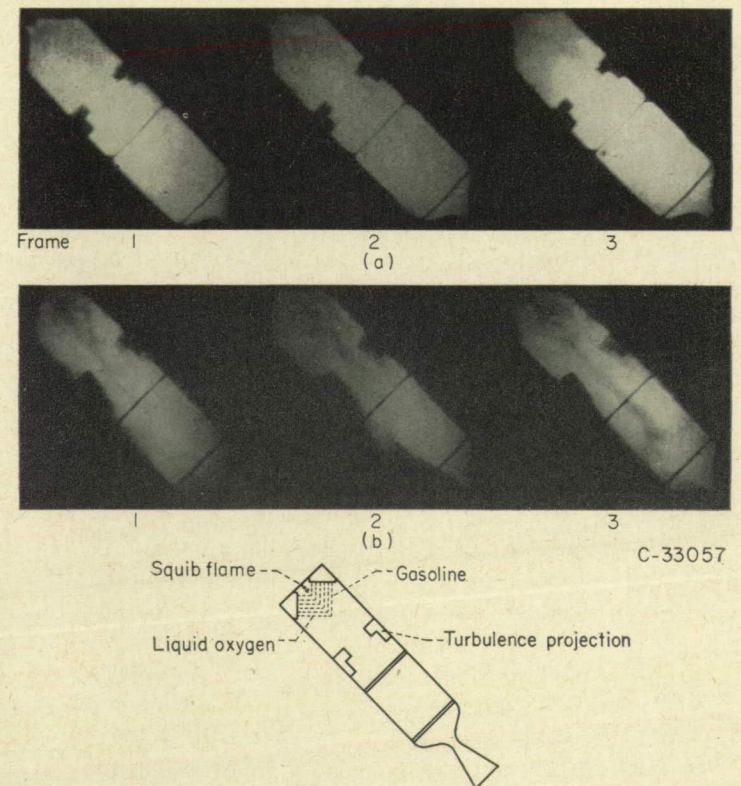


FIGURE 10.—Combustion of liquid oxygen and gasoline in transparent-sided rocket engine with single intersecting jets with turbulence projections. Run 2; camera speed, 3165 frames per second; thrust, 96 pounds.



(a) Run 7; camera speed, 1935 frames per second; thrust, 113 pounds.

(b) Run 8; camera speed, 2490 frames per second; thrust, 72 pounds.

FIGURE 11.—Combustion of liquid oxygen and gasoline in transparent-sided rocket engine with multiple intersecting jets with turbulence projections.

but the core or main stream of the jets ultimately shows downstream the basic pattern of the injector.

The magnitude of the small-scale lateral turbulence increased as the propellants moved toward the nozzle, but one basic pattern emerged: The number of dark zones corresponded exactly to the number of injector fuel zones and probably were caused by highly concentrated unreacted fuel. Also, the broad zones of combustion had a somewhat indistinct darkened core which probably was highly concen-

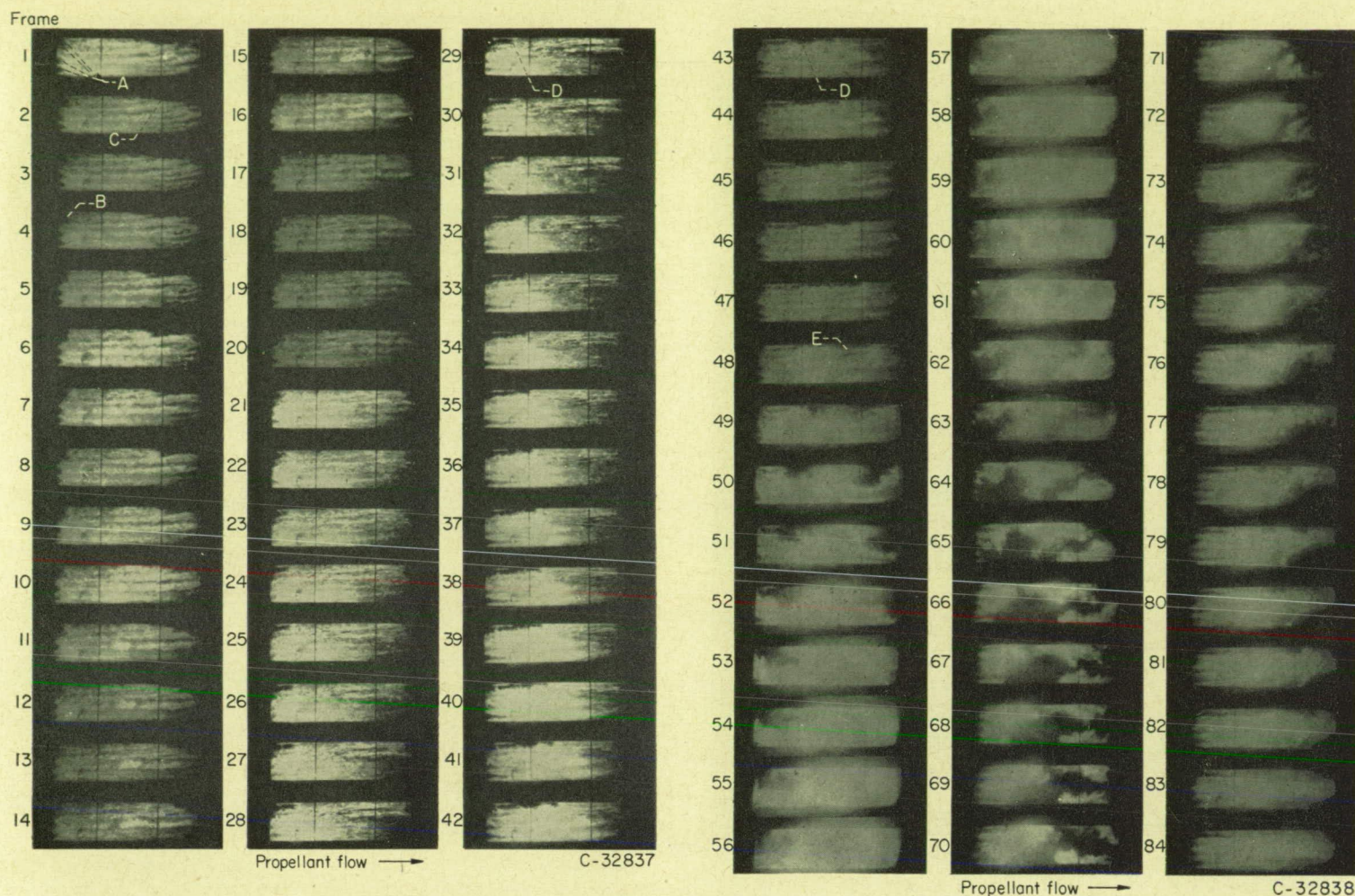


FIGURE 12.—Combustion of liquid oxygen and heptane in transparent-sided rocket engine with 16-parallel-jet injector. Run 14; thrust, approximately 79 pounds; camera speed at frame 1, 2500 frames per second; camera speed at frame 84, 2525 frames per second.

trated unused oxygen. The sides of the bright zones were obvious continuations of the eight high-intensity zones mentioned earlier. Reference 1 also reports that striations occurred in some systems, that the dark areas corresponded to fuel zones, and that the bright areas corresponded to oxidant zones. This over-all appearance leads to the postulation that fuel-rich mixtures of oxygen-hydrocarbons do not burn as well as oxidant-rich mixtures. It is not known what these mixtures were, but it seems reasonable from observation of the smooth streaming flow that the center cores of both the oxidant and the fuel zones were richer than the normal off-stoichiometric mixtures usually considered. The preceding postulation is further substantiated by an explosion, mentioned later in **IRREGULAR COMBUSTION PHENOMENA**, in which an unburned slug appears to come from a fuel zone.

BURN PATTERNS IN PLASTIC WINDOWS

Most of the photographs of burn patterns in plastic windows show a great deal of stratification at the nozzle throat. The plastic was burned to a lesser depth behind the dark streaks than behind the bright areas. This effect is shown in figure 15, in which a photograph of the combustion during one of the runs is compared with the plastic window from that run. On the bottom side of the nozzle throat, the

flame burned completely through the $\frac{1}{4}$ -inch plate and slightly into the backing plate; on the top side, the plastic did not burn through. The combustion photograph shows that the top side, which has the less luminous area, was not burned so deeply. Thus, nonuniform flow through the nozzle with the possibility of asymmetric thrust forces is indicated.

The luminosity of the gases decreased very rapidly as the gases passed into the divergent section of the nozzle probably because of a decrease in both temperature and density. Even though the flame outside the rocket engine appeared very bright to the eye, it was not bright enough to record on the high-speed photographs. Figure 15 further shows that in the divergent section of the nozzle a pattern in which the Mach lines are discernible was burned into the plastic windows.

The plastic windows alone form an interesting record of the run, for they show variations in the injector action by burning to a greater depth in line with some of the propellant streams. If such burning occurs more on one plate than on the other, poor alignment of the propellant streams is indicated. The sides also show some of the flow variations, particularly around the turbulence projections and in the nozzle region. In most cases, the nozzle throats were burned deeper at the edges than in the middle of the plastic surface,

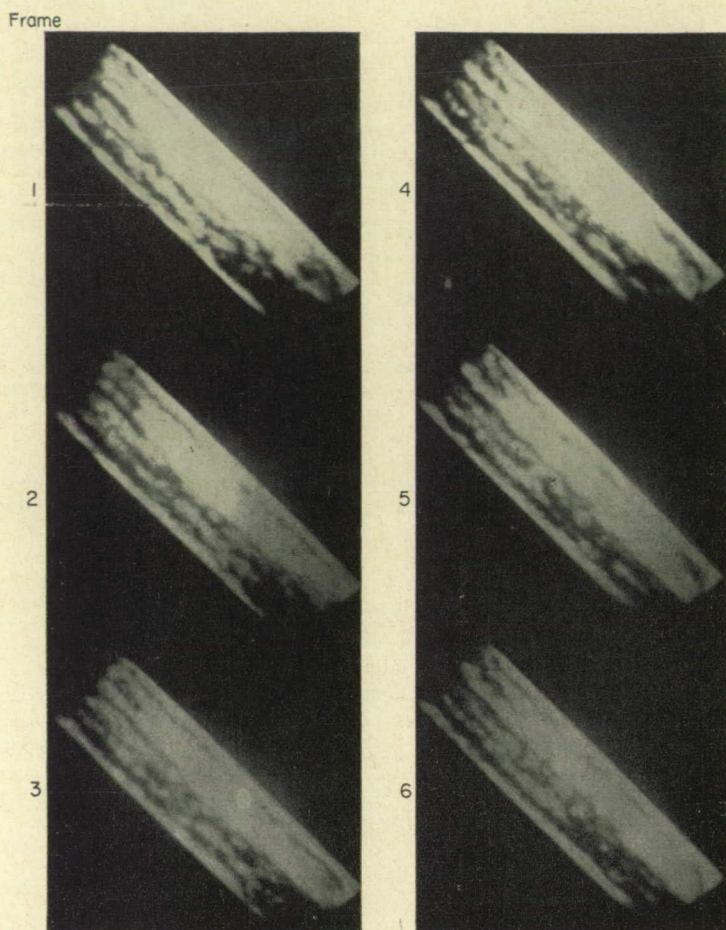


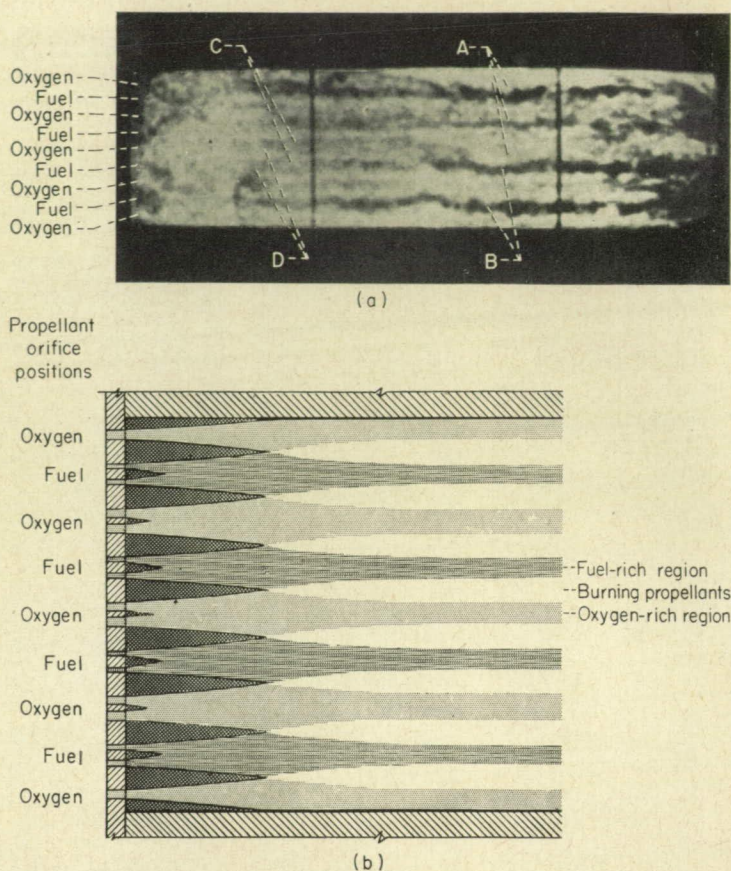
FIGURE 13.—Full chamber view of combustion of liquid oxygen and heptane in transparent-sided rocket engine with 31-parallel-jet injector. Run 15; thrust, 134 pounds; camera speed, 1845 frames per second.

which indicated either higher velocities or higher temperatures at the edges. In the divergent section of the nozzle, the Mach angle can be measured from the pattern formed there and gives an additional check on the calculated conditions. In the runs presented herein, however, conditions varied too greatly from the beginning to the end of the run for such a measurement to have much value.

PERFORMANCE

Thrust measurement.—It was found that engine performance data could not be measured with sufficient accuracy to make comparisons among the various injector configurations used. The short operating time, the lack of rapid-response instrumentation (except rate of change of pressure), the intermittent recording of chamber pressure and flow, the consumption of plastic during a run (up to 20 percent of the total flow), and the rapid erosion of the throat of the plastic nozzle were all contributing factors in preventing accurate performance measurements. Table I shows approximate performance data for use as a guide in studying the combustion photographs. Since thrust was measured continuously, it was used as a qualitative indication of performance, although this use can be misleading when there are large variations in chamber pressure and flows.

Temperature calculations.—For many of the runs, measurement of the velocity of the propellant gases was possible in the combustion chamber from observations of the move-



(a) Enlargement of frame from run 14, 63 seconds after start of combustion. Approximately one-tenth actual size of engine window and magnified 10 times from negative.

(b) Approximately two-thirds actual size of 16-parallel-jet injector. FIGURE 14.—Schematic illustration of propellant spray configuration showing primary zones of mixing (light areas).

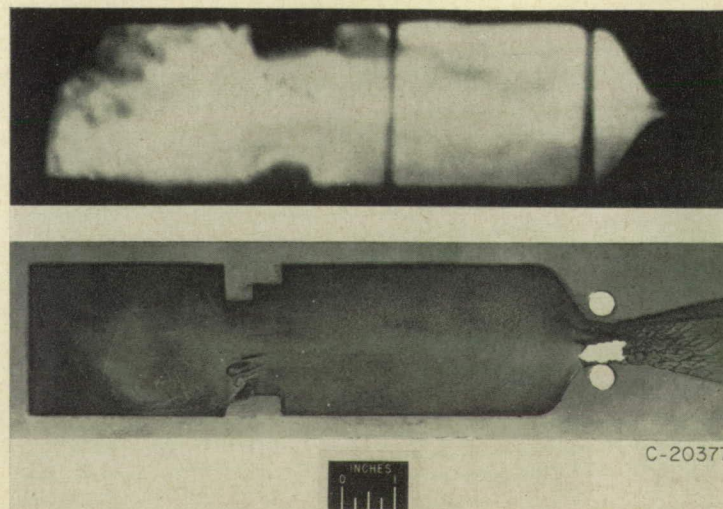
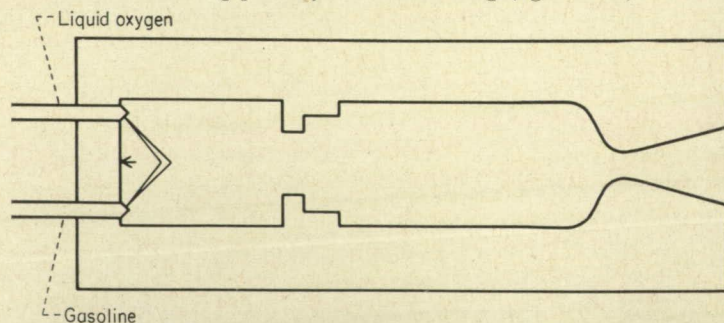


FIGURE 15.—Plastic plate after run of 1.9 seconds and combustion taken through same plate. Run 2.

TABLE I.—APPROXIMATE PERFORMANCE DATA

Run	Injector	Hydrocarbon propellant	Fuel flow, lb/sec	Oxidant flow, lb/sec	Thrust, lb	Combustion- chamber pressure, P_c , lb/sq in. abs
Engine 1 (plastic nozzle)						
1	Single intersecting jets...	Gasoline.....	---	---	75	-----
2	Single intersecting jets with turbulence pro- jections.	Gasoline.....	0.16	0.33	96	180
3	Multiple intersecting jets.	Gasoline.....	0.17	0.50	112	223
4			.17	.34	84	230
5			.17	.35	91	215
6			.16	.45	110	242
7	Multiple intersecting jets with turbulence projections.	Gasoline.....	0.18	0.35	113	-----
8			.18	.23	72	173
9	2 Pairs of intersecting jets with splash plates.	Gasoline.....	0.18	0.24	100	196
10			.18	.42	115	246
11	2 Pairs of intersecting jets.	Gasoline.....	0.18	0.24	73	176
12			.16	.21	65	164
Engine 2 (copper nozzle)						
13	8 Pairs of intersecting jets.	Gasoline.....	0.15	0.39	90	155
14	16 Parallel jets.....	Heptane.....	0.14	0.32	79	>200
15	31 Parallel jets.....	Heptane.....	0.16	0.34	>100	>300

ment of a nonluminous pocket. With the use of this value, together with measured values for combustion-chamber pressure, weight flow, and area in accordance with the following equation, the ratio of the combustion temperature to the molecular weight of the products T_c/m can be determined:

$$\frac{T_c}{m} = \frac{V_c P_c A_c}{WR} \quad (1)$$

where

T_c combustion-chamber temperature
 m average molecular weight of combustion gases
 V_c gas velocity in combustion chamber
 P_c combustion-chamber pressure
 A_c combustion-chamber cross-sectional area
 W total propellant weight flow
 R universal gas constant

In most of these experiments, the cross-sectional area and the combustion-chamber pressure continually changed and there was insufficient timing correlation to determine values for the pressure and the area at the exact time for which the velocity was measured. These measurement uncertainties allow only an approximate measurement of T_c/m by the use of equation (1). The ratio T_c/m can be determined from other measured quantities; thus

$$\frac{T_c}{m} = \left(V_c \frac{A_c}{A_t} \right)^2 \left[\frac{M^2(\gamma-1)+2}{\gamma+1} \right]^{-\frac{(\gamma+1)}{(\gamma-1)}} \frac{1}{\gamma g R} \quad (2)$$

where

A_t nozzle-throat area
 M combustion-chamber Mach number
 γ ratio of specific heats
 g acceleration due to gravity

Equation (2) can be derived from Bernoulli's equation by assuming an isentropic process. The ratio of specific heats cannot be experimentally measured, but the use of the

theoretical value of the ratio of specific heats introduces only a small error. For most rocket engines, the effect of combustion-chamber Mach number is so small that it can be neglected. In the first 12 experiments reported, the nozzle-throat area varied greatly during the runs, and consequently the use of equation (2) to determine T_c/m is also subject to measurement uncertainties. The following table presents the ratio of the measured values of T_c/m as determined by the use of equations (1) and (2) to the theoretical values of T_c/m :

Run	Injector system	Theoretical T_c/m	Measured T_c/m	
			Theoretical	Equation
			Equation (1)	Equation (2)
5	Multiple intersecting jets without turbulence projections.....	250	0.46	0.39
11	Two pairs of intersecting jets with splash plates.....	260	0.71	0.65
15	16 Parallel jets.....	267	0.59	0.58
16	31 Parallel jets.....	257	0.72	0.48

The areas used for the values shown in the table were those for the start of the run for the transparent nozzle (runs 4 and 10). Other values used were taken at the maximum combustion-chamber pressure. During each run with the transparent nozzle, the nozzle-throat area changed by a factor of 2; hence the measurement uncertainties for the variables of equation (1) were probably less than those for equation (2). By the use of equation (1), the results show that the measured T_c/m varies from about 40 to about 70 percent of the theoretical T_c/m .

IRREGULAR COMBUSTION PHENOMENA

STARTING CHARACTERISTICS

Some starts were smooth; that is, the combustion spread in a steady manner throughout the chamber. Other runs were erratic; at the start of these runs, a small flame appeared in the vicinity of the intersection of the propellant jets. This flame swelled and diminished in an irregular and uneven manner during the time it was extending to fill the combustion chamber. In many runs, about 1/2 second was required for the flame to fill the entire chamber and to become relatively stable. Some times the start consisted of a series of explosions, as shown by the sequence of photographs in figure 16. Explosive starts frequently caused bursting of the windows; this same phenomenon could damage metal chambers.

EXPLOSIONS

Starting explosions can exist in a mild form, as illustrated in figure 16, or as more disastrous starts, which are difficult to record and to evaluate. In figure 16 the burning appears to be normally smooth during the first cycle, which peaks at frame 5. However, the burning in the second and third cycles, which peak at frames 27 and 51, respectively, was explosive. The spatial luminosity change between frames 25 and 26 is faster than the movement of propellants through the same space (normally in the range of 100 to 300 ft/sec). Therefore, it seems reasonable to conclude that an unburned mixture was conditioned to become explosive and that some unrecorded disturbances triggered the combustion

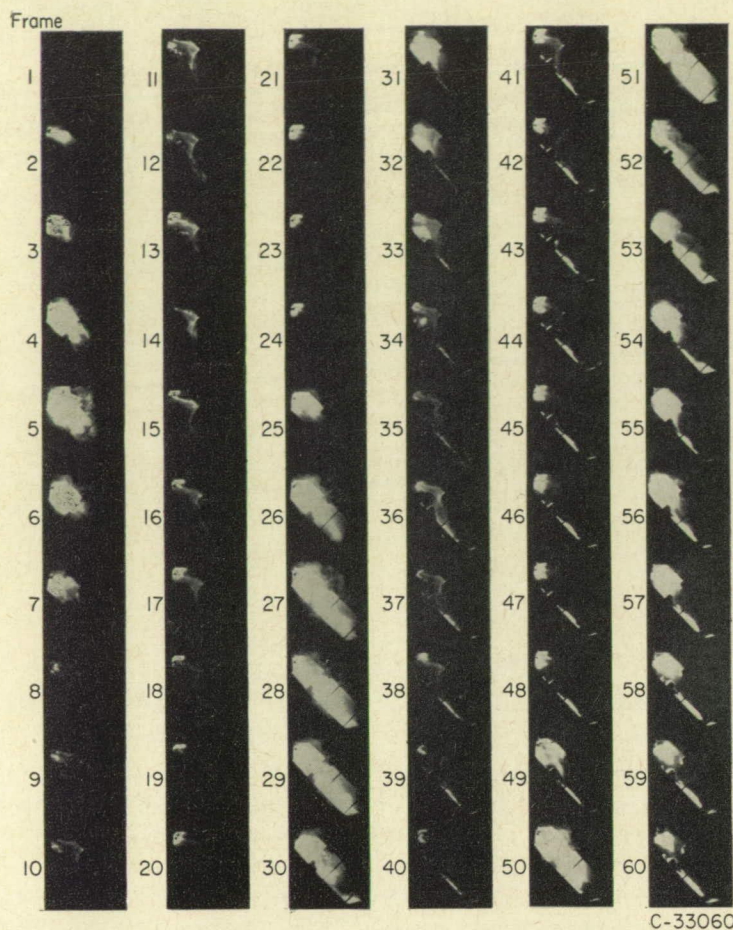


FIGURE 16.—Starting explosions of liquid oxygen and gasoline in transparent-sided rocket engine with multiple intersecting jets with turbulence projections. Run 8; camera speed at frame 1, 602 frames per second; camera speed at frame 60, 695 frames per second.

almost simultaneously throughout all parts of the chamber or triggered the mixture by a wave compression.

Another sequence in which the start was explosive is shown in figure 9, frames 1 to 6. Since only the igniter flame appears in frame 3 and the entire chamber is luminous by frame 5, no upper limit of spatial velocity can be computed; but a flame velocity of about 500 feet per second or greater is indicated because the flame fills the first 7 inches of the chamber of frame 4.

Explosions from previously mentioned improper mixing may occur during the middle of a run, as shown by figure 12. Projection of the sequence showed that the start was cyclic but that smooth streaming combustion existed by the first frame of figure 12. Four dark areas, A in frame 1, project into the chamber area and pulsate longitudinally, sometimes becoming quite small and other times quite extensive. One or both of the top dark areas indicate malfunctioning of the injector, as evidenced by propellant circulation in that area and shown by B in frame 4. Momentary malfunction may or may not be serious; for example, a slug of apparently unburned propellant passed harmlessly through the chamber (indicated in frame 2 of fig. 12 by C).

Soon afterwards another slug of unburned propellant started through the chamber as shown, for example, by D in frames 29 and 43, but this slug exploded when it reached position E in frame 48. Apparently, the second slug had

sufficient ignition delay to become explosive. The combustion mechanism or rate appeared to be changed, as evidenced by the increased luminosity. The injector flow was disturbed by the explosion but quickly reestablished itself (frames 61 to 63); combustion rapidly followed the reestablishment of the injector flow, the explosion gases were swept from the chamber, and the system appeared to be normal by frame 84, except for occasional slugs of unburned propellant that did not explode (typical of that shown by F in frame 78). The sequence showing the midrun explosion indicates a condition that also may cause destructive damage to engines.

From the photographs it appears that the variation of ignition lag from improper mixing indicates a propellant mixture ratio that can cause starting explosions, midrun explosions, and other transient phenomena.

COMBUSTION OSCILLATIONS

Low-frequency oscillations, commonly known as "chugging," are illustrated in the sequence of figure 17. The phenomenon of chugging was first encountered and perceived through the technique of transparent chamber photography. Figure 17 shows $2\frac{1}{2}$ cycles of chugging with a frequency of approximately 98 cycles per second. Another sequence, not

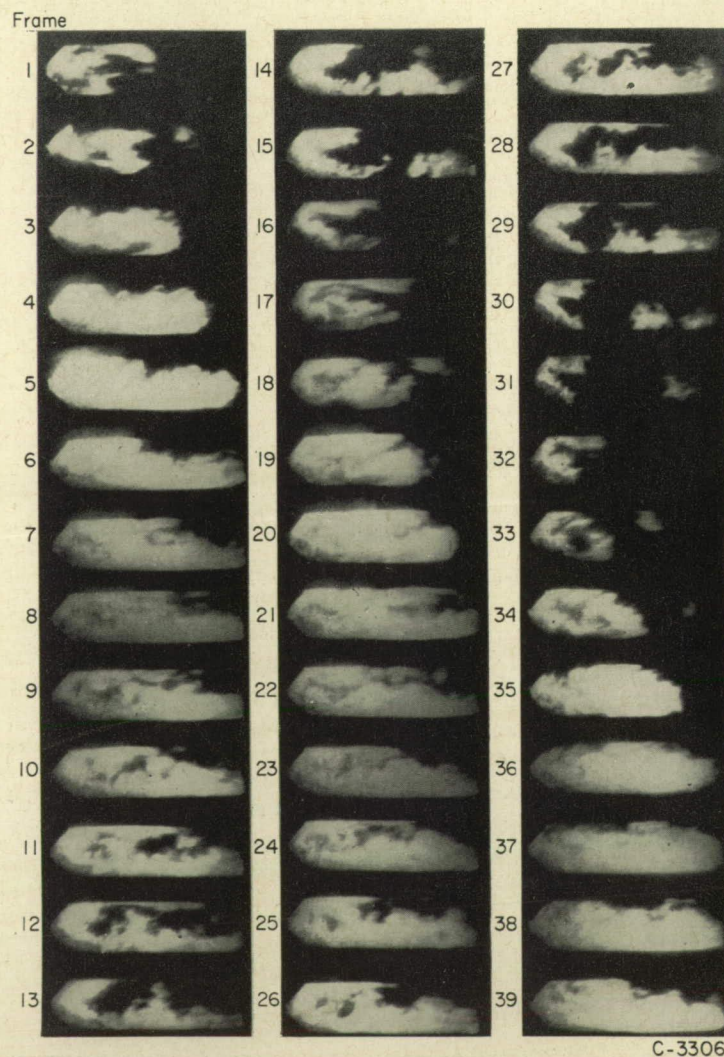


FIGURE 17.—Low-frequency oscillations or "chugging." Chugging frequency, 98 cycles per second; camera speed, 1500 frames per second.

illustrated, was analyzed and found to give a frequency of 103 cycles per second. Chugging is characterized by arrested flow within the chamber and sometimes by flow reversal. Analysis of propellant flows and chamber combustion parameters (ref. 4) has revealed a logical explanation of chugging. The analysis shows chugging to be an out-of-phase coupling between the fluid flow of the propellant feed system and the combustion process in the rocket chamber. Low propellant pressure drop across the injector aggravates chugging, which usually can be alleviated by use of an injector design that incorporates the use of higher pressure drop. During the start of many of the runs, a distinct oscillation of 100 to 300 cycles per second existed. This oscillation corresponded to a brightening and dimming throughout the entire combustion chamber, as illustrated in figure 14.

High-frequency oscillations have also been encountered. When T_c/m is known, computation of the speed of sound through the combustion gases and thus of the various natural frequencies of the chamber is possible. By use of an average value of T_c/m as obtained by means of equation (1) and a theoretical value for the ratio of specific heats, the following

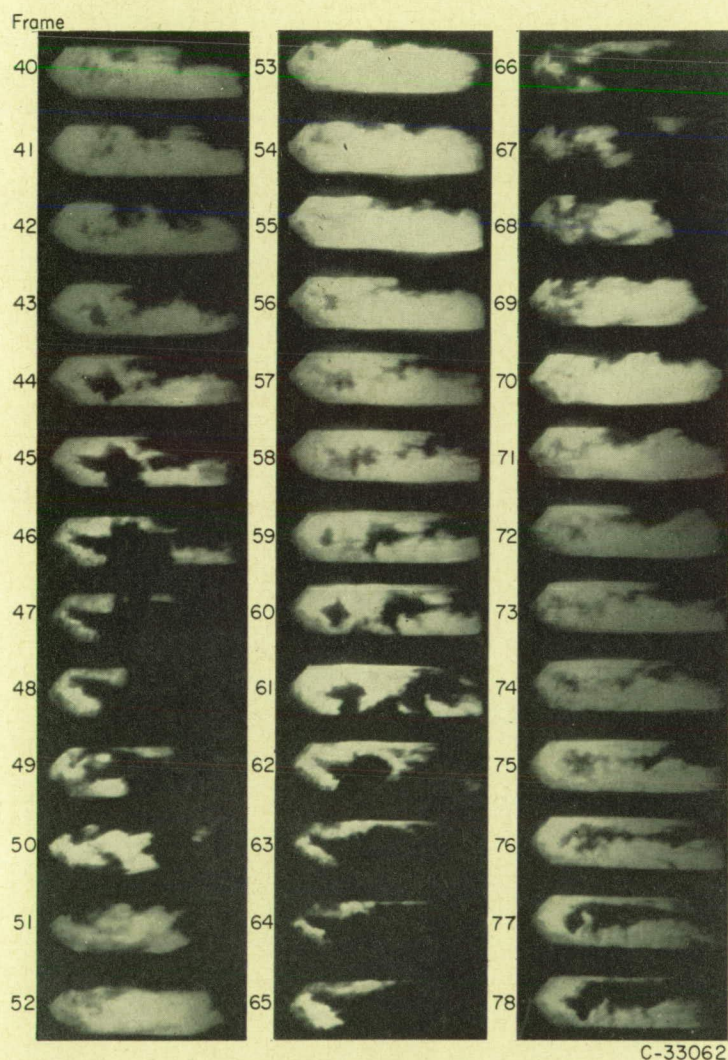
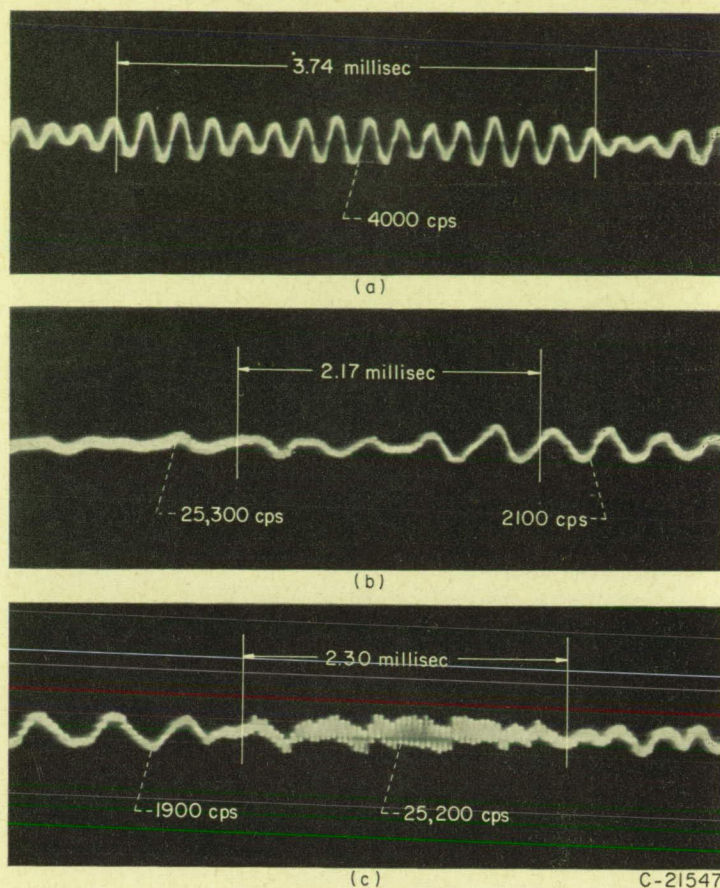


FIGURE 17—Concluded. Low-frequency oscillations or "chugging." Chugging frequency, 98 cycles per second; camera speed, 1500 frames per second.



- (a) Before start of run.
(b) After start of run.
(c) Near end of run.

FIGURE 18.—Combustion-chamber-pressure oscillation records with a magnetostriiction-type pressure pickup. Run 6.

values were determined for the natural frequencies of the combustion chamber:

Dimension	Frequency, cps
Length.....	1800
Width.....	6000
Thickness.....	33,000

In the combustion-chamber-pressure oscillation records, two vibration frequencies were prominent throughout the runs: one at approximately 1900 cycles per second and the other at approximately 25,000 cycles per second. These oscillations, therefore, approximately correspond to the natural frequencies for the length and thickness of the chamber. Samples of the various oscillation records are shown in figure 18. Occasionally a frequency of about 4000 cycles per second was recorded, and this frequency was found to be the natural frequency of the thrust stand.

SUMMARY OF RESULTS

A technique was developed that used high-speed, motion-picture photography to study combustion in a 100-pound-thrust, transparent-sided, two-dimensional rocket engine. Oxygen and either gasoline or heptane were introduced into the chamber through a variety of injectors, and motion pictures of the burning process along with simultaneous recordings of operational and oscillatory data were taken.

The following results were obtained:

1. All the injectors showed nonuniformity of combustion.
2. Turbulence projections used with the injectors increased the apparent mixing and circulation of propellants.
3. An increase in the number of holes of the parallel-jet injectors tended to increase the uniformity of combustion.
4. Plastic-window patterns provided additional information regarding gas-flow paths and qualitative indications of surface-temperature variations.
5. The ratio of the combustion temperature to the molecular weight of the products, calculated from gas velocities obtained from the combustion photographs, was computed to be 40 to 70 percent of the theoretical value.
6. Variation of ignition delay from improper mixing indicates a propellant mixture ratio that can cause starting explosions, midrun explosions, and other short-duration transient phenomena.
7. Low-frequency oscillations of approximately 100 cycles per second were recorded during some runs. Some of the starts gave frequencies of 100 to 300 cycles per second.
8. Combustion-chamber oscillations were recorded that

approximately corresponded to the resonant frequencies of the length and thickness of the combustion chamber.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, *May 22, 1953.*

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REPORT 1135

EQUATIONS, TABLES, AND CHARTS FOR COMPRESSIBLE FLOW ¹

By AMES RESEARCH STAFF

SUMMARY

This report, which is a revision and extension of NACA TN 1428, presents a compilation of equations, tables, and charts useful in the analysis of high-speed flow of a compressible fluid. The equations provide relations for continuous one-dimensional flow, normal and oblique shock waves, and Prandtl-Meyer expansions for both perfect and imperfect gases. The tables present useful dimensionless ratios for continuous one-dimensional flow and for normal shock waves as functions of Mach number for air considered as a perfect gas. One series of charts presents the characteristics of the flow of air (considered a perfect gas) for oblique shock waves and for cones in a supersonic air stream. A second series shows the effects of caloric imperfections on continuous one-dimensional flow and on the flow through normal and oblique shock waves.

INTRODUCTION

The practical analysis of compressible flow involves frequent application of a few basic results. A convenient compilation of equations, tables, and charts embodying these results is therefore of great assistance in both research and design. The present report makes one of the first such compilations (ref. 1) more readily available in a revised and extended form. The revisions include a complete rewriting of the lists of equations, as well as the correction of certain typographical errors which appeared in the earlier work. The extensions are primarily in the directions dictated by increasing flight speeds, that is, to higher Mach numbers and to higher temperatures with the accompanying gaseous imperfections.

Compilations similar to those of reference 1 have been given in other publications, as, for example, references 2 through 6. These references have been utilized in extending the tables and charts to higher values of the Mach number. The extension to imperfect gases is based on the relations presented in references 7 and 8.

SYMBOLS AND NOTATION

PRIMARY SYMBOLS

a	speed of sound
A	cross-sectional area of stream tube or channel

C_N	normal-force coefficient for cones, $\frac{\text{normal force}}{q_\infty S_b}$
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
h	enthalpy per unit mass, $u + pv$
l	characteristic reference length
M	Mach number, $\frac{V}{a}$
p	pressure ²
q	dynamic pressure, $\rho V^2/2$
q	heat added per unit mass
R	gas constant
R	Reynolds number, $\frac{\rho V l}{\mu}$
S_b	base area of cone
s	entropy per unit mass
T	absolute temperature ²
u	internal energy per unit mass
v	specific volume, $\frac{1}{\rho}$
u, v	velocity components parallel and perpendicular respectively, to free-stream flow direction
$\tilde{u}, \tilde{v}$	velocity components normal and tangential, respectively, to oblique shock wave
V	speed of flow
V_m	maximum speed obtainable by expanding to zero absolute temperature
w	external work performed per unit mass
α	angle of attack
β	$\sqrt{ M^2 - 1 }$
γ	ratio of specific heats, $\frac{c_p}{c_v}$
δ	angle of flow deflection across an oblique shock wave
θ	shock-wave angle measured from upstream flow direction
Θ	molecular vibrational-energy constant
μ	Mach angle, $\sin^{-1} \frac{1}{M}$
μ	absolute viscosity
ν	Prandtl-Meyer angle (angle through which a supersonic stream is turned to expand from $M=1$ to $M>1$)

¹ Supersedes NACA TN 1428, "Notes and Tables for Use in the Analysis of Supersonic Flow" by the Staff of the Ames 1- by 3-foot Supersonic Wind-Tunnel Section, 1947.

² When used without subscripts, p , ρ , and T denote static pressure, static density, and static temperature, respectively.

ξ	pressure ratio across a shock wave, $\frac{p_2}{p_1}$
ρ	mass density ²
σ	semivertex angle of cone

SUBSCRIPTS

∞	free-stream conditions
1	conditions just upstream of a shock wave
2	conditions just downstream of a shock wave
t	total conditions (i. e., conditions that would exist if the gas were brought to rest isentropically)
*	critical conditions (i. e., conditions where the local speed is equal to the local speed of sound)
c	conditions on the surface of a cone
r	reference (or datum) values
perf	quantity evaluated for a gas which is both thermally and calorically perfect
therm perf	quantity evaluated for a gas which is thermally perfect but calorically imperfect
$()_p$	derivative evaluated at constant pressure
$()_s$	derivative evaluated at constant entropy
$()_T$	derivative evaluated at constant temperature
$()_v$	derivative evaluated at constant specific volume
$()_{rev}$	quantity evaluated over a reversible path

NOTATION

The notation in brackets [] after many of the equations signifies that the equation is valid only within certain limitations. For example:

[perf]	means that the equation is restricted to a gas which is both thermally and calorically perfect. (By "thermally perfect" it is meant that the gas obeys the thermal equation of state $p = \rho RT$. By "calorically perfect" it is meant that the specific heats c_p and c_v are constant.)
[therm perf]	means that the only restriction on the gas is that it must be thermally perfect. Equations so marked may be used for calorically imperfect gases. (They are, of course, also valid for completely perfect gases.)
[isen]	means that the flow process must take place isentropically. Equations so marked may not be applied to the flow across a shock wave.
[adiab]	means that the only restriction on the flow process is that it must take place adiabatically—that is, without heat transfer. (Such a flow process may or may not be isentropic depending on whether it is or is not reversible.) Equations so marked may be applied to the flow across a shock wave.

An equation without notation has no restrictions beyond those basic to the study of thermodynamics and/or inviscid compressible flow.

FUNDAMENTAL RELATIONS

THERMODYNAMICS

THERMAL EQUATIONS OF STATE

A thermal equation of state is an equation of the form

$$p = p(v, T) \quad (1)$$

Several of the more commonly used thermal equations of state are the following:

Equation for thermally perfect gas

$$p = \frac{RT}{v} = \rho RT \text{ [therm perf]} \quad (2)$$

or

$$\frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0 \text{ [therm perf]} \quad (3)$$

Equations for thermally imperfect gas

Van der Waals' equation (ref. 9)

$$p = \frac{RT}{v-b} - \frac{a}{v^2} \quad (4)$$

where a is the intermolecular-force constant and b is the molecular-size constant (see ref. 9, pp. 390 et seq. for numerical values).

Berthelot's equation (ref. 7)

$$p = \frac{RT}{v-b} - \frac{c}{v^2T} \quad (5)$$

where b is the molecular-size constant and c is the intermolecular-force constant (see ref. 7 for numerical values).

Beattie-Bridgeman equation (ref. 10)

$$p = \frac{RT}{v^2} \left(1 - \frac{c}{vT^3} \right) \left[v + B_0 \left(1 - \frac{b}{v} \right) \right] - \frac{A_0}{v^2} \left(1 - \frac{a}{v} \right) \quad (6)$$

where a , A_0 , b , B_0 , and c are constants for a given gas (see ref. 10, p. 270 for numerical values).

CALORIC EQUATION OF STATE

A caloric equation of state is an equation of the form

$$u = u(v, T) \quad (7)$$

It can be shown that

$$du = c_v dT + \left[T \left(\frac{\partial p}{\partial T} \right)_v - p \right] dv \quad (8a)$$

$$du = c_v dT \text{ [therm perf]} \quad (8b)$$

If the gas is calorically perfect—that is, the specific heats are constant—equation (8b) can be integrated to obtain

$$u = c_v T + u_r \text{ [perf]} \quad (9)$$

² When used without subscripts, p , ρ , and T denote static pressure, static density, and static temperature, respectively.

ENERGY RELATIONS

The law of conservation of energy gives

$$\left. \begin{aligned} dq &= du + dw \quad (\text{first law of thermodynamics}) \\ &= du + p dv = dh - v dp \end{aligned} \right\} \quad (10a)$$

$$\left. \begin{aligned} dq &= c_p dT + p dv \\ &= c_p dT - v dp \end{aligned} \right\} \quad [\text{therm perf}] \quad (10b)$$

SPECIFIC HEATS

The specific heats at constant pressure and constant volume are defined by

$$c_p \equiv \left(\frac{\partial q}{\partial T} \right)_p = \left(\frac{\partial h}{\partial T} \right)_p \quad (11)$$

$$c_v \equiv \left(\frac{\partial q}{\partial T} \right)_v = \left(\frac{\partial u}{\partial T} \right)_v \quad (12)$$

It can be shown that

$$c_p - c_v = \left[\left(\frac{\partial u}{\partial v} \right)_T + p \right] \left(\frac{\partial v}{\partial T} \right)_p = -T \frac{\left(\frac{\partial p}{\partial T} \right)_v^2}{\left(\frac{\partial p}{\partial v} \right)_T} \quad (13a)$$

$$c_p - c_v = R \quad [\text{therm perf}] \quad (13b)$$

The ratio of specific heats is defined as

$$\gamma \equiv \frac{c_p}{c_v} \quad (14)$$

According to the kinetic theory of gases, for many gases over a moderate range of temperature,

$$\gamma = \frac{n+2}{n} \quad (15)$$

where n is the number of effective degrees of freedom of the gas molecule. Useful relations for thermally perfect gases are

$$c_p = \frac{dh}{dT} = c_v + R = \frac{\gamma R}{\gamma - 1} \quad [\text{therm perf}] \quad (16)$$

$$c_v = \frac{du}{dT} = c_p - R = \frac{R}{\gamma - 1} \quad [\text{therm perf}] \quad (17)$$

ENTHALPY

The enthalpy of a gas is defined by

$$h \equiv u + pv \quad (18)$$

It follows that

$$\begin{aligned} dh &= du + p dv + v dp = dq + v dp \\ &= \left[c_v + v \left(\frac{\partial p}{\partial T} \right)_v \right] dT + \left[v \left(\frac{\partial p}{\partial v} \right)_T + T \left(\frac{\partial p}{\partial T} \right)_v \right] dv \end{aligned} \quad (19a)$$

$$dh = (c_v + R) dT = c_p dT \quad [\text{therm perf}] \quad (19b)$$

$$h = (c_v + R) T + u_r = c_p T + u_r \quad [\text{perf}] \quad (20)$$

ENTROPY

The entropy is defined by

$$ds \equiv \left(\frac{dq}{T} \right)_{rev} \quad (21)$$

It follows that

$$ds = \left(\frac{du + dw}{T} \right)_{rev} = \left(\frac{du + p dv}{T} \right)_{rev} = c_v \frac{dT}{T} + \left(\frac{\partial p}{\partial T} \right)_v dv \quad (22a)$$

$$\left. \begin{aligned} ds &= c_v \frac{dT}{T} + R \frac{dv}{v} \\ &= c_v \frac{dT}{T} - R \frac{dp}{p} \\ &= c_p \frac{dT}{T} - R \frac{dp}{p} \\ &= c_v \frac{dp}{p} - c_p \frac{dp}{p} \end{aligned} \right\} \quad [\text{therm perf}] \quad (22b)$$

$$\left. \begin{aligned} s - s_r &= c_v \ln \frac{T}{T_r} - R \ln \frac{p}{p_r} \\ &= c_p \ln \frac{T}{T_r} - R \ln \frac{p}{p_r} \\ &= c_v \ln \frac{p}{p_r} - c_p \ln \frac{p}{p_r} \end{aligned} \right\} \quad [\text{perf}] \quad (23a)$$

$$\left. \begin{aligned} s - s_r &= c_v \ln \frac{T/T_r}{(\rho/\rho_r)^{\gamma-1}} \\ &= c_p \ln \frac{T/T_r}{(p/p_r)^{(\gamma-1)/\gamma}} \\ &= c_v \ln \frac{p/p_r}{(\rho/\rho_r)^\gamma} \end{aligned} \right\} \quad [\text{perf}] \quad (23b)$$

$$\frac{p}{\rho^\gamma} = \frac{p_r}{\rho_r^\gamma} e^{(s-s_r)/c_v} \quad [\text{perf}] \quad (24)$$

The second law of thermodynamics requires that

$$s - s_r \geq 0 \quad [\text{adiab}] \quad (25)$$

CONTINUOUS ONE-DIMENSIONAL FLOW

BASIC EQUATIONS AND DEFINITIONS

The basic equations for the continuous flow of an inviscid non-heat-conducting gas along a streamline are as follows:

Thermal equation of state

$$\frac{p}{\rho} = RT \quad [\text{therm perf}] \quad (26)$$

Dynamic equation

$$\frac{1}{\rho} dp + V dV = 0 \quad (27)$$

Energy equation

$$\left. \begin{aligned} du + d\left(\frac{p}{\rho}\right) + V dV &= 0 \\ dh + V dV &= 0 \end{aligned} \right\} \text{ [abiab]} \quad (28a)$$

$$\left. \begin{aligned} c_p dT + V dV &= 0 \\ \frac{\gamma}{\gamma-1} d\left(\frac{p}{\rho}\right) + V dV &= 0 \end{aligned} \right\} \text{ [adiab, therm perf]} \quad (28b)$$

Additional useful variables are defined as follows:

Speed of sound

$$a \equiv \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\gamma \left(\frac{\partial p}{\partial \rho}\right)_T} \quad (29a)$$

$$= \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma R T} \quad \text{[therm perf]} \quad (29b)$$

$$\begin{aligned} &\cong 49.0 \sqrt{T} \text{ ft/sec for air} \\ &\text{if } T \text{ is in degrees Rankine} \\ &(\text{=degrees Fahrenheit} + 459.6) \end{aligned} \quad (29c)$$

Mach number

$$M \equiv \frac{V}{a} \quad (30)$$

Dynamic pressure

$$q \equiv \frac{1}{2} \rho V^2 \quad (31a)$$

$$= \frac{\gamma}{2} p M^2 \quad \text{[therm perf]} \quad (31b)$$

INTEGRATED FORMS OF ENERGY EQUATION

The energy equation (28) can be integrated at once to obtain

$$h + \frac{V^2}{2} = \text{constant} = h_t \quad \text{[adiab]} \quad (32a)$$

$$\left. \begin{aligned} c_p T + \frac{V^2}{2} &= c_p T_t \\ \frac{\gamma}{\gamma-1} \left(\frac{p}{\rho}\right) + \frac{V^2}{2} &= \frac{\gamma}{\gamma-1} \left(\frac{p_t}{\rho_t}\right) \\ \frac{a^2}{\gamma-1} + \frac{V^2}{2} &= \frac{a_t^2}{\gamma-1} \\ \frac{a^2}{\gamma-1} + \frac{V^2}{2} &= \frac{1}{2} \left(\frac{\gamma+1}{\gamma-1}\right) a_*^2 \\ \frac{a^2}{\gamma-1} + \frac{V^2}{2} &= \frac{V_m^2}{2} \end{aligned} \right\} \text{ [adiab, perf]} \quad (32b)$$

The three reference speeds a_t , a_* , and V_m are related by

$$\left. \begin{aligned} \left(\frac{a_t}{a_*}\right)^2 &= \frac{\gamma+1}{2} \\ \left(\frac{V_m}{a_*}\right)^2 &= \frac{\gamma+1}{\gamma-1} \\ \left(\frac{V_m}{a_t}\right)^2 &= \frac{2}{\gamma-1} \end{aligned} \right\} \text{ [adiab, perf]} \quad (33)$$

PRESSURE-DENSITY RELATION

From equations (27) and (28b) it follows that

$$\frac{p}{\rho^\gamma} = \text{constant} = \frac{p_t}{\rho_t^\gamma} \quad \text{[isen, perf]} \quad (34)$$

from which

$$\frac{p}{p_t} = \left(\frac{\rho}{\rho_t}\right)^\gamma = \left(\frac{T}{T_t}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{a}{a_t}\right)^{\frac{2\gamma}{\gamma-1}} \quad \text{[isen, perf]} \quad (35)$$

BERNOULLI'S EQUATION

Combination of equations (32b) and (35) gives Bernoulli's equation for compressible flow in the form

$$\frac{\gamma}{\gamma-1} \left(\frac{p_t}{\rho_t}\right) \left(\frac{p}{p_t}\right)^{\frac{\gamma-1}{\gamma}} + \frac{V^2}{2} = \frac{\gamma}{\gamma-1} \left(\frac{p_t}{\rho_t}\right) \quad \text{[isen, perf]} \quad (36)$$

RELATIONS BETWEEN LOCAL AND FREE-STREAM CONDITIONS

With the aid of the foregoing equations it can be shown that

$$\frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2} M_\infty^2 \left[\left(\frac{V}{V_\infty}\right)^2 - 1 \right] \quad \text{[adiab, perf]} \quad (37)$$

$$\frac{p}{p_\infty} = \left\{ 1 - \frac{\gamma-1}{2} M_\infty^2 \left[\left(\frac{V}{V_\infty}\right)^2 - 1 \right] \right\}^{\frac{\gamma}{\gamma-1}} \quad \text{[isen, perf]} \quad (38)$$

$$\frac{\rho}{\rho_\infty} = \left\{ 1 - \frac{\gamma-1}{2} M_\infty^2 \left[\left(\frac{V}{V_\infty}\right)^2 - 1 \right] \right\}^{\frac{1}{\gamma-1}} \quad \text{[isen perf]} \quad (39)$$

In small-disturbance theory, where it is assumed that $(V - V_\infty) \ll V_\infty$, these equations take on the simplified form

$$\frac{T}{T_\infty} \cong 1 - (\gamma-1) M_\infty^2 \frac{V - V_\infty}{V_\infty} \quad \text{[adiab, perf]} \quad (40)$$

$$\frac{p}{p_\infty} \cong 1 - \gamma M_\infty^2 \frac{V - V_\infty}{V_\infty} \quad \text{[isen, perf]} \quad (41)$$

$$\frac{\rho}{\rho_\infty} \cong 1 - M_\infty^2 \frac{V - V_\infty}{V_\infty} \quad \text{[isen, perf]} \quad (42)$$

USEFUL RATIOS

On the basis of the above results, useful relations can be derived expressing various dimensionless ratios as functions of a single parameter. These relations are given below, grouped according to which of the various parameters (M , V/a_* , V/a_t , or V/V_m) is used as the independent variable.

In each case the second form of the equation applies for $\gamma = \frac{7}{5}$.

Parameter M .—

$$\frac{T}{T_t} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} = \left(1 + \frac{M^2}{5}\right)^{-1} \quad \text{[adiab, perf]} \quad (43)$$

$$\frac{p}{p_t} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma}{\gamma-1}} = \left(1 + \frac{M^2}{5}\right)^{-\frac{7}{2}} \quad \text{[isen, perf]} \quad (44)$$

$$\frac{\rho}{\rho_t} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{1}{\gamma-1}} = \left(1 + \frac{M^2}{5}\right)^{-\frac{5}{2}} \quad \text{[isen, perf]} \quad (45)$$

$$\frac{a}{a_t} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{1}{2}} = \left(1 + \frac{M^2}{5}\right)^{-\frac{1}{2}} \quad \text{[adiab, perf]} \quad (46)$$

$$\frac{q}{p} = \frac{\gamma}{2} M^2 = \frac{7}{10} M^2 \quad [\text{therm perf}] \quad (47)$$

$$\begin{aligned} \frac{q}{p_t} &= \frac{\gamma}{2} M^2 \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma}{\gamma-1}} \\ &= \frac{7}{10} M^2 \left(1 + \frac{M^2}{5}\right)^{-\frac{7}{2}} \quad [\text{isen, perf}] \quad (48) \end{aligned}$$

$$\begin{aligned} \left(\frac{V}{a_t}\right)^2 &= M^2 \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} \\ &= M^2 \left(1 + \frac{M^2}{5}\right)^{-1} \quad [\text{adiab, perf}] \quad (49) \end{aligned}$$

$$\begin{aligned} \left(\frac{V}{a_*}\right)^2 &= \frac{\gamma+1}{2} M^2 \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} \\ &= \frac{6M^2}{5} \left(1 + \frac{M^2}{5}\right)^{-1} \quad [\text{adiab, perf}] \quad (50) \end{aligned}$$

$$\begin{aligned} \left(\frac{V}{V_m}\right)^2 &= \frac{\gamma-1}{2} M^2 \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} \\ &= \frac{M^2}{5} \left(1 + \frac{M^2}{5}\right)^{-1} \quad [\text{adiab, perf}] \quad (51) \end{aligned}$$

Parameter $\frac{V}{a_*}$.—

$$\frac{T}{T_t} = 1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2 = 1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2 \quad [\text{adiab, perf}] \quad (52)$$

$$\begin{aligned} \frac{p}{p_t} &= \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2\right]^{\frac{\gamma}{\gamma-1}} \\ &= \left[1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2\right]^{\frac{7}{2}} \quad [\text{isen, perf}] \quad (53) \end{aligned}$$

$$\begin{aligned} \frac{\rho}{\rho_t} &= \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= \left[1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \quad (54) \end{aligned}$$

$$\begin{aligned} \frac{a}{a_t} &= \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2\right]^{\frac{1}{2}} \\ &= \left[1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2\right]^{\frac{1}{2}} \quad [\text{adiab, perf}] \quad (55) \end{aligned}$$

$$\begin{aligned} \frac{q}{p} &= \frac{\gamma}{\gamma+1} \left(\frac{V}{a_*}\right)^2 \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2\right]^{-1} \\ &= \frac{7}{12} \left(\frac{V}{a_*}\right)^2 \left[1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2\right]^{-1} \quad [\text{adiab, perf}] \quad (56) \end{aligned}$$

$$\begin{aligned} \frac{q}{p_t} &= \frac{\gamma}{\gamma+1} \left(\frac{V}{a_*}\right)^2 \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= \frac{7}{12} \left(\frac{V}{a_*}\right)^2 \left[1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \quad (57) \end{aligned}$$

$$\begin{aligned} M^2 &= \frac{2}{\gamma+1} \left(\frac{V}{a_*}\right)^2 \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2\right]^{-1} \\ &= \frac{5}{6} \left(\frac{V}{a_*}\right)^2 \left[1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2\right]^{-1} \quad [\text{adiab, perf}] \quad (58) \end{aligned}$$

$$\left(\frac{V}{a_t}\right)^2 = \frac{2}{\gamma+1} \left(\frac{V}{a_*}\right)^2 = \frac{5}{6} \left(\frac{V}{a_*}\right)^2 \quad [\text{adiab, perf}] \quad (59)$$

$$\left(\frac{V}{V_m}\right)^2 = \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2 = \frac{1}{6} \left(\frac{V}{a_*}\right)^2 \quad [\text{adiab, perf}] \quad (60)$$

Parameter $\frac{V}{a_t}$.—

$$\frac{T}{T_t} = 1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2 = 1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2 \quad [\text{adiab, perf}] \quad (61)$$

$$\begin{aligned} \frac{p}{p_t} &= \left[1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2\right]^{\frac{\gamma}{\gamma-1}} \\ &= \left[1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2\right]^{\frac{7}{2}} \quad [\text{isen, perf}] \quad (62) \end{aligned}$$

$$\begin{aligned} \frac{\rho}{\rho_t} &= \left[1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= \left[1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \quad (63) \end{aligned}$$

$$\begin{aligned} \frac{a}{a_t} &= \left[1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2\right]^{\frac{1}{2}} \\ &= \left[1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2\right]^{\frac{1}{2}} \quad [\text{adiab, perf}] \quad (64) \end{aligned}$$

$$\begin{aligned} \frac{q}{p} &= \frac{\gamma}{2} \left(\frac{V}{a_t}\right)^2 \left[1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2\right]^{-1} \\ &= \frac{7}{10} \left(\frac{V}{a_t}\right)^2 \left[1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2\right]^{-1} \quad [\text{adiab, perf}] \quad (65) \end{aligned}$$

$$\begin{aligned} \frac{q}{p_t} &= \frac{\gamma}{2} \left(\frac{V}{a_t}\right)^2 \left[1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= \frac{7}{10} \left(\frac{V}{a_t}\right)^2 \left[1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \quad (66) \end{aligned}$$

$$\begin{aligned} M^2 &= \left(\frac{V}{a_t}\right)^2 \left[1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2\right]^{-1} \\ &= \left(\frac{V}{a_t}\right)^2 \left[1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2\right]^{-1} \quad [\text{adiab, perf}] \quad (67) \end{aligned}$$

$$\left(\frac{V}{a_*}\right)^2 = \frac{\gamma+1}{2} \left(\frac{V}{a_t}\right)^2 = \frac{6}{5} \left(\frac{V}{a_t}\right)^2 \quad [\text{adiab, perf}] \quad (68)$$

$$\left(\frac{V}{V_m}\right)^2 = \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2 = \frac{1}{5} \left(\frac{V}{a_t}\right)^2 \quad [\text{adiab, perf}] \quad (69)$$

Parameter $\frac{V}{V_m}$.—

$$\frac{T}{T_t} = 1 - \left(\frac{V}{V_m}\right)^2 \quad [\text{adiab, perf}] \quad (70)$$

$$\frac{p}{p_t} = \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{\gamma}{\gamma-1}} = \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{7}{2}} \quad [\text{isen, perf}] \quad (71)$$

$$\frac{\rho}{\rho_t} = \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{1}{\gamma-1}} = \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \quad (72)$$

$$\frac{a}{a_t} = \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{1}{2}} \quad [\text{adiab, perf}] \quad (73)$$

$$\begin{aligned} \frac{q}{p} &= \frac{\gamma}{\gamma-1} \left(\frac{V}{V_m}\right)^2 \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{-1} \\ &= \frac{7}{2} \left(\frac{V}{V_m}\right)^2 \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{-1} \quad [\text{adiab, perf}] \end{aligned} \quad (74)$$

$$\begin{aligned} \frac{q}{p_t} &= \frac{\gamma}{\gamma-1} \left(\frac{V}{V_m}\right)^2 \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= \frac{7}{2} \left(\frac{V}{V_m}\right)^2 \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \end{aligned} \quad (75)$$

$$\begin{aligned} M^2 &= \frac{2}{\gamma+1} \left(\frac{V}{V_m}\right)^2 \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{-1} \\ &= \frac{5}{6} \left(\frac{V}{V_m}\right)^2 \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{-1} \quad [\text{adiab, perf}] \end{aligned} \quad (76)$$

$$\left(\frac{V}{a_t}\right)^2 = \frac{2}{\gamma-1} \left(\frac{V}{V_m}\right)^2 = 5 \left(\frac{V}{V_m}\right)^2 \quad [\text{adiab, perf}] \quad (77)$$

$$\left(\frac{V}{a_*}\right)^2 = \frac{\gamma+1}{\gamma-1} \left(\frac{V}{V_m}\right)^2 = 6 \left(\frac{V}{V_m}\right)^2 \quad [\text{adiab, perf}] \quad (78)$$

Tables I and II list numerical values of the following ratios with Mach number M as the independent variable:

$$\frac{p}{p_t}, \frac{\rho}{\rho_t}, \frac{T}{T_t}, \frac{q}{p_t}, \frac{V}{a_*}$$

STREAM-TUBE-AREA RELATIONS

If it is assumed that the density and speed are uniform across any section of a given stream tube, then the equation of continuity is

$$\rho V A = \text{constant} = \rho_* a_* A_* \quad (79)$$

By combining this and certain of the foregoing equations, the area ratio A_*/A can be expressed as a function of any one of the four parameters used above. The final equations are

$$\begin{aligned} \frac{A_*}{A} &= \left(\frac{\gamma+1}{2}\right)^{\frac{1}{2(\gamma-1)}} M \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \\ &= \frac{216}{125} M \left(1 + \frac{M^2}{5}\right)^{-3} \quad [\text{isen, perf}] \end{aligned} \quad (80)$$

$$\begin{aligned} \frac{A_*}{A} &= \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} \left(\frac{V}{a_*}\right) \left[1 - \frac{\gamma-1}{\gamma+1} \left(\frac{V}{a_*}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= \left(\frac{6}{5}\right)^{\frac{5}{2}} \left(\frac{V}{a_*}\right) \left[1 - \frac{1}{6} \left(\frac{V}{a_*}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \end{aligned} \quad (81)$$

$$\begin{aligned} \frac{A_*}{A} &= \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \left(\frac{V}{a_t}\right) \left[1 - \frac{\gamma-1}{2} \left(\frac{V}{a_t}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= \frac{216}{125} \left(\frac{V}{a_t}\right) \left[1 - \frac{1}{5} \left(\frac{V}{a_t}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \end{aligned} \quad (82)$$

$$\begin{aligned} \frac{A_*}{A} &= \left(\frac{2}{\gamma-1}\right)^{\frac{1}{2}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \left(\frac{V}{V_m}\right) \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{1}{\gamma-1}} \\ &= 5^{\frac{1}{2}} \left(\frac{216}{125}\right) \left(\frac{V}{V_m}\right) \left[1 - \left(\frac{V}{V_m}\right)^2\right]^{\frac{5}{2}} \quad [\text{isen, perf}] \end{aligned} \quad (83)$$

Numerical values of A_*/A as a function of M are given in tables I and II.

Equation (79) combined with equations (26), (29b), (45), and (46) can be employed to obtain the mass-flow rate per unit area ρV along a stream tube as a function of Mach number, total temperature, and total pressure. Numerical values can be obtained conveniently from chart 1 where the variation with Mach number of the mass-flow rate per unit cross-sectional area is presented for various total temperatures and a total pressure of 1 pound per square inch absolute.

SHOCK WAVES

NORMAL SHOCK WAVES

BASIC EQUATIONS

The previous relations for isentropic flow are valid on either side of a shock wave, but not across it, because at the shock wave the flow quantities have discontinuities. Jump

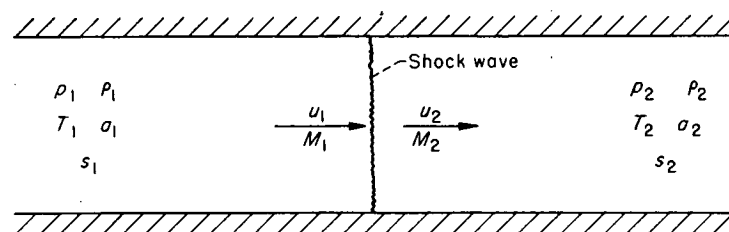


FIGURE 1.—Notation for normal shock wave.

conditions for a steady normal shock wave (fig. 1) result from requiring conservation of

$$\text{mass:} \quad \rho_1 u_1 = \rho_2 u_2 \quad (84)$$

$$\text{momentum:} \quad p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad (85)$$

$$\text{energy:}^3 \quad \frac{1}{2} u_1^2 + h_1 = \frac{1}{2} u_2^2 + h_2 \quad [\text{adiab}] \quad (86a)$$

³ The actual relation for conservation of energy is $\rho_1 u_1 \left(\frac{1}{2} u_1^2 + h_1\right) = \rho_2 u_2 \left(\frac{1}{2} u_2^2 + h_2\right)$; it reduces to the above form in view of equation (84).

$$\left. \begin{aligned} \frac{1}{2} u_1^2 + c_p T_1 &= \frac{1}{2} u_2^2 + c_p T_2 \\ \frac{1}{2} u_1^2 + \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} &= \frac{1}{2} u_2^2 + \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} \\ \frac{1}{2} u_1^2 + \frac{1}{\gamma-1} a_1^2 &= \frac{1}{2} u_2^2 + \frac{1}{\gamma-1} a_2^2 \end{aligned} \right\} \text{[adiab, perf]} \quad (86b)$$

together with the requirement that the entropy does not decrease:

$$\Delta s \equiv s_2 - s_1 \geq 0 \quad (87)$$

It follows immediately from the energy relation (86) that total enthalpy, total temperature, and total speed of sound are constant across the shock and hence (from the previous relations (33) for adiabatic flow) also the critical speed of sound and limiting speed:

$$h_{t_1} = h_{t_2} \quad \text{[adiab]} \quad (88a)$$

$$\left. \begin{aligned} T_{t_1} &= T_{t_2} \\ a_{t_1} &= a_{t_2} \\ a_{*1} &= a_{*2} \\ V_{m_1} &= V_{m_2} \end{aligned} \right\} \text{[adiab, perf]} \quad (88b)$$

Combining equations (84) to (86) leads to Prandtl's relation

$$u_1 u_2 = a_{*2}^2 = \frac{p_2 - p_1}{\rho_2 - \rho_1} \quad \text{[adiab, perf]} \quad (89)$$

which implies that the flow is supersonic ahead of the shock wave and subsonic behind (the reverse possibility is ruled out by the requirement of nondecreasing entropy), and to the Rankine-Hugoniot relations

$$\frac{p_2}{p_1} = \frac{(\gamma+1) \rho_2 - (\gamma-1) \rho_1}{(\gamma+1) \rho_1 - (\gamma-1) \rho_2} \quad \text{[adiab, perf]} \quad (90)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) p_2 + (\gamma-1) p_1}{(\gamma+1) p_1 + (\gamma-1) p_2} \quad \text{[adiab, perf]} \quad (91)$$

$$\frac{p_2 - p_1}{\rho_2 - \rho_1} = \gamma \frac{p_2 + p_1}{\rho_2 + \rho_1} \quad \text{[adiab, perf]} \quad (92)$$

USEFUL RELATIONS

Many relations for normal shock waves are conveniently expressed in terms of either upstream Mach number M_1 or the static-pressure ratio across the shock $\xi \equiv p_2/p_1$. The following relations apply to adiabatic flow of a completely perfect fluid. The last form of each equation holds for $\gamma=7/5$.

Parameter M_1 .—

$$\frac{p_2}{p_1} \equiv \xi = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} = \frac{7M_1^2 - 1}{6} \quad (93)$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_2^2} = \frac{a_{*2}^2}{a_{*1}^2} = \frac{(\gamma+1) M_1^2}{(\gamma-1) M_1^2 + 2} = \frac{6M_1^2}{M_1^2 + 5} \quad (94)$$

$$\frac{T_2}{T_1} = \frac{a_2^2}{a_1^2} = \frac{[2\gamma M_1^2 - (\gamma-1)] [(\gamma-1) M_1^2 + 2]}{(\gamma+1)^2 M_1^2} = \frac{(7M_1^2 - 1)(M_1^2 + 5)}{36M_1^2} \quad (95)$$

$$M_2^2 = \frac{(\gamma-1) M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)} = \frac{M_1^2 + 5}{7M_1^2 - 1} \quad (96)$$

$$\frac{p_2}{p_{t_1}} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \left[\frac{2}{(\gamma-1) M_1^2 + 2} \right]^{\frac{\gamma}{\gamma-1}} = \frac{7M_1^2 - 1}{6} \left(\frac{5}{M_1^2 + 5} \right)^{\frac{7}{2}} \quad (97)$$

$$\frac{p_2}{p_{t_2}} = \left[\frac{4\gamma M_1^2 - 2(\gamma-1)}{(\gamma+1)^2 M_1^2} \right]^{\frac{\gamma}{\gamma-1}} = \left[\frac{5(7M_1^2 - 1)}{36M_1^2} \right]^{\frac{7}{2}} \quad (98)$$

$$\begin{aligned} \frac{p_{t_2}}{p_{t_1}} &= \frac{\rho_{t_2}}{\rho_{t_1}} = e^{-\frac{\Delta s}{R}} \\ &= \left[\frac{(\gamma+1) M_1^2}{(\gamma-1) M_1^2 + 2} \right]^{\frac{\gamma}{\gamma-1}} \left[\frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)} \right]^{\frac{1}{\gamma-1}} \\ &= \left(\frac{6M_1^2}{M_1^2 + 5} \right)^{\frac{7}{2}} \left(\frac{6}{7M_1^2 - 1} \right)^{\frac{5}{2}} \quad (99) \end{aligned}$$

$$\begin{aligned} \frac{p_{t_2}}{p_1} &= \left[\frac{(\gamma+1) M_1^2}{2} \right]^{\frac{\gamma}{\gamma-1}} \left[\frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)} \right]^{\frac{1}{\gamma-1}} \\ &= \left(\frac{6M_1^2}{5} \right)^{\frac{7}{2}} \left(\frac{6}{7M_1^2 - 1} \right)^{\frac{5}{2}} \quad (100) \end{aligned}$$

(Rayleigh pitot formula)

$$\begin{aligned} \frac{\Delta s}{c_p} &= (\gamma-1) \frac{\Delta s}{R} = -(\gamma-1) \ln \left(\frac{p_{t_2}}{p_{t_1}} \right) \\ &= \ln \left[\frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \right] - \gamma \ln \left[\frac{(\gamma+1) M_1^2}{(\gamma-1) M_1^2 + 2} \right] \\ &= \ln \left(\frac{7M_1^2 - 1}{6} \right) - \frac{7}{5} \ln \left(\frac{6M_1^2}{M_1^2 + 5} \right) \quad (101) \end{aligned}$$

$$\frac{p_2 - p_1}{q_1} = \frac{4(M_1^2 - 1)}{(\gamma+1) M_1^2} = \frac{5(M_1^2 - 1)}{3M_1^2} \quad (102)$$

Numerical values from equations (93), (94), (95), (96), (99), and (100) (with $\gamma=7/5$) are given in table II.

For weak shock waves (M_1 only slightly greater than unity) the following series are useful:

$$\begin{aligned} \frac{p_{t_2}}{p_{t_1}} &= 1 - \frac{2\gamma}{3(\gamma+1)^2} (M_1^2 - 1)^3 + \frac{2\gamma^2}{(\gamma+1)^3} (M_1^2 - 1)^4 + \dots \\ &= 1 - \frac{35}{216} (M_1^2 - 1)^3 + \frac{245}{864} (M_1^2 - 1)^4 + \dots \quad (103) \end{aligned}$$

$$\begin{aligned} \frac{\Delta s}{R} &= \frac{1}{\gamma-1} \frac{\Delta s}{c_p} = \frac{2\gamma}{3(\gamma+1)^2} (M_1^2 - 1)^3 - \frac{2\gamma^2}{(\gamma+1)^3} (M_1^2 - 1)^4 + \dots \\ &= \frac{35}{216} (M_1^2 - 1)^3 - \frac{245}{864} (M_1^2 - 1)^4 + \dots \quad (104) \end{aligned}$$

Parameter $\xi \equiv p_2/p_1$.—

$$M_1^2 = \frac{(\gamma+1)\xi + (\gamma-1)}{2\gamma} = \frac{6\xi+1}{7} \quad (105)$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)\xi + (\gamma-1)}{(\gamma-1)\xi + (\gamma+1)} = \frac{6\xi+1}{\xi+6} \quad (106)$$

$$\frac{T_2}{T_1} = \frac{a_2^2}{a_1^2} = \xi \frac{(\gamma-1)\xi + (\gamma+1)}{(\gamma+1)\xi + (\gamma-1)} = \xi \frac{\xi+6}{6\xi+1} \quad (107)$$

$$M_2^2 = \frac{(\gamma-1)\xi + (\gamma+1)}{2\gamma\xi} = \frac{\xi+6}{7\xi} \quad (108)$$

$$\frac{p_2}{p_1} = \xi \frac{p_1}{p_1} = \xi \left\{ \frac{4\gamma}{(\gamma+1)[(\gamma-1)\xi + (\gamma+1)]} \right\}^{\frac{\gamma}{\gamma-1}} = \xi \left[\frac{35}{6(\xi+6)} \right]^{\frac{7}{2}} \quad (109)$$

$$\frac{p_2}{p_1} = \xi \frac{p_1}{p_1} = \left\{ \frac{4\gamma\xi}{(\gamma+1)[(\gamma+1)\xi + (\gamma-1)]} \right\}^{\frac{\gamma}{\gamma-1}} = \left[\frac{35\xi}{6(6\xi+1)} \right]^{\frac{7}{2}} \quad (110)$$

$$\frac{p_{t2}}{p_{t1}} = \frac{\rho_{t2}}{\rho_{t1}} = e^{-\frac{\Delta s}{R}} = \xi^{-\frac{1}{\gamma-1}} \left[\frac{(\gamma+1)\xi + (\gamma-1)}{(\gamma-1)\xi + (\gamma+1)} \right]^{\frac{\gamma}{\gamma-1}} \\ = \left(\frac{1}{\xi} \right)^{\frac{5}{2}} \left(\frac{6\xi+1}{\xi+6} \right)^{\frac{7}{2}} \quad (111)$$

$$\frac{\Delta s}{c_p} = (\gamma-1) \frac{\Delta s}{R} = -(\gamma-1) \ln \left(\frac{p_{t2}}{p_{t1}} \right) = \ln \xi - \gamma \ln \left[\frac{(\gamma+1)\xi + (\gamma-1)}{(\gamma-1)\xi + (\gamma+1)} \right] = \ln \xi - \frac{7}{5} \ln \left(\frac{6\xi+1}{\xi+6} \right) \quad (112)$$

For weak shock waves (ξ only slightly greater than unity)

$$\frac{p_{t2}}{p_{t1}} = 1 - \frac{\gamma+1}{12\gamma^2} (\xi-1)^3 + \frac{\gamma+1}{8\gamma^2} (\xi-1)^4 + \dots \\ = 1 - \frac{5}{49} (\xi-1)^3 + \frac{15}{98} (\xi-1)^4 + \dots \quad (113)$$

$$\frac{\Delta s}{R} = \frac{1}{\gamma-1} \frac{\Delta s}{c_p} = \frac{\gamma+1}{12\gamma^2} (\xi-1)^3 - \frac{\gamma+1}{8\gamma^2} (\xi-1)^4 + \dots \\ = \frac{5}{49} (\xi-1)^3 - \frac{15}{98} (\xi-1)^4 + \dots \quad (114)$$

In unsteady flow a normal shock wave acts at each instant as a steady shock. Hence all the above relations are valid across a moving normal shock wave if instantaneous velocities are measured relative to the shock.

OBLIQUE SHOCK WAVES

In general, a three-dimensional shock wave will be curved, and will separate two regions of nonuniform flow. However, the shock transition at each point takes place instantaneously, so that it is sufficient to consider an arbitrarily small neighborhood of the point. In such a neighborhood

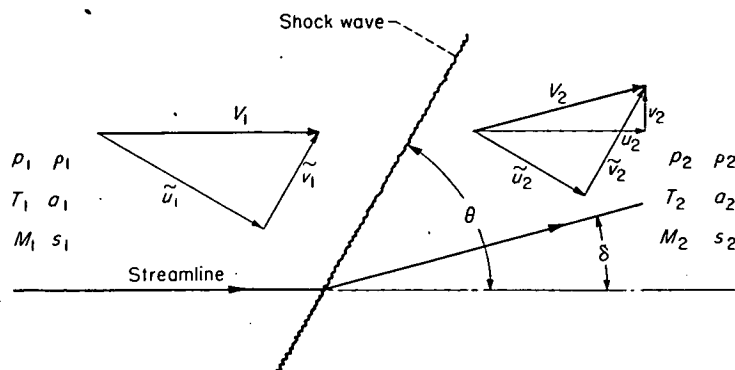


FIGURE 2.—Notation for oblique shock wave.

the shock wave may be regarded as plane to any desired degree of accuracy, and the flows on either side as uniform and parallel. Moreover, with the proper orientation of axes the flow is locally two-dimensional. Hence it is sufficient to consider a straight oblique shock wave in a uniform parallel two-dimensional stream, as shown in figure 2.

BASIC EQUATIONS

For a steady oblique shock wave, jump conditions result from requiring conservation of

$$\text{mass:} \quad \rho_1 \tilde{u}_1 = \rho_2 \tilde{u}_2 \quad (115)$$

$$\text{normal momentum:} \quad p_1 + \rho_1 \tilde{u}_1^2 = p_2 + \rho_2 \tilde{u}_2^2 \quad (116)$$

$$\text{tangential momentum:} \quad \rho_1 \tilde{u}_1 \tilde{v}_1 = \rho_2 \tilde{u}_2 \tilde{v}_2 \quad (117)$$

$$\text{energy:} \quad \frac{1}{2} (\tilde{u}_1^2 + \tilde{v}_1^2) + h_1 = \frac{1}{2} (\tilde{u}_2^2 + \tilde{v}_2^2) + h_2 \text{ [adiab]} \quad (118a)$$

$$\left. \begin{aligned} \frac{1}{2} (\tilde{u}_1^2 + \tilde{v}_1^2) + c_p T_1 &= \frac{1}{2} (\tilde{u}_2^2 + \tilde{v}_2^2) + c_p T_2 \\ \frac{1}{2} (\tilde{u}_1^2 + \tilde{v}_1^2) + \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} &= \frac{1}{2} (\tilde{u}_2^2 + \tilde{v}_2^2) + \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} \\ \frac{1}{2} (\tilde{u}_1^2 + \tilde{v}_1^2) + \frac{1}{\gamma-1} a_1^2 &= \frac{1}{2} (\tilde{u}_2^2 + \tilde{v}_2^2) + \frac{1}{\gamma-1} a_2^2 \end{aligned} \right\} \text{[adiab, perf]} \quad (118b)$$

together with the requirement that the entropy does not decrease:

$$\Delta s \equiv s_2 - s_1 \geq 0 \quad (119)$$

Again it follows from the energy relation (118) that total enthalpy, total temperature, and total speed of sound are constant across the shock and hence also the critical speed of sound and limiting speed:

$$h_{t1} = h_{t2} \text{ [adiab]} \quad (120)$$

$$\left. \begin{aligned} T_{t1} &= T_{t2} \\ a_{t1} &= a_{t2} \\ a_{*1} &= a_{*2} \\ V_{m1} &= V_{m2} \end{aligned} \right\} \text{[adiab, perf]} \quad (121)$$

* Compare remark for normal shock waves, footnote on page 618.

CONNECTION WITH NORMAL SHOCK

A comparison of equation (115) with (117) shows that the tangential velocity is constant across the shock wave:

$$\tilde{v}_1 = \tilde{v}_2 \quad [\text{adiab}] \quad (122)$$

so that the change in velocity is normal to the shock. It follows that

$$\frac{1}{2} \tilde{v}_1^2 = \frac{1}{2} \tilde{v}_2^2$$

so that the energy equation (118a) reduces to

$$\frac{1}{2} \tilde{u}_1^2 + h_1 = \frac{1}{2} \tilde{u}_2^2 + h_2 \quad [\text{adiab}] \quad (123)$$

Now equations (115), (116), and (123) involve only the component of velocity $\tilde{u}$ normal to the shock, and are identical with equations (84), (85), and (86) for normal shock waves. Hence an oblique shock wave acts as a normal shock to the component of flow perpendicular to it, while the tangential component is unchanged. This is also clear physically from the "sweepback principle" that the oblique flow is reduced to the normal flow by a uniform translation of axes (Galilean transformation).

Because the speed of sound depends on the tangential velocity, Prandtl's relation differs from that for normal shock waves (see ref. 11, pp. 302-303):

$$\tilde{u}_1 \tilde{u}_2 = a_*^2 - \frac{\gamma-1}{\gamma+1} \tilde{v}^2 \quad [\text{adiab, perf}] \quad (124)$$

where a_* and $\tilde{v}$ can be evaluated on either side of the shock.

The Rankine-Hugoniot relations are the same as for normal shock waves:

$$\frac{p_2}{p_1} = \frac{(\gamma+1)\rho_2 - (\gamma-1)\rho_1}{(\gamma+1)\rho_1 - (\gamma-1)\rho_2} \quad [\text{adiab, perf}] \quad (125)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2} \quad [\text{adiab, perf}] \quad (126)$$

$$\frac{p_2 - p_1}{\rho_2 - \rho_1} = \gamma \frac{p_2 + p_1}{\rho_2 + \rho_1} \quad [\text{adiab, perf}] \quad (127)$$

USEFUL RELATIONS

Because an oblique shock wave acts as a normal shock to the flow perpendicular to it, the previous relations for normal shocks (except those for ratios of static to total pressures) apply to oblique shocks if M_1 and M_2 are replaced by their normal components $M_1 \sin \theta$ and $M_2 \sin (\theta - \delta)$. This gives most of the following relations; the remainder are derived from them by using the kinematic condition that the stream turns through an angle δ , together with the previous isentropic-flow relations.

Parameters M_1 and θ .—

$$\frac{p_2}{p_1} = \xi = \frac{2\gamma M_1^2 \sin^2 \theta - (\gamma-1)}{\gamma+1} = \frac{7M_1^2 \sin^2 \theta - 1}{6} \quad (128)$$

$$\frac{\rho_2}{\rho_1} = \frac{\tilde{u}_1}{\tilde{u}_2} = \frac{(\gamma+1)M_1^2 \sin^2 \theta}{(\gamma-1)M_1^2 \sin^2 \theta + 2} = \frac{6M_1^2 \sin^2 \theta}{M_1^2 \sin^2 \theta + 5} \quad (129)$$

$$\begin{aligned} \frac{T_2}{T_1} = \frac{a_2^2}{a_1^2} &= \frac{[2\gamma M_1^2 \sin^2 \theta - (\gamma-1)][(\gamma-1)M_1^2 \sin^2 \theta + 2]}{(\gamma+1)^2 M_1^2 \sin^2 \theta} \\ &= \frac{(7M_1^2 \sin^2 \theta - 1)(M_1^2 \sin^2 \theta + 5)}{36M_1^2} \end{aligned} \quad (130)$$

$$M_2^2 \sin^2 (\theta - \delta) = \frac{(\gamma-1)M_1^2 \sin^2 \theta + 2}{2\gamma M_1^2 \sin^2 \theta - (\gamma-1)} = \frac{M_1^2 \sin^2 \theta + 5}{7M_1^2 \sin^2 \theta - 1} \quad (131)$$

$$\begin{aligned} M_2^2 &= \frac{(\gamma+1)^2 M_1^4 \sin^2 \theta - 4(M_1^2 \sin^2 \theta - 1)(\gamma M_1^2 \sin^2 \theta + 1)}{[2\gamma M_1^2 \sin^2 \theta - (\gamma-1)][(\gamma-1)M_1^2 \sin^2 \theta + 2]} \\ &= \frac{36M_1^4 \sin^2 \theta - 5(M_1^2 \sin^2 \theta - 1)(7M_1^2 \sin^2 \theta + 5)}{(7M_1^2 \sin^2 \theta - 1)(M_1^2 \sin^2 \theta + 5)} \end{aligned} \quad (132)$$

$$\frac{\tilde{u}_2}{V_1} = \frac{(\gamma-1)M_1^2 \sin^2 \theta + 2}{(\gamma+1)M_1^2 \sin^2 \theta} \sin \theta = \frac{M_1^2 \sin^2 \theta + 5}{6M_1^2 \sin^2 \theta} \sin \theta \quad (133)$$

$$\frac{\tilde{v}_2}{V_1} = \frac{\tilde{v}_1}{V_1} = \cos \theta \quad (134)$$

$$\frac{u_2}{V_1} = 1 - \frac{2(M_1^2 \sin^2 \theta - 1)}{(\gamma+1)M_1^2} = 1 - \frac{5(M_1^2 \sin^2 \theta - 1)}{6M_1^2} \quad (135)$$

$$\frac{v_2}{V_1} = \frac{2(M_1^2 \sin^2 \theta - 1)}{(\gamma+1)M_1^2} \cot \theta = \frac{5(M_1^2 \sin^2 \theta - 1)}{6M_1^2} \cot \theta \quad (136)$$

$$\begin{aligned} \frac{V_2^2}{V_1^2} &= 1 - 4 \frac{(M_1^2 \sin^2 \theta - 1)(\gamma M_1^2 \sin^2 \theta + 1)}{(\gamma+1)^2 M_1^4 \sin^2 \theta} \\ &= 1 - \frac{5}{36} \frac{(M_1^2 \sin^2 \theta - 1)(7M_1^2 \sin^2 \theta + 5)}{M_1^4 \sin^2 \theta} \end{aligned} \quad (137)$$

$$\begin{aligned} \cot \delta &= \tan \theta \left[\frac{(\gamma+1)M_1^2}{2(M_1^2 \sin^2 \theta - 1)} - 1 \right] \\ &= \tan \theta \left[\frac{6M_1^2}{5(M_1^2 \sin^2 \theta - 1)} - 1 \right] \end{aligned} \quad (138)$$

$$\tan \delta = \frac{2 \cot \theta (M_1^2 \sin^2 \theta - 1)}{2 + M_1^2 (\gamma + 1 - 2 \sin^2 \theta)} = \frac{5 \cot \theta (M_1^2 \sin^2 \theta - 1)}{5 + M_1^2 (6 - 5 \sin^2 \theta)} \quad (139a)$$

$$\begin{aligned} &= \frac{M_1^2 \sin 2\theta - 2 \cot \theta}{2 + M_1^2 (\gamma + \cos 2\theta)} = 5 \frac{M_1^2 \sin 2\theta - 2 \cot \theta}{10 + M_1^2 (7 + 5 \cos 2\theta)} \end{aligned} \quad (139b)$$

$$\begin{aligned} \frac{p_2}{p_{t_1}} &= \frac{2\gamma M_1^2 \sin^2 \theta - (\gamma-1)}{(\gamma+1)} \left[\frac{2}{(\gamma-1)M_1^2 + 2} \right]^{\frac{\gamma}{\gamma-1}} \\ &= \frac{7M_1^2 \sin^2 \theta - 1}{6} \left(\frac{5}{M_1^2 + 5} \right)^{7/2} \end{aligned} \quad (140)$$

$$\begin{aligned} \frac{p_2}{p_{t_2}} &= \left\{ 2 \frac{[2\gamma M_1^2 \sin^2 \theta - (\gamma-1)][(\gamma-1)M_1^2 \sin^2 \theta + 2]}{(\gamma+1)^2 M_1^2 \sin^2 \theta [(\gamma-1)M_1^2 + 2]} \right\}^{\frac{\gamma}{\gamma-1}} \\ &= \left[5 \frac{(7M_1^2 \sin^2 \theta - 1)(M_1^2 \sin^2 \theta + 5)}{36M_1^2 \sin^2 \theta (M_1^2 + 5)} \right]^{7/2} \end{aligned} \quad (141)$$

$$\frac{p_{t_2}}{p_{t_1}} = \frac{\rho_{t_2}}{\rho_{t_1}} = e^{-\frac{\Delta s}{R}}$$

$$= \left[\frac{(\gamma+1)M_1^2 \sin^2 \theta}{(\gamma-1)M_1^2 \sin^2 \theta + 2} \right]^{\frac{\gamma}{\gamma-1}} \left[\frac{\gamma+1}{2\gamma M_1^2 \sin^2 \theta - (\gamma-1)} \right]^{\frac{1}{\gamma-1}}$$

$$= \left(\frac{6M_1^2 \sin^2 \theta}{M_1^2 \sin^2 \theta + 5} \right)^{7/2} \left(\frac{6}{7M_1^2 \sin^2 \theta - 1} \right)^{5/2} \quad (142)$$

$$\frac{p_{t_2}}{p_1} = \left[\frac{\gamma+1}{2\gamma M_1^2 \sin^2 \theta - (\gamma-1)} \right]^{\frac{1}{\gamma-1}} \times$$

$$\left\{ \frac{(\gamma+1)M_1^2 \sin^2 \theta [(\gamma-1)M_1^2 + 2]}{2[(\gamma-1)M_1^2 \sin^2 \theta + 2]} \right\}^{\frac{\gamma}{\gamma-1}}$$

$$= \left(\frac{6}{7M_1^2 \sin^2 \theta - 1} \right)^{5/2} \left[\frac{6M_1^2 \sin^2 \theta (M_1^2 + 5)}{5(M_1^2 \sin^2 \theta + 5)} \right]^{7/2} \quad (143)$$

$$\frac{\Delta s}{c_v} = (\gamma-1) \frac{\Delta s}{R} = -(\gamma-1) \ln \left(\frac{p_{t_2}}{p_{t_1}} \right)$$

$$= \ln \left[\frac{2\gamma M_1^2 \sin^2 \theta - (\gamma-1)}{\gamma+1} \right] -$$

$$\gamma \ln \left[\frac{(\gamma+1)M_1^2 \sin^2 \theta}{(\gamma-1)M_1^2 \sin^2 \theta + 2} \right]$$

$$= \ln \left(\frac{7M_1^2 \sin^2 \theta - 1}{6} \right) - \frac{7}{5} \ln \left(\frac{6M_1^2 \sin^2 \theta}{M_1^2 \sin^2 \theta + 5} \right) \quad (144)$$

$$\frac{p_2 - p_1}{q_1} = \frac{4(M_1^2 \sin^2 \theta - 1)}{(\gamma+1)M_1^2} = \frac{5}{3} \frac{M_1^2 \sin^2 \theta - 1}{M_1^2} \quad (145)$$

Values of the following ratios for oblique shock waves can be read from table II, provided $M_1 \sin \theta$ is used instead of M_1 in the first column:

$$\frac{p_2}{p_1}, \frac{\rho_2}{\rho_1}, \frac{T_2}{T_1}, \frac{p_{t_2}}{p_{t_1}}$$

For weak shock waves ($M_1 \sin \theta$ only slightly greater than unity) the following series are obtained from equations (103) and (104) by replacing M_1 by $M_1 \sin \theta$:

$$\frac{p_{t_2}}{p_{t_1}} = 1 - \frac{2\gamma}{3(\gamma+1)^2} (M_1^2 \sin^2 \theta - 1)^3 + \frac{2\gamma^2}{(\gamma+1)^3} (M_1^2 \sin^2 \theta - 1)^4 + \dots$$

$$= 1 - \frac{35}{216} (M_1^2 \sin^2 \theta - 1)^3 + \frac{245}{864} (M_1^2 \sin^2 \theta - 1)^4 + \dots \quad (146)$$

$$\frac{\Delta s}{R} = \frac{1}{\gamma-1} \frac{\Delta s}{c_v} = \frac{2\gamma}{3(\gamma+1)^2} (M_1^2 \sin^2 \theta - 1)^3 -$$

$$\frac{2\gamma^2}{(\gamma+1)^3} (M_1^2 \sin^2 \theta - 1)^4 + \dots$$

$$= \frac{35}{216} (M_1^2 \sin^2 \theta - 1)^3 - \frac{245}{864} (M_1^2 \sin^2 \theta - 1)^4 + \dots \quad (147)$$

Parameters θ and δ .—

$$\frac{1}{M_1^2} = \sin^2 \theta - \frac{\gamma+1}{2} \frac{\sin \theta \sin \delta}{\cos(\theta-\delta)} = \sin^2 \theta - \frac{\gamma+1}{2} \frac{\tan \delta}{\tan \delta + \cot \theta}$$

$$= \sin^2 \theta - \frac{\gamma+1}{2} \frac{\tan \theta}{\tan \theta + \cot \delta} \quad (148a)$$

$$M_1^2 = \frac{2(\cot \theta + \tan \delta)}{\sin 2\theta - \tan \delta(\gamma + \cos 2\theta)}$$

$$= \frac{10(\cot \theta + \tan \delta)}{5 \sin 2\theta - \tan \delta(7 + 5 \cos 2\theta)} \quad (148b)$$

$$\frac{p_2 - p_1}{q_1} = 2 \frac{\sin \theta \sin \delta}{\cos(\theta-\delta)}$$

$$= 2 \frac{\tan \delta}{\tan \delta + \cot \theta} = 2 \frac{\tan \theta}{\tan \theta + \cot \delta} \quad (149a)$$

$$\frac{\rho_2 - \rho_1}{\rho_2} = \frac{\sin \delta}{\sin \theta \cos(\theta-\delta)} \quad (149b)$$

Parameters M_1 and δ .—

No convenient explicit relations exist. However, the value of $\sin^2 \theta$ can be found by solving the following cubic equation (ref. 12):

$$\sin^6 \theta + b \sin^4 \theta + c \sin^2 \theta + d = 0 \quad (150a)$$

where

$$\left. \begin{aligned} b &= -\frac{M_1^2 + 2}{M_1^2} - \gamma \sin^2 \delta \\ c &= \frac{2M_1^2 + 1}{M_1^4} + \left[\frac{(\gamma+1)^2}{4} + \frac{\gamma-1}{M_1^2} \right] \sin^2 \delta \\ d &= -\frac{\cos^2 \delta}{M_1^4} \end{aligned} \right\} \quad (150b)$$

The smallest of the three roots corresponds to a decrease in entropy and should be disregarded.

For weak shock waves (small deflections δ) the following series are useful (note that δ must be measured in radians):

$$\frac{p_2}{p_1} = 1 + \frac{\gamma M_1^2}{(M_1^2 - 1)^{1/2}} \delta + \gamma M_1^2 \frac{(\gamma+1)M_1^4 - 4(M_1^2 - 1)}{4(M_1^2 - 1)^2} \delta^2 +$$

$$\frac{\gamma M_1^2}{(M_1^2 - 1)^{7/2}} \left[\frac{(\gamma+1)^2}{32} M_1^8 - \frac{7+12\gamma-3\gamma^2}{24} M_1^6 + \right.$$

$$\left. \frac{3}{4} (\gamma+1)M_1^4 - M_1^2 + \frac{2}{3} \right] \delta^3 + \dots \quad (151)$$

$$\frac{p_2 - p_1}{q_1} = \frac{2}{(M_1^2 - 1)^{1/2}} \delta + \frac{(\gamma+1)M_1^4 - 4(M_1^2 - 1)}{2(M_1^2 - 1)^2} \delta^2 +$$

$$\frac{1}{(M_1^2 - 1)^{7/2}} \left[\frac{(\gamma+1)^2}{16} M_1^8 - \frac{7+12\gamma-3\gamma^2}{12} M_1^6 + \right.$$

$$\left. \frac{3}{2} (\gamma+1)M_1^4 - 2M_1^2 + \frac{4}{3} \right] \delta^3 + \dots \quad (152)$$

$$\frac{\rho_2}{\rho_1} = 1 + \frac{M_1^2}{(M_1^2 - 1)^{1/2}} \delta + M_1^2 \frac{(3 - \gamma)M_1^2(M_1^2 - 2) + 4}{4(M_1^2 - 1)^2} \delta^2 + \dots \quad (153)$$

$$\frac{T_2}{T_1} = 1 + \frac{(\gamma - 1)M_1^2}{(M_1^2 - 1)^{1/2}} \delta + (\gamma - 1)M_1^2 \frac{(\gamma + 1)M_1^4 - 2(M_1^2 + 2)(M_1^2 - 1)}{4(M_1^2 - 1)^2} \delta^2 + \dots \quad (154)$$

Since flow through weak shock waves is nearly isentropic, compressions through small angles can also be calculated with the aid of table II by regarding them as reversed Prandtl-Meyer expansions (see later section). The resulting numerical accuracy is greater than that obtained by retaining terms up to δ^2 in the above series, and nearly equal to that obtained by retaining terms up to δ^3 .

Charts 2, 3, and 4 show the variation of shock-wave angle, pressure coefficient across a shock wave, and downstream Mach number with flow-deflection angle for various upstream Mach numbers.

Parameter $\xi \equiv p_1/p_2$.—

$$M_1^2 \sin^2 \theta = \frac{(\gamma + 1)\xi + (\gamma - 1)}{2\gamma} = \frac{6\xi + 1}{7} \quad (155)$$

$$M_2^2 \sin^2 (\theta - \delta) = \frac{(\gamma - 1)\xi + (\gamma + 1)}{2\gamma\xi} = \frac{\xi + 6}{7\xi} \quad (156)$$

$$M_2^2 = \frac{M_1^2[(\gamma + 1)\xi + (\gamma - 1)] - 2(\xi^2 - 1)}{\xi[(\gamma - 1)\xi + (\gamma + 1)]} = \frac{M_1^2(6\xi + 1) - 5(\xi^2 - 1)}{\xi(\xi + 6)} \quad (157)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\xi + (\gamma - 1)}{(\gamma - 1)\xi + (\gamma + 1)} = \frac{6\xi + 1}{\xi + 6} \quad (158)$$

$$\frac{T_2}{T_1} = \frac{a_2^2}{a_1^2} = \xi \frac{(\gamma - 1)\xi + (\gamma + 1)}{(\gamma + 1)\xi + (\gamma - 1)} = \xi \frac{\xi + 6}{6\xi + 1} \quad (159)$$

$$\tan^2 \delta = \left(\frac{\xi - 1}{\gamma M_1^2 - \xi + 1} \right)^2 \frac{2\gamma M_1^2 - (\gamma - 1) - (\gamma + 1)\xi}{(\gamma + 1)\xi + (\gamma - 1)} = \left[\frac{5(\xi - 1)}{7M_1^2 - 5(\xi - 1)} \right]^2 \frac{7M_1^2 - (6\xi + 1)}{6\xi + 1} \quad (160)$$

$$\frac{p_{t_2}}{p_{t_1}} = \frac{\rho_{t_2}}{\rho_{t_1}} = e^{-\frac{\Delta s}{R}} = \left[\frac{(\gamma + 1)\xi + (\gamma - 1)}{(\gamma - 1)\xi + (\gamma + 1)} \right]^{\frac{\gamma}{\gamma - 1}} \xi^{-\frac{1}{\gamma - 1}} = \left(\frac{6\xi + 1}{\xi + 6} \right)^{7/2} \xi^{-5/2} \quad (161)$$

$$\frac{V_2^2}{V_1^2} = 1 - \frac{2(\xi^2 - 1)}{M_1^2[(\gamma + 1)\xi + (\gamma - 1)]} = 1 - \frac{5(\xi^2 - 1)}{M_1^2(6\xi + 1)} \quad (162)$$

For weak shock waves, equations (113) and (114) apply to oblique as well as normal shocks.

SHOCK POLAR

The velocities associated with an oblique shock wave are conveniently represented in the velocity-vector (hodograph) plane. For a given Mach number ahead of the shock wave, all possible velocity vectors behind the shock lie on a single curve.

Only the closed loop represents real shock waves with non-decreasing entropy, and forms Busemann's shock polar (fig. 3). Its equation is

$$v_2^2 = (V_1 - u_2)^2 \frac{u_2 - \frac{a_*^2}{V_1}}{\frac{2}{\gamma + 1} V_1 + \frac{a_*^2}{V_1} - u_2} \quad (163)$$

Other forms of this equation convenient for computation are, given V_1 and M_1 ,

$$\left(\frac{v_2}{V_1} \right)^2 = \left(1 - \frac{u_2}{V_1} \right)^2 \frac{(M_1^2 - 1) - \frac{\gamma + 1}{2} M_1^2 \left(1 - \frac{u_2}{V_1} \right)}{1 + \frac{\gamma + 1}{2} M_1^2 \left(1 - \frac{u_2}{V_1} \right)} = \left(1 - \frac{u_2}{V_1} \right)^2 \frac{5(M_1^2 - 1) - 6M_1^2 \left(1 - \frac{u_2}{V_1} \right)}{5 + 6M_1^2 \left(1 - \frac{u_2}{V_1} \right)} \quad (164a)$$

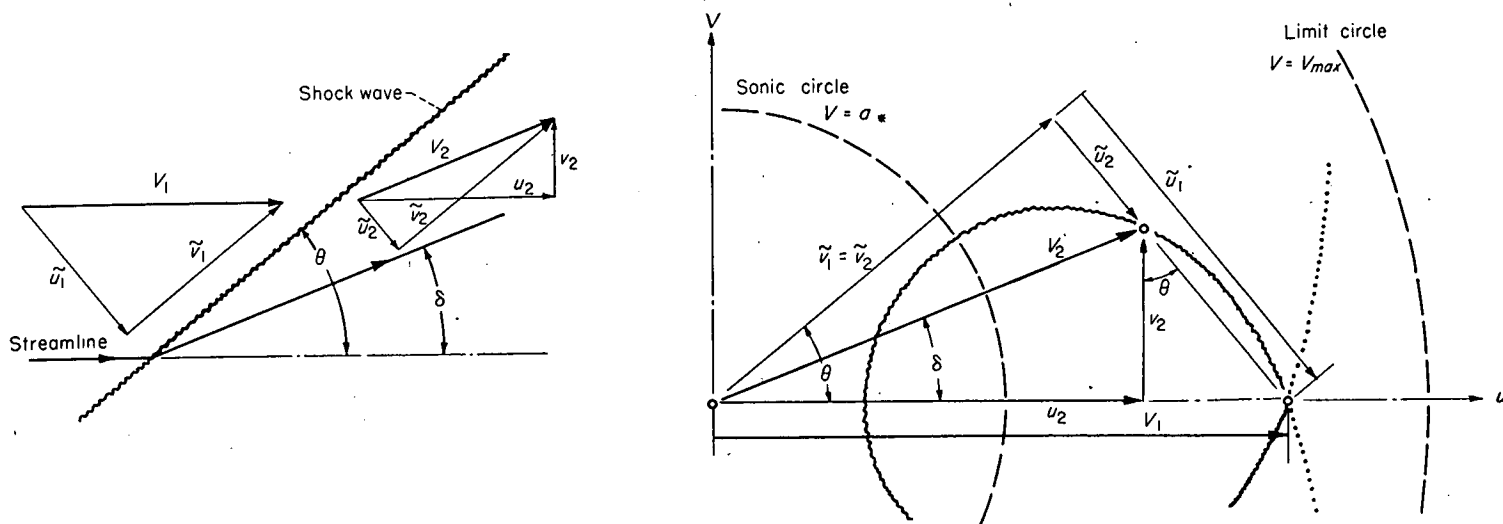


FIGURE 3.—Shock polar.

given a_* and V_1 ,

$$\left(\frac{V_2}{a_*}\right)^2 = \left(\frac{V_1}{a_*} - \frac{u_2}{a_*}\right)^2 \frac{\frac{V_1}{a_*} \frac{u_2}{a_*} - 1}{1 + \frac{2}{\gamma+1} \left(\frac{V_1}{a_*}\right)^2 - \frac{V_1}{a_*} \frac{u_2}{a_*}}$$

$$\left(\frac{V_1}{a_*} - \frac{u_2}{a_*}\right)^2 \frac{6 \left(\frac{V_1}{a_*} \frac{u_2}{a_*} - 1\right)}{5 \left(\frac{V_1}{a_*}\right)^2 - 6 \left(\frac{V_1}{a_*} \frac{u_2}{a_*} - 1\right)} \quad (164b)$$

and given V_1 and V_m ,

$$\left(\frac{v_2}{V_m}\right)^2 = \left(\frac{V_1}{V_m} - \frac{u_2}{V_m}\right)^2 \frac{\frac{V_1}{V_m} \frac{u_2}{V_m} - \frac{\gamma-1}{\gamma+1}}{\frac{2}{\gamma+1} \left(\frac{V_1}{V_m}\right)^2 + \frac{\gamma-1}{\gamma+1} - \frac{V_1}{V_m} \frac{u_2}{V_m}}$$

$$= \left(\frac{V_1}{V_m} - \frac{u_2}{V_m}\right)^2 \frac{\left(6 \frac{V_1}{V_m} \frac{u_2}{V_m} - 1\right)}{5 \left(\frac{V_1}{V_m}\right)^2 - \left(6 \frac{V_1}{V_m} \frac{u_2}{V_m} - 1\right)} \quad (164c)$$

The shock-wave angle θ and wedge angle δ are given in terms of the velocity components by

$$\tan \theta = \frac{V_1 - u_2}{v_2} = \frac{\tilde{u}_1}{\tilde{v}_1} \quad (165)$$

$$\tan \delta = \frac{v_2}{u_2} \quad (166)$$

The shock-wave angle θ_* for sonic flow behind the shock is found (by setting $M_2=1$ in eq. (132)) to be given by

$$\sin^2 \theta_* = \frac{1}{4\gamma M_1^2} \{(\gamma+1)M_1^2 - (3-\gamma) + \sqrt{(\gamma+1)[(\gamma+1)M_1^4 - 2(3-\gamma)M_1^2 + (\gamma+9)]}\}$$

$$= \frac{1}{7M_1^2} [3M_1^2 - 2 + \sqrt{3(3M_1^4 - 4M_1^2 + 13)}] \quad (167)$$

The shock-wave angle $\theta_{\delta_{max}}$ for maximum stream deflection behind the shock is given by

$$\sin^2 \theta_{\delta_{max}} = \frac{1}{4\gamma M_1^2} \{(\gamma+1)M_1^2 - 4 + \sqrt{(\gamma+1)[(\gamma+1)M_1^4 + 8(\gamma-1)M_1^2 + 16]}\}$$

$$= \frac{1}{7M_1^2} [3M_1^2 - 5 + \sqrt{3(3M_1^4 + 4M_1^2 + 20)}] \quad (168)$$

For small deflection angles (hence Mach numbers close to unity), the deflection angle (radians) for sonic flow behind the shock is given approximately in terms of the upstream Mach number by

$$\delta_* = \frac{1}{\sqrt{2}(\gamma+1)} \frac{(M_1^2-1)^{3/2}}{M_1^2} = 0.2946 \frac{(M_1^2-1)^{3/2}}{M_1^2} \quad (169)$$

The maximum stream deflection angle for a specified upstream Mach number is given approximately by

$$\delta_{max} = \frac{4}{3\sqrt{3}(\gamma+1)} \frac{(M_1^2-1)^{3/2}}{M_1^2} = 0.3208 \frac{(M_1^2-1)^{3/2}}{M_1^2} \quad (170)$$

In unsteady flow all the above relations are valid across a moving oblique shock wave if instantaneous velocities are measured relative to the shock.

SUPERSONIC FLOW PAST WEDGES AND CONES

A shock wave forms ahead of any body in supersonic flight and remains fixed relative to the body if the flight is steady. It stands ahead of blunt shapes, but may be attached to pointed shapes.

Just at the tip of a pointed airfoil or body of revolution the flow is the same as for the initially tangent wedge or cone. The bow wave is attached at sufficiently high Mach numbers for a wedge of semivertex angle δ less than $\sin^{-1}(1/\gamma) = 45.6^\circ$ for $\gamma=7/5$, and for a circular cone of semivertex angle σ less than 57.5° for $\gamma=1.405$. Below these limits, the wave is attached above a minimum Mach number whose dependence upon nose angle is shown for wedges and cones in figure 4. (These values can be applied to pointed airfoils and bodies of revolution which are not concave.) Also shown in figure 4 are the slightly higher Mach numbers above which the velocity behind the shock wave is supersonic, and for the cone the still higher Mach number above which the flow is supersonic even at the surface. (For wedges these last two coincide.) For thin wedges, these Mach numbers are given approximately by equations (169) and (170).

FLOW PAST WEDGES

If the bow shock wave is attached to a wedge, it is straight, and the flow behind the shock consists of uniform streams parallel to either face of the wedge. The flow pattern above the upper face (fig. 5) may be regarded as obtained from the straight oblique shock-wave pattern of figure 2 by replacing the streamline behind the shock wave with a solid wall. Flow quantities are determined by the oblique-shock-wave relations, equations (115) to (170). As noted previously, table II can also be applied if $M_1 \sin \theta$ is used in place of M_1 in the first column.

The flows above and below the wedge are independent, so that inclined wedges can be treated if neither face exceeds the attachment angle shown in figure 4. However, if the angle of attack exceeds the semivertex angle, the flow over the upper (leeward) surface is given by a Prandtl-Meyer expansion (see fig. 4) rather than by the shock relations.

It is clear from the shock polar (fig. 3) that two different shock waves and flow patterns are theoretically possible for a given wedge and Mach number. However, it is believed that only the weaker shock wave (larger u_2 and smaller θ) can occur attached to an isolated convex body.

Charts 2, 3, and 4 show the dependence of shock-wave angle, surface pressure coefficient, and downstream Mach number upon wedge angle for various free-stream Mach numbers.

FLOW PAST CONES

If the bow shock wave is attached to an uninclined circular cone, the shock wave too has the form of a circular cone. Flow quantities are constant on all concentric conical surfaces lying between the shock wave and the body, and so depend upon only one space variable. The transition across the shock wave is governed by the oblique-shock relations,

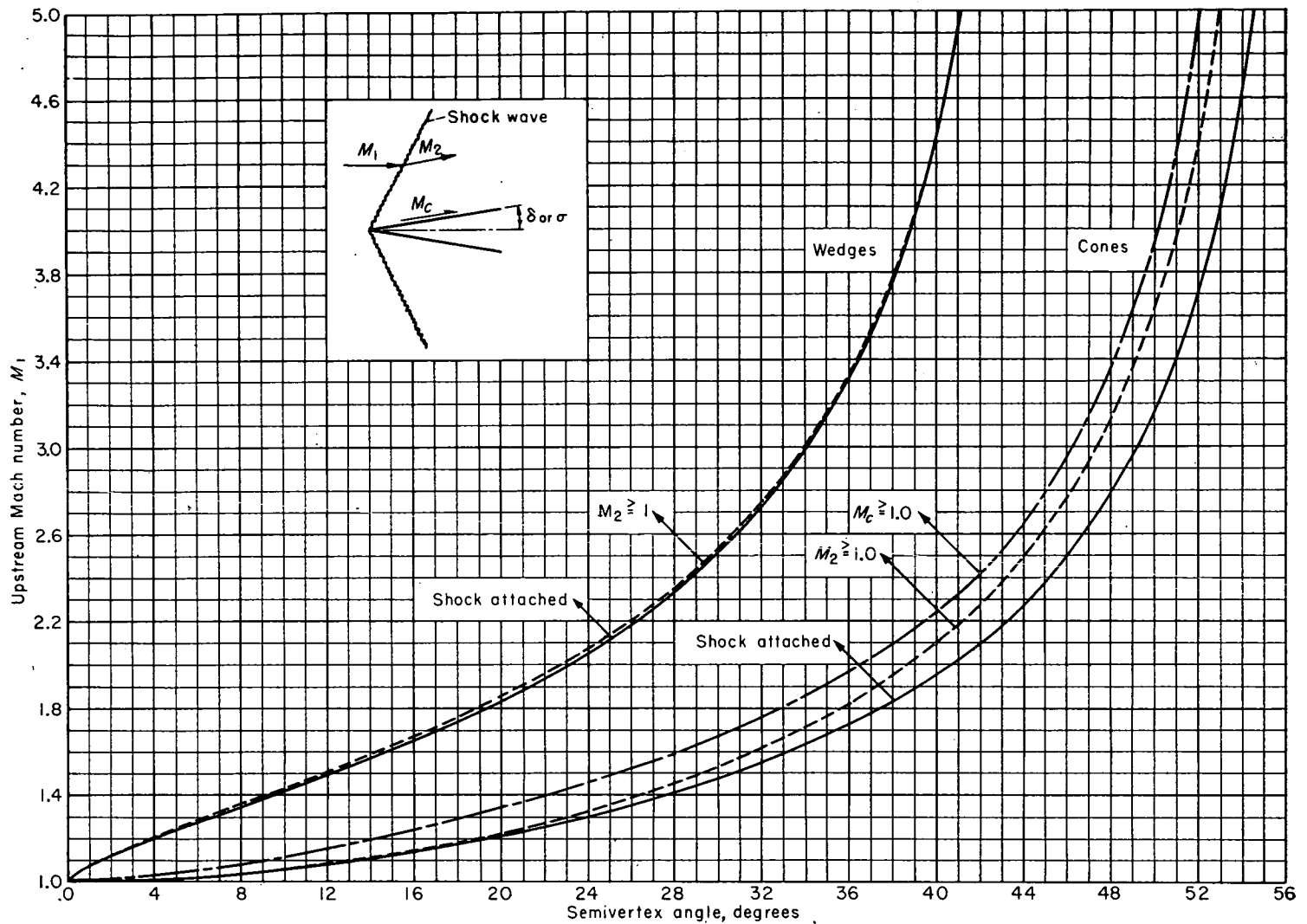


FIGURE 4.—Upstream Mach numbers for shock attachment and for supersonic flow behind shock wave on wedges and cones, and for supersonic flow at surface of cones.

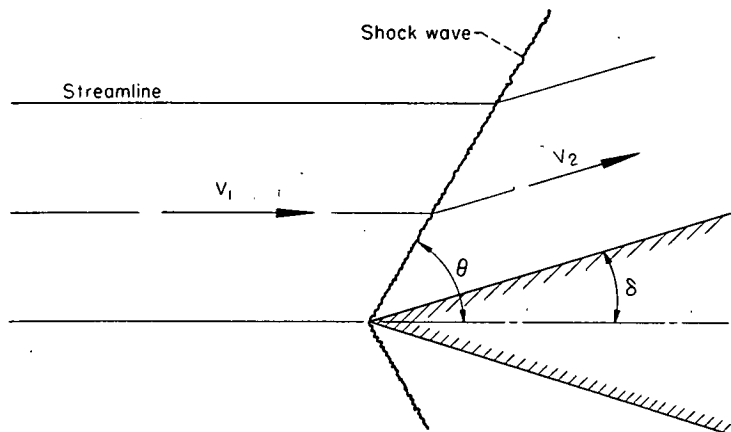


FIGURE 5.—Flow past a wedge.

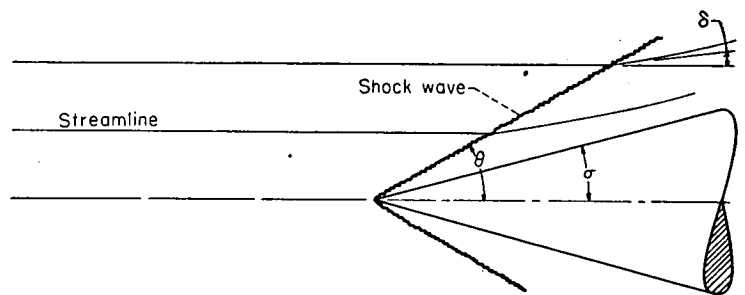


FIGURE 6.—Flow past a cone.

surface Mach number on cone semivertex angle for various free-stream Mach numbers.

The effects of slightly inclining a cone have been considered by Stone (ref. 13) and numerical results are tabulated in reference 14. Chart 8 shows the variation with Mach number of the initial slope of the normal-force curve for various cone angles. Stone has also sought an approximation for larger inclinations by retaining squares as well as first powers of angle of attack (ref. 15), and numerical results have been tabulated (ref. 16); however, these results are not free of error (see refs. 17 and 18).

PRANDTL-MEYER EXPANSION

A uniform two-dimensional supersonic stream flowing over a convex bend expands isentropically. Convenient relations are found by considering the special case of a stream at Mach number unity flowing around a sharp corner (fig. 7).

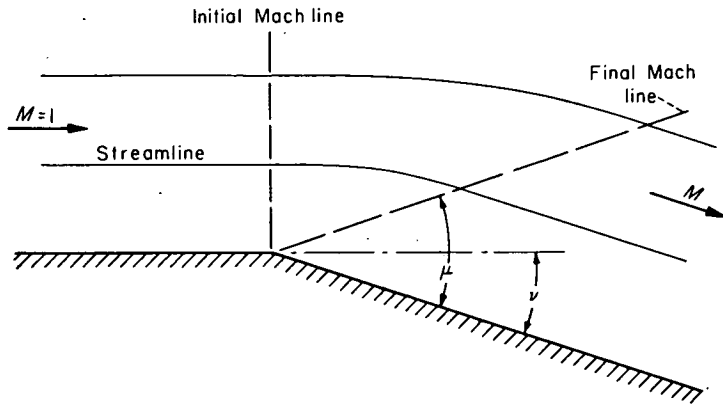


FIGURE 7.—Prandtl-Meyer expansion around a corner.

For a perfect gas, the Prandtl-Meyer angle ν through which the stream turns in expanding from $M=1$ to a supersonic Mach number M is

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}} (M^2-1) - (90^\circ - \mu) \quad (171a)$$

$$= \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}} (M^2-1) - \cos^{-1} \frac{1}{M} \quad (171b)$$

$$= \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}} (M^2-1) - \tan^{-1} \sqrt{M^2-1} \quad (171c)$$

(For $\gamma=7/5$, $\sqrt{\frac{\gamma+1}{\gamma-1}}=2.4495$, and $\sqrt{\frac{\gamma-1}{\gamma+1}}=0.40825$.) The maximum expansion angle, for $M=\infty$, is

$$\nu_{\max} = \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1 \right) \times 90^\circ = 130.45^\circ \text{ for } \gamma=7/5 \quad (172)$$

The ratio of static to total pressure; corresponding to Mach number M is given by

$$\left(\frac{p}{p_t} \right)^{\frac{\gamma-1}{\gamma}} = \frac{1}{\gamma+1} \left\{ 1 + \cos \left[2 \sqrt{\frac{\gamma-1}{\gamma+1}} (\nu + 90^\circ - \mu) \right] \right\} \quad (173a)$$

$$= \frac{1}{\gamma+1} \left\{ 1 + \cos \left[2 \sqrt{\frac{\gamma-1}{\gamma+1}} \left(\nu + \cos^{-1} \frac{1}{M} \right) \right] \right\} \quad (173b)$$

$$= \frac{1}{\gamma+1} \left\{ 1 + \cos \left[2 \sqrt{\frac{\gamma-1}{\gamma+1}} \left(\nu + \tan^{-1} \sqrt{M^2-1} \right) \right] \right\} \quad (173c)$$

which falls to zero as $\nu \rightarrow \nu_{\max}$. Numerical values of ν , μ , and p/p_t are given in table II as functions of M .

These relations and the values in table II apply to a uniform stream flowing past any convex surface in the ab-

sence of external disturbances. (They also give a very good approximation at all Mach numbers when, as on an airfoil, external disturbances arise only from interaction with a shock wave, and are disregarded.) If flow quantities are known at one point, the values at any second point can be read from table II by identifying the change in flow angle between the two points with $\Delta\nu = \nu_2 - \nu_1$, as indicated in figure 8.

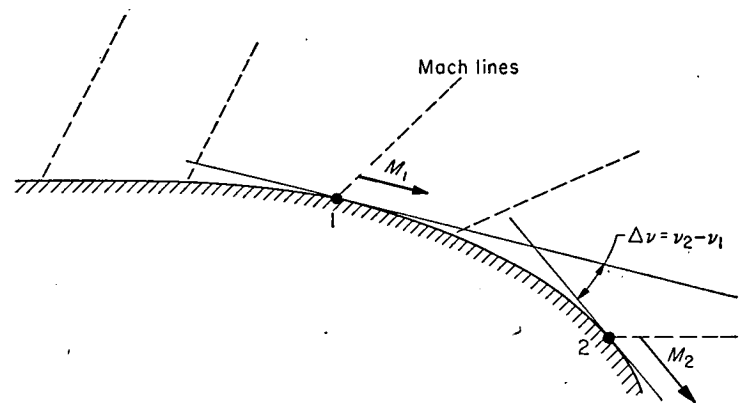


FIGURE 8.—Prandtl-Meyer expansion over a convex surface.

For expansions through small angles $\Delta\nu$, the ratio of final to initial static pressures is given by the following series ($\Delta\nu$ in radians):

$$\begin{aligned} \frac{p_2}{p_1} = 1 - \frac{\gamma M_1^2}{\sqrt{M_1^2-1}} (\Delta\nu) + \gamma M_1^2 \frac{(\gamma+1) M_1^4 - 4(M_1^2-1)}{4(M_1^2-1)^2} (\Delta\nu)^2 - \\ - \frac{\gamma M_1^2}{2(M_1^2-1)^{7/2}} \left[\frac{\gamma+1}{6} M_1^8 - \frac{5+7\gamma-2\gamma^2}{6} M_1^6 + \right. \\ \left. + \frac{5}{3} (\gamma+1) M_1^4 - 2M_1^2 + \frac{4}{3} \right] (\Delta\nu)^3 + \dots \quad (174) \end{aligned}$$

Up to and including the term in $(\Delta\nu)^2$ this series is identical with that for compression through an oblique shock wave (eq. (151) with $\delta = -\Delta\nu$).

IMPERFECT-GAS EFFECTS

Methods for calculating the flow of a calorically imperfect, thermally imperfect gas and a calorically imperfect, thermally perfect gas at temperatures up to 5000° R are described in this section. The equations presented are in substantially the same form as those given in references 7 and 8. Effects of gaseous imperfections, such as molecular dissociation, which become important at temperatures greater than about 5000° R are not considered.

Atmospheric and wind-tunnel air flows are of primary concern here. In such flows air generally exhibits only caloric imperfections to any appreciable degree. Consequently, numerical results are presented only for the flow of a calorically imperfect, thermally perfect diatomic gas.

THERMODYNAMICS

EQUATIONS OF STATE

The thermal equation of state used here for a calorically and thermally imperfect gas is the Berthelot equation

(eq. (5)). The thermal equation of state used for a calorically imperfect, thermally perfect gas is equation (2). The caloric equation of state used for a calorically and thermally imperfect gas is equation (8a). The caloric equation of state used for a calorically imperfect, thermally perfect gas is equation (8b).

SPECIFIC HEATS

The assumption of a simple harmonic vibrator is used to account for the contribution of the vibrational heat capacity to the specific heats. The equations for the specific heats at constant volume and constant pressure, respectively, are (see ref. 7)

$$c_v = (c_v)_{\text{perf}} \left\{ 1 + (\gamma_{\text{perf}} - 1) \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} + \frac{2c_p}{RT^2} \right] \right\} \quad (175)$$

$$c_v = (c_v)_{\text{perf}} \left\{ 1 + (\gamma_{\text{perf}} - 1) \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} \right] \right\} [\text{therm perf}] \quad (176)$$

$$c_p = (c_p)_{\text{perf}} \left\{ 1 + \frac{\gamma_{\text{perf}} - 1}{\gamma_{\text{perf}}} \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} + \frac{2c_p}{RT^2} \left(1 + \frac{2 - b_p}{1 - b_p} + \frac{c_p}{2RT^2} \right) \right] \right\} \quad (177)$$

$$c_p = (c_p)_{\text{perf}} \left\{ 1 + \frac{\gamma_{\text{perf}} - 1}{\gamma_{\text{perf}}} \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} \right] \right\} [\text{therm perf}] \quad (178)$$

The ratio of specific heats is then

$$\gamma = \gamma_{\text{perf}} \times \left[\frac{1 + \frac{\gamma_{\text{perf}} - 1}{\gamma_{\text{perf}}} \left\{ \left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} + \frac{2c_p}{RT^2} \left[1 + \frac{2 - b_p}{1 - b_p} + \frac{c_p}{2RT^2} \right] \right\}}{1 + (\gamma_{\text{perf}} - 1) \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} + \frac{2c_p}{RT^2} \right]} \right] \quad (179)$$

or, for a thermally perfect gas,

$$\gamma = 1 + \frac{\gamma_{\text{perf}} - 1}{1 + (\gamma_{\text{perf}} - 1) \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} \right]} [\text{therm perf}] \quad (180)$$

The following values of γ are for temperatures from 400° R to 5000° R, with $\Theta = 5500^\circ \text{R}$ (see ref. 7). For engineering purposes, these are a satisfactory approximation for air.

VARIATION OF RATIO OF SPECIFIC HEATS WITH TEMPERATURE

T, °R	γ	T, °R	γ	T, °R	γ
500	1.400	1300	1.361	2200	1.322
600	1.399	1400	1.355	2400	1.317
700	1.396	1500	1.349	2600	1.313
800	1.392	1600	1.344	2800	1.309
900	1.387	1700	1.339	3000	1.306
1000	1.381	1800	1.335	3500	1.301
1100	1.375	1900	1.331	4000	1.296
1200	1.368	2000	1.328	4500	1.293
				5000	1.294

CONTINUOUS ONE-DIMENSIONAL FLOW

BASIC EQUATIONS AND DEFINITIONS

Basic equations pertinent to this section are equations (26), (27), (28), (29), (30), and (31). The equations for the speed of sound are (see ref. 7)

$$a^2 = RT \left\{ \frac{1}{(1 - b_p)^2} - \frac{2c_p}{RT^2} + \frac{(\gamma_{\text{perf}} - 1) \left(\frac{c_p}{RT^2} + \frac{1}{1 - b_p} \right)^2}{1 + (\gamma_{\text{perf}} - 1) \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} + \frac{2c_p}{RT^2} \right]} \right\} \quad (181)$$

and

$$a^2 = RT \left\{ 1 + \frac{\gamma_{\text{perf}} - 1}{1 + (\gamma_{\text{perf}} - 1) \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} \right]} \right\} [\text{therm perf}] \quad (182)$$

INTEGRATED FORMS OF ENERGY EQUATION

The integrated forms of the energy equation are (see ref. 7)

$$V^2 = 2RT_i \left[\frac{1 - \frac{T}{T_i}}{\gamma_{\text{perf}} - 1} + \frac{\Theta}{T_i} \left(\frac{1}{e^{\Theta/T_i} - 1} - \frac{1}{e^{\Theta/T} - 1} \right) + \frac{2c}{RT_i} \left(\frac{p}{T} - \frac{p_i}{T_i} \right) + \frac{1}{RT_i} \left(\frac{p_i}{\rho_i} - \frac{p}{\rho} \right) \right] [\text{adiab}] \quad (183)$$

and

$$V^2 = 2RT_i \left[\frac{\gamma_{\text{perf}}}{\gamma_{\text{perf}} - 1} \left(1 - \frac{T}{T_i} \right) + \frac{\Theta}{T_i} \left(\frac{1}{e^{\Theta/T_i} - 1} - \frac{1}{e^{\Theta/T} - 1} \right) \right] [\text{adiab, therm perf}] \quad (184)$$

In terms of Mach number these equations become, respectively,

$$M^2 = \frac{2T_t \left[\frac{1 - \frac{T}{T_t}}{\gamma_{\text{pert}} - 1} + \frac{\Theta}{T_t} \left(\frac{1}{e^{\Theta/T_t} - 1} - \frac{1}{e^{\Theta/T} - 1} \right) + \frac{2c}{RT_t} \left(\frac{\rho}{T} - \frac{\rho_t}{T_t} \right) + \frac{1}{RT_t} \left(\frac{p_t}{\rho_t} - \frac{p}{\rho} \right) \right]}{T \left\{ \frac{1}{(1 - b\rho)^2} - \frac{2c\rho}{RT^2} + \frac{(\gamma_{\text{pert}} - 1) \left(\frac{c\rho}{RT^2} + \frac{1}{1 - b\rho} \right)^2}{1 + (\gamma_{\text{pert}} - 1) \left[\left(\frac{\Theta}{T} \right)^2 \frac{e^{\Theta/T}}{(e^{\Theta/T} - 1)^2} + \frac{2c\rho}{RT^2} \right]} \right\}} \quad [\text{adiab}] \quad (185)$$

and

$$M^2 = \frac{2T_t}{\gamma T} \left[\frac{\gamma_{\text{pert}}}{\gamma_{\text{pert}} - 1} \left(1 - \frac{T}{T_t} \right) + \frac{\Theta}{T_t} \left(\frac{1}{e^{\Theta/T_t} - 1} - \frac{1}{e^{\Theta/T} - 1} \right) \right] \quad [\text{adiab, therm perf}] \quad (186)$$

where γ is given by equation (180).

The variations of $\frac{\left(\frac{V}{a_*}\right)_{\text{therm perf}}}{\left(\frac{V}{a_*}\right)_{\text{pert}}}$ and $\frac{\left(\frac{T}{T_t}\right)_{\text{therm perf}}}{\left(\frac{T}{T_t}\right)_{\text{pert}}}$ with Mach number for several values of total temperature T_t are given in charts 9 and 10.

PRESSURE AND DENSITY RELATIONS

For isentropic flow, the relations between density and temperature are (see ref. 7)

$$\left(\frac{\rho}{\rho_t} \right) \left(\frac{1 - b\rho_t}{1 - b\rho} \right) = \left(\frac{e^{\Theta/T_t} - 1}{e^{\Theta/T} - 1} \right) \left(\frac{T}{T_t} \right)^{\frac{1}{\gamma_{\text{pert}} - 1}} \exp \left[\frac{c\rho_t}{RT_t^2} - \frac{c\rho}{RT^2} + \left(\frac{\Theta}{T} \right) \frac{e^{\Theta/T}}{e^{\Theta/T} - 1} - \left(\frac{\Theta}{T_t} \right) \frac{e^{\Theta/T_t}}{e^{\Theta/T_t} - 1} \right] \quad [\text{isen}] \quad (187)$$

and, for a thermally perfect gas,

$$\frac{\rho}{\rho_t} = \left(\frac{e^{\Theta/T_t} - 1}{e^{\Theta/T} - 1} \right) \left(\frac{T}{T_t} \right)^{\frac{1}{\gamma_{\text{pert}} - 1}} \exp \left[\left(\frac{\Theta}{T} \right) \frac{e^{\Theta/T}}{e^{\Theta/T} - 1} - \left(\frac{\Theta}{T_t} \right) \frac{e^{\Theta/T_t}}{e^{\Theta/T_t} - 1} \right] \quad [\text{isen, therm perf}] \quad (188)$$

The variation of $\frac{\left(\frac{\rho}{\rho_t}\right)_{\text{therm perf}}}{\left(\frac{\rho}{\rho_t}\right)_{\text{pert}}}$ with Mach number for several total temperatures is presented in chart 11.

For the isentropic flow of a thermally imperfect, calorically imperfect gas, the relation between pressure, density, and temperature can be obtained by a trial-and-error procedure using equations (5) and (187).⁵ For the isentropic flow of a thermally perfect gas, the relation between pressure and temperature is

$$\frac{p}{p_t} = \left(\frac{e^{\Theta/T_t} - 1}{e^{\Theta/T} - 1} \right) \left(\frac{T}{T_t} \right)^{\frac{\gamma_{\text{pert}}}{\gamma_{\text{pert}} - 1}} \exp \left[\left(\frac{\Theta}{T} \right) \frac{e^{\Theta/T}}{e^{\Theta/T} - 1} - \left(\frac{\Theta}{T_t} \right) \frac{e^{\Theta/T_t}}{e^{\Theta/T_t} - 1} \right] \quad [\text{isen, therm perf}] \quad (189)$$

The relation between dynamic and static pressure for a thermally imperfect gas can be obtained by a trial-and-error procedure using equations (5), (31a), (183), and (187). The relation between dynamic and static pressure for a thermally perfect gas can be obtained with equations (31b) and (186), and is

$$\frac{q}{p} = \frac{\gamma_{\text{pert}}}{\gamma_{\text{pert}} - 1} \left(\frac{T_t}{T} - 1 \right) + \frac{\Theta}{T} \left(\frac{1}{e^{\Theta/T_t} - 1} - \frac{1}{e^{\Theta/T} - 1} \right) \quad [\text{adiab, therm perf}] \quad (190)$$

The variations of $\frac{\left(\frac{p}{p_t}\right)_{\text{therm perf}}}{\left(\frac{p}{p_t}\right)_{\text{pert}}}$ and $\frac{\left(\frac{q}{p_t}\right)_{\text{therm perf}}}{\left(\frac{q}{p_t}\right)_{\text{pert}}}$ with Mach

number for several total temperatures are given in charts 12 and 13.

STREAM-TUBE-AREA RELATIONS

The stream-tube-area relation is given by equation (79), or, in more convenient form,

$$\frac{A}{A_*} = \frac{\rho_* a_*}{\rho a M} \quad (191)$$

This ratio can be evaluated for a thermally imperfect gas with the aid of equations (187), (181), (5), and (185), and for a thermally perfect gas with the aid of equations (188),

(182), and (186). The variation of $\frac{\left(\frac{A}{A_*}\right)_{\text{therm perf}}}{\left(\frac{A}{A_*}\right)_{\text{pert}}}$ with Mach

number for several values of total temperature is presented in chart 14.

⁵ In this, as in many of the cases to be presented, no direct solution for flow properties is possible if the gas exhibits both thermal and caloric imperfections. Approximate solutions of this type can be obtained, however, if the degree of imperfection is small (see ref. 7).

NORMAL SHOCK WAVES

The requirements for conservation of mass, momentum, and energy across a normal shock wave are given by equations (84), (85), and (86a). The energy relation can be written

$$\frac{u_2^2}{2} - \frac{u_1^2}{2} + \frac{R}{\gamma_{\text{pert}} - 1} (T_2 - T_1) - \left(\frac{2c\rho_2}{T_2} - \frac{2c\rho_1}{T_1} \right) + \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) + R\theta \left(\frac{1}{e^{\theta/T_2} - 1} - \frac{1}{e^{\theta/T_1} - 1} \right) = 0 \quad [\text{adiab}] \quad (192)$$

or, for a thermally perfect gas,

$$\frac{u_2^2}{2} - \frac{u_1^2}{2} + \left(\frac{\gamma_{\text{pert}}}{\gamma_{\text{pert}} - 1} \right) R(T_2 - T_1) + R\theta \left(\frac{1}{e^{\theta/T_2} - 1} - \frac{1}{e^{\theta/T_1} - 1} \right) = 0 \quad [\text{adiab, therm perf}] \quad (193)$$

No explicit equation has been found to relate the temperature downstream of a normal shock wave in thermally imperfect air to the upstream conditions. A trial-and-error procedure, starting with assumed values of ρ_2 and T_2 and involving equations (5), (84), (85), and (192), can be used to determine the downstream temperature.

For the flow of a thermally perfect gas, the simultaneous solution of equations (84), (85), (193), and (2) yields the following relation from which the temperature behind the shock wave can be found:

$$\left(u_1 + \frac{RT_1}{u_1} \right)^2 - \left(u_1 + \frac{RT_1}{u_1} \right) \sqrt{\left(u_1 + \frac{RT_1}{u_1} \right)^2 - 4RT_2 - 2RT_2 - 2u_1^2 + \left(\frac{\gamma_{\text{pert}}}{\gamma_{\text{pert}} - 1} \right) 4R(T_2 - T_1) + 4R\theta \left(\frac{1}{e^{\theta/T_2} - 1} - \frac{1}{e^{\theta/T_1} - 1} \right)} = 0 \quad [\text{adiab, therm perf}] \quad (194)$$

Since the total temperature T_t remains constant across a shock wave, other flow parameters behind the shock wave can be found with the aid of previously presented one-dimensional flow relations. The variations of

$$\frac{\left(\frac{T_2}{T_1} \right)_{\text{therm perf}}}{\left(\frac{T_2}{T_1} \right)_{\text{pert}}}, \frac{\left(\frac{\rho_2}{\rho_1} \right)_{\text{therm perf}}}{\left(\frac{\rho_2}{\rho_1} \right)_{\text{pert}}}, \frac{\left(\frac{p_1}{p_{t_2}} \right)_{\text{therm perf}}}{\left(\frac{p_1}{p_{t_2}} \right)_{\text{pert}}}, \frac{\left(\frac{p_2}{p_1} \right)_{\text{therm perf}}}{\left(\frac{p_2}{p_1} \right)_{\text{pert}}}, \frac{M_{2,\text{therm perf}}}{M_{2,\text{pert}}}, \text{ and } \frac{\left(\frac{p_{t_2}}{p_{t_1}} \right)_{\text{therm perf}}}{\left(\frac{p_{t_2}}{p_{t_1}} \right)_{\text{pert}}}$$

with upstream Mach number for several total temperatures are presented in charts 15 through 20, respectively.

OBLIQUE SHOCK WAVES

For a thermally imperfect gas, no simple equations can be found to relate the values of the flow parameters across oblique shock waves. In general, trial-and-error procedure, starting with assumed values of ρ_2 and T_2 , and involving the relations for the conservation of mass, momentum, and energy, must be used. (See eqs. (115), (116), (117), and (118a) as well as equations (5) and (183).) For a thermally perfect gas, the Mach number downstream of an oblique shock wave can be found with the aid of the energy equation (see eqs. (118a) and (186)), thus

$$M_2^2 = \frac{2T_1}{\gamma_2 T_2} \left[\frac{\gamma_1 M_1^2}{2} + \left(\frac{\gamma_{\text{pert}}}{\gamma_{\text{pert}} - 1} \right) \left(1 - \frac{T_2}{T_1} \right) + \frac{\theta}{T_1} \left(\frac{1}{e^{\theta/T_1} - 1} - \frac{1}{e^{\theta/T_2} - 1} \right) \right] \quad [\text{adiab, therm perf}] \quad (195)$$

where γ_1 and γ_2 are the functions of T_1 and T_2 , respectively, given by equation (180). The pressure ratio across the shock is given by

$$\frac{p_1}{p_2} = \frac{1}{2} \left\{ (1 + \gamma_2 M_2^2) - \frac{T_1}{T_2} (1 + \gamma_1 M_1^2) + \sqrt{\left[(1 + \gamma_2 M_2^2) - \frac{T_1}{T_2} (1 + \gamma_1 M_1^2) \right]^2 + 4 \frac{T_1}{T_2}} \right\} \quad [\text{adiab, therm perf}] \quad (196)$$

The density ratio can be determined from the equation of state (eq. (2)) with the aid of the pressure and temperature ratios. The shock-wave and deflection angles are given by (see ref. 8)

$$\sin^2 \theta = \frac{\left(\frac{\gamma_2}{\gamma_1} \right) \left(\frac{T_2}{T_1} \right) \left(\frac{M_2}{M_1} \right)^2 - 1}{\left(\frac{\rho_1}{\rho_2} \right)^2 - 1} \quad [\text{adiab, therm perf}] \quad (197)$$

and

$$\cot \delta = \tan \theta \left(\frac{\gamma_1 M_1^2}{p_2 - 1} - 1 \right) \quad [\text{adiab, therm perf}] \quad (198)$$

respectively.

The variation of θ with δ for various values of M_1 and T_1 is presented in chart 21. In addition, the variations of

$$\frac{(M_2)_{\text{therm perf}}}{(M_2)_{\text{pert}}} \text{ and } \frac{\left(\frac{p_2 - p_1}{q_1} \right)_{\text{therm perf}}}{\left(\frac{p_2 - p_1}{q_1} \right)_{\text{pert}}} \text{ with } \delta \text{ for various } M_1 \text{ and } T_1$$

are presented in charts 22 and 23.

Values of the ratios

$$\frac{p_2}{p_1}, \frac{\rho_2}{\rho_1}, \frac{T_2}{T_1}, \frac{p_{t_2}}{p_{t_1}}$$

for the flow of a thermally perfect gas across an oblique shock wave can be determined from the normal-shock relations,

provided that $M_1 \sin \theta$ is used instead of M_1 and that the static temperature T_1 just upstream of the shock wave is the same for the oblique shock wave as for the normal shock wave.

PRANDTL-MEYER EXPANSION

The Prandtl-Meyer angle for the flow of an imperfect gas can be found by graphically integrating the equation (see ref. 8)

$$\nu = - \int_{p_*}^p \frac{dp}{\rho V^2 \tan \mu} \quad [\text{isen}] \quad (199)$$

The relations between p , ρ , V , and μ can be found with the

aid of equations (5), (187), (183), and (185). For a thermally perfect gas this equation becomes (see, again, ref. 8)

$$\nu = - \int_{p_*}^p \frac{\sin 2\mu}{2\gamma p} dp \quad [\text{isen, therm perf}] \quad (200)$$

The relations between γ , p , and μ can be found with the aid of equations (180), (189), and (186) using the temperature as a parameter. The graphical integration of equation (200) has been carried out, and the variations of $\nu_{\text{therm perf}}$ and $\frac{\nu_{\text{therm perf}}}{\nu_{\text{perf}}}$ with Mach number for various values of total temperature are presented in chart 24.

APPENDIX A

VISCOSITY AND THERMODYNAMIC CONSTANTS FOR AIR

VISCOSITY

The viscosity of air is nearly independent of pressure; the variation with absolute temperature, between temperatures of about 300° R and 900° R, may be approximated by the formula

$$\frac{\mu}{\mu_r} = \left(\frac{T}{T_r} \right)^{0.76} \quad (\text{A1})$$

For a wider range of temperatures, between about 180° R and 3400° R, Sutherland's formula (see ref. 19) is more accurate:

$$\frac{\mu}{\mu_r} = \frac{T_r + 198.6}{T + 198.6} \left(\frac{T}{T_r} \right)^{3/2} \quad (\text{A2})$$

The viscosity of air, as determined from this relation, may be expressed as

$$\mu = 2.270 \frac{T^{3/2}}{T + 198.6} \times 10^{-8} \frac{\text{lb sec}}{\text{ft}^2} \quad (\text{A3})$$

This latter equation has been employed in the calculations of Reynolds number (chart 25).

THERMODYNAMIC CONSTANTS

The value of γ employed for air, when treated as a completely perfect gas, is 7/5. This simple value, which has been employed in table I, table II, charts 1 to 4, and chart 25, is a good approximation to the more precise values obtained from spectroscopic measurements (see ref. 20). Values of c_p , c_v , and R for air, consistent with the approximation $\gamma=7/5$, are

$$c_p = 6006 \text{ ft}^2/\text{sec}^2 \text{ } ^\circ\text{R}$$

$$c_v = 4290 \text{ ft}^2/\text{sec}^2 \text{ } ^\circ\text{R}$$

$$R = 1716 \text{ ft}^2/\text{sec}^2 \text{ } ^\circ\text{R}$$

APPENDIX B

REYNOLDS NUMBER

Reynolds number is defined as

$$R = \frac{\rho V l}{\mu} \quad (\text{B1})$$

For sea-level conditions,

$$R \approx 10,000 (V \text{ in mph}) (l \text{ in ft}) \quad (\text{B2})$$

In a wind tunnel (subsonic or supersonic), if isentropic expansion is assumed from a total pressure p_t and equation

(A2) is used for the variation of viscosity with temperature, the Reynolds number per unit reference length is given by

$$\frac{R}{l} = \frac{p_t M}{\mu_t} \sqrt{\frac{\gamma}{(\gamma-1)c_v T_t}} \left(\frac{T_t}{T} \right)^{\frac{\gamma-2}{\gamma-1}} \frac{\frac{T}{T_t} + \frac{198.6}{T_t}}{1 + \frac{198.6}{T_t}} \quad [\text{perf}] \quad (\text{B3})$$

The Reynolds number per unit length for $p_t=1$ psia has been plotted in chart 25 as a function of M for various total temperatures T_t .

APPENDIX C

PRESSURE CONVERSION FACTORS AND CONSTANTS

Multiply by to obtain	lb in. ²	lb ft ²	in. H ₂ O at 70° F	in. Hg at 70° F	cm. Hg at 70° F	Standard atmos- pheres
1 lb/in. ²	1	0.006944	0.03607	0.4892	0.1926	14.70
1 lb/ft ²	144	1	5.194	70.45	27.74	2117
1 in. H ₂ O (70° F)	27.73	.1925	1	13.56	5.340	407.6
1 in. Hg (70° F)	2.044	.01420	.07373	1	.3937	30.05
1 cm. Hg (70° F)	5.192	.03605	.1873	2.540	1	76.33
1 Standard atmosphere	.06804	.0004725	.002453	.03328	.01310	1

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TABLES

The tables that follow contain numerical values for certain quantities often required for the solution of problems in compressible flow. The symbols used in these tables are the same as those used in the preceding sections. For convenience, however, the symbols are redefined at the end of table II.

To conserve space, a modified computing-machine notation has been adopted to indicate the position of the decimal point in the tabulated quantities. The location of the decimal point is governed by the following rules:

- (a) A group of digits followed by $-_n$ indicates that the decimal point should be n places to the left of the first digit.

Example: $.3268 -_3 = .0003268$

- (b) A group of digits followed by $+_n$ indicates that the decimal point should be n places to the right of the last digit.

Example: $3268 +_3 = 3,268,000$

- (c) A group of digits without a suffix indicates that the decimal point is correctly located as printed.

TABLE I.—SUBSONIC FLOW

The ratios given by equations (43), (44), (45), (48), (50), and (83) are given as functions of Mach number. If, at a point in an isentropic flow, any one of these ratios or the Mach number is known, then all other ratios for that point can be read or interpolated from the table. In addition, the parameter $\beta = \sqrt{M^2 - 1}$, which is sometimes more convenient to use than the Mach number itself, is also tabulated.

TABLE II.—SUPERSONIC FLOW

The ratios given in table I for subsonic flow are also given in table II for supersonic flow. The Mach angle μ and the Prandtl-Meyer angle ν are also given as functions of Mach number. In addition to these point functions for isentropic flow, the normal-shock relations given by equations (93), (94), (95), (96), (99), and (100) are tabulated as functions of the Mach number M_1 ahead of the shock wave. Although these values are for normal shock waves, the values of p_2/p_1 , ρ_2/ρ_1 , T_2/T_1 , and p_{t2}/p_{t1} may also be used for oblique shock waves, provided $M_1 \sin \theta$ is used instead of M_1 in the first column.

TABLE I.—SUBSONIC FLOW

 $\gamma = 7/5$

M	$\frac{p}{p_t}$	$\frac{\rho}{\rho_t}$	$\frac{T}{T_t}$	β	$\frac{q}{p_t}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	M	$\frac{p}{p_t}$	$\frac{\rho}{\rho_t}$	$\frac{T}{T_t}$	β	$\frac{q}{p_t}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$
0	1.0000	1.0000	1.0000	1.0000	0	∞	0	0.50	0.8430	0.8852	0.9524	0.8660	0.1475	1.3398	0.53452
.01	.9999	1.0000	1.0000	1.0000	.7000 -4	57.8738	.01095	.51	.8374	.8809	.9506	.8602	.1525	1.3212	.54469
.02	.9997	.9998	.9999	.9998	.2799 -3	28.9421	.02191	.52	.8317	.8766	.9487	.8542	.1574	1.3034	.55483
.03	.9994	.9996	.9998	.9995	.6206 -3	19.3005	.03286	.53	.8259	.8723	.9468	.8480	.1624	1.2865	.56493
.04	.9989	.9992	.9997	.9992	.1119 -2	14.4815	.04381	.54	.8201	.8679	.9449	.8417	.1674	1.2703	.57501
.05	.9983	.9988	.9995	.9987	.1747 -2	11.5914	.05476	.55	.8142	.8634	.9430	.8352	.1724	1.2550	.58506
.06	.9975	.9982	.9993	.9982	.2514 -2	9.6659	.06570	.56	.8082	.8589	.9410	.8285	.1774	1.2403	.59507
.07	.9966	.9976	.9990	.9975	.3418 -2	8.2915	.07664	.57	.8022	.8544	.9390	.8216	.1825	1.2263	.60505
.08	.9955	.9968	.9987	.9968	.4460 -2	7.2616	.08758	.58	.7962	.8498	.9370	.8146	.1875	1.2130	.61501
.09	.9944	.9960	.9984	.9959	.5638 -2	6.4613	.09851	.59	.7901	.8451	.9349	.8074	.1925	1.2003	.62492
.10	.9930	.9950	.9980	.9950	.6951 -2	5.8218	.10944	.60	.7840	.8405	.9328	.8000	.1976	1.1882	.63481
.11	.9916	.9940	.9976	.9939	.8369 -2	5.2992	.12035	.61	.7778	.8357	.9307	.7924	.2026	1.1767	.64466
.12	.9900	.9928	.9971	.9928	.9979 -2	4.8643	.13126	.62	.7716	.8310	.9286	.7846	.2076	1.1657	.65448
.13	.9883	.9916	.9966	.9915	.1169 -1	4.4969	.14217	.63	.7654	.8262	.9265	.7766	.2127	1.1552	.66427
.14	.9864	.9903	.9961	.9902	.1353 -1	4.1824	.15306	.64	.7591	.8213	.9243	.7684	.2177	1.1452	.67402
.15	.9844	.9888	.9955	.9887	.1550 -1	3.9103	.16395	.65	.7528	.8164	.9221	.7599	.2227	1.1356	.68374
.16	.9823	.9873	.9949	.9871	.1760 -1	3.6727	.17482	.66	.7465	.8115	.9199	.7513	.2276	1.1265	.69342
.17	.9800	.9857	.9943	.9854	.1983 -1	3.4635	.18569	.67	.7401	.8066	.9176	.7424	.2326	1.1179	.70307
.18	.9776	.9840	.9936	.9837	.2217 -1	3.2779	.19654	.68	.7338	.8016	.9153	.7332	.2375	1.1097	.71268
.19	.9751	.9822	.9928	.9818	.2464 -1	3.1123	.20739	.69	.7274	.7966	.9131	.7238	.2424	1.1018	.72225
.20	.9725	.9803	.9921	.9798	.2723 -1	2.9635	.21822	.70	.7209	.7916	.9107	.7141	.2473	1.0944	.73179
.21	.9697	.9783	.9913	.9777	.2994 -1	2.8293	.22904	.71	.7145	.7865	.9084	.7042	.2521	1.0873	.74129
.22	.9668	.9762	.9904	.9755	.3276 -1	2.7076	.23984	.72	.7080	.7814	.9061	.6940	.2569	1.0806	.75076
.23	.9638	.9740	.9895	.9732	.3569 -1	2.5968	.25063	.73	.7016	.7763	.9037	.6834	.2617	1.0742	.76019
.24	.9607	.9718	.9886	.9705	.3874 -1	2.4956	.26141	.74	.6951	.7712	.9013	.6726	.2664	1.0681	.76958
.25	.9575	.9694	.9877	.9682	.4189 -1	2.4027	.27217	.75	.6886	.7660	.8989	.6614	.2711	1.0624	.77894
.26	.9541	.9670	.9867	.9656	.4515 -1	2.3173	.28291	.76	.6821	.7609	.8964	.6499	.2758	1.0570	.78825
.27	.9506	.9645	.9856	.9629	.4851 -1	2.2385	.29364	.77	.6756	.7557	.8940	.6380	.2804	1.0519	.79753
.28	.9470	.9619	.9846	.9600	.5197 -1	2.1656	.30435	.78	.6691	.7505	.8915	.6258	.2849	1.0471	.80677
.29	.9433	.9592	.9835	.9570	.5553 -1	2.0979	.31504	.79	.6625	.7452	.8890	.6131	.2894	1.0425	.81597
.30	.9395	.9564	.9823	.9539	.5919 -1	2.0351	.32572	.80	.6560	.7400	.8865	.6000	.2939	1.0382	.82514
.31	.9355	.9535	.9811	.9507	.6293 -1	1.9765	.33637	.81	.6495	.7347	.8840	.5864	.2983	1.0342	.83426
.32	.9315	.9506	.9799	.9474	.6677 -1	1.9219	.34701	.82	.6430	.7295	.8815	.5724	.3027	1.0305	.84335
.33	.9274	.9476	.9787	.9440	.7069 -1	1.8707	.35762	.83	.6365	.7242	.8789	.5578	.3069	1.0270	.85239
.34	.9231	.9445	.9774	.9404	.7470 -1	1.8229	.36822	.84	.6300	.7189	.8763	.5426	.3112	1.0237	.86140
.35	.9188	.9413	.9761	.9367	.7879 -1	1.7780	.37879	.85	.6235	.7136	.8737	.5268	.3153	1.0207	.87037
.36	.9143	.9380	.9747	.9330	.8295 -1	1.7358	.38935	.86	.6170	.7083	.8711	.5103	.3195	1.0179	.87929
.37	.9098	.9347	.9733	.9290	.8719 -1	1.6961	.39988	.87	.6106	.7030	.8685	.4931	.3235	1.0153	.88818
.38	.9052	.9313	.9719	.9250	.9149 -1	1.6587	.41039	.88	.6041	.6977	.8659	.4750	.3275	1.0129	.89703
.39	.9004	.9278	.9705	.9208	.9587 -1	1.6234	.42087	.89	.5977	.6924	.8632	.4560	.3314	1.0108	.90583
.40	.8956	.9243	.9690	.9165	.1003	1.5901	.43133	.90	.5913	.6870	.8606	.4359	.3352	1.0089	.91460
.41	.8907	.9207	.9675	.9121	.1048	1.5587	.44177	.91	.5849	.6817	.8579	.4146	.3390	1.0071	.92332
.42	.8857	.9170	.9659	.9075	.1094	1.5289	.45218	.92	.5785	.6764	.8552	.3919	.3427	1.0056	.93201
.43	.8807	.9132	.9643	.9028	.1140	1.5007	.46257	.93	.5721	.6711	.8525	.3676	.3464	1.0043	.94065
.44	.8755	.9094	.9627	.8980	.1187	1.4740	.47293	.94	.5658	.6658	.8498	.3412	.3500	1.0031	.94925
.45	.8703	.9055	.9611	.8930	.1234	1.4487	.48326	.95	.5595	.6604	.8471	.3122	.3534	1.0022	.95781
.46	.8650	.9016	.9594	.8879	.1281	1.4246	.49357	.96	.5532	.6551	.8444	.2800	.3569	1.0014	.96633
.47	.8596	.8976	.9577	.8827	.1329	1.4018	.50385	.97	.5469	.6498	.8416	.2431	.3602	1.0008	.97481
.48	.8541	.8935	.9560	.8773	.1378	1.3801	.51410	.98	.5407	.6445	.8389	.1990	.3635	1.0003	.98325
.49	.8486	.8894	.9542	.8717	.1426	1.3595	.52433	.99	.5345	.6392	.8361	.1411	.3667	1.0001	.99165
1.00	.5283	.6339	.8333	.0000	.3698	1.0000	1.00000	1.00	.5283	.6339	.8333	.0000	.3698	1.0000	1.00000

TABLE II.—SUPERSONIC FLOW

 $\gamma = 7/5$

$\frac{M}{M_1}$	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t_2}}{p_{t_1}}$	$\frac{p_1}{p_{t_1}}$
1.00	0.5283	0.6339	0.8333	0	0.3698	1.000	1.00000	0	90.00	1.000	1.000	1.000	1.000	1.000	0.5283
1.01	.5221	.6287	.8306	.1418	.3728	1.000	1.00831	.04473	81.93	.9901	1.023	1.017	1.007	1.000	.5221
1.02	.5160	.6234	.8278	.2010	.3758	1.000	1.01658	.1257	78.64	.9805	1.047	1.033	1.013	1.000	.5160
1.03	.5099	.6181	.8250	.2468	.3787	1.001	1.02481	.2294	76.14	.9712	1.071	1.050	1.020	1.000	.5100
1.04	.5039	.6129	.8222	.2857	.3815	1.001	1.03300	.3510	74.06	.9620	1.095	1.067	1.026	.9999	.5039
1.05	.4979	.6077	.8193	.3202	.3842	1.002	1.04114	.4874	72.25	.9531	1.120	1.084	1.033	.9999	.4980
1.06	.4919	.6024	.8165	.3516	.3869	1.003	1.04925	.6367	70.63	.9444	1.144	1.101	1.039	.9997	.4920
1.07	.4860	.5972	.8137	.3807	.3895	1.004	1.05731	.7973	69.16	.9360	1.169	1.118	1.046	.9996	.4861
1.08	.4800	.5920	.8108	.4079	.3919	1.005	1.06533	.9680	67.81	.9277	1.194	1.135	1.052	.9994	.4803
1.09	.4742	.5869	.8080	.4337	.3944	1.006	1.07331	1.148	66.55	.9196	1.219	1.152	1.059	.9992	.4746
1.10	.4684	.5817	.8052	.4583	.3967	1.008	1.08124	1.336	65.38	.9118	1.245	1.169	1.065	.9989	.4689
1.11	.4626	.5766	.8023	.4818	.3990	1.010	1.08913	1.532	64.28	.9041	1.271	1.186	1.071	.9986	.4632
1.12	.4568	.5714	.7994	.5044	.4011	1.011	1.09699	1.735	63.23	.8966	1.297	1.203	1.078	.9982	.4576
1.13	.4511	.5663	.7966	.5262	.4032	1.013	1.10479	1.944	62.25	.8892	1.323	1.221	1.084	.9978	.4521
1.14	.4455	.5612	.7937	.5474	.4052	1.015	1.11256	2.160	61.31	.8820	1.350	1.238	1.090	.9973	.4467
1.15	.4398	.5562	.7908	.5679	.4072	1.017	1.12029	2.381	60.41	.8750	1.376	1.255	1.097	.9967	.4413
1.16	.4343	.5511	.7879	.5879	.4090	1.020	1.12797	2.607	59.55	.8682	1.403	1.272	1.103	.9961	.4360
1.17	.4287	.5461	.7851	.6074	.4108	1.022	1.13561	2.839	58.73	.8615	1.430	1.290	1.109	.9953	.4307
1.18	.4232	.5411	.7822	.6264	.4125	1.025	1.14321	3.074	57.94	.8549	1.458	1.307	1.115	.9946	.4255
1.19	.4178	.5361	.7793	.6451	.4141	1.026	1.15077	3.314	57.18	.8485	1.485	1.324	1.122	.9937	.4204
1.20	.4124	.5311	.7764	.6633	.4157	1.030	1.15828	3.558	56.44	.8422	1.513	1.342	1.128	.9928	.4154
1.21	.4070	.5262	.7735	.6812	.4171	1.033	1.16575	3.806	55.74	.8360	1.541	1.359	1.134	.9918	.4104
1.22	.4017	.5213	.7706	.6989	.4185	1.037	1.17319	4.057	55.05	.8300	1.570	1.376	1.141	.9907	.4055
1.23	.3964	.5164	.7677	.7162	.4198	1.040	1.18057	4.312	54.39	.8241	1.598	1.394	1.147	.9896	.4006
1.24	.3912	.5115	.7648	.7332	.4211	1.043	1.18792	4.569	53.75	.8183	1.627	1.411	1.153	.9884	.3958

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
1.25	.3861	.5067	.7619	.7500	.4223	1.047	1.19523	4.830	53.13	.8126	1.656	1.429	1.159	.9871	.3911
1.26	.3809	.5019	.7590	.7666	.4233	1.050	1.20249	5.093	52.53	.8071	1.686	1.446	1.166	.9857	.3865
1.27	.3759	.4971	.7561	.7829	.4244	1.054	1.20972	5.359	51.94	.8016	1.715	1.463	1.172	.9842	.3819
1.28	.3708	.4923	.7532	.7990	.4253	1.058	1.21690	5.627	51.38	.7963	1.745	1.481	1.178	.9827	.3774
1.29	.3658	.4876	.7503	.8149	.4262	1.062	1.22404	5.898	50.82	.7911	1.775	1.498	1.185	.9811	.3729
1.30	.3609	.4829	.7474	.8307	.4270	1.066	1.23114	6.170	50.28	.7860	1.805	1.516	1.191	.9794	.3685
1.31	.3560	.4782	.7445	.8462	.4277	1.071	1.23819	6.445	49.76	.7809	1.835	1.533	1.197	.9776	.3642
1.32	.3512	.4736	.7416	.8616	.4283	1.075	1.24521	6.721	49.25	.7760	1.866	1.551	1.204	.9758	.3599
1.33	.3464	.4690	.7387	.8769	.4289	1.080	1.25218	7.000	48.75	.7712	1.897	1.568	1.210	.9738	.3557
1.34	.3417	.4644	.7358	.8920	.4294	1.084	1.25912	7.280	48.27	.7664	1.928	1.585	1.216	.9718	.3516
1.35	.3370	.4598	.7329	.9069	.4299	1.089	1.26601	7.561	47.79	.7618	1.960	1.603	1.223	.9697	.3475
1.36	.3323	.4553	.7300	.9217	.4303	1.094	1.27286	7.844	47.33	.7572	1.991	1.620	1.229	.9676	.3435
1.37	.3277	.4508	.7271	.9364	.4306	1.099	1.27968	8.128	46.88	.7527	2.023	1.638	1.235	.9653	.3395
1.38	.3232	.4463	.7242	.9510	.4308	1.104	1.28645	8.413	46.44	.7483	2.055	1.655	1.242	.9630	.3356
1.39	.3187	.4418	.7213	.9655	.4310	1.109	1.29318	8.699	46.01	.7440	2.087	1.672	1.248	.9607	.3317
1.40	.3142	.4374	.7184	.9798	.4311	1.115	1.29987	8.987	45.58	.7397	2.120	1.690	1.255	.9582	.3280
1.41	.3098	.4330	.7155	.9940	.4312	1.120	1.30652	9.276	45.17	.7355	2.153	1.707	1.261	.9557	.3242
1.42	.3055	.4287	.7126	1.008	.4312	1.126	1.31313	9.565	44.77	.7314	2.186	1.724	1.268	.9531	.3205
1.43	.3012	.4244	.7097	1.022	.4311	1.132	1.31970	9.855	44.37	.7274	2.219	1.742	1.274	.9504	.3169
1.44	.2969	.4201	.7069	1.036	.4310	1.138	1.32623	10.146	43.98	.7235	2.253	1.759	1.281	.9476	.3133
1.45	.2927	.4158	.7040	1.050	.4308	1.144	1.33272	10.438	43.60	.7196	2.286	1.776	1.287	.9448	.3098
1.46	.2886	.4116	.7011	1.064	.4306	1.150	1.33917	10.731	43.23	.7157	2.320	1.793	1.294	.9420	.3063
1.47	.2845	.4074	.6982	1.077	.4303	1.156	1.34558	11.023	42.86	.7120	2.354	1.811	1.300	.9390	.3029
1.48	.2804	.4032	.6954	1.091	.4299	1.161	1.35195	11.317	42.51	.7083	2.389	1.828	1.307	.9360	.2996
1.49	.2764	.3991	.6925	1.105	.4295	1.169	1.35828	11.611	42.16	.7047	2.423	1.845	1.314	.9329	.2962
1.50	.2724	.3950	.6897	1.118	.4290	1.176	1.36458	11.905	41.81	.7011	2.458	1.862	1.320	.9298	.2930
1.51	.2685	.3909	.6868	1.131	.4285	1.183	1.37083	12.200	41.47	.6976	2.493	1.879	1.327	.9266	.2898
1.52	.2646	.3869	.6840	1.145	.4279	1.190	1.37705	12.495	41.14	.6941	2.529	1.896	1.334	.9233	.2866
1.53	.2608	.3829	.6811	1.158	.4273	1.197	1.38322	12.790	40.81	.6907	2.564	1.913	1.340	.9200	.2835
1.54	.2570	.3789	.6783	1.171	.4266	1.204	1.38936	13.086	40.49	.6874	2.600	1.930	1.347	.9166	.2804
1.55	.2533	.3750	.6754	1.184	.4259	1.212	1.39546	13.381	40.18	.6841	2.636	1.947	1.354	.9132	.2773
1.56	.2496	.3710	.6726	1.197	.4252	1.219	1.40152	13.677	39.87	.6809	2.673	1.964	1.361	.9097	.2744
1.57	.2459	.3672	.6698	1.210	.4243	1.227	1.40755	13.973	39.56	.6777	2.709	1.981	1.367	.9061	.2714
1.58	.2423	.3633	.6670	1.223	.4235	1.234	1.41353	14.269	39.27	.6746	2.746	1.998	1.374	.9026	.2685
1.59	.2388	.3595	.6642	1.236	.4226	1.242	1.41948	14.564	38.97	.6715	2.783	2.015	1.381	.8989	.2656
1.60	.2353	.3557	.6614	1.249	.4216	1.250	1.42539	14.861	38.68	.6684	2.820	2.032	1.388	.8952	.2628
1.61	.2318	.3520	.6586	1.262	.4206	1.258	1.43127	15.156	38.40	.6655	2.857	2.049	1.395	.8915	.2600
1.62	.2284	.3483	.6558	1.275	.4196	1.267	1.43710	15.452	38.12	.6625	2.895	2.065	1.402	.8877	.2573
1.63	.2250	.3446	.6530	1.287	.4185	1.275	1.44290	15.747	37.84	.6596	2.933	2.082	1.409	.8838	.2546
1.64	.2217	.3409	.6502	1.300	.4174	1.284	1.44866	16.043	37.57	.6568	2.971	2.099	1.416	.8799	.2519
1.65	.2184	.3373	.6475	1.312	.4162	1.292	1.45439	16.338	37.31	.6540	3.010	2.115	1.423	.8760	.2493
1.66	.2151	.3337	.6447	1.325	.4150	1.301	1.46008	16.633	37.04	.6512	3.048	2.132	1.430	.8720	.2467
1.67	.2119	.3302	.6419	1.337	.4138	1.310	1.46573	16.928	36.78	.6485	3.087	2.148	1.437	.8680	.2442
1.68	.2088	.3266	.6392	1.350	.4125	1.319	1.47135	17.222	36.53	.6458	3.126	2.165	1.444	.8640	.2417
1.69	.2057	.3232	.6364	1.362	.4112	1.328	1.47693	17.516	36.28	.6431	3.165	2.181	1.451	.8598	.2392
1.70	.2026	.3197	.6337	1.375	.4098	1.338	1.48247	17.810	36.03	.6405	3.205	2.198	1.458	.8557	.2368
1.71	.1996	.3163	.6310	1.387	.4085	1.347	1.48798	18.103	35.79	.6380	3.245	2.214	1.466	.8516	.2344
1.72	.1966	.3129	.6283	1.399	.4071	1.357	1.49345	18.397	35.55	.6355	3.285	2.230	1.473	.8474	.2320
1.73	.1936	.3095	.6256	1.412	.4056	1.367	1.49889	18.689	35.31	.6330	3.325	2.247	1.480	.8431	.2296
1.74	.1907	.3062	.6229	1.424	.4041	1.376	1.50429	18.981	35.08	.6305	3.366	2.263	1.487	.8389	.2273
1.75	.1878	.3029	.6202	1.436	.4026	1.386	1.50966	19.273	34.85	.6281	3.406	2.279	1.495	.8346	.2251
1.76	.1850	.2996	.6175	1.448	.4011	1.397	1.51499	19.565	34.62	.6257	3.447	2.295	1.502	.8302	.2228
1.77	.1822	.2964	.6148	1.460	.3996	1.407	1.52029	19.855	34.40	.6234	3.488	2.311	1.509	.8259	.2206
1.78	.1794	.2931	.6121	1.473	.3980	1.418	1.52555	20.146	34.18	.6210	3.530	2.327	1.517	.8215	.2184
1.79	.1767	.2900	.6095	1.485	.3964	1.428	1.53078	20.436	33.96	.6188	3.571	2.343	1.524	.8171	.2163
1.80	.1740	.2868	.6068	1.497	.3947	1.439	1.53598	20.725	33.75	.6165	3.613	2.359	1.532	.8127	.2142
1.81	.1714	.2837	.6041	1.509	.3931	1.450	1.54114	21.014	33.54	.6143	3.655	2.375	1.539	.8082	.2121
1.82	.1688	.2806	.6015	1.521	.3914	1.461	1.54626	21.302	33.33	.6121	3.698	2.391	1.547	.8038	.2100
1.83	.1662	.2776	.5989	1.533	.3897	1.472	1.55136	21.590	33.12	.6099	3.740	2.407	1.554	.7993	.2080
1.84	.1637	.2745	.5963	1.545	.3879	1.484	1.55642	21.877	32.92	.6078	3.783	2.422	1.562	.7948	.2060
1.85	.1612	.2715	.5936	1.556	.3862	1.495	1.56145	22.163	32.72	.6057	3.826	2.438	1.569	.7902	.2040
1.86	.1587	.2686	.5910	1.568	.3844	1.507	1.56644	22.449	32.52	.6036	3.870	2.454	1.577	.7857	.2020
1.87	.1563	.2656	.5884	1.580	.3826	1.519	1.57140	22.735	32.33	.6016	3.913	2.469	1.585	.7811	.2001
1.88	.1539	.2627	.5859	1.592	.3808	1.531	1.57633	23.019	32.13	.5996	3.957	2.485	1.592	.7765	.1982
1.89	.1516	.2598	.5833	1.604	.3790	1.543	1.58123	23.303	31.94	.5976	4.001	2.500	1.600	.7720	.1963
1.90	.1492	.2570	.5807	1.616	.3771	1.555	1.58609	23.586	31.76	.5956	4.045	2.516	1.608	.7674	.1945
1.91	.1470	.2542	.5782	1.627	.3753	1.568	1.59092	23.869	31.57	.5937	4.089	2.531	1.616	.7627	.1927
1.92	.1447	.2514	.5756	1.639	.3734	1.580	1.59572	24.151	31.39	.5918	4.134	2.546	1.624	.7581	.1909
1.93	.1425	.2486	.5731	1.651	.3715	1.593	1.60049	24.432	31.21	.5899	4.179	2.562	1.631	.7535	.1891
1.94	.1403	.2459	.5705	1.662	.3696	1.606	1.60523	24.712	31.03	.5880	4.224	2.577	1.639	.7488	.1873
1.95	.1381	.2432	.5680	1.674	.3677	1.619	1.60993	24.992	30.85	.5862	4.270	2.592	1.647	.7442	.1856
1.96	.1360	.2405	.5655	1.686	.3657	1.633	1.61460	25.271	30.68	.5844	4.315	2.607	1.655	.7395	.1839
1.97	.1339	.2378	.5630	1.697	.3638	1.646	1.61925	25.549	30.51	.5826	4.361	2.622	1.663	.7349	.1822
1.98	.1318	.2352	.5605	1.709	.3618	1.660	1.62386	25.827	30.33	.5808	4.407	2.637	1.671	.7302	.1806
1.99	.1298	.2326	.5580	1.720	.3598	1.674	1.62844	26.104	30.17	.5791	4.453	2.652	1.679		

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
2.15	.1011	.1946	.5196	1.903	.3272	1.919	1.69774	30.425	27.72	.5540	5.226	2.882	1.813	.6511	.1553
2.16	.9956	.1925	.5173	1.915	.3252	1.935	1.70183	30.689	27.58	.5525	5.277	2.896	1.822	.6464	.1540
2.17	.9802	.1903	.5150	1.926	.3231	1.953	1.70589	30.951	27.44	.5511	5.327	2.910	1.831	.6419	.1527
2.18	.9649	.1882	.5127	1.937	.3210	1.970	1.70992	31.212	27.30	.5498	5.378	2.924	1.839	.6373	.1514
2.19	.9500	.1861	.5104	1.948	.3189	1.987	1.71393	31.473	27.17	.5484	5.429	2.938	1.848	.6327	.1502
2.20	.9352	.1841	.5081	1.960	.3169	2.005	1.71791	31.732	27.04	.5471	5.480	2.951	1.857	.6281	.1489
2.21	.9207	.1820	.5059	1.971	.3148	2.023	1.72187	31.991	26.90	.5457	5.531	2.965	1.866	.6236	.1476
2.22	.9064	.1800	.5036	1.982	.3127	2.041	1.72579	32.250	26.77	.5444	5.583	2.978	1.875	.6191	.1464
2.23	.8923	.1780	.5014	1.993	.3106	2.059	1.72970	32.507	26.64	.5431	5.636	2.992	1.883	.6145	.1452
2.24	.8785	.1760	.4991	2.004	.3085	2.078	1.73357	32.763	26.51	.5418	5.687	3.005	1.892	.6100	.1440
2.25	.8648	.1740	.4969	2.016	.3065	2.096	1.73742	33.018	26.39	.5406	5.740	3.019	1.901	.6055	.1428
2.26	.8514	.1721	.4947	2.027	.3044	2.115	1.74125	33.273	26.26	.5393	5.792	3.032	1.910	.6011	.1417
2.27	.8382	.1702	.4925	2.038	.3023	2.134	1.74504	33.527	26.14	.5381	5.845	3.045	1.919	.5966	.1405
2.28	.8251	.1683	.4903	2.049	.3003	2.154	1.74882	33.780	26.01	.5368	5.898	3.058	1.929	.5921	.1394
2.29	.8123	.1664	.4881	2.060	.2982	2.173	1.75257	34.032	25.89	.5356	5.951	3.071	1.938	.5877	.1382
2.30	.7997	.1646	.4859	2.071	.2961	2.193	1.75629	34.283	25.77	.5344	6.005	3.085	1.947	.5833	.1371
2.31	.7873	.1628	.4837	2.082	.2941	2.213	1.75999	34.533	25.65	.5332	6.059	3.098	1.956	.5789	.1360
2.32	.7751	.1609	.4816	2.093	.2920	2.233	1.76366	34.783	25.53	.5321	6.113	3.110	1.965	.5745	.1349
2.33	.7631	.1592	.4794	2.104	.2900	2.254	1.76731	35.031	25.42	.5309	6.167	3.123	1.974	.5702	.1338
2.34	.7512	.1574	.4773	2.116	.2879	2.274	1.77093	35.279	25.30	.5297	6.222	3.136	1.984	.5658	.1328
2.35	.7396	.1556	.4752	2.127	.2859	2.295	1.77453	35.526	25.18	.5286	6.276	3.149	1.993	.5615	.1317
2.36	.7281	.1539	.4731	2.138	.2839	2.316	1.77811	35.771	25.07	.5275	6.331	3.162	2.002	.5572	.1307
2.37	.7168	.1522	.4709	2.149	.2818	2.338	1.78166	36.017	24.96	.5264	6.386	3.174	2.012	.5529	.1297
2.38	.7057	.1505	.4688	2.160	.2798	2.359	1.78519	36.261	24.85	.5253	6.442	3.187	2.021	.5486	.1286
2.39	.6948	.1488	.4668	2.171	.2778	2.381	1.78869	36.504	24.73	.5242	6.497	3.199	2.031	.5444	.1276
2.40	.6840	.1472	.4647	2.182	.2758	2.403	1.79218	36.746	24.62	.5231	6.553	3.212	2.040	.5401	.1266
2.41	.6734	.1456	.4626	2.193	.2738	2.425	1.79563	36.988	24.52	.5221	6.609	3.224	2.050	.5359	.1257
2.42	.6630	.1439	.4606	2.204	.2718	2.448	1.79907	37.229	24.41	.5210	6.666	3.237	2.059	.5317	.1247
2.43	.6527	.1424	.4585	2.215	.2698	2.471	1.80248	37.469	24.30	.5200	6.722	3.249	2.069	.5276	.1237
2.44	.6426	.1408	.4565	2.226	.2678	2.494	1.80587	37.708	24.19	.5189	6.779	3.261	2.079	.5234	.1228
2.45	.6327	.1392	.4544	2.237	.2658	2.517	1.80924	37.946	24.09	.5179	6.836	3.273	2.088	.5193	.1218
2.46	.6229	.1377	.4524	2.248	.2639	2.540	1.81258	38.183	23.99	.5169	6.894	3.285	2.098	.5152	.1209
2.47	.6133	.1362	.4504	2.259	.2619	2.564	1.81591	38.420	23.88	.5159	6.951	3.298	2.108	.5111	.1200
2.48	.6038	.1346	.4484	2.269	.2599	2.588	1.81921	38.655	23.78	.5149	7.009	3.310	2.118	.5071	.1191
2.49	.5945	.1332	.4464	2.280	.2580	2.612	1.82249	38.890	23.68	.5140	7.067	3.321	2.128	.5030	.1182
2.50	.5853	.1317	.4444	2.291	.2561	2.637	1.82574	39.124	23.58	.5130	7.125	3.333	2.138	.4990	.1173
2.51	.5762	.1302	.4425	2.302	.2541	2.661	1.82898	39.357	23.48	.5120	7.183	3.345	2.147	.4950	.1164
2.52	.5674	.1288	.4405	2.313	.2522	2.686	1.83219	39.589	23.38	.5111	7.242	3.357	2.157	.4911	.1155
2.53	.5586	.1274	.4386	2.324	.2503	2.712	1.83538	39.820	23.28	.5102	7.301	3.369	2.167	.4871	.1147
2.54	.5500	.1260	.4366	2.335	.2484	2.737	1.83855	40.050	23.18	.5092	7.360	3.380	2.177	.4832	.1138
2.55	.5415	.1246	.4347	2.346	.2465	2.763	1.84170	40.280	23.09	.5083	7.420	3.392	2.187	.4793	.1130
2.56	.5332	.1232	.4328	2.357	.2446	2.789	1.84483	40.509	22.99	.5074	7.479	3.403	2.198	.4754	.1122
2.57	.5250	.1218	.4309	2.367	.2427	2.815	1.84794	40.736	22.91	.5065	7.539	3.415	2.208	.4715	.1113
2.58	.5169	.1205	.4289	2.378	.2409	2.842	1.85103	40.963	22.81	.5056	7.599	3.426	2.218	.4677	.1105
2.59	.5090	.1192	.4271	2.389	.2390	2.869	1.85410	41.189	22.71	.5047	7.659	3.438	2.228	.4639	.1097
2.60	.5012	.1179	.4252	2.400	.2371	2.896	1.85714	41.415	22.62	.5039	7.720	3.449	2.238	.4601	.1089
2.61	.4935	.1166	.4233	2.411	.2353	2.923	1.86017	41.639	22.53	.5030	7.781	3.460	2.249	.4564	.1081
2.62	.4859	.1153	.4214	2.422	.2335	2.951	1.86318	41.863	22.44	.5022	7.842	3.471	2.259	.4526	.1074
2.63	.4784	.1140	.4196	2.432	.2317	2.979	1.86616	42.086	22.35	.5013	7.903	3.483	2.269	.4489	.1066
2.64	.4711	.1128	.4177	2.443	.2298	3.007	1.86913	42.307	22.26	.5005	7.965	3.494	2.280	.4452	.1058
2.65	.4639	.1115	.4159	2.454	.2280	3.036	1.87208	42.529	22.17	.4996	8.026	3.505	2.290	.4416	.1051
2.66	.4568	.1103	.4141	2.465	.2262	3.065	1.87501	42.749	22.08	.4988	8.088	3.516	2.301	.4379	.1043
2.67	.4498	.1091	.4122	2.476	.2245	3.094	1.87792	42.968	22.00	.4980	8.150	3.527	2.311	.4343	.1036
2.68	.4429	.1079	.4104	2.486	.2227	3.123	1.88081	43.187	21.91	.4972	8.213	3.537	2.322	.4307	.1028
2.69	.4362	.1067	.4086	2.497	.2209	3.153	1.88368	43.405	21.82	.4964	8.275	3.548	2.332	.4271	.1021
2.70	.4295	.1056	.4068	2.508	.2192	3.183	1.88653	43.621	21.74	.4956	8.338	3.559	2.343	.4236	.1014
2.71	.4229	.1044	.4051	2.519	.2174	3.213	1.88936	43.838	21.65	.4949	8.401	3.570	2.354	.4201	.1007
2.72	.4165	.1033	.4033	2.530	.2157	3.244	1.89218	44.053	21.57	.4941	8.465	3.580	2.364	.4166	.1000
2.73	.4102	.1022	.4015	2.540	.2140	3.275	1.89497	44.267	21.49	.4933	8.528	3.591	2.375	.4131	.9989
2.74	.4039	.1010	.3998	2.551	.2123	3.306	1.89775	44.481	21.41	.4926	8.592	3.601	2.386	.4097	.9980
2.75	.3978	.1000	.3980	2.562	.2106	3.338	1.90051	44.694	21.32	.4918	8.656	3.612	2.397	.4062	.9972
2.76	.3917	.9985	.3963	2.572	.2089	3.370	1.90325	44.906	21.24	.4911	8.721	3.622	2.407	.4028	.9964
2.77	.3858	.9978	.3945	2.583	.2072	3.402	1.90598	45.117	21.16	.4903	8.785	3.633	2.418	.3994	.9956
2.78	.3799	.9971	.3928	2.594	.2055	3.434	1.90868	45.327	21.08	.4896	8.850	3.643	2.429	.3961	.9949
2.79	.3742	.9966	.3911	2.605	.2039	3.467	1.91137	45.537	21.00	.4889	8.915	3.653	2.440	.3928	.9942
2.80	.3685	.9961	.3894	2.615	.2022	3.500	1.91404	45.746	20.92	.4882	8.980	3.664	2.451	.3895	.9935
2.81	.3629	.9956	.3877	2.626	.2006	3.534	1.91669	45.954	20.85	.4875	9.045	3.674	2.462	.3862	.9928
2.82	.3574	.9951	.3860	2.637	.1990	3.567	1.91933	46.161	20.77	.4868	9.111	3.684	2.473	.3829	.9921
2.83	.3520	.9946	.3844	2.647	.1973	3.601	1.92195	46.368	20.69	.4861	9.177	3.694	2.484	.3797	.9914
2.84	.3467	.9941	.3827	2.658	.1957	3.636	1.92455	46.573	20.62	.4854	9.243	3.704	2.496	.3765	.9907
2.85	.3415	.9936	.3810	2.669	.1941	3.671	1.92714	46.778	20.54	.4847	9.310	3.714	2.507	.3733	.9900
2.86	.3363	.9931	.3794	2.679	.1926	3.706	1.92970	46.982	20.47	.4840	9.376	3.724	2.518	.3701	.9893
2.87	.3312	.9926	.3777	2.690	.1910	3.741	1.93225	47.185	20.39	.4833	9.443	3.734	2.529	.3670	.9886
2.88	.3263	.9921	.3761	2.701	.1894	3.777	1.93479	47.388	20.32	.4827	9.510	3.744	2.540	.3639	.9879
2.89	.3213	.9916	.3745	2.711	.1879	3.813	1.93731	47.589	20.24	.48					

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
3.05	.2526	.7226	.3496	2.881	.1645	4.441	1.97547	50.713	19.14	.4723	10.69	3.902	2.738	.3145	.8032
3.06	.2489	.7149	.3481	2.892	.1631	4.483	1.97772	50.902	19.07	.4717	10.76	3.911	2.750	.3118	.7982
3.07	.2452	.7074	.3466	2.903	.1618	4.526	1.97997	51.090	19.01	.4712	10.83	3.920	2.762	.3091	.7932
3.08	.2416	.6999	.3452	2.913	.1604	4.570	1.98219	51.277	18.95	.4706	10.90	3.929	2.774	.3065	.7882
3.09	.2380	.6925	.3437	2.924	.1591	4.613	1.98441	51.464	18.88	.4701	10.97	3.938	2.786	.3038	.7833
3.10	.2345	.6852	.3422	2.934	.1577	4.657	1.98661	51.650	18.82	.4695	11.05	3.947	2.799	.3012	.7785
3.11	.2310	.6779	.3408	2.945	.1564	4.702	1.98879	51.835	18.76	.4690	11.12	3.955	2.811	.2986	.7737
3.12	.2276	.6708	.3393	2.955	.1551	4.747	1.99097	52.020	18.69	.4685	11.19	3.964	2.823	.2960	.7689
3.13	.2243	.6637	.3379	2.966	.1538	4.792	1.99313	52.203	18.63	.4679	11.26	3.973	2.835	.2935	.7642
3.14	.2210	.6568	.3365	2.977	.1525	4.838	1.99527	52.386	18.57	.4674	11.34	3.981	2.848	.2910	.7595
3.15	.2177	.6499	.3351	2.987	.1512	4.884	1.99740	52.569	18.51	.4669	11.41	3.990	2.860	.2885	.7549
3.16	.2146	.6430	.3337	2.998	.1500	4.930	1.99952	52.751	18.45	.4664	11.48	3.998	2.872	.2860	.7503
3.17	.2114	.6363	.3323	3.008	.1487	4.977	2.00162	52.931	18.39	.4659	11.56	4.006	2.885	.2835	.7457
3.18	.2083	.6296	.3309	3.019	.1475	5.025	2.00372	53.112	18.33	.4654	11.63	4.015	2.897	.2811	.7412
3.19	.2053	.6231	.3295	3.029	.1462	5.073	2.00579	53.292	18.27	.4648	11.71	4.023	2.909	.2786	.7367
3.20	.2023	.6165	.3281	3.040	.1450	5.121	2.00786	53.470	18.21	.4643	11.78	4.031	2.922	.2762	.7323
3.21	.1993	.6101	.3267	3.050	.1438	5.170	2.00991	53.648	18.15	.4639	11.85	4.039	2.935	.2738	.7279
3.22	.1964	.6037	.3253	3.061	.1426	5.219	2.01195	53.826	18.09	.4634	11.93	4.048	2.947	.2715	.7235
3.23	.1936	.5975	.3240	3.071	.1414	5.268	2.01398	54.003	18.03	.4629	12.01	4.056	2.960	.2691	.7192
3.24	.1908	.5912	.3226	3.082	.1402	5.319	2.01599	54.179	17.98	.4624	12.08	4.064	2.972	.2668	.7149
3.25	.1880	.5851	.3213	3.092	.1390	5.369	2.01799	54.355	17.92	.4619	12.16	4.072	2.985	.2645	.7107
3.26	.1853	.5790	.3199	3.103	.1378	5.420	2.01998	54.529	17.86	.4614	12.23	4.080	2.998	.2622	.7065
3.27	.1826	.5730	.3186	3.113	.1367	5.472	2.02196	54.703	17.81	.4610	12.31	4.088	3.011	.2600	.7023
3.28	.1799	.5671	.3173	3.124	.1355	5.523	2.02392	54.877	17.75	.4605	12.38	4.096	3.023	.2577	.6982
3.29	.1773	.5612	.3160	3.134	.1344	5.576	2.02587	55.050	17.70	.4600	12.46	4.104	3.036	.2555	.6941
3.30	.1748	.5554	.3147	3.145	.1332	5.629	2.02781	55.222	17.64	.4596	12.54	4.112	3.049	.2533	.6900
3.31	.1722	.5497	.3134	3.155	.1321	5.682	2.02974	55.393	17.58	.4591	12.62	4.120	3.062	.2511	.6860
3.32	.1698	.5440	.3121	3.166	.1310	5.736	2.03165	55.564	17.53	.4587	12.69	4.128	3.075	.2489	.6820
3.33	.1673	.5384	.3108	3.176	.1299	5.790	2.03356	55.734	17.48	.4582	12.77	4.135	3.088	.2468	.6781
3.34	.1649	.5329	.3095	3.187	.1288	5.845	2.03545	55.904	17.42	.4578	12.85	4.143	3.101	.2446	.6741
3.35	.1625	.5274	.3082	3.197	.1277	5.900	2.03733	56.073	17.37	.4573	12.93	4.151	3.114	.2425	.6702
3.36	.1602	.5220	.3069	3.208	.1266	5.956	2.03920	56.241	17.31	.4569	13.00	4.158	3.127	.2404	.6664
3.37	.1579	.5166	.3057	3.218	.1255	6.012	2.04106	56.409	17.26	.4565	13.08	4.166	3.141	.2383	.6626
3.38	.1557	.5113	.3044	3.229	.1245	6.069	2.04290	56.576	17.21	.4560	13.16	4.173	3.154	.2363	.6588
3.39	.1534	.5061	.3032	3.239	.1234	6.126	2.04474	56.742	17.16	.4556	13.24	4.181	3.167	.2342	.6550
3.40	.1512	.5009	.3019	3.250	.1224	6.184	2.04666	56.907	17.10	.4552	13.32	4.188	3.180	.2322	.6513
3.41	.1491	.4958	.3007	3.260	.1214	6.242	2.04857	57.073	17.05	.4548	13.40	4.196	3.194	.2302	.6476
3.42	.1470	.4908	.2995	3.271	.1203	6.301	2.05047	57.237	17.00	.4544	13.48	4.203	3.207	.2282	.6439
3.43	.1449	.4858	.2982	3.281	.1193	6.360	2.05196	57.401	16.95	.4540	13.56	4.211	3.220	.2263	.6403
3.44	.1428	.4808	.2970	3.291	.1183	6.420	2.05374	57.564	16.90	.4535	13.64	4.218	3.234	.2243	.6367
3.45	.1408	.4759	.2958	3.302	.1173	6.480	2.05551	57.726	16.85	.4531	13.72	4.225	3.247	.2224	.6331
3.46	.1388	.4711	.2946	3.312	.1163	6.541	2.05727	57.888	16.80	.4527	13.80	4.232	3.261	.2205	.6296
3.47	.1368	.4663	.2934	3.323	.1153	6.602	2.05901	58.050	16.75	.4523	13.88	4.240	3.274	.2186	.6261
3.48	.1349	.4616	.2922	3.333	.1144	6.664	2.06075	58.210	16.70	.4519	13.96	4.247	3.288	.2167	.6226
3.49	.1330	.4569	.2910	3.344	.1134	6.727	2.06247	58.370	16.65	.4515	14.04	4.254	3.301	.2148	.6191
3.50	.1311	.4523	.2899	3.354	.1124	6.790	2.06419	58.530	16.60	.4512	14.13	4.261	3.315	.2129	.6157
3.51	.1293	.4478	.2887	3.365	.1115	6.853	2.06589	58.689	16.55	.4508	14.21	4.268	3.329	.2111	.6123
3.52	.1274	.4433	.2875	3.375	.1105	6.917	2.06759	58.847	16.51	.4504	14.29	4.275	3.343	.2093	.6089
3.53	.1256	.4388	.2864	3.385	.1096	6.982	2.06927	59.004	16.46	.4500	14.37	4.282	3.356	.2075	.6056
3.54	.1239	.4344	.2852	3.396	.1087	7.047	2.07094	59.162	16.41	.4496	14.45	4.289	3.370	.2057	.6023
3.55	.1221	.4300	.2841	3.406	.1078	7.113	2.07261	59.318	16.36	.4492	14.54	4.296	3.384	.2039	.5990
3.56	.1204	.4257	.2829	3.417	.1069	7.179	2.07426	59.474	16.31	.4489	14.62	4.303	3.398	.2022	.5957
3.57	.1188	.4214	.2818	3.427	.1059	7.246	2.07590	59.629	16.27	.4485	14.70	4.309	3.412	.2004	.5925
3.58	.1171	.4172	.2806	3.437	.1051	7.313	2.07754	59.784	16.22	.4481	14.79	4.316	3.426	.1987	.5892
3.59	.1155	.4131	.2795	3.448	.1042	7.382	2.07916	59.938	16.17	.4478	14.87	4.323	3.440	.1970	.5861
3.60	.1138	.4089	.2784	3.458	.1033	7.450	2.08077	60.091	16.13	.4474	14.95	4.330	3.454	.1953	.5829
3.61	.1123	.4049	.2773	3.469	.1024	7.519	2.08238	60.244	16.08	.4471	15.04	4.336	3.468	.1936	.5798
3.62	.1107	.4008	.2762	3.479	.1016	7.589	2.08397	60.397	16.04	.4467	15.12	4.343	3.482	.1920	.5767
3.63	.1092	.3968	.2751	3.490	.1007	7.659	2.08556	60.549	15.99	.4463	15.21	4.350	3.496	.1903	.5736
3.64	.1076	.3929	.2740	3.500	.9984	7.730	2.08713	60.700	15.95	.4460	15.29	4.356	3.510	.1887	.5705
3.65	.1062	.3890	.2729	3.510	.9900	7.802	2.08870	60.851	15.90	.4456	15.38	4.363	3.525	.1871	.5675
3.66	.1047	.3852	.2718	3.521	.9817	7.874	2.09026	61.000	15.86	.4453	15.46	4.369	3.539	.1855	.5645
3.67	.1032	.3813	.2707	3.531	.9734	7.947	2.09180	61.150	15.81	.4450	15.55	4.376	3.553	.1839	.5615
3.68	.1018	.3776	.2697	3.542	.9652	8.020	2.09334	61.299	15.77	.4446	15.63	4.382	3.568	.1823	.5585
3.69	.1004	.3739	.2686	3.552	.9570	8.094	2.09487	61.447	15.72	.4443	15.72	4.388	3.582	.1807	.5556
3.70	.9903	.3702	.2675	3.562	.9490	8.169	2.09639	61.595	15.68	.4439	15.81	4.395	3.596	.1792	.5526
3.71	.9767	.3665	.2665	3.573	.9410	8.244	2.09790	61.743	15.64	.4436	15.89	4.401	3.611	.1777	.5497
3.72	.9633	.3629	.2654	3.583	.9331	8.320	2.09941	61.889	15.59	.4433	15.98	4.408	3.625	.1761	.5469
3.73	.9500	.3594	.2644	3.593	.9253	8.397	2.10090	62.036	15.55	.4430	16.07	4.414	3.640	.1746	.5440
3.74	.9370	.3558	.2633	3.604	.9175	8.474	2.10238	62.181	15.51	.4426	16.15	4.420	3.654	.1731	.5412
3.75	.9242	.3524	.2623	3.614	.9098	8.552	2.10386	62.326	15.47	.4423	16.24	4.426	3.669	.1717	.5384
3.76	.9116	.3489	.2613	3.625	.9021	8.630	2.10533	62.471	15.42	.4420	16.33	4.432	3.684	.1702	.5356
3.77	.8991	.3455	.2602	3.635	.8945	8.709	2.10679	62.615	15.38	.4417	16.42	4.439	3.698	.1687	.5328
3.78	.8869	.3421	.2592	3.645	.8870	8.789	2.10824	62.758	15.34	.4414	16.50	4.445	3.713	.1673	.5301
3.79	.8748	.3388	.2582	3.656	.8796	8.870	2.10968	62.901	15.30	.4					

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	r	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
3.95	.7042 -1	.2902 -1	.2427	3.821	.7691 -1	10.25	2.13163	65.118	14.67	.4363	18.04	4.544	3.969	.1448	.4865 -1
3.96	.6948 -1	.2874 -1	.2418	3.832	.7627 -1	10.34	2.13294	65.253	14.63	.4360	18.13	4.549	3.985	.1435	.4841 -1
3.97	.6855 -1	.2846 -1	.2408	3.842	.7563 -1	10.44	2.13424	65.386	14.59	.4358	18.22	4.555	4.000	.1423	.4817 -1
3.98	.6764 -1	.2819 -1	.2399	3.852	.7500 -1	10.53	2.13553	65.520	14.55	.4355	18.31	4.560	4.016	.1411	.4793 -1
3.99	.6675 -1	.2793 -1	.2390	3.863	.7438 -1	10.62	2.13681	65.652	14.52	.4352	18.41	4.566	4.031	.1399	.4770 -1
4.00	.6586 -1	.2766 -1	.2381	3.873	.7376 -1	10.72	2.13809	65.785	14.48	.4350	18.50	4.571	4.047	.1388	.4747 -1
4.01	.6499 -1	.2740 -1	.2372	3.883	.7315 -1	10.81	2.13936	65.917	14.44	.4347	18.59	4.577	4.062	.1376	.4723 -1
4.02	.6413 -1	.2714 -1	.2363	3.894	.7255 -1	10.91	2.14062	66.048	14.40	.4344	18.69	4.582	4.078	.1364	.4700 -1
4.03	.6328 -1	.2688 -1	.2354	3.904	.7194 -1	11.01	2.14188	66.179	14.37	.4342	18.78	4.588	4.094	.1353	.4678 -1
4.04	.6245 -1	.2663 -1	.2345	3.914	.7133 -1	11.11	2.14312	66.309	14.33	.4339	18.88	4.593	4.110	.1342	.4655 -1
4.05	.6163 -1	.2638 -1	.2336	3.925	.7076 -1	11.21	2.14436	66.439	14.30	.4336	18.97	4.598	4.125	.1330	.4633 -1
4.06	.6082 -1	.2613 -1	.2327	3.935	.7017 -1	11.31	2.14560	66.569	14.26	.4334	19.06	4.604	4.141	.1319	.4610 -1
4.07	.6002 -1	.2589 -1	.2319	3.945	.6959 -1	11.41	2.14682	66.698	14.22	.4331	19.16	4.609	4.157	.1308	.4588 -1
4.08	.5923 -1	.2564 -1	.2310	3.956	.6902 -1	11.51	2.14804	66.826	14.19	.4329	19.25	4.614	4.173	.1297	.4566 -1
4.09	.5845 -1	.2540 -1	.2301	3.966	.6845 -1	11.61	2.14926	66.954	14.15	.4326	19.35	4.619	4.189	.1286	.4544 -1
4.10	.5769 -1	.2516 -1	.2293	3.976	.6788 -1	11.71	2.15046	67.082	14.12	.4324	19.45	4.624	4.205	.1276	.4523 -1
4.11	.5694 -1	.2493 -1	.2284	3.986	.6732 -1	11.82	2.15166	67.209	14.08	.4321	19.54	4.630	4.221	.1265	.4501 -1
4.12	.5619 -1	.2470 -1	.2275	3.997	.6677 -1	11.92	2.15285	67.336	14.05	.4319	19.64	4.635	4.237	.1254	.4480 -1
4.13	.5546 -1	.2447 -1	.2267	4.007	.6622 -1	12.03	2.15404	67.462	14.01	.4316	19.73	4.640	4.253	.1244	.4459 -1
4.14	.5474 -1	.2424 -1	.2258	4.017	.6568 -1	12.14	2.15522	67.588	13.98	.4314	19.83	4.645	4.269	.1234	.4438 -1
4.15	.5403 -1	.2401 -1	.2250	4.028	.6514 -1	12.24	2.15639	67.713	13.94	.4311	19.93	4.650	4.285	.1223	.4417 -1
4.16	.5333 -1	.2379 -1	.2242	4.038	.6460 -1	12.35	2.15756	67.838	13.91	.4309	20.02	4.655	4.301	.1213	.4396 -1
4.17	.5264 -1	.2357 -1	.2233	4.048	.6407 -1	12.46	2.15871	67.963	13.88	.4306	20.12	4.660	4.318	.1203	.4375 -1
4.18	.5195 -1	.2335 -1	.2225	4.059	.6354 -1	12.57	2.15987	68.087	13.84	.4304	20.22	4.665	4.334	.1193	.4355 -1
4.19	.5128 -1	.2313 -1	.2217	4.069	.6302 -1	12.68	2.16101	68.210	13.81	.4302	20.32	4.670	4.350	.1183	.4334 -1
4.20	.5062 -1	.2292 -1	.2208	4.079	.6251 -1	12.79	2.16215	68.333	13.77	.4299	20.41	4.675	4.367	.1173	.4314 -1
4.21	.4997 -1	.2271 -1	.2200	4.090	.6200 -1	12.90	2.16329	68.456	13.74	.4297	20.51	4.680	4.383	.1164	.4294 -1
4.22	.4932 -1	.2250 -1	.2192	4.100	.6149 -1	13.02	2.16442	68.578	13.71	.4295	20.61	4.685	4.399	.1154	.4274 -1
4.23	.4869 -1	.2229 -1	.2184	4.110	.6098 -1	13.13	2.16554	68.700	13.67	.4292	20.71	4.690	4.416	.1144	.4255 -1
4.24	.4806 -1	.2209 -1	.2176	4.120	.6049 -1	13.25	2.16665	68.821	13.64	.4290	20.81	4.694	4.432	.1135	.4235 -1
4.25	.4745 -1	.2189 -1	.2168	4.131	.5999 -1	13.36	2.16776	68.942	13.61	.4288	20.91	4.699	4.449	.1126	.4215 -1
4.26	.4684 -1	.2169 -1	.2160	4.141	.5950 -1	13.48	2.16886	69.063	13.58	.4286	21.01	4.704	4.466	.1116	.4196 -1
4.27	.4624 -1	.2149 -1	.2152	4.151	.5902 -1	13.60	2.16996	69.183	13.54	.4283	21.11	4.709	4.482	.1107	.4177 -1
4.28	.4565 -1	.2129 -1	.2144	4.162	.5854 -1	13.72	2.17105	69.302	13.51	.4281	21.20	4.713	4.499	.1098	.4158 -1
4.29	.4507 -1	.2110 -1	.2136	4.172	.5806 -1	13.83	2.17214	69.422	13.48	.4279	21.30	4.718	4.516	.1089	.4139 -1
4.30	.4449 -1	.2090 -1	.2129	4.182	.5759 -1	13.95	2.17321	69.541	13.45	.4277	21.41	4.723	4.532	.1080	.4120 -1
4.31	.4393 -1	.2071 -1	.2121	4.192	.5712 -1	14.08	2.17429	69.659	13.42	.4275	21.51	4.728	4.549	.1071	.4101 -1
4.32	.4337 -1	.2052 -1	.2113	4.203	.5666 -1	14.20	2.17535	69.777	13.38	.4272	21.61	4.732	4.566	.1062	.4082 -1
4.33	.4282 -1	.2034 -1	.2105	4.213	.5620 -1	14.32	2.17642	69.895	13.35	.4270	21.71	4.737	4.583	.1054	.4064 -1
4.34	.4228 -1	.2015 -1	.2098	4.223	.5574 -1	14.45	2.17747	70.012	13.32	.4268	21.81	4.741	4.600	.1045	.4046 -1
4.35	.4174 -1	.1997 -1	.2090	4.233	.5529 -1	14.57	2.17852	70.128	13.29	.4266	21.91	4.746	4.617	.1036	.4027 -1
4.36	.4121 -1	.1979 -1	.2083	4.244	.5484 -1	14.70	2.17956	70.245	13.26	.4264	22.01	4.751	4.633	.1028	.4009 -1
4.37	.4069 -1	.1961 -1	.2075	4.254	.5440 -1	14.82	2.18060	70.361	13.23	.4262	22.11	4.755	4.651	.1020	.3991 -1
4.38	.4018 -1	.1944 -1	.2067	4.264	.5396 -1	14.95	2.18163	70.476	13.20	.4260	22.22	4.760	4.668	.1011	.3973 -1
4.39	.3968 -1	.1926 -1	.2060	4.275	.5352 -1	15.08	2.18266	70.591	13.17	.4258	22.32	4.764	4.685	.1003	.3956 -1
4.40	.3918 -1	.1909 -1	.2053	4.285	.5309 -1	15.21	2.18368	70.706	13.14	.4255	22.42	4.768	4.702	.09948	.3938 -1
4.41	.3868 -1	.1892 -1	.2045	4.295	.5266 -1	15.34	2.18470	70.820	13.11	.4253	22.52	4.773	4.719	.09867	.3921 -1
4.42	.3820 -1	.1875 -1	.2038	4.305	.5224 -1	15.47	2.18571	70.934	13.08	.4251	22.63	4.777	4.736	.09787	.3903 -1
4.43	.3772 -1	.1858 -1	.2030	4.316	.5182 -1	15.61	2.18671	71.048	13.05	.4249	22.73	4.782	4.753	.09707	.3886 -1
4.44	.3725 -1	.1841 -1	.2023	4.326	.5140 -1	15.74	2.187714	71.161	13.02	.4247	22.83	4.786	4.771	.09628	.3869 -1
4.45	.3678 -1	.1825 -1	.2016	4.336	.5099 -1	15.87	2.188708	71.274	12.99	.4245	22.94	4.790	4.788	.09550	.3852 -1
4.46	.3633 -1	.1808 -1	.2009	4.346	.5058 -1	16.01	2.189697	71.386	12.96	.4243	23.04	4.795	4.805	.09473	.3835 -1
4.47	.3587 -1	.1792 -1	.2002	4.357	.5017 -1	16.15	2.190681	71.498	12.93	.4241	23.14	4.799	4.823	.09396	.3818 -1
4.48	.3543 -1	.1776 -1	.1994	4.367	.4977 -1	16.28	2.191659	71.610	12.90	.4239	23.25	4.803	4.840	.09320	.3801 -1
4.49	.3499 -1	.1761 -1	.1987	4.377	.4937 -1	16.42	2.192632	71.721	12.87	.4237	23.35	4.808	4.858	.09244	.3785 -1
4.50	.3455 -1	.1745 -1	.1980	4.387	.4898 -1	16.56	2.193603	71.832	12.84	.4236	23.46	4.812	4.875	.09170	.3768 -1
4.51	.3412 -1	.1729 -1	.1973	4.398	.4859 -1	16.70	2.194563	71.942	12.81	.4234	23.56	4.816	4.893	.09096	.3752 -1
4.52	.3370 -1	.1714 -1	.1966	4.408	.4820 -1	16.84	2.195520	72.052	12.78	.4232	23.67	4.820	4.910	.09022	.3735 -1
4.53	.3329 -1	.1699 -1	.1959	4.418	.4781 -1	16.99	2.196473	72.162	12.75	.4230	23.77	4.824	4.928	.08950	.3719 -1
4.54	.3288 -1	.1684 -1	.1952	4.428	.4743 -1	17.13	2.197420	72.271	12.73	.4228	23.88	4.829	4.946	.08878	.3703 -1
4.55	.3247 -1	.1669 -1	.1945	4.439	.4706 -1	17.28	2.198363	72.380	12.70	.4226	23.99	4.833	4.963	.08806	.3687 -1
4.56	.3207 -1	.1654 -1	.1938	4.449	.4668 -1	17.42	2.199300	72.489	12.67	.4224	24.09	4.837	4.981	.08735	.3671 -1
4.57	.3168 -1	.1640 -1	.1932	4.459	.4631 -1	17.57	2.200233	72.597	12.64	.4222	24.20	4.841	4.999	.08665	.3656 -1
4.58	.3129 -1	.1625 -1	.1925	4.469	.4594 -1	17.72	2.201160	72.705	12.61	.4220	24.31	4.845	5.017	.08596	.3640 -1
4.59	.3090 -1	.1611 -1	.1918	4.480	.4558 -1	17.87	2.202033	72.812	12.58	.4219	24.41	4.849	5.034	.08527	.3624 -1
4.60	.3053 -1	.1597 -1	.1911	4.490	.4522 -1	18.02	2.203000	72.919	12.56	.4217	24.52	4.853	5.052	.08459	.3609 -1
4.61	.3015 -1	.1583 -1	.1905	4.500	.4486 -1	18.17	2.203913	73.026	12.53	.4215	24.63	4.857	5.070	.08391	.3593 -1
4.62	.2978 -1	.1569 -1	.1898	4.510	.4450 -1	18.32	2.204822	73.132	12.50	.4213	24.74	4.861	5.088	.08324	.3578 -1
4.63	.2942 -1	.1556 -1	.1891	4.521	.4415 -1	18.48	2.205725	73.238	12.47	.4211	24.84	4.865	5.106	.08257	.3563 -1
4.64	.2906 -1	.1542 -1	.1885	4.531	.4380 -1	18.63	2.206624	73.344	12.45	.4210	24.95	4.869	5.124	.08192	.3548 -1
4.65	.2871 -1	.1529 -1	.1878	4.541	.4345 -1	18.79	2.207518	73.449	12.42	.4208	25.06	4.873	5		

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A^*}$	$\frac{V}{a^*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
4.85	.2255 -2	.1287 -1	.1753	4.746	.3714 -1	22.15	2.224455	75.482	11.90	.4175	27.28	4.948	5.512	.6036 -1	.3252 -1
4.86	.2229 -1	.1276 -1	.1747	4.756	.3685 -1	22.33	2.225257	75.580	11.87	.4173	27.39	4.952	5.531	.6082 -1	.3239 -1
4.87	.2202 -1	.1265 -1	.1741	4.766	.3657 -1	22.51	2.226055	75.678	11.85	.4172	27.50	4.955	5.550	.6128 -1	.3226 -1
4.88	.2177 -1	.1254 -1	.1735	4.776	.3628 -1	22.70	2.226848	75.775	11.83	.4170	27.62	4.959	5.569	.6175 -1	.3213 -1
4.89	.2151 -1	.1244 -1	.1729	4.787	.3600 -1	22.88	2.227638	75.872	11.80	.4169	27.73	4.962	5.588	.6222 -1	.3200 -1
4.90	.2126 -1	.1233 -1	.1724	4.797	.3573 -1	23.07	2.228424	75.969	11.78	.4167	27.85	4.966	5.607	.6270 -1	.3187 -1
4.91	.2101 -1	.1223 -1	.1718	4.807	.3545 -1	23.25	2.229206	76.066	11.75	.4165	27.96	4.969	5.626	.6318 -1	.3174 -1
4.92	.2076 -1	.1213 -1	.1712	4.817	.3518 -1	23.44	2.229984	76.162	11.73	.4164	28.07	4.973	5.646	.6367 -1	.3161 -1
4.93	.2052 -1	.1202 -1	.1706	4.828	.3491 -1	23.63	2.230758	76.258	11.70	.4163	28.19	4.976	5.665	.6416 -1	.3149 -1
4.94	.2028 -1	.1192 -1	.1700	4.838	.3464 -1	23.82	2.231528	76.353	11.68	.4161	28.30	4.980	5.684	.6465 -1	.3136 -1
4.95	.2004 -1	.1182 -1	.1695	4.848	.3437 -1	24.02	2.232294	76.449	11.66	.4160	28.42	4.983	5.703	.6515 -1	.3124 -1
4.96	.1981 -1	.1173 -1	.1689	4.858	.3411 -1	24.21	2.233056	76.544	11.63	.4158	28.54	4.987	5.723	.6566 -1	.3111 -1
4.97	.1957 -1	.1163 -1	.1683	4.868	.3385 -1	24.41	2.233815	76.638	11.61	.4157	28.65	4.990	5.742	.6617 -1	.3099 -1
4.98	.1935 -1	.1153 -1	.1678	4.879	.3359 -1	24.60	2.234570	76.732	11.58	.4155	28.77	4.993	5.761	.6668 -1	.3087 -1
4.99	.1912 -1	.1144 -1	.1672	4.889	.3333 -1	24.80	2.235321	76.826	11.56	.4154	28.88	4.997	5.781	.6720 -1	.3075 -1
5.00	.1890 -1	.1134 -1	.1667	4.899	.3308 -1	25.00	2.236068	76.920	11.54	.4152	29.00	5.000	5.800	.6772 -1	.3062 -1
5.01	.1868 -1	.1125 -1	.1661	4.909	.3282 -1	25.20	2.236811	77.013	11.51	.4151	29.12	5.003	5.820	.6824 -1	.3051 -1
5.02	.1847 -1	.1115 -1	.1656	4.919	.3257 -1	25.40	2.237551	77.106	11.49	.4149	29.23	5.007	5.839	.6877 -1	.3039 -1
5.03	.1825 -1	.1106 -1	.1650	4.930	.3233 -1	25.61	2.238287	77.199	11.47	.4148	29.35	5.010	5.859	.6930 -1	.3027 -1
5.04	.1804 -1	.1097 -1	.1645	4.940	.3208 -1	25.81	2.239020	77.291	11.44	.4147	29.47	5.013	5.878	.6984 -1	.3015 -1
5.05	.1783 -1	.1088 -1	.1639	4.950	.3184 -1	26.02	2.239749	77.385	11.42	.4145	29.59	5.016	5.898	.7038 -1	.3003 -1
5.06	.1763 -1	.1079 -1	.1634	4.960	.3159 -1	26.22	2.240474	77.477	11.40	.4144	29.70	5.020	5.918	.7093 -1	.2991 -1
5.07	.1742 -1	.1070 -1	.1628	4.970	.3135 -1	26.43	2.241195	77.568	11.38	.4142	29.82	5.023	5.937	.7148 -1	.2980 -1
5.08	.1722 -1	.1061 -1	.1623	4.981	.3112 -1	26.64	2.241914	77.660	11.35	.4141	29.94	5.026	5.957	.7204 -1	.2968 -1
5.09	.1703 -1	.1053 -1	.1618	4.991	.3088 -1	26.86	2.242628	77.751	11.33	.4140	30.06	5.029	5.977	.7260 -1	.2957 -1
5.10	.1683 -1	.1044 -1	.1612	5.001	.3065 -1	27.07	2.243339	77.841	11.31	.4138	30.18	5.033	5.997	.7317 -1	.2945 -1
5.11	.1664 -1	.1035 -1	.1607	5.011	.3042 -1	27.28	2.244047	77.931	11.29	.4137	30.30	5.036	6.016	.7374 -1	.2934 -1
5.12	.1645 -1	.1027 -1	.1602	5.021	.3019 -1	27.50	2.244751	78.021	11.26	.4136	30.42	5.039	6.036	.7432 -1	.2923 -1
5.13	.1626 -1	.1019 -1	.1597	5.032	.2996 -1	27.72	2.245451	78.111	11.24	.4134	30.54	5.042	6.056	.7491 -1	.2911 -1
5.14	.1608 -1	.1010 -1	.1591	5.042	.2973 -1	27.94	2.246148	78.201	11.22	.4133	30.66	5.045	6.076	.7550 -1	.2900 -1
5.15	.1589 -1	.1002 -1	.1586	5.052	.2951 -1	28.16	2.246842	78.290	11.20	.4132	30.78	5.048	6.096	.7610 -1	.2889 -1
5.16	.1571 -1	.9939 -2	.1581	5.062	.2929 -1	28.38	2.247532	78.379	11.18	.4130	30.90	5.051	6.117	.7670 -1	.2878 -1
5.17	.1553 -1	.9858 -2	.1576	5.072	.2907 -1	28.60	2.248219	78.468	11.15	.4129	31.02	5.054	6.137	.7730 -1	.2867 -1
5.18	.1536 -1	.9778 -2	.1571	5.083	.2885 -1	28.83	2.248903	78.556	11.13	.4128	31.14	5.058	6.157	.7791 -1	.2856 -1
5.19	.1518 -1	.9699 -2	.1566	5.093	.2863 -1	29.06	2.249583	78.645	11.11	.4126	31.26	5.061	6.177	.7852 -1	.2845 -1
5.20	.1501 -1	.9620 -2	.1561	5.103	.2842 -1	29.28	2.250260	78.733	11.09	.4125	31.38	5.064	6.197	.7913 -1	.2834 -1
5.21	.1484 -1	.9543 -2	.1555	5.113	.2821 -1	29.51	2.250934	78.820	11.07	.4124	31.50	5.067	6.217	.7975 -1	.2824 -1
5.22	.1468 -1	.9466 -2	.1550	5.123	.2799 -1	29.74	2.251604	78.908	11.04	.4123	31.62	5.070	6.238	.8037 -1	.2813 -1
5.23	.1451 -1	.9389 -2	.1545	5.134	.2778 -1	29.98	2.252271	78.995	11.02	.4121	31.75	5.073	6.258	.8100 -1	.2803 -1
5.24	.1435 -1	.9314 -2	.1540	5.144	.2758 -1	30.21	2.252935	79.081	11.00	.4120	31.87	5.076	6.278	.8163 -1	.2792 -1
5.25	.1419 -1	.9239 -2	.1536	5.154	.2737 -1	30.45	2.253596	79.167	10.98	.4119	31.99	5.079	6.299	.8226 -1	.2782 -1
5.26	.1403 -1	.9165 -2	.1531	5.164	.2717 -1	30.68	2.254254	79.254	10.96	.4118	32.11	5.082	6.319	.8290 -1	.2771 -1
5.27	.1387 -1	.9092 -2	.1526	5.174	.2697 -1	30.92	2.254908	79.340	10.94	.4116	32.24	5.085	6.340	.8354 -1	.2761 -1
5.28	.1372 -1	.9019 -2	.1521	5.184	.2677 -1	31.16	2.255559	79.426	10.92	.4115	32.36	5.088	6.360	.8418 -1	.2750 -1
5.29	.1356 -1	.8947 -2	.1516	5.195	.2657 -1	31.41	2.256207	79.511	10.90	.4114	32.48	5.090	6.381	.8483 -1	.2740 -1
5.30	.1341 -1	.8875 -2	.1511	5.205	.2637 -1	31.65	2.256852	79.597	10.88	.4113	32.61	5.093	6.401	.8548 -1	.2730 -1
5.31	.1326 -1	.8805 -2	.1506	5.215	.2617 -1	31.89	2.257494	79.681	10.86	.4112	32.73	5.096	6.422	.8614 -1	.2720 -1
5.32	.1311 -1	.8734 -2	.1501	5.225	.2598 -1	32.14	2.258133	79.765	10.83	.4110	32.85	5.099	6.443	.8680 -1	.2710 -1
5.33	.1297 -1	.8665 -2	.1497	5.235	.2579 -1	32.39	2.258779	79.850	10.81	.4109	32.98	5.102	6.464	.8746 -1	.2700 -1
5.34	.1282 -1	.8596 -2	.1492	5.246	.2560 -1	32.64	2.259401	79.934	10.79	.4108	33.10	5.105	6.484	.8813 -1	.2690 -1
5.35	.1268 -1	.8528 -2	.1487	5.256	.2541 -1	32.89	2.260031	80.018	10.77	.4107	33.23	5.108	6.505	.8880 -1	.2680 -1
5.36	.1254 -1	.8461 -2	.1482	5.266	.2522 -1	33.14	2.260658	80.101	10.75	.4106	33.35	5.111	6.526	.8948 -1	.2670 -1
5.37	.1240 -1	.8394 -2	.1478	5.276	.2504 -1	33.40	2.261281	80.185	10.73	.4104	33.48	5.113	6.547	.9016 -1	.2660 -1
5.38	.1227 -1	.8327 -2	.1473	5.286	.2485 -1	33.66	2.261902	80.268	10.71	.4103	33.60	5.116	6.568	.9085 -1	.2650 -1
5.39	.1213 -1	.8262 -2	.1468	5.296	.2467 -1	33.91	2.262520	80.351	10.69	.4102	33.73	5.119	6.589	.9154 -1	.2641 -1
5.40	.1200 -1	.8197 -2	.1464	5.307	.2449 -1	34.17	2.263135	80.434	10.67	.4101	33.85	5.122	6.610	.9224 -1	.2631 -1
5.41	.1187 -1	.8132 -2	.1459	5.317	.2431 -1	34.44	2.263747	80.515	10.65	.4100	33.98	5.125	6.631	.9294 -1	.2621 -1
5.42	.1174 -1	.8068 -2	.1454	5.327	.2413 -1	34.70	2.264356	80.597	10.63	.4099	34.11	5.127	6.652	.9365 -1	.2612 -1
5.43	.1161 -1	.8005 -2	.1450	5.337	.2395 -1	34.97	2.264962	80.680	10.61	.4098	34.23	5.130	6.673	.9436 -1	.2602 -1
5.44	.1148 -1	.7942 -2	.1445	5.347	.2378 -1	35.23	2.265566	80.760	10.59	.4096	34.36	5.133	6.694	.9508 -1	.2593 -1
5.45	.1135 -1	.7880 -2	.1441	5.357	.2361 -1	35.50	2.266166	80.842	10.57	.4095	34.49	5.136	6.715	.9580 -1	.2583 -1
5.46	.1123 -1	.7818 -2	.1436	5.368	.2344 -1	35.77	2.266764	80.923	10.55	.4094	34.61	5.138	6.737	.9653 -1	.2574 -1
5.47	.1111 -1	.7757 -2	.1432	5.378	.2326 -1	36.04	2.267359	81.004	10.53	.4093	34.74	5.141	6.758	.9726 -1	.2565 -1
5.48	.1099 -1	.7697 -2	.1427	5.388	.2309 -1	36.32	2.267951	81.084	10.51	.4092	34.87	5.144	6.779	.9799 -1	.2556 -1
5.49	.1087 -1	.7637 -2	.1423	5.398	.2293 -1	36.59	2.268540	81.165	10.50	.4091	35.00	5.146	6.800	.9873 -1	.2546 -1
5.50	.1075 -1	.7578 -2	.1418	5.408	.2276 -1	36.87	2.269127	81.245	10.48	.4090	35.13	5.149	6.822	.9947 -1	.2537 -1
5.51	.1063 -1	.7519 -2	.1414	5.418	.2260 -1	37.15	2.269711	81.324	10.46	.4089	35.25	5.152	6.843	.1000 -1	.2528 -1
5.52	.1052 -1	.7460 -2	.1410	5.429	.2243 -1	37.43	2.270292	81.404	10.44	.4088	35.38	5.154	6.865	.1000 -1	.2519 -1
5.53	.1040 -1	.7403 -2	.1405	5.439	.2227 -1	37.71	2.270870	81.484	10.42	.4086	35.51	5.157	6.886	.1000 -1	.2510 -1
5.54	.1029 -1	.7345 -2	.1401	5.449	.2211 -1	38.00	2.271446	81.563	10.40	.4085	35.64	5.159	6.90		

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A^*}$	$\frac{V}{a^*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$					
5.75	.8216	→	.6254	→	.1314	5.662	.1902	→	44.40	2.282942	83.169	10.02	.4064	38.41	5.212	7.369	.3536	→	.2324	→
5.76	.8130	→	.6207	→	.1310	5.673	.1888	→	44.72	2.283462	83.243	9.998	.4063	38.54	5.214	7.392	.3510	→	.2316	→
5.77	.8044	→	.6161	→	.1306	5.683	.1875	→	45.05	2.283980	83.317	9.980	.4062	38.68	5.217	7.414	.3486	→	.2308	→
5.78	.7960	→	.6114	→	.1302	5.693	.1862	→	45.38	2.284496	83.391	9.963	.4061	38.81	5.219	7.436	.3461	→	.2300	→
5.79	.7876	→	.6069	→	.1298	5.703	.1848	→	45.72	2.285009	83.463	9.946	.4060	38.94	5.221	7.459	.3436	→	.2292	→
5.80	.7794	→	.6023	→	.1294	5.713	.1835	→	46.05	2.285520	83.537	9.928	.4059	39.08	5.224	7.481	.3412	→	.2284	→
5.81	.7713	→	.5978	→	.1290	5.723	.1823	→	46.39	2.286029	83.609	9.911	.4059	39.22	5.226	7.504	.3388	→	.2277	→
5.82	.7632	→	.5934	→	.1286	5.733	.1810	→	46.72	2.286535	83.683	9.894	.4058	39.35	5.228	7.527	.3364	→	.2269	→
5.83	.7553	→	.5889	→	.1282	5.744	.1797	→	47.07	2.287040	83.755	9.877	.4057	39.49	5.231	7.549	.3340	→	.2261	→
5.84	.7474	→	.5846	→	.1279	5.754	.1784	→	47.41	2.287542	83.827	9.860	.4056	39.62	5.233	7.572	.3317	→	.2254	→
5.85	.7396	→	.5802	→	.1275	5.764	.1772	→	47.75	2.288041	83.899	9.842	.4055	39.76	5.235	7.595	.3293	→	.2246	→
5.86	.7320	→	.5759	→	.1271	5.774	.1760	→	48.10	2.288539	83.971	9.826	.4054	39.90	5.237	7.618	.3270	→	.2238	→
5.87	.7244	→	.5716	→	.1267	5.784	.1747	→	48.45	2.289034	84.042	9.809	.4053	40.03	5.240	7.640	.3247	→	.2231	→
5.88	.7169	→	.5674	→	.1263	5.794	.1735	→	48.80	2.289527	84.114	9.792	.4052	40.17	5.242	7.663	.3225	→	.2223	→
5.89	.7095	→	.5632	→	.1260	5.804	.1723	→	49.15	2.290018	84.185	9.775	.4051	40.31	5.244	7.686	.3202	→	.2216	→
5.90	.7021	→	.5590	→	.1256	5.815	.1711	→	49.51	2.290507	84.257	9.758	.4050	40.45	5.246	7.709	.3180	→	.2208	→
5.91	.6949	→	.5549	→	.1252	5.825	.1699	→	49.86	2.290993	84.327	9.742	.4049	40.58	5.249	7.732	.3157	→	.2201	→
5.92	.6877	→	.5508	→	.1249	5.835	.1687	→	50.22	2.291477	84.398	9.725	.4049	40.72	5.251	7.755	.3135	→	.2194	→
5.93	.6807	→	.5468	→	.1245	5.845	.1676	→	50.59	2.291960	84.468	9.708	.4048	40.86	5.253	7.778	.3113	→	.2186	→
5.94	.6737	→	.5428	→	.1241	5.855	.1664	→	50.95	2.292440	84.539	9.692	.4047	41.00	5.255	7.801	.3092	→	.2179	→
5.95	.6668	→	.5388	→	.1238	5.865	.1652	→	51.32	2.292918	84.609	9.675	.4046	41.14	5.257	7.824	.3070	→	.2172	→
5.96	.6599	→	.5348	→	.1234	5.876	.1641	→	51.68	2.293394	84.679	9.659	.4045	41.28	5.260	7.847	.3049	→	.2165	→
5.97	.6532	→	.5309	→	.1230	5.886	.1630	→	52.05	2.293867	84.748	9.643	.4044	41.41	5.262	7.871	.3028	→	.2157	→
5.98	.6465	→	.5270	→	.1227	5.896	.1618	→	52.43	2.294339	84.817	9.626	.4043	41.55	5.264	7.894	.3007	→	.2150	→
5.99	.6399	→	.5232	→	.1223	5.906	.1607	→	52.80	2.294809	84.887	9.610	.4042	41.69	5.266	7.917	.2986	→	.2143	→
6.00	.6334	→	.5194	→	.1220	5.916	.1596	→	53.18	2.295276	84.955	9.594	.4042	41.83	5.268	7.941	.2965	→	.2136	→
6.01	.6269	→	.5156	→	.1216	5.926	.1585	→	53.56	2.295742	85.025	9.578	.4041	41.97	5.270	7.964	.2945	→	.2129	→
6.02	.6205	→	.5118	→	.1212	5.936	.1574	→	53.94	2.296205	85.095	9.562	.4040	42.11	5.273	7.987	.2924	→	.2122	→
6.03	.6142	→	.5081	→	.1209	5.947	.1563	→	54.32	2.296667	85.162	9.546	.4039	42.25	5.275	8.011	.2904	→	.2115	→
6.04	.6080	→	.5044	→	.1205	5.957	.1553	→	54.71	2.297126	85.230	9.530	.4038	42.40	5.277	8.034	.2884	→	.2108	→
6.05	.6018	→	.5008	→	.1202	5.967	.1542	→	55.10	2.297583	85.297	9.514	.4037	42.54	5.279	8.058	.2864	→	.2101	→
6.06	.5957	→	.4971	→	.1198	5.977	.1531	→	55.49	2.298039	85.366	9.498	.4037	42.68	5.281	8.081	.2844	→	.2094	→
6.07	.5897	→	.4935	→	.1195	5.987	.1521	→	55.88	2.298492	85.433	9.482	.4036	42.82	5.283	8.105	.2825	→	.2088	→
6.08	.5838	→	.4900	→	.1191	5.997	.1511	→	56.28	2.298944	85.500	9.467	.4035	42.96	5.285	8.129	.2806	→	.2081	→
6.09	.5779	→	.4864	→	.1188	6.007	.1500	→	56.68	2.299393	85.568	9.451	.4034	43.10	5.287	8.152	.2786	→	.2074	→
6.10	.5721	→	.4829	→	.1185	6.017	.1490	→	57.08	2.299841	85.635	9.435	.4033	43.25	5.289	8.176	.2767	→	.2067	→
6.11	.5663	→	.4795	→	.1181	6.028	.1480	→	57.48	2.300286	85.702	9.420	.4033	43.39	5.291	8.200	.2748	→	.2061	→
6.12	.5606	→	.4760	→	.1178	6.038	.1470	→	57.88	2.300730	85.768	9.404	.4032	43.53	5.293	8.223	.2730	→	.2054	→
6.13	.5550	→	.4726	→	.1174	6.048	.1460	→	58.29	2.301172	85.834	9.389	.4031	43.67	5.295	8.247	.2711	→	.2047	→
6.14	.5494	→	.4692	→	.1171	6.058	.1450	→	58.70	2.301612	85.901	9.373	.4030	43.82	5.297	8.271	.2692	→	.2041	→
6.15	.5439	→	.4658	→	.1168	6.068	.1440	→	59.11	2.302050	85.967	9.358	.4029	43.96	5.299	8.295	.2674	→	.2034	→
6.16	.5385	→	.4625	→	.1164	6.078	.1430	→	59.53	2.302486	86.033	9.343	.4029	44.10	5.301	8.319	.2656	→	.2028	→
6.17	.5331	→	.4592	→	.1161	6.088	.1421	→	59.94	2.302920	86.099	9.327	.4028	44.25	5.303	8.343	.2638	→	.2021	→
6.18	.5278	→	.4559	→	.1158	6.099	.1411	→	60.36	2.303353	86.164	9.312	.4027	44.39	5.305	8.367	.2620	→	.2015	→
6.19	.5225	→	.4527	→	.1154	6.109	.1402	→	60.79	2.303783	86.229	9.297	.4026	44.54	5.307	8.391	.2602	→	.2008	→
6.20	.5173	→	.4495	→	.1151	6.119	.1392	→	61.21	2.304212	86.295	9.282	.4025	44.68	5.309	8.415	.2584	→	.2002	→
6.21	.5122	→	.4463	→	.1148	6.129	.1383	→	61.64	2.304639	86.360	9.267	.4025	44.82	5.311	8.439	.2567	→	.1995	→
6.22	.5071	→	.4431	→	.1144	6.139	.1373	→	62.07	2.305064	86.424	9.252	.4024	44.97	5.313	8.464	.2550	→	.1989	→
6.23	.5021	→	.4400	→	.1141	6.149	.1364	→	62.50	2.305487	86.489	9.237	.4023	45.12	5.315	8.488	.2532	→	.1983	→
6.24	.4971	→	.4369	→	.1138	6.159	.1355	→	62.93	2.305908	86.554	9.222	.4022	45.26	5.317	8.512	.2515	→	.1977	→
6.25	.4922	→	.4338	→	.1135	6.169	.1346	→	63.37	2.306328	86.618	9.207	.4022	45.41	5.319	8.536	.2498	→	.1970	→
6.26	.4874	→	.4307	→	.1132	6.180	.1337	→	63.81	2.306746	86.683	9.192	.4021	45.55	5.321	8.561	.2482	→	.1964	→
6.27	.4825	→	.4277	→	.1128	6.190	.1328	→	64.25	2.307162	86.746	9.177	.4020	45.70	5.323	8.585	.2465	→	.1958	→
6.28	.4778	→	.4246	→	.1125	6.200	.1319	→	64.69	2.307576	86.810	9.163	.4019	45.84	5.325	8.610	.2448	→	.1952	→
6.29	.4731	→	.4217	→	.1122	6.210	.1310	→	65.14	2.307989	86.874	9.148	.4019	45.99	5.327	8.634	.2432	→	.1945	→
6.30	.4684	→	.4187	→	.1119	6.220	.1302	→	65.59	2.308400	86.937	9.133	.4018	46.14	5.329	8.658	.2416	→	.1939	→
6.31	.4638	→	.4158	→	.1116	6.230	.1293	→	66.04	2.308809	87.000	9.119	.4017	46.29	5.331	8.683	.2399	→	.1933	→
6.32	.4593	→	.4128	→	.1113	6.240	.1284	→	66.50	2.309216	87.063	9.104	.4016	46.43	5.332	8.708	.2383	→	.1927	→
6.33	.4548	→	.4100	→	.1109	6.251	.1276	→	66.95	2.309622	87.126	9.090	.4016	46.58	5.334	8.732	.2367	→	.1921	→
6.34	.4504	→	.4071	→	.1106	6.261	.1267	→	67.41	2.310026	87.189	9.075	.4015	46.73	5.336	8.757	.2352	→	.1915	→
6.35	.4460	→	.4042	→	.1103	6.271	.1259	→	67.88	2.310429	87.251	9.061	.4014	46.88	5.338	8.781	.2336	→	.1909	→
6.36	.4416	→	.4014	→	.1100	6.281	.1250	→	68.34	2.310828	87.313	9.046	.4014	47.02	5.340	8.806	.2320	→	.1903	→
6.37	.4373	→	.3986	→	.1097	6.291	.1242	→	68.81	2.311227										

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
6.65	.3341 -	.3289 -	.1016	6.574	.1034 -	83.03	2.321750	89.049	8.649	.3994	51.43	5.391	9.540	.1918 -	.1742 -
6.66	.3309 -	.3267 -	.1013	6.584	.1028 -	83.58	2.322104	89.106	8.636	.3993	51.58	5.392	9.566	.1905 -	.1737 -
6.67	.3278 -	.3245 -	.1010	6.595	.1021 -	84.13	2.322456	89.164	8.623	.3993	51.74	5.394	9.592	.1893 -	.1732 -
6.68	.3247 -	.3223 -	.1008	6.605	.1014 -	84.68	2.322807	89.221	8.610	.3992	51.89	5.395	9.618	.1881 -	.1727 -
6.69	.3217 -	.3201 -	.1005	6.615	.1008 -	85.24	2.323157	89.278	8.597	.3992	52.05	5.397	9.644	.1869 -	.1721 -
6.70	.3187 -	.3180 -	.1002	6.625	.1001 -	85.80	2.323505	89.335	8.584	.3991	52.21	5.399	9.670	.1857 -	.1716 -
6.71	.3157 -	.3158 -	.9995 -	6.635	.9950 -	86.37	2.323852	89.391	8.571	.3990	52.36	5.400	9.696	.1845 -	.1711 -
6.72	.3127 -	.3137 -	.9968 -	6.645	.9886 -	86.94	2.324198	89.448	8.558	.3990	52.52	5.402	9.722	.1833 -	.1706 -
6.73	.3098 -	.3116 -	.9942 -	6.655	.9823 -	87.51	2.324542	89.504	8.545	.3989	52.68	5.403	9.748	.1821 -	.1701 -
6.74	.3069 -	.3096 -	.9915 -	6.665	.9761 -	88.08	2.324884	89.561	8.532	.3988	52.83	5.405	9.775	.1810 -	.1696 -
6.75	.3041 -	.3075 -	.9889 -	6.676	.9699 -	88.66	2.325226	89.617	8.520	.3988	52.99	5.407	9.801	.1798 -	.1691 -
6.76	.3013 -	.3055 -	.9862 -	6.686	.9637 -	89.24	2.325566	89.673	8.507	.3987	53.15	5.408	9.827	.1786 -	.1686 -
6.77	.2985 -	.3034 -	.9836 -	6.696	.9576 -	89.82	2.325904	89.729	8.494	.3987	53.31	5.410	9.853	.1775 -	.1681 -
6.78	.2957 -	.3014 -	.9810 -	6.706	.9515 -	90.41	2.326242	89.784	8.482	.3986	53.46	5.411	9.880	.1764 -	.1677 -
6.79	.2930 -	.2994 -	.9784 -	6.716	.9454 -	91.00	2.326578	89.840	8.469	.3986	53.62	5.413	9.906	.1753 -	.1671 -
6.80	.2902 -	.2974 -	.9758 -	6.726	.9395 -	91.59	2.326912	89.895	8.457	.3985	53.78	5.415	9.933	.1741 -	.1667 -
6.81	.2876 -	.2955 -	.9732 -	6.736	.9335 -	92.19	2.327245	89.950	8.444	.3984	53.94	5.416	9.959	.1730 -	.1662 -
6.82	.2849 -	.2935 -	.9706 -	6.746	.9276 -	92.79	2.327577	90.005	8.432	.3984	54.10	5.418	9.986	.1719 -	.1657 -
6.83	.2823 -	.2916 -	.9681 -	6.756	.9218 -	93.39	2.327908	90.060	8.419	.3983	54.26	5.419	10.01	.1709 -	.1652 -
6.84	.2797 -	.2897 -	.9655 -	6.767	.9160 -	94.00	2.328237	90.116	8.407	.3983	54.42	5.421	10.04	.1698 -	.1647 -
6.85	.2771 -	.2878 -	.9630 -	6.777	.9102 -	94.61	2.328565	90.170	8.394	.3982	54.58	5.422	10.07	.1687 -	.1643 -
6.86	.2746 -	.2859 -	.9604 -	6.787	.9045 -	95.22	2.328892	90.225	8.382	.3981	54.74	5.424	10.09	.1676 -	.1638 -
6.87	.2720 -	.2840 -	.9579 -	6.797	.8988 -	95.83	2.329217	90.279	8.370	.3981	54.90	5.425	10.12	.1666 -	.1633 -
6.88	.2696 -	.2821 -	.9554 -	6.807	.8931 -	96.45	2.329541	90.333	8.357	.3980	55.06	5.427	10.15	.1655 -	.1628 -
6.89	.2671 -	.2803 -	.9529 -	6.817	.8875 -	97.08	2.329864	90.387	8.345	.3980	55.22	5.428	10.17	.1645 -	.1624 -
6.90	.2646 -	.2785 -	.9504 -	6.827	.8820 -	97.70	2.330186	90.441	8.333	.3979	55.38	5.430	10.20	.1634 -	.1619 -
6.91	.2622 -	.2766 -	.9479 -	6.837	.8764 -	98.33	2.330506	90.495	8.321	.3979	55.54	5.431	10.23	.1624 -	.1614 -
6.92	.2598 -	.2745 -	.9454 -	6.847	.8710 -	98.96	2.330825	90.549	8.309	.3978	55.70	5.433	10.25	.1614 -	.1610 -
6.93	.2575 -	.2730 -	.9430 -	6.857	.8655 -	99.60	2.331143	90.602	8.297	.3977	55.86	5.434	10.28	.1604 -	.1605 -
6.94	.2551 -	.2713 -	.9405 -	6.868	.8601 -	100.2	2.331460	90.655	8.285	.3977	56.02	5.436	10.31	.1594 -	.1601 -
6.95	.2528 -	.2695 -	.9380 -	6.878	.8548 -	100.9	2.331775	90.709	8.273	.3976	56.19	5.437	10.33	.1584 -	.1596 -
6.96	.2505 -	.2677 -	.9356 -	6.888	.8495 -	101.5	2.332089	90.762	8.261	.3976	56.35	5.439	10.36	.1574 -	.1592 -
6.97	.2482 -	.2660 -	.9332 -	6.898	.8442 -	102.2	2.332402	90.815	8.249	.3975	56.51	5.440	10.39	.1564 -	.1587 -
6.98	.2460 -	.2643 -	.9307 -	6.908	.8389 -	102.8	2.332714	90.867	8.237	.3975	56.67	5.442	10.42	.1554 -	.1582 -
6.99	.2438 -	.2626 -	.9283 -	6.918	.8337 -	103.5	2.333024	90.920	8.225	.3974	56.84	5.443	10.44	.1545 -	.1578 -
7.00	.2416 -	.2609 -	.9259 -	6.928	.8286 -	104.1	2.333333	90.973	8.213	.3974	57.00	5.444	10.47	.1535 -	.1574 -
7.01	.2394 -	.2592 -	.9235 -	6.938	.8234 -	104.8	2.333641	91.026	8.201	.3973	57.16	5.446	10.50	.1526 -	.1569 -
7.02	.2372 -	.2575 -	.9211 -	6.948	.8183 -	105.5	2.333948	91.078	8.190	.3973	57.33	5.447	10.52	.1516 -	.1565 -
7.03	.2351 -	.2559 -	.9188 -	6.959	.8133 -	106.1	2.334254	91.130	8.178	.3972	57.49	5.449	10.55	.1507 -	.1560 -
7.04	.2330 -	.2542 -	.9164 -	6.969	.8082 -	106.8	2.334558	91.182	8.166	.3971	57.66	5.450	10.58	.1497 -	.1556 -
7.05	.2309 -	.2526 -	.9140 -	6.979	.8032 -	107.5	2.334862	91.234	8.153	.3971	57.82	5.452	10.61	.1488 -	.1551 -
7.06	.2288 -	.2510 -	.9117 -	6.989	.7983 -	108.2	2.335164	91.286	8.141	.3970	57.98	5.453	10.63	.1479 -	.1547 -
7.07	.2267 -	.2494 -	.9093 -	6.999	.7934 -	108.9	2.335465	91.337	8.131	.3970	58.15	5.454	10.66	.1470 -	.1543 -
7.08	.2247 -	.2478 -	.9070 -	7.009	.7885 -	109.5	2.335765	91.389	8.120	.3969	58.31	5.456	10.69	.1461 -	.1538 -
7.09	.2227 -	.2462 -	.9047 -	7.019	.7837 -	110.2	2.336063	91.440	8.108	.3969	58.48	5.457	10.72	.1452 -	.1534 -
7.10	.2207 -	.2446 -	.9024 -	7.029	.7789 -	110.9	2.336361	91.492	8.097	.3968	58.65	5.459	10.74	.1443 -	.1530 -
7.11	.2187 -	.2430 -	.9001 -	7.039	.7741 -	111.6	2.336657	91.543	8.085	.3968	58.81	5.460	10.77	.1434 -	.1525 -
7.12	.2168 -	.2415 -	.8978 -	7.049	.7693 -	112.3	2.336952	91.594	8.074	.3967	58.98	5.461	10.80	.1425 -	.1521 -
7.13	.2149 -	.2400 -	.8955 -	7.060	.7646 -	113.0	2.337246	91.645	8.062	.3967	59.14	5.463	10.83	.1416 -	.1517 -
7.14	.2130 -	.2384 -	.8932 -	7.070	.7600 -	113.7	2.337539	91.695	8.051	.3966	59.31	5.464	10.85	.1408 -	.1513 -
7.15	.2111 -	.2369 -	.8909 -	7.080	.7553 -	114.5	2.337831	91.746	8.040	.3966	59.48	5.465	10.88	.1399 -	.1509 -
7.16	.2092 -	.2354 -	.8886 -	7.090	.7507 -	115.2	2.338122	91.796	8.028	.3965	59.64	5.467	10.91	.1390 -	.1504 -
7.17	.2073 -	.2339 -	.8864 -	7.100	.7461 -	115.9	2.338412	91.847	8.017	.3965	59.81	5.468	10.94	.1382 -	.1500 -
7.18	.2055 -	.2324 -	.8841 -	7.110	.7416 -	116.6	2.338700	91.897	8.006	.3964	59.98	5.470	10.97	.1374 -	.1496 -
7.19	.2037 -	.2310 -	.8819 -	7.120	.7371 -	117.3	2.338988	91.947	7.995	.3964	60.15	5.471	10.99	.1365 -	.1492 -
7.20	.2019 -	.2295 -	.8797 -	7.130	.7326 -	118.1	2.339274	91.997	7.984	.3963	60.31	5.472	11.02	.1357 -	.1488 -
7.21	.2001 -	.2281 -	.8774 -	7.140	.7281 -	118.8	2.339559	92.047	7.972	.3963	60.48	5.474	11.05	.1349 -	.1484 -
7.22	.1983 -	.2266 -	.8752 -	7.150	.7237 -	119.6	2.339843	92.097	7.961	.3962	60.65	5.475	11.08	.1340 -	.1480 -
7.23	.1966 -	.2252 -	.8730 -	7.161	.7194 -	120.3	2.340127	92.146	7.950	.3962	60.82	5.476	11.11	.1332 -	.1476 -
7.24	.1949 -	.2238 -	.8708 -	7.171	.7150 -	121.0	2.340409	92.196	7.939	.3961	60.99	5.478	11.13	.1324 -	.1472 -
7.25	.1932 -	.2224 -	.8686 -	7.181	.7107 -	121.8	2.340690	92.245	7.928	.3961	61.16	5.479	11.16	.1316 -	.1468 -
7.26	.1915 -	.2210 -	.8664 -	7.191	.7064 -	122.5	2.340969	92.294	7.917	.3960	61.33	5.480	11.19	.1308 -	.1464 -
7.27	.1898 -	.2196 -	.8643 -	7.201	.7021 -	123.3	2.341248	92.343	7.906	.3960	61.50	5.481	11.22	.1300 -	.1460 -
7.28	.1881 -	.2182 -	.8621 -	7.211	.6979 -	124.1	2.341526	92.392	7.895	.3959	61.66	5.483	11.25	.1292 -	.1456 -
7.29	.1865 -	.2169 -	.8599 -	7.221	.6937 -	124.8	2.341803	92.441	7.884	.3959	61.83	5.484	11.28	.1285 -	.1452 -
7.30	.1848 -	.2155 -	.8578 -	7.231	.6896 -	125.6	2.342079	92.490	7.874	.3958	62.01	5.485	11.30	.1277 -	.1448 -
7.31	.1832 -	.2142 -	.8556 -	7.241	.6854 -	126.4	2.342353	92.538	7.863	.3958	62.18	5.487	11.33	.1269 -	.1444 -
7.32	.1816 -	.2128 -	.8535 -	7.251	.6813 -	127.2	2.342627	92.587	7.852	.3957	62.35	5.488	11.36	.1262 -	.1440 -
7.33	.1801 -	.2115 -	.8514 -	7.261	.6772 -	127.9	2.342900	92.635	7.841	.3957	62.52	5.489	11.39	.1254 -	.1436 -
7.34	.1785 -	.2102 -	.8492 -	7.272	.6731 -	128.7	2.343171	92.684	7.830	.3956	62.69	5.490	11.42	.1246 -	.1432 -
7.35	.1769 -	.2089 -	.8471 -	7.282	.6691 -	129.5	2.343442	92.732	7.820	.3956	62.86	5.492	11.45	.1239 -	.1428 -
7.36	.1754 -	.2076 -	.8450 -	7.292	.6651 -	130.3	2.343711	92.780							

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
7.55	.1489 -1	.1847 -1	.8064 -1	7.483	.5942 -2	146.2	2.348648	93.670	7.611	.3947	66.34	5.516	12.03	.1100 -1	.1354 -1
7.56	.1477 -1	.1836 -1	.8045 -1	7.494	.5908 -1	147.0	2.348899	93.716	7.601	.3946	66.51	5.517	12.06	.1093 -1	.1351 -1
7.57	.1464 -1	.1824 -1	.8025 -1	7.504	.5873 -1	147.9	2.349148	93.762	7.591	.3946	66.69	5.518	12.09	.1087 -1	.1347 -1
7.58	.1452 -1	.1813 -1	.8006 -1	7.514	.5839 -1	148.8	2.349397	93.807	7.581	.3946	66.87	5.520	12.11	.1081 -1	.1343 -1
7.59	.1439 -1	.1802 -1	.7986 -1	7.524	.5805 -1	149.7	2.349644	93.853	7.571	.3945	67.04	5.521	12.14	.1074 -1	.1340 -1
7.60	.1427 -1	.1792 -1	.7967 -1	7.534	.5771 -1	150.6	2.349891	93.898	7.561	.3945	67.22	5.522	12.17	.1068 -1	.1336 -1
7.61	.1415 -1	.1781 -1	.7948 -1	7.544	.5737 -1	151.5	2.350137	93.943	7.551	.3944	67.40	5.523	12.20	.1062 -1	.1333 -1
7.62	.1403 -1	.1770 -1	.7928 -1	7.554	.5704 -1	152.4	2.350382	93.988	7.541	.3944	67.58	5.524	12.23	.1056 -1	.1329 -1
7.63	.1391 -1	.1759 -1	.7909 -1	7.564	.5671 -1	153.3	2.350626	94.033	7.531	.3943	67.75	5.525	12.26	.1049 -1	.1326 -1
7.64	.1380 -1	.1749 -1	.7890 -1	7.574	.5638 -1	154.2	2.350869	94.078	7.521	.3943	67.93	5.527	12.29	.1043 -1	.1322 -1
7.65	.1368 -1	.1738 -1	.7871 -1	7.584	.5605 -1	155.1	2.351112	94.123	7.511	.3943	68.11	5.528	12.32	.1037 -1	.1319 -1
7.66	.1357 -1	.1728 -1	.7852 -1	7.594	.5572 -1	156.0	2.351353	94.168	7.501	.3942	68.29	5.529	12.35	.1031 -1	.1316 -1
7.67	.1345 -1	.1717 -1	.7833 -1	7.605	.5540 -1	157.0	2.351594	94.212	7.491	.3942	68.47	5.530	12.38	.1025 -1	.1312 -1
7.68	.1334 -1	.1707 -1	.7815 -1	7.615	.5508 -1	157.9	2.351834	94.257	7.482	.3941	68.65	5.531	12.41	.1019 -1	.1309 -1
7.69	.1323 -1	.1697 -1	.7796 -1	7.625	.5476 -1	158.8	2.352072	94.301	7.472	.3941	68.83	5.532	12.44	.1013 -1	.1305 -1
7.70	.1312 -1	.1687 -1	.7777 -1	7.635	.5445 -1	159.8	2.352310	94.345	7.462	.3941	69.01	5.533	12.47	.1008 -1	.1302 -1
7.71	.1301 -1	.1677 -1	.7759 -1	7.645	.5413 -1	160.7	2.352548	94.389	7.452	.3940	69.18	5.534	12.50	.1002 -1	.1299 -1
7.72	.1290 -1	.1667 -1	.7740 -1	7.655	.5382 -1	161.7	2.352784	94.433	7.443	.3940	69.36	5.536	12.53	.9959 -2	.1295 -1
7.73	.1279 -1	.1657 -1	.7722 -1	7.665	.5351 -1	162.6	2.353019	94.477	7.433	.3939	69.55	5.537	12.56	.9902 -2	.1292 -1
7.74	.1269 -1	.1647 -1	.7703 -1	7.675	.5320 -1	163.6	2.353254	94.521	7.423	.3939	69.73	5.538	12.59	.9845 -2	.1289 -1
7.75	.1258 -1	.1637 -1	.7685 -1	7.685	.5290 -1	164.5	2.353488	94.565	7.414	.3939	69.91	5.539	12.62	.9788 -2	.1285 -1
7.76	.1248 -1	.1627 -1	.7667 -1	7.695	.5259 -1	165.5	2.353721	94.608	7.404	.3938	70.09	5.540	12.65	.9732 -2	.1282 -1
7.77	.1237 -1	.1618 -1	.7648 -1	7.705	.5229 -1	166.5	2.353953	94.652	7.395	.3938	70.27	5.541	12.68	.9676 -2	.1279 -1
7.78	.1227 -1	.1608 -1	.7630 -1	7.715	.5199 -1	167.4	2.354184	94.695	7.385	.3937	70.45	5.542	12.71	.9620 -2	.1276 -1
7.79	.1217 -1	.1599 -1	.7612 -1	7.726	.5170 -1	168.4	2.354415	94.739	7.375	.3937	70.63	5.543	12.74	.9565 -2	.1272 -1
7.80	.1207 -1	.1589 -1	.7594 -1	7.736	.5140 -1	169.4	2.354644	94.782	7.366	.3937	70.81	5.544	12.77	.9510 -2	.1269 -1
7.81	.1197 -1	.1580 -1	.7576 -1	7.746	.5111 -1	170.4	2.354873	94.825	7.356	.3936	71.00	5.545	12.80	.9456 -2	.1266 -1
7.82	.1187 -1	.1571 -1	.7558 -1	7.756	.5082 -1	171.4	2.355101	94.868	7.347	.3936	71.18	5.547	12.83	.9402 -2	.1263 -1
7.83	.1177 -1	.1561 -1	.7540 -1	7.766	.5053 -1	172.4	2.355328	94.911	7.338	.3935	71.36	5.548	12.86	.9348 -2	.1259 -1
7.84	.1168 -1	.1552 -1	.7523 -1	7.776	.5024 -1	173.4	2.355555	94.954	7.328	.3935	71.54	5.549	12.89	.9295 -2	.1256 -1
7.85	.1158 -1	.1543 -1	.7505 -1	7.786	.4995 -1	174.4	2.355780	94.996	7.319	.3935	71.73	5.550	12.92	.9242 -2	.1253 -1
7.86	.1149 -1	.1534 -1	.7487 -1	7.796	.4967 -1	175.4	2.356005	95.039	7.309	.3934	71.91	5.551	12.96	.9189 -2	.1250 -1
7.87	.1139 -1	.1525 -1	.7470 -1	7.806	.4939 -1	176.4	2.356229	95.082	7.300	.3934	72.09	5.552	12.99	.9137 -2	.1247 -1
7.88	.1130 -1	.1516 -1	.7452 -1	7.816	.4911 -1	177.5	2.356453	95.124	7.291	.3933	72.28	5.553	13.02	.9085 -2	.1244 -1
7.89	.1121 -1	.1507 -1	.7435 -1	7.826	.4883 -1	178.5	2.356675	95.166	7.281	.3933	72.46	5.554	13.05	.9033 -2	.1241 -1
7.90	.1111 -1	.1498 -1	.7417 -1	7.836	.4855 -1	179.5	2.356897	95.208	7.272	.3933	72.65	5.555	13.08	.8982 -2	.1237 -1
7.91	.1102 -1	.1490 -1	.7400 -1	7.847	.4828 -1	180.5	2.357118	95.251	7.263	.3932	72.83	5.556	13.11	.8931 -2	.1234 -1
7.92	.1093 -1	.1481 -1	.7383 -1	7.857	.4801 -1	181.6	2.357338	95.293	7.254	.3932	73.01	5.557	13.14	.8880 -2	.1231 -1
7.93	.1084 -1	.1472 -1	.7365 -1	7.867	.4774 -1	182.6	2.357557	95.334	7.245	.3932	73.20	5.558	13.17	.8830 -2	.1228 -1
7.94	.1076 -1	.1464 -1	.7348 -1	7.877	.4747 -1	183.7	2.357776	95.376	7.235	.3931	73.38	5.559	13.20	.8780 -2	.1225 -1
7.95	.1067 -1	.1455 -1	.7331 -1	7.887	.4720 -1	184.7	2.357994	95.418	7.226	.3931	73.57	5.560	13.23	.8731 -2	.1222 -1
7.96	.1058 -1	.1447 -1	.7314 -1	7.897	.4693 -1	185.8	2.358211	95.460	7.217	.3930	73.76	5.561	13.26	.8682 -2	.1219 -1
7.97	.1050 -1	.1438 -1	.7297 -1	7.907	.4667 -1	186.9	2.358427	95.501	7.208	.3930	73.94	5.562	13.29	.8633 -2	.1216 -1
7.98	.1041 -1	.1430 -1	.7280 -1	7.917	.4641 -1	188.0	2.358642	95.542	7.199	.3930	74.13	5.563	13.33	.8584 -2	.1213 -1
7.99	.1033 -1	.1422 -1	.7263 -1	7.927	.4615 -1	189.0	2.358857	95.584	7.190	.3929	74.31	5.564	13.36	.8536 -2	.1210 -1
8.00	.1024 -1	.1414 -1	.7246 -1	7.937	.4589 -1	190.1	2.359071	95.625	7.181	.3929	74.50	5.565	13.39	.8488 -2	.1207 -1
8.01	.1016 -1	.1405 -1	.7230 -1	7.947	.4563 -1	191.2	2.359285	95.666	7.172	.3929	74.69	5.566	13.42	.8440 -2	.1204 -1
8.02	.1008 -1	.1397 -1	.7213 -1	7.957	.4538 -1	192.3	2.359497	95.707	7.163	.3928	74.87	5.567	13.45	.8393 -2	.1201 -1
8.03	.1000 -1	.1389 -1	.7196 -1	7.967	.4512 -1	193.4	2.359709	95.748	7.154	.3928	75.06	5.568	13.48	.8346 -2	.1198 -1
8.04	.9916 -1	.1381 -1	.7180 -1	7.978	.4487 -1	194.5	2.359920	95.789	7.145	.3927	75.25	5.569	13.51	.8299 -2	.1195 -1
8.05	.9837 -1	.1373 -1	.7163 -1	7.988	.4462 -1	195.6	2.360130	95.830	7.136	.3927	75.44	5.570	13.54	.8253 -2	.1192 -1
8.06	.9758 -1	.1365 -1	.7147 -1	7.998	.4437 -1	196.7	2.360340	95.871	7.127	.3927	75.62	5.571	13.57	.8207 -2	.1189 -1
8.07	.9679 -1	.1358 -1	.7130 -1	8.008	.4413 -1	197.8	2.360549	95.911	7.118	.3926	75.81	5.572	13.61	.8161 -2	.1186 -1
8.08	.9602 -1	.1350 -1	.7114 -1	8.018	.4388 -1	199.0	2.360757	95.951	7.109	.3926	76.00	5.573	13.64	.8115 -2	.1183 -1
8.09	.9525 -1	.1342 -1	.7097 -1	8.028	.4364 -1	200.1	2.360965	95.992	7.100	.3926	76.19	5.574	13.67	.8070 -2	.1180 -1
8.10	.9449 -1	.1334 -1	.7081 -1	8.038	.4339 -1	201.2	2.361172	96.032	7.092	.3925	76.38	5.575	13.70	.8025 -2	.1177 -1
8.11	.9373 -1	.1327 -1	.7065 -1	8.048	.4315 -1	202.4	2.361378	96.073	7.083	.3925	76.57	5.576	13.73	.7981 -2	.1174 -1
8.12	.9298 -1	.1319 -1	.7049 -1	8.058	.4292 -1	203.5	2.361583	96.112	7.074	.3925	76.76	5.577	13.76	.7937 -2	.1172 -1
8.13	.9224 -1	.1312 -1	.7033 -1	8.068	.4268 -1	204.6	2.361788	96.153	7.065	.3924	76.95	5.578	13.80	.7893 -2	.1169 -1
8.14	.9150 -1	.1304 -1	.7017 -1	8.078	.4244 -1	205.8	2.361992	96.193	7.057	.3924	77.14	5.579	13.83	.7849 -2	.1166 -1
8.15	.9078 -1	.1297 -1	.7001 -1	8.088	.4221 -1	207.0	2.362195	96.233	7.048	.3924	77.33	5.580	13.86	.7805 -2	.1163 -1
8.16	.9005 -1	.1289 -1	.6985 -1	8.098	.4197 -1	208.1	2.362397	96.272	7.039	.3923	77.52	5.581	13.89	.7762 -2	.1160 -1
8.17	.8934 -1	.1282 -1	.6969 -1	8.109	.4174 -1	209.3	2.362599	96.312	7.031	.3923	77.71	5.582	13.92	.7719 -2	.1157 -1
8.18	.8863 -1	.1275 -1	.6953 -1	8.119	.4151 -1	210.5	2.362800	96.352	7.022	.3923	77.90	5.583	13.95	.7677 -2	.1155 -1
8.19	.8793 -1	.1267 -1	.6937 -1	8.129	.4129 -1	211.7	2.363001	96.391	7.013	.3922	78.09	5.584	13.99	.7634 -2	.1152 -1
8.20	.8723 -1	.1260 -1	.6921 -1	8.139	.4106 -1	212.8	2.363201	96.430	7.005	.3922	78.28	5.585	14.02	.7592 -2	.1149 -1
8.21	.8654 -1	.1253 -1	.6906 -1	8.149	.4083 -1	214.0	2.363400	96.470	6.996	.3921	78.47	5.586	14.05	.7551 -2	.1146 -1
8.22	.8586 -1	.1246 -1	.6890 -1	8.159	.4061 -1	215.2	2.363598	96.509	6.988	.3921	78.66	5.587	14.08	.7509 -2	.1143 -1
8.23	.8518 -1	.1239 -1	.6874 -1	8.169	.4039 -1	216.4	2.363796	96.548	6.979	.3921	78.86	5.588	14.11	.7468 -2	.1141 -1
8.2															

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_1}{p_1}$	$\frac{\rho_1}{\rho_1}$	$\frac{T_1}{T_1}$	$\frac{p_{t_1}}{p_{t_1}}$	$\frac{p_{t_2}}{p_{t_2}}$
8.45	.7170	.1096	.6544	8.391	.3584	244.4	2.367983	97.388	6.797	.3914	83.14	5.607	14.83	.6625	.1082
8.46	.7115	.1090	.6530	8.401	.3565	245.7	2.368166	97.424	6.788	.3913	83.33	5.608	14.86	.6589	.1080
8.47	.7060	.1084	.6515	8.411	.3545	247.0	2.368348	97.462	6.780	.3913	83.53	5.609	14.89	.6554	.1077
8.48	.7006	.1078	.6501	8.421	.3526	248.4	2.368530	97.499	6.772	.3913	83.73	5.610	14.93	.6519	.1075
8.49	.6952	.1072	.6487	8.431	.3508	249.7	2.368712	97.536	6.764	.3912	83.93	5.611	14.96	.6484	.1072
8.50	.6898	.1066	.6472	8.441	.3489	251.1	2.368892	97.573	6.756	.3912	84.13	5.612	14.99	.6449	.1070
8.51	.6846	.1060	.6458	8.451	.3470	252.5	2.369072	97.609	6.748	.3912	84.32	5.613	15.02	.6415	.1067
8.52	.6793	.1054	.6444	8.461	.3452	253.8	2.369252	97.646	6.740	.3911	84.52	5.613	15.05	.6380	.1065
8.53	.6741	.1048	.6430	8.471	.3433	255.2	2.369431	97.683	6.732	.3911	84.72	5.614	15.09	.6346	.1062
8.54	.6690	.1043	.6416	8.481	.3415	256.6	2.369609	97.719	6.725	.3911	84.92	5.615	15.12	.6313	.1060
8.55	.6638	.1037	.6402	8.491	.3397	258.0	2.369787	97.756	6.717	.3911	85.12	5.616	15.16	.6279	.1057
8.56	.6588	.1031	.6388	8.501	.3379	259.4	2.369964	97.792	6.709	.3910	85.32	5.617	15.19	.6246	.1055
8.57	.6538	.1026	.6374	8.511	.3361	260.8	2.370140	97.828	6.701	.3910	85.52	5.618	15.22	.6212	.1052
8.58	.6488	.1020	.6360	8.522	.3343	262.2	2.370316	97.865	6.693	.3910	85.72	5.618	15.26	.6179	.1050
8.59	.6438	.1015	.6346	8.532	.3326	263.6	2.370492	97.901	6.685	.3909	85.92	5.619	15.29	.6147	.1048
8.60	.6390	.1009	.6332	8.542	.3308	265.0	2.370667	97.937	6.677	.3909	86.12	5.620	15.32	.6114	.1045
8.61	.6341	.1004	.6319	8.552	.3291	266.4	2.370841	97.973	6.670	.3909	86.32	5.621	15.36	.6082	.1043
8.62	.6293	.1001	.6305	8.562	.3273	267.9	2.371015	98.009	6.662	.3909	86.52	5.622	15.39	.6050	.1040
8.63	.6245	.0997	.6291	8.572	.3256	269.3	2.371188	98.045	6.654	.3908	86.72	5.623	15.42	.6018	.1038
8.64	.6198	.0993	.6277	8.582	.3239	270.8	2.371360	98.081	6.646	.3908	86.92	5.623	15.46	.5986	.1035
8.65	.6151	.0989	.6264	8.592	.3222	272.2	2.371532	98.116	6.639	.3908	87.13	5.624	15.49	.5954	.1033
8.66	.6105	.0987	.6250	8.602	.3205	273.7	2.371704	98.152	6.631	.3907	87.33	5.625	15.53	.5923	.1031
8.67	.6059	.0984	.6237	8.612	.3188	275.1	2.371875	98.187	6.623	.3907	87.53	5.626	15.56	.5892	.1028
8.68	.6013	.0982	.6223	8.622	.3171	276.6	2.372045	98.223	6.616	.3907	87.73	5.627	15.59	.5861	.1026
8.69	.5968	.0980	.6210	8.632	.3155	278.1	2.372215	98.258	6.608	.3906	87.94	5.627	15.63	.5830	.1024
8.70	.5923	.0978	.6197	8.642	.3138	279.6	2.372384	98.293	6.600	.3906	88.14	5.628	15.66	.5799	.1021
8.71	.5878	.0977	.6183	8.652	.3122	281.1	2.372553	98.329	6.593	.3906	88.34	5.629	15.69	.5769	.1019
8.72	.5834	.0975	.6170	8.662	.3105	282.6	2.372721	98.364	6.585	.3906	88.54	5.630	15.73	.5739	.1017
8.73	.5790	.0973	.6157	8.673	.3089	284.1	2.372889	98.399	6.578	.3905	88.75	5.631	15.76	.5709	.1014
8.74	.5747	.0971	.6143	8.683	.3073	285.6	2.373056	98.434	6.570	.3905	88.95	5.631	15.80	.5679	.1012
8.75	.5704	.0969	.6130	8.693	.3057	287.1	2.373222	98.469	6.562	.3905	89.16	5.632	15.83	.5649	.1010
8.76	.5661	.0967	.6117	8.703	.3041	288.6	2.373388	98.504	6.555	.3904	89.36	5.633	15.86	.5620	.1007
8.77	.5619	.0965	.6104	8.713	.3025	290.1	2.373554	98.539	6.547	.3904	89.57	5.634	15.90	.5590	.1005
8.78	.5577	.0963	.6091	8.723	.3010	291.7	2.373719	98.573	6.540	.3904	89.77	5.635	15.93	.5561	.1003
8.79	.5536	.0961	.6078	8.733	.2994	293.2	2.373883	98.608	6.532	.3904	89.97	5.635	15.97	.5532	.1001
8.80	.5494	.0959	.6065	8.743	.2978	294.8	2.374047	98.642	6.525	.3903	90.18	5.636	16.00	.5504	.9983
8.81	.5453	.0957	.6052	8.753	.2963	296.3	2.374210	98.677	6.518	.3903	90.39	5.637	16.04	.5475	.9960
8.82	.5413	.0955	.6039	8.763	.2948	297.9	2.374373	98.711	6.510	.3903	90.59	5.638	16.07	.5447	.9938
8.83	.5373	.0953	.6026	8.773	.2932	299.5	2.374535	98.745	6.503	.3903	90.80	5.638	16.10	.5418	.9916
8.84	.5333	.0951	.6014	8.783	.2917	301.0	2.374697	98.780	6.495	.3902	91.00	5.639	16.14	.5390	.9893
8.85	.5293	.0949	.6001	8.793	.2902	302.6	2.374859	98.814	6.488	.3902	91.21	5.640	16.17	.5362	.9871
8.86	.5254	.0947	.5988	8.803	.2887	304.2	2.375019	98.848	6.481	.3902	91.42	5.641	16.21	.5335	.9849
8.87	.5215	.0945	.5975	8.813	.2872	305.8	2.375180	98.882	6.473	.3901	91.62	5.641	16.24	.5307	.9827
8.88	.5177	.0943	.5962	8.824	.2857	307.4	2.375339	98.916	6.466	.3901	91.83	5.642	16.28	.5280	.9805
8.89	.5139	.0941	.5950	8.834	.2843	309.0	2.375499	98.950	6.459	.3901	92.04	5.643	16.31	.5253	.9783
8.90	.5101	.0939	.5938	8.844	.2828	310.6	2.375657	98.984	6.451	.3901	92.25	5.644	16.35	.5226	.9761
8.91	.5063	.0937	.5925	8.854	.2814	312.3	2.375816	99.018	6.444	.3900	92.45	5.645	16.38	.5199	.9739
8.92	.5026	.0935	.5913	8.864	.2799	313.9	2.375973	99.051	6.437	.3900	92.66	5.645	16.41	.5172	.9718
8.93	.4989	.0933	.5901	8.874	.2785	315.5	2.376131	99.085	6.430	.3900	92.87	5.646	16.45	.5145	.9696
8.94	.4952	.0931	.5888	8.884	.2771	317.2	2.376287	99.119	6.422	.3900	93.08	5.647	16.48	.5119	.9675
8.95	.4916	.0929	.5875	8.894	.2756	318.8	2.376444	99.152	6.415	.3899	93.29	5.647	16.52	.5093	.9653
8.96	.4880	.0927	.5863	8.904	.2742	320.5	2.376599	99.186	6.408	.3899	93.50	5.648	16.55	.5067	.9631
8.97	.4844	.0925	.5851	8.914	.2728	322.1	2.376755	99.219	6.401	.3899	93.70	5.649	16.59	.5041	.9610
8.98	.4809	.0923	.5838	8.924	.2714	323.8	2.376909	99.252	6.394	.3899	93.91	5.650	16.62	.5015	.9589
8.99	.4773	.0921	.5826	8.934	.2701	325.5	2.377064	99.286	6.387	.3898	94.12	5.650	16.66	.4989	.9567
9.00	.4739	.0919	.5814	8.944	.2687	327.2	2.377217	99.319	6.379	.3898	94.33	5.651	16.69	.4964	.9546
9.01	.4704	.0917	.5802	8.954	.2673	328.9	2.377371	99.352	6.372	.3898	94.54	5.652	16.73	.4939	.9525
9.02	.4670	.0915	.5790	8.964	.2660	330.6	2.377524	99.384	6.365	.3897	94.75	5.653	16.76	.4913	.9504
9.03	.4636	.0913	.5778	8.974	.2646	332.3	2.377676	99.417	6.358	.3897	94.96	5.653	16.80	.4888	.9483
9.04	.4602	.0911	.5766	8.985	.2633	334.0	2.377828	99.451	6.351	.3897	95.18	5.654	16.83	.4864	.9462
9.05	.4569	.0909	.5754	8.995	.2619	335.7	2.377979	99.483	6.344	.3897	95.39	5.655	16.87	.4839	.9441
9.06	.4535	.0907	.5742	9.005	.2606	337.5	2.378130	99.516	6.337	.3896	95.60	5.656	16.90	.4814	.9421
9.07	.4503	.0905	.5730	9.015	.2593	339.2	2.378281	99.549	6.330	.3896	95.81	5.656	16.94	.4790	.9400
9.08	.4470	.0903	.5718	9.025	.2580	340.9	2.378431	99.581	6.323	.3896	96.02	5.657	16.97	.4766	.9380
9.09	.4438	.0901	.5706	9.035	.2567	342.7	2.378580	99.614	6.316	.3896	96.23	5.658	17.01	.4742	.9359
9.10	.4405	.0899	.5694	9.045	.2554	344.5	2.378729	99.646	6.309	.3895	96.45	5.658	17.05	.4718	.9338
9.11	.4374	.0897	.5682	9.055	.2541	346.2	2.378878	99.679	6.302	.3895	96.66	5.659	17.08	.4694	.9318
9.12	.4342	.0895	.5671	9.065	.2528	348.0	2.379026	99.711	6.295	.3895	96.87	5.660	17.12	.4670	.9298
9.13	.4311	.0893	.5659	9.075	.2515	349.8	2.379174	99.743	6.288	.3895	97.08	5.660	17.15	.4646	.9277
9.14	.4280	.0891	.5647	9.085	.2503	351.6	2.379321	99.775	6.281	.3894	97.30	5.661	17.19	.4623	.9257
9.15	.4249	.0889	.5636	9.095	.2490	353.4	2.379468	99.807	6.274	.3894	97.51	5.662	17.22	.4600	.9237
9.16	.4218	.0887	.5624	9.105	.2478	355.2	2.379614	99.840	6.268	.3894	97.72	5.663	17.26	.4577	.9217
9.17	.4188	.0885	.5612	9.115	.2465	357.0	2.379760	99.872	6.261	.3894	97.94	5.663	17.29	.4554	.9197
9.18	.4158	.0883	.5601	9.125	.2453	358.8	2.379905	99.904	6.254	.3893	98.15	5.664	17.33	.4531	.9177
9.19	.4128	.0													

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
9.35	.3683 -	.6807 -	.5410 -	9.296	.2254 -	390.9	2.382311	100.436	6.140	.3889	101.8	5.675	17.94	.4162 -	.8849 -
9.36	.3657 -	.6773 -	.5399 -	9.306	.2243 -	392.9	2.382448	100.467	6.133	.3889	102.0	5.676	17.98	.4142 -	.8828 -
9.37	.3631 -	.6739 -	.5388 -	9.316	.2232 -	394.8	2.382585	100.498	6.127	.3889	102.3	5.677	18.01	.4121 -	.8809 -
9.38	.3605 -	.6705 -	.5377 -	9.327	.2221 -	396.8	2.382722	100.529	6.120	.3889	102.5	5.677	18.05	.4101 -	.8791 -
9.39	.3580 -	.6671 -	.5366 -	9.337	.2210 -	398.8	2.382859	100.559	6.113	.3888	102.7	5.678	18.09	.4081 -	.8773 -
9.40	.3555 -	.6638 -	.5356 -	9.347	.2199 -	400.8	2.382995	100.590	6.107	.3888	102.9	5.679	18.12	.4061 -	.8754 -
9.41	.3530 -	.6604 -	.5345 -	9.357	.2188 -	402.8	2.383130	100.620	6.100	.3888	103.1	5.679	18.16	.4041 -	.8736 -
9.42	.3505 -	.6571 -	.5334 -	9.367	.2177 -	404.8	2.383265	100.651	6.094	.3888	103.4	5.680	18.20	.4021 -	.8718 -
9.43	.3481 -	.6538 -	.5323 -	9.377	.2167 -	406.8	2.383400	100.681	6.087	.3888	103.6	5.681	18.23	.4001 -	.8699 -
9.44	.3456 -	.6506 -	.5313 -	9.387	.2156 -	408.8	2.383534	100.711	6.081	.3887	103.8	5.681	18.27	.3982 -	.8681 -
9.45	.3432 -	.6473 -	.5302 -	9.397	.2146 -	410.9	2.383668	100.742	6.074	.3887	104.0	5.682	18.31	.3962 -	.8662 -
9.46	.3408 -	.6441 -	.5291 -	9.407	.2135 -	412.9	2.383802	100.772	6.068	.3887	104.2	5.683	18.34	.3943 -	.8644 -
9.47	.3384 -	.6409 -	.5281 -	9.417	.2125 -	414.9	2.383935	100.802	6.062	.3887	104.5	5.683	18.38	.3924 -	.8626 -
9.48	.3361 -	.6377 -	.5270 -	9.427	.2114 -	417.0	2.384068	100.832	6.055	.3886	104.7	5.684	18.42	.3904 -	.8607 -
9.49	.3337 -	.6345 -	.5260 -	9.437	.2104 -	419.1	2.384200	100.862	6.049	.3886	104.9	5.684	18.45	.3885 -	.8589 -
9.50	.3314 -	.6313 -	.5249 -	9.447	.2094 -	421.1	2.384332	100.892	6.042	.3886	105.1	5.685	18.49	.3866 -	.8572 -
9.51	.3291 -	.6282 -	.5239 -	9.457	.2084 -	423.2	2.384464	100.922	6.036	.3886	105.3	5.686	18.53	.3848 -	.8554 -
9.52	.3268 -	.6251 -	.5228 -	9.467	.2073 -	425.3	2.384595	100.952	6.030	.3886	105.6	5.686	18.57	.3829 -	.8536 -
9.53	.3246 -	.6220 -	.5218 -	9.477	.2063 -	427.4	2.384726	100.981	6.023	.3885	105.8	5.687	18.60	.3810 -	.8518 -
9.54	.3223 -	.6189 -	.5208 -	9.487	.2053 -	429.5	2.384856	101.011	6.017	.3885	106.0	5.688	18.64	.3792 -	.8500 -
9.55	.3201 -	.6158 -	.5197 -	9.498	.2043 -	431.6	2.384986	101.041	6.011	.3885	106.2	5.688	18.68	.3773 -	.8483 -
9.56	.3179 -	.6128 -	.5187 -	9.508	.2034 -	433.7	2.385116	101.070	6.004	.3885	106.5	5.689	18.71	.3755 -	.8465 -
9.57	.3157 -	.6098 -	.5177 -	9.518	.2024 -	435.9	2.385245	101.100	5.998	.3884	106.7	5.689	18.75	.3737 -	.8447 -
9.58	.3135 -	.6067 -	.5167 -	9.528	.2014 -	438.0	2.385374	101.129	5.992	.3884	106.9	5.690	18.79	.3719 -	.8431 -
9.59	.3113 -	.6037 -	.5156 -	9.538	.2004 -	440.2	2.385502	101.159	5.985	.3884	107.1	5.691	18.83	.3701 -	.8412 -
9.60	.3092 -	.6008 -	.5146 -	9.548	.1995 -	442.3	2.385630	101.188	5.979	.3884	107.4	5.691	18.86	.3683 -	.8394 -
9.61	.3070 -	.5978 -	.5136 -	9.558	.1985 -	444.5	2.385758	101.217	5.973	.3884	107.6	5.692	18.90	.3665 -	.8376 -
9.62	.3049 -	.5949 -	.5126 -	9.568	.1975 -	446.7	2.385885	101.247	5.967	.3883	107.8	5.692	18.94	.3647 -	.8359 -
9.63	.3028 -	.5919 -	.5116 -	9.578	.1966 -	448.8	2.386012	101.276	5.960	.3883	108.0	5.693	18.98	.3630 -	.8343 -
9.64	.3007 -	.5890 -	.5106 -	9.588	.1956 -	451.0	2.386139	101.305	5.954	.3883	108.3	5.694	19.01	.3612 -	.8325 -
9.65	.2987 -	.5861 -	.5096 -	9.598	.1947 -	453.2	2.386265	101.334	5.948	.3883	108.5	5.694	19.05	.3595 -	.8308 -
9.66	.2966 -	.5833 -	.5086 -	9.608	.1938 -	455.4	2.386391	101.363	5.942	.3883	108.7	5.695	19.09	.3578 -	.8291 -
9.67	.2946 -	.5804 -	.5076 -	9.618	.1928 -	457.7	2.386516	101.392	5.936	.3882	108.9	5.695	19.13	.3560 -	.8275 -
9.68	.2926 -	.5776 -	.5066 -	9.628	.1919 -	459.9	2.386641	101.421	5.930	.3882	109.2	5.696	19.16	.3543 -	.8257 -
9.69	.2906 -	.5747 -	.5056 -	9.638	.1910 -	462.1	2.386766	101.450	5.923	.3882	109.4	5.697	19.20	.3526 -	.8240 -
9.70	.2886 -	.5719 -	.5046 -	9.648	.1901 -	464.4	2.386890	101.479	5.917	.3882	109.6	5.697	19.24	.3510 -	.8224 -
9.71	.2866 -	.5691 -	.5036 -	9.658	.1892 -	466.6	2.387014	101.507	5.911	.3882	109.8	5.698	19.28	.3493 -	.8206 -
9.72	.2847 -	.5664 -	.5026 -	9.668	.1883 -	468.9	2.387138	101.536	5.905	.3881	110.1	5.698	19.31	.3476 -	.8190 -
9.73	.2827 -	.5636 -	.5016 -	9.678	.1874 -	471.2	2.387261	101.564	5.899	.3881	110.3	5.699	19.35	.3459 -	.8174 -
9.74	.2808 -	.5609 -	.5007 -	9.689	.1865 -	473.4	2.387384	101.593	5.893	.3881	110.5	5.700	19.39	.3443 -	.8155 -
9.75	.2789 -	.5581 -	.4997 -	9.699	.1856 -	475.7	2.387507	101.621	5.887	.3881	110.7	5.700	19.43	.3427 -	.8139 -
9.76	.2770 -	.5554 -	.4987 -	9.709	.1847 -	478.0	2.387629	101.650	5.881	.3880	111.0	5.701	19.47	.3410 -	.8123 -
9.77	.2751 -	.5527 -	.4977 -	9.719	.1838 -	480.3	2.387751	101.678	5.875	.3880	111.2	5.701	19.50	.3394 -	.8107 -
9.78	.2733 -	.5501 -	.4968 -	9.729	.1830 -	482.6	2.387872	101.707	5.869	.3880	111.4	5.702	19.54	.3378 -	.8090 -
9.79	.2714 -	.5474 -	.4958 -	9.739	.1821 -	485.0	2.387993	101.735	5.863	.3880	111.7	5.703	19.58	.3362 -	.8073 -
9.80	.2696 -	.5447 -	.4949 -	9.749	.1812 -	487.3	2.388114	101.763	5.857	.3880	111.9	5.703	19.62	.3346 -	.8057 -
9.81	.2677 -	.5421 -	.4939 -	9.759	.1804 -	489.6	2.388234	101.791	5.851	.3879	112.1	5.704	19.66	.3330 -	.8040 -
9.82	.2659 -	.5395 -	.4929 -	9.769	.1795 -	492.0	2.388354	101.820	5.845	.3879	112.3	5.704	19.69	.3314 -	.8025 -
9.83	.2641 -	.5369 -	.4920 -	9.779	.1787 -	494.4	2.388474	101.848	5.839	.3879	112.6	5.705	19.73	.3298 -	.8008 -
9.84	.2624 -	.5343 -	.4910 -	9.789	.1778 -	496.7	2.388593	101.876	5.833	.3879	112.8	5.705	19.77	.3283 -	.7992 -
9.85	.2606 -	.5317 -	.4901 -	9.799	.1770 -	499.1	2.388712	101.904	5.827	.3879	113.0	5.706	19.81	.3267 -	.7977 -
9.86	.2588 -	.5292 -	.4891 -	9.809	.1762 -	501.5	2.388831	101.932	5.821	.3878	113.3	5.707	19.85	.3252 -	.7960 -
9.87	.2571 -	.5266 -	.4882 -	9.819	.1753 -	503.9	2.388949	101.960	5.815	.3878	113.5	5.707	19.89	.3237 -	.7944 -
9.88	.2554 -	.5241 -	.4873 -	9.829	.1745 -	506.3	2.389067	101.987	5.809	.3878	113.7	5.708	19.92	.3221 -	.7928 -
9.89	.2537 -	.5216 -	.4863 -	9.839	.1737 -	508.7	2.389185	102.015	5.803	.3878	113.9	5.708	19.96	.3206 -	.7912 -
9.90	.2520 -	.5191 -	.4854 -	9.849	.1729 -	511.2	2.389302	102.043	5.797	.3878	114.2	5.709	20.00	.3191 -	.7896 -
9.91	.2503 -	.5166 -	.4845 -	9.859	.1720 -	513.6	2.389419	102.070	5.792	.3877	114.4	5.709	20.04	.3176 -	.7880 -
9.92	.2486 -	.5141 -	.4835 -	9.869	.1712 -	516.0	2.389536	102.098	5.786	.3877	114.6	5.710	20.08	.3161 -	.7864 -
9.93	.2469 -	.5117 -	.4826 -	9.880	.1704 -	518.5	2.389652	102.126	5.780	.3877	114.9	5.710	20.12	.3146 -	.7848 -
9.94	.2453 -	.5092 -	.4817 -	9.890	.1696 -	521.0	2.389768	102.153	5.774	.3877	115.1	5.711	20.15	.3132 -	.7831 -
9.95	.2436 -	.5068 -	.4808 -	9.900	.1689 -	523.4	2.389884	102.180	5.768	.3877	115.3	5.712	20.19	.3117 -	.7817 -
9.96	.2420 -	.5044 -	.4798 -	9.910	.1681 -	525.9	2.389999	102.208	5.762	.3877	115.6	5.712	20.23	.3102 -	.7801 -
9.97	.2404 -	.5020 -	.4789 -	9.920	.1673 -	528.4	2.390114	102.235	5.756	.3876	115.8	5.713	20.27	.3088 -	.7785 -
9.98	.2388 -	.4996 -	.4780 -	9.930	.1665 -	530.9	2.390229	102.262	5.751	.3876	116.0	5.713	20.31	.3073 -	.7770 -
9.99	.2372 -	.4972 -	.4771 -	9.940	.1657 -	533.4	2.390343	102.290	5.745	.3876	116.3	5.714	20.35	.3059 -	.7755 -
10.00	.2356 -	.4948 -	.4762 -	9.950	.1649 -	535.9	2.390457	102.317	5.739	.3876	116.5	5.714	20.39	.3045 -	.7739 -
10.02	.2325 -	.4901 -	.4744 -	9.970	.1634 -	541.0	2.390684	102.37	5.728	.3875	117.0	5.715	20.47	.3016 -	.7708 -
10.04	.2294 -	.4855 -	.4726 -	9.990	.1619 -	546.1	2.390910	102.42	5.716	.3875	117.4	5.717	20.54	.2988 -	.7678 -
10.06	.2264 -	.4809 -	.4708 -	10.01	.1604 -	551.3	2.391134	102.48	5.705	.3875	117.9	5.718	20.62	.2961 -	.7648 -
10.08	.2234 -	.4764 -	.4690 -	10.03	.1589 -	556.4	2.391358	102.53	5.693	.3874	118.4	5.719	20.70	.2934 -	.7616 -
10.10	.2205 -	.4719 -	.4673 -	10.05	.1575 -	561.7	2.391579	102.59	5.682	.3874	118.9	5.720	20.78	.2906 -	.7588 -
10.12	.2														

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
10.50	.1701 -4	.3920 -3	.4338 -1	10.45	.1313 -2	875.0	2.395766	103.61	5.465	.3867	128.5	5.740	22.38	.2422 -2	.7022 -1
10.52	.1679 -4	.3885 -3	.4323 -1	10.47	.1301 -2	681.1	2.395964	103.66	5.455	.3867	129.0	5.741	22.46	.2400 -2	.6996 -1
10.54	.1658 -4	.3850 -3	.4307 -1	10.49	.1289 -2	687.3	2.396160	103.71	5.444	.3866	129.4	5.742	22.54	.2379 -2	.6968 -1
10.56	.1637 -4	.3815 -3	.4291 -1	10.51	.1278 -2	693.5	2.396355	103.76	5.434	.3866	129.9	5.743	22.63	.2358 -2	.6942 -1
10.58	.1616 -4	.3780 -3	.4276 -1	10.53	.1267 -2	699.7	2.396550	103.81	5.424	.3866	130.4	5.744	22.71	.2337 -2	.6917 -1
10.60	.1596 -4	.3747 -3	.4260 -1	10.55	.1255 -2	706.0	2.396743	103.86	5.413	.3865	130.9	5.744	22.79	.2317 -2	.6891 -1
10.62	.1576 -4	.3713 -3	.4245 -1	10.57	.1244 -2	712.3	2.396935	103.90	5.403	.3865	131.4	5.745	22.87	.2296 -2	.6866 -1
10.64	.1556 -4	.3680 -3	.4230 -1	10.59	.1233 -2	718.7	2.397126	103.96	5.393	.3865	131.9	5.746	22.96	.2276 -2	.6839 -1
10.66	.1537 -4	.3647 -3	.4215 -1	10.61	.1223 -2	725.2	2.397316	104.01	5.383	.3864	132.4	5.747	23.04	.2256 -2	.6814 -1
10.68	.1518 -4	.3614 -3	.4200 -1	10.63	.1212 -2	731.6	2.397505	104.05	5.373	.3864	132.9	5.748	23.12	.2236 -2	.6789 -1
10.70	.1499 -4	.3582 -3	.4185 -1	10.65	.1201 -2	738.2	2.397693	104.10	5.363	.3864	133.4	5.749	23.21	.2216 -2	.6763 -1
10.72	.1480 -4	.3550 -3	.4170 -1	10.67	.1191 -2	744.8	2.397880	104.14	5.353	.3863	133.9	5.750	23.29	.2197 -2	.6737 -1
10.74	.1462 -4	.3518 -3	.4155 -1	10.69	.1180 -2	751.4	2.398066	104.19	5.343	.3863	134.4	5.751	23.37	.2178 -2	.6712 -1
10.76	.1444 -4	.3487 -3	.4140 -1	10.71	.1170 -2	758.1	2.398251	104.24	5.333	.3863	134.9	5.752	23.46	.2159 -2	.6688 -1
10.78	.1426 -4	.3456 -3	.4125 -1	10.73	.1160 -2	764.8	2.398435	104.29	5.323	.3862	135.4	5.753	23.54	.2140 -2	.6663 -1
10.80	.1408 -4	.3426 -3	.4111 -1	10.75	.1150 -2	771.5	2.398618	104.33	5.313	.3862	135.9	5.753	23.62	.2121 -2	.6638 -1
10.82	.1391 -4	.3395 -3	.4096 -1	10.77	.1140 -2	778.3	2.398801	104.38	5.303	.3862	136.4	5.754	23.71	.2103 -2	.6614 -1
10.84	.1374 -4	.3365 -3	.4081 -1	10.79	.1130 -2	785.2	2.398982	104.43	5.293	.3862	136.9	5.755	23.79	.2085 -2	.6589 -1
10.86	.1357 -4	.3336 -3	.4067 -1	10.81	.1120 -2	792.1	2.399162	104.48	5.283	.3861	137.4	5.756	23.88	.2067 -2	.6565 -1
10.88	.1340 -4	.3306 -3	.4053 -1	10.83	.1110 -2	799.1	2.399341	104.52	5.274	.3861	137.9	5.757	23.96	.2048 -2	.6542 -1
10.90	.1324 -4	.3277 -3	.4038 -1	10.85	.1101 -2	806.1	2.399519	104.57	5.264	.3861	138.5	5.758	24.05	.2031 -2	.6518 -1
10.92	.1307 -4	.3249 -3	.4024 -1	10.87	.1091 -2	813.1	2.399697	104.61	5.254	.3860	139.0	5.759	24.13	.2013 -2	.6494 -1
10.94	.1291 -4	.3220 -3	.4010 -1	10.89	.1082 -2	820.3	2.399873	104.66	5.245	.3860	139.5	5.759	24.21	.1996 -2	.6469 -1
10.96	.1276 -4	.3192 -3	.3996 -1	10.91	.1073 -2	827.4	2.400049	104.71	5.235	.3860	140.0	5.760	24.30	.1979 -2	.6447 -1
10.98	.1260 -4	.3165 -3	.3982 -1	10.93	.1064 -2	834.6	2.400223	104.75	5.225	.3860	140.5	5.761	24.39	.1962 -2	.6424 -1
11.00	.1245 -4	.3137 -3	.3968 -1	10.95	.1054 -2	841.9	2.400397	104.80	5.216	.3859	141.0	5.762	24.47	.1945 -2	.6400 -1
11.02	.1230 -4	.3109 -3	.3954 -1	10.97	.1045 -2	849.2	2.400570	104.85	5.206	.3859	141.5	5.763	24.56	.1929 -2	.6376 -1
11.04	.1215 -4	.3083 -3	.3941 -1	11.00	.1036 -2	856.6	2.400741	104.89	5.197	.3859	142.0	5.764	24.64	.1912 -2	.6354 -1
11.06	.1200 -4	.3056 -3	.3927 -1	11.02	.1028 -2	864.0	2.400912	104.93	5.188	.3858	142.5	5.764	24.73	.1896 -2	.6330 -1
11.08	.1186 -4	.3030 -3	.3913 -1	11.04	.1019 -2	871.5	2.401082	104.98	5.178	.3858	143.1	5.765	24.81	.1880 -2	.6308 -1
11.10	.1171 -4	.3003 -3	.3900 -1	11.06	.1010 -2	879.0	2.401252	105.02	5.169	.3858	143.6	5.766	24.90	.1864 -2	.6286 -1
11.12	.1157 -4	.2978 -3	.3886 -1	11.08	.1002 -2	886.6	2.401420	105.06	5.159	.3858	144.1	5.767	24.99	.1848 -2	.6263 -1
11.14	.1143 -4	.2952 -3	.3873 -1	11.10	.9932 -2	894.2	2.401587	105.11	5.150	.3857	144.6	5.768	25.08	.1832 -2	.6241 -1
11.16	.1130 -4	.2927 -3	.3860 -1	11.12	.9847 -2	901.9	2.401754	105.16	5.141	.3857	145.1	5.768	25.16	.1817 -2	.6218 -1
11.18	.1116 -4	.2902 -3	.3846 -1	11.14	.9765 -2	909.6	2.401919	105.20	5.132	.3857	145.7	5.769	25.25	.1801 -2	.6197 -1
11.20	.1103 -4	.2877 -3	.3833 -1	11.16	.9683 -2	917.4	2.402084	105.24	5.123	.3856	146.2	5.770	25.33	.1786 -2	.6174 -1
11.22	.1090 -4	.2852 -3	.3820 -1	11.18	.9602 -2	925.2	2.402248	105.28	5.113	.3856	146.7	5.771	25.42	.1771 -2	.6152 -1
11.24	.1077 -4	.2828 -3	.3807 -1	11.20	.9521 -2	933.1	2.402412	105.33	5.104	.3856	147.2	5.772	25.51	.1756 -2	.6131 -1
11.26	.1064 -4	.2804 -3	.3794 -1	11.22	.9440 -2	941.1	2.402574	105.37	5.095	.3856	147.8	5.772	25.60	.1742 -2	.6108 -1
11.28	.1051 -4	.2780 -3	.3781 -1	11.24	.9362 -2	949.1	2.402735	105.42	5.086	.3855	148.3	5.773	25.69	.1727 -2	.6087 -1
11.30	.1039 -4	.2756 -3	.3768 -1	11.26	.9283 -2	957.1	2.402896	105.46	5.077	.3855	148.8	5.774	25.77	.1712 -2	.6066 -1
11.32	.1026 -4	.2733 -3	.3755 -1	11.28	.9206 -2	965.3	2.403056	105.50	5.068	.3855	149.3	5.775	25.86	.1698 -2	.6044 -1
11.34	.1014 -4	.2710 -3	.3743 -1	11.30	.9130 -2	973.5	2.403215	105.55	5.059	.3855	149.9	5.775	25.95	.1684 -2	.6023 -1
11.36	.1002 -4	.2687 -3	.3730 -1	11.32	.9054 -2	981.6	2.403373	105.59	5.050	.3854	150.4	5.776	26.04	.1670 -2	.6002 -1
11.38	.9905 -3	.2664 -3	.3717 -1	11.34	.8979 -2	989.9	2.403531	105.63	5.041	.3854	150.9	5.777	26.12	.1656 -2	.5981 -1
11.40	.9788 -3	.2642 -3	.3705 -1	11.36	.8904 -2	998.3	2.403687	105.67	5.032	.3854	151.5	5.778	26.21	.1642 -2	.5959 -1
11.42	.9673 -3	.2620 -3	.3692 -1	11.38	.8830 -2	1007	2.403843	105.71	5.024	.3854	152.0	5.779	26.30	.1629 -2	.5939 -1
11.44	.9559 -3	.2598 -3	.3680 -1	11.40	.8757 -2	1015	2.403998	105.75	5.015	.3853	152.5	5.779	26.39	.1615 -2	.5918 -1
11.46	.9447 -3	.2576 -3	.3668 -1	11.42	.8685 -2	1024	2.404152	105.80	5.006	.3853	153.1	5.780	26.48	.1602 -2	.5897 -1
11.48	.9337 -3	.2554 -3	.3655 -1	11.44	.8613 -2	1032	2.404306	105.84	4.997	.3853	153.6	5.781	26.57	.1589 -2	.5877 -1
11.50	.9228 -3	.2533 -3	.3643 -1	11.46	.8543 -2	1041	2.404459	105.88	4.989	.3853	154.1	5.781	26.66	.1575 -2	.5858 -1
11.52	.9120 -3	.2512 -3	.3631 -1	11.48	.8472 -2	1050	2.404610	105.92	4.980	.3852	154.7	5.782	26.75	.1562 -2	.5836 -1
11.54	.9014 -3	.2491 -3	.3619 -1	11.50	.8403 -2	1058	2.404762	105.97	4.971	.3852	155.2	5.783	26.84	.1550 -2	.5816 -1
11.56	.8909 -3	.2470 -3	.3607 -1	11.52	.8334 -2	1067	2.404912	106.01	4.963	.3852	155.7	5.784	26.93	.1537 -2	.5797 -1
11.58	.8806 -3	.2450 -3	.3595 -1	11.54	.8266 -2	1076	2.405062	106.05	4.954	.3852	156.3	5.784	27.02	.1525 -2	.5776 -1
11.60	.8704 -3	.2430 -3	.3583 -1	11.56	.8199 -2	1085	2.405211	106.09	4.945	.3851	156.8	5.785	27.11	.1512 -2	.5757 -1
11.62	.8604 -3	.2409 -3	.3571 -1	11.58	.8132 -2	1094	2.405359	106.13	4.937	.3851	157.4	5.786	27.20	.1500 -2	.5737 -1
11.64	.8505 -3	.2390 -3	.3559 -1	11.60	.8066 -2	1103	2.405506	106.17	4.928	.3851	157.9	5.787	27.29	.1488 -2	.5717 -1
11.66	.8406 -3	.2370 -3	.3547 -1	11.62	.8000 -2	1112	2.405653	106.21	4.920	.3851	158.5	5.787	27.38	.1476 -2	.5698 -1
11.68	.8310 -3	.2350 -3	.3536 -1	11.64	.7935 -2	1121	2.405799	106.25	4.912	.3850	159.0	5.788	27.47	.1464 -2	.5678 -1
11.70	.8215 -3	.2331 -3	.3524 -1	11.66	.7871 -2	1130	2.405944	106.29	4.903	.3850	159.5	5.789	27.56	.1452 -2	.5659 -1
11.72	.8121 -3	.2312 -3	.3512 -1	11.68	.7808 -2	1140	2.406089	106.33	4.895	.3850	160.1	5.789	27.65	.1440 -2	.5639 -1
11.74	.8027 -3	.2293 -3	.3501 -1	11.70	.7744 -2	1149	2.406233	106.37	4.886	.3850	160.6	5.790	27.74	.1428 -2	.5620 -1
11.76	.7935 -3	.2274 -3	.3489 -1	11.72	.7682 -2	1158	2.406376	106.41	4.878	.3849	161.2	5.791	27.84	.1417 -2	.5601 -1
11.78	.7845 -3	.2256 -3	.3478 -1	11.74	.7620 -2	1168	2.406518	106.45	4.870	.3849	161.7	5.791	27.93	.1405 -2	.5583 -1
11.80	.7755 -3	.2237 -3	.3466 -1	11.76	.7559 -2	1177	2.406660	106.49	4.861	.3849	162.3	5.792	28.02	.1394 -2	.5564 -1
11.82	.7667 -3	.2219 -3	.3455 -1	11.78	.7498 -2	1187	2.406801	106.53	4.853	.3849	162.8	5.793	28.11	.1383 -2	.5544 -1
11.84	.7580 -3	.2201 -3	.3444 -1	11.80	.7438 -2	1197	2.406942	106.57	4.845	.3848	163.4	5.793	28.20	.1372 -2	.5526 -1
11.86	.7494 -3	.2183 -3	.3433 -1	11.82	.7379 -2	1206	2.407081	106.61	4.837	.3848	163.9	5.7			

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
12.30	.5857 -3	.1831 -3	.3199 -1	12.26	.6202 -3	1437	2.409989	107.44	4.663	.3843	176.3	5.808	30.36	.1144 -2	.5122 -2
12.32	.5792 -3	.1816 -3	.3189 -1	12.28	.6184 -3	1448	2.410115	107.48	4.656	.3843	176.9	5.809	30.46	.1135 -2	.5105 -2
12.34	.5729 -3	.1802 -3	.3179 -1	12.30	.6167 -3	1460	2.410239	107.51	4.648	.3843	177.5	5.809	30.55	.1126 -2	.5088 -2
12.36	.5667 -3	.1788 -3	.3169 -1	12.32	.6150 -3	1471	2.410363	107.55	4.641	.3843	178.1	5.810	30.65	.1117 -2	.5072 -2
12.38	.5605 -3	.1774 -3	.3159 -1	12.34	.6133 -3	1482	2.410486	107.59	4.633	.3843	178.6	5.810	30.75	.1109 -2	.5056 -2
12.40	.5544 -3	.1760 -3	.3149 -1	12.36	.5967 -3	1494	2.410609	107.62	4.626	.3842	179.2	5.811	30.84	.1100 -2	.5039 -2
12.42	.5484 -3	.1747 -3	.3140 -1	12.38	.5921 -3	1506	2.410731	107.66	4.618	.3842	179.8	5.812	30.94	.1092 -2	.5023 -2
12.44	.5424 -3	.1733 -3	.3130 -1	12.40	.5876 -3	1517	2.410853	107.69	4.611	.3842	180.4	5.812	31.04	.1083 -2	.5007 -2
12.46	.5365 -3	.1720 -3	.3120 -1	12.42	.5831 -3	1529	2.410974	107.73	4.603	.3842	181.0	5.813	31.13	.1075 -2	.4991 -2
12.48	.5307 -3	.1706 -3	.3110 -1	12.44	.5786 -3	1541	2.411094	107.77	4.596	.3842	181.5	5.813	31.23	.1067 -2	.4975 -2
12.50	.5250 -3	.1693 -3	.3101 -1	12.46	.5742 -3	1553	2.411214	107.80	4.589	.3841	182.1	5.814	31.33	.1059 -2	.4960 -2
12.52	.5193 -3	.1680 -3	.3091 -1	12.48	.5698 -3	1565	2.411333	107.84	4.581	.3841	182.7	5.815	31.42	.1051 -2	.4944 -2
12.54	.5137 -3	.1667 -3	.3082 -1	12.50	.5655 -3	1577	2.411452	107.87	4.574	.3841	183.3	5.815	31.52	.1043 -2	.4927 -2
12.56	.5082 -3	.1654 -3	.3072 -1	12.52	.5612 -3	1589	2.411571	107.90	4.567	.3841	183.9	5.816	31.62	.1035 -2	.4912 -2
12.58	.5028 -3	.1642 -3	.3063 -1	12.54	.5570 -3	1601	2.411688	107.94	4.559	.3841	184.5	5.816	31.72	.1027 -2	.4897 -2
12.60	.4973 -3	.1629 -3	.3053 -1	12.56	.5527 -3	1614	2.411805	107.98	4.552	.3840	185.1	5.817	31.81	.1019 -2	.4881 -2
12.62	.4920 -3	.1617 -3	.3044 -1	12.58	.5486 -3	1626	2.411922	108.01	4.545	.3840	185.6	5.817	31.91	.1011 -2	.4865 -2
12.64	.4868 -3	.1604 -3	.3035 -1	12.60	.5444 -3	1639	2.412038	108.05	4.538	.3840	186.2	5.818	32.01	.1004 -2	.4850 -2
12.66	.4816 -3	.1592 -3	.3025 -1	12.62	.5403 -3	1651	2.412154	108.08	4.530	.3840	186.8	5.819	32.11	.09961 -2	.4835 -2
12.68	.4764 -3	.1580 -3	.3016 -1	12.64	.5362 -3	1664	2.412269	108.12	4.523	.3840	187.4	5.819	32.21	.09885 -2	.4820 -2
12.70	.4714 -3	.1568 -3	.3007 -1	12.66	.5322 -3	1676	2.412383	108.15	4.516	.3839	188.0	5.820	32.31	.09810 -2	.4805 -2
12.72	.4663 -3	.1556 -3	.2998 -1	12.68	.5282 -3	1689	2.412497	108.18	4.509	.3839	188.6	5.820	32.41	.09737 -2	.4790 -2
12.74	.4614 -3	.1544 -3	.2989 -1	12.70	.5242 -3	1702	2.412611	108.22	4.502	.3839	189.2	5.821	32.50	.09664 -2	.4775 -2
12.76	.4565 -3	.1532 -3	.2979 -1	12.72	.5203 -3	1715	2.412723	108.25	4.495	.3839	189.8	5.821	32.60	.09591 -2	.4760 -2
12.78	.4517 -3	.1521 -3	.2970 -1	12.74	.5164 -3	1728	2.412836	108.29	4.488	.3839	190.4	5.822	32.70	.09520 -2	.4745 -2
12.80	.4469 -3	.1509 -3	.2961 -1	12.76	.5126 -3	1741	2.412948	108.32	4.481	.3839	191.0	5.822	32.80	.09448 -2	.4730 -2
12.82	.4422 -3	.1498 -3	.2952 -1	12.78	.5087 -3	1754	2.413059	108.35	4.474	.3838	191.6	5.823	32.90	.09378 -2	.4715 -2
12.84	.4376 -3	.1487 -3	.2944 -1	12.80	.5050 -3	1767	2.413170	108.39	4.467	.3838	192.2	5.823	33.00	.09308 -2	.4701 -2
12.86	.4329 -3	.1475 -3	.2935 -1	12.82	.5012 -3	1781	2.413280	108.42	4.460	.3838	192.8	5.824	33.10	.09239 -2	.4686 -2
12.88	.4284 -3	.1464 -3	.2926 -1	12.84	.4975 -3	1794	2.413390	108.45	4.453	.3838	193.4	5.825	33.20	.09170 -2	.4672 -2
12.90	.4239 -3	.1453 -3	.2917 -1	12.86	.4938 -3	1807	2.413500	108.49	4.446	.3838	194.0	5.825	33.30	.09102 -2	.4657 -2
12.92	.4195 -3	.1442 -3	.2908 -1	12.88	.4901 -3	1821	2.413609	108.52	4.439	.3837	194.6	5.826	33.40	.09035 -2	.4643 -2
12.94	.4151 -3	.1432 -3	.2900 -1	12.90	.4865 -3	1835	2.413717	108.55	4.432	.3837	195.2	5.826	33.50	.08968 -2	.4629 -2
12.96	.4107 -3	.1421 -3	.2891 -1	12.92	.4829 -3	1848	2.413825	108.59	4.425	.3837	195.8	5.827	33.60	.08902 -2	.4614 -2
12.98	.4065 -3	.1410 -3	.2882 -1	12.94	.4794 -3	1862	2.413932	108.62	4.419	.3837	196.4	5.827	33.70	.08836 -2	.4600 -2
13.00	.4023 -3	.1400 -3	.2874 -1	12.96	.4759 -3	1876	2.414039	108.65	4.412	.3837	197.0	5.828	33.81	.08771 -2	.4586 -2
13.02	.3981 -3	.1389 -3	.2865 -1	12.98	.4723 -3	1890	2.414146	108.69	4.405	.3837	197.6	5.828	33.91	.08706 -2	.4572 -2
13.04	.3939 -3	.1379 -3	.2857 -1	13.00	.4689 -3	1904	2.414252	108.72	4.398	.3836	198.2	5.829	34.01	.08642 -2	.4559 -2
13.06	.3898 -3	.1369 -3	.2848 -1	13.02	.4655 -3	1918	2.414357	108.75	4.391	.3836	198.8	5.829	34.11	.08580 -2	.4544 -2
13.08	.3858 -3	.1359 -3	.2840 -1	13.04	.4620 -3	1933	2.414462	108.78	4.385	.3836	199.4	5.830	34.21	.08517 -2	.4530 -2
13.10	.3818 -3	.1349 -3	.2831 -1	13.06	.4586 -3	1947	2.414567	108.82	4.378	.3836	200.1	5.830	34.31	.08453 -2	.4517 -2
13.12	.3779 -3	.1339 -3	.2823 -1	13.08	.4553 -3	1961	2.414671	108.85	4.371	.3836	200.7	5.831	34.42	.08392 -2	.4503 -2
13.14	.3740 -3	.1329 -3	.2814 -1	13.10	.4520 -3	1976	2.414775	108.88	4.365	.3836	201.3	5.831	34.52	.08331 -2	.4489 -2
13.16	.3701 -3	.1319 -3	.2806 -1	13.12	.4487 -3	1990	2.414878	108.91	4.358	.3835	201.9	5.832	34.62	.08271 -2	.4475 -2
13.18	.3663 -3	.1309 -3	.2798 -1	13.14	.4454 -3	2005	2.414981	108.94	4.351	.3835	202.5	5.832	34.72	.08210 -2	.4462 -2
13.20	.3626 -3	.1300 -3	.2790 -1	13.16	.4422 -3	2020	2.415083	108.97	4.345	.3835	203.1	5.833	34.82	.08151 -2	.4448 -2
13.22	.3589 -3	.1290 -3	.2781 -1	13.18	.4390 -3	2034	2.415185	109.01	4.338	.3835	203.7	5.833	34.93	.08091 -2	.4435 -2
13.24	.3552 -3	.1281 -3	.2773 -1	13.20	.4358 -3	2049	2.415286	109.04	4.332	.3835	204.4	5.834	35.03	.08032 -2	.4422 -2
13.26	.3516 -3	.1271 -3	.2765 -1	13.22	.4327 -3	2064	2.415387	109.07	4.325	.3835	205.0	5.834	35.13	.07974 -2	.4409 -2
13.28	.3480 -3	.1262 -3	.2757 -1	13.24	.4296 -3	2079	2.415488	109.10	4.319	.3834	205.6	5.835	35.24	.07918 -2	.4395 -2
13.30	.3444 -3	.1253 -3	.2749 -1	13.26	.4264 -3	2095	2.415588	109.13	4.312	.3834	206.2	5.835	35.34	.07860 -2	.4382 -2
13.32	.3409 -3	.1244 -3	.2741 -1	13.28	.4234 -3	2110	2.415687	109.16	4.306	.3834	206.8	5.836	35.44	.07802 -2	.4369 -2
13.34	.3374 -3	.1235 -3	.2733 -1	13.30	.4203 -3	2125	2.415786	109.20	4.299	.3834	207.5	5.836	35.55	.07747 -2	.4356 -2
13.36	.3340 -3	.1226 -3	.2725 -1	13.32	.4173 -3	2141	2.415886	109.23	4.293	.3834	208.1	5.837	35.65	.07691 -2	.4342 -2
13.38	.3306 -3	.1217 -3	.2717 -1	13.34	.4143 -3	2156	2.415983	109.26	4.286	.3834	208.7	5.837	35.76	.07636 -2	.4330 -2
13.40	.3273 -3	.1208 -3	.2709 -1	13.36	.4113 -3	2172	2.416081	109.29	4.280	.3833	209.3	5.838	35.86	.07582 -2	.4316 -2
13.42	.3240 -3	.1199 -3	.2701 -1	13.38	.4084 -3	2188	2.416179	109.32	4.273	.3833	210.0	5.838	35.96	.07527 -2	.4304 -2
13.44	.3207 -3	.1191 -3	.2694 -1	13.40	.4055 -3	2204	2.416276	109.35	4.267	.3833	210.6	5.838	36.07	.07474 -2	.4291 -2
13.46	.3175 -3	.1182 -3	.2686 -1	13.42	.4026 -3	2219	2.416373	109.38	4.261	.3833	211.2	5.839	36.17	.07420 -2	.4278 -2
13.48	.3143 -3	.1174 -3	.2678 -1	13.44	.3997 -3	2236	2.416469	109.41	4.254	.3833	211.8	5.839	36.28	.07367 -2	.4265 -2
13.50	.3111 -3	.1165 -3	.2670 -1	13.46	.3969 -3	2252	2.416565	109.44	4.248	.3833	212.5	5.840	36.38	.07315 -2	.4253 -2
13.52	.3080 -3	.1157 -3	.2663 -1	13.48	.3941 -3	2268	2.416660	109.47	4.242	.3832	213.1	5.840	36.49	.07263 -2	.4241 -2
13.54	.3049 -3	.1149 -3	.2655 -1	13.50	.3913 -3	2284	2.416754	109.51	4.235	.3832	213.7	5.841	36.59	.07212 -2	.4228 -2
13.56	.3019 -3	.1140 -3	.2647 -1	13.52	.3885 -3	2300	2.416849	109.54	4.229	.3832	214.4	5.841	36.70	.07161 -2	.4216 -2
13.58	.2988 -3	.1132 -3	.2640 -1	13.54	.3858 -3	2317	2.416944	109.57	4.223	.3832	215.0	5.842	36.80	.07109 -2	.4204 -2
13.60	.2958 -3	.1124 -3	.2632 -1	13.56	.3830 -3	2334	2.417037	109.59	4.217	.3832	215.6	5.842	36.91	.07059 -2	.4191 -2
13.62	.2929 -3	.1116 -3	.2625 -1	13.58	.3803 -3	2350	2.417131	109.62	4.211	.3832	216.3	5.843	37.02	.07009 -2	.4179 -2
13.64	.2900 -3	.1108 -3	.2617 -1	13.60	.3777 -3	2367	2.417224	109.65	4.204	.3832	216.9	5.843	37.12	.06960 -2	.4166 -2
13.66	.2871 -3	.1100 -3	.2610 -1	13.62	.3750 -3	2384	2.417316	109.69	4.198	.3831	217.5				

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	V a_*	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{P_{t_2}}{P_{t_1}}$	$\frac{p_1}{p_{t_2}}$	
14.10	.2313 -3	.9427 -1	.2453 -1	14.06	.3219 -3	2780	2.4192569	110.32	4.067	.3828	231.8	5.853	39.60	.5931 -3	.3900 -2
14.12	.2290 -3	.9362 -1	.2447 -1	14.08	.3197 -3	2799	2.4193409	110.35	4.061	.3828	232.4	5.853	39.71	.5890 -3	.3889 -2
14.14	.2268 -3	.9297 -1	.2440 -1	14.10	.3175 -3	2818	2.4194246	110.38	4.055	.3828	233.1	5.854	39.82	.5849 -3	.3878 -2
14.16	.2246 -3	.9233 -1	.2433 -1	14.13	.3153 -3	2838	2.4195079	110.41	4.050	.3828	233.8	5.854	39.93	.5810 -3	.3867 -2
14.18	.2225 -3	.9170 -1	.2426 -1	14.15	.3132 -3	2857	2.4195909	110.44	4.044	.3828	234.4	5.854	40.04	.5770 -3	.3856 -2
14.20	.2204 -3	.9108 -1	.2420 -1	14.17	.3111 -3	2877	2.4196735	110.46	4.038	.3828	235.1	5.855	40.15	.5732 -3	.3845 -2
14.22	.2183 -3	.9045 -1	.2413 -1	14.19	.3089 -3	2897	2.4197558	110.49	4.033	.3827	235.7	5.855	40.26	.5693 -3	.3834 -2
14.24	.2162 -3	.8983 -1	.2406 -1	14.21	.3068 -3	2916	2.4198378	110.52	4.027	.3827	236.4	5.856	40.37	.5654 -3	.3824 -2
14.26	.2141 -3	.8922 -1	.2400 -1	14.23	.3048 -3	2936	2.4199194	110.54	4.021	.3827	237.1	5.856	40.48	.5616 -3	.3813 -2
14.28	.2121 -3	.8861 -1	.2393 -1	14.25	.3027 -3	2956	2.4200007	110.57	4.016	.3827	237.7	5.856	40.60	.5578 -3	.3802 -2
14.30	.2100 -3	.8800 -1	.2387 -1	14.27	.3006 -3	2977	2.4200816	110.60	4.010	.3827	238.4	5.857	40.71	.5540 -3	.3791 -2
14.32	.2080 -3	.8740 -1	.2380 -1	14.29	.2986 -3	2997	2.4201622	110.63	4.004	.3827	239.1	5.857	40.82	.5503 -3	.3780 -2
14.34	.2061 -3	.8682 -1	.2374 -1	14.31	.2967 -3	3017	2.4202425	110.65	3.999	.3827	239.7	5.858	40.93	.5466 -3	.3771 -2
14.36	.2041 -3	.8623 -1	.2367 -1	14.33	.2946 -3	3038	2.4203225	110.68	3.993	.3827	240.4	5.858	41.04	.5429 -3	.3760 -2
14.38	.2022 -3	.8565 -1	.2361 -1	14.35	.2927 -3	3058	2.4204021	110.71	3.988	.3826	241.1	5.858	41.15	.5393 -3	.3750 -2
14.40	.2003 -3	.8506 -1	.2355 -1	14.37	.2907 -3	3079	2.4204815	110.74	3.982	.3826	241.8	5.859	41.26	.5357 -3	.3739 -2
14.42	.1984 -3	.8449 -1	.2348 -1	14.39	.2888 -3	3100	2.4205604	110.76	3.977	.3826	242.4	5.859	41.38	.5321 -3	.3729 -2
14.44	.1965 -3	.8392 -1	.2342 -1	14.41	.2869 -3	3121	2.4206391	110.79	3.971	.3826	243.1	5.860	41.49	.5285 -3	.3719 -2
14.46	.1947 -3	.8335 -1	.2335 -1	14.43	.2849 -3	3142	2.4207175	110.81	3.966	.3826	243.8	5.860	41.60	.5250 -3	.3708 -2
14.48	.1929 -3	.8280 -1	.2329 -1	14.45	.2830 -3	3163	2.4207955	110.84	3.960	.3826	244.5	5.860	41.71	.5215 -3	.3698 -2
14.50	.1910 -3	.8224 -1	.2323 -1	14.47	.2812 -3	3184	2.4208732	110.87	3.955	.3826	245.1	5.861	41.83	.5180 -3	.3688 -2
14.52	.1892 -3	.8168 -1	.2317 -1	14.49	.2793 -3	3206	2.4209506	110.90	3.949	.3825	245.8	5.861	41.94	.5146 -3	.3677 -2
14.54	.1875 -3	.8114 -1	.2310 -1	14.51	.2774 -3	3227	2.4210277	110.92	3.944	.3825	246.5	5.861	42.05	.5111 -3	.3668 -2
14.56	.1857 -3	.8059 -1	.2304 -1	14.53	.2756 -3	3249	2.4211045	110.95	3.938	.3825	247.2	5.862	42.17	.5078 -3	.3657 -2
14.58	.1840 -3	.8005 -1	.2298 -1	14.55	.2737 -3	3271	2.4211810	110.97	3.933	.3825	247.8	5.862	42.28	.5044 -3	.3647 -2
14.60	.1823 -3	.7952 -1	.2292 -1	14.57	.2720 -3	3292	2.4212572	111.00	3.927	.3825	248.5	5.863	42.39	.5011 -3	.3638 -2
14.62	.1806 -3	.7899 -1	.2286 -1	14.59	.2702 -3	3314	2.4213330	111.03	3.922	.3825	249.2	5.863	42.51	.4978 -3	.3628 -2
14.64	.1789 -3	.7847 -1	.2280 -1	14.61	.2684 -3	3336	2.4214086	111.05	3.917	.3825	249.9	5.863	42.62	.4945 -3	.3618 -2
14.66	.1772 -3	.7794 -1	.2274 -1	14.63	.2666 -3	3359	2.4214838	111.08	3.911	.3825	250.6	5.864	42.73	.4912 -3	.3608 -2
14.68	.1756 -3	.7743 -1	.2268 -1	14.65	.2649 -3	3381	2.4215588	111.10	3.906	.3825	251.3	5.864	42.85	.4880 -3	.3598 -2
14.70	.1739 -3	.7691 -1	.2262 -1	14.67	.2631 -3	3404	2.4216335	111.13	3.901	.3824	251.9	5.864	42.96	.4847 -3	.3588 -2
14.72	.1723 -3	.7640 -1	.2256 -1	14.69	.2614 -3	3426	2.4217078	111.16	3.895	.3824	252.6	5.865	43.08	.4816 -3	.3578 -2
14.74	.1707 -3	.7590 -1	.2250 -1	14.71	.2597 -3	3449	2.4217819	111.18	3.890	.3824	253.3	5.865	43.19	.4784 -3	.3569 -2
14.76	.1692 -3	.7540 -1	.2244 -1	14.73	.2580 -3	3472	2.4218557	111.21	3.885	.3824	254.0	5.865	43.31	.4753 -3	.3559 -2
14.78	.1676 -3	.7490 -1	.2238 -1	14.75	.2563 -3	3494	2.4219292	111.23	3.880	.3824	254.7	5.866	43.42	.4722 -3	.3550 -2
14.80	.1660 -3	.7440 -1	.2232 -1	14.77	.2546 -3	3518	2.4220023	111.26	3.874	.3824	255.4	5.866	43.54	.4691 -3	.3540 -2
14.82	.1645 -3	.7392 -1	.2226 -1	14.79	.2530 -3	3541	2.4220752	111.28	3.869	.3824	256.1	5.866	43.65	.4660 -3	.3531 -2
14.84	.1630 -3	.7343 -1	.2220 -1	14.81	.2513 -3	3564	2.4221479	111.31	3.864	.3824	256.8	5.867	43.77	.4630 -3	.3521 -2
14.86	.1615 -3	.7295 -1	.2214 -1	14.83	.2497 -3	3588	2.4222202	111.34	3.859	.3823	257.5	5.867	43.88	.4600 -3	.3512 -2
14.88	.1600 -3	.7247 -1	.2208 -1	14.85	.2480 -3	3611	2.4222922	111.36	3.853	.3823	258.2	5.868	44.00	.4570 -3	.3502 -2
14.90	.1586 -3	.7199 -1	.2203 -1	14.87	.2464 -3	3635	2.4223640	111.38	3.848	.3823	258.9	5.868	44.11	.4540 -3	.3493 -2
14.92	.1571 -3	.7153 -1	.2197 -1	14.89	.2449 -3	3659	2.4224355	111.41	3.843	.3823	259.5	5.868	44.23	.4511 -3	.3483 -2
14.94	.1557 -3	.7106 -1	.2191 -1	14.91	.2433 -3	3683	2.4225066	111.43	3.838	.3823	260.2	5.869	44.35	.4481 -3	.3474 -2
14.96	.1543 -3	.7059 -1	.2185 -1	14.93	.2417 -3	3707	2.4225776	111.46	3.833	.3823	260.9	5.869	44.46	.4452 -3	.3465 -2
14.98	.1529 -3	.7014 -1	.2180 -1	14.95	.2401 -3	3731	2.4226482	111.48	3.828	.3823	261.6	5.869	44.58	.4424 -3	.3456 -2
15.00	.1515 -3	.6968 -1	.2174 -1	14.97	.2386 -3	3755	2.4227186	111.51	3.823	.3823	262.3	5.870	44.69	.4395 -3	.3446 -2
15.02	.1501 -3	.6923 -1	.2168 -1	14.99	.2371 -3	3779	2.4227886	111.53	3.817	.3823	263.0	5.870	44.81	.4367 -3	.3437 -2
15.04	.1487 -3	.6878 -1	.2163 -1	15.01	.2355 -3	3804	2.4228585	111.56	3.812	.3822	263.7	5.870	44.93	.4339 -3	.3428 -2
15.06	.1474 -3	.6833 -1	.2157 -1	15.03	.2340 -3	3829	2.4229280	111.59	3.807	.3822	264.4	5.871	45.05	.4311 -3	.3419 -2
15.08	.1461 -3	.6789 -1	.2151 -1	15.05	.2325 -3	3854	2.4229973	111.61	3.802	.3822	265.1	5.871	45.16	.4283 -3	.3410 -2
15.10	.1447 -3	.6745 -1	.2146 -1	15.07	.2310 -3	3879	2.4230663	111.63	3.797	.3822	265.9	5.871	45.28	.4256 -3	.3401 -2
15.12	.1434 -3	.6702 -1	.2140 -1	15.09	.2296 -3	3904	2.4231350	111.66	3.792	.3822	266.6	5.872	45.40	.4229 -3	.3392 -2
15.14	.1421 -3	.6658 -1	.2135 -1	15.11	.2281 -3	3929	2.4232035	111.68	3.787	.3822	267.3	5.872	45.52	.4201 -3	.3383 -2
15.16	.1409 -3	.6615 -1	.2129 -1	15.13	.2266 -3	3955	2.4232717	111.71	3.782	.3822	268.0	5.872	45.63	.4175 -3	.3374 -2
15.18	.1396 -3	.6573 -1	.2124 -1	15.15	.2252 -3	3980	2.4233396	111.73	3.777	.3822	268.7	5.873	45.75	.4148 -3	.3365 -2
15.20	.1383 -3	.6531 -1	.2118 -1	15.17	.2237 -3	4005	2.4234073	111.76	3.772	.3822	269.4	5.873	45.87	.4122 -3	.3357 -2
15.22	.1371 -3	.6489 -1	.2113 -1	15.19	.2223 -3	4032	2.4234747	111.78	3.767	.3821	270.1	5.873	45.99	.4096 -3	.3347 -2
15.24	.1359 -3	.6447 -1	.2107 -1	15.21	.2209 -3	4057	2.4235419	111.80	3.762	.3821	270.8	5.874	46.11	.4070 -3	.3339 -2
15.26	.1347 -3	.6406 -1	.2102 -1	15.23	.2195 -3	4083	2.4236088	111.83	3.757	.3821	271.5	5.874	46.22	.4044 -3	.3330 -2
15.28	.1335 -3	.6365 -1	.2097 -1	15.25	.2181 -3	4110	2.4236754	111.85	3.752	.3821	272.2	5.874	46.34	.4018 -3	.3321 -2
15.30	.1323 -3	.6325 -1	.2091 -1	15.27	.2167 -3	4135	2.4237418	111.88	3.748	.3821	272.9	5.875	46.46	.3992 -3	.3313 -2
15.32	.1311 -3	.6284 -1	.2086 -1	15.29	.2154 -3	4162	2.4238079	111.90	3.743	.3821	273.7	5.875	46.58	.3967 -3	.3304 -2
15.34	.1299 -3	.6244 -1	.2081 -1	15.31	.2140 -3	4189	2.4238738	111.92	3.738	.3821	274.4	5.875	46.70	.3942 -3	.3295 -2
15.36	.1288 -3	.6204 -1	.2075 -1	15.33	.2127 -3	4215	2.4239394	111.95	3.733	.3821	275.1	5.876	46.82	.3917 -3	.3287 -2
15.38	.1276 -3	.6165 -1	.2070 -1	15.35	.2113 -3	4242	2.4240048	111.97	3.728	.3821	275.8	5.876	46.94	.3893 -3	.3278 -2
15.40	.1265 -3	.6126 -1	.2065 -1	15.37	.2100 -3	4269	2.4240699	112.00	3.723	.3820	276.5	5.876	47.06	.3868 -3	.3270 -2
15.42	.1254 -3	.6087 -1	.2060 -1	15.39	.2087 -3	4296	2.4241348	112.02	3.718	.3820	277.2	5.876	47.18	.3844 -3	.3262 -2
15.44	.1243 -3	.6049 -1	.2054 -1	15.41	.2074 -3	4323	2.4241994	112.04	3.714	.3820	277.9	5.877	47.30	.3820 -3	.3253 -2
15.46	.1232 -3	.6010 -1	.2049 -1	15.43	.2061 -3	4351	2.4242638	112.06	3.709	.38					

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
15.90	.1016 -3	.5238 -1	.1939 -1	15.87	.1798 -3	4990	2.4256206	112.57	3.606	.3818	294.8	5.884	50.10	.3311 -3	.3068 -2
15.92	.1007 -3	.5206 -1	.1935 -1	15.89	.1787 -3	5020	2.4256797	112.59	3.601	.3818	295.5	5.884	50.23	.3291 -3	.3060 -2
15.94	.9986 -3	.5174 -1	.1930 -1	15.91	.1776 -3	5051	2.4257385	112.61	3.597	.3818	296.3	5.884	50.35	.3271 -3	.3053 -2
15.96	.9989 -3	.5142 -1	.1925 -1	15.93	.1765 -3	5082	2.4257971	112.63	3.592	.3818	297.0	5.885	50.47	.3251 -3	.3045 -2
15.98	.9815 -3	.5111 -1	.1920 -1	15.95	.1754 -3	5113	2.4258555	112.66	3.588	.3818	297.8	5.885	50.60	.3232 -3	.3037 -2
16.00	.9731 -3	.5079 -1	.1916 -1	15.97	.1744 -3	5145	2.4259137	112.68	3.583	.3817	298.5	5.885	50.72	.3212 -3	.3030 -2
16.02	.9647 -3	.5048 -1	.1911 -1	15.99	.1733 -3	5176	2.4259717	112.70	3.579	.3817	299.3	5.885	50.85	.3192 -3	.3022 -2
16.04	.9565 -3	.5017 -1	.1906 -1	16.01	.1723 -3	5208	2.4260295	112.72	3.574	.3817	300.0	5.886	50.97	.3173 -3	.3015 -2
16.06	.9484 -3	.4987 -1	.1902 -1	16.03	.1712 -3	5239	2.4260871	112.74	3.570	.3817	300.7	5.886	51.10	.3154 -3	.3007 -2
16.08	.9404 -3	.4957 -1	.1897 -1	16.05	.1702 -3	5271	2.4261444	112.76	3.566	.3817	301.5	5.886	51.22	.3135 -3	.3000 -2
16.10	.9323 -3	.4926 -1	.1892 -1	16.07	.1692 -3	5304	2.4262015	112.79	3.561	.3817	302.3	5.887	51.35	.3116 -3	.2992 -2
16.12	.9244 -3	.4896 -1	.1888 -1	16.09	.1681 -3	5336	2.4262585	112.81	3.557	.3817	303.0	5.887	51.47	.3097 -3	.2985 -2
16.14	.9165 -3	.4867 -1	.1883 -1	16.11	.1671 -3	5369	2.4263152	112.83	3.552	.3817	303.8	5.887	51.60	.3079 -3	.2977 -2
16.16	.9089 -3	.4838 -1	.1879 -1	16.13	.1661 -3	5401	2.4263717	112.85	3.548	.3817	304.5	5.887	51.72	.3060 -3	.2970 -2
16.18	.9011 -3	.4808 -1	.1874 -1	16.15	.1651 -3	5434	2.4264280	112.87	3.543	.3817	305.3	5.888	51.85	.3042 -3	.2963 -2
16.20	.8936 -3	.4779 -1	.1870 -1	16.17	.1642 -3	5466	2.4264841	112.89	3.539	.3817	306.0	5.888	51.97	.3024 -3	.2955 -2
16.22	.8860 -3	.4751 -1	.1865 -1	16.19	.1632 -3	5499	2.4265400	112.91	3.535	.3816	306.8	5.888	52.10	.3005 -3	.2948 -2
16.24	.8784 -3	.4724 -1	.1861 -1	16.21	.1622 -3	5533	2.4265958	112.94	3.530	.3816	307.5	5.888	52.23	.2987 -3	.2941 -2
16.26	.8712 -3	.4694 -1	.1856 -1	16.23	.1612 -3	5566	2.4266513	112.96	3.526	.3816	308.3	5.889	52.35	.2969 -3	.2934 -2
16.28	.8638 -3	.4665 -1	.1852 -1	16.25	.1603 -3	5600	2.4267066	112.98	3.522	.3816	309.0	5.889	52.48	.2952 -3	.2926 -2
16.30	.8565 -3	.4637 -1	.1847 -1	16.27	.1593 -3	5634	2.4267617	113.00	3.517	.3816	309.8	5.889	52.61	.2934 -3	.2919 -2
16.32	.8494 -3	.4609 -1	.1843 -1	16.29	.1584 -3	5667	2.4268166	113.02	3.513	.3816	310.6	5.889	52.73	.2916 -3	.2912 -2
16.34	.8423 -3	.4582 -1	.1838 -1	16.31	.1574 -3	5701	2.4268713	113.04	3.509	.3816	311.3	5.890	52.86	.2899 -3	.2905 -2
16.36	.8352 -3	.4554 -1	.1834 -1	16.33	.1565 -3	5735	2.4269258	113.06	3.504	.3816	312.1	5.890	52.99	.2882 -3	.2898 -2
16.38	.8283 -3	.4527 -1	.1830 -1	16.35	.1556 -3	5770	2.4269801	113.08	3.500	.3816	312.9	5.890	53.12	.2865 -3	.2891 -2
16.40	.8213 -3	.4500 -1	.1825 -1	16.37	.1546 -3	5804	2.4270342	113.11	3.496	.3816	313.6	5.891	53.24	.2848 -3	.2884 -2
16.42	.8144 -3	.4473 -1	.1821 -1	16.39	.1537 -3	5839	2.4270881	113.13	3.492	.3816	314.4	5.891	53.37	.2831 -3	.2877 -2
16.44	.8077 -3	.4447 -1	.1816 -1	16.41	.1528 -3	5874	2.4271418	113.15	3.487	.3815	315.2	5.891	53.50	.2814 -3	.2870 -2
16.46	.8009 -3	.4420 -1	.1812 -1	16.43	.1519 -3	5910	2.4271954	113.17	3.483	.3815	315.9	5.891	53.63	.2798 -3	.2863 -2
16.48	.7942 -3	.4394 -1	.1808 -1	16.45	.1510 -3	5945	2.4272487	113.19	3.479	.3815	316.7	5.892	53.75	.2781 -3	.2856 -2
16.50	.7876 -3	.4367 -1	.1803 -1	16.47	.1501 -3	5980	2.4273019	113.21	3.475	.3815	317.5	5.892	53.88	.2765 -3	.2849 -2
16.52	.7811 -3	.4341 -1	.1799 -1	16.49	.1492 -3	6016	2.4273548	113.23	3.470	.3815	318.2	5.892	54.01	.2749 -3	.2842 -2
16.54	.7747 -3	.4316 -1	.1795 -1	16.51	.1484 -3	6051	2.4274076	113.25	3.466	.3815	319.0	5.892	54.14	.2732 -3	.2836 -2
16.56	.7682 -3	.4290 -1	.1791 -1	16.53	.1475 -3	6087	2.4274602	113.27	3.462	.3815	319.8	5.893	54.27	.2716 -3	.2828 -2
16.58	.7620 -3	.4265 -1	.1786 -1	16.55	.1466 -3	6123	2.4275126	113.29	3.458	.3815	320.6	5.893	54.40	.2700 -3	.2822 -2
16.60	.7556 -3	.4240 -1	.1782 -1	16.57	.1457 -3	6160	2.4275648	113.31	3.454	.3815	321.3	5.893	54.53	.2685 -3	.2814 -2
16.62	.7493 -3	.4215 -1	.1778 -1	16.59	.1449 -3	6196	2.4276169	113.33	3.449	.3815	322.1	5.893	54.66	.2669 -3	.2808 -2
16.64	.7432 -3	.4190 -1	.1774 -1	16.61	.1440 -3	6233	2.4276687	113.35	3.445	.3815	322.9	5.894	54.78	.2653 -3	.2801 -2
16.66	.7372 -3	.4166 -1	.1770 -1	16.63	.1432 -3	6268	2.4277204	113.37	3.441	.3815	323.7	5.894	54.91	.2638 -3	.2795 -2
16.68	.7311 -3	.4141 -1	.1765 -1	16.65	.1424 -3	6306	2.4277719	113.39	3.437	.3814	324.4	5.894	55.04	.2622 -3	.2788 -2
16.70	.7250 -3	.4117 -1	.1761 -1	16.67	.1415 -3	6343	2.4278232	113.41	3.433	.3814	325.2	5.894	55.17	.2607 -3	.2781 -2
16.72	.7191 -3	.4093 -1	.1757 -1	16.69	.1407 -3	6380	2.4278743	113.43	3.429	.3814	326.0	5.895	55.30	.2592 -3	.2774 -2
16.74	.7132 -3	.4069 -1	.1753 -1	16.71	.1399 -3	6417	2.4279252	113.45	3.425	.3814	326.8	5.895	55.43	.2577 -3	.2768 -2
16.76	.7074 -3	.4045 -1	.1749 -1	16.73	.1391 -3	6455	2.4279760	113.47	3.421	.3814	327.6	5.895	55.56	.2562 -3	.2762 -2
16.78	.7016 -3	.4021 -1	.1745 -1	16.75	.1383 -3	6493	2.4280266	113.49	3.417	.3814	328.3	5.895	55.69	.2547 -3	.2755 -2
16.80	.6959 -3	.3998 -1	.1741 -1	16.77	.1375 -3	6531	2.4280770	113.51	3.413	.3814	329.1	5.896	55.82	.2532 -3	.2748 -2
16.82	.6902 -3	.3974 -1	.1737 -1	16.79	.1367 -3	6570	2.4281272	113.53	3.408	.3814	329.9	5.896	55.96	.2517 -3	.2742 -2
16.84	.6846 -3	.3951 -1	.1733 -1	16.81	.1359 -3	6607	2.4281772	113.55	3.404	.3814	330.7	5.896	56.09	.2503 -3	.2735 -2
16.86	.6790 -3	.3928 -1	.1729 -1	16.83	.1351 -3	6647	2.4282271	113.57	3.400	.3814	331.5	5.896	56.22	.2488 -3	.2729 -2
16.88	.6735 -3	.3905 -1	.1725 -1	16.85	.1343 -3	6685	2.4282768	113.59	3.396	.3814	332.3	5.897	56.35	.2474 -3	.2722 -2
16.90	.6680 -3	.3883 -1	.1721 -1	16.87	.1336 -3	6724	2.4283264	113.61	3.392	.3814	333.1	5.897	56.48	.2460 -3	.2716 -2
16.92	.6626 -3	.3860 -1	.1717 -1	16.89	.1328 -3	6763	2.4283757	113.63	3.388	.3813	333.8	5.897	56.61	.2446 -3	.2709 -2
16.94	.6572 -3	.3838 -1	.1713 -1	16.91	.1320 -3	6802	2.4284249	113.65	3.384	.3813	334.6	5.897	56.74	.2432 -3	.2703 -2
16.96	.6520 -3	.3816 -1	.1709 -1	16.93	.1313 -3	6841	2.4284739	113.67	3.380	.3813	335.4	5.898	56.88	.2418 -3	.2697 -2
16.98	.6467 -3	.3794 -1	.1705 -1	16.95	.1305 -3	6881	2.4285228	113.69	3.376	.3813	336.2	5.898	57.01	.2404 -3	.2690 -2
17.00	.6415 -3	.3772 -1	.1701 -1	16.97	.1298 -3	6920	2.4285714	113.71	3.372	.3813	337.0	5.898	57.14	.2390 -3	.2684 -2
17.02	.6364 -3	.3750 -1	.1697 -1	16.99	.1290 -3	6960	2.4286199	113.73	3.368	.3813	337.8	5.898	57.27	.2376 -3	.2678 -2
17.04	.6311 -3	.3728 -1	.1693 -1	17.01	.1283 -3	7001	2.4286683	113.75	3.364	.3813	338.6	5.898	57.40	.2362 -3	.2671 -2
17.06	.6261 -3	.3707 -1	.1689 -1	17.03	.1276 -3	7042	2.4287164	113.77	3.360	.3813	339.4	5.899	57.54	.2349 -3	.2665 -2
17.08	.6211 -3	.3686 -1	.1685 -1	17.05	.1268 -3	7081	2.4287645	113.79	3.356	.3813	340.2	5.899	57.67	.2336 -3	.2659 -2
17.10	.6161 -3	.3665 -1	.1681 -1	17.07	.1261 -3	7122	2.4288123	113.81	3.353	.3813	341.0	5.899	57.80	.2322 -3	.2653 -2
17.12	.6111 -3	.3644 -1	.1677 -1	17.09	.1254 -3	7163	2.4288600	113.83	3.349	.3813	341.8	5.899	57.94	.2309 -3	.2646 -2
17.14	.6063 -3	.3623 -1	.1674 -1	17.11	.1247 -3	7204	2.4289075	113.85	3.345	.3813	342.6	5.900	58.07	.2296 -3	.2641 -2
17.16	.6014 -3	.3602 -1	.1670 -1	17.13	.1240 -3	7246	2.4289548	113.87	3.341	.3813	343.4	5.900	58.20	.2283 -3	.2634 -2
17.18	.5966 -3	.3581 -1	.1666 -1	17.15	.1233 -3	7287	2.4290020	113.88	3.337	.3812	344.2	5.900	58.34	.2270 -3	.2628 -2
17.20	.5918 -3	.3561 -1	.1662 -1	17.17	.1226 -3	7329	2.4290490	113.90	3.333	.3812	345.0	5.900	58.47	.2257 -3	.2622 -2
17.22	.5871 -3	.3541 -1	.1658 -1	17.19	.1219 -3	7371	2.4290959	113.92	3.329	.3812	345.8	5.901	58.60	.2245 -3	.2616 -2
17.24	.5824 -3	.3520 -1	.1654 -1	17.21	.1212 -3	7413	2.4291426	113.94	3.325	.3812	346.6	5.901	58.74	.2232 -3	.2610 -2
17.26	.5779 -3	.3501 -1	.1651 -1	17.23	.1205 -3	7454	2.4291891	113.96	3.321						

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
17.70	.4859 -	.3093 -	.1571 -	17.67	.1066 -	8434	2.4301741	114.36	3.239	.3811	365.3	5.906	61.86	.1962 -	.2476 -
17.72	.4821 -	.3076 -	.1567 -	17.69	.1060 -	8481	2.4302172	114.38	3.235	.3810	366.2	5.906	62.00	.1951 -	.2471 -
17.74	.4783 -	.3059 -	.1564 -	17.71	.1054 -	8529	2.4302601	114.40	3.231	.3810	367.0	5.906	62.14	.1941 -	.2465 -
17.76	.4747 -	.3042 -	.1561 -	17.73	.1048 -	8575	2.4303029	114.42	3.228	.3810	367.8	5.906	62.28	.1930 -	.2460 -
17.78	.4710 -	.3025 -	.1557 -	17.75	.1042 -	8623	2.4303455	114.44	3.224	.3810	368.7	5.907	62.41	.1919 -	.2454 -
17.80	.4674 -	.3009 -	.1554 -	17.77	.1037 -	8670	2.4303880	114.45	3.221	.3810	369.5	5.907	62.55	.1909 -	.2449 -
17.82	.4637 -	.2992 -	.1550 -	17.79	.1031 -	8719	2.4304304	114.47	3.217	.3810	370.3	5.907	62.69	.1898 -	.2443 -
17.84	.4602 -	.2975 -	.1547 -	17.81	.1025 -	8767	2.4304726	114.49	3.213	.3810	371.1	5.907	62.83	.1888 -	.2437 -
17.86	.4566 -	.2959 -	.1543 -	17.83	.1020 -	8815	2.4305147	114.51	3.210	.3810	372.0	5.907	62.97	.1878 -	.2432 -
17.88	.4531 -	.2943 -	.1540 -	17.85	.1014 -	8864	2.4305566	114.52	3.206	.3810	372.8	5.908	63.11	.1867 -	.2427 -
17.90	.4496 -	.2926 -	.1537 -	17.87	.1009 -	8913	2.4305984	114.54	3.203	.3810	373.7	5.908	63.25	.1857 -	.2421 -
17.92	.4462 -	.2910 -	.1533 -	17.89	.1003 -	8962	2.4306401	114.56	3.199	.3810	374.5	5.908	63.39	.1847 -	.2416 -
17.94	.4428 -	.2895 -	.1530 -	17.91	.9976 -	9010	2.4306816	114.58	3.195	.3810	375.3	5.908	63.53	.1837 -	.2411 -
17.96	.4394 -	.2879 -	.1526 -	17.93	.9921 -	9060	2.4307230	114.59	3.192	.3810	376.2	5.908	63.67	.1827 -	.2405 -
17.98	.4361 -	.2863 -	.1523 -	17.95	.9868 -	9109	2.4307642	114.61	3.188	.3810	377.0	5.909	63.80	.1817 -	.2400 -
18.00	.4328 -	.2848 -	.1520 -	17.97	.9815 -	9159	2.4308053	114.63	3.185	.3810	377.8	5.909	63.94	.1807 -	.2395 -
18.02	.4294 -	.2832 -	.1516 -	17.99	.9760 -	9210	2.4308463	114.65	3.181	.3809	378.7	5.909	64.08	.1797 -	.2389 -
18.04	.4261 -	.2816 -	.1513 -	18.01	.9707 -	9260	2.4308872	114.66	3.178	.3809	379.5	5.909	64.23	.1788 -	.2384 -
18.06	.4229 -	.2801 -	.1510 -	18.03	.9655 -	9311	2.4309279	114.68	3.174	.3809	380.4	5.909	64.37	.1778 -	.2379 -
18.08	.4197 -	.2786 -	.1507 -	18.05	.9602 -	9362	2.4309685	114.70	3.171	.3809	381.2	5.910	64.51	.1768 -	.2373 -
18.10	.4165 -	.2771 -	.1503 -	18.07	.9552 -	9411	2.4310089	114.72	3.167	.3809	382.1	5.910	64.65	.1759 -	.2368 -
18.12	.4134 -	.2756 -	.1500 -	18.09	.9500 -	9463	2.4310492	114.73	3.164	.3809	382.9	5.910	64.79	.1749 -	.2363 -
18.14	.4102 -	.2741 -	.1497 -	18.11	.9448 -	9515	2.4310894	114.75	3.160	.3809	383.7	5.910	64.93	.1740 -	.2357 -
18.16	.4071 -	.2726 -	.1494 -	18.13	.9398 -	9566	2.4311295	114.77	3.157	.3809	384.6	5.910	65.07	.1731 -	.2353 -
18.18	.4041 -	.2711 -	.1490 -	18.15	.9349 -	9617	2.4311694	114.78	3.153	.3809	385.4	5.911	65.21	.1721 -	.2348 -
18.20	.4010 -	.2696 -	.1487 -	18.17	.9297 -	9671	2.4312092	114.80	3.150	.3809	386.3	5.911	65.35	.1712 -	.2342 -
18.22	.3979 -	.2682 -	.1484 -	18.19	.9247 -	9723	2.4312488	114.82	3.146	.3809	387.1	5.911	65.49	.1703 -	.2337 -
18.24	.3949 -	.2667 -	.1481 -	18.21	.9198 -	9775	2.4312884	114.83	3.143	.3809	388.0	5.911	65.64	.1694 -	.2332 -
18.26	.3920 -	.2653 -	.1477 -	18.23	.9148 -	9828	2.4313278	114.85	3.139	.3809	388.8	5.911	65.78	.1685 -	.2327 -
18.28	.3890 -	.2639 -	.1474 -	18.25	.9099 -	9881	2.4313671	114.87	3.136	.3809	389.7	5.912	65.92	.1676 -	.2321 -
18.30	.3861 -	.2625 -	.1471 -	18.27	.9052 -	9933	2.4314062	114.89	3.133	.3809	390.5	5.912	66.06	.1667 -	.2317 -
18.32	.3832 -	.2611 -	.1468 -	18.29	.9003 -	9987	2.4314452	114.90	3.129	.3809	391.4	5.912	66.20	.1658 -	.2312 -
18.34	.3803 -	.2596 -	.1465 -	18.31	.8954 -	1004	2.4314841	114.92	3.126	.3808	392.3	5.912	66.35	.1649 -	.2306 -
18.36	.3775 -	.2583 -	.1462 -	18.33	.8907 -	1010	2.4315229	114.94	3.122	.3808	393.1	5.912	66.49	.1640 -	.2302 -
18.38	.3747 -	.2569 -	.1459 -	18.35	.8861 -	1015	2.4315616	114.95	3.119	.3808	394.0	5.913	66.63	.1632 -	.2297 -
18.40	.3718 -	.2555 -	.1455 -	18.37	.8812 -	1020	2.4316001	114.97	3.115	.3808	394.8	5.913	66.78	.1623 -	.2291 -
18.42	.3691 -	.2541 -	.1452 -	18.39	.8765 -	1026	2.4316385	114.99	3.112	.3808	395.7	5.913	66.92	.1614 -	.2287 -
18.44	.3663 -	.2528 -	.1449 -	18.41	.8719 -	1031	2.4316768	115.00	3.109	.3808	396.5	5.913	67.06	.1606 -	.2281 -
18.46	.3636 -	.2514 -	.1446 -	18.43	.8673 -	1037	2.4317149	115.02	3.105	.3808	397.4	5.913	67.21	.1597 -	.2277 -
18.48	.3609 -	.2501 -	.1443 -	18.45	.8628 -	1042	2.4317530	115.04	3.102	.3808	398.3	5.913	67.35	.1589 -	.2272 -
18.50	.3582 -	.2488 -	.1440 -	18.47	.8582 -	1048	2.4317909	115.05	3.099	.3808	399.1	5.914	67.49	.1580 -	.2267 -
18.52	.3555 -	.2475 -	.1437 -	18.49	.8536 -	1054	2.4318287	115.07	3.095	.3808	400.0	5.914	67.64	.1572 -	.2262 -
18.54	.3530 -	.2462 -	.1434 -	18.51	.8492 -	1059	2.4318664	115.08	3.092	.3808	400.9	5.914	67.78	.1564 -	.2257 -
18.56	.3503 -	.2448 -	.1431 -	18.53	.8446 -	1065	2.4319039	115.10	3.089	.3808	401.7	5.914	67.93	.1555 -	.2252 -
18.58	.3477 -	.2436 -	.1428 -	18.55	.8403 -	1070	2.4319413	115.12	3.085	.3808	402.6	5.914	68.07	.1547 -	.2248 -
18.60	.3452 -	.2423 -	.1425 -	18.57	.8359 -	1076	2.4319787	115.13	3.082	.3808	403.5	5.915	68.21	.1539 -	.2243 -
18.62	.3426 -	.2410 -	.1422 -	18.59	.8315 -	1082	2.4320159	115.15	3.079	.3808	404.3	5.915	68.36	.1531 -	.2238 -
18.64	.3400 -	.2397 -	.1419 -	18.61	.8270 -	1088	2.4320529	115.17	3.075	.3808	405.2	5.915	68.50	.1523 -	.2233 -
18.66	.3375 -	.2384 -	.1416 -	18.63	.8226 -	1093	2.4320899	115.18	3.072	.3807	406.1	5.915	68.65	.1515 -	.2228 -
18.68	.3351 -	.2372 -	.1413 -	18.65	.8185 -	1099	2.4321267	115.20	3.069	.3807	406.9	5.915	68.79	.1507 -	.2224 -
18.70	.3326 -	.2359 -	.1410 -	18.67	.8142 -	1105	2.4321635	115.21	3.065	.3807	407.8	5.915	68.94	.1499 -	.2219 -
18.72	.3301 -	.2347 -	.1407 -	18.69	.8099 -	1111	2.4322001	115.23	3.062	.3807	408.7	5.916	69.09	.1491 -	.2214 -
18.74	.3278 -	.2335 -	.1404 -	18.71	.8058 -	1116	2.4322366	115.25	3.059	.3807	409.6	5.916	69.23	.1484 -	.2209 -
18.76	.3253 -	.2322 -	.1401 -	18.73	.8015 -	1122	2.4322729	115.26	3.056	.3807	410.4	5.916	69.38	.1476 -	.2205 -
18.78	.3230 -	.2310 -	.1398 -	18.75	.7974 -	1128	2.4323092	115.28	3.052	.3807	411.3	5.916	69.52	.1468 -	.2200 -
18.80	.3206 -	.2298 -	.1395 -	18.77	.7931 -	1134	2.4323454	115.29	3.049	.3807	412.2	5.916	69.67	.1460 -	.2195 -
18.82	.3182 -	.2286 -	.1392 -	18.79	.7890 -	1140	2.4323814	115.31	3.046	.3807	413.1	5.917	69.82	.1453 -	.2191 -
18.84	.3159 -	.2274 -	.1389 -	18.81	.7849 -	1146	2.4324173	115.33	3.043	.3807	413.9	5.917	69.96	.1445 -	.2186 -
18.86	.3136 -	.2262 -	.1386 -	18.83	.7809 -	1152	2.4324531	115.34	3.039	.3807	414.8	5.917	70.11	.1438 -	.2182 -
18.88	.3113 -	.2251 -	.1383 -	18.85	.7768 -	1158	2.4324888	115.36	3.036	.3807	415.7	5.917	70.26	.1430 -	.2177 -
18.90	.3090 -	.2239 -	.1380 -	18.87	.7727 -	1164	2.4325244	115.38	3.033	.3807	416.6	5.917	70.40	.1423 -	.2172 -
18.92	.3068 -	.2227 -	.1378 -	18.89	.7687 -	1170	2.4325599	115.39	3.030	.3807	417.5	5.917	70.55	.1416 -	.2167 -
18.94	.3046 -	.2216 -	.1375 -	18.91	.7649 -	1176	2.4325953	115.41	3.027	.3807	418.3	5.918	70.70	.1408 -	.2163 -
18.96	.3024 -	.2204 -	.1372 -	18.93	.7608 -	1182	2.4326305	115.42	3.023	.3807	419.2	5.918	70.84	.1401 -	.2158 -
18.98	.3002 -	.2193 -	.1369 -	18.95	.7570 -	1188	2.4326657	115.44	3.020	.3807	420.1	5.918	70.99	.1394 -	.2154 -
19.00	.2980 -	.2181 -	.1366 -	18.97	.7530 -	1195	2.4327007	115.45	3.017	.3806	421.0	5.918	71.14	.1387 -	.2149 -
19.02	.2959 -	.2170 -	.1363 -	18.99	.7492 -	1201	2.4327356	115.47	3.014	.3806	421.9	5.918	71.29	.1379 -	.2145 -
19.04	.2937 -	.2159 -	.1361 -	19.01	.7454 -	1207	2.4327705	115.48	3.011	.3806	422.8	5.918	71.44	.1372 -	.2141 -
19.06	.2915 -	.2148 -	.1358 -	19.03	.7414 -	1213	2.4328052	115.50	3.008	.3806	423.7	5.919	71.58	.1365 -	.2136 -
19.08	.2894 -	.2136 -	.1355 -	19.05	.7376 -	1220	2.4328398	115.52	3.004	.3806	424.6	5.919	71.73	.1358 -	.2131 -
19.10	.2874 -	.2125 -	.1352 -	19.07	.7338 -	1226	2.4328743	115.53	3.001	.3806	425.5	5.919	71.88	.1351 -	.2127 -
19.12	.2														

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
19.50	.2491 -	.1919 -	.1298 -	19.47	.6630 -	1357 +	2.4335424	115.83	2.940	.3805	443.5	5.922	74.88	.1221 -	.2041 -
19.52	.2473 -	.1909 -	.1295 -	19.49	.6595 -	1365 +	2.4335747	115.85	2.937	.3805	444.4	5.922	75.03	.1214 -	.2036 -
19.54	.2455 -	.1900 -	.1293 -	19.51	.6563 -	1371 +	2.4336070	115.86	2.934	.3805	445.3	5.922	75.19	.1208 -	.2032 -
19.56	.2438 -	.1890 -	.1290 -	19.53	.6530 -	1378 +	2.4336391	115.88	2.931	.3805	446.2	5.923	75.34	.1202 -	.2028 -
19.58	.2421 -	.1881 -	.1287 -	19.55	.6497 -	1385 +	2.4336712	115.89	2.928	.3805	447.1	5.923	75.49	.1196 -	.2024 -
19.60	.2404 -	.1871 -	.1285 -	19.57	.6464 -	1392 +	2.4337031	115.91	2.925	.3805	448.0	5.923	75.64	.1190 -	.2020 -
19.62	.2387 -	.1862 -	.1282 -	19.59	.6432 -	1399 +	2.4337350	115.92	2.922	.3805	448.9	5.923	75.80	.1184 -	.2016 -
19.64	.2371 -	.1853 -	.1280 -	19.61	.6401 -	1406 +	2.4337667	115.94	2.919	.3805	449.9	5.923	75.95	.1178 -	.2012 -
19.66	.2354 -	.1843 -	.1277 -	19.63	.6369 -	1413 +	2.4337984	115.95	2.916	.3805	450.8	5.923	76.10	.1173 -	.2008 -
19.68	.2337 -	.1834 -	.1275 -	19.65	.6336 -	1420 +	2.4338300	115.97	2.913	.3805	451.7	5.924	76.25	.1167 -	.2003 -
19.70	.2321 -	.1825 -	.1272 -	19.67	.6306 -	1427 +	2.4338615	115.98	2.910	.3805	452.6	5.924	76.41	.1161 -	.2000 -
19.72	.2305 -	.1816 -	.1269 -	19.69	.6274 -	1435 +	2.4338928	116.00	2.907	.3805	453.5	5.924	76.56	.1155 -	.1995 -
19.74	.2289 -	.1807 -	.1267 -	19.72	.6243 -	1442 +	2.4339241	116.01	2.904	.3805	454.5	5.924	76.71	.1149 -	.1991 -
19.76	.2273 -	.1798 -	.1264 -	19.74	.6213 -	1449 +	2.4339553	116.03	2.901	.3804	455.4	5.924	76.87	.1144 -	.1988 -
19.78	.2257 -	.1788 -	.1262 -	19.76	.6180 -	1456 +	2.4339864	116.04	2.898	.3804	456.3	5.924	77.02	.1138 -	.1983 -
19.80	.2241 -	.1780 -	.1259 -	19.78	.6150 -	1464 +	2.4340174	116.05	2.895	.3804	457.2	5.924	77.17	.1132 -	.1979 -
19.82	.2226 -	.1771 -	.1257 -	19.80	.6120 -	1471 +	2.4340483	116.07	2.892	.3804	458.1	5.925	77.33	.1127 -	.1975 -
19.84	.2210 -	.1762 -	.1254 -	19.82	.6090 -	1478 +	2.4340792	116.08	2.889	.3804	459.1	5.925	77.48	.1121 -	.1971 -
19.86	.2195 -	.1753 -	.1252 -	19.84	.6059 -	1486 +	2.4341099	116.10	2.886	.3804	460.0	5.925	77.64	.1116 -	.1967 -
19.88	.2179 -	.1745 -	.1249 -	19.86	.6029 -	1493 +	2.4341405	116.11	2.883	.3804	460.9	5.925	77.79	.1110 -	.1963 -
19.90	.2165 -	.1736 -	.1247 -	19.88	.6001 -	1500 +	2.4341711	116.13	2.880	.3804	461.9	5.925	77.95	.1105 -	.1960 -
19.92	.2150 -	.1727 -	.1244 -	19.90	.5971 -	1508 +	2.4342015	116.14	2.878	.3804	462.8	5.925	78.10	.1099 -	.1956 -
19.94	.2135 -	.1719 -	.1242 -	19.92	.5941 -	1515 +	2.4342319	116.15	2.875	.3804	463.7	5.926	78.26	.1094 -	.1952 -
19.96	.2120 -	.1710 -	.1240 -	19.94	.5913 -	1523 +	2.4342622	116.17	2.872	.3804	464.6	5.926	78.41	.1088 -	.1948 -
19.98	.2105 -	.1702 -	.1237 -	19.96	.5883 -	1530 +	2.4342924	116.18	2.869	.3804	465.6	5.926	78.57	.1083 -	.1944 -
20.00	.2091 -	.1694 -	.1235 -	19.98	.5855 -	1538 +	2.4343225	116.20	2.866	.3804	466.5	5.926	78.72	.1078 -	.1940 -
20.20	.1952 -	.1612 -	.1211 -	20.18	.5574 -	1615 +	2.4346186	116.34	2.838	.3803	475.9	5.927	80.29	.1026 -	.1902 -
20.40	.1823 -	.1536 -	.1187 -	20.38	.5311 -	1695 +	2.4349062	116.47	2.810	.3803	485.4	5.929	81.86	.0978 -	.1865 -
20.60	.1704 -	.1463 -	.1165 -	20.58	.5062 -	1779 +	2.4351855	116.61	2.782	.3802	494.9	5.930	83.46	.0930 -	.1829 -
20.80	.1594 -	.1395 -	.1143 -	20.78	.4827 -	1866 +	2.4354569	116.74	2.756	.3802	504.6	5.932	85.07	.0887 -	.1794 -
21.00	.1492 -	.1331 -	.1121 -	20.98	.4606 -	1956 +	2.4357207	116.87	2.729	.3802	514.3	5.933	86.69	.08478 -	.1760 -
21.20	.1397 -	.1270 -	.1100 -	21.18	.4396 -	2049 +	2.4359772	117.00	2.704	.3801	524.2	5.934	88.34	.08091 -	.1727 -
21.40	.1309 -	.1212 -	.1080 -	21.38	.4197 -	2147 +	2.4362265	117.12	2.678	.3801	534.1	5.935	89.99	.07725 -	.1695 -
21.60	.1227 -	.1158 -	.1060 -	21.58	.4009 -	2248 +	2.4364690	117.24	2.654	.3800	544.2	5.936	91.66	.07380 -	.1663 -
21.80	.1151 -	.1106 -	.1041 -	21.78	.3830 -	2353 +	2.4367050	117.36	2.629	.3800	554.3	5.938	93.35	.07052 -	.1633 -
22.00	.1081 -	.1057 -	.1023 -	21.98	.3662 -	2461 +	2.4369346	117.48	2.605	.3800	564.5	5.939	95.05	.06742 -	.1604 -
22.20	.1015 -	.1011 -	.1004 -	22.18	.3502 -	2574 +	2.4371581	117.60	2.582	.3799	574.8	5.940	96.77	.06447 -	.1575 -
22.40	.9541 -	.9670 -	.9867 -	22.38	.3351 -	2690 +	2.4373757	117.71	2.559	.3799	585.2	5.941	98.51	.06168 -	.1547 -
22.60	.8971 -	.9253 -	.9694 -	22.58	.3207 -	2811 +	2.4375876	117.82	2.536	.3799	595.7	5.942	100.3	.05904 -	.1520 -
22.80	.8439 -	.8858 -	.9527 -	22.78	.3071 -	2936 +	2.4377940	117.93	2.514	.3798	606.3	5.943	102.0	.05653 -	.1493 -
23.00	.7943 -	.8483 -	.9363 -	22.98	.2941 -	3065 +	2.4379951	118.04	2.492	.3798	617.0	5.944	103.8	.05414 -	.1467 -
23.20	.7480 -	.8127 -	.9204 -	23.18	.2818 -	3199 +	2.4381911	118.15	2.470	.3798	627.8	5.945	105.6	.05188 -	.1442 -
23.40	.7048 -	.7789 -	.9049 -	23.38	.2701 -	3338 +	2.4383821	118.25	2.449	.3797	638.7	5.946	107.4	.04972 -	.1418 -
23.60	.6644 -	.7467 -	.8897 -	23.58	.2590 -	3481 +	2.4385683	118.36	2.429	.3797	649.6	5.947	109.2	.04768 -	.1394 -
23.80	.6266 -	.7161 -	.8750 -	23.78	.2485 -	3630 +	2.4387499	118.46	2.408	.3797	660.7	5.948	111.1	.04573 -	.1370 -
24.00	.5913 -	.6871 -	.8606 -	23.98	.2384 -	3783 +	2.4389270	118.56	2.388	.3796	671.8	5.948	112.9	.04388 -	.1348 -
24.20	.5582 -	.6594 -	.8465 -	24.18	.2288 -	3942 +	2.4390998	118.65	2.368	.3796	683.1	5.949	114.8	.04211 -	.1325 -
24.40	.5272 -	.6330 -	.8328 -	24.38	.2197 -	4106 +	2.4392683	118.75	2.349	.3796	694.4	5.950	116.7	.04044 -	.1304 -
24.60	.4981 -	.6079 -	.8195 -	24.58	.2110 -	4275 +	2.4394328	118.85	2.330	.3796	705.9	5.951	118.6	.03884 -	.1283 -
24.80	.4709 -	.5839 -	.8064 -	24.78	.2027 -	4450 +	2.4395934	118.94	2.311	.3795	717.4	5.952	120.5	.03731 -	.1262 -
25.00	.4454 -	.5611 -	.7937 -	24.98	.1948 -	4631 +	2.4397502	119.03	2.292	.3795	729.0	5.952	122.5	.03586 -	.1242 -
25.20	.4214 -	.5394 -	.7812 -	25.18	.1873 -	4817 +	2.4399033	119.12	2.274	.3795	740.7	5.953	124.4	.03447 -	.1222 -
25.40	.3989 -	.5187 -	.7690 -	25.38	.1801 -	5009 +	2.4400528	119.21	2.256	.3795	752.5	5.954	126.4	.03315 -	.1203 -
25.60	.3777 -	.4988 -	.7572 -	25.58	.1733 -	5208 +	2.4401988	119.30	2.239	.3794	764.4	5.955	128.4	.03189 -	.1184 -
25.80	.3578 -	.4800 -	.7456 -	25.78	.1667 -	5412 +	2.4403415	119.38	2.221	.3794	776.4	5.955	130.4	.03069 -	.1166 -
26.00	.3391 -	.4619 -	.7342 -	25.98	.1605 -	5624 +	2.4404809	119.47	2.204	.3794	788.5	5.956	132.4	.02953 -	.1148 -
26.20	.3216 -	.4447 -	.7231 -	26.18	.1548 -	5841 +	2.4406172	119.55	2.187	.3794	800.7	5.957	134.4	.02844 -	.1131 -
26.40	.3050 -	.4282 -	.7123 -	26.38	.1495 -	6066 +	2.4407504	119.63	2.171	.3794	813.0	5.957	136.5	.02739 -	.1114 -
26.60	.2894 -	.4125 -	.7017 -	26.58	.1444 -	6297 +	2.4408806	119.71	2.154	.3793	825.3	5.958	138.5	.02638 -	.1097 -
26.80	.2747 -	.3974 -	.6913 -	26.78	.1391 -	6535 +	2.4410080	119.79	2.138	.3793	837.8	5.959	140.6	.02542 -	.1081 -
27.00	.2609 -	.3830 -	.6812 -	26.98	.1331 -	6781 +	2.4411325	119.87	2.123	.3793	850.3	5.959	142.7	.02450 -	.1065 -
27.20	.2479 -	.3692 -	.6713 -	27.18	.1284 -	7033 +	2.4412544	119.95	2.107	.3793	863.0	5.960	144.8	.02362 -	.1049 -
27.40	.2355 -	.3560 -	.6616 -	27.38	.1238 -	7294 +	2.4413736	120.03	2.092	.3793	875.7	5.960	146.9	.02278 -	.1034 -
27.60	.2239 -	.3434 -	.6521 -	27.58	.1194 -	7562 +	2.4414902	120.10	2.076	.3792	888.6	5.961	149.1	.02197 -	.1019 -
27.80	.2130 -	.3313 -	.6428 -	27.78	.1152 -	7837 +	2.4416043	120.18	2.061	.3792	901.5	5.961	151.2	.02120 -	.1005 -
28.00	.2026 -	.3197 -	.6337 -	27.98	.1112 -	8121 +	2.4417160	120.25	2.047	.3792	914.5	5.962	153.4	.02046 -	.9902 -
28.20	.1928 -	.3086 -	.6248 -	28.18	.1073 -	8413 +	2.4418254	120.32	2.032	.3792	927.6	5.963	155.6	.01975 -	.9762 -
28.40	.1836 -	.2979 -	.6161 -	28.38	.1036 -	8713 +	2.4419325	120.39	2.018	.3792	940.8	5.963	157.8	.01907 -	.9626 -
28.60	.1748 -	.2877 -	.6076 -	28.58	.1001 -	9022 +	2.4420373	120.46	2.004	.3792	954.1	5.964	160.0	.01842 -	.9491 -
28.80	.1665 -	.2779 -	.5992 -	28.78	.9669 -	9340 +	2.4421400	120.53	1.990	.3791	967.5	5.964	162.2	.01779 -	.9360 -
29.00	.1587 -	.2685 -	.5910 -	28.98	.9343 -	9666 +									

TABLE II.—SUPERSONIC FLOW—Continued

 $\gamma=7/5$

M or M_1	$\frac{p}{p_1}$	$\frac{\rho}{\rho_1}$	$\frac{T}{T_1}$	β	$\frac{q}{p_1}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
33.00	.6454 -3	.1412 -3	.4570 -2	32.98	.4920 -3	1837 +2	2.4438858	121.79	1.737	.3789	1270	5.973	212.7	.9053 -3	.7129 -3
33.20	.6188 -3	.1370 -3	.4516 -2	33.18	.4774 -3	1893 +2	2.4439529	121.84	1.726	.3788	1286	5.973	215.3	.8785 -3	.7044 -3
33.40	.5935 -3	.1330 -3	.4462 -2	33.39	.4634 -3	1950 +2	2.4440188	121.89	1.716	.3788	1301	5.973	217.9	.8527 -3	.6860 -3
33.60	.5692 -3	.1291 -3	.4409 -2	33.59	.4499 -3	2009 +2	2.4440835	121.94	1.706	.3788	1317	5.974	220.5	.8277 -3	.6578 -3
33.80	.5462 -3	.1253 -3	.4358 -2	33.79	.4368 -3	2069 +2	2.4441471	121.99	1.695	.3788	1333	5.974	223.1	.8037 -3	.6296 -3
34.00	.5242 -3	.1217 -3	.4307 -2	33.99	.4242 -3	2131 +2	2.4442095	122.04	1.685	.3788	1349	5.974	225.7	.7805 -3	.6016 -3
34.20	.5032 -3	.1182 -3	.4257 -2	34.19	.4120 -3	2194 +2	2.4442709	122.09	1.676	.3788	1364	5.975	228.4	.7581 -3	.5738 -3
34.40	.4832 -3	.1148 -3	.4208 -2	34.39	.4002 -3	2259 +2	2.4443312	122.14	1.666	.3788	1380	5.975	231.0	.7364 -3	.5461 -3
34.60	.4640 -3	.1116 -3	.4159 -2	34.59	.3889 -3	2325 +2	2.4443905	122.19	1.656	.3788	1397	5.975	233.7	.7155 -3	.5186 -3
34.80	.4457 -3	.1084 -3	.4112 -2	34.79	.3779 -3	2392 +2	2.4444488	122.24	1.647	.3788	1413	5.975	236.4	.6952 -3	.4911 -3
35.00	.4283 -3	.1054 -3	.4065 -2	34.99	.3672 -3	2462 +2	2.4445060	122.28	1.637	.3788	1429	5.976	239.1	.6757 -3	.4638 -3
35.20	.4116 -3	.1024 -3	.4019 -2	35.19	.3570 -3	2532 +2	2.4445623	122.33	1.628	.3787	1445	5.976	241.9	.6568 -3	.4367 -3
35.40	.3957 -3	.9956 -3	.3974 -2	35.39	.3471 -3	2605 +2	2.4446177	122.37	1.619	.3787	1462	5.976	244.6	.6385 -3	.4107 -3
35.60	.3804 -3	.9960 -3	.3930 -2	35.59	.3375 -3	2679 +2	2.4446721	122.42	1.610	.3787	1478	5.976	247.4	.6210 -3	.3857 -3
35.80	.3658 -3	.9414 -3	.3886 -2	35.79	.3282 -3	2754 +2	2.4447256	122.47	1.601	.3787	1495	5.977	250.2	.6039 -3	.3609 -3
36.00	.3519 -3	.9156 -3	.3843 -2	35.99	.3192 -3	2832 +2	2.4447783	122.51	1.592	.3787	1512	5.977	252.9	.5874 -3	.3369 -3
36.20	.3386 -3	.8907 -3	.3801 -2	36.19	.3106 -3	2911 +2	2.4448300	122.55	1.583	.3787	1529	5.977	255.8	.5714 -3	.3132 -3
36.40	.3258 -3	.8666 -3	.3760 -2	36.39	.3022 -3	2992 +2	2.4448810	122.60	1.574	.3787	1546	5.977	258.6	.5560 -3	.2906 -3
36.60	.3136 -3	.8433 -3	.3719 -2	36.59	.2941 -3	3075 +2	2.4449311	122.64	1.566	.3787	1563	5.978	261.4	.5410 -3	.2689 -3
36.80	.3019 -3	.8207 -3	.3679 -2	36.79	.2862 -3	3159 +2	2.4449803	122.68	1.557	.3787	1580	5.978	264.3	.5265 -3	.2481 -3
37.00	.2907 -3	.7988 -3	.3639 -2	36.99	.2786 -3	3246 +2	2.4450288	122.72	1.549	.3787	1597	5.978	267.1	.5126 -3	.2281 -3
37.20	.2800 -3	.7777 -3	.3600 -2	37.19	.2712 -3	3334 +2	2.4450765	122.77	1.540	.3787	1614	5.978	270.0	.4990 -3	.2090 -3
37.40	.2697 -3	.7572 -3	.3562 -2	37.39	.2641 -3	3424 +2	2.4451235	122.81	1.532	.3787	1632	5.979	272.9	.4858 -3	.1906 -3
37.60	.2598 -3	.7373 -3	.3524 -2	37.59	.2572 -3	3516 +2	2.4451697	122.85	1.524	.3787	1649	5.979	275.8	.4732 -3	.1730 -3
37.80	.2504 -3	.7181 -3	.3487 -2	37.79	.2504 -3	3611 +2	2.4452152	122.89	1.516	.3786	1667	5.979	278.8	.4608 -3	.1564 -3
38.00	.2414 -3	.6995 -3	.3451 -2	37.99	.2440 -3	3706 +2	2.4452599	122.93	1.508	.3786	1685	5.979	281.7	.4489 -3	.1407 -3
38.20	.2327 -3	.6814 -3	.3415 -2	38.19	.2377 -3	3805 +2	2.4453040	122.97	1.500	.3786	1702	5.980	284.7	.4373 -3	.1259 -3
38.40	.2244 -3	.6639 -3	.3379 -2	38.39	.2316 -3	3905 +2	2.4453474	123.00	1.492	.3786	1720	5.980	287.7	.4260 -3	.1123 -3
38.60	.2164 -3	.6469 -3	.3345 -2	38.59	.2257 -3	4007 +2	2.4453901	123.04	1.485	.3786	1738	5.980	290.7	.4152 -3	.1000 -3
38.80	.2087 -3	.6305 -3	.3310 -2	38.79	.2199 -3	4112 +2	2.4454321	123.08	1.477	.3786	1756	5.980	293.7	.4046 -3	.889 -3
39.00	.2013 -3	.6145 -3	.3277 -2	38.99	.2144 -3	4219 +2	2.4454735	123.12	1.469	.3786	1774	5.980	296.7	.3944 -3	.789 -3
39.20	.1943 -3	.5991 -3	.3243 -2	39.19	.2090 -3	4327 +2	2.4455143	123.16	1.462	.3786	1793	5.981	299.7	.3845 -3	.698 -3
39.40	.1875 -3	.5841 -3	.3211 -2	39.39	.2038 -3	4438 +2	2.4455545	123.19	1.454	.3786	1811	5.981	302.8	.3749 -3	.616 -3
39.60	.1810 -3	.5695 -3	.3178 -2	39.59	.1987 -3	4552 +2	2.4455940	123.23	1.447	.3786	1829	5.981	305.9	.3655 -3	.542 -3
39.80	.1748 -3	.5554 -3	.3147 -2	39.79	.1938 -3	4667 +2	2.4456330	123.27	1.440	.3786	1848	5.981	309.0	.3565 -3	.476 -3
40.00	.1688 -3	.5417 -3	.3115 -2	39.99	.1890 -3	4785 +2	2.4456714	123.30	1.433	.3786	1867	5.981	312.1	.3477 -3	.416 -3
40.20	.1630 -3	.5284 -3	.3084 -2	40.19	.1844 -3	4906 +2	2.4457092	123.34	1.425	.3786	1885	5.982	315.2	.3392 -3	.363 -3
40.40	.1574 -3	.5155 -3	.3054 -2	40.39	.1799 -3	5028 +2	2.4457464	123.37	1.418	.3786	1904	5.982	318.3	.3309 -3	.316 -3
40.60	.1521 -3	.5029 -3	.3024 -2	40.59	.1755 -3	5154 +2	2.4457831	123.41	1.411	.3786	1923	5.982	321.5	.3229 -3	.273 -3
40.80	.1470 -3	.4908 -3	.2995 -2	40.79	.1713 -3	5281 +2	2.4458193	123.44	1.404	.3785	1942	5.982	324.6	.3151 -3	.234 -3
41.00	.1420 -3	.4789 -3	.2966 -2	40.99	.1671 -3	5412 +2	2.4458549	123.48	1.398	.3785	1961	5.982	327.8	.3075 -3	.200 -3
41.20	.1373 -3	.4675 -3	.2937 -2	41.19	.1631 -3	5544 +2	2.4458901	123.51	1.391	.3785	1980	5.982	331.0	.3001 -3	.171 -3
41.40	.1327 -3	.4563 -3	.2909 -2	41.39	.1592 -3	5680 +2	2.4459247	123.54	1.384	.3785	2000	5.983	334.2	.2929 -3	.146 -3
41.60	.1283 -3	.4455 -3	.2881 -2	41.59	.1555 -3	5818 +2	2.4459588	123.58	1.377	.3785	2019	5.983	337.4	.2860 -3	.123 -3
41.80	.1241 -3	.4349 -3	.2854 -2	41.79	.1518 -3	5959 +2	2.4459924	123.61	1.371	.3785	2038	5.983	340.7	.2793 -3	.102 -3
42.00	.1201 -3	.4247 -3	.2827 -2	41.99	.1482 -3	6102 +2	2.4460256	123.64	1.364	.3785	2058	5.983	343.9	.2727 -3	.084 -3
42.20	.1161 -3	.4148 -3	.2800 -2	42.19	.1448 -3	6248 +2	2.4460583	123.67	1.358	.3785	2078	5.983	347.2	.2663 -3	.068 -3
42.40	.1124 -3	.4051 -3	.2774 -2	42.39	.1414 -3	6397 +2	2.4460905	123.71	1.351	.3785	2097	5.983	350.5	.2602 -3	.054 -3
42.60	.1087 -3	.3957 -3	.2748 -2	42.59	.1381 -3	6549 +2	2.4461223	123.74	1.345	.3785	2117	5.984	353.8	.2541 -3	.042 -3
42.80	.1052 -3	.3866 -3	.2722 -2	42.79	.1349 -3	6704 +2	2.4461536	123.77	1.339	.3785	2137	5.984	357.1	.2483 -3	.032 -3
43.00	.1019 -3	.3777 -3	.2697 -2	42.99	.1318 -3	6861 +2	2.4461845	123.80	1.333	.3785	2157	5.984	360.5	.2426 -3	.024 -3
43.20	.9861 -3	.3691 -3	.2672 -2	43.19	.1288 -3	7022 +2	2.4462150	123.83	1.326	.3785	2177	5.984	363.8	.2370 -3	.018 -3
43.40	.9548 -3	.3607 -3	.2648 -2	43.39	.1259 -3	7186 +2	2.4462451	123.86	1.320	.3785	2197	5.984	367.2	.2316 -3	.014 -3
43.60	.9246 -3	.3525 -3	.2623 -2	43.59	.1230 -3	7352 +2	2.4462747	123.89	1.314	.3785	2218	5.984	370.6	.2264 -3	.011 -3
43.80	.8956 -3	.3445 -3	.2600 -2	43.79	.1203 -3	7522 +2	2.4463039	123.92	1.308	.3785	2238	5.984	374.0	.2213 -3	.009 -3
44.00	.8676 -3	.3368 -3	.2576 -2	43.99	.1176 -3	7694 +2	2.4463328	123.95	1.302	.3785	2259	5.985	377.4	.2163 -3	.008 -3
44.20	.8405 -3	.3293 -3	.2553 -2	44.19	.1150 -3	7870 +2	2.4463612	123.98	1.296	.3785	2279	5.985	380.8	.2115 -3	.007 -3
44.40	.8144 -3	.3219 -3	.2530 -2	44.39	.1124 -3	8049 +2	2.4463893	124.01	1.291	.3785	2300	5.985	384.3	.2068 -3	.006 -3
44.60	.7893 -3	.3148 -3	.2507 -2	44.59	.1099 -3	8232 +2	2.4464170	124.04	1.285	.3785	2321	5.985	387.7	.2022 -3	.005 -3
44.80	.7650 -3	.3078 -3	.2485 -2	44.79	.1075 -3	8418 +2	2.4464443	124.07	1.279	.3784	2341	5.985	391.2	.1977 -3	.004 -3
45.00	.7416 -3	.3011 -3	.2463 -2	44.99	.1051 -3	8606 +2	2.4464713	124.10	1.273	.3784	2362	5.985	394.7	.1934 -3	.003 -3
45.20	.7190 -3	.2945 -3	.2441 -2	45.19	.1028 -3	8798 +2	2.4464979	124.12	1.268	.3784	2383	5.985	398.2	.1892 -3	.003 -3
45.40	.6971 -3	.2881 -3	.2420 -2	45.39	.1006 -3	8995 +2	2.4465241	124.15	1.262	.3784	2405	5.986	401.7	.1851 -3	.002 -3
45.60	.6760 -3	.2818 -3	.2399 -2	45.59	.9840 -3	9194 +2	2.4465500	124.18	1.257	.3784	2426	5.986	405.3	.1810 -3	.002 -3
45.80	.6557 -3	.2758 -3	.2378 -2	45.79	.9629 -3	9396 +2	2.4465756	124.21	1.251	.3784	2447	5.986	408.8	.1771 -3	.002 -3
46.00	.6361 -3	.2698 -3	.2357 -2	45.99	.9422 -3	9603 +2	2.4466009	124.23	1.246	.3784	2469	5.986	412.4	.1733 -3	.002 -3
46.20	.6171 -3	.2641 -3	.2337 -2	46.19	.9220 -3	9813 +2	2.4466258	124.26	1.240	.3784	2490	5.986	416.0	.1696 -3	.002 -3
46.40	.5987 -3	.2584 -3	.2317 -2	46.39	.9023 -3	1003 +2	2.4466504	124.29	1.235	.3784	2512	5.986	419.6	.1660 -3	.002 -3
46.60	.5810 -3	.2529 -3	.2297 -2	46.59	.8832 -3	1024 +2	2								

TABLE II.—SUPERSONIC FLOW—Concluded

$\gamma=7/5$

M or M_1	$\frac{p}{p_t}$	$\frac{\rho}{\rho_t}$	$\frac{T}{T_t}$	β	$\frac{q}{p_t}$	$\frac{A}{A_*}$	$\frac{V}{a_*}$	ν	μ	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{t2}}{p_{t1}}$	$\frac{p_1}{p_{t2}}$
55.00	.1826 -9	.1106 -6	.1650 -2	54.99	.3866 -6	2341 +3	2.4474679	125.25	1.042	.3783	3529	5.990	589.1	.7111 -6	.2567 -3
56.00	.1609 -9	.1011 -6	.1592 -2	55.99	.3533 -6	2562 +3	2.4475394	125.34	1.023	.3783	3659	5.990	610.7	.6499 -6	.2476 -3
57.00	.1422 -9	.9256 -7	.1537 -2	56.99	.3235 -6	2798 +3	2.4476071	125.43	1.005	.3783	3790	5.991	632.7	.5950 -6	.2390 -3
58.00	.1259 -9	.8485 -7	.1484 -2	57.99	.2965 -6	3052 +3	2.4476714	125.52	.9879	.3783	3925	5.991	655.1	.5455 -6	.2308 -3
59.00	.1118 -9	.7791 -7	.1434 -2	58.99	.2723 -6	3324 +3	2.4477325	125.60	.9712	.3782	4061	5.991	677.8	.5009 -6	.2231 -3
60.00	.9937 -10	.7165 -7	.1387 -2	59.99	.2504 -6	3615 +3	2.4477905	125.68	.9550	.3782	4200	5.992	700.9	.4606 -6	.2157 -3
61.00	.8852 -10	.6596 -7	.1342 -2	60.99	.2306 -6	3926 +3	2.4478457	125.76	.9393	.3782	4341	5.992	724.5	.4241 -6	.2087 -3
62.00	.7900 -10	.6082 -7	.1299 -2	61.99	.2126 -6	4258 +3	2.4478982	125.84	.9241	.3782	4485	5.992	748.4	.3911 -6	.2020 -3
63.00	.7065 -10	.5615 -7	.1258 -2	62.99	.1963 -6	4612 +3	2.4479483	125.91	.9095	.3782	4630	5.993	772.7	.3611 -6	.1957 -3
64.00	.6328 -10	.5190 -7	.1219 -2	63.99	.1814 -6	4990 +3	2.4479961	125.98	.8953	.3782	4779	5.993	797.4	.3338 -6	.1896 -3
65.00	.5678 -10	.4803 -7	.1182 -2	64.99	.1679 -6	5391 +3	2.4480416	126.05	.8815	.3782	4929	5.993	822.5	.3089 -6	.1838 -3
66.00	.5103 -10	.4451 -7	.1147 -2	65.99	.1556 -6	5818 +3	2.4480851	126.12	.8682	.3782	5082	5.993	847.9	.2863 -6	.1783 -3
67.00	.4594 -10	.4129 -7	.1113 -2	66.99	.1444 -6	6271 +3	2.4481267	126.18	.8552	.3782	5237	5.993	873.8	.2655 -6	.1730 -3
68.00	.4141 -10	.3834 -7	.1080 -2	67.99	.1340 -6	6754 +3	2.4481665	126.24	.8426	.3782	5395	5.994	900.1	.2466 -6	.1679 -3
69.00	.3740 -10	.3565 -7	.1049 -2	68.99	.1246 -6	7264 +3	2.4482045	126.30	.8304	.3782	5554	5.994	926.7	.2293 -6	.1631 -3
70.00	.3382 -10	.3318 -7	.1019 -2	69.99	.1160 -6	7804 +3	2.4482410	126.36	.8185	.3782	5717	5.994	953.7	.2134 -6	.1585 -3
71.00	.3062 -10	.3091 -7	.9909 -3	70.99	.1081 -6	8378 +3	2.4482759	126.42	.8070	.3782	5881	5.994	981.1	.1988 -6	.1540 -3
72.00	.2777 -10	.2852 -7	.9636 -3	71.99	.1008 -6	8984 +3	2.4483093	126.48	.7958	.3782	6048	5.994	1009	.1854 -6	.1498 -3
73.00	.2522 -10	.2690 -7	.9374 -3	72.99	.9406 -7	9625 +3	2.4483414	126.53	.7849	.3781	6217	5.994	1037	.1730 -6	.1457 -3
74.00	.2293 -10	.2513 -7	.9122 -3	73.99	.8789 -7	1030 +4	2.4483722	126.59	.7742	.3781	6389	5.995	1066	.1617 -6	.1418 -3
75.00	.2088 -10	.2351 -7	.8881 -3	74.99	.8220 -7	1102 +4	2.4484018	126.64	.7639	.3781	6562	5.995	1095	.1512 -6	.1381 -3
76.00	.1903 -10	.2200 -7	.8649 -3	75.99	.7693 -7	1177 +4	2.4484302	126.69	.7539	.3781	6739	5.995	1124	.1415 -6	.1345 -3
77.00	.1737 -10	.2061 -7	.8426 -3	76.99	.7207 -7	1256 +4	2.4484576	126.74	.7441	.3781	6917	5.995	1154	.1326 -6	.1310 -3
78.00	.1587 -10	.1932 -7	.8212 -3	77.99	.6757 -7	1340 +4	2.4484838	126.78	.7345	.3781	7098	5.995	1184	.1243 -6	.1276 -3
79.00	.1451 -10	.1813 -7	.8005 -3	78.99	.6341 -7	1428 +4	2.4485091	126.83	.7253	.3781	7281	5.995	1215	.1166 -6	.1244 -3
80.00	.1329 -10	.1703 -7	.7806 -3	79.99	.5954 -7	1521 +4	2.4485335	126.88	.7162	.3781	7467	5.995	1245	.1095 -6	.1214 -3
81.00	.1219 -10	.1600 -7	.7615 -3	80.99	.5596 -7	1618 +4	2.4485569	126.92	.7074	.3781	7654	5.995	1277	.1030 -6	.1184 -3
82.00	.1118 -10	.1505 -7	.7431 -3	81.99	.5264 -7	1720 +4	2.4485795	126.96	.6987	.3781	7845	5.995	1308	.9682 -7	.1155 -3
83.00	.1027 -10	.1417 -7	.7253 -3	82.99	.4954 -7	1828 +4	2.4486013	127.00	.6903	.3781	8037	5.996	1341	.9113 -7	.1127 -3
84.00	.9448 -11	.1334 -7	.7081 -3	83.99	.4667 -7	1940 +4	2.4486223	127.05	.6821	.3781	8232	5.996	1373	.8585 -7	.1101 -3
85.00	.8697 -11	.1258 -7	.6916 -3	84.99	.4399 -7	2059 +4	2.4486426	127.09	.6741	.3781	8429	5.996	1406	.8092 -7	.1075 -3
86.00	.8014 -11	.1186 -7	.6756 -3	85.99	.4149 -7	2182 +4	2.4486622	127.12	.6662	.3781	8629	5.996	1439	.7632 -7	.1050 -3
87.00	.7391 -11	.1120 -7	.6602 -3	86.99	.3916 -7	2312 +4	2.4486811	127.16	.6586	.3781	8830	5.996	1473	.7204 -7	.1026 -3
88.00	.6823 -11	.1058 -7	.6452 -3	87.99	.3699 -7	2448 +4	2.4486994	127.20	.6511	.3781	9035	5.996	1507	.6804 -7	.1003 -3
89.00	.6305 -11	.9995 -8	.6308 -3	88.99	.3496 -7	2590 +4	2.4487170	127.24	.6438	.3781	9241	5.996	1541	.6431 -7	.9805 -4
90.00	.5831 -11	.9452 -8	.6169 -3	89.99	.3306 -7	2739 +4	2.4487341	127.27	.6366	.3781	9450	5.996	1576	.6082 -7	.9588 -4
91.00	.5397 -11	.8944 -8	.6034 -3	90.99	.3129 -7	2894 +4	2.4487506	127.31	.6296	.3781	9661	5.996	1611	.5755 -7	.9378 -4
92.00	.5000 -11	.8469 -8	.5904 -3	91.99	.2962 -7	3057 +4	2.4487666	127.34	.6228	.3781	9875	5.997	1647	.5450 -7	.9175 -4
93.00	.4636 -11	.8023 -8	.5778 -3	92.99	.2807 -7	3226 +4	2.4487820	127.38	.6160	.3781	1009	5.997	1683	.5163 -7	.8978 -4
94.00	.4302 -11	.7606 -8	.5656 -3	93.99	.2661 -7	3403 +4	2.4487970	127.41	.6095	.3781	1031	5.997	1719	.4894 -7	.8790 -4
95.00	.3995 -11	.7214 -8	.5537 -3	94.99	.2524 -7	3588 +4	2.4488115	127.44	.6031	.3781	1053	5.997	1756	.4642 -7	.8605 -4
96.00	.3712 -11	.6846 -8	.5422 -3	95.99	.2395 -7	3781 +4	2.4488255	127.47	.5968	.3781	1075	5.997	1793	.4405 -7	.8427 -4
97.00	.3453 -11	.6501 -8	.5311 -3	96.99	.2274 -7	3982 +4	2.4488392	127.50	.5907	.3781	1098	5.997	1831	.4183 -7	.8254 -4
98.00	.3214 -11	.6176 -8	.5204 -3	97.99	.2161 -7	4191 +4	2.4488524	127.53	.5847	.3781	1121	5.997	1869	.3974 -7	.8087 -4
99.00	.2993 -11	.5870 -8	.5099 -3	98.99	.2054 -7	4410 +4	2.4488652	127.56	.5787	.3781	1143	5.997	1907	.3778 -7	.7923 -4
100.00	.2790 -11	.5583 -8	.4998 -3	100.00	.1953 -7	4637 +4	2.4488776	127.59	.5730	.3781	1167	5.997	1945	.3593 -7	.7765 -4

NOTATIONS FOR TABLES I AND II

 M or M_1 local Mach number or Mach number upstream of a normal shock wave $\frac{p}{p_t}$ ratio of static pressure to total pressure $\frac{\rho}{\rho_t}$ ratio of static density to total density $\frac{T}{T_t}$ ratio of static temperature to total temperature β $\sqrt{M^2-1}$ $\frac{q}{p_t}$ ratio of dynamic pressure, $\frac{1}{2} \rho V^2$, to total pressure $\frac{A}{A_*}$ ratio of local cross-sectional area of an isentropic stream tube to cross-sectional area at the point where $M=1$ $\frac{V}{a_*}$ ratio of local speed to speed of sound at the point where $M=1$ ν Prandtl-Meyer angle (angle through which a supersonic stream is turned to expand from $M=1$ to $M>1$), deg μ Mach angle, $\sin^{-1} \frac{1}{M}$, deg M_2 Mach number downstream of a normal shock wave $\frac{p_2}{p_1}$ static pressure ratio across a normal shock wave $\frac{\rho_2}{\rho_1}$ static density ratio across a normal shock wave $\frac{T_2}{T_1}$ static temperature ratio across a normal shock wave $\frac{p_{t2}}{p_{t1}}$ total pressure ratio across a normal shock wave $\frac{p_1}{p_{t2}}$ ratio of static pressure upstream of a normal shock wave to total pressure downstream

CHARTS

The charts that follow present numerical values of certain physical quantities that are functions of two variables and hence are cumbersome to tabulate. These charts are designed to provide accuracy to three significant figures.

Charts 1 through 8 and chart 25 are for a perfect gas. The values presented in charts 1 through 4 and chart 25 were calculated for a ratio of specific heats of 7/5. The values presented in charts 5 through 8 were taken from references 6 and 14 and are for a ratio of specific heats of 1.405.

Charts 9 through 24 provide correction factors to account for the effects of caloric imperfections on the quantities tabulated in tables I and II and plotted in charts 2, 3, and 4.

On many charts, points corresponding to static temperatures of 5000° R and 100° R (−360° F) have been indicated. These temperatures represent very approximately the limits of validity of the charts. Exact limits cannot be stated simply as they depend on pressure as well as temperature. At temperatures near 5000° R dissociation effects, which were neglected in the calculations, can be significant at high altitudes though perhaps not at sea level. At temperatures less than about 100° R, air may condense at the pressures encountered in many wind tunnels.

On the Reynolds number chart (chart 25), points corresponding to a static temperature of 180° R (−280° F) also are indicated since this is the lowest temperature for which experimental viscosity data have been obtained. At temperatures much lower than −280° F, Sutherland's equation (A2) may significantly underestimate the true viscosity.

The contents of the charts are as follows:

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2. Variation of shock-wave angle with flow-deflection angle for various upstream Mach numbers. Perfect gas, $\gamma=7/5$	654
3. Variation of pressure coefficient across shock waves with flow-deflection angle for various upstream Mach numbers. Perfect gas, $\gamma=7/5$	656
4. Variation of Mach number downstream of a shock wave with flow-deflection angle for various upstream Mach numbers. Perfect gas, $\gamma=7/5$	658
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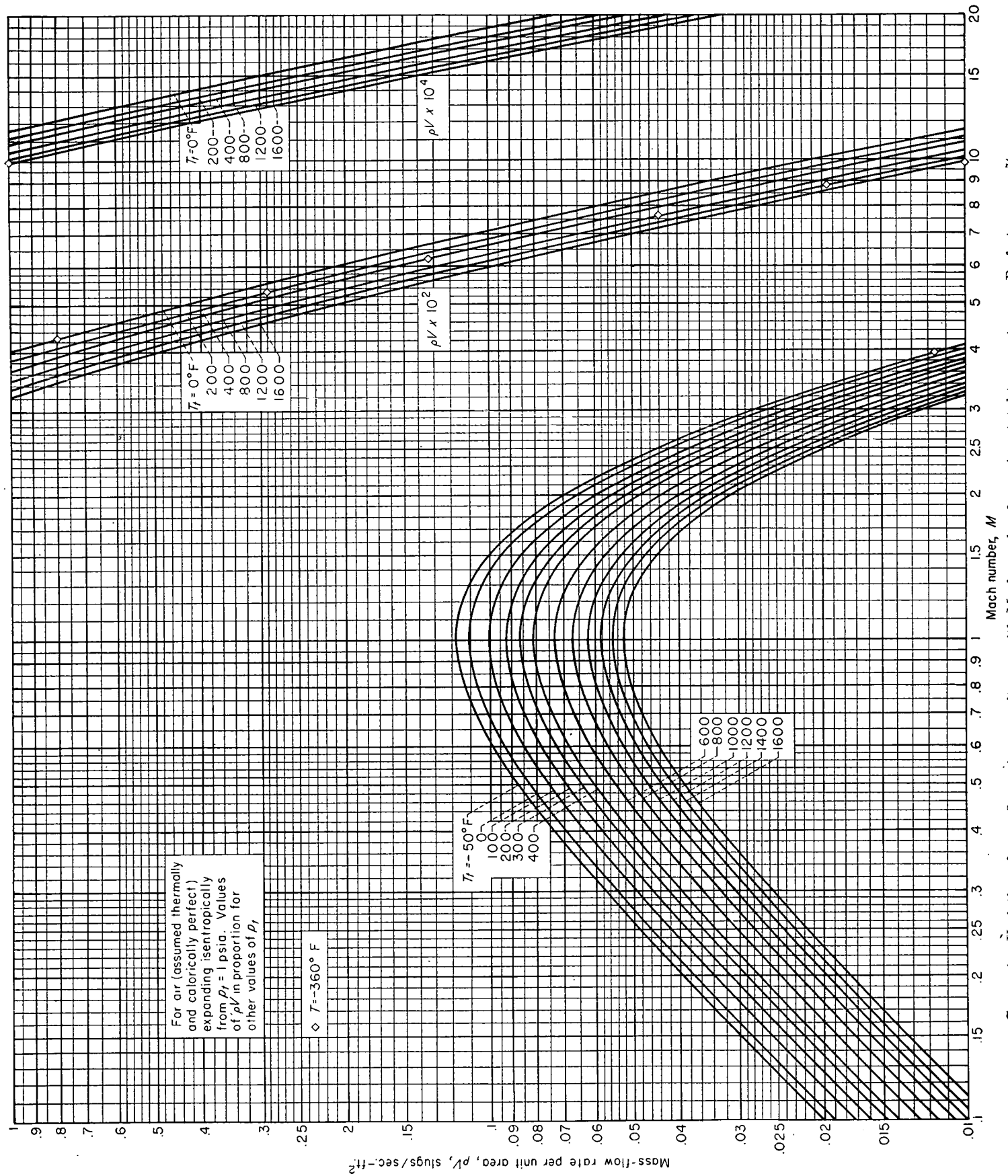


CHART 1.—Variation of mass-flow rate per unit area with Mach number for various total temperatures. Perfect gas. $\gamma = 1.4$.

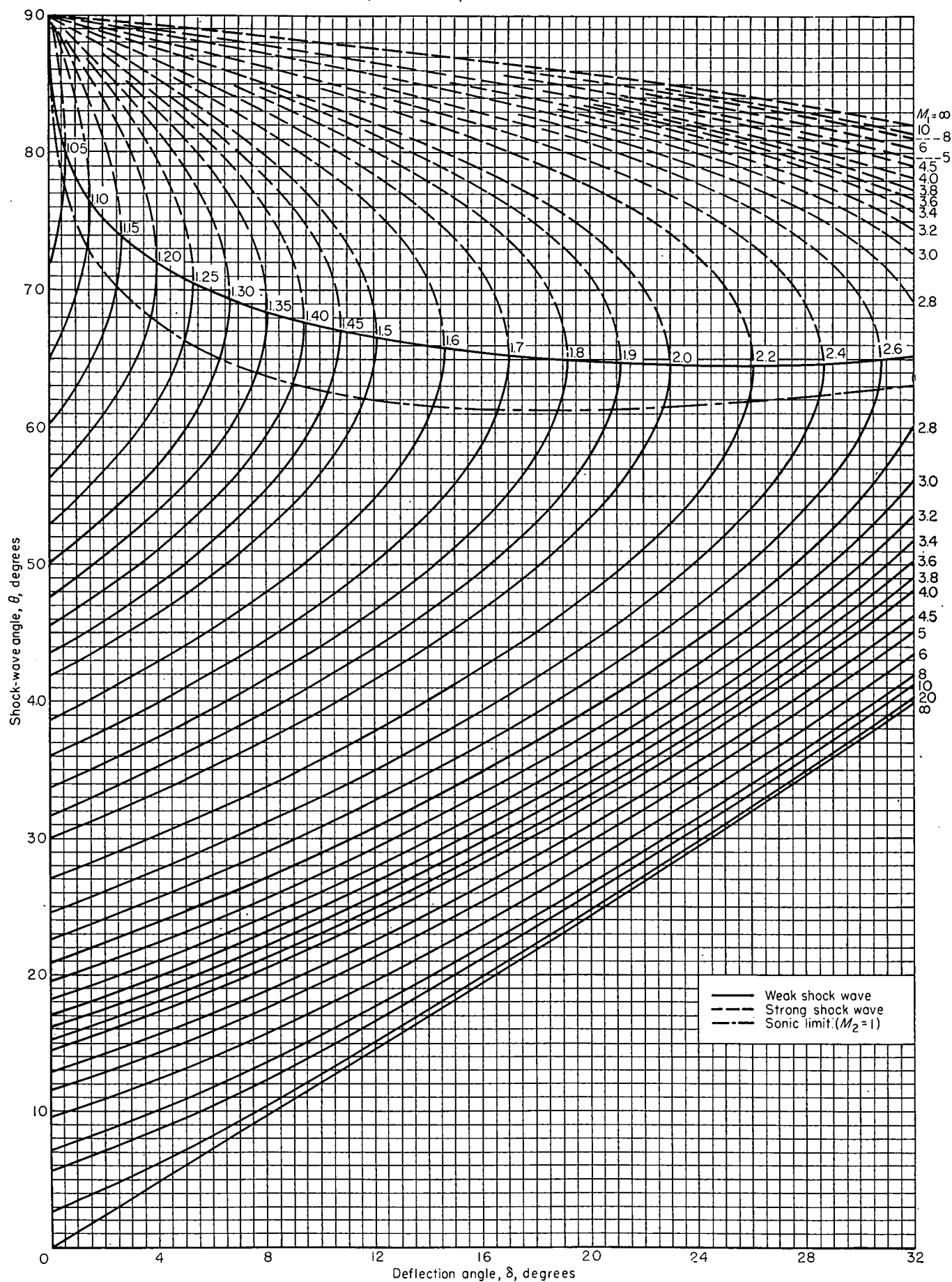


CHART 2.—Variation of shock-wave angle with flow-deflection angle for various upstream Mach numbers Perfect gas, $\gamma = \frac{7}{5}$.

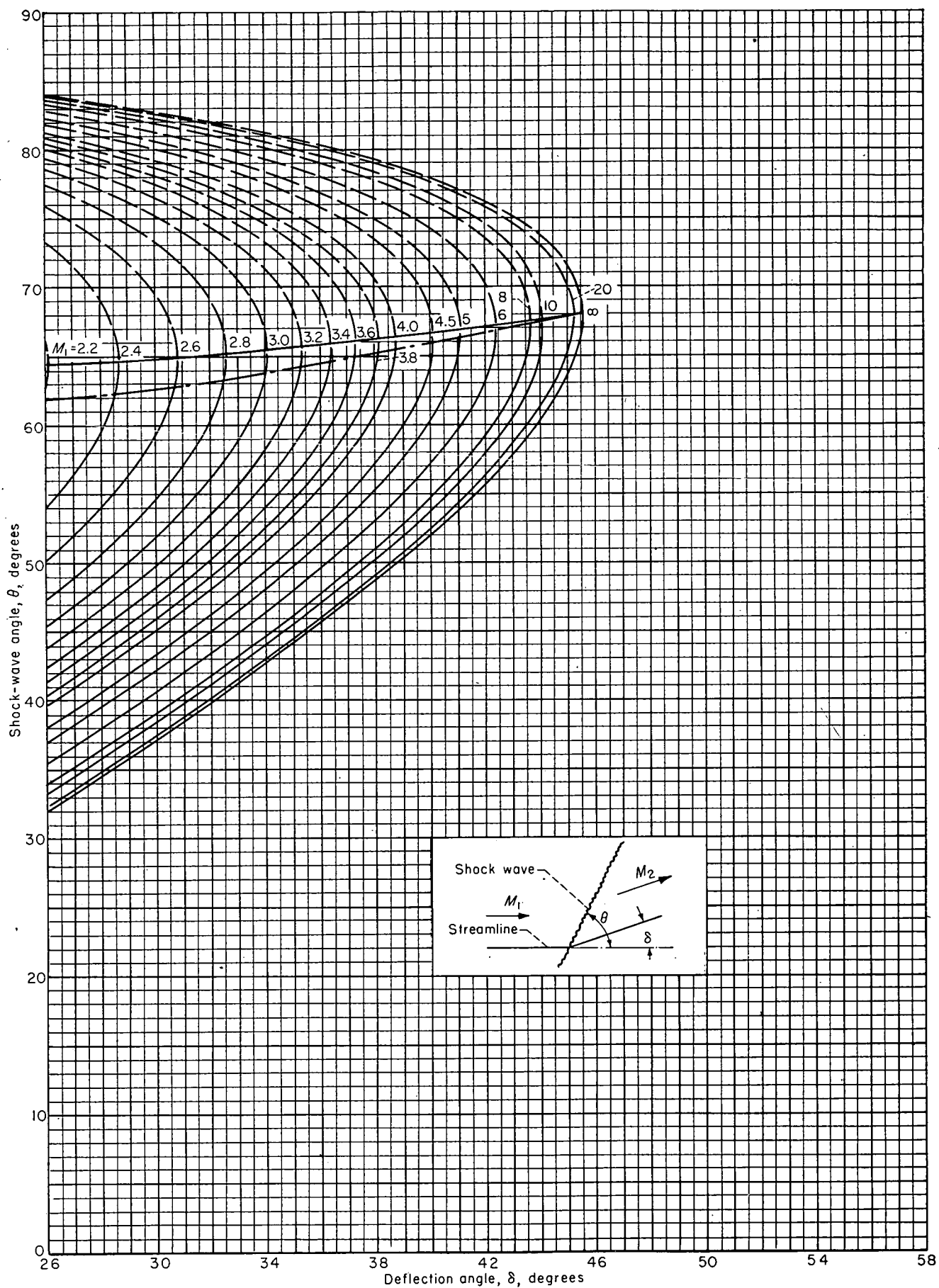


CHART 2.—Concluded

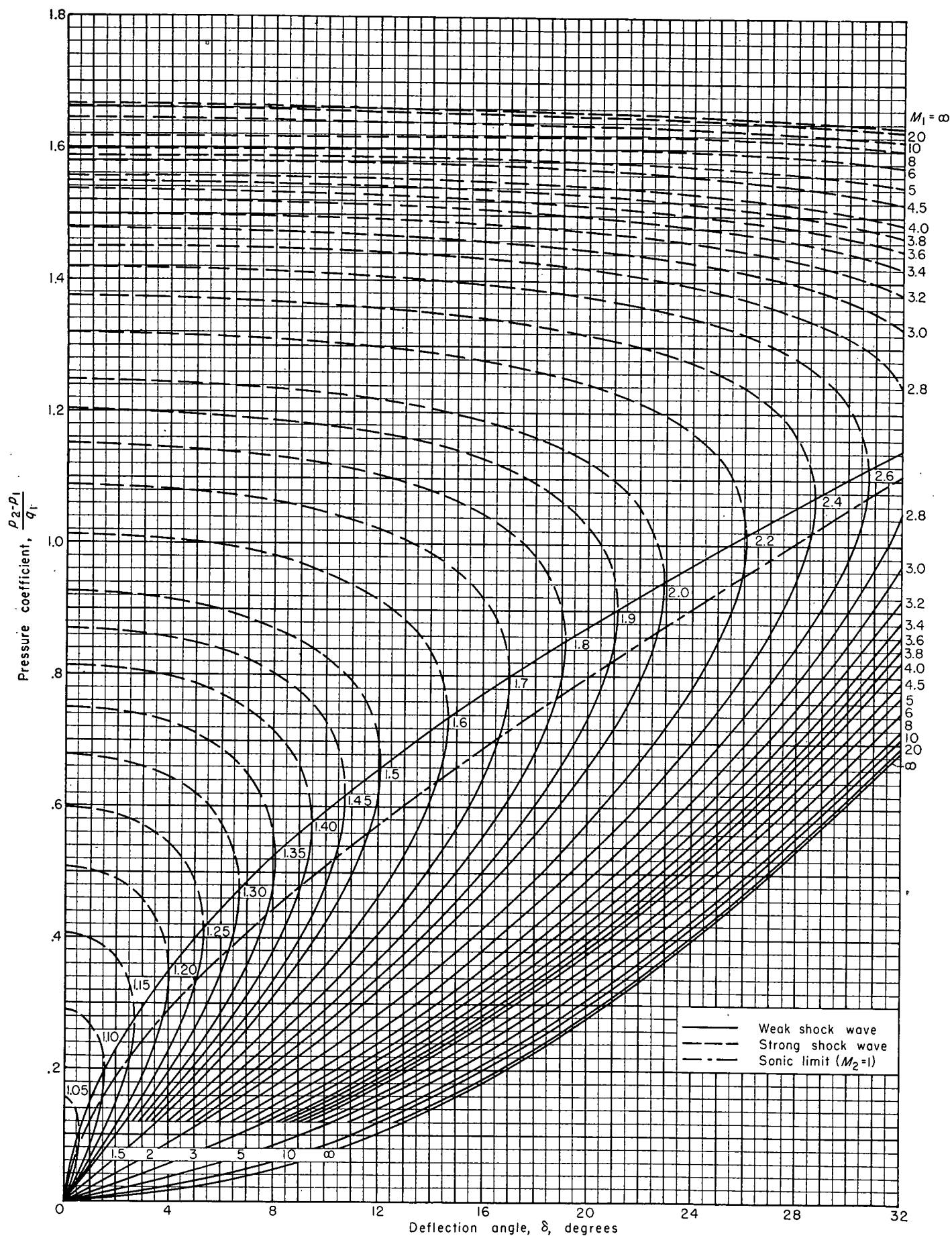


CHART 3.—Variation of pressure coefficient across shock waves with flow-deflection angle for various upstream Mach numbers. Perfect gas, $\gamma = \frac{7}{5}$.

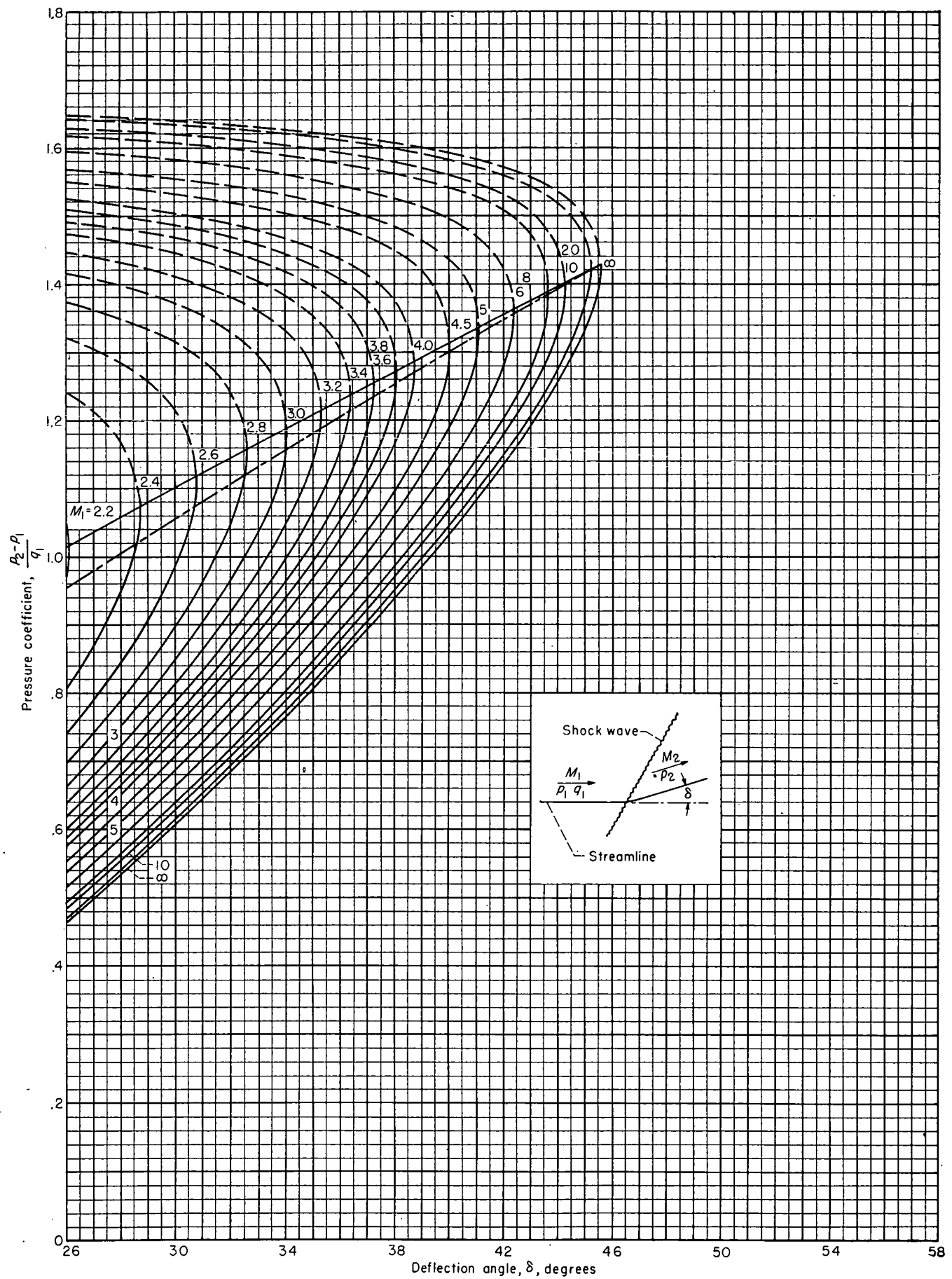


CHART 3.—Concluded

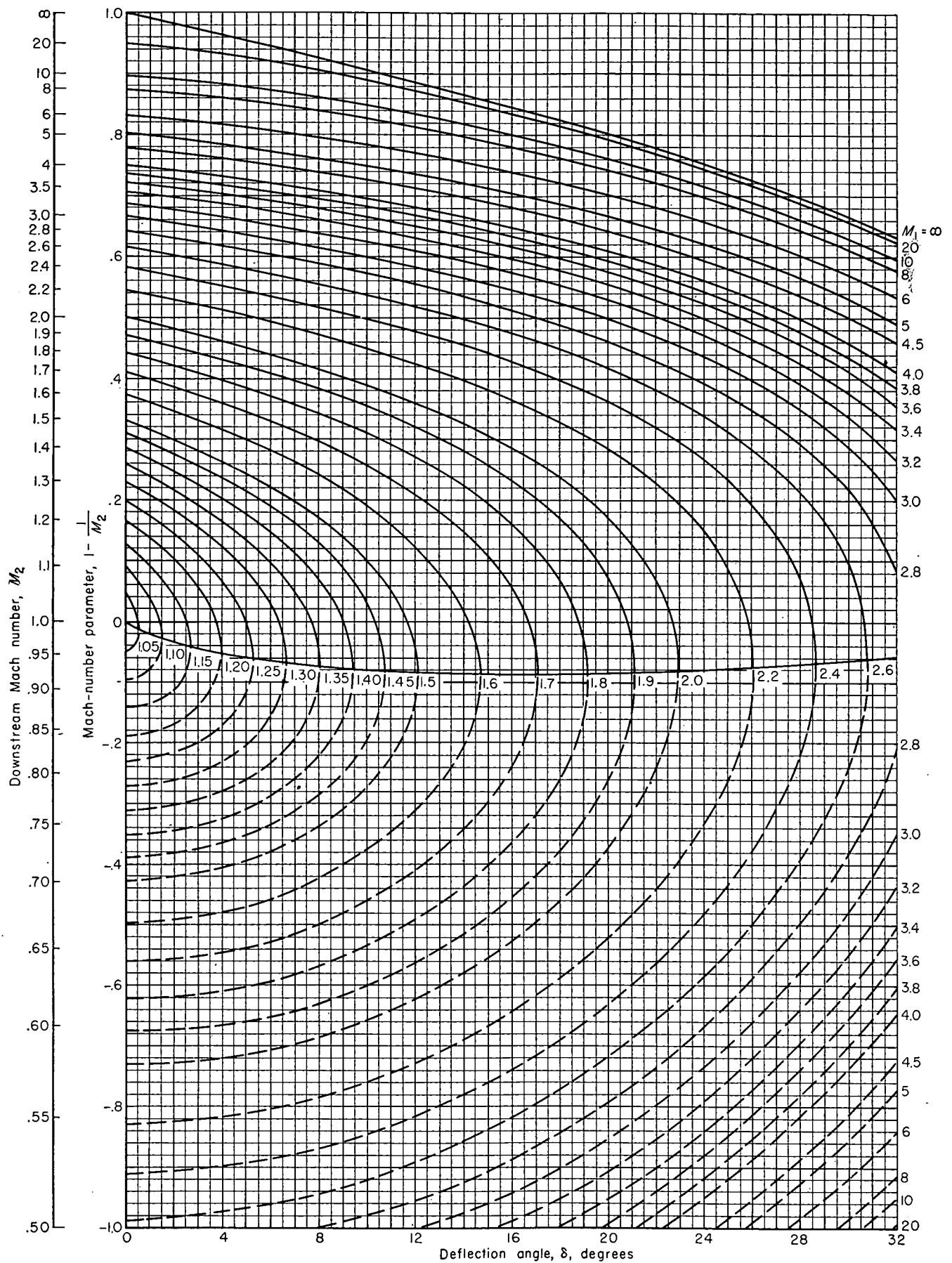


CHART 4.—Variation of Mach number downstream of a shock wave with flow-deflection angle for various upstream Mach numbers. Perfect gas, $\gamma = \frac{7}{5}$.

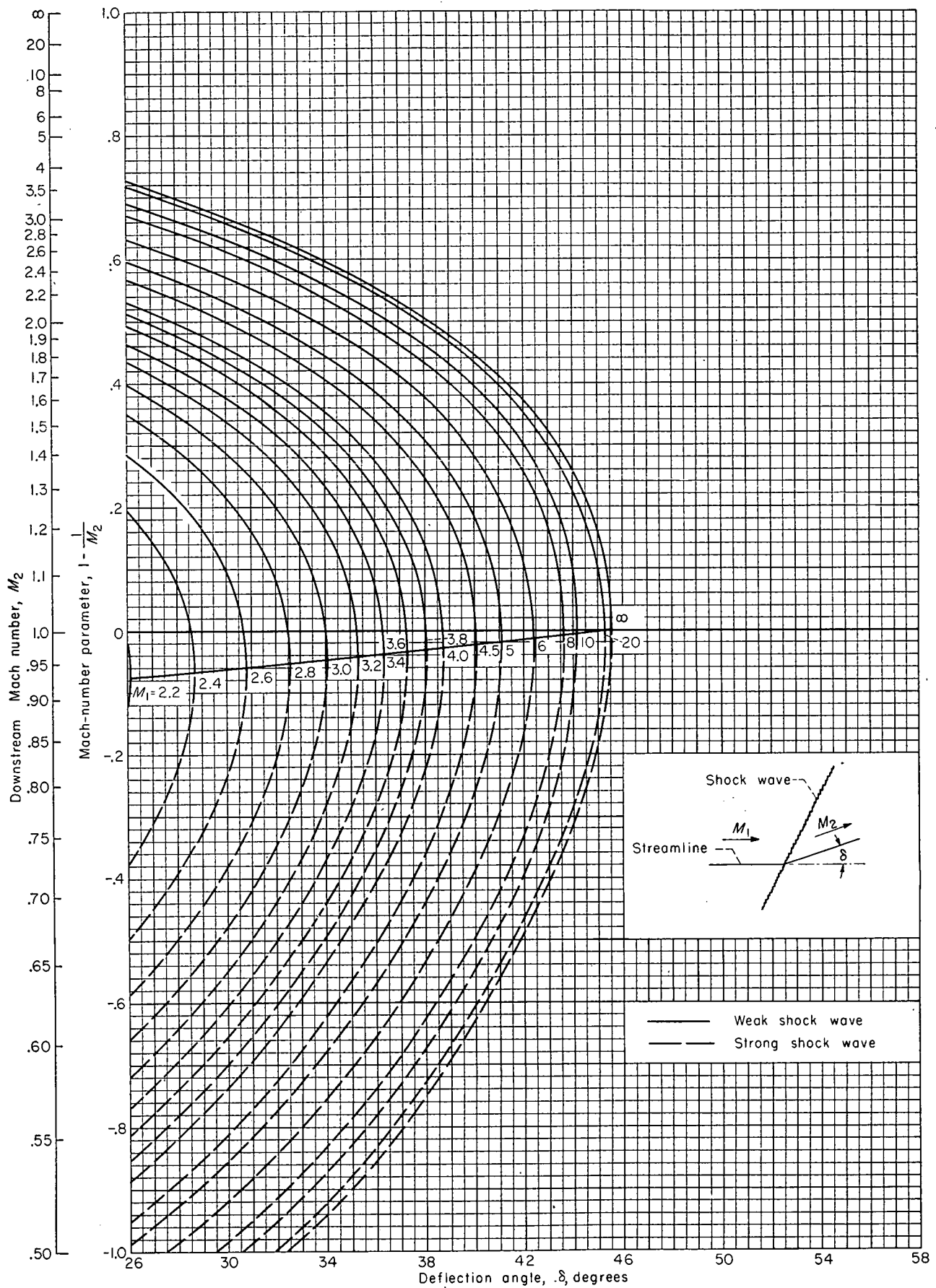


CHART 4.—Concluded

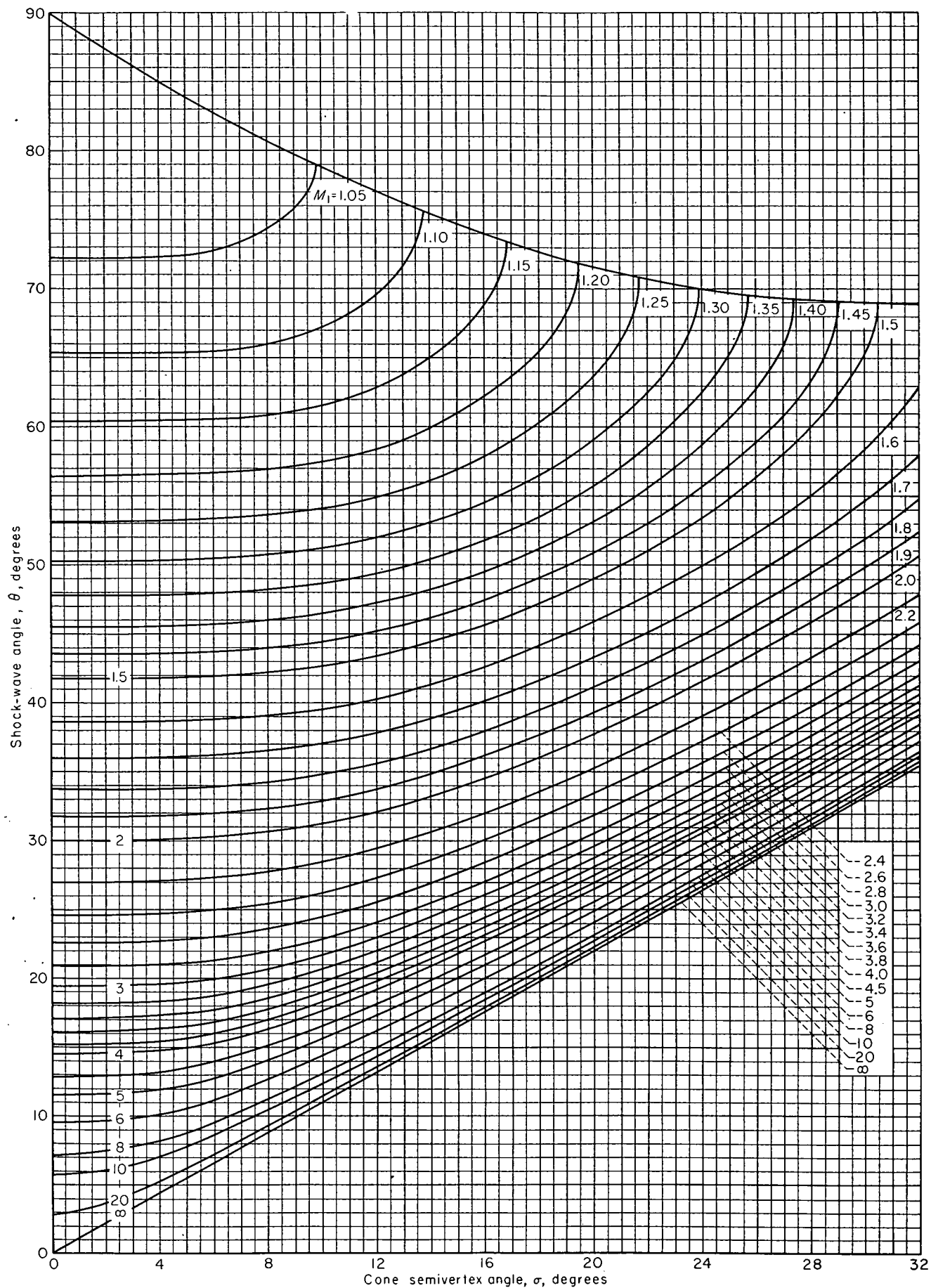


CHART 5.—Variation of shock-wave angle with cone semivertex angle for various upstream Mach numbers. Perfect gas, $\gamma=1.405$.

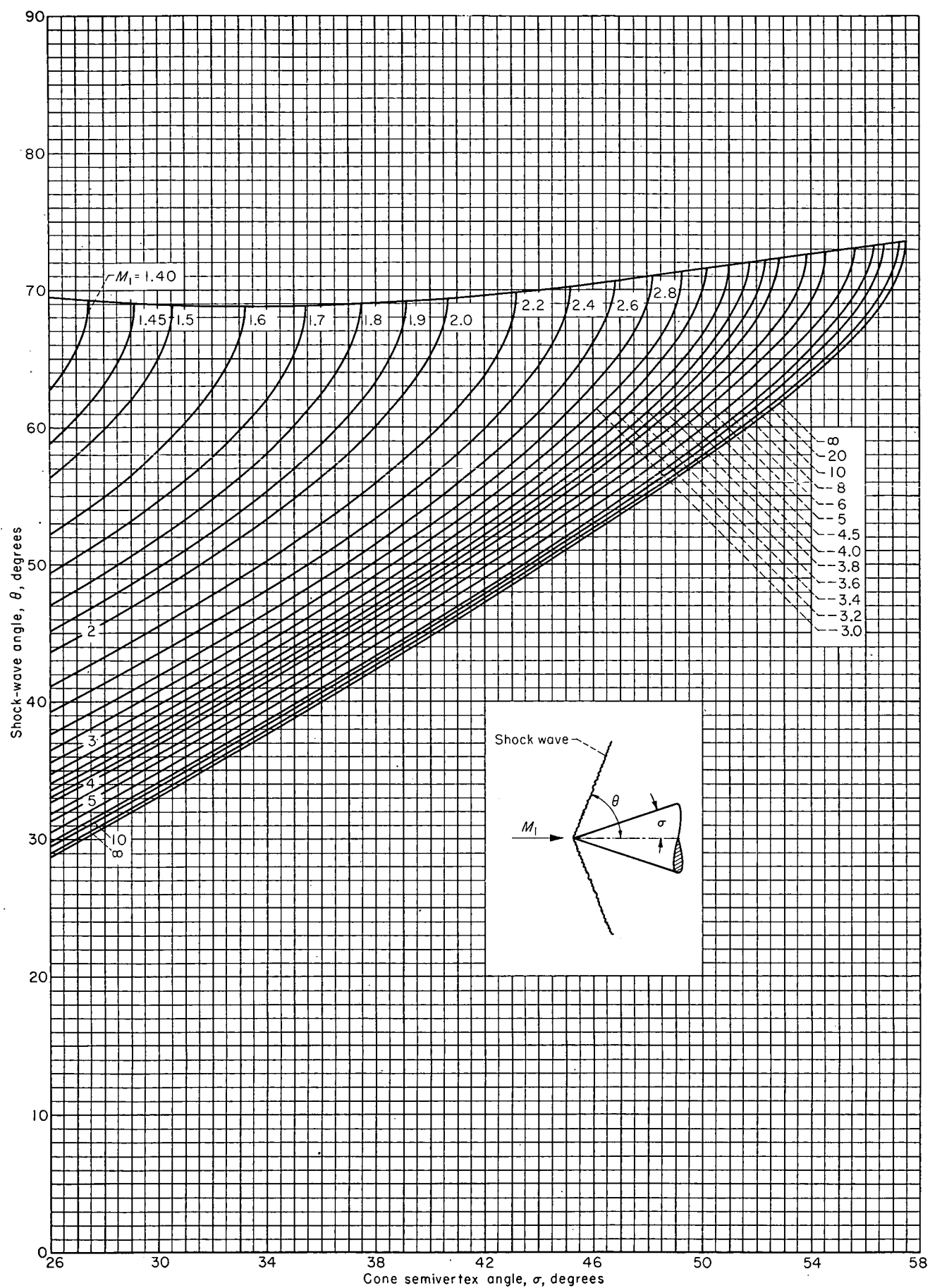


CHART 5.—Concluded

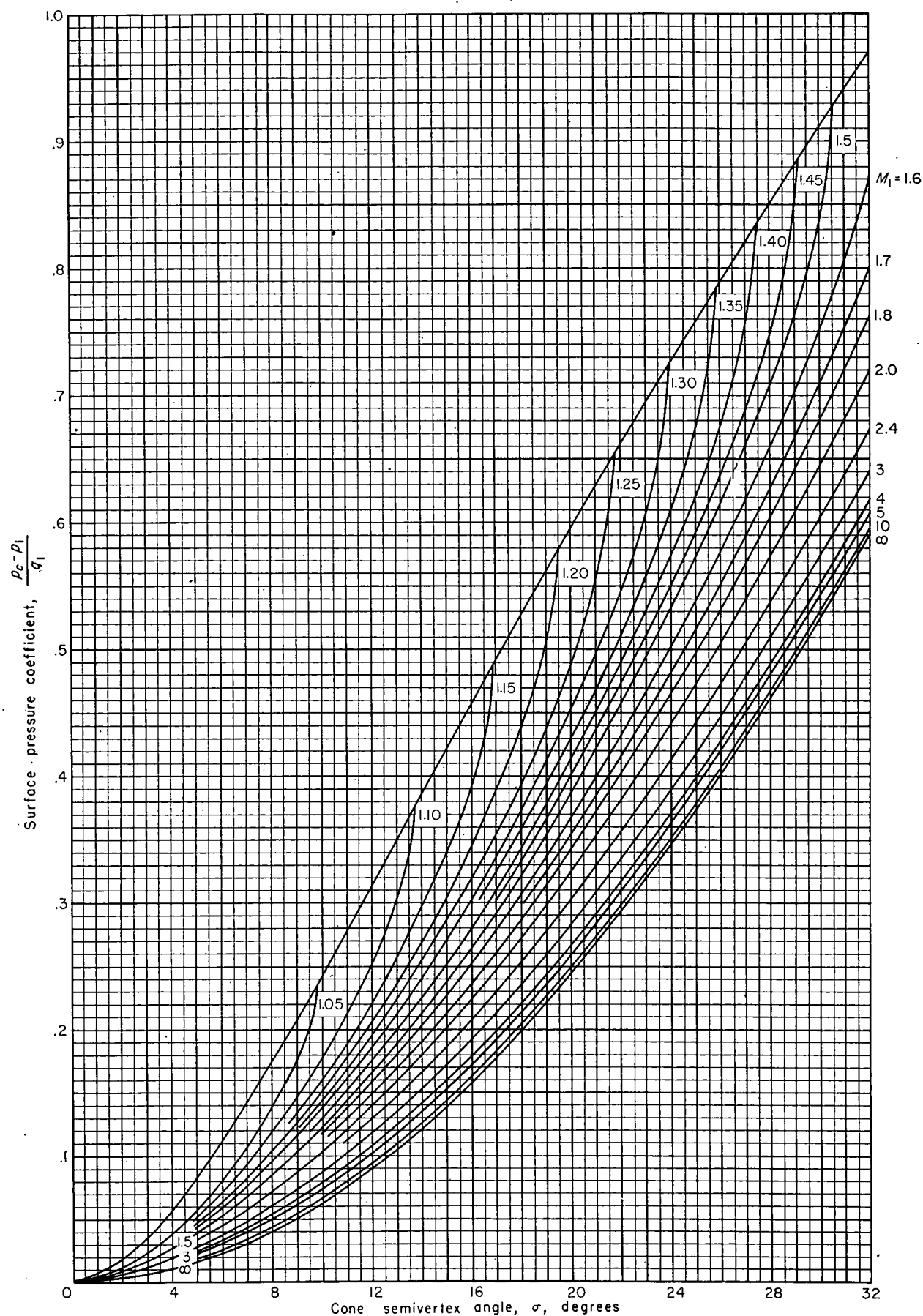


CHART 6.—Variation of surface pressure coefficient with cone semivertex angle for various upstream Mach numbers. Perfect gas, $\gamma = 1.405$.

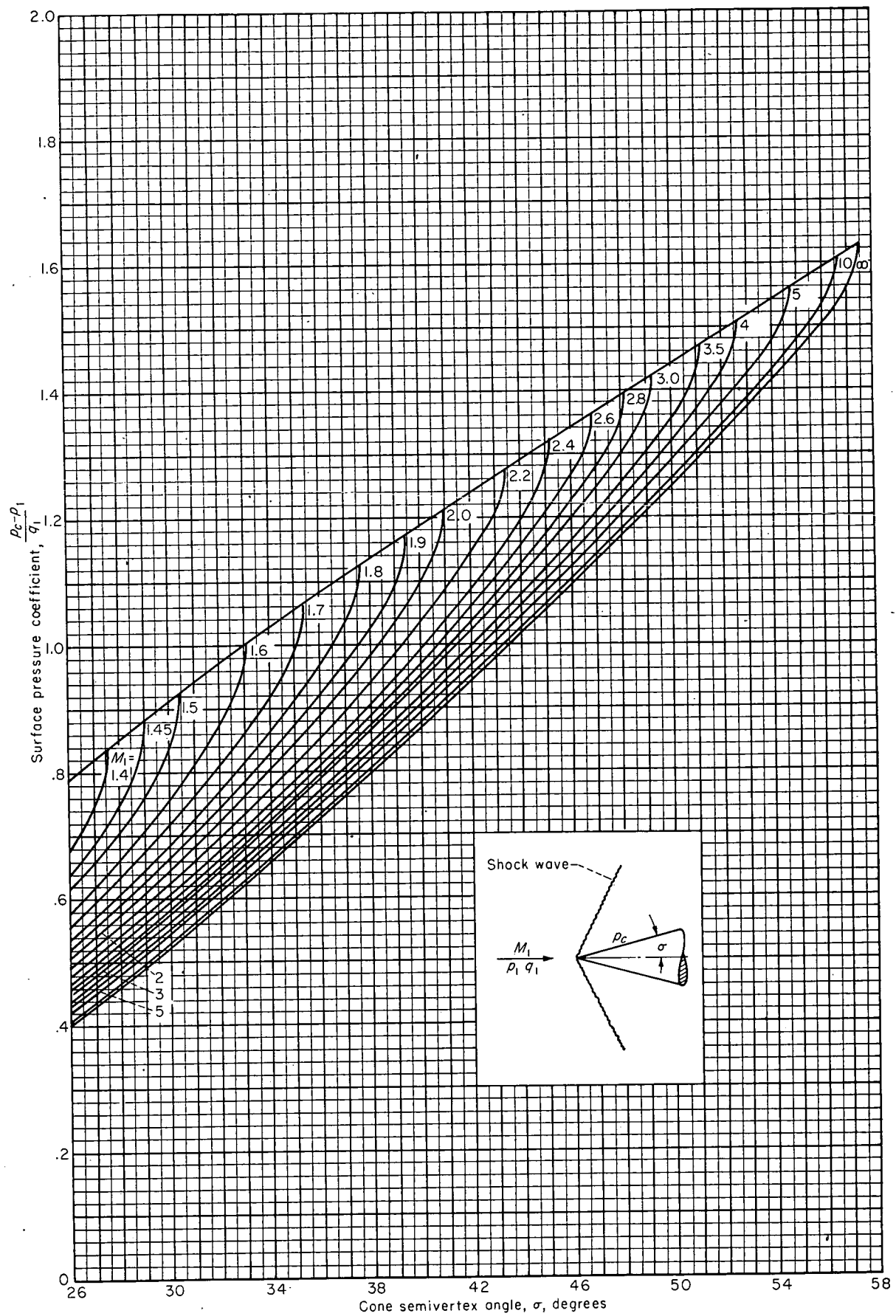


CHART 6.—Concluded

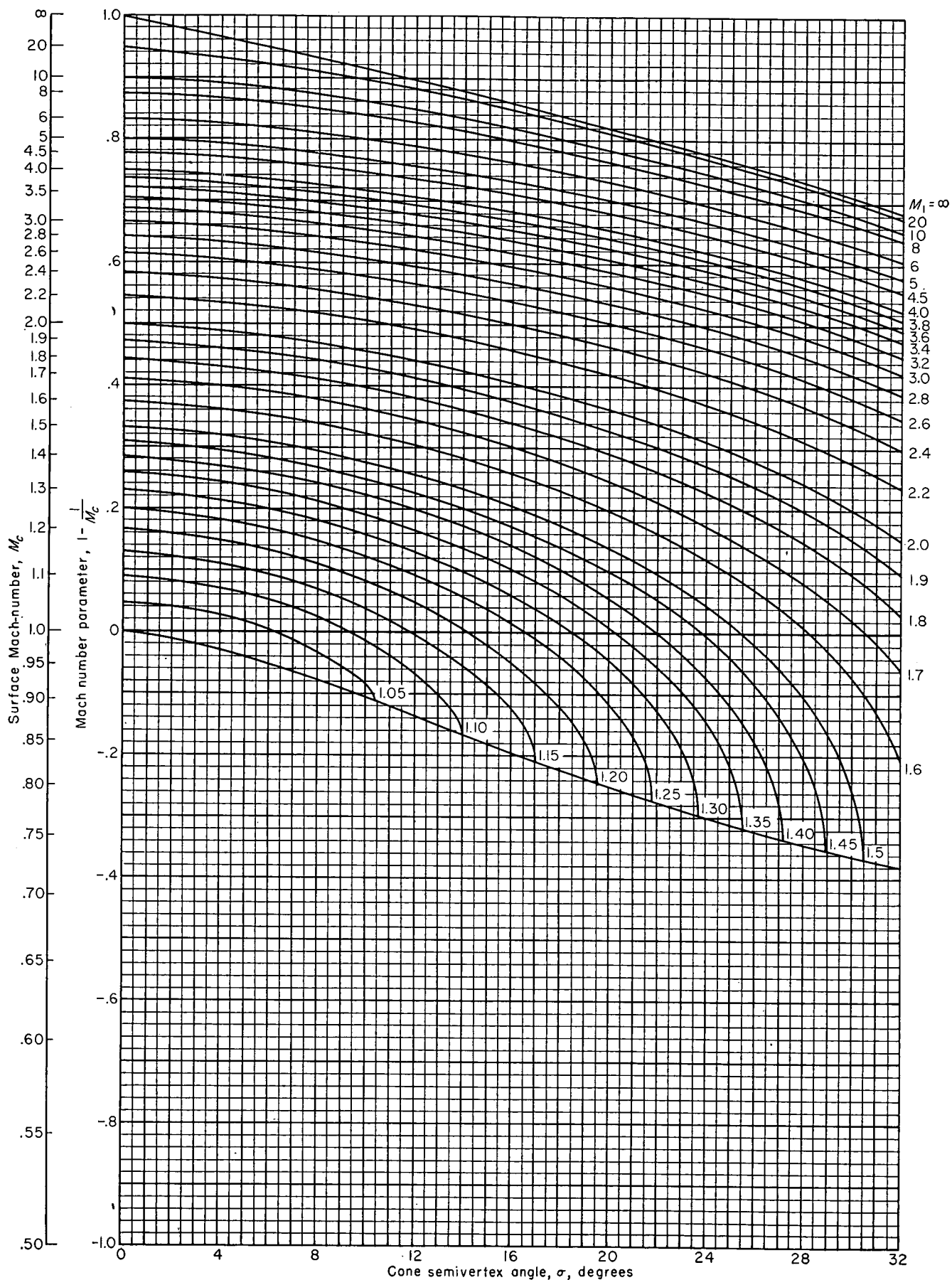


CHART 7.—Variation of Mach number at the surface of a cone with cone semivertex angle for various upstream Mach numbers. Perfect gas. $\gamma = 1.405$.

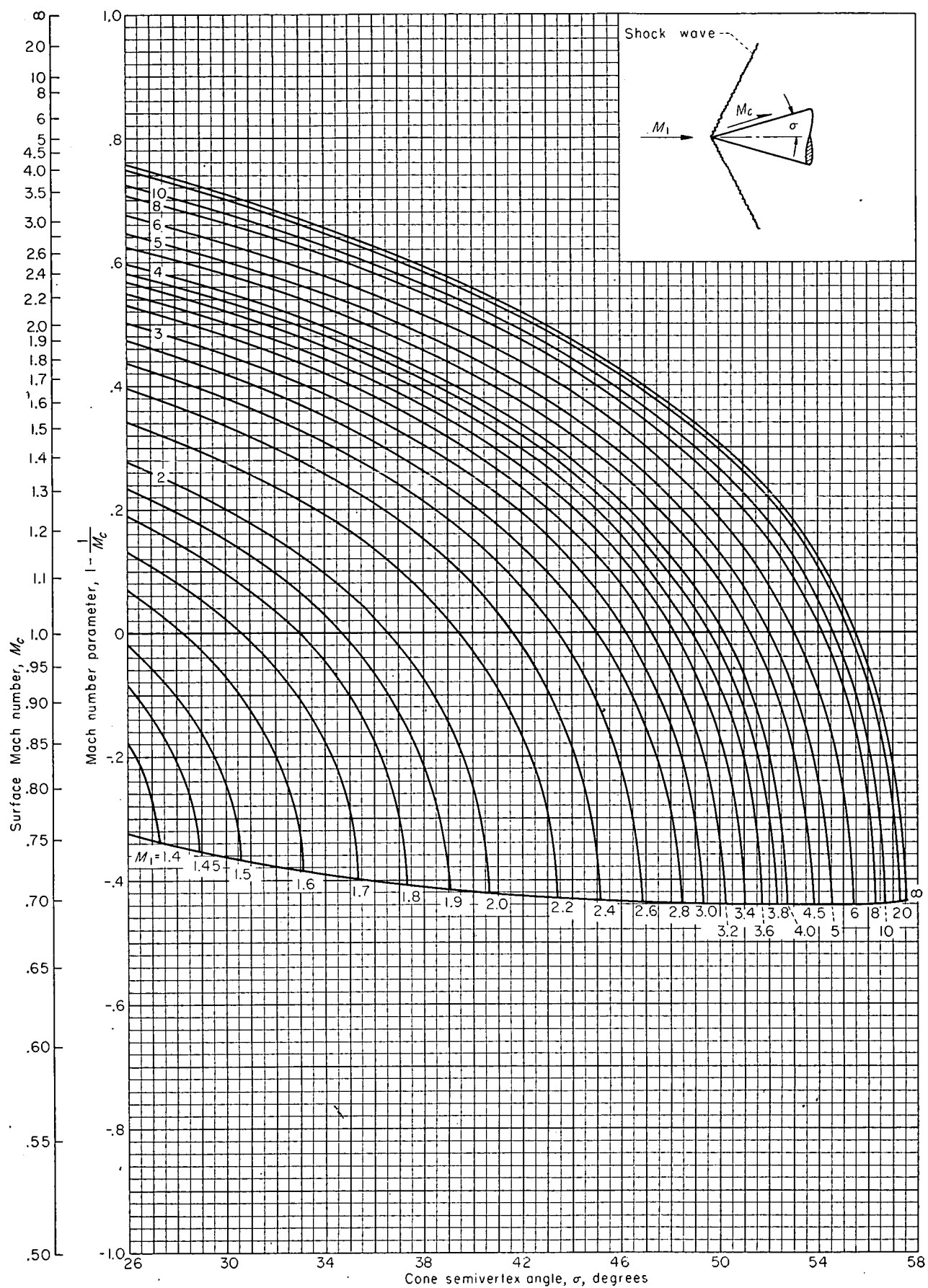


CHART 7.—Concluded

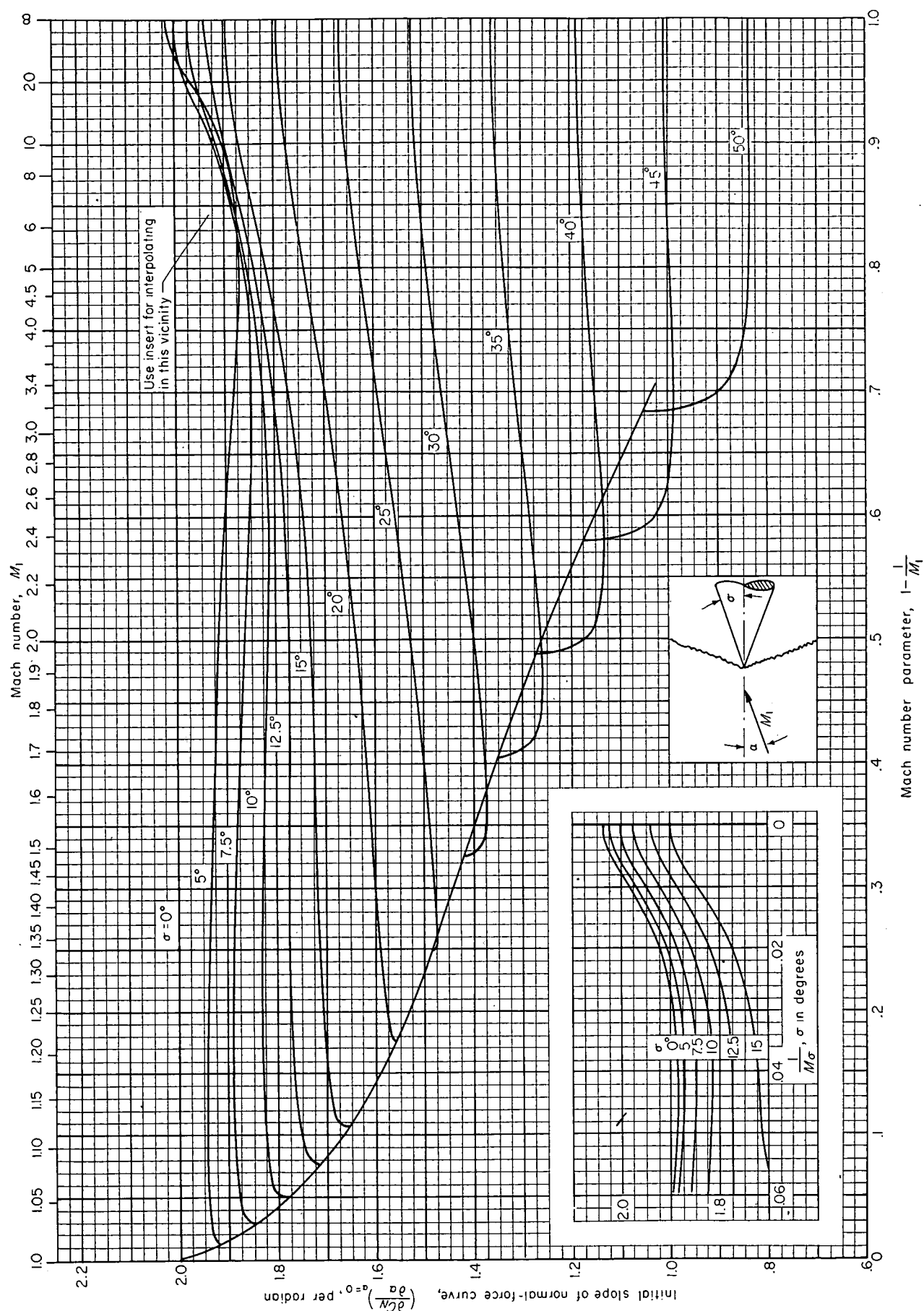


CHART 8.—Variation of the initial slope of the normal-force curve with upstream Mach number for various cone semivertex angles. Perfect gas, $\gamma = 1.405$.

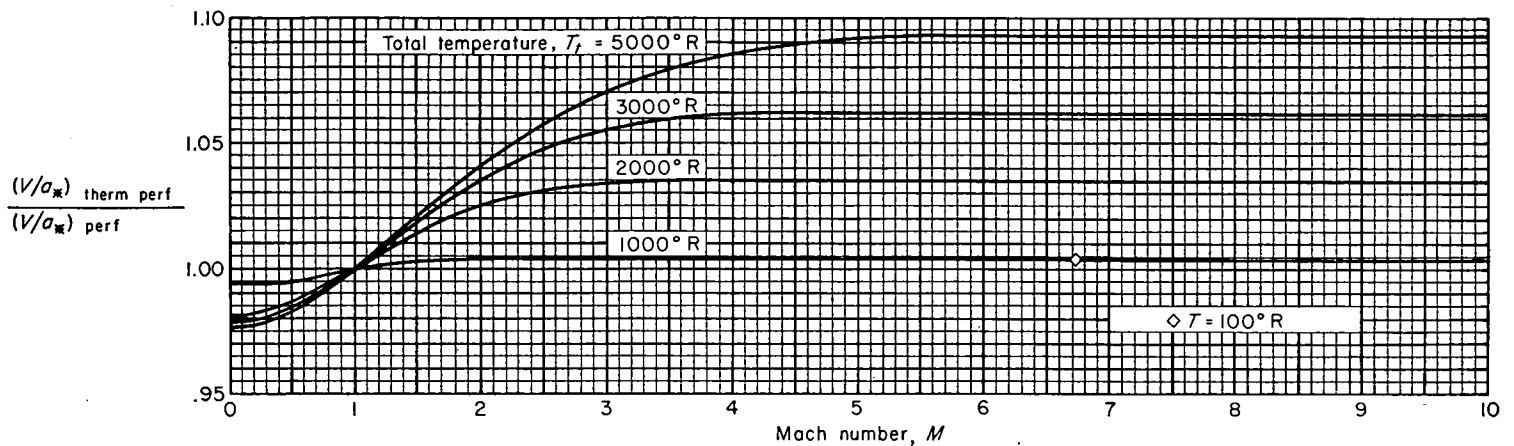


CHART 9.—Effect of caloric imperfections on the ratio of local speed to speed of sound at the point where $M=1$.

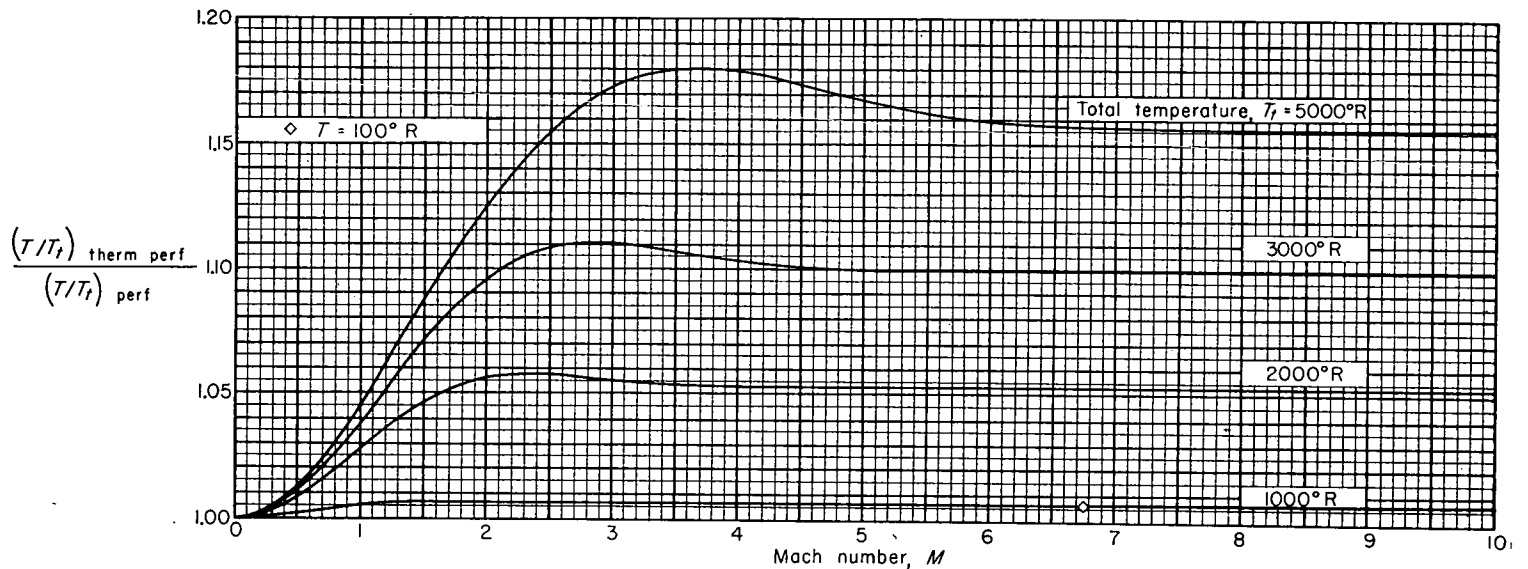


CHART 10.—Effect of caloric imperfections on the ratio of static temperature to total temperature.

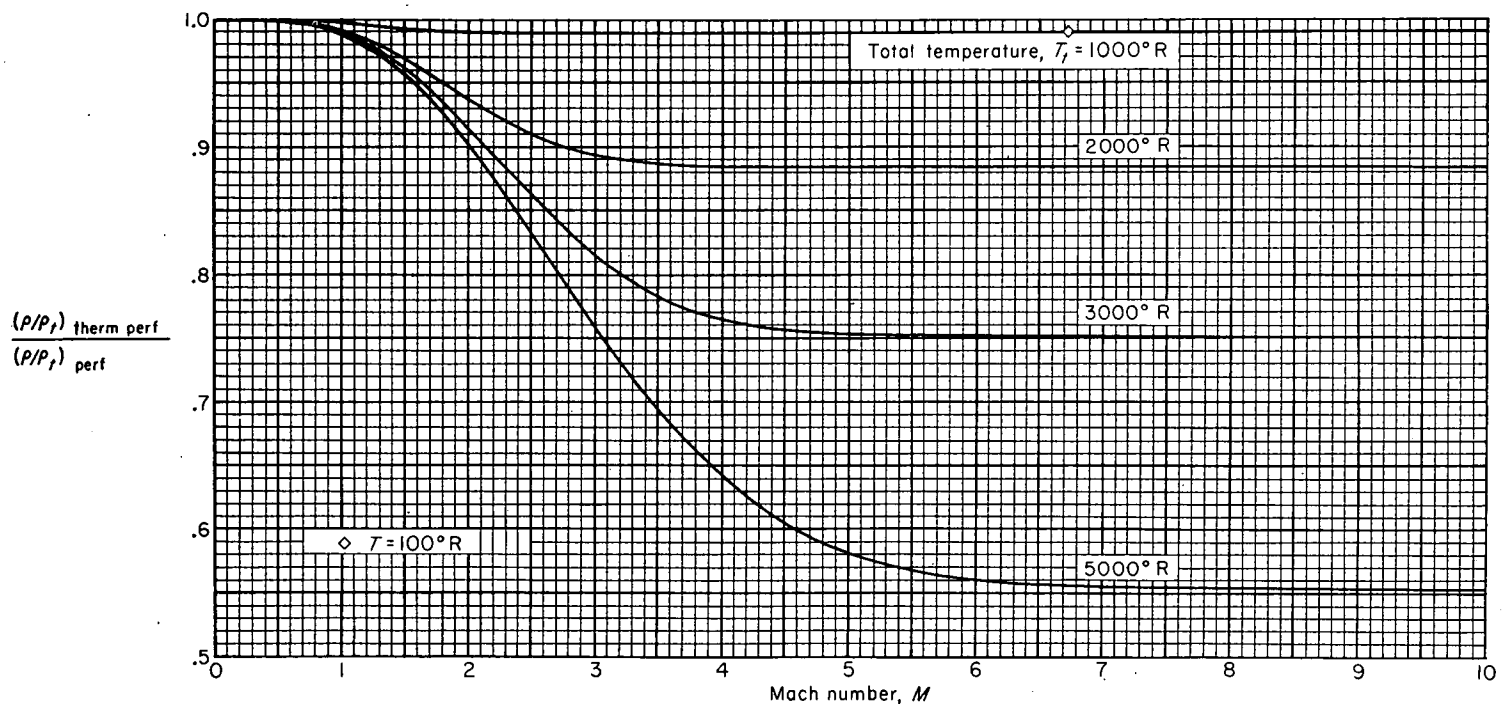


CHART 11.—Effect of caloric imperfections on the ratio of static density to total density.

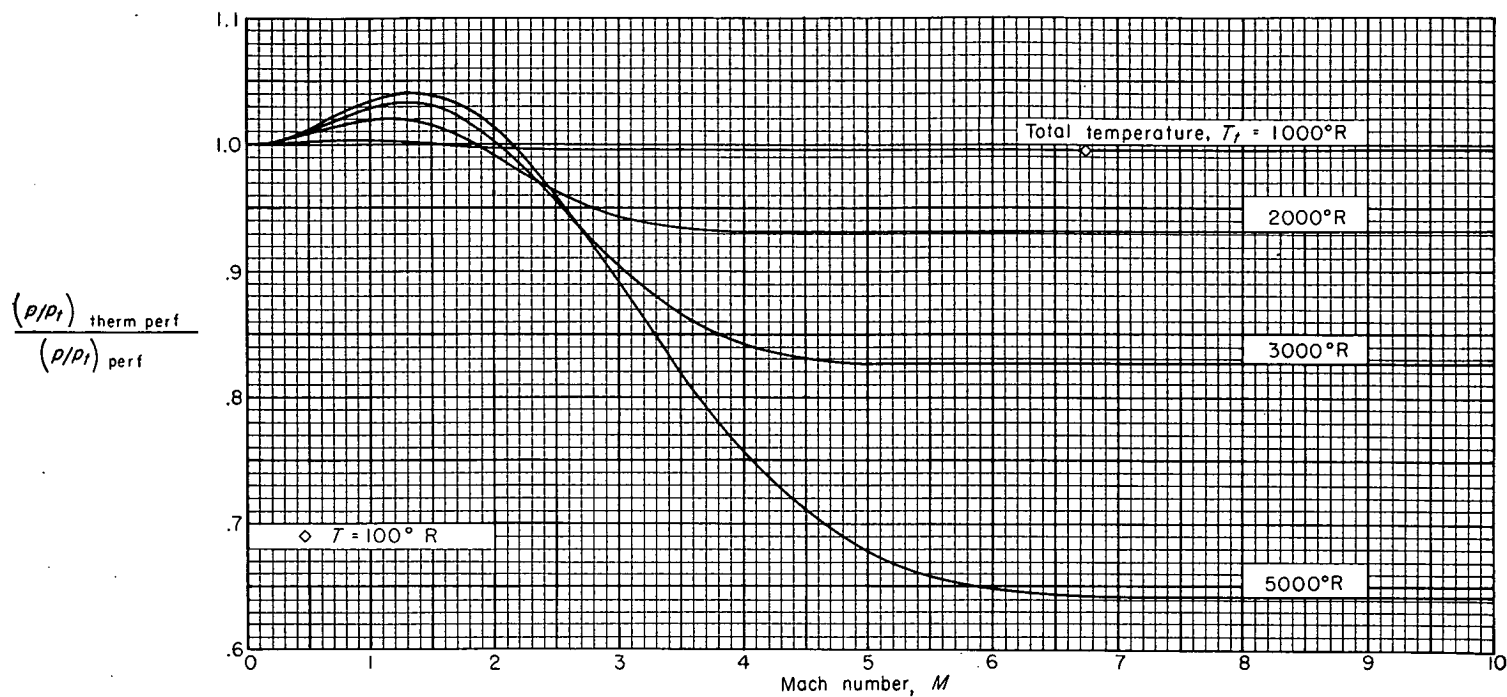


CHART 12.—Effect of caloric imperfections on the ratio of static pressure to total pressure.

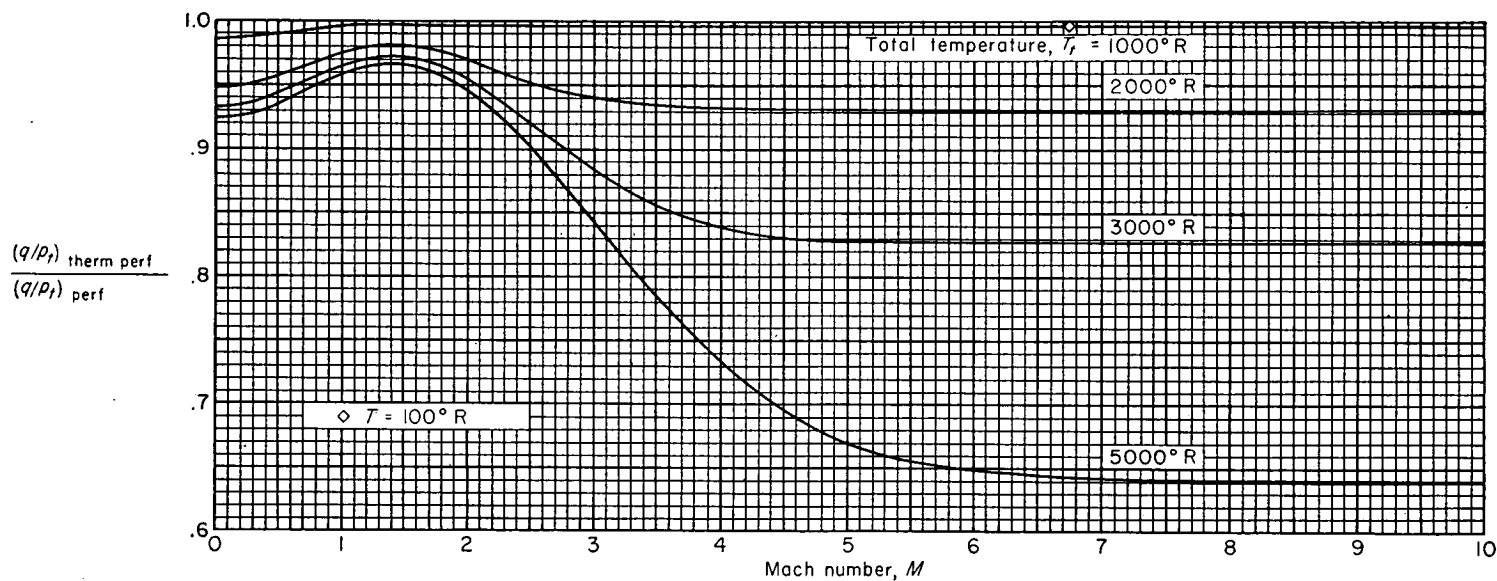


CHART 13.—Effect of caloric imperfections on the ratio of dynamic pressure to total pressure.

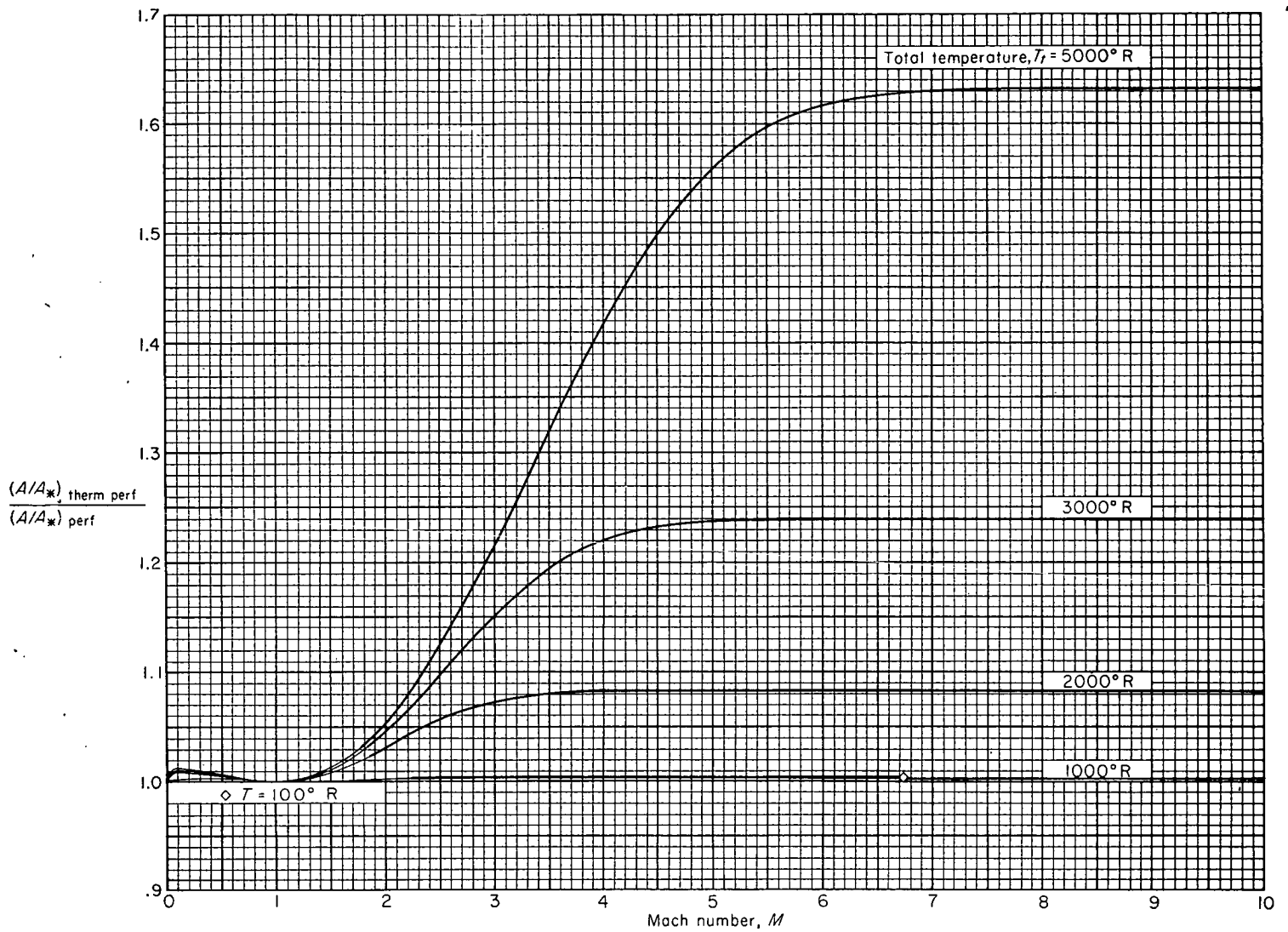


CHART 14.—Effect of caloric imperfections on the ratio of local cross-sectional area of a stream tube to the cross-sectional area at the point where $M=1$.

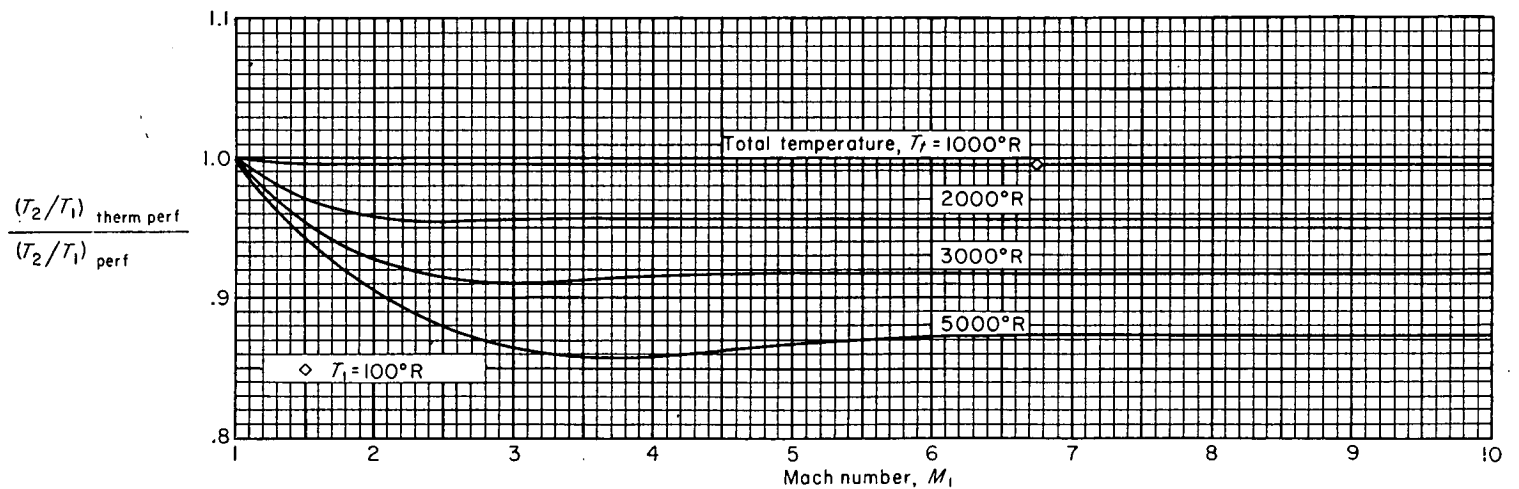


CHART 15.—Effect of caloric imperfections on the static-temperature ratio across a normal shock wave.

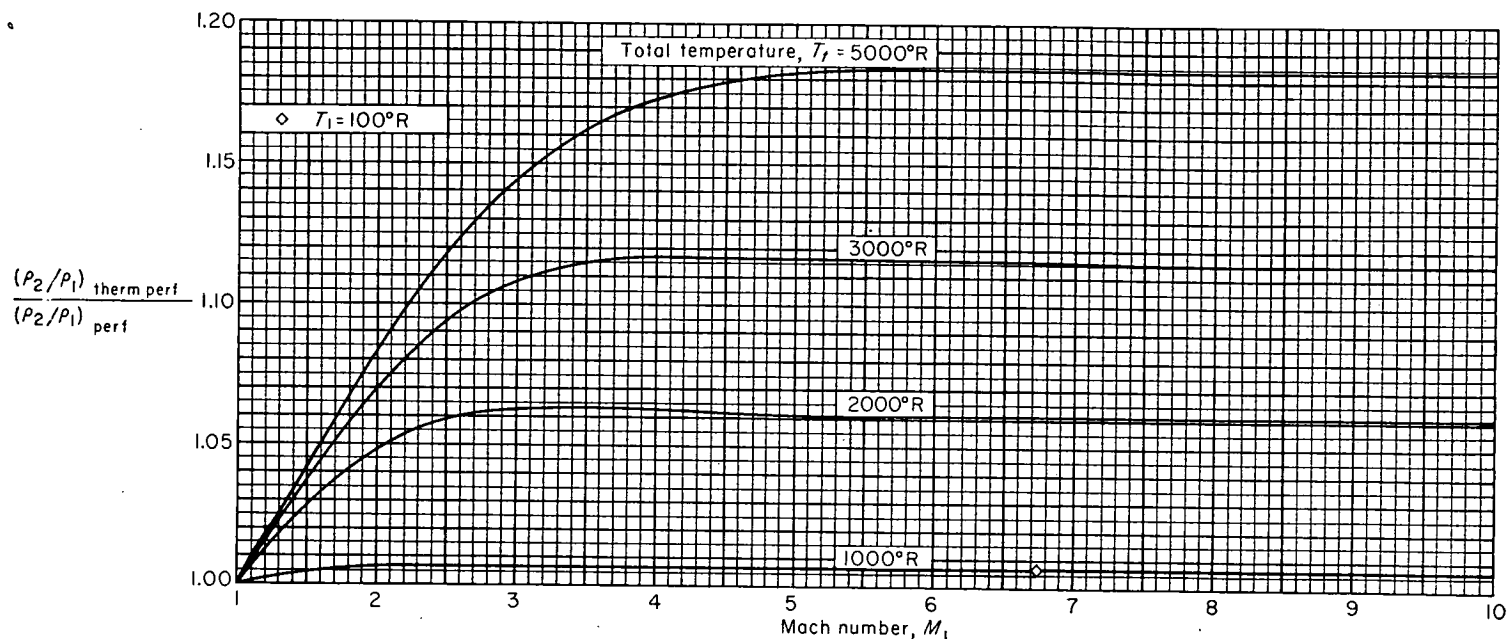


CHART 16.—Effect of caloric imperfections on the static-density ratio across a normal shock wave.

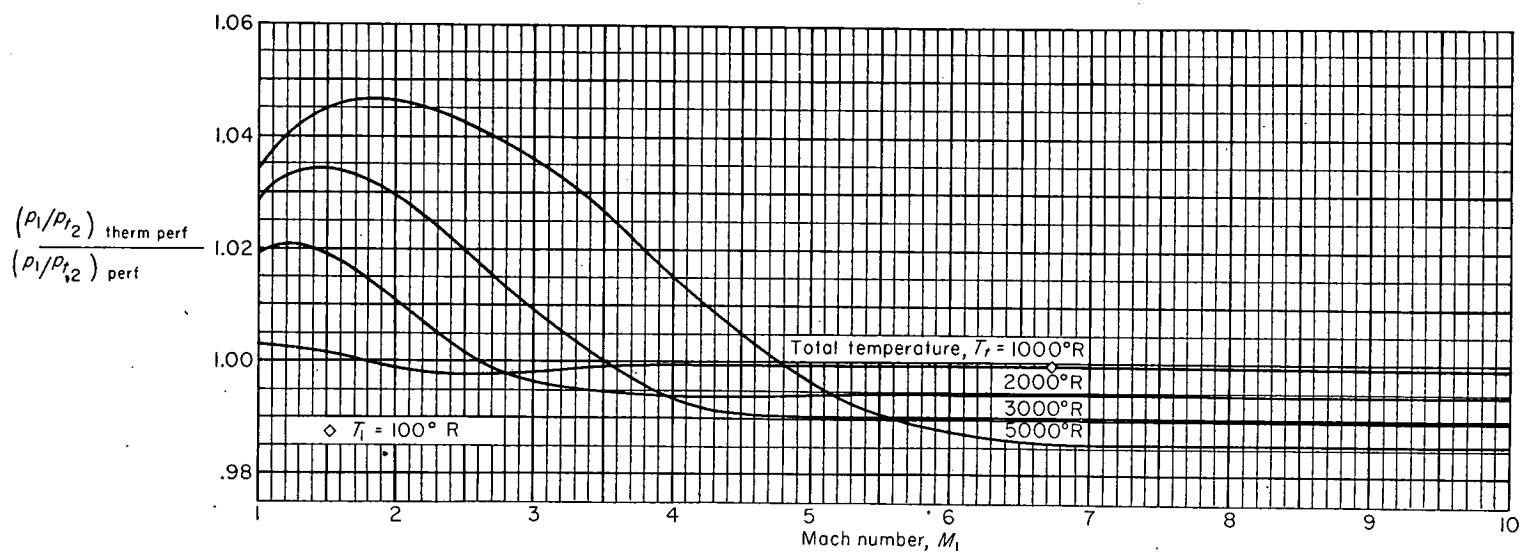


CHART 17.—Effect of caloric imperfections on the ratio of static pressure upstream of a normal shock wave to total pressure downstream.

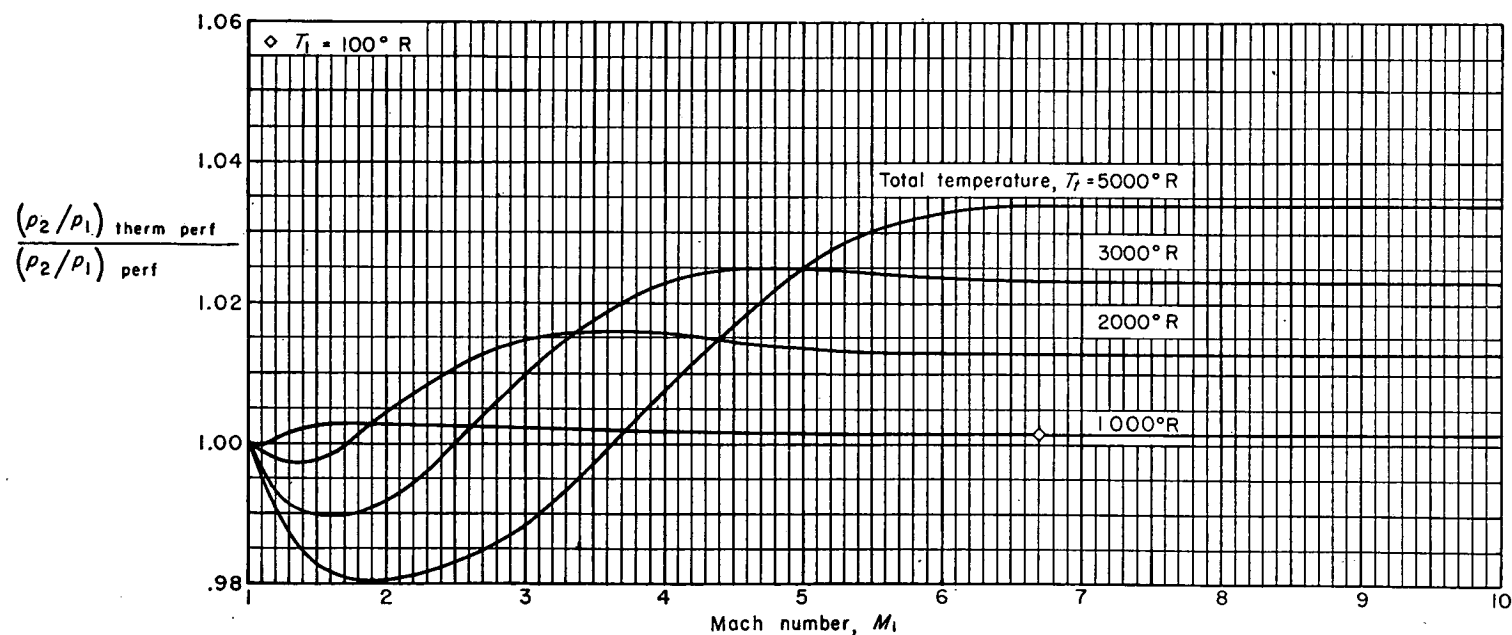


CHART 18.—Effect of caloric imperfections on the static-pressure ratio across a normal shock wave.

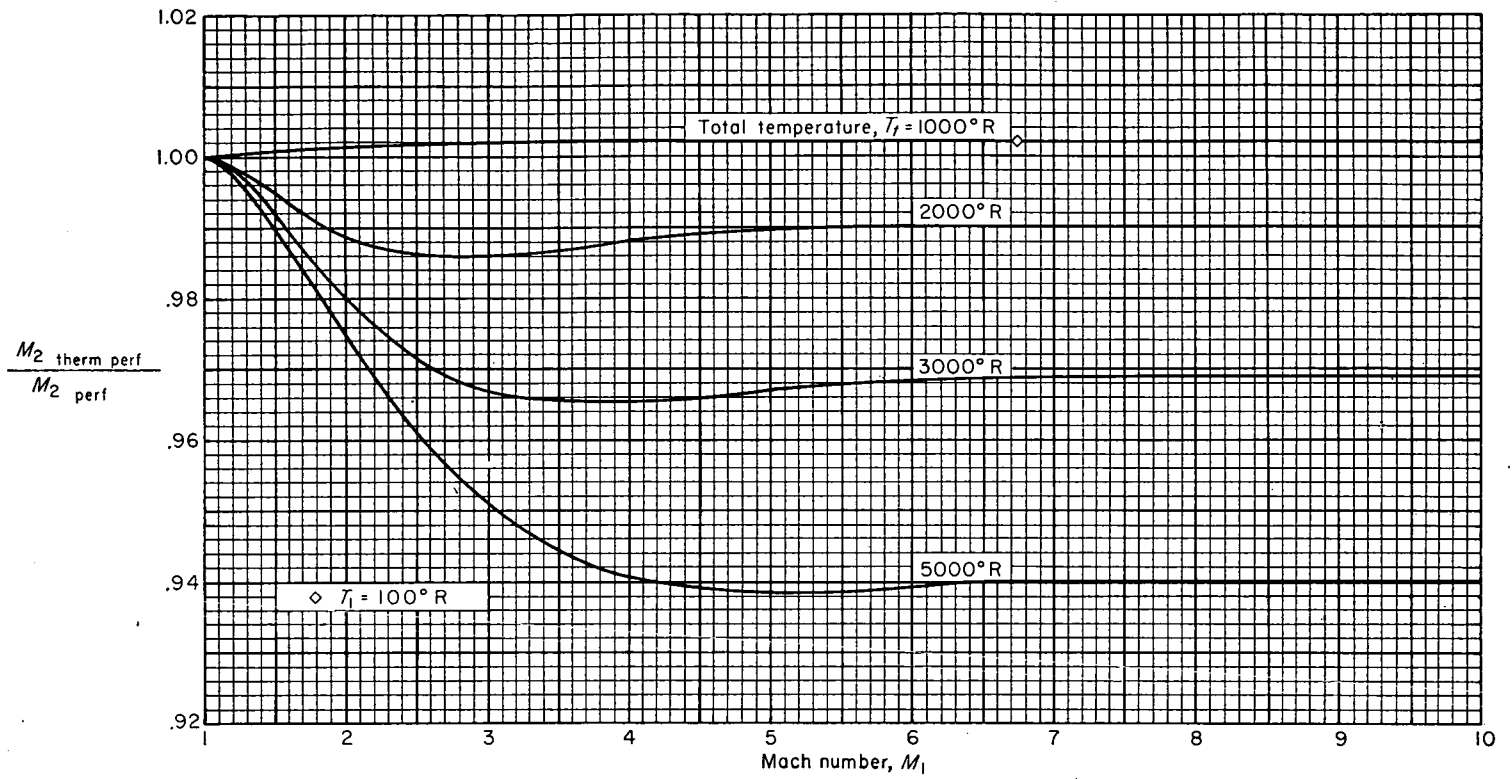


CHART 19.—Effect of caloric imperfections on the Mach number downstream of a normal shock wave.

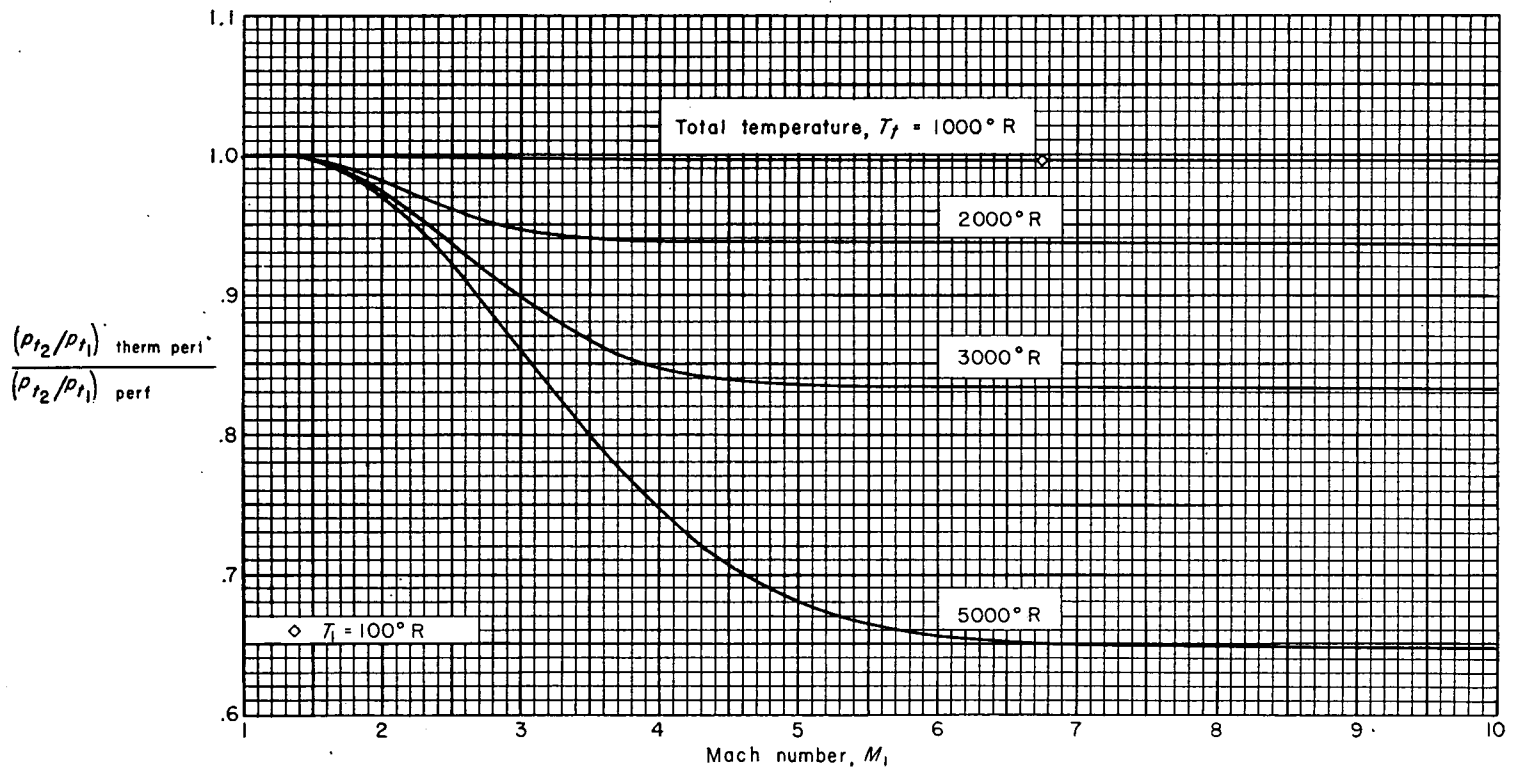


CHART 20.—Effect of caloric imperfections on the total-pressure ratio across a normal shock wave.

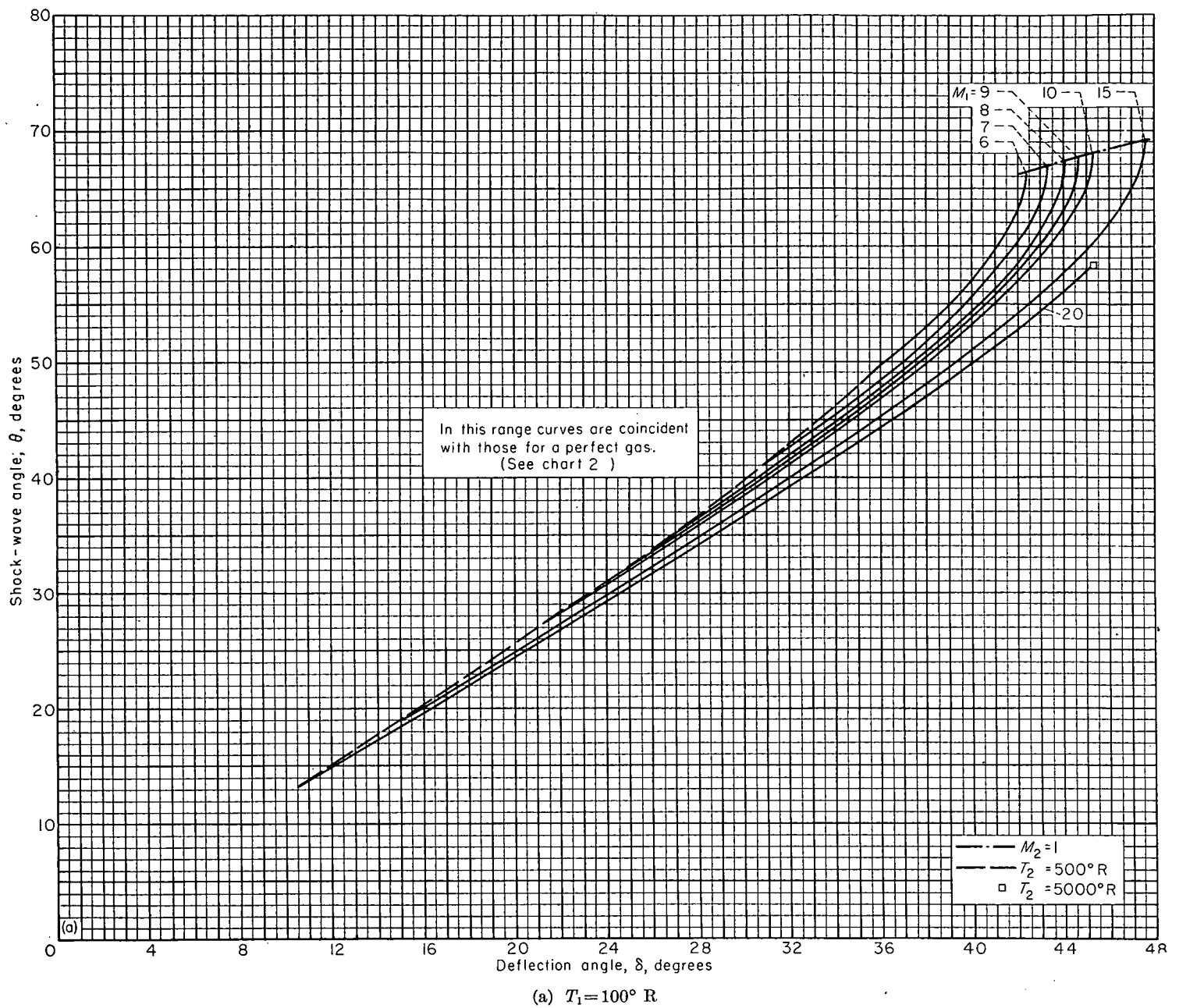


CHART 21.—Effect of caloric imperfections on the variation with flow-deflection angle of the shock-wave angle for a weak oblique shock wave.

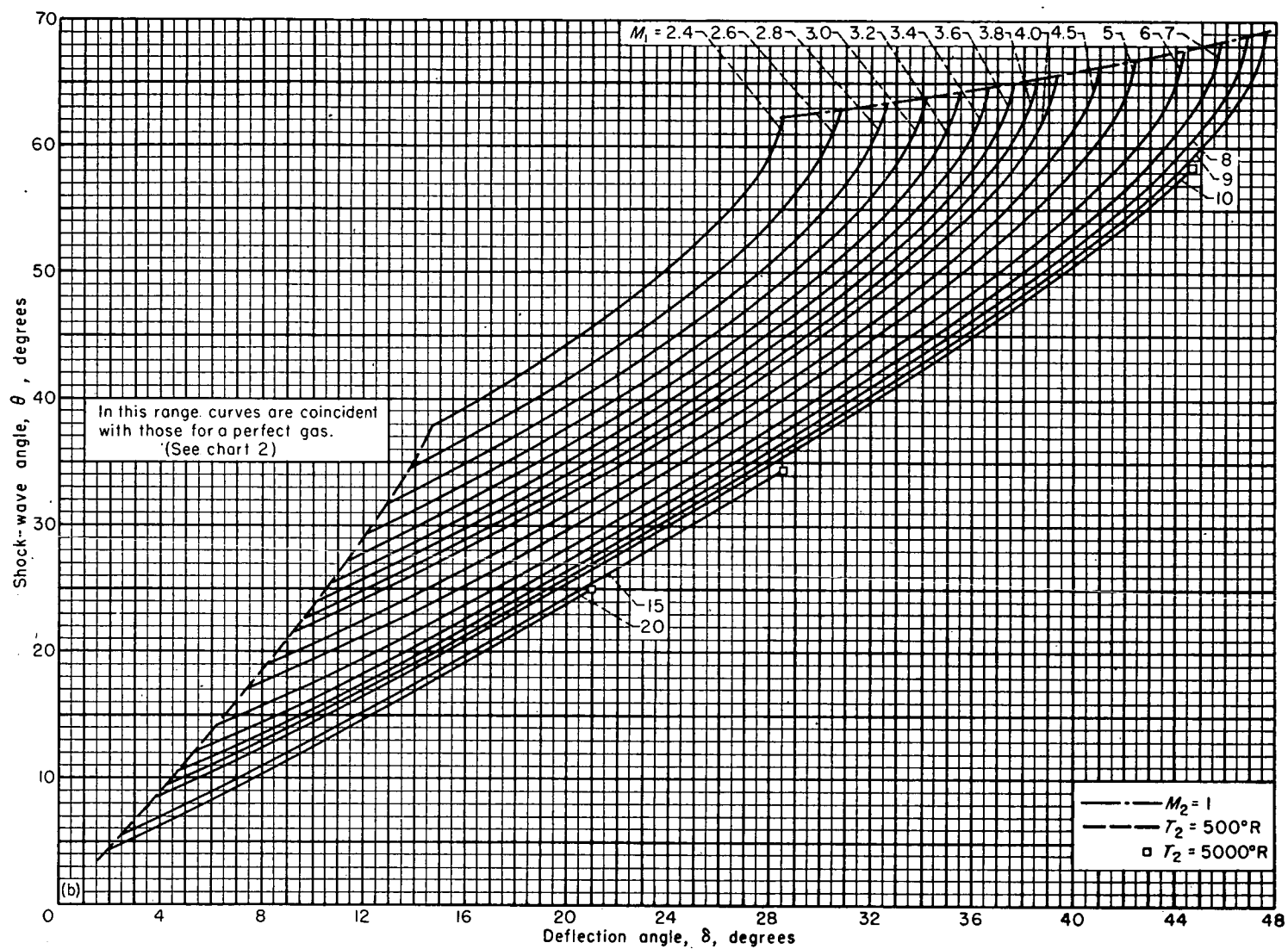
(b) $T_1 = 390^\circ\text{R}$

CHART 21.—Continued

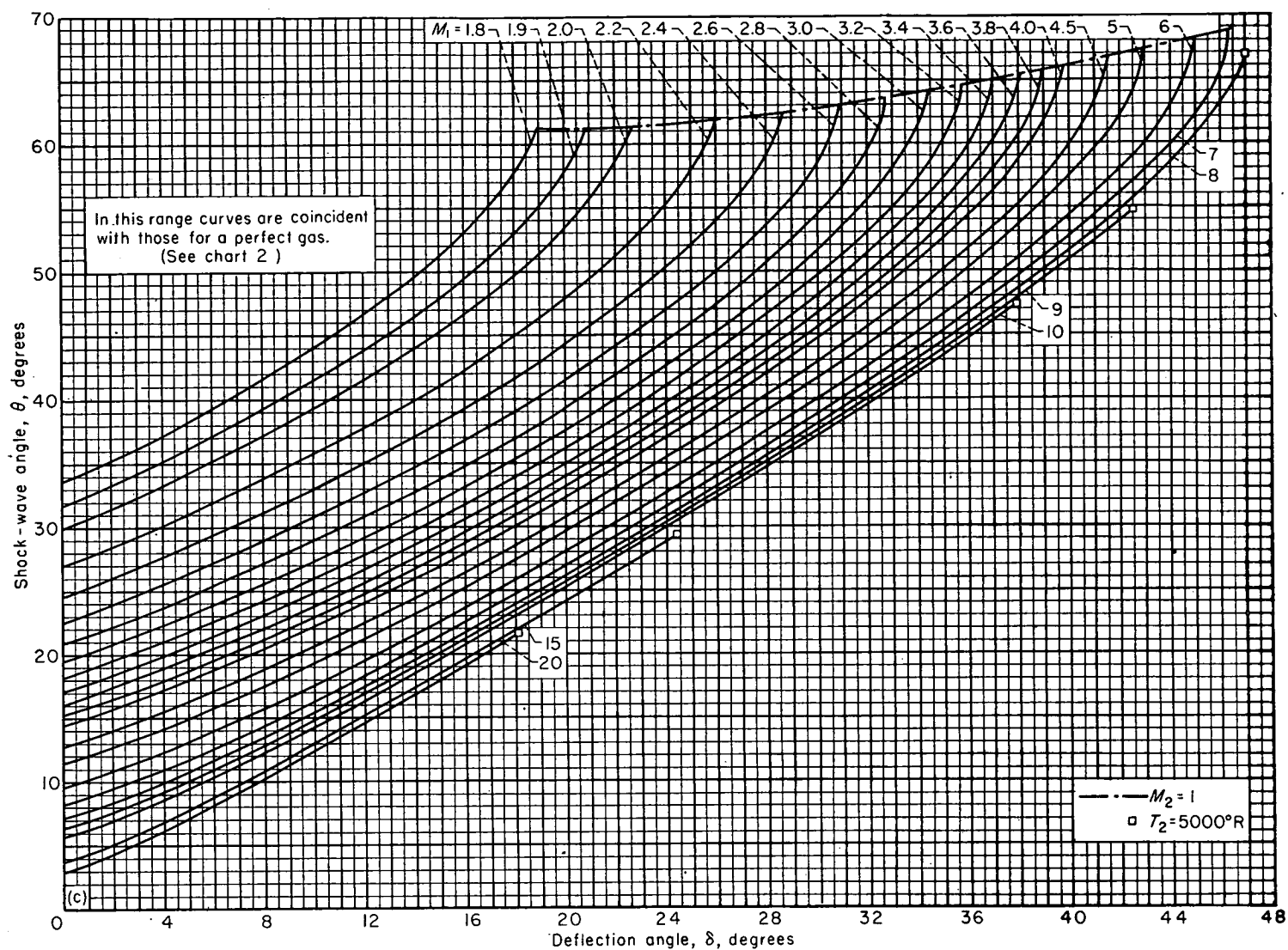
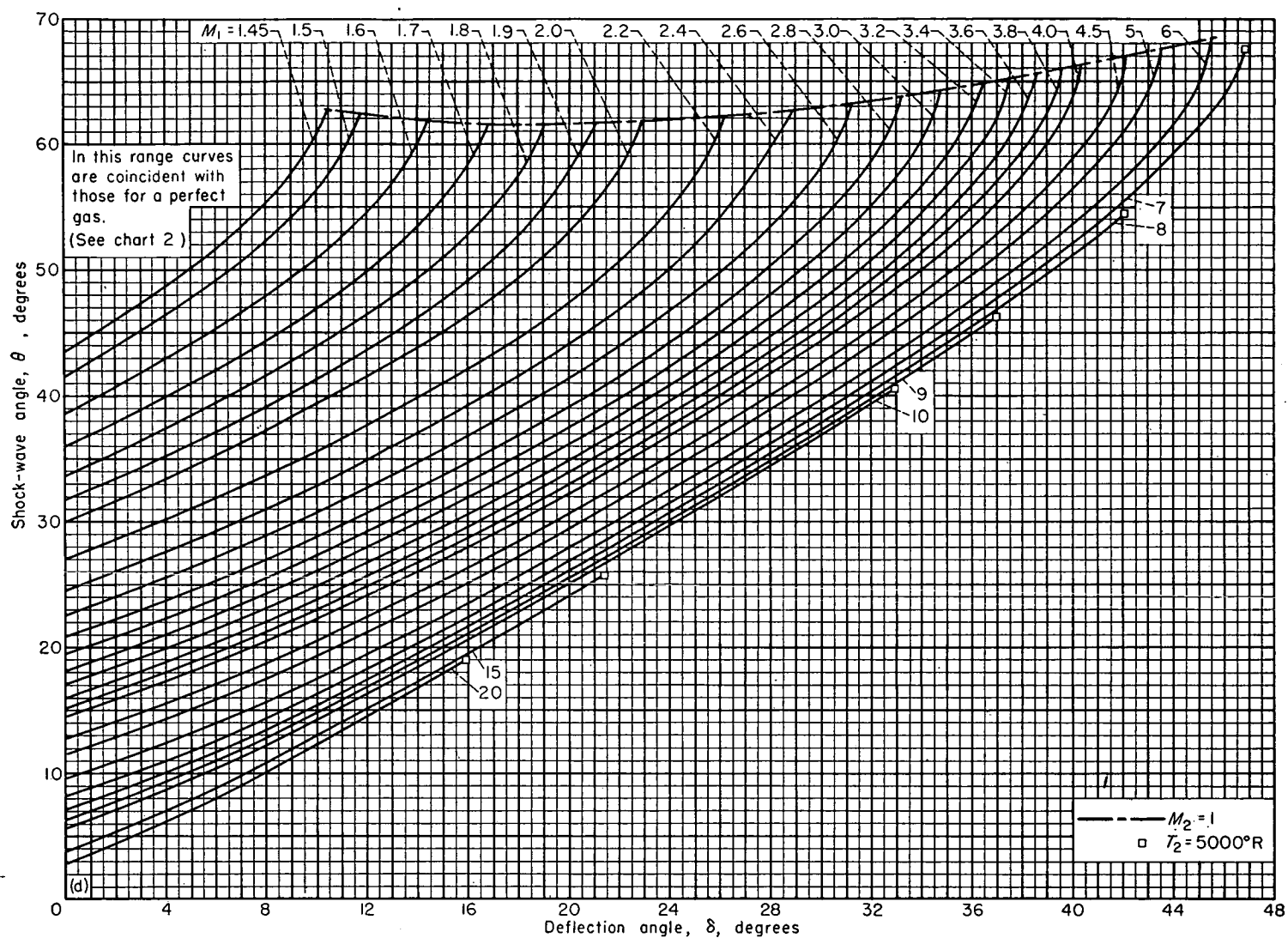


CHART 21.—Continued



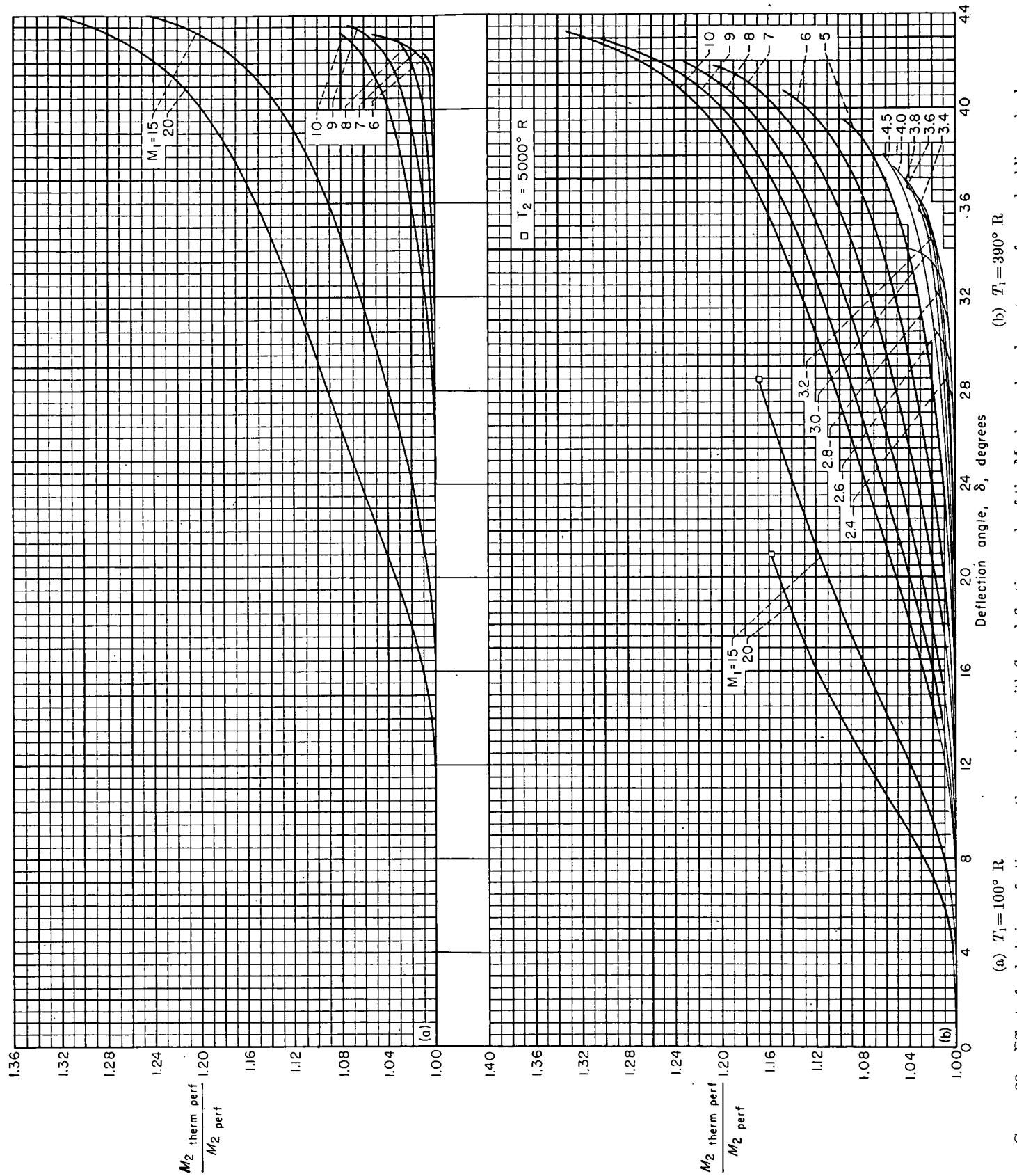


CHART 22.—Effect of caloric imperfections on the variation of Mach number downstream of a weak oblique shock wave.

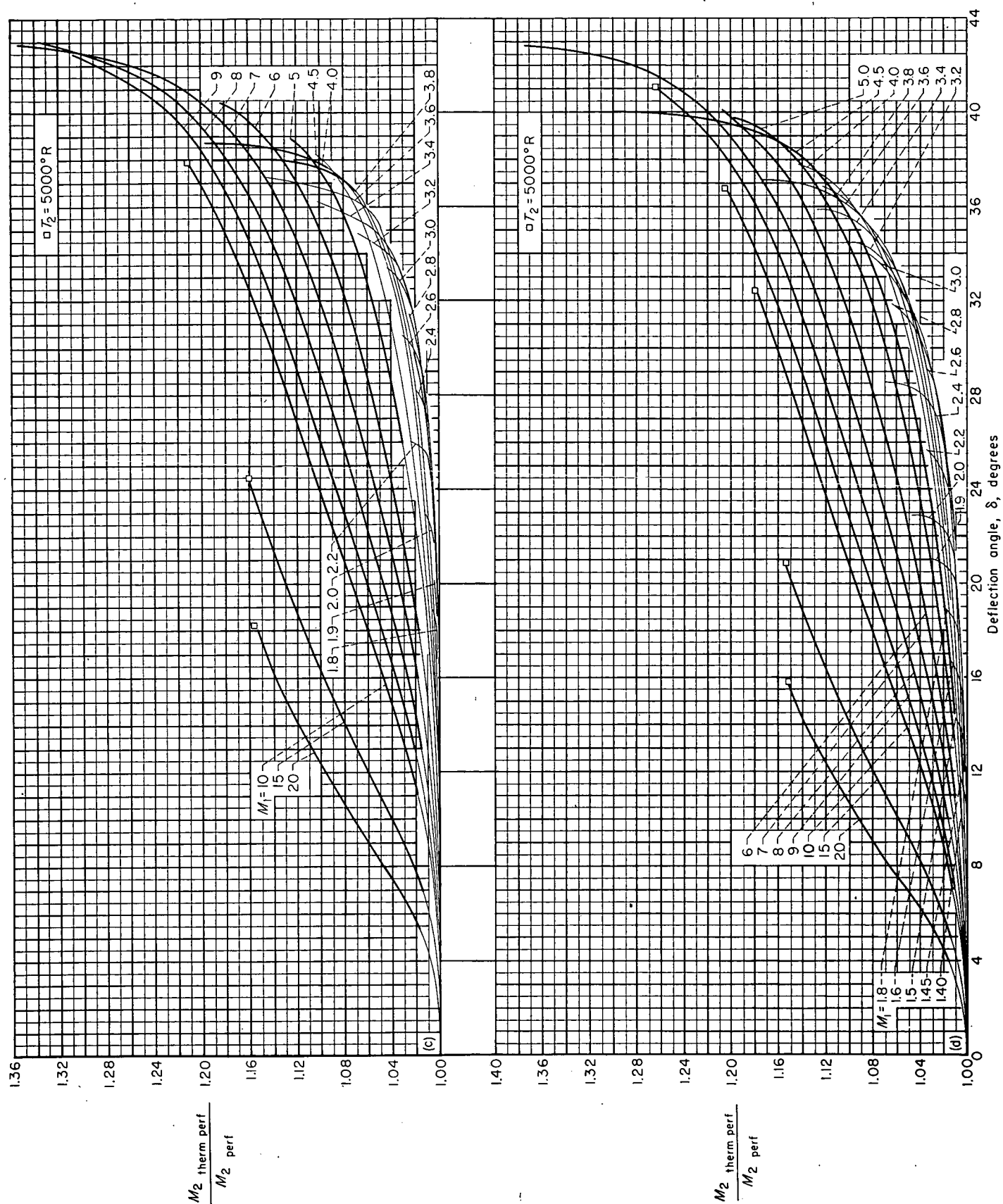
(d) $T_1 = 630^\circ \text{R}$ (c) $T_1 = 500^\circ \text{R}$

CHART 22.—Concluded

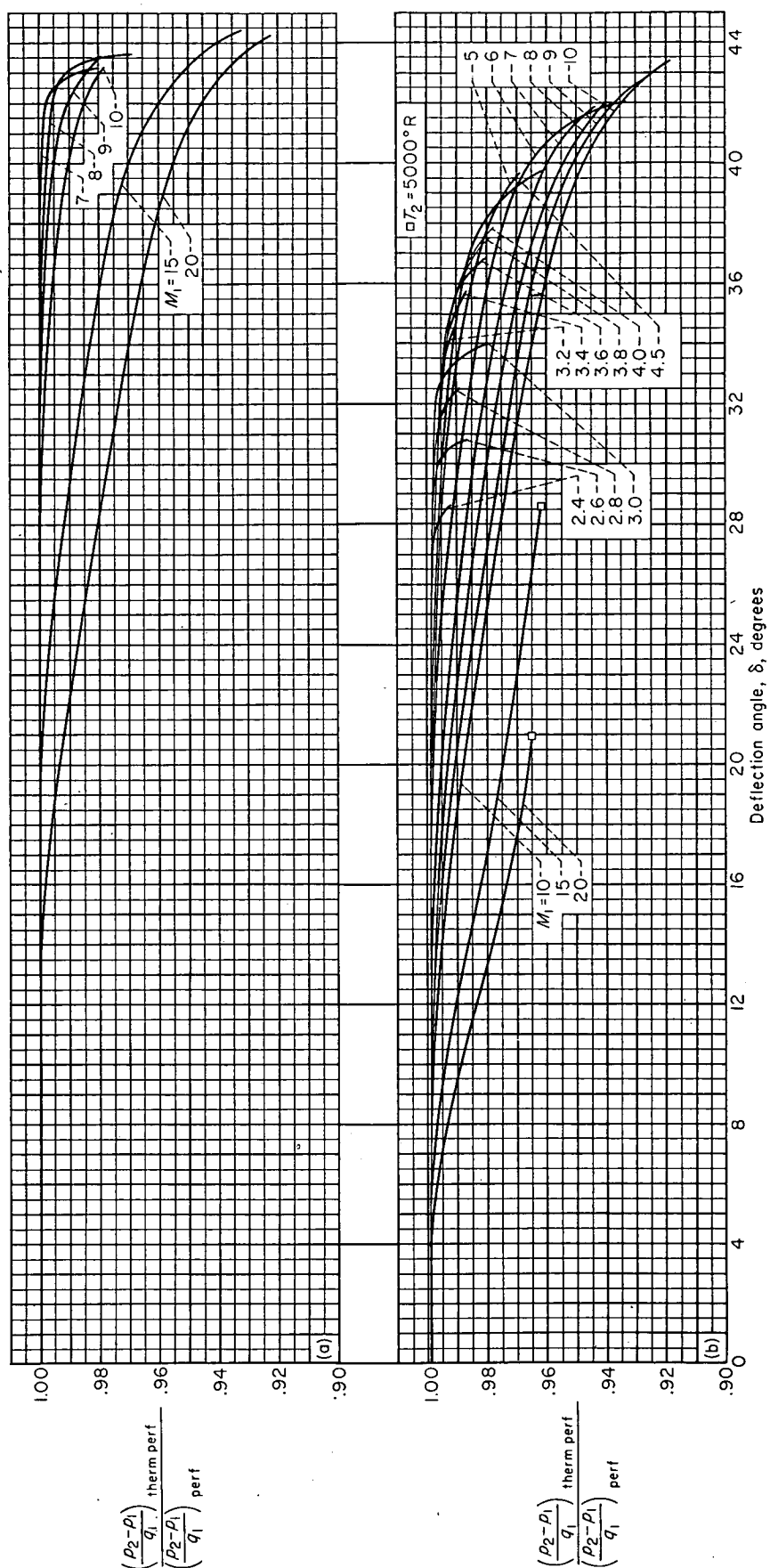


CHART 23.—Effect of caloric imperfections on the variation with flow-deflection angle of the pressure coefficient across a weak oblique shock wave.

(a) $T_1 = 100^\circ \text{ R}$
 (b) $T_1 = 390^\circ \text{ R}$

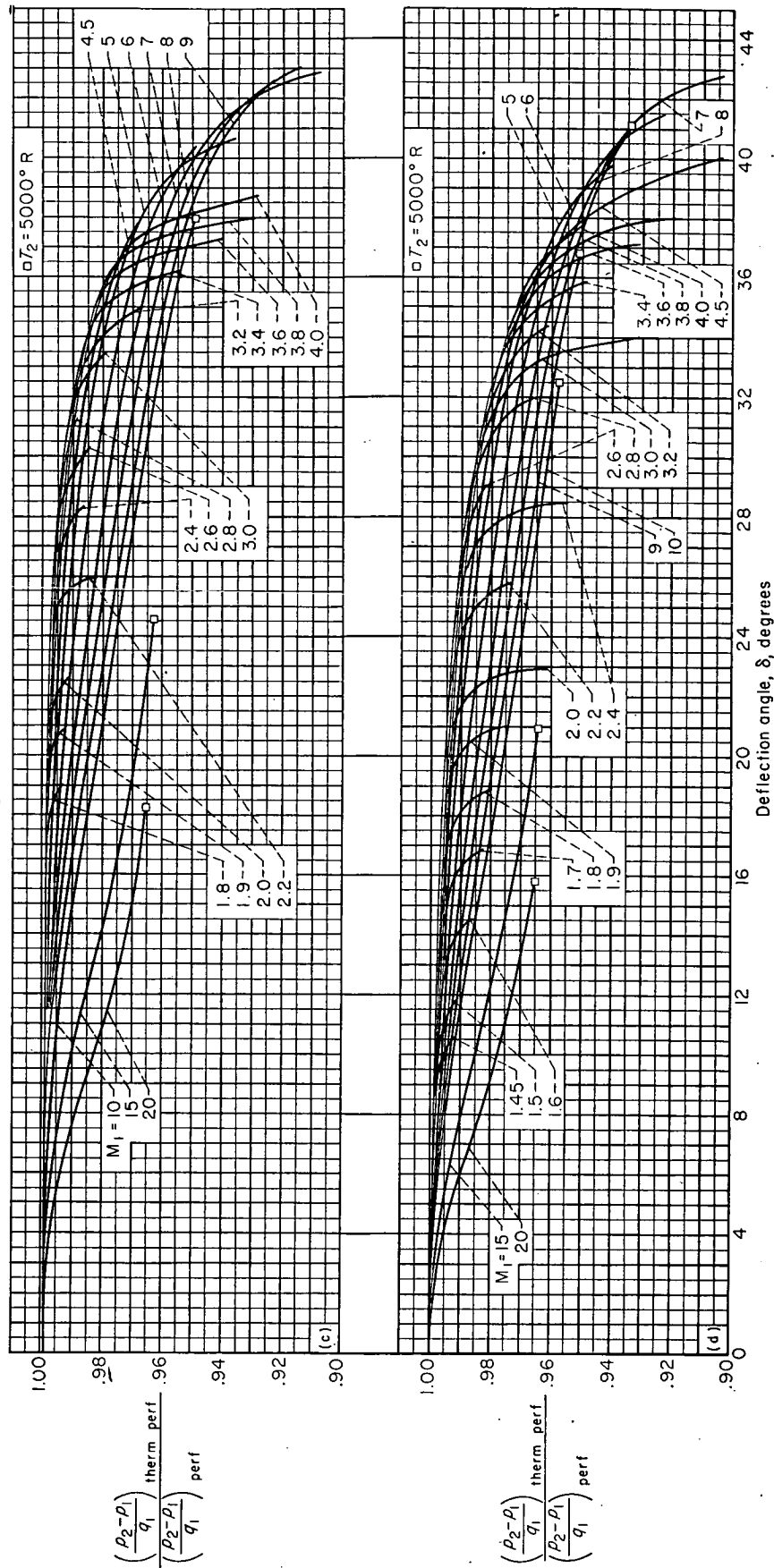
(c) $T_1 = 500^\circ \text{ R}$ (d) $T_1 = 630^\circ \text{ R}$

CHART 23.—Concluded

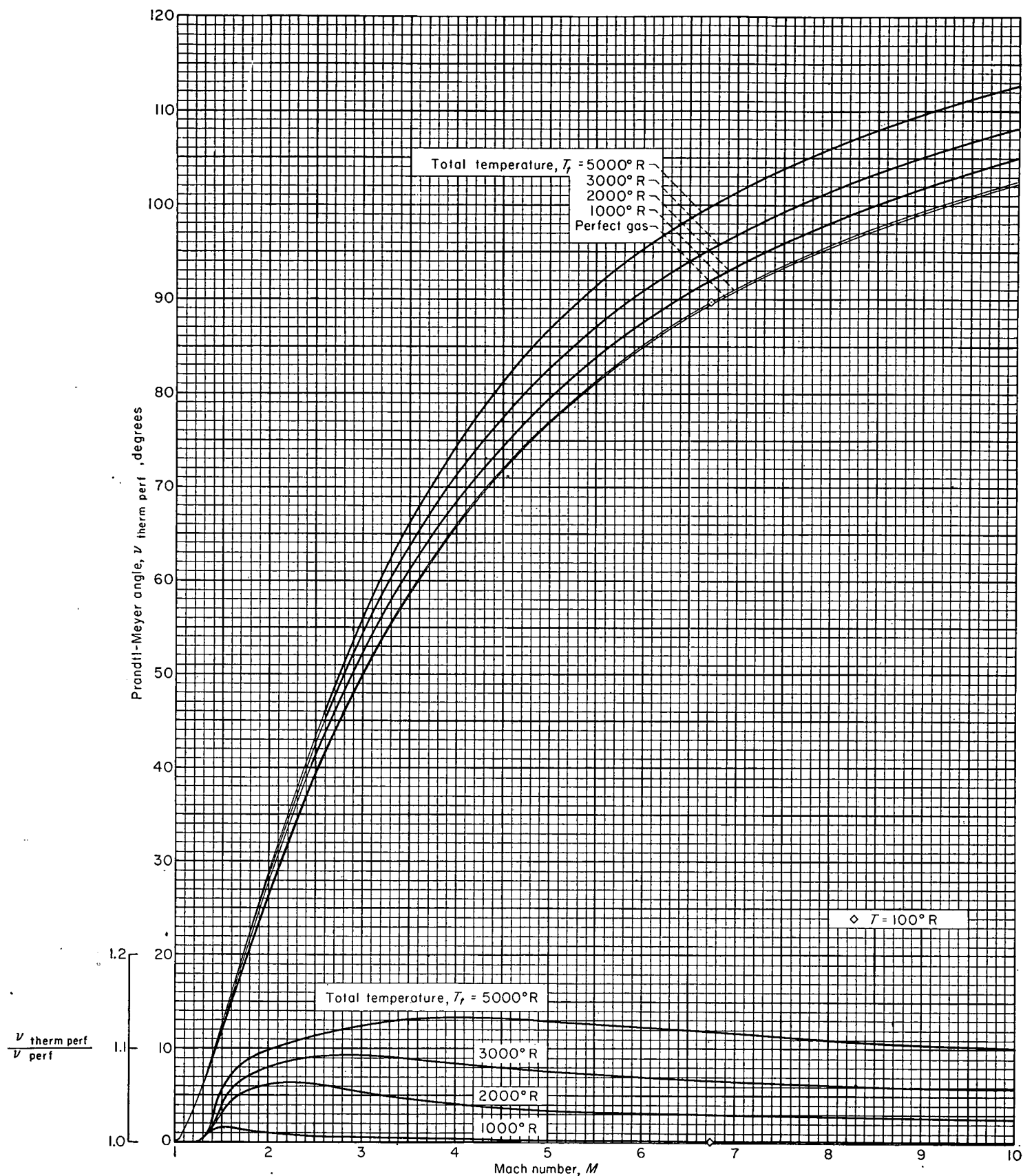


CHART 24.—Effect of caloric imperfections on the Prandtl-Meyer angle.

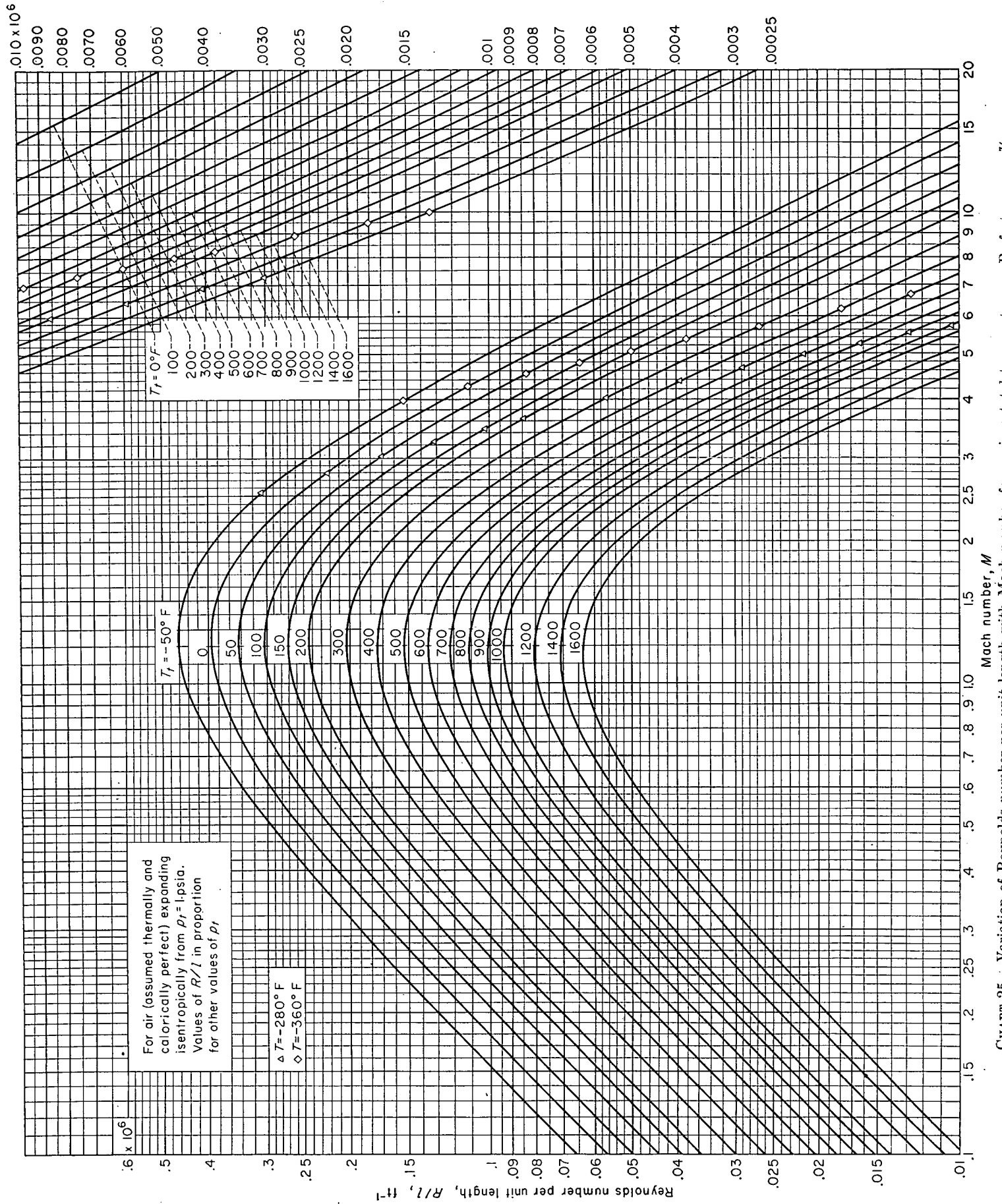


CHART 25.—Variation of Reynolds number per unit length with Mach number for various total temperatures. Perfect gas, $\gamma = 1.4$.

REPORT 1136

ESTIMATION OF THE MAXIMUM ANGLE OF SIDESLIP FOR DETERMINATION OF VERTICAL-TAIL LOADS IN ROLLING MANEUVERS ¹

By RALPH W. STONE, JR.

SUMMARY

Recent experiences have indicated that angles of sideslip in rolling maneuvers may be critical in the design of vertical tails for current research airplanes having weight distributed mainly along the fuselage. Previous investigations have indicated the seriousness of the problem for the World War II type of airplane. Some preliminary calculations for airplanes of current design, particularly with weight distributed primarily along the fuselage, are made herein.

The results of this study indicate that existing simplified expressions for calculating maximum sideslip angles to determine the vertical-tail loads in rolling maneuvers are not generally applicable to airplanes of current design. A general solution of the three linearized lateral equations of motion, including product-of-inertia terms, will usually indicate with sufficient accuracy the sideslip angles expected in aileron rolls from trimmed flight. In rolling pullouts, however, where the pitching velocity is large, consideration of cross-couple inertia terms in the equations of motion is necessary to obtain the sideslip angles accurately. The inclusion of the equation of the pitching motion seems desirable along with the lateral equations of motion in order to obtain the influence of pitching in the cross-couple inertia terms of the lateral equations. Pitching oscillations started during rolling maneuvers will be influenced by cross-couple inertia moments in pitch and may cause large variations in angle of attack which affect the horizontal-tail loads.

INTRODUCTION

Large angles of sideslip and resultant large vertical-tail loads have been encountered in a flight of a high-speed swept-wing research airplane and with models of two designs flown by the Langley Pilotless Aircraft Research Division. All three configurations rolled abruptly while pitching up. In the flight of one model, the vertical tail, which was designed by conventional methods, was lost during the rolling maneuver. The motion for all flights appeared to be essentially a rolling about the *X* body axis while at high angles of attack. The airplane and both models were representative of airplane configurations with weight distributed mainly

along the fuselage such that the moments of inertia in pitch and yaw were much larger than the moment of inertia in roll. Thus, with regard to inertia, the airplane and models were much more prone to rolling than to yawing or pitching. The maneuvers mentioned were apparently uncontrolled and were possibly the result of the stall of one wing before the other, but the rates of roll were not abnormally high. From general considerations the existing techniques for determining critical design vertical-tail loads seem to be somewhat inadequate for some current airplane designs and mass distributions.

The problem of determining critical vertical-tail loads in rolling maneuvers had been recognized for conventional airplanes of the past decade (refs. 1 to 3). These investigations indicated that vertical-tail loads can be calculated with sufficient accuracy provided the sideslip angle, rudder deflection, and dynamic pressure are known. The investigations also presented simplified expressions for estimating the maximum sideslip angle in rolling maneuvers. The results determined by the simplified expression of reference 3 were in close agreement with results found by more rigorous expressions and with flight results for airplanes of World War II type flying in that period (1946). Examination of the simplified expressions indicates that certain assumptions and limitations were made regarding the values of some aerodynamic derivatives and ranges of mass distributions considered. Subsequent studies have indicated that these assumptions generally are not applicable for current airplane designs similar to existing research airplanes.

The vertical-tail load in a sideslip is proportional to the value of the sideslip angle and, as has been assumed in the previous work, is assumed to be a criterion for vertical-tail design. The purpose of this report is to present the results of preliminary estimations of the sideslip angle in rolling maneuvers for which the assumptions and limitations used in the simplified expressions of previous work are not included. Some preliminary estimations are also included to determine whether limitation of the motion in a rolling maneuver to the three lateral degrees of freedom, as has previously been the practice, can seriously influence the sideslip estimations.

¹ Supersedes NACA TN 2633, "Estimation of the Maximum Angle of Sideslip for Determination of Vertical-Tail Loads in Rolling Maneuvers" by Ralph W. Stone, Jr., 1952.

COEFFICIENTS AND SYMBOLS

The motions presented herein were calculated about either of two systems of axes, the stability axes and the body axes, the use of either system depending on convenience. A diagram of both systems of axes showing the positive directions of the forces and moments is presented in figure 1. The coefficients and symbols presented may be considered to apply to either system of axes except where noted. Aerodynamic derivatives, normally available relative to stability axes, were transposed by the methods of reference 4 when body axes were used.

C_L	lift coefficient, $\frac{L}{\frac{1}{2} \rho V^2 S}$
C_Y	lateral-force coefficient, $\frac{Y}{\frac{1}{2} \rho V^2 S}$
C_l	rolling-moment coefficient, $\frac{L'}{\frac{1}{2} \rho V^2 S b}$
C_m	pitching-moment coefficient, $\frac{M}{\frac{1}{2} \rho V^2 S \bar{c}}$
C_n	yawing-moment coefficient, $\frac{N}{\frac{1}{2} \rho V^2 S b}$
ΔC_l	increment of rolling-moment coefficient caused by aileron deflection
ΔC_n	increment of yawing-moment coefficient caused by aileron deflection

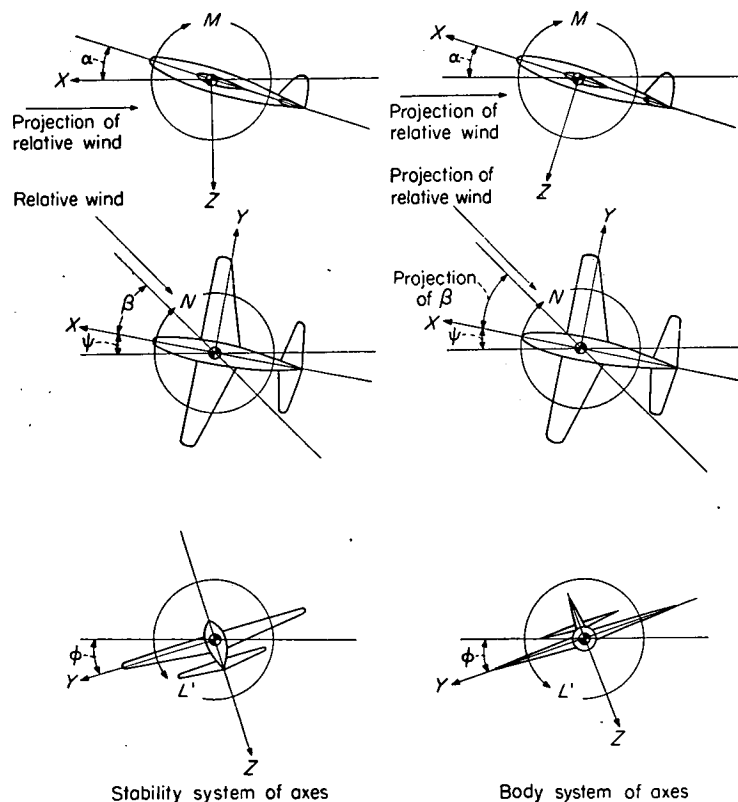


FIGURE 1.—Sketch depicting the stability and body systems of axes. Each view presents a plane of the axes system as viewed along the third axis.

L	lift, lb
Y	lateral force, lb
L'	rolling moment, ft-lb
M	pitching moment, ft-lb
N	yawing moment, ft-lb
S	wing area, sq ft
b	wing span, ft
$\bar{c}$	mean aerodynamic chord, ft
ρ	air density, slugs/cu ft
V	velocity, fps
$I_{x_0}, I_{y_0}, I_{z_0}$	moments of inertia about X , Y , and Z principal axes, respectively, slug-ft ²
I_x, I_y, I_z	moments of inertia about X , Y , and Z stability axes, respectively, slug-ft ²
I_{xz}	product of inertia (negative when principal axis is inclined above the flight path), slug-ft ²
μ_b	relative density coefficient based on span, $m/\rho S b$
m	mass of airplane, W/g , slugs
W	weight of airplane, lb
W_Y	component of weight along Y -axis
g	acceleration due to gravity, 32.2 ft/sec ²
n	normal acceleration divided by acceleration due to gravity
α	angle of attack (assumed to be equal to $\tan^{-1}(\frac{w}{u})$ in the body system of axes), radians except when otherwise noted
$\Delta \alpha$	increment of angle of attack from trimmed condition
β	angle of sideslip, $\sin^{-1} \frac{v}{V}$, radians except when otherwise noted
u, v, w	components of velocity V along the X , Y , and Z body axes, respectively; v is also component of V along Y stability axis, fps
$\dot{v}$	rate of change of v with time
θ	angle of pitch, radians except when otherwise noted
ψ	angle of yaw, radians except when otherwise noted
ϕ	angle of roll, radians except when otherwise noted
$\dot{\phi}$ or p	rolling angular velocity, radians per second except when otherwise noted
$\dot{\theta}$ or q	pitching angular velocity, radians per second except when otherwise noted
$\dot{\psi}$ or r	yawing angular velocity, radians per second except when otherwise noted
$\dot{\beta}$	rate of change of angle of sideslip with time
$\ddot{\phi}$	rate of change of rolling angular velocity with time
$\ddot{\psi}$	rate of change of yawing angular velocity with time
$\ddot{\theta}$	rate of change of pitching angular velocity with time

$$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

$$C_{l_p} = \frac{\partial C_l}{\partial \frac{pb}{2V}}$$

$$C_{n_p} = \frac{\partial C_n}{\partial \frac{pb}{2V}}$$

$$C_{l_r} = \frac{\partial C_l}{\partial \frac{rb}{2V}}$$

$$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$$

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}$$

$$C_{m_q} = \frac{\partial C_m}{\partial \frac{qc}{2V}}$$

GENERAL CONSIDERATIONS

Consideration has been given to various existing methods for calculating angles of sideslip in rolling maneuvers. Limitations in these procedures which may critically influence the sideslip angles for airplanes of current design have been investigated and are discussed briefly herein.

Simplified expressions, as has previously been noted, are presented in previous studies for the determination of the maximum sideslip angle in rolling maneuvers. The simplified expression of reference 1 gives the sideslip angle necessary to balance combined yawing moments caused by the ailerons (ΔC_n) and by rolling (C_{n_p}). The expression of reference 3 is the same as that of reference 1 but with an analytic empirical factor of 2 and is

$$\beta_{max} = \frac{1}{4} \frac{\Delta C_l}{C_{l_p}} \frac{C_L}{C_{n_\beta}} \quad (1)$$

Stability studies subsequent to the study of reference 3, such as references 5 and 6, have indicated that the assumptions used in this simplified expression are not applicable for some current configurations. Also, the range of mass parameters used to evaluate this expression was limited; the ratio of values of I_z/I_x studied in reference 3 was from $1\frac{1}{3}$ to $3\frac{1}{3}$, whereas some current high-speed airplane designs have ratios of the order of 5 to 12. The effect of these differences for current airplane designs should be evaluated to justify any further use of the simplified expression (eq. (1)).

A more rigorous expression used to set up design charts for the maximum sideslip angle in rolling maneuvers is also presented in reference 3. This expression is based on the linearized lateral equations of motion in which the product-of-inertia terms have been omitted. With the advent of the fuselage-heavy loadings the product-of-inertia terms will be large, particularly at high angles of attack, and may influence the motion and the maximum angle of sideslip in rolling maneuvers. An evaluation of the effects of products of inertia on the maximum sideslip angles estimated in rolling maneuvers for airplanes of current designs therefore should be made.

Another point for consideration in estimating sideslip angles in rolling maneuvers, particularly when rolls occur in pull-ups, may be cross-couple inertia moments which exist when both lateral and longitudinal motions occur together. The effects of cross-couple inertia moments may be particularly important when the weight is distributed primarily along the fuselage. It is believed therefore that the effects of these cross-couple inertia moments on the sideslip angle in rolling maneuvers also should be evaluated.

METHOD OF ANALYSIS

In order to evaluate the effects of the various parameters and changes in parameters which may influence estimates of sideslip angles in rolling maneuvers of current airplane configurations, a preliminary study of rolling maneuvers of two airplane configurations was made. Calculations were made of the variation of sideslip angle with time in rolling maneuvers by an analytic solution of the linearized lateral equations of motion, both with and without product-of-inertia terms, and by a step-by-step integration of equations of motion that are more complete than the linearized lateral equations of motion. The maximum angle of sideslip was also calculated by use of the simplified expression of reference 3 (eq. (1)).

The three linearized lateral equations of motion used, with product-of-inertia terms included, are

$$\left. \begin{aligned} I_x \ddot{\phi} - I_{xz} \ddot{\psi} - \left(C_{l_\beta} \beta + C_{l_p} \frac{\dot{\phi} b}{2V} + C_{l_r} \frac{\dot{\psi} b}{2V} + \Delta C_l \right) \frac{1}{2} \rho V^2 S b &= 0 \\ I_z \ddot{\psi} - I_{xz} \ddot{\phi} - \left(C_{n_\beta} \beta + C_{n_p} \frac{\dot{\phi} b}{2V} + C_{n_r} \frac{\dot{\psi} b}{2V} + \Delta C_n \right) \frac{1}{2} \rho V^2 S b &= 0 \\ m V (\dot{\beta} + \dot{\psi}) - \left(\frac{C_L}{n} \phi + C_{Y_\beta} \beta \right) \frac{1}{2} \rho V^2 S &= 0 \end{aligned} \right\} \quad (2)$$

Solution of these equations was made relative to the stability system of axes.

In order to evaluate the effects of cross-couple inertia terms on rolling and yawing motions, the pitching velocity must be included; therefore a fourth degree of freedom is necessary, the pitching degree of freedom. It was assumed that changes in accelerations along the X- and Z-axes would not be sufficient in the time necessary to roll 90° to influence the resultant motion greatly. Because the cross-couple inertia terms are nonlinear, an analytic solution was not

possible and a step-by-step integration was made. The step-by-step method used was Euler's method, briefly outlined in reference 7. Euler's dynamical equations for the four degrees of freedom are

$$\left. \begin{aligned} L' &= I_{x_0} \ddot{\phi} - (I_{y_0} - I_{z_0}) \dot{\theta} \dot{\psi} \\ M &= I_{y_0} \ddot{\theta} - (I_{z_0} - I_{x_0}) \dot{\phi} \dot{\psi} \\ N &= I_{z_0} \ddot{\psi} - (I_{x_0} - I_{y_0}) \dot{\phi} \dot{\theta} \\ Y &= m(\dot{v} + u\dot{\psi} - w\dot{\phi}) - W_Y \end{aligned} \right\} \quad (3)$$

Solution of these equations was made relative to the body axes (assumed to be the principal axes); therefore product-of-inertia terms do not appear. These equations as used in the step-by-step integration were written as follows:

$$I_{x_0} \ddot{\phi} - (I_{y_0} - I_{z_0}) \dot{\theta} \dot{\psi} - \left(C_{l_p} \beta + C_{l_p} \frac{\dot{\phi} b}{2V} + C_{l_r} \frac{\dot{\psi} b}{2V} + \Delta C_l \right) \frac{1}{2} \rho V^2 S b = 0 \quad (4a)$$

$$I_{y_0} \ddot{\theta} - (I_{z_0} - I_{x_0}) \dot{\phi} \dot{\psi} - \left(C_{m_a} \Delta \alpha + C_{m_q} \frac{\dot{\theta} \bar{c}}{2V} \right) \frac{1}{2} \rho V^2 S \bar{c} = 0 \quad (4b)$$

$$I_{z_0} \ddot{\psi} - (I_{x_0} - I_{y_0}) \dot{\phi} \dot{\theta} - \left(C_{n_p} \beta + C_{n_p} \frac{\dot{\phi} b}{2V} + C_{n_r} \frac{\dot{\psi} b}{2V} + \Delta C_n \right) \frac{1}{2} \rho V^2 S b = 0 \quad (4c)$$

$$mV(\cos \beta \dot{\beta} + \dot{\psi} \cos \alpha - \dot{\phi} \sin \alpha) - W \sin(\phi \cos \alpha + \psi \sin \alpha) - C_{Y\beta} \beta \frac{1}{2} \rho V^2 S = 0 \quad (4d)$$

The expression for the weight component in the side-force equation (4d) is approximate but is considered to be sufficiently accurate provided the angle of attack does not become excessively large or have large variations.

In order to evaluate the angle of attack for use in equations (4), an additional equation was used whereby the angle of attack was estimated for each step of the calculations.

For all calculations made the maneuver was considered to be initiated by an abrupt aileron deflection, the rudder being held fixed. The aileron deflection was considered to be constant throughout the maneuver. The motion was presumed to take place approximately in a horizontal plane for all calculations. These assumptions are conservative in that they result in somewhat larger estimated sideslip angles than those that would be obtained in actual flight where a finite time is required to reach maximum aileron deflections or where the motion is not in a horizontal plane, as may be true particularly for a roll in a pull-up. For the case of the pull-up maneuver the assumption was made that the initial pitching velocity had no influence on changes in the angle of attack but that only additional pitching velocities affected this angle.

CALCULATIONS

The sideslip angles were calculated by each of the various methods for two airplanes having different stability derivatives. The aerodynamic and physical parameters for the two airplanes are listed in table I. The aerodynamic parameters and stability derivatives for airplane A (table I) are

those which might be representative of an airplane having a long fuselage and a short-span thin wing. It should be noted that the stability derivatives C_{n_p} and C_{l_p} are relatively large. The aerodynamic characteristics for airplane B (table I) were taken from configuration 1 of reference 3; the necessary pitching derivatives were assumed. For this case the values of C_{n_p} and C_{l_p} were considerably smaller than those of airplane A. In order to make computations of sideslip for a condition similar to one for configuration 1 of reference 3, the value of ΔC_l of airplane B was taken to make $\frac{\Delta C_l}{C_{l_p}} \frac{C_L}{C_{n_p}}$ equal to 120 and the value of ΔC_n was taken as equal to $\frac{\Delta C_l}{C_{l_p}} \frac{C_L}{16}$. The value of C_{n_p} used for these calculations was assumed to be constant throughout the rolling motion. The effect of aileron deflection on the rotary wash of the wing at the tail was not included in the value of C_{n_p} used. For specific cases, however, consideration of this effect should be made. In addition, because the motions considered herein have accelerations in roll, a lag in the rotary wash of the wing at the tail will exist and will affect the value of the moment $C_{n_p} \frac{pb}{2V}$ acting at any given instant. Consideration of this effect should be made in a specific case. A discussion of the rotary wash of the wing on the tail is given in reference 6.

The mass characteristics used for the calculations on each of the two airplanes are listed in table II. Two loadings

TABLE I.—AERODYNAMIC AND DIMENSIONAL CHARACTERISTICS

[Aerodynamic characteristics are referred to stability axes]

	Airplane A	Airplane B
Wing area, sq ft.....	166.5	248
Wing span, ft.....	22.7	38.3
Mean aerodynamic chord, ft.....	7.84	6.87
C_{l_p} , per degree.....	-0.0032	-0.0010
C_{n_p} , per degree.....	0.0065	0.00040
$C_{Y\beta}$, per degree.....	-0.015	-0.0075
C_{l_p} , per radian.....	-0.225	-0.455
C_{n_p} , per radian.....	-0.130	-0.044
C_{l_r} , per radian.....	0.235	0.198
C_{n_r} , per radian.....	-1.090	-0.0669
C_{m_q} , per radian.....	-9.000	-9.000
C_{m_a} , per degree.....	-0.0167	-0.0174
ΔC_l	0.0197	0.0242
ΔC_n	-0.0035	-0.00299

TABLE II.—MASS CHARACTERISTICS

Loading	Weight, lb	Airplane relative density coefficient, μ_b	Moments of inertia, slug-ft ²		
			I_{x_0}	I_{y_0}	I_{z_0}
Airplane A					
1	20, 828	71. 9	5, 381	63, 971	65, 559
2	20, 828	71. 9	34, 676	34, 676	65, 559
Airplane B					
1	21, 800	30	14, 900	26, 000	39, 750
2	21, 800	30	5, 381	63, 971	65, 559

were used for each airplane. For airplane A loading 1 was such that $I_z \approx 12I_x$ and $I_y - I_x \approx 11I_x$ and loading 2 was such that $I_z \approx 2I_x$ and $I_x - I_y = 0$. Loading 1 for airplane B was taken for configuration 1 of reference 3 for $\mu_b = 30$; a pitching moment of inertia was assumed. Loading 2 for airplane B was similar to loading 1 of airplane A.

Estimations of the sideslip angles in rolling maneuvers were made for each of the various methods for the flight conditions of table III.

The calculations were based on sea-level air density ρ of 0.002378. The motion for each flight condition was calculated by each of the various methods through approximately 90° of roll. For the step-by-step solutions the aerodynamic derivatives, referred to the body axes, were assumed to be constant and thus independent of angle of attack for the range of angle of attack obtained. For the analytic solution of the lateral equations of motion the angle of attack is assumed to be constant and thus, of course, the aerodynamic derivatives are also constant.

The results for the step-by-step solutions are presented in figures 2 to 6 for the various conditions listed in table III. Shown in figures 2 to 6 are the variations of angular displacement and angular velocity about the three body axes with time, as well as the variation of the angle of sideslip and the angle of attack with time. The results for the analytic solutions of the three lateral equations of motion both with and without the effects of products of inertia of the variation of the sideslip angle with time for the various conditions listed in table III are presented in figures 7 to 11. The variation of sideslip angle with time for the step-by-step solutions is also presented in figures 7 to 11 for comparative purposes.

In the step-by-step procedure, an increment of time is used which in general should be relatively small. A sufficiently small increment of time should be chosen so as to obtain the proper result. In general, large time increments tend to indicate a less stable motion and thus may indicate larger maximum values of sideslip than may actually exist. If the motions tend to be irregular, smaller increments of time may be necessary than when the variations of the motion are small. As an example of the effect of different time increments, the trimmed flight solution for airplane A, loading 1, was briefly studied for three time increments; the effects on the sideslip angle are shown in figure 12. For the step-by-step calculations presented herein, brief studies were made of the effects of time increments and sufficiently small values were used so that the maximum sideslip angle may be considered to be accurate within $\frac{1^\circ}{2}$.

TABLE III.—FLIGHT CONDITIONS FOR CALCULATIONS

Airplane	Loading	Flight condition	Initial angle of attack, deg	C_L	V , fps
A	1	Aileron roll from trimmed flight	10	0.6	419
A	2	Aileron roll from trimmed flight	10	.6	419
A	1	6g pullout	13	.78	900
B	1	Aileron roll from trimmed flight	12	.9	286
B	2	Aileron roll from trimmed flight	12	.9	286

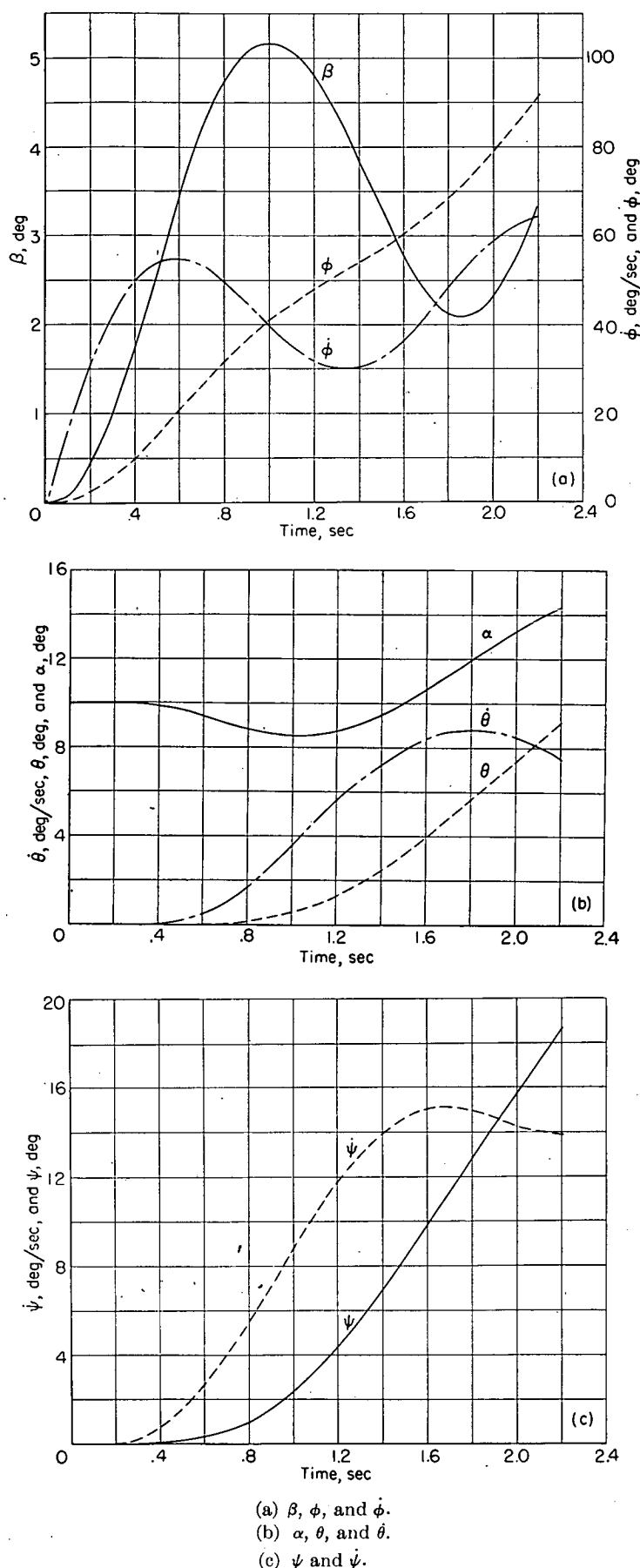
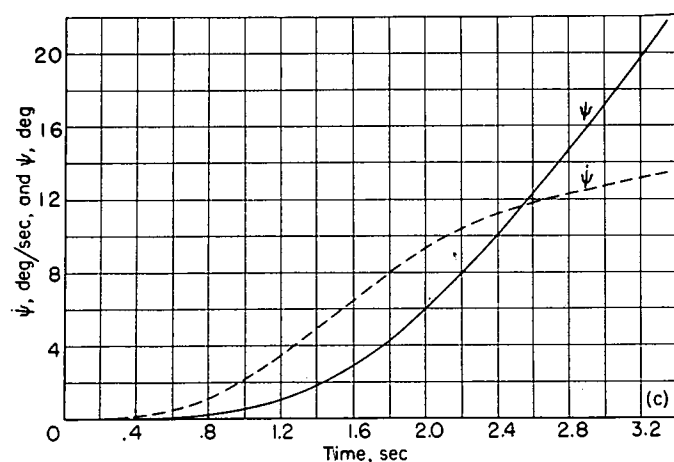
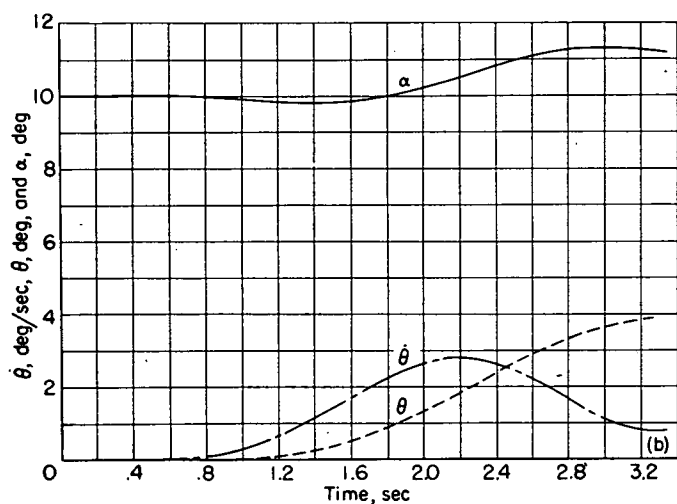
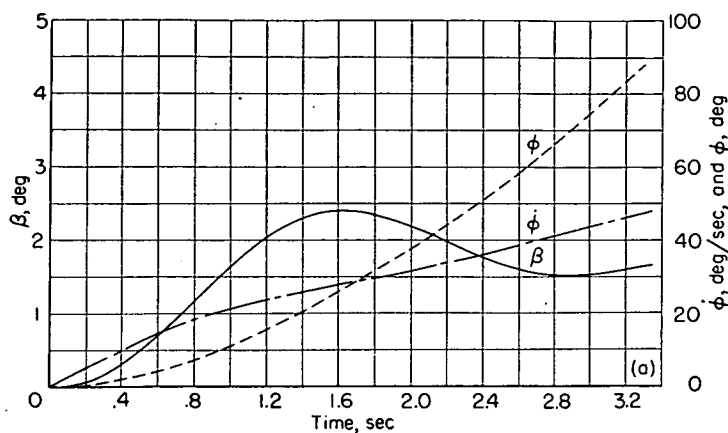
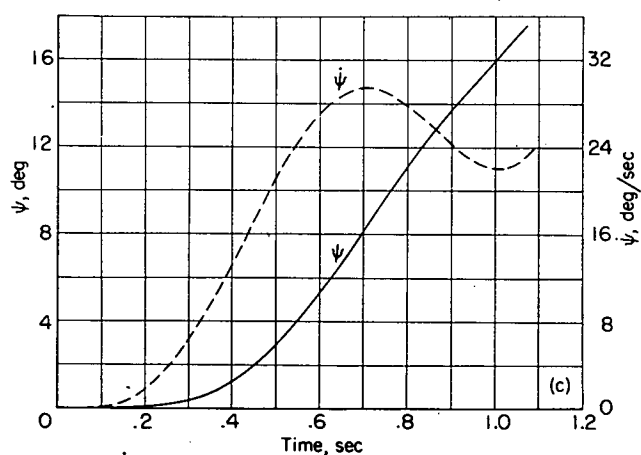
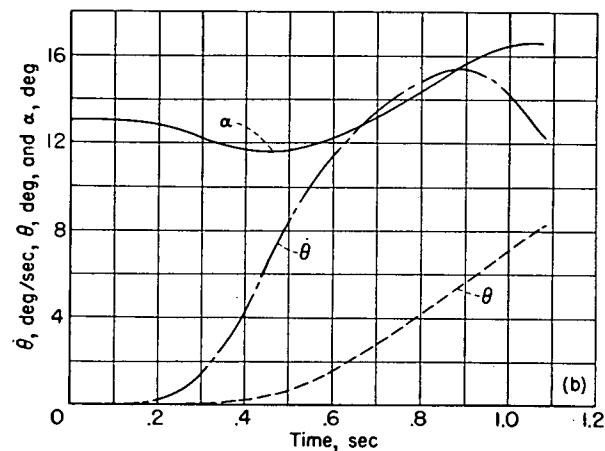
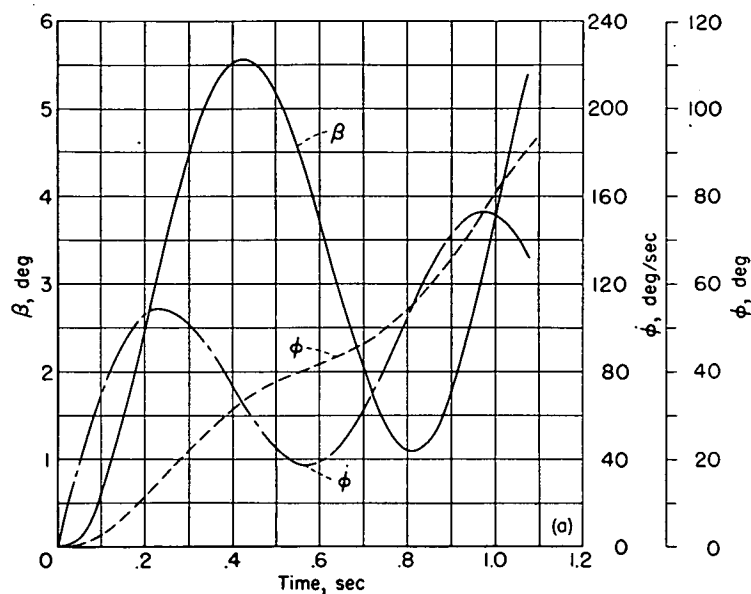


FIGURE 2.—Motion of airplane A, heavily loaded along the fuselage (loading 1), when rolled from trimmed flight at an angle of attack of 10° . Step-by-step method.



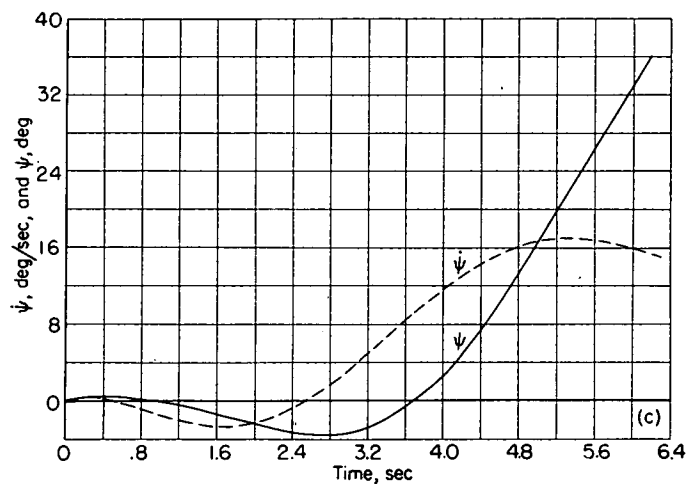
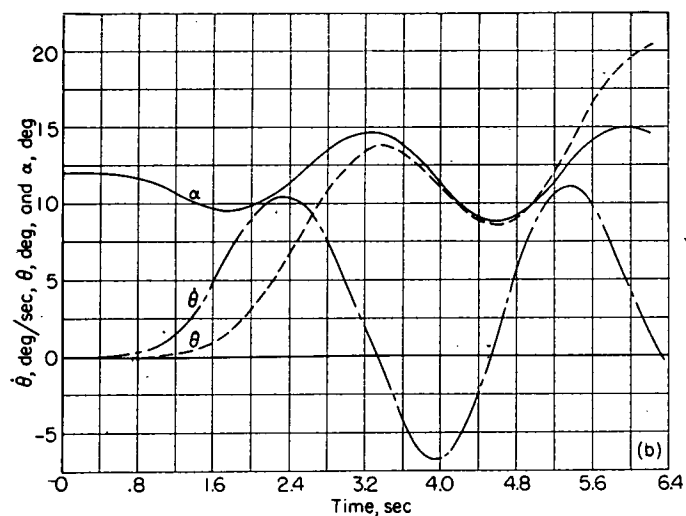
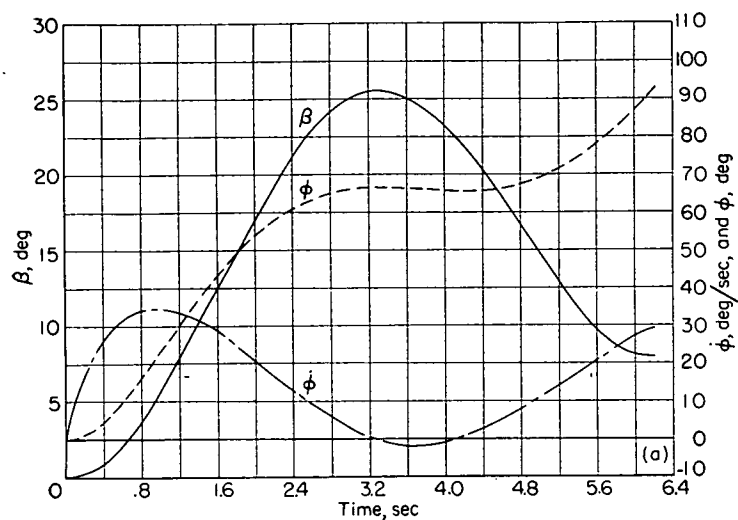
(a) β , ϕ , and $\dot{\phi}$.
 (b) α , θ , and $\dot{\theta}$.
 (c) ψ and $\dot{\psi}$.

FIGURE 3.—Motion of airplane A, equally loaded along the wings and the fuselage (loading 2), when rolled from trimmed flight at an angle of attack of 10° . Step-by-step method.



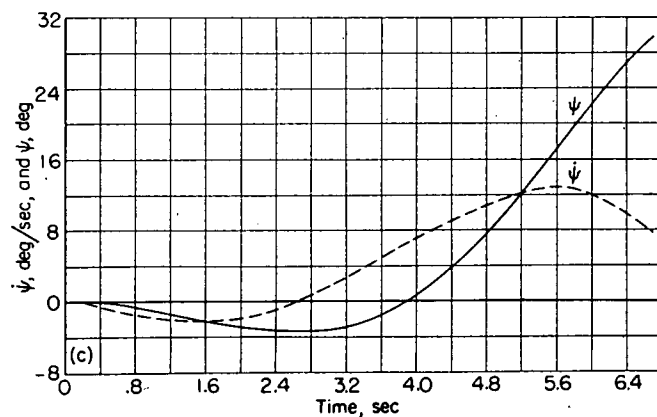
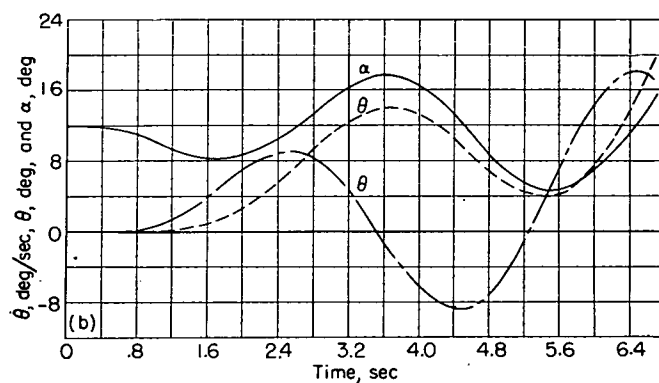
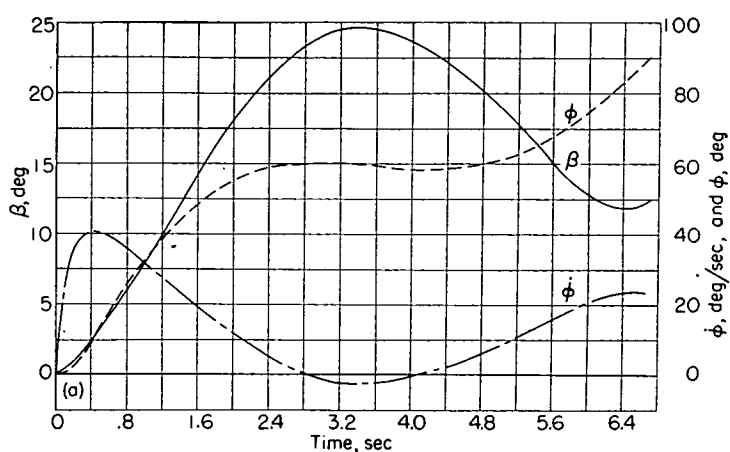
(a) β , ϕ , and $\dot{\phi}$.
 (b) α , θ , and $\dot{\theta}$, where $\dot{\theta}$ is an increment added to the initial pitching velocity.
 (c) ψ and $\dot{\psi}$.

FIGURE 4.—Motion of airplane A, heavily loaded along the fuselage (loading 1), when rolled in a $6g$ pull-out. Step-by-step method.



(a) β , ϕ , and $\dot{\phi}$.
(b) α , θ , and $\dot{\theta}$.
(c) ψ and $\dot{\psi}$.

FIGURE 5.—Motion of airplane B, normal mass distribution (loading 1), when rolled from trimmed flight at an angle of attack of 12° . Step-by-step method.



(a) β , ϕ , and $\dot{\phi}$.
(b) α , θ , and $\dot{\theta}$.
(c) ψ and $\dot{\psi}$.

FIGURE 6.—Motion of airplane B, heavily loaded along the fuselage (loading 2), when rolled from trimmed flight at an angle of attack of 12° . Step-by-step method.

RESULTS AND DISCUSSION

AIRPLANE A

The results for airplane A with the mass distributed mostly along the fuselage (loading 1) show that by the step-by-step method a maximum sideslip of about $5\frac{1}{2}^\circ$ was obtained (fig. 2). The solution of the linearized lateral equations of motion, including product-of-inertia terms, gave a maximum sideslip angle of about $4\frac{3}{4}^\circ$; whereas the solution of the linearized lateral equations without product-of-inertia terms gave a maximum sideslip angle of $4\frac{1}{2}^\circ$ (fig. 7). Solution of the simplified expression of reference 3 (eq. (1)) gives a result for the maximum sideslip angle of only 2° .

The results for airplane A for the second loading where I_x and I_y were equal and each approximately one-half of I_z show a maximum angle of sideslip of about $2\frac{1}{2}^\circ$ by the step-by-step calculations (fig. 3). The solution of the linearized lateral equation of motion, including product-of-inertia terms, gave a maximum angle of sideslip of about $2\frac{1}{4}^\circ$ and the solution without product-of-inertia terms gave

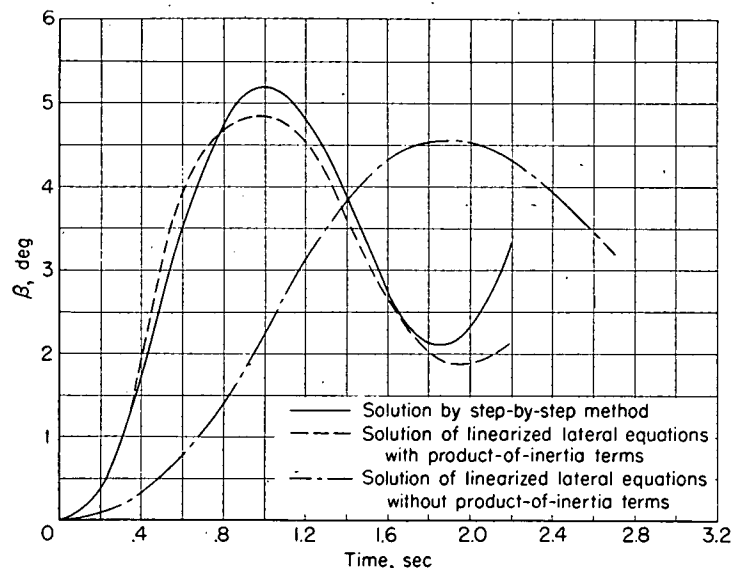


FIGURE 7.—Comparison of a solution by step-by-step method and a solution of linearized lateral equations of the variation of angle of sideslip with time for airplane A, loading 1, when rolled from trimmed flight at an angle of attack of 10° .

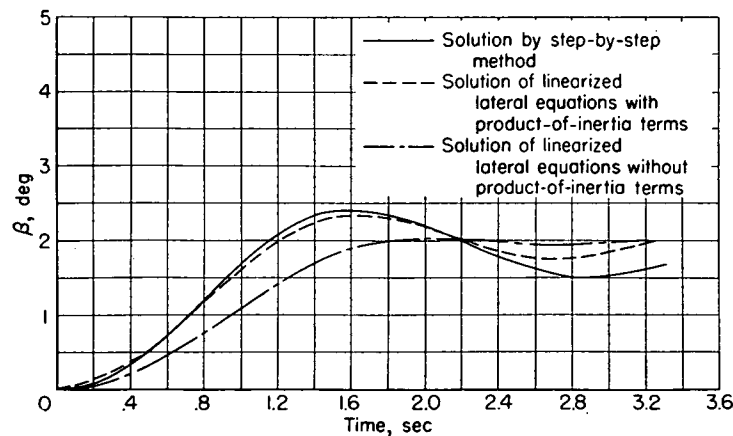


FIGURE 8.—Comparison of a solution by step-by-step method and a solution of linearized lateral equations of the variation of angle of sideslip with time for airplane A, loading 2, when rolled from trimmed flight at an angle of attack of 10° .

a maximum sideslip angle of approximately 2° (fig. 8). The simplified expression of reference 3, equation (1), gives a value of maximum sideslip angle of 2° .

The results for the 6g pull-up for airplane A, loading 1, (fig. 4) are for a relatively high velocity of 900 feet per second (Mach number of approximately 0.83). (The stability derivatives in table I for airplane A were used without consideration of any compressibility effects.) A maximum angle of sideslip of approximately $5\frac{1}{2}^\circ$ was obtained by the step-by-step method. The linearized lateral equations of motion, including product-of-inertia terms, gave a maximum angle of sideslip of about $4\frac{1}{2}^\circ$ (fig. 9); whereas the linearized lateral equations excluding product-of-inertia terms gave a maximum sideslip angle of only $2\frac{1}{2}^\circ$. The simplified expression of reference 3 (eq. (1)) gives a maximum

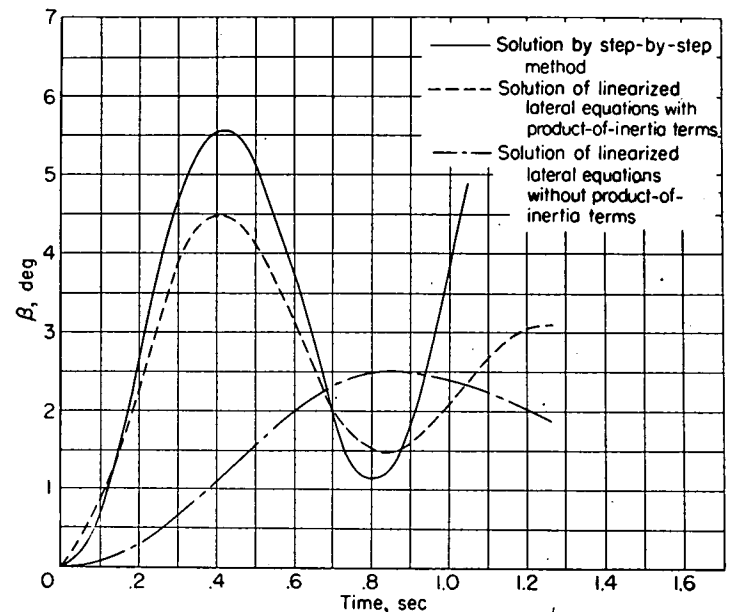


FIGURE 9.—Comparison of a solution by step-by-step method and a solution of linearized lateral equations of the variation of angle of sideslip with time for airplane A, loading 1, when rolled in a 6g pullout.

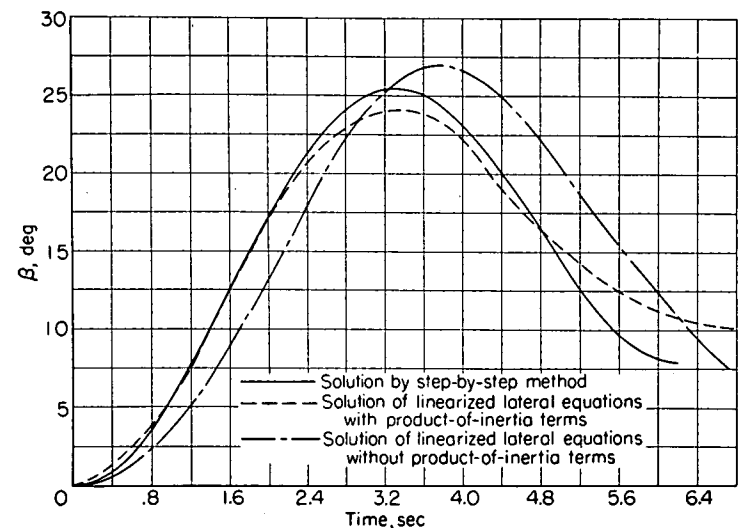


FIGURE 10.—Comparison of a solution by step-by-step method and a solution of linearized lateral equations of the variation of angle of sideslip with time for airplane B, loading 1, when rolled from trimmed flight at an angle of attack of 12° .

value of sideslip angle for this case of about $2\frac{1}{2}^\circ$. It may be of interest to note that the value of maximum sideslip angle obtained for the rolling pull-up by the more complete methods was somewhat in excess of the maximum values for which similar recent research airplanes have been designed.

A summarization of these results for the maximum angles of sideslip for each airplane condition and method of calculation is presented in table IV. The values listed are approximate in that they have been rounded to the nearest $\frac{1}{4}^\circ$.

The results of airplane A for aileron rolls in trimmed flight (loadings 1 and 2) by the step-by-step method and by a solution of the linearized lateral equations, including product-of-inertia terms, compare favorably both in the maximum sideslip and its variation with time (figs. 7 and 8). It appears, therefore, that the pitching velocities which are included in the step-by-step solutions did not appreciably influence the sideslip angles through the cross-couple inertia moments in yaw $(I_X - I_Y)\dot{\theta}\dot{\phi}$ and roll $(I_Y - I_Z)\dot{\theta}\dot{\psi}$. The solution of the linearized lateral equations of motion, including product-of-inertia terms, therefore appears to be adequate for estimating the maximum angle of sideslip when the pitching motion is relatively small.

For the case of the 6g pull-up, however, the linearized solution underestimated the maximum sideslip angle obtained by the step-by-step solution by approximately 20 percent and the variation of sideslip angle with time (fig. 9) is somewhat different for the solution of the linearized lateral equations (including product-of-inertia terms) and for the step-by-step solution. The periods of the motion for the two solutions are similar but the damping characteristics appear to be considerably different. This difference is in agreement with reference 8 which indicates that in steady rolling cross-couple inertia moments cause changes in stability when the directional and longitudinal stabilities are different. The differences indicate some influence of the

TABLE IV.—SUMMARY OF MAXIMUM SIDESLIP ANGLES OBTAINED IN ROLLING MANEUVERS FOR THE VARIOUS CONDITIONS CONSIDERED AND BY THE VARIOUS METHODS USED

Loading	Flight condition	Sideslip angle, deg, obtained by—			
		Step-by-step solution	Solution of linearized lateral equations with product-of-inertia terms	Solution of linearized lateral equations without product-of-inertia terms	Solution of simplified expression (eq. (1))
Airplane A					
1	Aileron roll from trimmed level flight	5½	4½	4½	2
2	Aileron roll from trimmed level flight	2½	2½	2	2
1	Roll from 6g pullout	5½	4½	2½	2½
Airplane B					
1	Aileron roll from trimmed level flight	25½	24	27	30
2	Aileron roll from trimmed level flight	24½	23½	30	30

pitching motion on the lateral motion through the cross-couple inertia moments resulting from the pitching, these moments being $(I_X - I_Y)\dot{\phi}\dot{\theta}$ and $(I_Y - I_Z)\dot{\psi}\dot{\theta}$ for yawing and rolling, respectively. This effect for the pull-up case appears to be the result of the much larger pitching velocity associated with the pullout maneuver than existed for the aileron rolls from trimmed flight where the agreement by the two solutions was good both for the maximum sideslip as well as its variation with time (figs. 7 and 8). It appears therefore that, for rolls in pullouts when the pitching velocity is large, sideslip angles should be calculated by the more complete step-by-step method of the nonlinear equations of motion which include cross-couple inertia terms.

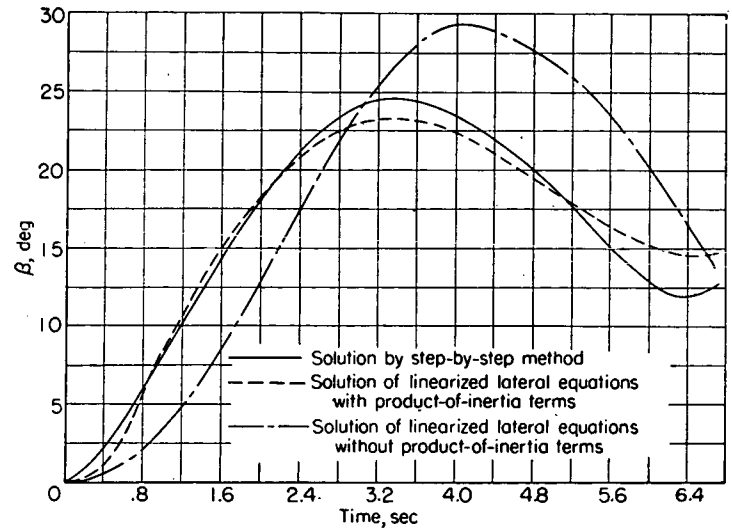


FIGURE 11.—Comparison of a solution by step-by-step method and a solution of linearized lateral equations of the variation of angle of sideslip with time for airplane B, loading 2, when rolled from trimmed flight at an angle of attack of 12° .

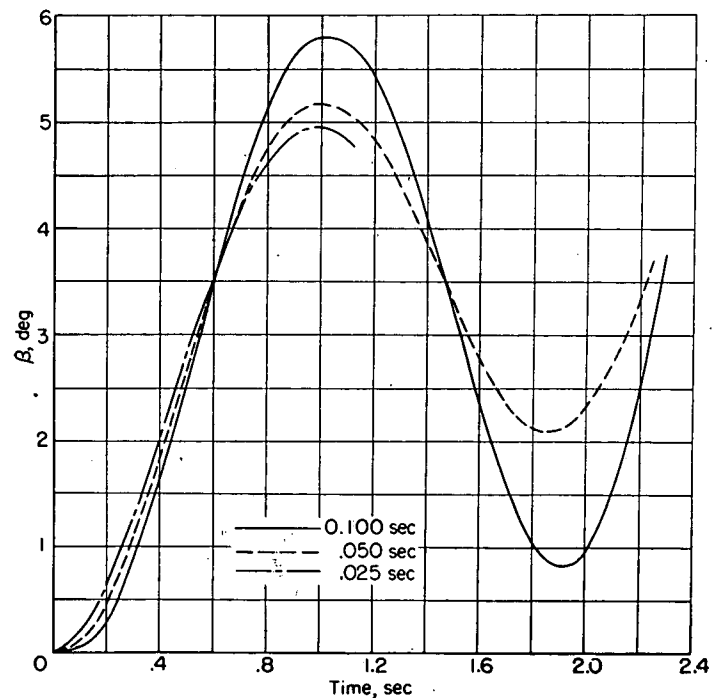


FIGURE 12.—Effects of different time increments on the variation with time and the maximum value of sideslip angle for airplane A, loading 1, when rolling from trimmed flight at an angle of attack of 10° . Step-by-step method.

The rate of roll $\dot{\phi}$ for airplane A, loading 1, both for the aileron roll from trimmed flight and the $6g$ pullout (figs. 2 and 4) varied considerably during part of the motion, apparently being dependent upon the sideslip angle and upon C_{l_p} (the dihedral effect). A maximum value of $pb/2V$ of about 0.03 was obtained for both cases; whereas the value of $\Delta C_l/C_{l_p}$, the steady-state value of $pb/2V$ for the rolling degree of freedom, would be about 0.088 for both cases. For loading 2 (where the weight is more heavily distributed along the wing than for loading 1) the airplane rolled slower than for the first loading and the maximum value of $pb/2V$ attained was about 0.023 as compared again with the value of $\Delta C_l/C_{l_p}$ of 0.088. Since the simplified expression of reference 3 (eq. (1)) is based on a substitution of $\Delta C_l/C_{l_p}$ for $pb/2V$, agreement in these parameters appears essential to the valid use of equation (1). For airplane A these parameters were appreciably different. For loading 1 either for the aileron roll from trimmed flight or the $6g$ pullout, the simplified expression appreciably underestimated the value of maximum sideslip angle. Because of differences in the values of $\Delta C_l/C_{l_p}$ and $pb/2V$ the agreement indicated by the simplified expression and other methods of calculation for airplane A, loading 2 (aileron roll from trimmed flight), appears to be only coincidental. It appears, therefore, that the simplified expression is not generally applicable for airplanes of the type of airplane A.

The effects of products of inertia on the maximum angle of sideslip and its variation with time are shown by the comparison of the solutions of the lateral equations both with and without products of inertia in figures 7 to 9 for the various conditions calculated for airplane A. Solutions without the product-of-inertia terms underestimated the maximum sideslip angle as obtained by the more complete methods of calculation. The effect of the products of inertia on the sideslip angle appears to be primarily an effect on the yawing moment, where the component of moment is $\dot{\phi}I_{xz}$. In general this effect (for negative values of I_{xz} as exist for the cases considered herein) is to increase the maximum sideslip angle, provided $\dot{\phi}$ is positive prior to the time the maximum sideslip angle is reached. It is also of interest to note that products of inertia had an appreciable influence on the period of the motion as well as the damping of the motion for this airplane as calculated by the linearized lateral equations, both the period and time to damp of the oscillations being shortened by the products of inertia (table V).

As has previously been indicated, the pitching motion and the resulting cross-couple inertia moments do not appreciably influence the motion or the maximum sideslip angle for the aileron rolls from trimmed flight (figs. 7 and 8). The differences between the results for loadings 1 and 2 thus seem to be caused by the differences in the value of I_x , loading 2 leading to a smaller value of maximum sideslip. Because

of the increased value of I_x for the second loading, the airplane rolled slower than for the first loading; hence at a given instant of time $pb/2V$ was smaller and the moment $C_{n_p} \frac{pb}{2V}$ which was in a direction to increase the sideslip angle was smaller than for the first loading. Airplanes which roll fast may therefore tend to encounter larger values of maximum sideslip than those which roll slow.

As an extreme example of the effects of inertia, consider a case of infinite inertia about the Z - and Y -axes and some finite inertia about the X -axis. Rolling about the X body axis would cause maximum sideslip angles equal to the initial angle of attack, and the angle of attack would vary through a range of plus and minus the initial value of angle of attack. It is apparent, therefore, that the initial angle of attack is important to any study made and that the most serious condition would be one of the rolling pullout where the angle of attack is large and the speed is high.

As is implied in the previous discussion, when the rolling motion of an airplane is considered about the body axis, increased roll results in decreased angle of attack. Because the airplane has static stability (and a finite value of I_y), the airplane tended to maintain its original trim angle for airplane A, and a pitching oscillation was started (figs. 2 (b), 3 (b), and 4 (b)). The cross-couple inertia moment in pitch ($I_z - I_x$) $\dot{\phi}\dot{\psi}$ for all cases for airplane A was in a direction to cause the airplane to trim at angle of attack greater than the original trim angle because $\dot{\phi}$ and $\dot{\psi}$ were both always positive, as is $I_z - I_x$.

AIRPLANE B

The results of step-by-step calculations for airplane B, loading 1 (configuration 1 of ref. 3), gave a value of maximum angle of sideslip of approximately $25\frac{1}{2}^\circ$ (fig. 5). The solution

TABLE V.—LATERAL OSCILLATORY STABILITY FOR CONDITIONS CALCULATED

Flight condition	Loading	Product-of-inertia effects included		Product-of-inertia effects excluded	
		Period of oscillation, sec	Time to damp to one-half amplitude, sec	Period of oscillation, sec	Time to damp to one-half amplitude, sec
Airplane A					
Aileron roll from trimmed level flight	1	1.98	1.85	2.83	78.1
Aileron roll from trimmed level flight	2	2.83	3.62	2.95	3.80
Roll from 6g pullout	1	.84	.57	1.34	2.36
Airplane B					
Aileron roll from trimmed level flight	1	6.61	5.52	6.85	42.7
Aileron roll from trimmed level flight	2	8.40	2.89	7.95	22.5

of the linearized lateral equations of motion, including product-of-inertia terms, gave a maximum sideslip angle of about 24° ; whereas the solution of the linearized lateral equations of motion without product-of-inertia terms gave a maximum sideslip angle of about 27° (fig. 10). The simplified expression of reference 3 (eq. (1)) gives a value of approximately 30° .

The results for the second loading for airplane B (fig. 6) indicate that a maximum sideslip angle of about $24\frac{1}{2}^\circ$ was attained by the step-by-step method. The solution of the linearized lateral equations of motion, including product-of-inertia terms, gave a maximum sideslip angle of about $23\frac{1}{4}^\circ$ which compares favorably with the value of $24\frac{1}{2}^\circ$ obtained by the step-by-step method. The variation of sideslip angle with time also compares favorably for the two methods (fig. 11). A solution of the linearized lateral equations, excluding product-of-inertia effects, gave a maximum sideslip angle of about 30° . The simplified expression of reference 3 (eq. (1)) gives a maximum sideslip angle of about 30° for this case.

These results of approximate maximum sideslip angles for airplane B are listed in table IV. As was indicated for airplane A, the values in table IV have been rounded to the nearest $\frac{1}{4}^\circ$.

As was the case for airplane A for aileron rolls from trimmed flight the agreement was good between the maximum sideslip angle and the variation of sideslip angle with time between the step-by-step solution and the solution of the linearized lateral equations including product-of-inertia terms for airplane B (figs. 10 and 11). These results also indicate that the pitching motion and resulting cross-couple inertia terms $(I_x - I_y)\dot{\theta}\dot{\phi}$ and $(I_y - I_z)\dot{\theta}\dot{\psi}$ for rolls from trimmed flight are not sufficient to influence the maximum sideslip angle.

For airplane B the rolling velocity $\dot{\phi}$ varied widely because of the dihedral effect resulting from the large sideslip angles. Maximum values of $pb/2V$ of 0.041 and 0.048 were obtained for loadings 1 and 2, respectively. The value of $\Delta C_l/C_{l_p}$, the steady-state value of $pb/2V$ for the rolling degree of freedom, was 0.053. Substitution of $\Delta C_l/C_{l_p}$ for $pb/2V$ in the simplified expression of reference 3 (eq. (1)) therefore seems to be more nearly accurate for this airplane than it was for airplane A and the agreement of the simplified expression with the more complete solutions may be considered better for this airplane than for airplane A. It is of interest to note, however, that the simplified expression overestimated the maximum sideslip angle for airplane B, whereas it had underestimated the values for airplane A.

For both loadings for airplane B, the solution of the linearized lateral equations, excluding product-of-inertia terms, leads to larger maximum sideslip angles than were obtained when products of inertia were included (figs. 10 and 11). This result occurs primarily because the rolling acceleration was negative for some time prior to the time

the maximum sideslip angle was reached and the moment $I_{xz}\ddot{\phi}$ in the yawing-moment equation was such as to reduce the sideslip angle.

The results for airplane B showed little influence of loading on the maximum sideslip angle (figs. 5 and 6) for aileron rolls from trimmed flight. The differences in loadings 1 and 2 for airplane B were a change in the value of both I_x and I_z (table II). Since the small value of C_{n_β} for airplane B led to large values of sideslip angles, the dihedral effect $C_{l_\beta}\beta$ became a predominant moment in roll and the resultant rates of roll for both loadings for most of the motion were not appreciably different. The moment $C_{n_p} \frac{pb}{2V}$ was, therefore, similar for both loadings, this similarity contributing in part to the agreement in sideslip angles for the two loadings.

For both loadings on airplane B, an oscillation in angle of attack was started about the initial trim angle of attack, the oscillation tending to be divergent (figs. 5(b) and 6(b)). Maximum deviations of $3\frac{1}{2}^\circ$ and $7\frac{1}{2}^\circ$ from the trim angle of attack were obtained for loadings 1 and 2, respectively. For this airplane the cross-couple inertia moment in pitch $(I_z - I_x)\dot{\phi}\dot{\psi}$ changed sign during the motion in that the values of $\dot{\phi}$ and $\dot{\psi}$ changed sign; whereas for airplane A these values were of constant sign. This variation of sign of this moment may have augmented the oscillation in angle of attack. It appears that, if the motion were allowed to progress, larger variations in angle of attack and even negative angles of attack may be encountered. Variations of angle of attack of this type as encountered in rolling maneuvers may be problems for consideration in horizontal-tail designs.

Because accelerations along the Z-axis were not considered, the effect of $\partial C_L/\partial \alpha$ was neglected. The effect of $\partial C_{m_\alpha}/\partial \alpha$ was also neglected, since this derivative was omitted from the pitching equation. Inclusion of these factors would have caused a somewhat more heavily damped pitching oscillation and somewhat smaller variations in angle of attack than are presented. The effects of pitching caused by changes in angle of attack on the sideslip angle, through the cross-couple inertia moments, have, however, been shown to be relatively small; whereas the effect of an initial pitching velocity (which is not influenced by the omissions discussed) as in the 6g pull-up for airplane A does have some influence on the sideslip angles.

COMPARISON OF AIRPLANES A AND B

As has previously been noted, the second loading for airplane B is nearly the same as loading 1 for airplane A; therefore, the significant difference in the results in figures 2 and 6 (maximum values of β of $5\frac{1}{4}^\circ$ and $24\frac{1}{2}^\circ$, respectively) is caused by differences in the aerodynamic forces and moments acting. One primary difference for these two cases is the value of C_{n_β} (directional-stability derivative). For low

values of $C_{n\beta}$ (airplane B) large values of sideslip angle were obtained (about 25°); whereas for large values of $C_{n\beta}$ (airplane A) small values of maximum sideslip angle were obtained (about 5°). It appears, therefore, that $C_{n\beta}$ is a critical parameter for vertical-tail design.

As has previously been noted, changes in loading for airplane A had an appreciable effect on the maximum angle of sideslip; whereas changes in loading for airplane B had relatively little influence on the maximum sideslip angle. For airplane A, for which the angles of sideslip were relatively small (of the order of 5°) because of the large value of $C_{n\beta}$, the change in loading caused primarily a change in the rate of roll and the yawing moments $C_{n_p} \frac{pb}{2V}$ were appreciably different. This moment contributed to the differences in sideslip angle. For airplane B the relatively small value of the directional-stability derivative $C_{n\beta}$ allowed the airplane to reach large sideslip angles, and the dihedral effect arising from these large sideslip angles caused the rates of roll to be small and similar for both loadings such that the contributions of $C_{n_p} \frac{pb}{2V}$ were similar, as were the resultant sideslip angles.

For airplanes of current design for high-speed high-altitude flight, the trends in aerodynamic characteristics, particularly increasing values of $C_{n\beta}$, appear to be such as to cause smaller sideslip angles in rolling maneuvers than were encountered with the World War II type of airplane. The changes in aerodynamic characteristics appear, therefore, to cause a change in the order of magnitude of the maximum sideslip angles. It is important to note, however, that vertical-tail sizes as well as airspeeds have tended to increase for these current designs and hence the vertical-tail loads may be large in spite of the smaller sideslip angles. Changes in mass distribution appear to influence critically the maximum sideslip angle only for airplanes of current design where the sideslip angles may be relatively small.

CONCLUSIONS

The results of the investigation presented herein give the following indications with regard to sideslip angles and resultant vertical-tail loads in rolling maneuvers for current high-speed airplanes:

1. Existing simplified expressions for calculating maximum sideslip angles in rolling maneuvers will greatly underestimate the maximum sideslip angle for some conditions.
2. Solution of the three linearized lateral equations of

motion, including product-of-inertia terms, will generally indicate with sufficient accuracy the sideslip angles expected in aileron rolls from trimmed flight.

3. In rolling pullouts where the pitching velocity is large, inclusion of the equation of pitching motion along with the lateral equations of motion and consideration of cross-couple inertia terms is necessary to obtain the maximum sideslip angles accurately.

4. Trends in aerodynamic characteristics, particularly increasing values of $C_{n\beta}$ (the rate of change of the yawing-moment coefficient due to sideslip), appear to be such as to cause smaller maximum sideslip angles than were encountered in the past although the vertical-tail loads may be large because of the higher airspeeds. For the case of large $C_{n\beta}$, variations in mass distribution may critically affect the maximum sideslip angle.

5. Pitching oscillations started during the rolling motion will be influenced by cross-couple inertia moments and may cause large variations in angle of attack which affect the horizontal-tail loads.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., December 7, 1951.

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REPORT 1137

INITIAL RESULTS OF INSTRUMENT-FLYING TRIALS CONDUCTED IN A SINGLE-ROTOR HELICOPTER¹

By ALMER D. CRIM, JOHN P. REEDER, and JAMES B. WHITTEN

SUMMARY

Instrument-flying trials have been conducted in a single-rotor helicopter, the maneuver stability of which could be changed from satisfactory to unsatisfactory. The results indicated that existing longitudinal flying-qualities requirements based on contact flight were adequate for instrument flight at speeds above that for minimum power. However, lateral-directional problems were encountered at low speeds and during precision maneuvers.

The adequacy, for helicopter use, of standard airplane instruments was also investigated, and the conclusion was reached that special instruments would be desirable under all conditions and necessary for sustained low-speed instrument flight.

INTRODUCTION

If the capabilities of the helicopter are to be fully realized, instrument and night flight must be readily accomplished. Since comparatively little instrument flying has been attempted with helicopters, the Langley Aeronautical Laboratory has undertaken a flight investigation to determine whether the flying-qualities requirements for helicopters suggested in reference 1 are adequate for instrument flight and whether any unknown or unusual problems exist. In addition, information was sought as to whether special flight instruments are necessary for successful instrument flying in rotary-wing aircraft. The initial results of this program are given in the present report.

TEST HELICOPTER AND METHODS

CONFIGURATION AND MODIFICATIONS

The single-rotor helicopter used in this investigation is shown in figure 1. An additional set of controls, a flight-instrument panel, and a cloth hood (fig. 2) were installed in the rear cockpit to enable the pilot to fly solely by instruments.

The flight instruments provided (fig. 3) were those that are normally considered adequate for an airplane and included a directional gyro, an artificial horizon, and a turn-and-bank indicator, all of which were electrically driven. The artificial horizon was somewhat more sensitive in pitch

than a standard instrument, 27° providing full-scale deflection. The trim range of this instrument was kept within desirable limits by tilting the entire instrument panel approximately 6° to compensate for the nose-down flight attitude of the helicopter.

For test purposes the maneuver stability, or the presence or absence of a tendency to diverge in pitch (see ref. 2), was changed by means of horizontal tail surfaces linked to the longitudinal cyclic control. Two configurations were used: tail-off, wherein the helicopter did not meet the longitudinal requirements of reference 1, and tail-on, wherein it satisfied these criteria.

Inasmuch as the test helicopter was equipped with irreversible servocontrols, which give no stick-force gradient, it was possible to introduce artificial "feel" by means of spring-loaded centering devices installed in the lateral and longitudinal cyclic control systems.

MANEUVERS

Three different flight maneuvers were used for purposes of this investigation: (1) straight and level flight for 1 minute followed by left and right 90° turns; (2) pattern C (shown in fig. 4), and (3) ground-controlled approach (GCA).

Precision GCA was flown at 65 to 70 knots (lower speeds were not permitted because of traffic conditions) with both the tail-on and tail-off configurations. Pattern C was also flown with both configurations. The level-flight-plus-turn pattern was performed not only with the tail on and off but also with the artificial horizon covered and uncovered, at various airspeeds ranging from 15 to 75 knots. No effort was made to find smooth air, and all flights were made in light to moderate turbulence.

In general, each maneuver was performed by two pilots who were experienced in helicopters and held airplane instrument ratings but who had not previously flown helicopters under instrument conditions.

Standard NACA recording instruments were used to obtain records of control position, altitude, airspeed, manifold pressure, stick force, and yawing velocity.

¹ Supersedes NACA TN 2721, "Initial Results of Instrument-Flying Trials Conducted in a Single-Rotor Helicopter" by Almer D. Crim, John P. Reeder, and James B. Whitten, 1952.



FIGURE 1.—Helicopter used in instrument-flying trials.



FIGURE 2.—Test helicopter with instrument-flying hood.



FIGURE 3.—Flight-instrument panel installed in rear cockpit.

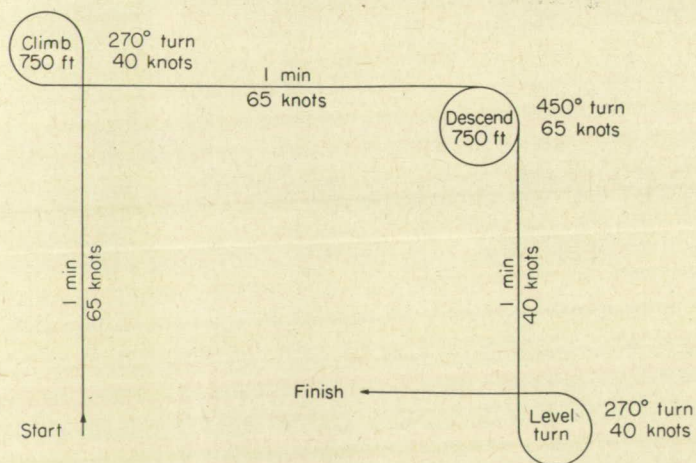


FIGURE 4.—Instrument-flying pattern C.

RESULTS AND DISCUSSION

MANEUVER STABILITY

Maneuver stability is an important flying-qualities requirement for contact flight of helicopters (see ref. 2). As pointed out in reference 3, a helicopter which lacks such stability can exhibit a rapid and dangerous divergence in pitch if the pilot allows his attention to be diverted. The applicability

of this requirement to instrument flying was investigated by means of hooded flights in which the maneuver stability was changed from unsatisfactory to satisfactory by the addition of the previously mentioned tail surfaces. The results indicate that although the accuracy with which any given maneuver could be flown was about the same with the tail on or off, the effort lessened for the stable configuration as the speed increased above about 45 knots, whereas a reverse trend in this respect was shown with tail off, the effort and concentration required increasing with speed. At 25 knots little difference was noted between the two configurations. Although the records provide no measure of the mental effort and concentration required, the greater physical effort at 75 knots for the unstable as compared with the stable configuration is shown in the flight records of figure 5 in the form of greater frequency and amplitude of control motion.

DIRECTIONAL CONTROL

Under certain instrument-flying conditions, holding a given heading appears to be a greater problem than maintaining airspeed or altitude. Although the pilots had no particular difficulty in flying a given course under cruising conditions, they reported that at low airspeeds large deviations in heading occurred. Also, during ground-controlled

approaches, where precise headings and small corrections are necessary, a disproportionate amount of attention was required to maintain headings within reasonable limits. A major reason for the difficulty in heading control is that the rate of turn for a given bank angle goes up as speed is reduced, becoming rather high at the lower helicopter speeds. Inadvertent deviations in bank resulted in relatively large heading changes while the pilot was scanning the instrument panel, particularly when other corrections to flight attitude had to be made. Flight records show larger and more frequent control motions and an increase in yawing-velocity variation for the low-speed and GCA maneuvers.

The yawing-velocity records are of particular interest. At almost all speeds and under most flight conditions the yawing velocity varied in a characteristic oscillatory manner; the amplitude changed with flight condition but the period did not vary materially (fig. 6). Harmonic analysis of several such records revealed the predominant motion to have a period of between 3 and 5 seconds. This period was such that it might have been related to the time required for the pilot to scan the instrument panel. However, when the helicopter was intentionally disturbed by rudder kicks, the resulting oscillation, although usually damped, had approximately the same frequency as that for controlled flight; therefore, the

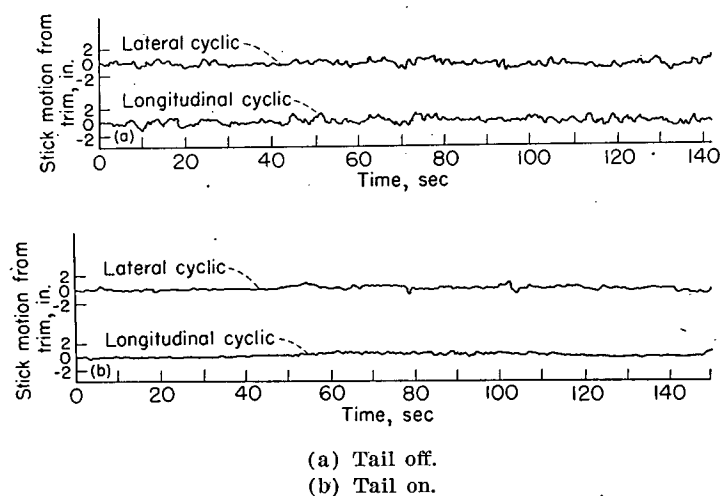


FIGURE 5.—Comparison of cyclic control motion, tail on and tail off, during level-flight-plus-turn maneuver at 75 knots.

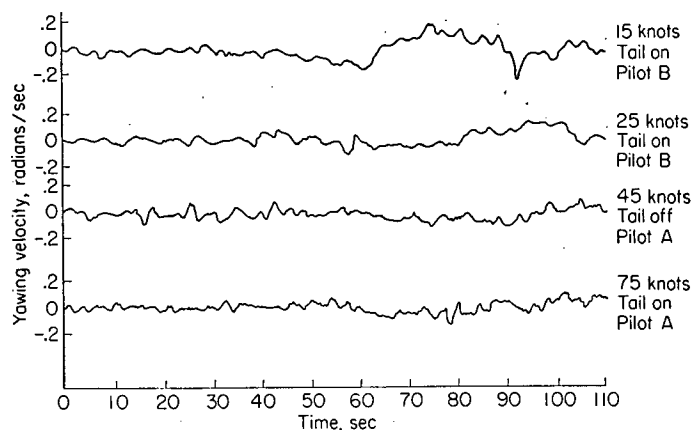


FIGURE 6.—Yawing-velocity variations during level-flight-plus-turn maneuvers.

motion was probably not induced by the pilot. Typical oscillations following an intentional disturbance are shown in figure 7. The significance of this lateral-directional oscillation in relation to the pilots' difficulties is not clear at the present time. However, since the helicopter was almost continually being disturbed, either by small control motions or by atmospheric turbulence, and since under instrument-flying conditions the pilot can devote only a part of his attention to heading indication, oscillations having a relatively short period would probably add to the problem of maintaining directional control.

SIMULATED INSTRUMENT FAILURE

In order that the feasibility of helicopter instrument flying in event of failure of the artificial horizon might be determined, several of the level-flight-plus-turn maneuvers were made with the face of this instrument covered. In the unstable configuration (tail-off), the uncertainty felt by the pilots at the higher speeds with the horizon covered was indicated by continuous manipulation of the controls in contrast to the relatively little motion employed when the horizon was available. Also, larger deviations in flight path occurred when the horizon was not available.

For the stable configuration (tail-on), the flight records show little difference between maneuvers made with the horizon covered and uncovered. At cruising speeds, heading, altitude, and airspeed were maintained with about the same accuracy and with little or no more control movement than when all instruments were available. Even in the stable configuration, however, the pilots stated that prolonged flight without the artificial horizon was undesirable except as an emergency measure because of the high degree of concentration and mental effort required when flying without this instrument.

EFFECTS OF FORWARD SPEED

The degree of difficulty encountered in these helicopter instrument-flying trials depended to a large extent upon the forward speed at which they were attempted. At speeds in the vicinity of cruising, all the maneuvers previously listed were successfully accomplished. By means of close and constant attention to flight instruments, altitude, airspeed, and heading were maintained within reasonable limits. Although more difficult than in an airplane, relatively complex

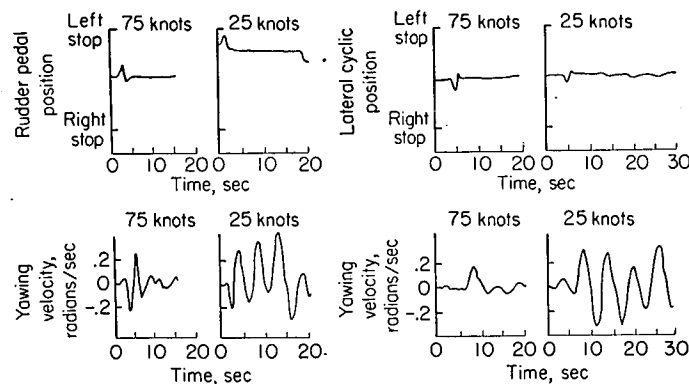
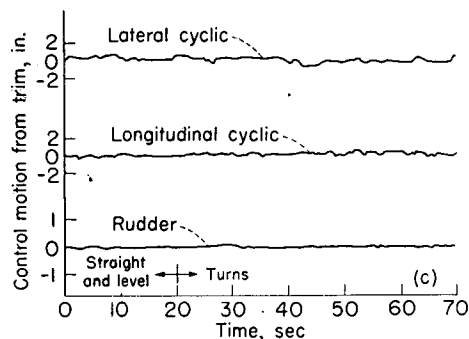
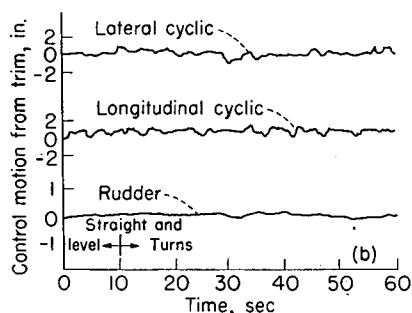
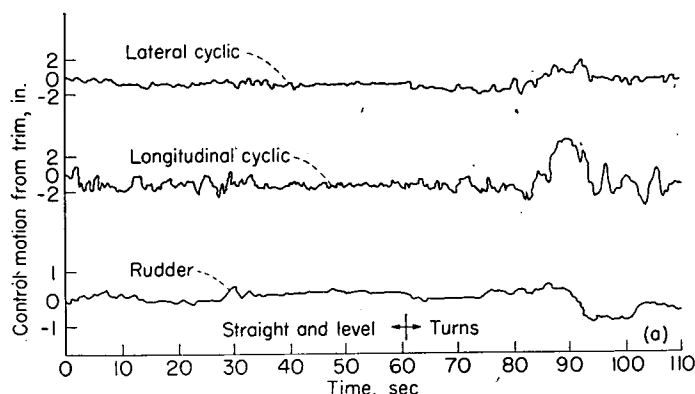


FIGURE 7.—Oscillations resulting from control displacements at two different airspeeds.

maneuvers such as pattern C were accomplished with the helicopter, and ground-controlled approaches were consistently made down to altitudes as low as 50 feet. The maneuvers were successful with both the tail-off and tail-on configurations, although, as previously mentioned, the stable condition required less effort on the part of the pilot.

As forward speed was decreased below that for minimum power, the helicopter became increasingly difficult to fly on instruments. Larger and more frequent control motions were necessary, and greater deviations from the desired flight path occurred. The increased control motions at low speed can be seen in figure 8, which is a comparison of flight records at various airspeeds. Although no sharp dividing line seems to exist between practical and impractical forward speeds, the pilots' opinion was that steady flight below 25 knots would be possible only for short periods of time.

Some of the difficulties encountered below 25 knots are



(a) 15 knots.

(b) 45 knots.

(c) 75 knots.

FIGURE 8.—Comparison of control motion at various airspeeds during level-flight-plus-turn maneuvers, tail-on configuration.

apparently due to errors in the present airspeed indicating system. Steady indications are difficult to maintain, even during visual flight, and fluctuations due to yaw and pitch have been observed. Also, the pilots reported that at these lower speeds unsteady conditions were often encountered, similar to those found in the vortex-ring state of operation during descents at low forward velocity with partial power.

There are other problems peculiar to the low-speed region. For example, at low speeds very small angles of bank produce high rates of turn, so that precise lateral-directional control is difficult. In addition, the pilot experiences little or no normal acceleration during maneuvers at low speeds, whereas at higher speeds such accelerations are believed to be an aid in detecting changes in the flight path. There is also the fact that at speeds below that for minimum power the relationship between power required and speed is just opposite to that normally expected, and an additional burden is thus placed upon the pilot.

CONTROL FORCES

The data of reference 3 indicate that the pilot should be able to trim steady forces to zero and that he should also have a force gradient opposing displacement of the controls. These requirements seem particularly desirable for instrument flight, since in the present trials the pilots objected to small out-of-trim forces that had not been apparent to them in contact flight.

Various force gradients were tried in the longitudinal and lateral control systems of the helicopter in the stable configuration. For this helicopter, the pilots found that stick-force gradients of about 2 pounds per inch longitudinally and 1 pound per inch laterally were satisfactory. This longitudinal gradient was too light, however, to provide satisfactory maneuvering forces. The preload necessary to overcome friction (about 1 lb laterally and 4 lb longitudinally) and provide positive stick centering was found to be objectionable since it had to be overcome each time the stick was moved from trim. A trimming device is considered to be necessary since the pilots objected to out-of-trim forces as low as 1 pound.

POSSIBLE MEANS FOR IMPROVEMENT

STABILITY AND CONTROL CHARACTERISTICS

The results of the present investigation have shown that improving the maneuver stability in accordance with the requirements of reference 1 decreases the effort required of the pilot in maintaining a given flight path. For the test helicopter, this improvement was accomplished by means of tail surfaces linked to the longitudinal cyclic control. However, depending upon the basic configuration, there are other methods, such as those suggested in reference 2, which might work equally well for helicopters lacking in maneuver stability.

The lateral-directional problem seems less likely to have a simple solution. One possibility for improvement would be to increase the damping in roll and thus reduce the sensitivity to gusts. Another suggestion has been to reduce both the directional-control sensitivity and the weathercock stability

in order to avoid overcontrolling and to decrease the sensitivity to lateral gusts, although this procedure may conflict with practical requirements. A third solution is to use an autopilot, either to control the helicopter directly or, in effect, to modify its stability derivatives. This approach is feasible but involves a substantial weight penalty and does not alter the fact that, in event of autopilot failure, inherent stability of the helicopter would be highly desirable.

The previously mentioned directional oscillation requires further investigation to determine the extent to which it contributes to the pilot's difficulties. An increase in either period or damping of this motion might reduce directional-control difficulties. However, no basis for evaluating the importance of this item has as yet been obtained.

FLIGHT INSTRUMENTS

Since the fuselage attitude of a helicopter is independent of the lifting-rotor position, the conventional artificial horizon does not always provide a reliable indication of flight-path change and does not supply the pilot with information that will allow him to anticipate displacement of the helicopter from trim conditions. Instruments which indicate the attitude of the rotor with respect to the horizon would appear to offer solutions to problems of this type. Another approach is to provide the pilot both attitude and rate indication. Flight data obtained with an instrument which combines fuselage pitch attitude and rate of change of attitude have indicated that a combination of rate and attitude information, possibly about all three axes of the helicopter, might be desirable. There are also strong indications that an instrument which combines information usually obtained from several sources might reduce considerably the difficulty of helicopter instrument flying.

Another possibility for improvement exists in regard to airspeed indication, since at very low speeds reliable airspeed information is difficult to obtain in the helicopter. In addition to usual problems, the variation of inflow through the rotor makes the avoidance of installation errors difficult, and gusts, even of small magnitude, can produce a large percentage of the indicated reading. A satisfactory instrument would probably have to average or damp the gust velocities and be relatively insensitive to variations in yaw and pitch.

CONCLUSIONS

As a result of instrument-flying trials conducted with a single-rotor helicopter, the following conclusions were drawn:

1. Existing longitudinal flying-qualities requirements for helicopter appeared to be adequate for instrument flying at speeds above that for minimum power.

2. Changing the maneuver stability from unsatisfactory to satisfactory markedly reduced the effort required of the pilot to maintain a given flight path under instrument conditions. In addition, the danger due to divergent tendencies was removed.

3. During precision maneuvers, such as ground-controlled approaches, and for low-speed flight in general, control of heading appeared to be a greater problem than maintaining altitude or airspeed. Much of this difficulty was due to the fact that, at usual helicopter speeds, small angles of bank result in high rates of turn. The lightly damped lateral-directional oscillation, which is not always apparent to the pilot, requires further investigation to determine the extent to which it contributes to the pilot's difficulties.

4. With standard airplane instruments, normal instrument-flying maneuvers were possible in the helicopter at speeds from about 45 to 75 knots. However, close and constant attention to flight instruments was necessary. Increasing difficulty was encountered at lower speeds, and flight below 25 knots was possible only for very short periods.

5. All unbalanced control forces, even those of small magnitude, were objectionable during instrument flight and means must be provided for trimming such forces to zero about all axes.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *March 11, 1952.*

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REPORT 1138

STUDY OF INADVERTENT SPEED INCREASES IN TRANSPORT OPERATION¹

By HENRY A. PEARSON

SUMMARY

Some factors relating to inadvertent speed and Mach number increases in transport operation are discussed with the object of indicating the manner in which they might vary with different qualities of the airplane and the minimum margins required to guard against reaching unsafe values. The speed increments and the margins required under several assumed conditions are investigated. The results indicate that, on a percentage basis, smaller margins should be required of high-speed airplanes than of low-speed airplanes to prevent overspeeding in inadvertent maneuvers. The possibility of exceeding placard speed in prolonged descents is illustrated by computations for typical transport airplanes. Equations are suggested that allow estimates to be made of the necessary speed margins.

INTRODUCTION

In order to guard against inadvertent increases in airspeed, flight regulations limit the effective airplane cruising speed to a fixed percentage (80 percent) of the design or demonstrated speed. For lack of better information, the same limit has been applied to the Mach number. As a result of these regulations, low-altitude propeller-driven airplanes tend to be limited by the indicated-airspeed placard so that a margin of 20 percent is maintained on the speed for structurally safe flight but with the possibility of a greater margin on the Mach number where adverse compressibility effects occur. On the other hand high-altitude high-speed jet-powered transports might be expected to be limited in cruising by the Mach number placard which results in a 20 percent margin on Mach number and a greater margin on the structurally safe indicated speed.

Plans for the development of turbojet transports have renewed interest in the problem of selecting satisfactory limits for airplane operating speeds that will insure against exceeding values of either Mach number or the dynamic pressure for which the airplane can be expected to remain controllable and structurally sound. In order for such transports to be economically feasible, however, they must necessarily be operated nearer the maximum level-flight speed than the older propeller-driven airplanes and excessive margins cannot be tolerated.

For these reasons reexamination of the problem of selecting the limiting operating speeds appears desirable. This report presents an analysis for determining the margins required under several assumed conditions and is confined mainly to the physical aspects of the problem.

SYMBOLS

The following symbols are used throughout the present report:

A	aspect ratio, b^2/S
a	speed of sound, fps
b	wing span, ft
C_D	airplane drag coefficient, Drag/qS
C_{D_0}	profile-drag coefficient
C_L	airplane lift coefficient, Lift/qS
C_{L_α}	airplane lift-curve slope per radian
$C_{L_{\alpha_t}}$	tailplane lift-curve slope per radian
c	wing chord, ft
C_{m_0}	pitching-moment coefficient of wing-fuselage combination at zero lift
α_t	tail angle of attack, radians
d	horizontal distance from given object, or distance between tail-off aerodynamic center and center of gravity, ft
$\frac{d\epsilon}{d\alpha}$	downwash factor
e	span efficiency factor
f, f_0	compressibility factors defined in reference 1
g	acceleration of gravity, 32.2 ft/sec ²
M	Mach number, V/a
q	dynamic pressure, $\frac{1}{2} \rho V^2$, lb/sq ft
S	wing area, sq ft
S_t	tail area, sq ft
T	engine thrust, lb
t	time, sec
U	true gust velocity, fps
V	true velocity, fps except with subscript mph
V_c	calibrated airspeed (the airspeed related to differential pressure by the accepted standard adiabatic formula used in the calibration of differential-pressure airspeed indicators and equal to true airspeed for standard sea-level conditions), fps
W	airplane weight, lb
w_p	weight of shifted payload, lb
x_p	distance through which payload is shifted, ft
x_t	distance from airplane center of gravity to tail hinge line, ft
L/D	airplane lift-drag ratio
h	altitude, ft

¹ Supersedes NACA TN 2638, "Study of Inadvertent Speed Increases in Transport Operation" by Henry A. Pearson, 1952.

Δh	altitude change, ft
Δn	incremental load factor
ΔV	velocity increase, fps
$\Delta h/\Delta t$ or dh/dt	rate of descent, fps
$\Delta V/\Delta t$ or dV/dt	rate of change of velocity, ft/sec ²
α	angle of attack, radians
δ	elevator angle, radians
γ	flight-path angle, radians
ρ	mass density of air, slugs/cu ft
ρ_0	mass density of air at sea level, slugs/cu ft

CONDITIONS FOR WHICH SPEED MARGINS ARE REQUIRED

Some of the conditions which may be considered as leading to inadvertent increases in airspeed are listed as follows:

- (1) Increases in speed and Mach number resulting from maneuvers made either to avoid obstacles or from a sudden failure of automatic pilot or booster system
- (2) Increases in speed and Mach number resulting from encountering gusts during cruising
- (3) Increase in speed and Mach number due to a forward shift in passengers or payload
- (4) Mach number margin required to permit maneuvering without reaching the buffeting boundary
- (5) Mach number changes resulting from traversing areas with temperature inversions
- (6) Increase in speed and Mach number associated with carrying out a planned descent from altitude.

AVOIDANCE OF OBSTACLES

If an airplane were required to execute a rapid push-down pull-up maneuver in order to pass under an obstacle on a collision course, an increment in speed and Mach number would be gained during both the push-down part of the maneuver and the recovery to level flight. For example, if a small altitude loss were necessary in order safely to clear an obstacle, an equal altitude change would be required to return to level flight providing both the push-down and pull-up were made with equal rapidity. If all of the potential energy represented by the combined altitude change were converted into kinetic energy, the equation for this limiting case would be

$$\frac{1}{2} \frac{W}{g} [(V + \Delta V)^2 - V^2] = W \Delta h \quad (1)$$

Expanding $(V + \Delta V)^2$, dropping the second-order term, and dividing through by V^2 reduces the expression to

$$\frac{\Delta V}{V} = \frac{g \Delta h}{V^2} \quad (2)$$

From the relation $V = Ma$ another form of equation (2) is

$$\frac{\Delta M}{M} = \frac{g \Delta h}{M^2 a^2} \quad (2a)$$

Thus, to a first approximation, the maximum possible percentage increase in velocity incurred in clearing a given obstacle would vary inversely as the square of the initial

speed or Mach number. Since the obstacle to be avoided would most likely be another airplane, the minimum altitude loss would be of the order of 50 feet and the minimum total altitude change, including an equally rapid recovery, would be of the order of 100 feet. From equation (2) the percentage change in speed for this amount of altitude change is determined to be a 1-percent increase for an airplane traveling 388 miles per hour and a 5-percent increase for an airplane traveling 173 miles per hour. Thus, speed increments resulting from avoiding collisions would be of consequence only if the total altitude loss were larger, as would be the case if the pilot tried to clear by more than 50 feet or failed to recover as rapidly after clearing the obstacle. The margin in speed required to guard against this possibility should vary inversely with the cruising speed.

Of some interest, perhaps, in such a maneuver is the small distance over which the flight path can be changed without imposing large loads on the airplane. If it is assumed that a load-factor increment of Δn could be instantaneously applied, the greatest deviation which can be made to the flight path would be given by the following equation

$$\Delta h = \frac{g d^2}{2 V^2} \Delta n = \frac{g t^2}{2} \Delta n \quad (3)$$

where

Δn acceleration increment in g units

t time during which load increment is applied

d distance at which object to be cleared is first sighted

For example, if an object were initially seen at the distance corresponding to that traveled in 2 seconds and a load-factor increment of 2 were instantaneously applied, a deviation of only 128 feet in the altitude could be made under this assumption. The fact that the pilot could not react immediately plus the fact that the airplane does not respond instantaneously to an instantaneous elevator impulse may reduce the value in this case to about one-quarter, or from 128 to 32 feet.

It appears from this example that the increase in speed due to avoiding obstacles would be important only if the minimum altitude changes were made larger; however, unless a collision path was recognized in time, the altitude changes involved would be small.

AUTOPILOT OR BOOSTER FAILURE

The altitude change and increase in speed if an automatic pilot or a booster system were to fail suddenly are also related by equations (1) to (3). In this case, however, both the time during which the acceleration increment acts and the pilot reaction time may be longer because of the unexpectedness of the failure, with the result that both the altitude losses and the increases in speed would be larger. The percentage increase would again vary inversely with the square of the initial velocity and would be given by the equations

$$\frac{\Delta V}{V} = \frac{g^2 t^2 \Delta n}{2 V^2} \quad (4a)$$

$$\frac{\Delta M}{M} = \frac{g^2 t^2 \Delta n}{2 a^2 M^2} \quad (4b)$$

If the sudden failure is assumed to impose a load-factor increment Δn of -2 (a value that would normally be within the strength limits of the wing), equations (4) may be re-written

$$\frac{\Delta V}{V} \cong 1000 \left(\frac{t}{V} \right)^2 \quad (5a)$$

$$\frac{\Delta V}{M} \cong 1000 \left(\frac{t}{Ma} \right)^2 \quad (5b)$$

so that, if a maximum fractional increase of not more than 0.1 in speed is to be obtained, the ratio t/V should be less than $1/100$. For example, for a transport traveling 350 feet per second, recovery should be started within $3\frac{1}{2}$ seconds. Inasmuch as the time for initiating a recovery would be a constant and independent of the initial speed, it appears that a smaller speed margin should be required for high-speed airplanes than for low-speed airplanes to guard against an autopilot failure.

SYMMETRICAL NORMAL GUST

Equation (3) also serves as a starting point for arriving at estimates of the margins required to cover the speed gained in encountering gusts normal or parallel to the flight path. Assume that an airplane with neutral stability encounters a negative gust normal to the flight path for which the intensity increases linearly to a peak in 10 chords ($10c$), or a distance H , and decreases thereafter linearly to 0 in the same distance. This type of gust-intensity distribution is one commonly investigated in gust-load calculations. If unsteady lift and alleviation defects are omitted, the following simple load-factor relations apply

$$\Delta n_{max} = \frac{C_{L_a} \rho U V}{2 W/S} = 2 \Delta n_{av} \quad (6)$$

The altitude loss for this case is

$$\Delta h = \frac{\Delta n_{av} g t^2}{2}$$

where

$$t = \frac{20c}{V}$$

so that

$$\Delta h = \frac{1}{2} \frac{C_{L_a} \rho U V g (20c)^2}{4 W/S V^2} \quad (7)$$

and from equations (2) and (7)

$$\frac{\Delta V}{V} = \frac{\Delta M}{M} = \frac{1}{8} C_{L_a} \frac{U}{V^3} \frac{\rho g^2}{W/S} (20c)^2 = \frac{50 C_{L_a} U \rho g^2 c^2}{W/S V^3} \quad (8)$$

In this case the percentage increase in speed or Mach number varies inversely with the cube of the initial airspeed or Mach number, directly with the square of the distance occupied by the gusts, and directly with the gust velocity. If typical maximum values are substituted in equation (6), it may be seen that the percentage increase in speed in encountering a single normal gust is likely to be small, providing the airspeed is not initially so low that the gust causes the airplane to stall. For a succession of down gusts the increase in speed would be larger than for a single down gust but even so the

percentage gain would be less with the faster airplane than with the slower one.

SYMMETRICAL HORIZONTAL GUST

For horizontal gusts that occur in level flight, the increases in airspeed and Mach number would be instantaneous and are likely to be relatively higher than those due to the normal gust. In this case the fractional increase in airspeed or Mach number is given directly by the equation

$$\frac{\Delta V}{V} = \frac{\Delta M}{M} = \frac{U}{V} \quad (9)$$

A horizontal gust would usually be of relatively short duration and would not be expected to be too critical in its effects; however, in a descent from altitude through an area of wind shear, more prolonged gust effects might be encountered. Although the magnitudes of the velocity and the gradients in such an area of wind shear are not definitely known, it appears that the gust velocity for use in equation (9) is either less than or, at most, equal to the value of 50 feet per second commonly used.

SHIFTING PAYLOAD

A gain in speed or in Mach number can occur if a shift in weight were to occur as when several passengers walk from the rear to the front of the cabin and the pilot fails to retrim the airplane. If it is assumed (a) that the aerodynamic coefficients are linear and do not vary with Mach number over the range being considered, (b) that the speed gain occurs in an atmosphere of constant density equal to that of the initial altitude, and (c) that the pilot does not move either the elevator or throttle before a new equilibrium condition is established, then the following derivation may be made. By designating the initial conditions by the subscript 1 and new trim conditions by subscript 2, the equilibrium equations can be written as

$$W = C_{L_a} \alpha_1 q_1 S \quad (10)$$

$$\left(C_{m_0} \frac{c}{x_t} + C_{L_a} \alpha_1 \frac{d_1}{x_t} - C_{L_{a_i}} \alpha_{i1} \frac{S_i}{S} - C_{L_{b_i}} \delta_1 \frac{S_i}{S} \right) q_1 S = 0 \quad (11)$$

After a change in moment $w_p x_p$ due to a shift in payload the new equilibrium equations are

$$W = C_{L_a} \alpha_2 q_2 S \quad (12)$$

$$\left(C_{m_0} \frac{c}{x_t} + C_{L_a} \alpha_2 \frac{d_2}{x_t} - C_{L_{a_i}} \alpha_{i2} \frac{S_i}{S} - C_{L_{b_i}} \delta_2 \frac{S_i}{S} \right) q_2 S = 0 \quad (13)$$

where the distance d_2 is related to the distance d_1 through the equation

$$d_2 = d_1 - \frac{w_p x_p}{W} \quad (14)$$

If equation (13) is subtracted from equation (11) and if $qS \neq 0$, there is obtained

$$C_{L_a} \left(\alpha_1 \frac{d_1}{x_t} - \alpha_2 \frac{d_2}{x_t} \right) - C_{L_{a_i}} \frac{S_i}{S} (\alpha_{i1} - \alpha_{i2}) = 0 \quad (15)$$

By noting that

$$\alpha_{t_1} = \left(1 - \frac{d\epsilon}{d\alpha}\right) \alpha_1$$

and that

$$\frac{\alpha_2}{\alpha_1} = \frac{\alpha_{t_2}}{\alpha_{t_1}}$$

equation (15) can be written as

$$\left[C_{L_{\alpha}} \frac{d_1}{x_i} - C_{L_{\alpha_i}} \frac{S_i}{S} \left(1 - \frac{d\epsilon}{d\alpha}\right) \right] \alpha_1 - \left[C_{L_{\alpha}} \frac{d_2}{x_i} - C_{L_{\alpha_i}} \frac{S_i}{S} \left(1 - \frac{d\epsilon}{d\alpha}\right) \right] \alpha_2 = 0 \quad (16)$$

By using equations (10) and (12), equation (16) can be rearranged as follows:

$$\frac{\alpha_1}{\alpha_2} = \frac{q_1}{q_2} = \frac{V_2^2}{V_1^2} = \frac{M_2^2}{M_1^2} = \frac{\left[C_{L_{\alpha}} \frac{d_2}{x_i} - C_{L_{\alpha_i}} \frac{S_i}{S} \left(1 - \frac{d\epsilon}{d\alpha}\right) \right]}{\left[C_{L_{\alpha}} \frac{d_1}{x_i} - C_{L_{\alpha_i}} \frac{S_i}{S} \left(1 - \frac{d\epsilon}{d\alpha}\right) \right]} \quad (17)$$

Substituting d_2 (from eq. (14)) into equation (17) and rearranging gives

$$\frac{V_2}{V_1} = \frac{M_2}{M_1} = \sqrt{1 + \frac{\left(\frac{w_p}{W}\right) \frac{x_p}{x_i}}{C_{L_{\alpha_i}} \frac{S_i}{S} \left(1 - \frac{d\epsilon}{d\alpha}\right) - \frac{d_1}{x_i}}} \quad (18)$$

If V_2 is set equal to $V_1 + \Delta V$ and if second-order terms are neglected,

$$\frac{\Delta V}{V_1} = \frac{\Delta M}{M_1} \approx \frac{\frac{1}{2} \left(\frac{w_p}{W}\right) \left(\frac{x_p}{x_i}\right)}{C_{L_{\alpha_i}} \frac{S_i}{S} \left(1 - \frac{d\epsilon}{d\alpha}\right) - \frac{d_1}{x_i}} \quad (19)$$

Equation (19) indicates that the percentage increase in speed is independent of the initial speed and depends on the aerodynamic and geometric characteristics of the airplane. It shows that the ratio $\Delta V/V$ increases as the ratio of the change in trim moment to the airplane moment about the hinge line increases $\left(\frac{w_p x_p}{W x_i}\right)$ and that the ratio $\Delta V/V$ increases as the static stability (given by the denominator) decreases. The speed increase given by equation (19) is the equilibrium value reached under the assumption made and would correspond to a condition in which the airplane is in a slight but steady glide angle. The transient values of $\Delta V/V$ may be lower or higher than the trim value given by equation (19) depending upon when the pilot takes appropriate action to retrim. The largest value of $\Delta V/V$ should occur at a time about equal to one-quarter of the period of the phugoid oscillation of the airplane $\left(\text{roughly } \frac{V_{mph}}{16} \text{ sec}\right)$. Since this motion is lightly damped, the maximum value reached could be very nearly equal to twice that given by equation (19).

As an example of the quantities involved for the case of a typical present-day four-engine transport, a change in either $\Delta V/V$ or $\Delta M/M$ equal to 0.025 would be obtained from equation (19) if 500 pounds of passengers moved from the most rearward position to the front seats.

MARGINS REQUIRED FOR MANEUVERING

Most current airplanes capable of operating in or near the transonic speed range are limited to operation below what is commonly called a buffeting boundary. A typical buffeting boundary for a fighter airplane, designated airplane 1, is shown in figure 1. The data for this curve are taken from reference 2. The part of the curve to the right of the dashed vertical line is associated with a separation caused by compressibility on some main component of the airplane; whereas that part of the curve to the left of the vertical line is the usual $C_{L_{max}}$ curve. Although the boundary to the right of the vertical line can be and has been crossed during special tests with small military airplanes, to do so not only subjects the airplane to large oscillating forces but also places it in a region where stability and control difficulties occur. At present a quantitative evaluation of either the forces or handling qualities of airplanes beyond or at this boundary is not possible. For these reasons, transport airplanes should not be operated at the boundary and some margin appears to be necessary to permit mild maneuvers at cruising speed without the possibility of reaching the buffeting boundary.

In the high-speed cruising range, which is mainly of interest in the present study, the slopes of the available buffeting boundaries are relatively steep. Thus, some idea of the margins which should be maintained can be obtained, even though, as stated previously, the boundary cannot be predicted too accurately. Information on the buffeting boundaries was obtained by analyzing the results for several airplanes, including that given in figure 1, to show the margin required in Mach number in order that a steady 45° bank could be executed at cruising speed without exceeding the boundary. Execution of a 45° bank without loss of altitude requires that the airplane lift coefficient be increased by the factor $\sqrt{2}$. Thus, if, for example, in figure 1 the original

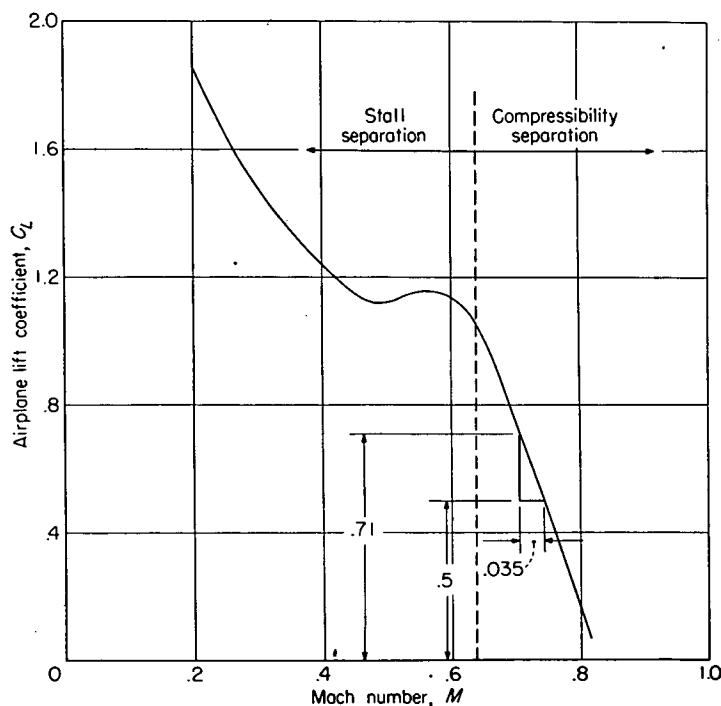


FIGURE 1.—Typical buffeting boundary for high-speed fighter airplane 1.

cruising lift coefficient C_L were 0.5 the lift coefficient in the bank would be approximately 0.71. As shown in figure 1, the margin in Mach number required would then be about 0.035. This point is illustrated in figure 2 by the circular symbol on the curve for airplane 1. Repeating this procedure at other lift coefficients and for other airplanes gave the results shown in figure 2 in which the abscissa represents the over-all airplane C_L at cruising speed and the ordinate is the Mach number margin from the buffeting boundary that should be maintained if mild maneuvering at cruising speed is to be permitted. In general, the Mach number at which the airplane would buffet in level flight is best determined from flight demonstration tests and the margins of figure 2 would be applied to this Mach number. For the present at least, a conservative estimate of the margin required to permit maneuvering appears to be given by any one of the following:

$$\Delta M = \frac{C_L}{12} \text{ allows } 30^\circ \text{ bank}$$

$$\Delta M = \frac{C_L}{10} \text{ allows } 45^\circ \text{ bank}$$

$$\Delta M = \frac{C_L}{7} \text{ allows } 60^\circ \text{ bank}$$

The relation $\Delta M = \frac{C_L}{10}$ represents an envelope of most of the data shown in figure 2 for the 45° bank. The relations given for other angles of bank were determined in a similar manner.

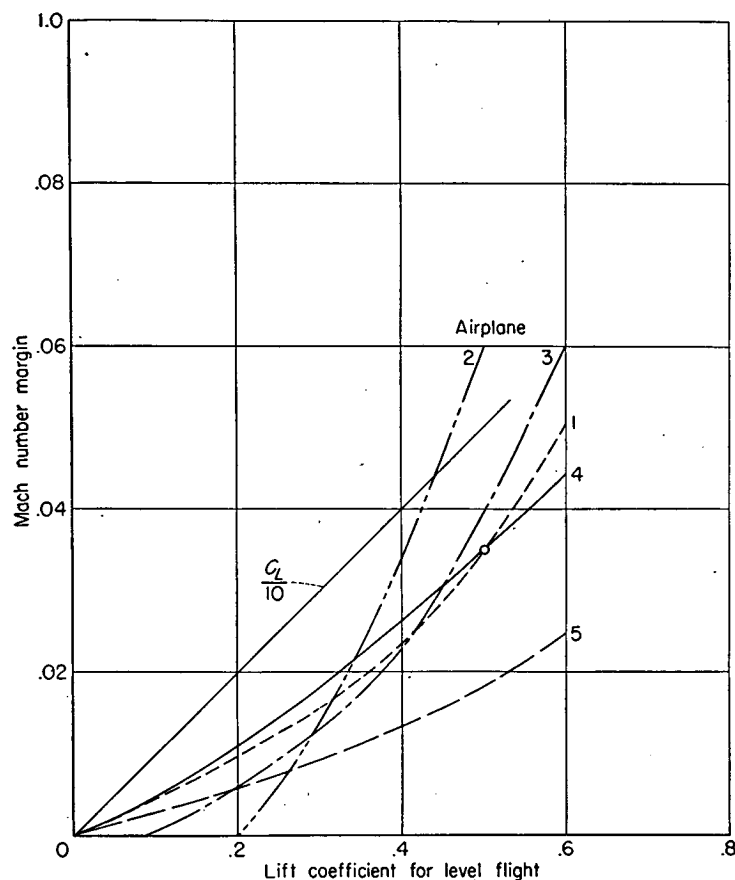


FIGURE 2.—Mach number margin required to execute 45° bank without crossing the buffet boundary.

MARGIN REQUIRED FOR TEMPERATURE INVERSION

Temperature inversions are known to exist in the atmosphere and the altitude range over which inversions may occur varies from hundreds to thousands of feet. Within such inversions fairly localized gradients of 10° F per thousand feet are not uncommon. A change in the temperature of 10° F corresponds to a change of about 1.3 percent in the speed of sound.

During steep descents with small airplanes operating near critical Mach number, adverse compressibility effects such as buffeting or stability changes are sometimes inadvertently encountered. These occurrences have been correlated with measured temperature inversions so that, in some types of research testing, a margin of about 0.015 in Mach number has been necessary in order to avoid inadvertently reaching the buffeting boundary. Since the airplanes on which such experience has been obtained had critical Mach numbers of about 0.75, a suitable margin to guard against the effects of a temperature inversion during a descent may be obtained from the relation

$$\frac{\Delta M}{M} = 0.02$$

The constant 0.02 in this relation would be associated with a somewhat larger temperature gradient than the 10° F mentioned earlier.

SPEED GAINS DURING PROLONGED DESCENTS

Statistical data have shown that the probability of exceeding the placard speeds is greatest in prolonged descents. Some of the reasons for such overspeeding (exceeding placard speed), such as meeting schedules or encountering an emergency, are obvious, whereas other less obvious reasons could conceivably be linked with the operation of either the engine or cabin pressurization system.

Overspeeding in the case of an emergency cannot be rationalized as a pilot would take whatever risks were required. Even introducing automatically operated devices such as brake flaps would not positively prevent overspeeding unless these flaps provided sufficient braking to keep the terminal velocity in a steep dive below the placard value.

Because jet engines must be operated at a higher percentage of power at high altitudes than piston engines in order to avoid a "flame out," this characteristic could offer an excuse for overspeeding in a descent in case the engines could not be restarted easily. In such a case a pilot, if pressed for time, might not tolerate the low rates of descent which would be forced on him by operating the engine at a relatively high percentage of power. Similarly, an airplane having a cabin pressurized by an exhaust-driven turbosupercharger might also offer an excuse for overspeeding since some engine power would be required during a descent in order to maintain cabin pressure. The obvious remedies in both these instances would be to provide positive means of restarting jet engines in flight and the avoidance of pressurization by exhaust-driven superchargers; otherwise, air brakes would be necessary in order to compensate for the undesirable engine thrust and the weight component in a descent.

In addition to these somewhat unusual and possibly outmoded cases, overspeeding might also occur if the pilot were

to follow some fixed plan of descent without making due allowances for airplane characteristics. Conceivably, a descent from altitude could be made according to a number of predetermined plans such as: at constant indicated airspeed, at constant Mach number, at a constant true airspeed, at constant glide-path angle, at constant rate of change of absolute altitude, or at constant rate of change of cabin pressure altitude. In each of these plans there would be a steady decrease in the potential and total energy involved. All plans, however, would be characterized by conditions which could be treated by equations (1) to (3). For instance, the initial phase of the descent before reaching steady conditions may be considered transient as may be any subsequent deviations from the main path due to overcontrolling or inattentiveness on the part of the pilot. These additional speed changes over and above that called for by the adopted plan can be treated as before by using equation (2) or its equivalent

$$\frac{dV}{dt} = -g \frac{dh}{V dt} \quad (20)$$

In equation (2) the maximum possible percentage increase for a given Δh below the intended descent path varied inversely as the square of the average speed along the path. Equation (20) indicates that the rate of change of speed is directly proportional to the rate of descent and inversely proportional to the speed; thus the higher the initial speed the greater the time available for the pilot to prevent a unit increase in speed by detecting and checking a unit rate of descent above the intended value.

The equations of motion for some of the plans of descent which might be used are given in general form, where the asterisk is used with the symbols to identify the parameters held constant.

For a descent at a constant small glide-path angle for which the weight may be considered equal to the lift, the equation of motion is

$$\frac{dV}{dt} = g \left(\sin \gamma^* + \frac{T}{W} - \frac{1}{L/D} \right) \quad (21)$$

Since by definition

$$\sin \gamma = \frac{dh/dt}{V} = \frac{dh/dt}{Ma} \quad (22)$$

and, from reference 1,

$$V_c = V \frac{f_0}{f} \sqrt{\frac{\rho}{\rho_0}} \quad (23)$$

the equation of motion for a descent at a constant calibrated airspeed is (calibrated airspeed is the pilots' indicated airspeed when the airspeed system has no error)

$$\frac{dV}{dt} = g \left(\frac{dh/dt}{V_c^*} \frac{f_0}{f} \sqrt{\frac{\rho}{\rho_0}} + \frac{T}{W} - \frac{1}{L/D} \right) \quad (24)$$

Similarly, for a descent at constant Mach number the equation is

$$\frac{dV}{dt} = g \left(\frac{dh/dt}{M^* a} + \frac{T}{W} - \frac{1}{L/D} \right) \quad (25)$$

and for a constant rate of change of altitude based on standard conditions

$$\frac{dV}{dt} = g \left[\frac{(dh/dt)^*}{V} + \frac{T}{W} - \frac{1}{L/D} \right] \quad (26)$$

Alternate forms of equations (21) to (26) may be obtained by substituting relations involving the coefficients of lift and drag

$$D = C_D q S = C_D \frac{\rho}{2} V^2 S = C_D \frac{\rho a^2}{2} M^2 S \quad (27)$$

where C_D may be expressed as

$$C_D = C_{D_0} + \frac{C_L^2}{\pi A} = C_{D_0} + \left(\frac{W \cos \gamma}{q S} \right)^2 \frac{1}{\pi A} \quad (28)$$

In general T/W will be some function of V and h for a given engine speed or throttle setting and L/D will be a function of Mach number M and C_L .

ILLUSTRATIVE EXAMPLES AND APPLICATION OF RESULTS

AIRPLANE CHARACTERISTICS AND CONDITIONS

In order to integrate equations (21), (24), (25), and (26) to obtain the speed gain with a given descent plan, a step-by-step solution must be made. Since several of the principal variables are either nonlinear or are complicated functions of other variables, an infinite number of solutions would exist so that no general charts can be given. However, in order to illustrate the potential speed gains that may occur in following various plans for descent, examples are given for three typical transport airplanes designated airplanes A, B, and C. The airplane characteristics and conditions assumed to illustrate the application of the formulas are summarized in table I.

Airplane A is representative of a propeller-driven airplane of about ten years ago with a nonpressurized cabin. In the example, this airplane is assumed to start a descent from

TABLE I.—PERTINENT AIRPLANE CHARACTERISTICS AND CONDITIONS USED IN EXAMPLES

Quantity	Airplane		
	A	B	C
Weight, lb.	25, 000	85, 000	125, 000
Wing loading, lb/sq ft.	25	55	75
Thrust/Weight.	$18/V_{mph}$	$23.1/V_{mph}$	$19,000-0.3H$
Initial conditions:			
Altitude, ft.	10, 000	20, 000	30, 000
V , mph.	200	350	500
M	0. 272	0. 495	0. 738
V_c , mph.	172. 6	259. 2	230. 1
q , lb/sq ft.	75. 5	166. 5	238. 0
γ , radians.	^a 0. 0284	0. 0324	0. 0227
	^b 0. 0568	-----	0. 0454
$\Delta h/\Delta t$, ft/min.	^a 500	1000	1000
	^b 1000	-----	2000

^a Descent shown both with engine power assumed constant and with zero thrust.

^b Descent shown with zero thrust only.

10,000 feet with an initial true airspeed of 200 miles per hour for two thrust conditions. In one case the engine power and propeller efficiency are assumed to be constant during the descent so that the thrust-weight ratio varies inversely with the speed according to the relation $\frac{T}{W} = \frac{18}{V_{mph}}$. In the other case the thrust is assumed to be zero. The variation of L/D with airplane lift coefficient C_L is given in figure 3 (a) where the initial condition from which the computations were started is represented by the circular point. Since the speed range of airplane A is quite low, no Mach number effects on the L/D curve were assumed. Also since it was assumed that no cabin pressurization was used, the rate of change of altitude was kept low for the various descent plans.

Airplane B is a pressurized-cabin, propeller-driven airplane typical of some present-day transports. For this airplane the descent was assumed to start at 20,000 feet from a true cruising speed of 350 miles per hour. Rates of descent of 1000 and 2000 feet per minute were assumed. As with airplane A, two cases were considered: one with zero power and one in which the engine power and propeller efficiency were assumed constant in the descent. For this case, the total thrust-weight ratio was given by $\frac{T}{W} = \frac{23.1}{V_{mph}}$. The assumed variation of L/D with C_L and Mach number is given by the curves in figure 3 (b), which were derived from tests of a current transport configuration. From supplementary curves it was established that critical Mach number occurred around $M=0.65$; the initial conditions are represented by the point on the $M=0.50$ curve.

Airplane C used in the examples is an assumed swept-wing, pressurized-cabin, turbojet airplane capable of cruising at 30,000 feet at 500 miles per hour or a Mach number of about 0.75. The L/D curves for this airplane are given in figure 3 (c). The critical Mach number for airplane C was established as being around 0.81. For the case of flight with power on, the variation of thrust for the speed and altitude range of interest was assumed to be given by the equation $T=19000-0.3h$. Results of wind-tunnel model tests and jet engine tests were used to estimate the L/D and thrust relations.

For airplane A, which was not pressurized, the value $\frac{dh}{dt}=500$ feet per minute is slightly above the upper limit permissible in current practice regarding the effect of rate of change of pressure on passenger comfort, whereas the 1000 feet per minute might apply with only crew members aboard. For airplane B in which the cabin was assumed to be pressurized to 10,000 feet, the rate of descent was chosen as 1,000 feet per minute which would give a total descent time of 20 minutes. With this rate, a total time of about 23 minutes would be required to raise the cabin pressure from 10,000 feet to sea level at a cabin rate not exceeding that corresponding to a change of 0.4 inch of mercury per minute. (A rate not exceeding 0.4 inch of mercury per minute represents current practice. This rate corresponds to values of dh/dt equal to 370, 500, 700, and 1000 feet per minute at sea level, 10,000, 20,000, and 30,000 feet, respectively.) For airplane C also with the 10,000-foot pressurized cabin, descent rates of 1,000 and 2,000 feet per minute were chosen.

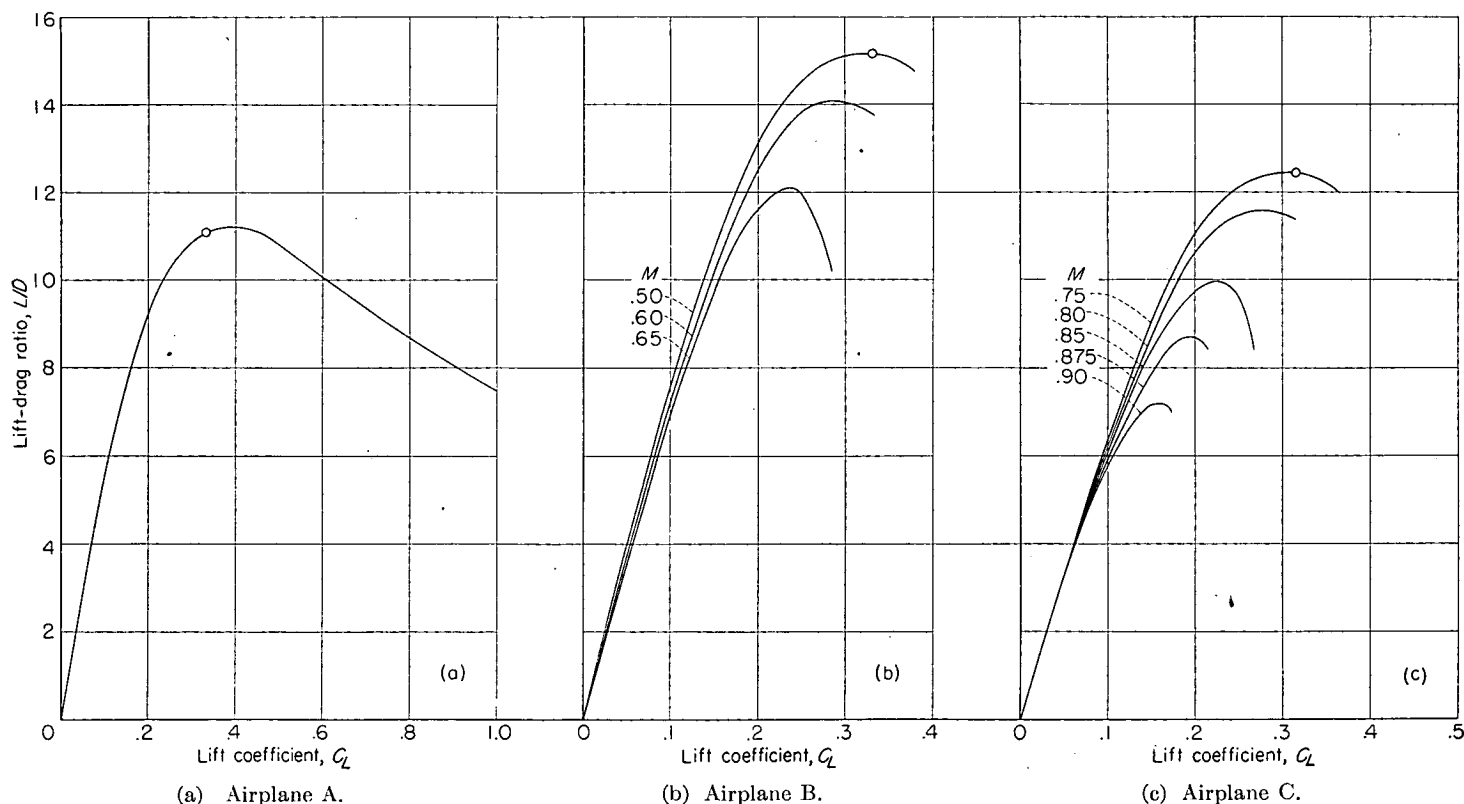


FIGURE 3.—Variation of lift-drag ratio with lift coefficient for airplanes of example.

At the lower rate, the cabin and outside pressure would just be equalized shortly before landing, whereas at the higher rate some ground time would be required.

CALCULATIONS FOR SPECIFIC DESCENT PLANS

Calculations were made for the various specific descent plans for airplanes A, B, and C and the results are presented in figures 4 to 6. In the computations the transition from the cruising condition to the specific condition planned for the descent was assumed to be instantaneous. In the figures, the altitude is plotted against calibrated airspeed for each plan of descent. Ticks are added to each curve to indicate the elapsed time in minutes. The values of dh/dt used in the calculations were based on values intended to provide reasonable passenger comfort and adequate cabin pressurization.

Descent at constant V_c .—A descent at constant calibrated airspeed would be represented in figures 4 to 6 by vertical lines. For the range of conditions considered, constant calibrated airspeed corresponds closely to a constant dynamic pressure q . Therefore, during such a descent the airplane lift coefficient and lift-drag ratios would remain nearly constant except for Mach number effects on these quantities.

The difference between a descent at a constant calibrated airspeed and one at a constant dynamic pressure q , if such

Descent plan	With thrust	Zero thrust
$\gamma = 0.0284$ radian	A	E
$\gamma = 0.0568$ radian		F
$\frac{dh}{dt} = 500$ ft/min	B	G
$\frac{dh}{dt} = 1000$ ft/min		H
$M = 0.272$	C	C
$q = 75.5$ lb/sq ft	D	D

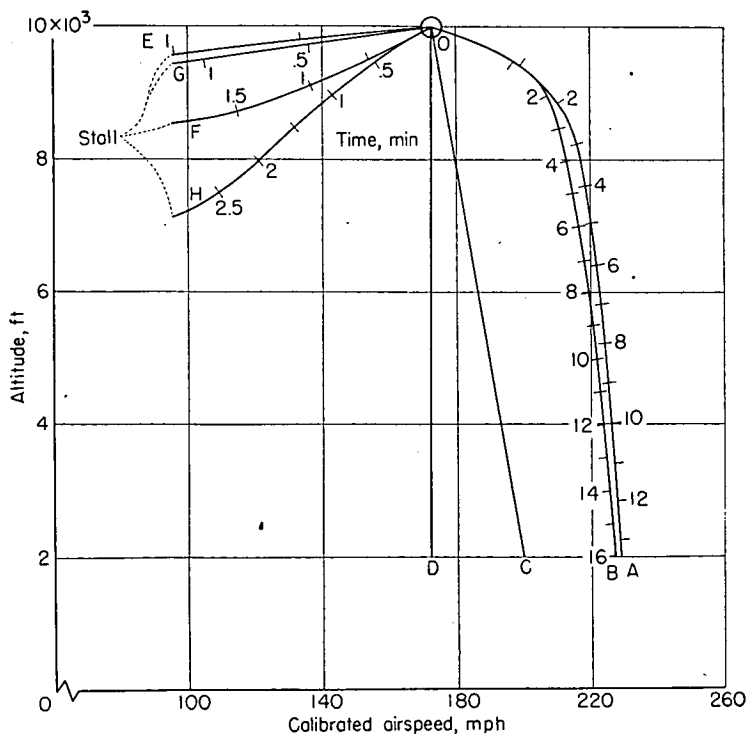


FIGURE 4.—Velocity-altitude relations for various descent plans with airplane A.

a descent could be made, may be obtained by noting the deviation of line D in figures 4 to 6 from a vertical line through the initial point. From the deviations shown, it appears that in a descent at constant calibrated airspeed the dynamic pressure would be expected to increase slightly.

Descent at constant M .—Curve C of figures 4 to 6 shows that the descent at constant M would result in an increase in calibrated speed and hence in the dynamic pressure q . On a percentage basis, the increase in calibrated speed is successively greater with airplanes A, B, and C mainly because the altitude range covered is greater. Regardless of the initial cruising speed, the increase in true airspeed during a constant Mach number descent in a standard atmosphere would not exceed 10 percent; however, for airplane C, the value of q would increase about $3\frac{1}{2}$ times during the descent from 30,000 to 2,000 feet. This increase in q for descent at constant Mach number would probably be excessive from structural considerations, so that this plan is a less practical one than the descent at constant calibrated airspeed.

Descent plan	
$\gamma = 0.0324$ radian	A
$\frac{dh}{dt} = 1000$ ft/min	B
$M = 0.495$	C
$q = 166.5$ lb/sq ft	D

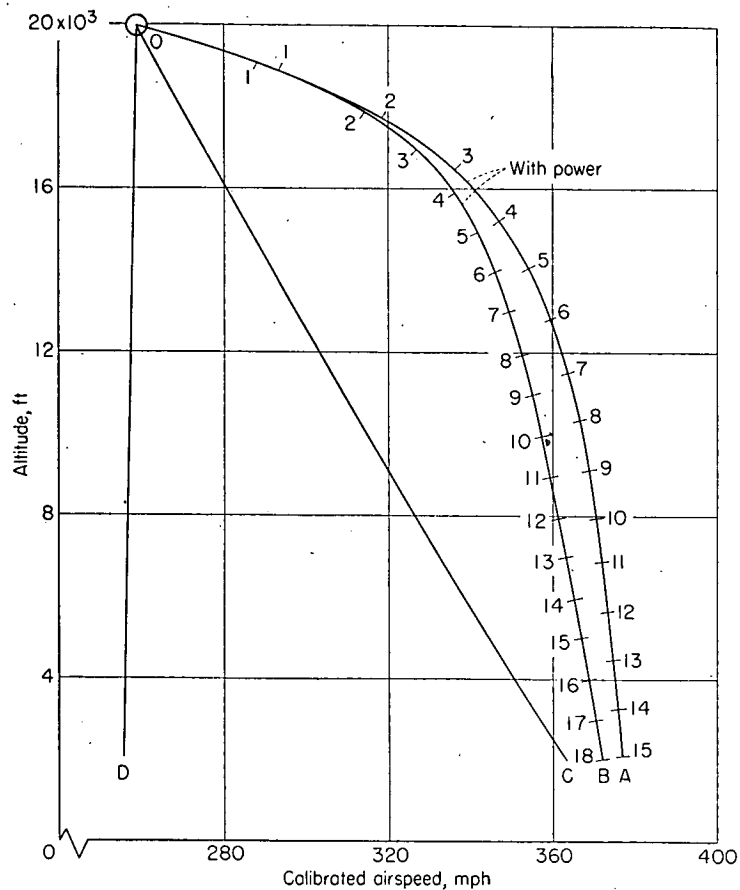


FIGURE 5.—Velocity-altitude relations for various descent plans with airplane B.

The curves given in figures 4 to 6 for the descent at constant M apply to either the zero thrust or power-on condition, since the pilot would adjust the throttle and glide angle as required in order to maintain the values selected. In fact, equations (24) and (25) cannot be integrated for the power-on cases considered in this report unless throttling or some brake devices are used.

Descent at constant rate of change of altitude.—The computations for airplane A (curve B, fig. 4) show that, with constant power setting, the calibrated speed in a descent at constant rate of change of altitude would be increased by about 24 percent in 5 minutes, of which about half would occur within the first minute. For the power-off case (curves G and H) the airplane would stall in trying to maintain a constant rate of change of altitude of either 500 or 1000 feet per minute. Thus it appears that, in the event of sudden engine failure, rates of descent higher than 1000 feet per minute would necessarily prevail regardless of passenger comfort.

The fact that the increases in speed measured in present-day transport operations have generally been less than 10 percent means that some throttling is used during the de-

scend. For airplane A, computations would show that if the engines were immediately throttled to about two-thirds of the cruising power (that is, about four-ninths of rated power) a descent could be made at about 500 feet per minute without a substantial increase in speed.

The increase in speed for airplanes B and C at the constant rates of descent chosen are, on a percentage basis, about the same as for airplane A with about half the final maximum increase occurring in the first minute. Thus, it appears that, if the present placard speed which limits cruising operation to 80 percent of the design or demonstrated speed is to be raised with future transports, provisions must be made for reducing the "effective" airplane L/D ratio either by engine throttling or by use of aerodynamic braking.

Descent along constant flight-path angle.—The curves labeled A, E, and F in figures 4 to 6 for descents at constant flight-path angle indicate slightly greater increases in speed and consequently greater rates of descent than the curves for constant rate of descent (curves B, G, and H), even though the flight-path angle γ was selected on the basis that it be equal to the rate of descent divided by the initial airspeed. Of interest are the small glide-path angles involved which seldom exceed more than 2° . These small angles are in approximate agreement with statistical measurements which have seldom indicated glide-path angles in a descent of over 5° .

COMPOSITE PLAN

Although the specific plans discussed would probably not be followed throughout a descent without modifications, they are useful in indicating the problem and in pointing out safe procedures to be followed.

It is possible that, unless the cabin were capable of being pressurized to sea-level pressure up to the highest cruising altitude, passenger comfort not only could influence the type of descent plan but also could affect the placard speeds. As stated previously, present practice is to limit the rate of change of cabin pressure to about 0.4 inches of mercury per minute which corresponds to a value of $\frac{dh}{dt} = 1000$ feet per minute at 30,000 feet and 370 feet per minute at sea level.

For structural reasons future transports will continue to be designed to withstand some maximum dynamic pressure q or its equivalent in airspeed. This maximum dynamic pressure could either be one which is arbitrarily selected as a design point or one to which the airplane must be demonstrated. Transport airplanes of the immediate future will also be limited by some Mach number which is not to be exceeded if stability and control troubles as well as buffeting are to be avoided. These considerations will in general require that a composite plan of descent be adopted.

In order to illustrate these limits some of the results for airplane C given in figures 3 (c) and 6 can be used. It is assumed that wind-tunnel tests of a model or flight demonstrations have shown that, at low lift coefficients, adverse compressibility effects begin at $M=0.81$ (represented by C') and that this value of M should not be exceeded. The structure is assumed to have been designed or demonstrated

Descent plan	
$\gamma = 0.0227$ radian	A
$\gamma = 0.0454$ radian	E
$\frac{dh}{dt} = 1000$ ft/min	B
$\frac{dh}{dt} = 2000$ ft/min	F
$M = 0.738$	C
$q = 238$ lb/sq ft	D

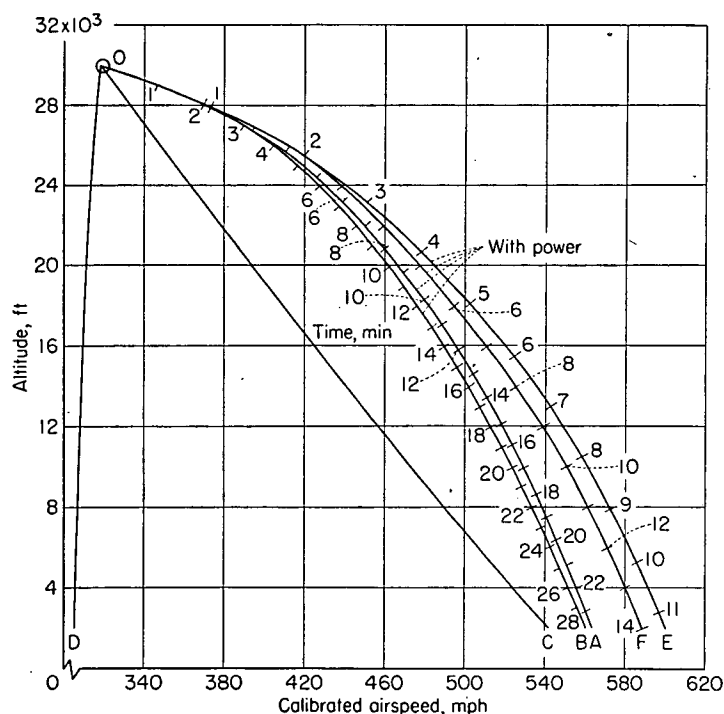


FIGURE 6.—Velocity-altitude relations for various descent plans with airplane C.

to withstand the loads at a dynamic pressure corresponding to a calibrated airspeed of 422 miles per hour. These extreme operational limits which should not be exceeded are shown in figure 7 by the heavy dashed line having two segments and labeled "design limit." The short upper segment is a part of the curve representing the 0.81 Mach number limit, whereas the lower segment is a part of the curve representing the dynamic-pressure limit. For comparison, lines B and C from figure 6 are also shown from which it is seen that neither the descent at a constant value of $\frac{dh}{dt}=1000$ nor at a constant $M=0.738$ could be followed throughout without exceeding these limits.

REQUIRED MARGINS

In order to allow for the possibility of inadvertent increases in speed and Mach number, some margin from the limits shown by the heavy dashed line is required. Although equations have already been given from which the margins required under a single condition may be obtained, the question arises as to the probability of separate events occurring together. The speed gains resulting from avoiding obstacles, from encountering normal down gusts, and from shifting payload have been omitted in determining the required margins because either they have been shown to be small or the probability of these events occurring simultaneously with other more important events is remote. Deliberate overspeeding beyond established placard speeds

has also been omitted because of its psychological aspect. With these possibilities eliminated, the combinations of events that might have reasonable probability of occurrence are

(1) A mild maneuver during descent at operational speed (30° bank) in a region of temperature inversion where horizontal gusts of moderate size (15 fps) exist

(2) Autopilot failure resulting in $-2g$ increment in load factor with 5 seconds delay in recovery in a region of moderate horizontal gusts.

Consideration of these possibilities indicates that a guide to the required margins in Mach number from the operational limits might be obtained from the equation

$$\Delta M = \frac{C_L}{12} + 15 \left(\frac{M}{V} \right)_{\text{cruise}} + 0.02 M_{\text{crit}} + 0.02 \quad (29)$$

where the first term is the margin on the buffet boundary at the airplane cruising lift coefficient, the second term is a margin allowing for a 15-foot-per-second horizontal gust at cruising speed, the third term allows for a possible temperature inversion, and the last term takes into consideration the spread in critical Mach number for a series of airplanes of the same type.

The reduction from the design indicated airspeed is based on the second possibility and would be given by an equation of the type

$$\Delta V = 45 + \frac{25000}{V_{\text{design}}} \quad (30)$$

where the first term is a combined one allowing for horizontal gusts and variations between airplanes and the second term allows for the possibility of an autopilot failure. In equation (30) V_{design} is considered to be the true airspeed in feet per second corresponding to the design indicated airspeed at the lowest altitude at which the autopilot would be used.

If these suggested equations were applied to airplane C, the reduction from the critical Mach number of 0.810 would be 0.072 or 8.9 percent and would yield a maximum operational Mach number of 0.738. If 10,000 feet is assumed to be the lowest altitude in which flight with the autopilot would occur, the reduction from the true design airspeed of 484 miles per hour would be 80 feet per second or about 55 miles per hour. This new airspeed (484 mph - 55 mph) would correspond to an indicated airspeed of 374 miles per hour, which represents a margin of 11.5 percent on the design indicated speed of 422 miles per hour. The operational limits obtained in this manner are given by the solid heavy line in figure 7.

It should be remembered that the constants appearing in equations (29) and (30) for calculating the required margins are estimated and should be checked at the first opportunity with flight experiences.

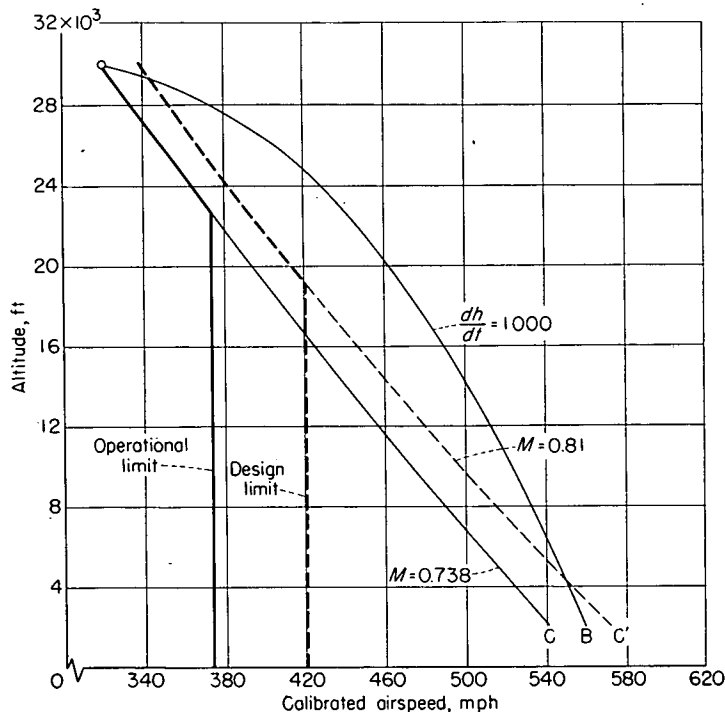


FIGURE 7.—Operational and design limits for composite descent. (Airplane C.)

CONCLUDING REMARKS

As a result of a study to estimate the speed margins that should be allowed to provide for inadvertent speed increases in transport operation, the following trends are indicated:

1. As the cruising speeds of transports increase, the percentage margins required to avoid inadvertent speed gains caused by gusts, autopilot failure, and so forth should decrease.

2. The descent plan of future transport airplanes will probably be a composite one in which the Mach number will be used to furnish the limit in the beginning of the descent and the indicated airspeed will furnish the limit during the later stages.

3. The necessity of including aerodynamic-braking devices will become increasingly important with future transports if reasonable rates of descent are to be attained without large

increases in either the airspeed or Mach number. The size and the projected area of such brakes should, however, be coordinated with the requirements of passenger comfort and the type of cabin pressurization used.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *November 16, 1951.*

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REPORT 1139

CHARTS AND APPROXIMATE FORMULAS FOR THE ESTIMATION OF AEROELASTIC EFFECTS ON THE LATERAL CONTROL OF SWEEPED AND UNSWEEPED WINGS¹

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SUMMARY

Charts and approximate formulas are presented for the estimation of static aeroelastic effects on the spanwise lift distribution, rolling-moment coefficient, and rate of roll due to the deflection of ailerons on swept and unswept wings at subsonic and supersonic speeds. Some design considerations brought out by the results of this report are discussed.

This report treats the lateral-control case in a manner similar to that employed in NACA Report 1140 for the symmetric-flight case and is intended to be used in conjunction with NACA Report 1140 and the charts and formulas presented therein.

INTRODUCTION

The lateral control and maneuverability of a wing are important design considerations. These characteristics may be affected to a significant extent by aeroelastic action, particularly at high dynamic pressures and in the case of thin wings, swept wings, and wings designed for low wing loadings, because the operation of ailerons and spoilers usually creates aerodynamic forces which deform the wing.

As a result of these deformations, the angles of attack along the span often change in such a manner as to produce lifts which oppose the rolling moment of the aileron or spoiler; furthermore, these lifts cause additional deformations which may again reduce the rolling moment, and so on, until equilibrium is reached. Wing flexibility may thus cause a serious loss in the control power; in fact, if the dynamic pressure of the airstream is sufficiently high, the aileron rolling moment may be completely nullified. The speed and dynamic pressure at this condition are often referred to as the aileron reversal speed and reversal dynamic pressure, because at higher dynamic pressures the controls would have to be reversed in order to roll the airplane. When wing flexibility causes a loss in lateral control, there is also usually a loss in the rolling maneuverability, which may be expressed as the wing-tip helix angle due to rolling and is affected by changes in both the control power and the damping in roll. These aeroelastic effects on the lateral control and maneuverability have to be taken into account in the design of a wing.

Several methods are available for calculating these effects (ref. 1, for instance), but, since these effects depend

on the structural characteristics of the wing, which are not accurately known in advance of its design, the relatively large amount of time required for even the most efficient of these methods militates against their use in connection with preliminary design calculations. A need exists, therefore, for means of estimating quickly some of the more important aeroelastic effects on lateral control with an accuracy that is sufficient for preliminary design purposes.

The related problem of estimating static aeroelastic effects on the magnitude and spanwise distribution of the lift in symmetric flight has been treated by the charts and approximate formulas presented in reference 2. The present report consists of an extension of the analysis of reference 2 to the lateral-control case. Inasmuch as the static aeroelastic equations are linear, the results presented in the two reports may be superimposed. Included in the present report are approximate formulas for the estimation of the static aeroelastic effects on the spanwise lift distribution, rolling moment, and rate of roll due to aileron deflections on swept and unswept wings at subsonic and supersonic speeds. Also presented are summary charts which indicate whether a given design is likely to be affected by losses in lateral control. By means of these charts and approximate formulas as well as those of reference 2, the conventional procedure of designing a wing on the basis of certain strength criteria, checking it for aeroelastic phenomena, and then reinforcing it, when necessary, to meet the stiffness requirements imposed by these phenomena can often be simplified greatly, inasmuch as the effect of some of these phenomena can be estimated in advance of design.

In order to keep the length of the report to a minimum and to avoid a repetition of much of the material presented in reference 2, the present report has been written in such a manner as to facilitate its joint use with reference 2 rather than to make it entirely self-contained. The use of the charts and approximate formulas presented herein is described and the limitations of the charts and the light they shed on some design problems are discussed. A numerical example is included to illustrate the use of the approximate formulas of this report. A brief description of the calculations on which the charts and approximate formulas are based is contained in the appendix to supplement the more detailed derivations in references 1 and 2.

¹ Previously released as NACA TN 2747, "Charts and Approximate Formulas for the Estimation of Aeroelastic Effects on the Lateral Control of Swept and Unswept Wings" by Kenneth A. Foss and Franklin W. Diederich, 1952.

SYMBOLS

A	aspect ratio, b^2/S	n	design load factor
A_A	swept-span aspect ratio, $A/\cos^2 \Lambda$	$\dot{p}$	rolling acceleration, radians/sec ²
a	location of section aerodynamic center measured from leading edge, fraction of chord	$pb/2V$	wing-tip helix angle due to roll, radians
b	wing span, in.	q	dynamic pressure, lb/sq ft
c	chord (measured perpendicular to elastic axis), in.	q^*	dimensionless dynamic pressure, $\frac{q}{144} \frac{C_{L_{\alpha_e}} e_1 c_r^2 s_i^2 \cos \Lambda}{(GJ)_r}$; the dimensionless dynamic pressure ϵq^* is identical to q^* except that e_1 is replaced by e_2
c_a	aileron chord, in.	$\bar{q}$	dimensionless dynamic pressure, $\frac{q}{144} \frac{C_{L_{\alpha_e}} c_r s_i^3 \sin \Lambda}{(EI)_r}$
$c_{l_{\alpha}}$	section lift-curve slope per radian	S	total wing area, sq in.
$C_{L_{\alpha_e}}$	effective wing lift-curve slope per radian	s	distance along elastic axis measured from wing root, in.
C_l	rolling-moment coefficient	s^*	dimensionless distance along elastic axis, s/s_i
C_{l_p}	damping-in-roll derivative	T	accumulated torque about elastic axis, in-lb
C_{l_s}	rolling-moment coefficient due to aileron deflection	t_i	distributed torque due to inertia loading, in-lb/in.
cp_s	location of chordwise center of pressure of lift produced by aileron deflection measured from leading edge, fraction of chord	V	airspeed, ft/sec
d	dimensionless sweep parameter, $\frac{(GJ)_r}{(EI)_r} \tan^2 \Lambda$	W	design gross weight of airplane, lb
E	Young's modulus of elasticity, lb/sq in.	W_s	weight of primary structure of both wings, lb
e	location of elastic axis measured from leading edge, fraction of chord	y	lateral coordinate, in.
e_1	dimensionless moment arm of section lift about elastic axis, $e-a$	α	angle of attack in planes parallel to plane of symmetry, radians
e_2	dimensionless moment arm of lift due to aileron deflection about elastic axis, $cp_s - e$	α_s	angle of attack equivalent to unit aileron deflection, radians
F_r	root-stiffness function given in equation (B25) of reference 2	Γ	angle of local dihedral, radians; or spanwise slope of normal displacement of elastic axis
F_B	allowable bending stress, lb/sq in.	δ	aileron deflection measured in planes parallel to airstream, radians
f_a	dimensionless function of distance along span used in approximate formulas for angle of attack due to aeroelastic action of ailerons	ϵ	moment-arm ratio, e_2/e_1
G	modulus of rigidity, lb/sq in.	η_a, η_b	structural-effectiveness factors defined in equation (15) of reference 2
h	wing thickness, in.	$\eta_1, \dots, \eta_{21}$	structural-effectiveness factors defined in table 1 of reference 2
I	section bending moment of inertia, in. ⁴	Λ	angle of sweepback at elastic axis
$\bar{I}$	mass moment of inertia of entire airplane about its longitudinal axis, in. ⁴	λ	wing taper ratio, c_t/c_r
$\bar{I}_w$	mass moment of inertia of both wings about longitudinal axis of airplane, in. ⁴	ρ	density, slugs/cu ft
J	section twisting moment of inertia, in. ⁴	φ	angle of structural twist in planes perpendicular to elastic axis, radians
$K_1, K_2, \dots, K_7$	dimensionless parameters used in approximate formulas for dimensionless dynamic pressures at aileron reversal given in table I	ω	tip stiffness ratio, $\frac{(EI)_t}{(EI)_r}$
k	dimensionless sweep parameter, $\frac{s_i}{e_1 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda$; the dimensionless parameter k/ϵ is identical to k except that e_1 is replaced by e_2	I_a	unit step function of distance along span
l	lift per unit distance along span, lb/in.	Subscripts:	
L'_w	rolling moment on both wings, in-lb	D	at divergence
M	bending moment about an axis perpendicular to elastic axis, in-lb	i	at inboard end of aileron; inertia, in equation (36b)
M_o	free-stream Mach number	o	at outboard end of aileron; except in M_o
		R	at aileron reversal
		r	at wing root
		s	structural (due to structural deformation)
		t	at wing tip
		0	rigid wing (for $q^* = \epsilon q^* = \bar{q} = 0$)

USE OF THE CHARTS AND APPROXIMATE FORMULAS

SUMMARY OF METHOD AND SCOPE OF THE CALCULATIONS ON WHICH THE CHARTS AND APPROXIMATE FORMULAS ARE BASED

A brief description of the method and scope of the calculations is given here to indicate the limitations of the charts and approximate formulas. A detailed description of the method is given in the appendix.

Most of these calculations were performed by an extension, based on reference 1, of the matrix method described in appendix A of reference 2. This method consists in solving the differential equations descriptive of an elastically deformed wing under aerodynamic loadings by numerical methods employing matrix techniques. Treated by this method were wings with three taper ratios (1, 0.5, 0.2), one aileron configuration (50 percent semispan, outboard), two types of stiffness distributions, several values of the sweep parameters k and d which include sweptforward, unswept, and sweptback wings, and several values of the section moment-arm ratio ϵ and the dynamic-pressure ratio q/q_D . Calculated for each case were the dynamic pressure at aileron reversal and the changes in the spanwise lift distribution and rolling moment due to aileron deflection. For the constant-chord—constant-stiffness wings, calculations were also performed by an extension of the analytic method described in appendix A of reference 2, which consists in solving the differential equations exactly for these relatively simple cases; these calculations were made for two aileron spans and several values of the parameters k , d , ϵ , and q/q_D . In all cases the ratio of the aileron chord to the wing chord was constant along the span of the aileron.

Some approximations have been made in the calculations concerning the aerodynamic induction effects, the root rotations, and the stiffness distribution, primarily in order to hold the number of variables considered in the analysis to a minimum and to make the results more generally applicable.

Aerodynamic induction effects at subsonic speeds are taken into account by an overall reduction of the strip-theory loading and, in the matrix calculations, by rounding off the strip-theory loading at the tip (see refs. 1 and 2); for supersonic speeds strip theory is used with a small reduction at the tip in the matrix calculations. This approximation has made it unnecessary to consider explicitly the effects of aspect ratio, sweep, and Mach number on the rigid-wing lift distribution; the effects of these parameters on the total lift and on the aeroelastic increment to the lift distribution have been taken into account.

The rigid-body rotations imparted to a swept wing by its triangular root portion vary among different designs in a largely unpredictable manner. They have therefore been taken into account only by the use of an effective root, the selection of which in any given case is discussed briefly in a subsequent section.

The spanwise distributions of the bending and torsional stiffnesses depend on the detailed design of the wing and cannot be generalized easily. The stiffness distributions used in the calculations of aeroelastic effects were obtained from the constant-stress concept outlined in appendix B of reference 2, which constitutes an effort to relate the stiff-

ness of a wing to its strength on the basis of the following assumptions:

(1) The combined bending and torsional stresses are constant along the span.

(2) The bending and torsional stresses are combined in such a manner that the sum of the ratio of the actual to the allowable bending stress and the ratio of the actual to the allowable torsion stress is equal to unity when the margin of safety is zero.

(3) The structure is of the thin-skin, stringer-reinforced shell type and its main features do not vary along the span; for instance, the number of spars and their chordwise locations are constant along the span.

(4) At the design condition the spanwise distribution of the applied loading is proportional to the chord.

Also used in the calculations were stiffness distributions which vary as the fourth power of the chord, as do those of solid wings and wings with geometrically similar cross sections; as pointed out in a subsequent section, the results of these calculations can be used to estimate aeroelastic phenomena of some wings which have large cutouts or which for some other reason do not have stiffness distributions represented fairly closely by those of the constant-stress type.

All calculations are based on the assumptions that twisting is resisted primarily by the torsion cells of the wing structure and that the wing deformations can be estimated by means of the elementary theories of bending and torsion about an elastic axis.

SELECTION OF PARAMETERS

Geometric parameters.—The geometric parameters used in the analysis are defined in figure 1. The location of the

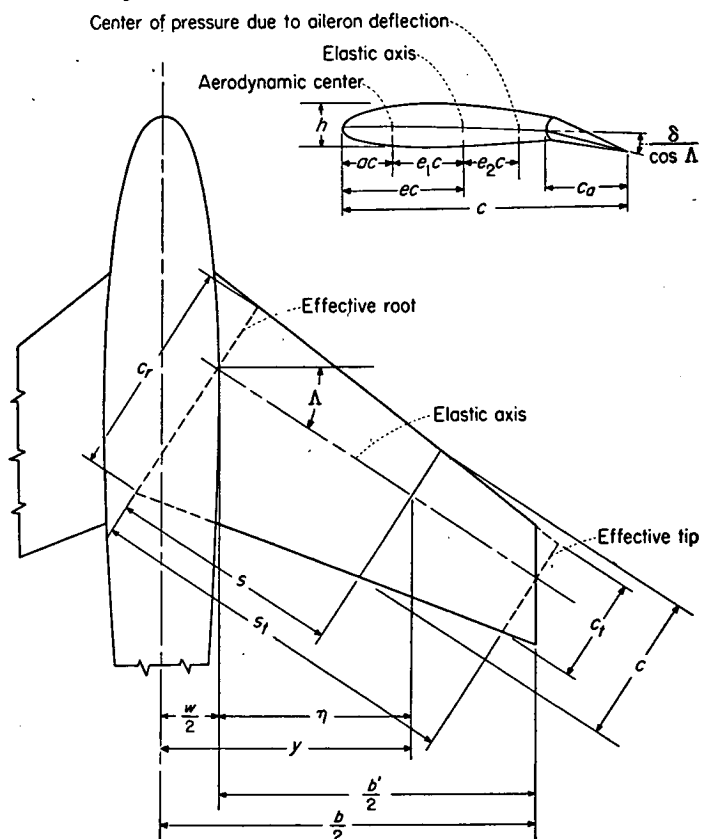


FIGURE 1.—Definitions of geometric parameters.

effective root indicated in this figure is discussed in reference 2. In the present report the angle of aileron deflection δ is defined as being measured in planes parallel to the air-stream. This angle is equal to the product of the angle of rotation of the aileron about the hinge axis and the cosine of the sweep angle of the hinge axis.

Although most of the charts and approximate formulas are based on a half-span outboard aileron ($s_i^*=0.5$, $s_o^*=1$), the results of the analysis of the uniform wings with full-span ailerons ($s_i^*=0$, $s_o^*=1$) and "tip ailerons" ($s_i^*\rightarrow 1$, $s_o^*=1$) indicate that, except for the angle-of-attack distributions, the results based on a half-span aileron may be expected to be valid for outboard ailerons having spans which differ considerably from one-half.

Aerodynamic parameters.—The aerodynamic parameters which enter the analysis are the effective wing lift-curve slope, the location of the wing aerodynamic center, the location of the chordwise center of pressure due to aileron deflection, and the angle of attack equivalent to unit aileron deflection. An effective wing lift-curve slope $C_{L_{\alpha_e}}$, applicable to basic lift distributions due to built-in twist, aileron deflection, roll, or aeroelastic twist, is approximately given at subsonic speeds by the relation

$$C_{L_{\alpha_e}} = c_{l_{\alpha}} \frac{A \cos \Lambda}{A + 4 \frac{c_{l_{\alpha}}}{2\pi} \cos \Lambda} \quad (1)$$

where $c_{l_{\alpha}}$ is approximately given by

$$c_{l_{\alpha}} = \frac{2\pi}{\sqrt{1 - M_o^2 \cos^2 \Lambda}} \quad (2)$$

The basis of equation (1) is explained in reference 2. At supersonic speeds (more specifically, for supersonic leading and trailing edges), the effective wing lift-curve slope is approximately

$$C_{L_{\alpha_e}} = \frac{4 \cos \Lambda}{\sqrt{M_o^2 \cos^2 \Lambda - 1}} \quad (3)$$

provided M_o is greater than $1/\cos \Lambda$. If M_o is greater than 1 but less than and not too close to $1/\cos \Lambda$, equations (1) and (2) may be used in the absence of better information; however, the results obtained for this range of Mach numbers should be used with caution.

The lift-curve slopes given by equations (1) and (3) should not be confused with the rigid-wing lift-curve slope or the damping-in-roll derivative; they are merely effective values suitable for aeroelastic calculations. Values of the rigid-wing lift-curve slope and the damping-in-roll derivative can be used in conjunction with the methods of the present report and of reference 2 to obtain the values of the flexible-wing lift-curve slope and the damping-in-roll derivative because means are presented herein and in reference 2 for estimating the ratios of the flexible-wing to the rigid-wing values. For this purpose any experimental information concerning the rigid-wing values can be used; if none is available, references 3, 4, and 5 may be used at supersonic speeds and reference 6, at subsonic speeds.

The local aerodynamic centers are assumed to be at a constant fraction of the chord from the leading edge, so that they are all equal to the wing aerodynamic center as a fraction

of the mean aerodynamic chord. The moment arm e_1 is then given by the relation

$$e_1 = e - a \quad (4)$$

The local centers of pressure of the lift due to aileron deflection are also assumed to be a constant fraction of the chord from the leading edge; and the moment arm e_2 is then given by

$$e_2 = cp_{\delta} - e \quad (5)$$

Theoretical two-dimensional values of the parameter cp_{δ} are presented in figure 2 for both trailing-edge and leading-edge ailerons at subsonic and supersonic speeds. At subsonic speeds the effect of finite span is to shift the center of pressure rearward. An appropriate value for this rearward shift may be estimated from the following relation, based on lifting-line theory for unswept elliptic wings with full-span ailerons:

$$cp_{\delta_{III}} - cp_{\delta_{II}} = \frac{4}{A} \left(cp_{\delta_{II}} - \frac{1}{4} \right)$$

where the subscripts II and III refer, respectively, to two- and three-dimensional values. The use of the swept-span aspect ratio A_{Λ} in place of A should serve to extend this approximate relation to swept wings.

Theoretical two-dimensional values of α_s , the angle of attack equivalent to unit aileron deflection, are also given in figure 2. At low aspect ratios the values of α_s for subsonic speeds tend to be higher than these two-dimensional values; as the aspect ratio approaches 0, α_s approaches 1, at least in the case of wings without reentrant trailing edges. Experimental values of both α_s and cp_{δ} are preferable to theoretical values if they are available. For spoilers, experimentally obtained values have to be used.

The effective lift-curve slope and the values of the section moment arms vary with the free-stream Mach number; hence, the appropriate values must be used at each flight condition for which aeroelastic calculations are made.

The airspeeds at which the lateral-control aeroelastic phenomena are of interest enter the calculations in the form of the corresponding dynamic pressures. These dynamic pressures, in turn, are expressed in dimensionless form by means of the relations

$$q^* = \frac{q}{144} \frac{C_{L_{\alpha_e}} e_1 c_r^2 s_i^2 \cos \Lambda}{(GJ)_r} \quad (6)$$

$$\epsilon q^* = \frac{q}{144} \frac{C_{L_{\alpha_e}} e_2 c_r^2 s_i^2 \cos \Lambda}{(GJ)_r} \quad (7)$$

or

$$\bar{q} = \frac{q}{144} \frac{C_{L_{\alpha_e}} c_r s_i^3 \sin \Lambda}{(EI)_r} \quad (8)$$

The ratios of these quantities,

$$k = \frac{\bar{q}}{q^*} = \frac{s_i}{e_1 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda \quad (9)$$

and

$$\frac{k}{\epsilon} = \frac{\bar{q}}{\epsilon q^*} = \frac{s_i}{e_2 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda \quad (10)$$

are independent of the dynamic pressure and are very useful

for analyzing the aeroelastic behavior of swept wings. Two other dimensionless parameters, which are independent of the dynamic pressure, enter the calculations:

$$\epsilon = \frac{e_2}{e_1} \quad (11)$$

and

$$d = \frac{(GJ)_r}{(EI)_r} \tan^2 \Lambda \quad (12)$$

Structural parameters.—For the purposes of an aeroelastic analysis the wing structure is characterized by the location of the elastic axis, the magnitude and distribution of the bending and torsional stiffnesses EI and GJ , and the magnitude of the rigid-body rotations imparted to the wing by its root (taken into account in this report only by the location of an effective root). The selection of these structural parameters is discussed in reference 2.

PRELIMINARY SURVEY OF LOSS IN LATERAL CONTROL

The information contained in some of the charts and approximate formulas presented in the following sections of this report has been summarized in figures 3 to 5 for the purpose of ascertaining in advance of more detailed estimates, if desired, whether the aeroelastic phenomena considered in this report are likely to affect the design of the wing structure. This preliminary survey is not essential to any of the further calculations but may show them to be unnecessary in some cases. These figures pertain to constant-stress wings with half-span outboard ailerons.

The charts of figures 3 (a), 4 (a), and 5 (a) pertain to wings of taper ratios 0.2, 0.5, and 1.0, respectively, with the moment arm e_2 equal to 0 (corresponding roughly to subsonic flow

conditions and elastic-axis locations fairly far back on the wing). These figures show the dynamic-pressure parameter q^* defined by either equation (6) or

$$q^* = \frac{(1+\lambda)^2}{18432} \frac{C_{L_{\alpha_e}} e_1 \cos \Lambda}{\frac{G}{F_B} \frac{nW}{S} \frac{h_r}{c_r} F_r \eta_a} q \quad (13)$$

plotted against the sweep parameter k defined by equation (9) or

$$k = \frac{1+\lambda}{2} \frac{G}{E} \eta_b \frac{A_\Lambda}{\eta_{19} e_1} \tan \Lambda \quad (14)$$

for several values of the aileron effectiveness parameter $C_{l_b}/C_{l_{b0}}$ and for two values of the stiffness parameter

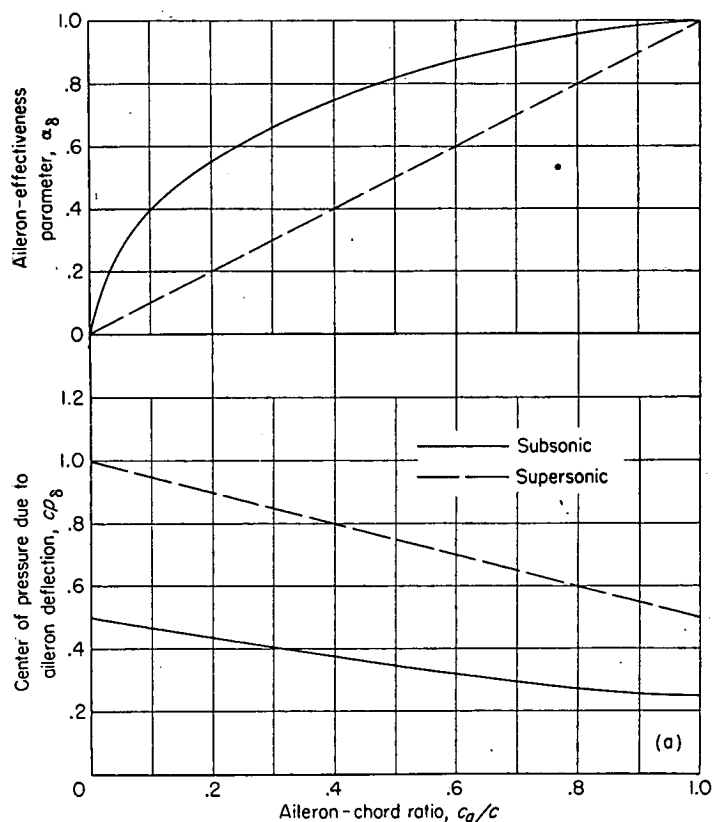
$$\frac{d}{k^2} = \frac{(EI)_r}{(GJ)_r} \left(\frac{e_1 c_r}{s_t} \right)^2 \quad (15)$$

or

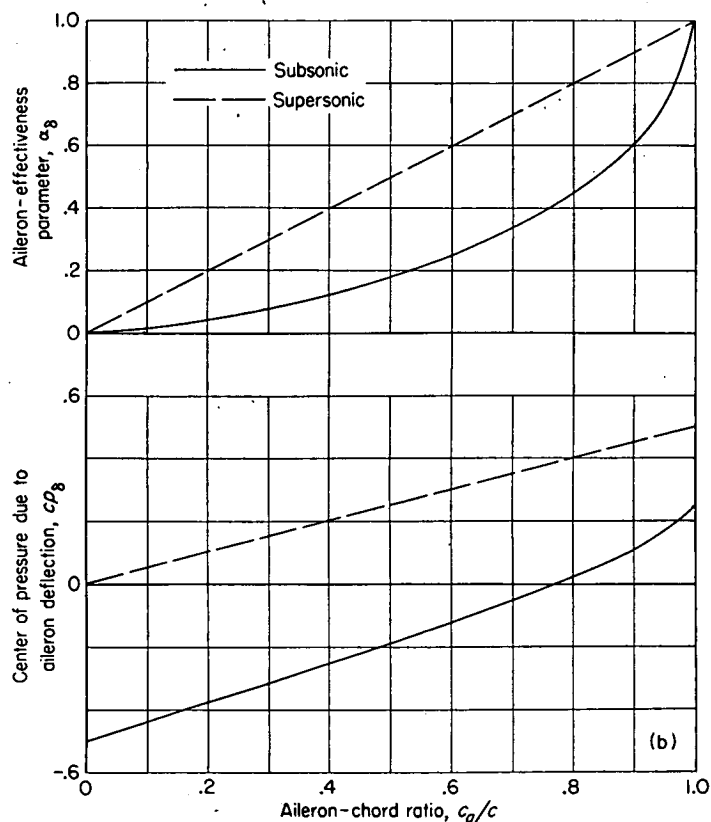
$$\frac{d}{k^2} = \frac{32}{(1+\lambda)^2} \frac{E}{G} \left(\frac{\eta_{19} e_1}{A_\Lambda} \right)^2 \frac{\eta_6 \eta_7^2}{\eta_8 \eta_9 \eta_{15} \eta_b^2} \quad (16)$$

where F_r is a root-stiffness parameter, and η_a , η_b , η_6 , η_7 , η_8 , η_9 , η_{15} , and η_{19} are structural-effectiveness factors defined in reference 2. (If, at the time a preliminary survey of aeroelastic effects is to be made, no information whatever concerning the wing stiffness is available, eqs. (13), (14), and (16) may be used; if an estimate of the root stiffnesses $(GJ)_r$ and $(EI)_r$ is available, eqs. (6), (9), and (15) may be used.)

The charts of figures 3 (b), 4 (b), and 5 (b) are the same as those of figures 3 (a), 4 (a), and 5 (a), respectively, except that these charts are for wings with the moment-arm ratio ϵ equal to unity (corresponding roughly to subsonic conditions with the elastic axis fairly far forward on the wing).



(a) Trailing-edge ailerons.



(b) Leading-edge ailerons.

FIGURE 2.—Theoretical two-dimensional values of the aileron-force parameters.

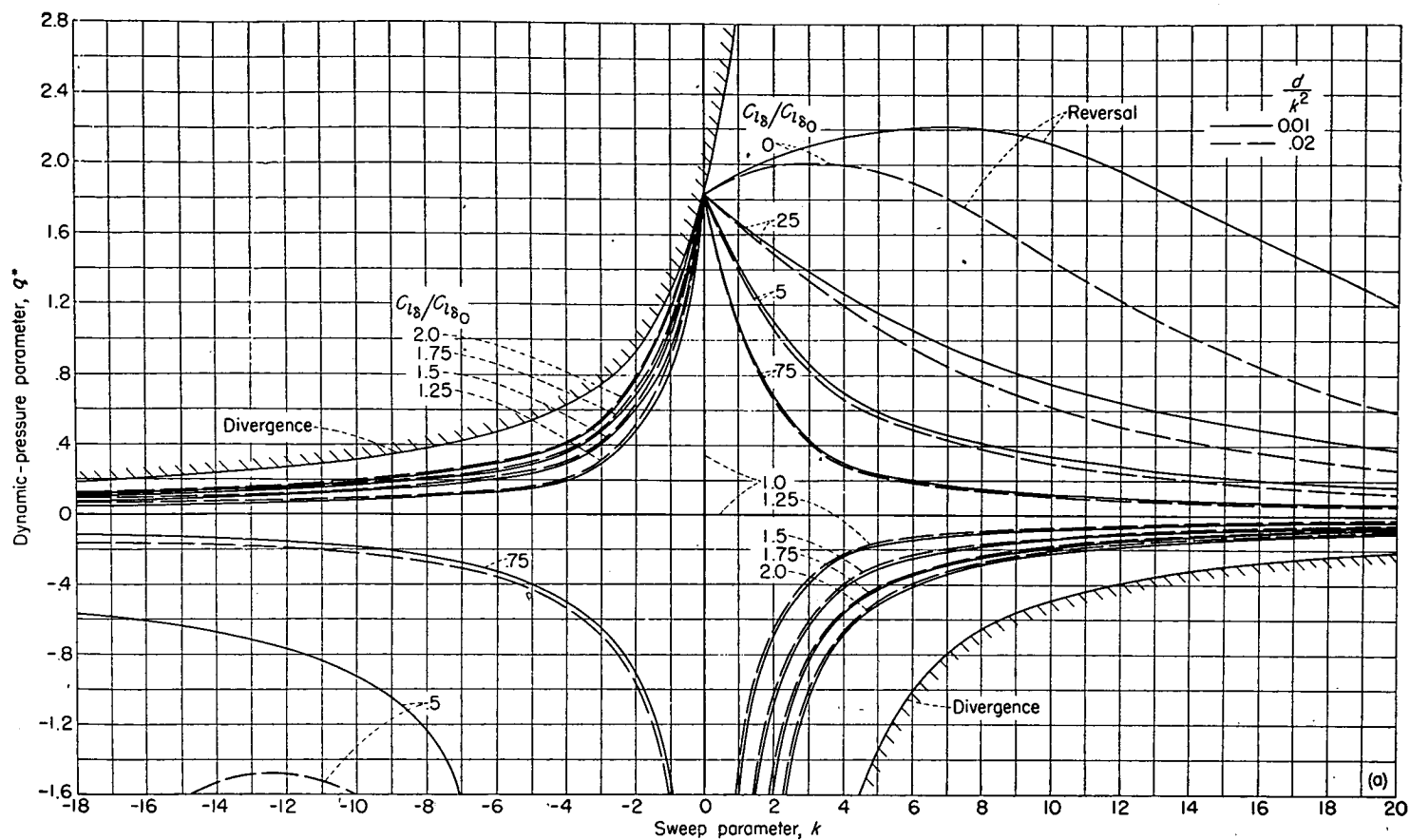
(a) Wings with moment arm $e_2=0$.

FIGURE 3.—Charts for a preliminary survey of lateral control for flexible wings of taper ratio 0.2.

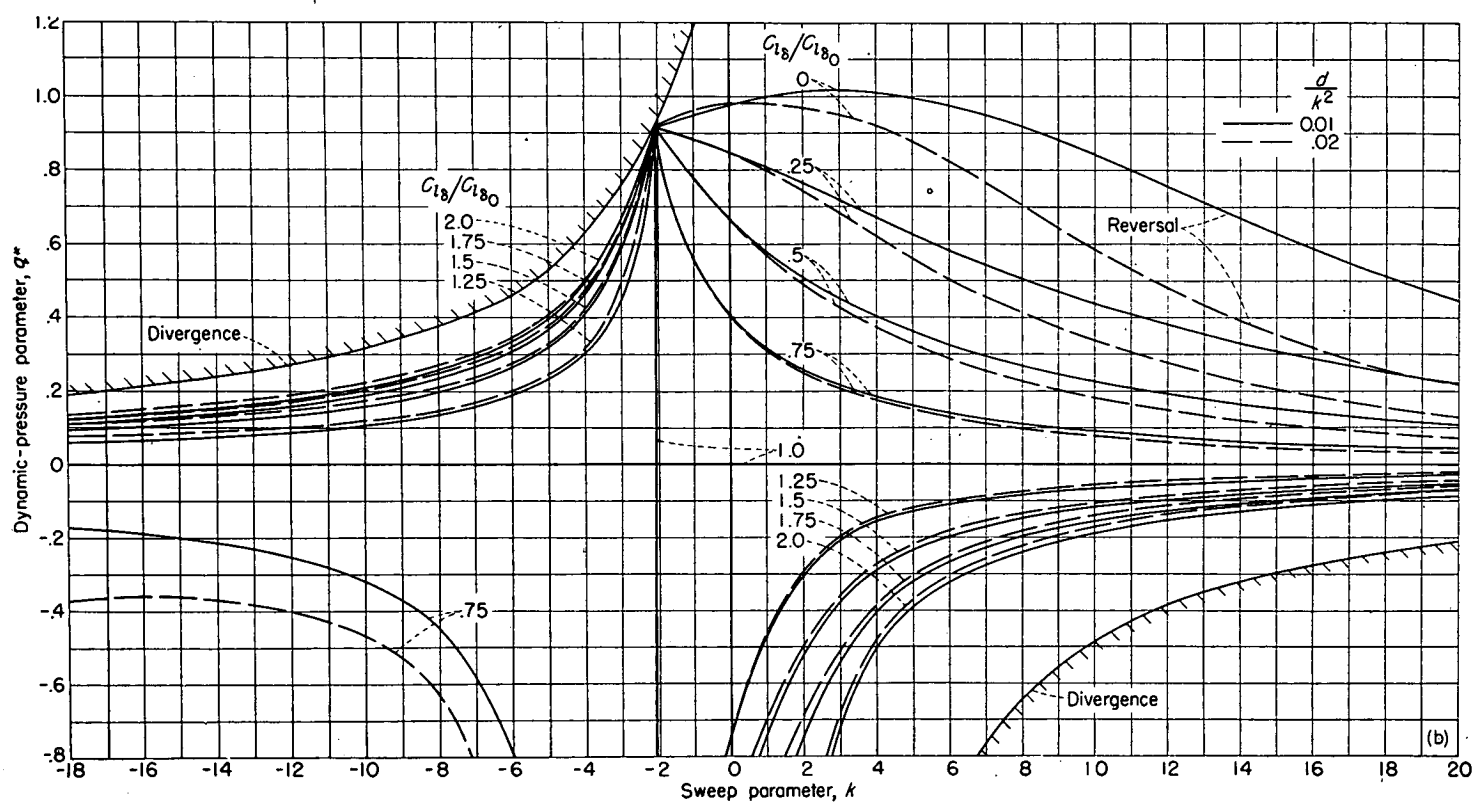
(b) Wings with moment-arm ratio $\epsilon=1.0$.

FIGURE 3.—Continued.

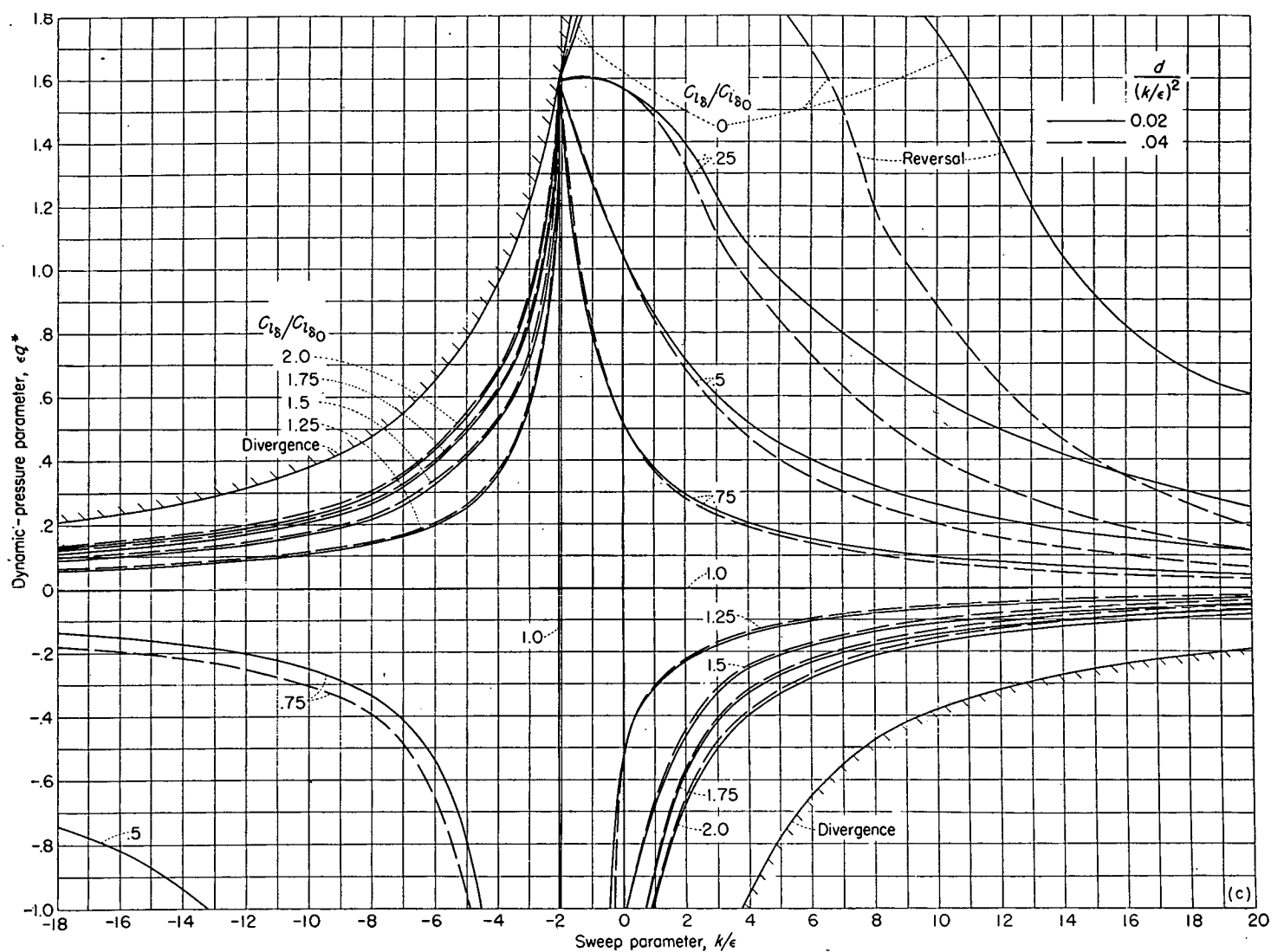
(c) Wings with moment arm $e_1=0$.

FIGURE 3.—Concluded.

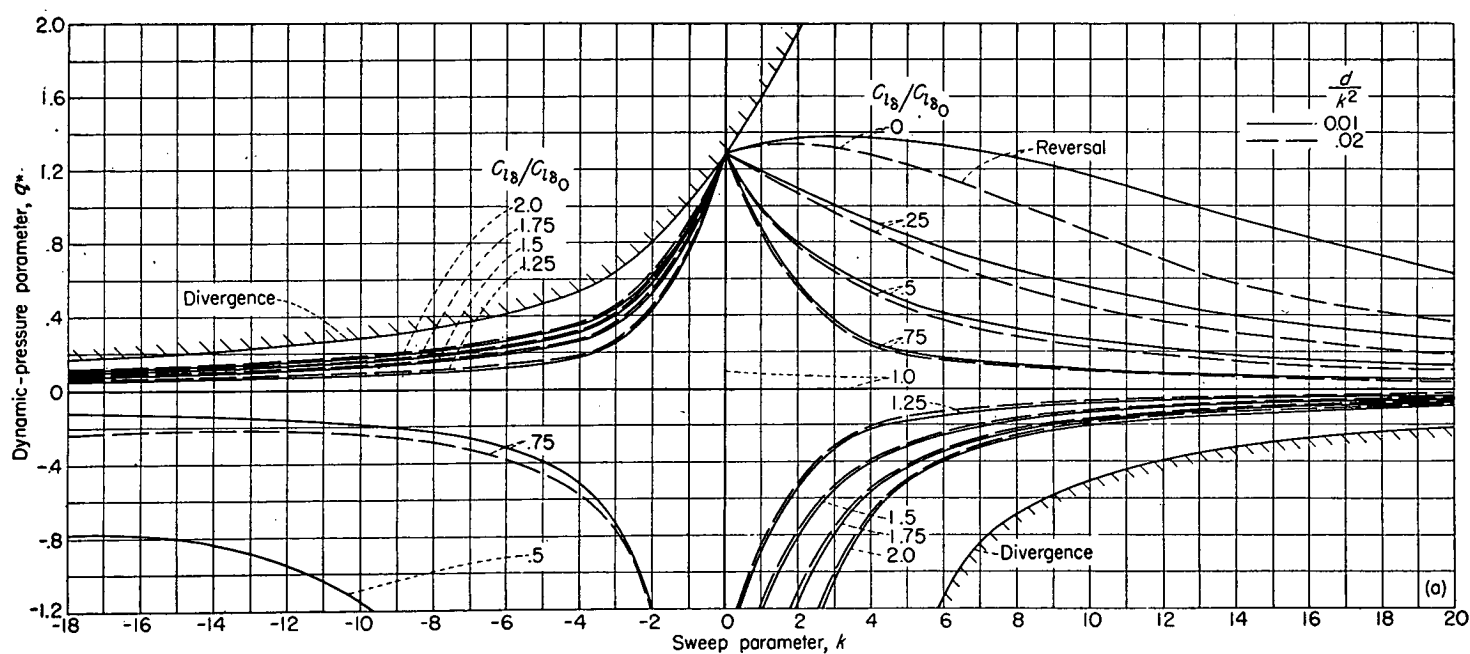
(a) Wings with moment arm $e_2=0$.

FIGURE 4.—Charts for a preliminary survey of lateral control for flexible wings of taper ratio 0.5.

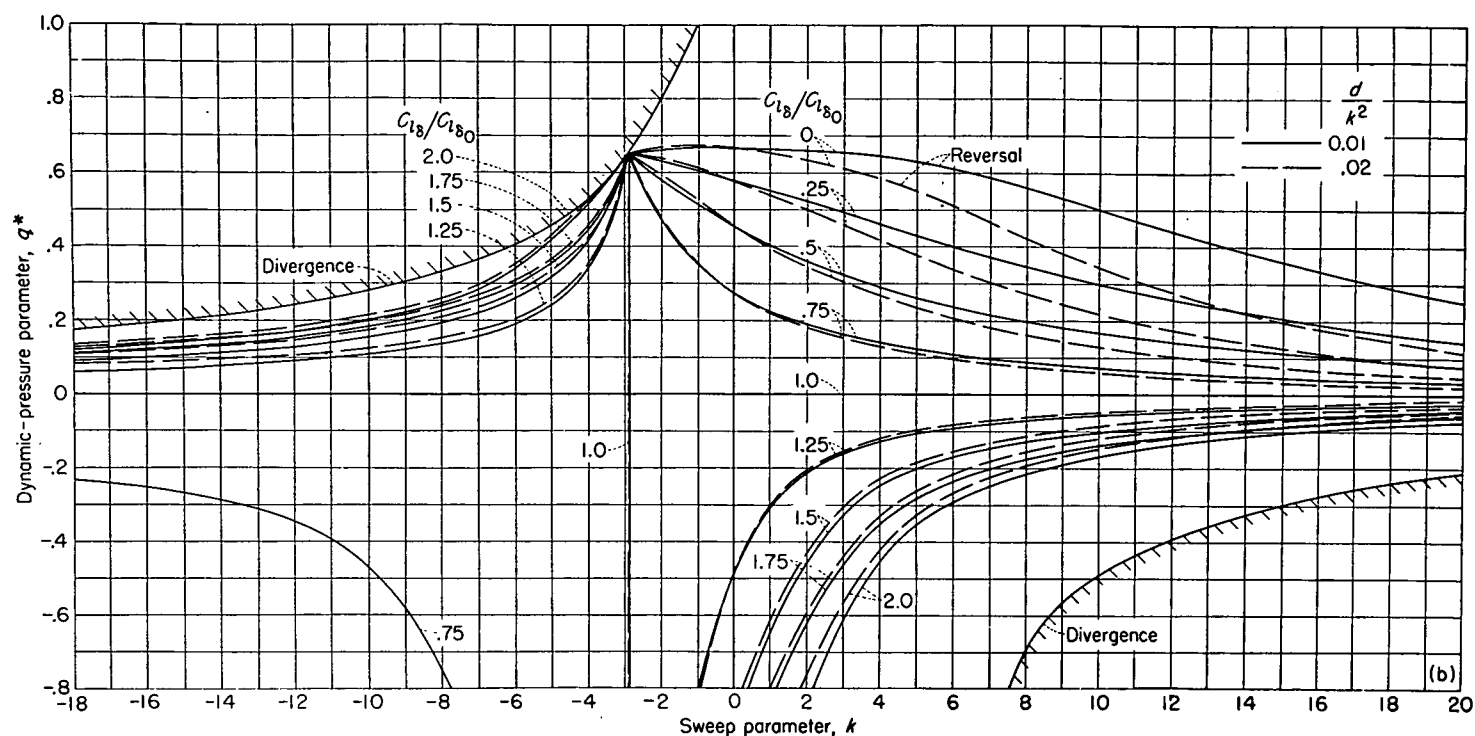
(b) Wings with moment-arm ratio $e = 1.0$.

FIGURE 4.—Continued.

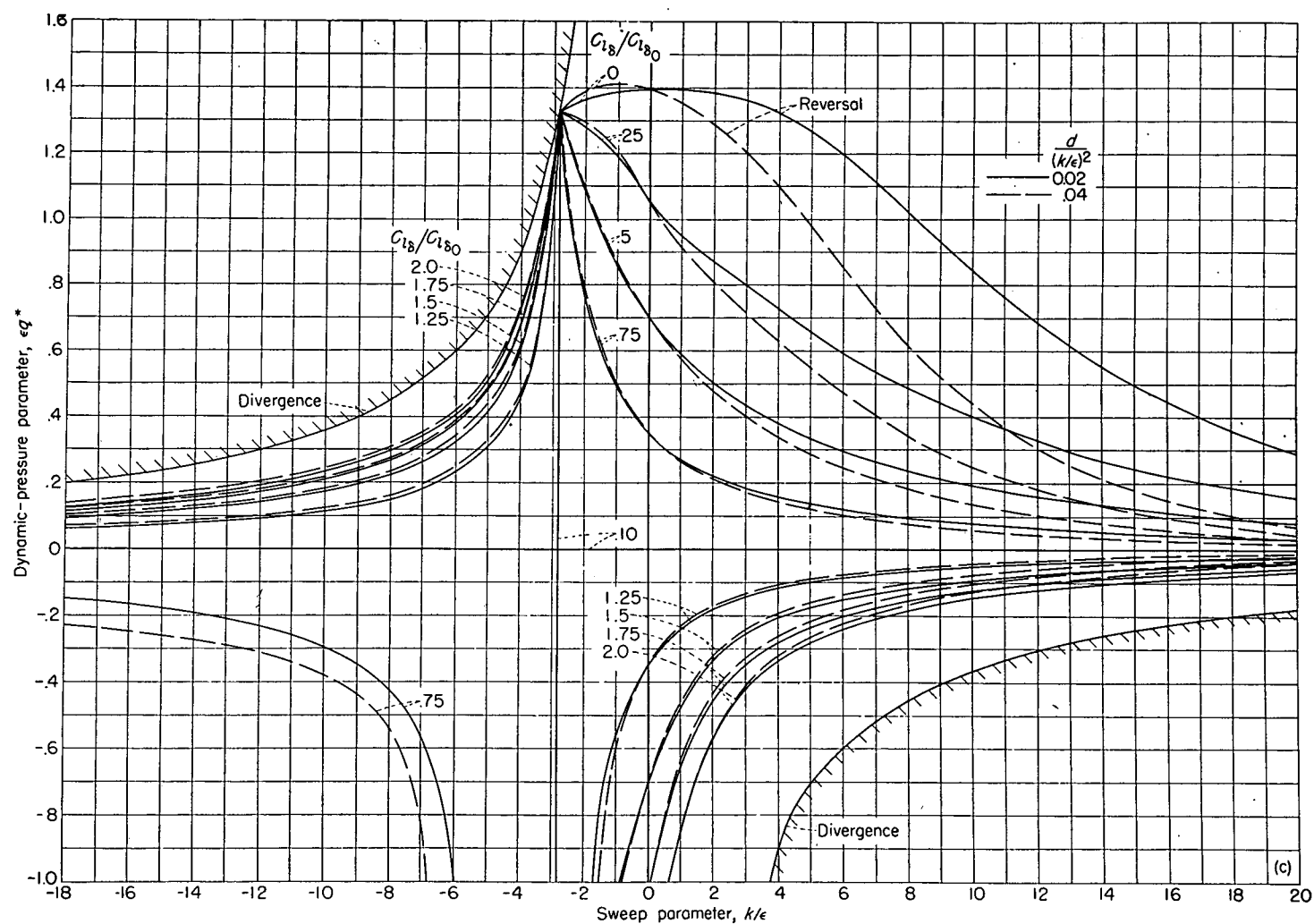
(c) Wings with moment arm $e_1 = 0$.

FIGURE 4.—Concluded.

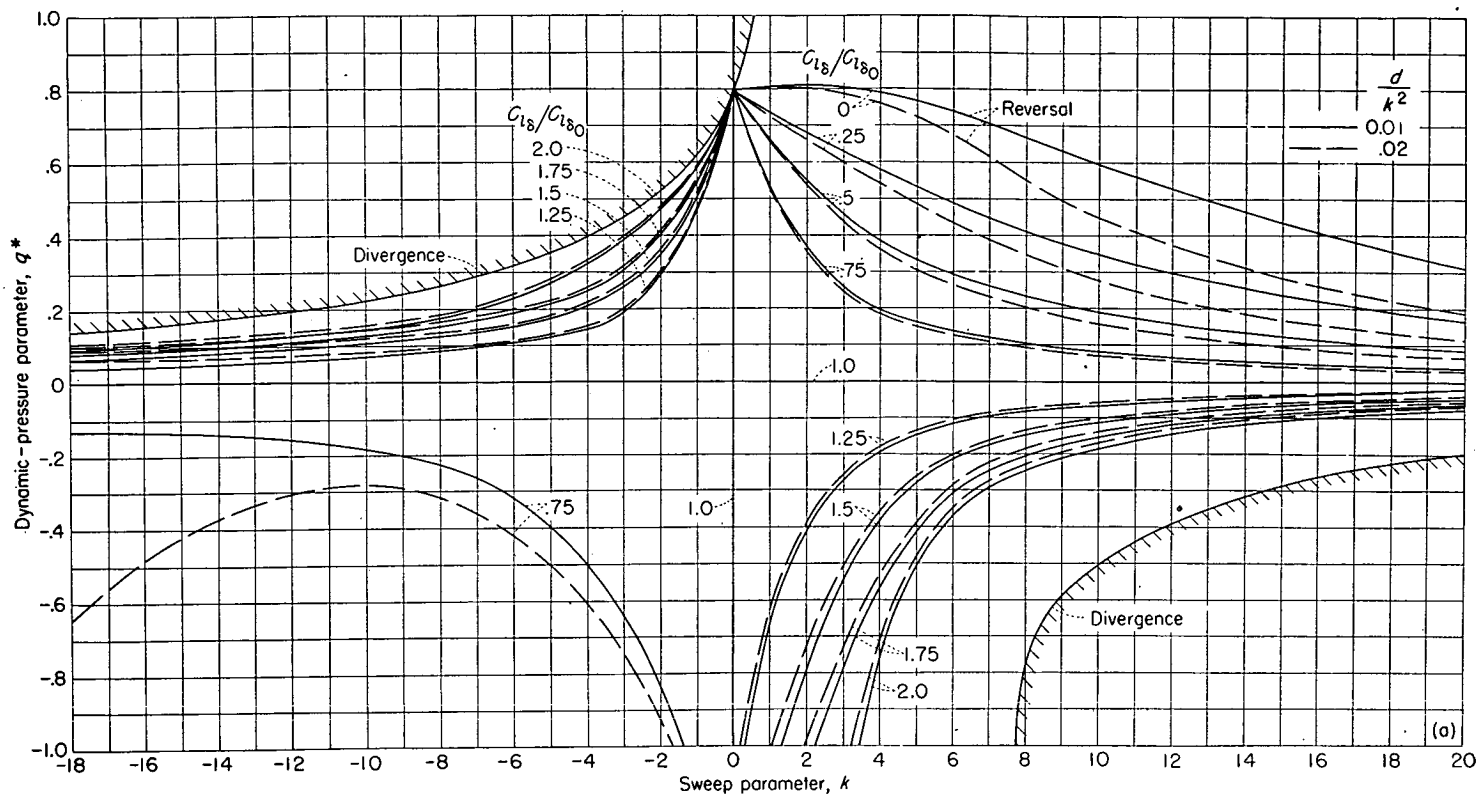
(a) Wings with moment arm $e_2=0$.

FIGURE 5.—Charts for a preliminary survey of lateral control for flexible wings of taper ratio 1.0.

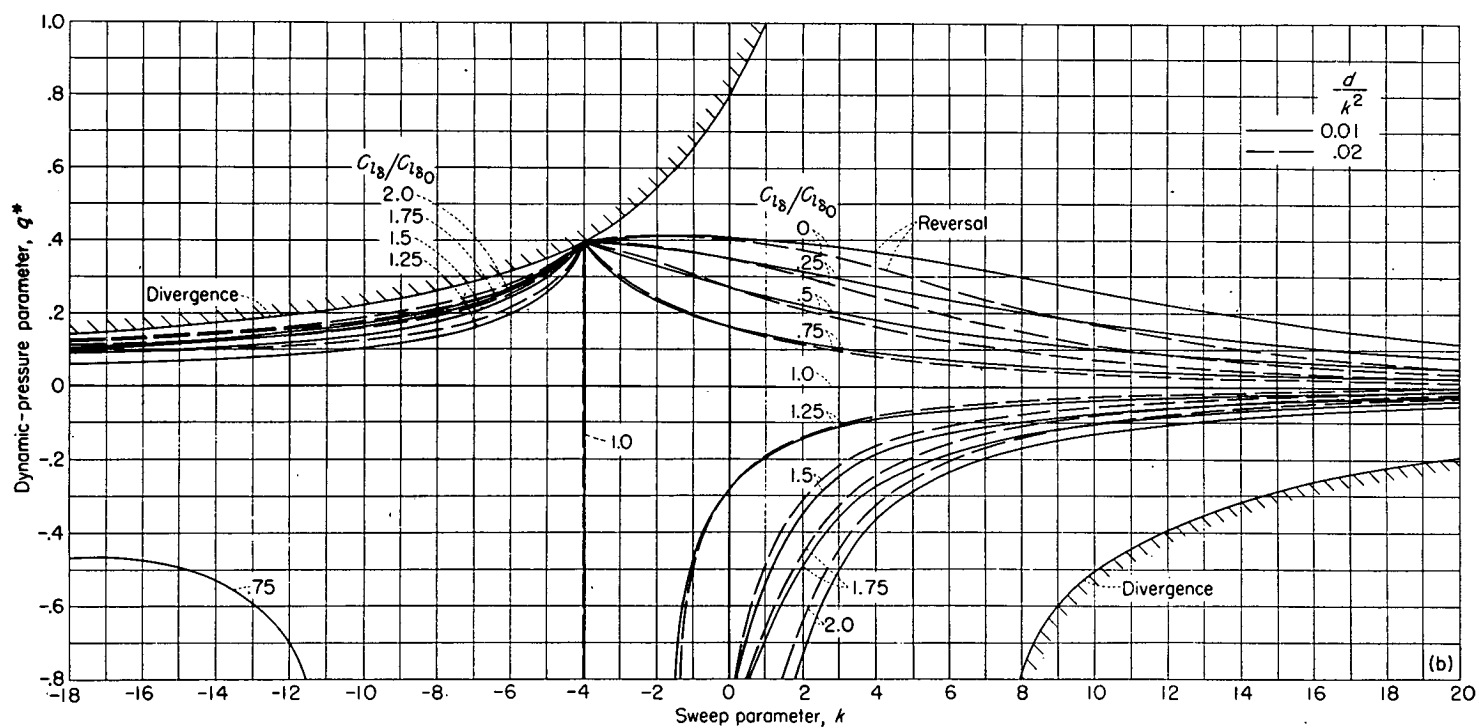
(b) Wings with moment-arm ratio $\epsilon=1.0$.

FIGURE 5.—Continued.

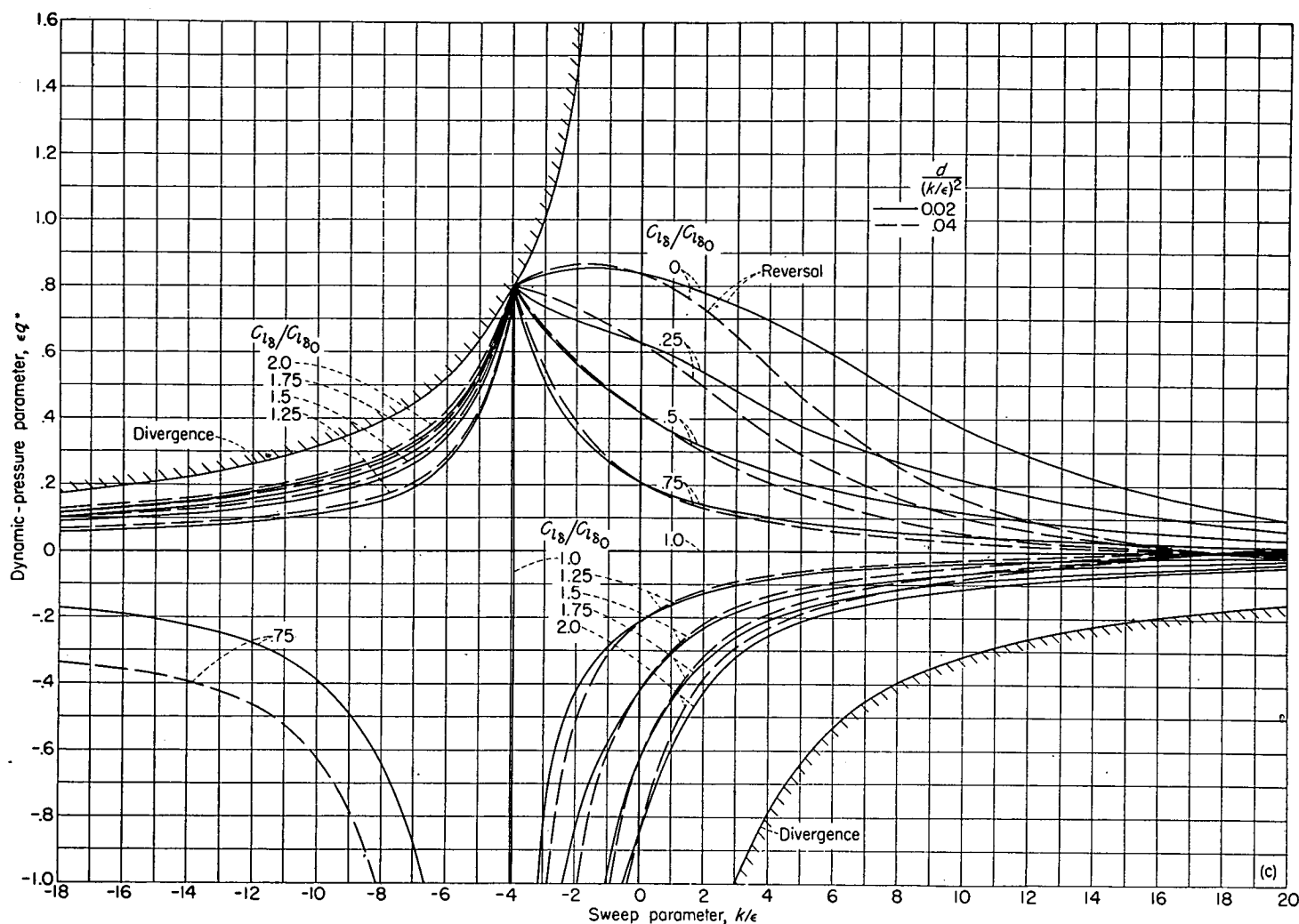
(c) Wings with moment arm $e_1=0$.

FIGURE 5.—Concluded.

The charts of figures 3 (c), 4 (c), and 5 (c) pertain to wings with the moment arm e_1 equal to zero (corresponding roughly to supersonic flow conditions). These figures show the dynamic-pressure parameter ϵq^* defined by either equation (7) or

$$\epsilon q^* = \frac{(1+\lambda)^2}{18432} \frac{C_{L\alpha} e_2 \cos \Lambda}{\frac{G}{F_B} \frac{nW}{S} \frac{h_r}{c_r} F_r \eta_a} q \quad (17)$$

plotted against the sweep parameter k/ϵ defined by equation (10) or

$$\frac{k}{\epsilon} = \frac{1+\lambda}{2} \frac{G}{E} \eta_b \frac{A_\Lambda}{\eta_{19} e_2} \tan \Lambda \quad (18)$$

for two values of the stiffness parameter

$$\frac{d}{(k/\epsilon)^2} = \frac{(EI)_r}{(GJ)_r} \left(\frac{e_2 c_r}{s_t} \right)^2 \quad (19)$$

or

$$\frac{d}{(k/\epsilon)^2} = \frac{32}{(1+\lambda)^2} \frac{E}{G} \left(\frac{\eta_{19} e_2}{A_\Lambda} \right)^2 \frac{\eta_6 \eta_7^2}{\eta_8 \eta_9 \eta_{15} \eta_b^2} \quad (20)$$

The various lines of the charts of figures 3 to 5 designate the conditions at which a wing designed on the basis of strength considerations alone is likely to encounter changes in aileron rolling moment by various amounts due to wing flexibility. The lines for zero rolling moment designate the conditions at aileron reversal. These charts should be used in conjunction with the preliminary survey charts in reference 2. The significance of the four quadrants of figures 3 (a), 3 (b), 4 (a), 4 (b), 5 (a), and 5 (b) is the same as that of the four quadrants of figures 2 (a), 2 (b), 2 (c), and 2 (d) of reference 2 and is discussed in the section "Preliminary Survey of Aeroelastic Behavior" of reference 2. The significance of the quadrants of figures 3 (c), 4 (c), and 5 (c) of the present report can be analyzed in the same way, except that the moment arm e_2 takes the place of the moment e_1 . These three figures are analogous to figure 2 (e) of reference 2 in that they pertain to the case $e_1=0$. (Although k is infinite when e_1 is zero, which happens when the section aerodynamic centers are on the elastic axis, the parameter k/ϵ is not infinite in that case, except if e_2 also happens to be zero, a condition which can be realized only with a full-chord aileron.)

After it has been found, through the use of the charts in reference 2, to what extent the wing design is affected by such aeroelastic phenomena as divergence and aerodynamic-center shift, the same procedure can be used with the preliminary survey charts in this report to ascertain whether the wing design is likely to be affected by lateral-control difficulties. If these charts indicate the likelihood of significant aeroelastic effects on the aileron rolling moment, further calculations are desirable. The charts and approximate formulas of this report may be used for the preliminary calculations; once the structure has been designed, more refined methods, such as that of reference 1, may be used.

CALCULATION OF THE AEROELASTIC PHENOMENA RELATED TO LATERAL CONTROL

Analysis of the many solutions for the aeroelastic phenomena considered in this report obtained by the methods given in the appendix shows that the data can be summarized by means of approximate formulas. These formulas involve the aerodynamic, geometric, and structural parameters of the wing through the dimensionless parameters k , k/ϵ , ϵ , and d (eqs. (9), (10), (11), and (12), respectively) and through a series of constants K_1 to K_7 . The constants are functions of the taper ratio, stiffness distribution, and aileron span and are given in table I. As in reference 2, the form of these approximate formulas has been guided by considerations based on an idealized semi-rigid wing, and the actual values of the constants K_3 to K_7 (K_1 and K_2 have been given in ref. 2) were obtained by fitting the solution for the functions B_1 to B_6 defined in the appendix by equations (A18) and (A27) to their approximate expressions, equations (A32).

Dynamic pressure at aileron reversal.—The solutions for the aileron reversal speed obtained by the methods given in the appendix can be summarized by approximate formulas which give the dimensionless parameters q^*_{R} , $(\epsilon q^*)_R$, or $\bar{q}_R$ —that is, the values of the parameters defined in equations (6), (7), and (8) which correspond to the value of the dynamic pressure at aileron reversal—in terms of the parameters k , k/ϵ , ϵ , and d defined by equations (9), (10), (11), and (12), respectively.

An approximate formula for q^*_{R} is

$$q^*_{R} = \frac{K_1 \left(1 - K_3 \frac{\epsilon}{k} d \right)}{1 + (K_4 + K_2 K_3 d) \epsilon + K_5 d + K_6 \frac{\epsilon}{k} d + K_7 k} \quad (21)$$

For very small values of the section moment arm e_1 and the resulting large values of the parameters ϵ and k , the following alternative forms of equation (21) are more convenient to use:

$$(\epsilon q^*)_R = \frac{K_1 \left(1 - K_3 \frac{\epsilon}{k} d \right)}{\frac{1}{\epsilon} + K_4 + K_2 K_3 d + K_5 \frac{1}{\epsilon} d + K_6 \frac{1}{\epsilon} \frac{\epsilon}{k} d + K_7 \frac{k}{\epsilon}} \quad (22)$$

and

$$\bar{q}_R = \frac{K_1 \left(1 - K_3 \frac{\epsilon}{k} d \right)}{\frac{1}{k} + (K_4 + K_2 K_3 d) \frac{\epsilon}{k} + K_5 \frac{d}{k} + K_6 \frac{\epsilon}{k} \frac{d}{k} + K_7} \quad (23)$$

When the angle of sweep is zero, equations (21) and (22) reduce, respectively, to

$$q^*_{R} = \frac{K_1}{1 + K_4 \epsilon} \quad (24)$$

and

$$(\epsilon q^*)_R = \frac{K_1}{K_4 + \frac{1}{\epsilon}} \quad (25)$$

and when the moment arm e_1 is zero, as it may be in supersonic flow, equations (22) and (23) reduce to

$$(\epsilon q^*)_R = \frac{K_1 \left(1 - K_3 \frac{\epsilon}{k} d \right)}{K_4 + K_2 K_3 d + K_7 \frac{k}{\epsilon}} \quad (26)$$

and

$$\bar{q}_R = \frac{K_1 \left(1 - K_3 \frac{\epsilon}{k} d \right)}{K_4 + K_2 K_3 d \frac{\epsilon}{k} + K_7} \quad (27)$$

The constants K_1 to K_7 are given in table I for wings having half-span outboard ailerons and taper ratios of 0.2, 0.5, and 1.0, for the two different types of stiffness distributions. These constants were found from the results of the numerical matrix method derived in the appendix. Also given in table I are these constants for uniform wings having full-span ailerons and tip ailerons calculated from the results of the analytic integration method of the appendix. Since values of the constants K_1 to K_7 are given for three aileron

TABLE I.—VALUES OF THE COEFFICIENTS K_1 TO K_7

λ	GJ and EI	$\frac{(h/c)_t}{(h/c)_r}$	s_i^*	s_o^*	K_1	K_2	K_3	K_4	K_5	K_6	K_7
1.0	Vary as c^4	1.0	0	1.0	2.47	0.390	2.000	1.028	0.615	-0.285	-0.020
1.0		1.0	.5	1.0	2.58	.381	1.362	.972	.715	.194	.028
1.0		1.0	→1	1.0	2.47	.390	1.000	.822	.850	.310	.129
.5		1.0	.5	1.0	2.83	.480	.891	1.009	.643	.159	.025
.2		1.0	.5	1.0	2.92	.590	.633	1.035	.567	.134	-.016
1.0	Given by constant-stress criterion	1.0	.5	1.0	.795	.252	1.362	.946	.430	-.011	-.010
.5		1.0	.5	1.0	1.287	.357	.891	.923	.410	-.025	-.031
.2		1.0	.5	1.0	1.830	.480	.633	.872	.392	-.047	-.053
.5		.5	.5	1.0	.928	.310	.891	.852	.335	-.105	-.051
.5	Increased beyond values required by constant-stress criterion	1.0	.5	1.0	1.700	.398	.891	.975	.433	-.006	-.015

spans in the case of the uniform wing, they may be interpolated to yield values for other aileron spans. No calculations were made for other than half-span ailerons on tapered wings; nevertheless, as pointed out previously, q_R calculated for half-span ailerons should be reasonably valid for outboard ailerons having spans which differ considerably from one-half.

No aileron-reversal calculations have been made for swept wings with inboard ailerons. However, for unswept uniform wings the dynamic pressure at aileron reversal has been calculated for the limiting case of a wing with an inboard aileron of vanishingly small span by operational methods similar to those described in the appendix in connection with the calculations for uniform wings. The value of q^*_R obtained in this manner is shown as a function of ϵ in figure 6. Also shown in figure 6 are the values of q^*_R for full-span and for tip ailerons. For small values of ϵ , such as are likely to be encountered in subsonic flight, there is little difference in the values of q^*_R for the three aileron configurations, but at large values of ϵ , such as are likely to be encountered at high-supersonic speeds, there is some difference between them; the aileron reversal speed is highest for the tip aileron and lowest for the inboard aileron. However, these conclusions may not be valid for nonuniform or swept wings.

With the values of q^*_R , $(\epsilon q^*)_R$, and $\bar{q}_R$ given by equations (21), (22), and (23) and the definitions of these parameters given by equations (6), (7), and (8), the values of q at aileron reversal may be determined. If desired, the corresponding airspeed may be determined from the relation

$$V_R = \sqrt{\frac{q_R}{\rho/2}} \quad (28)$$

If both q_D (as obtained from ref. 2, for instance) and q_R are positive and q_R is greater than q_D , there is no actual aileron reversal speed; in fact, the rolling moment due to aileron deflection will tend to increase with dynamic pressure until divergence is reached.

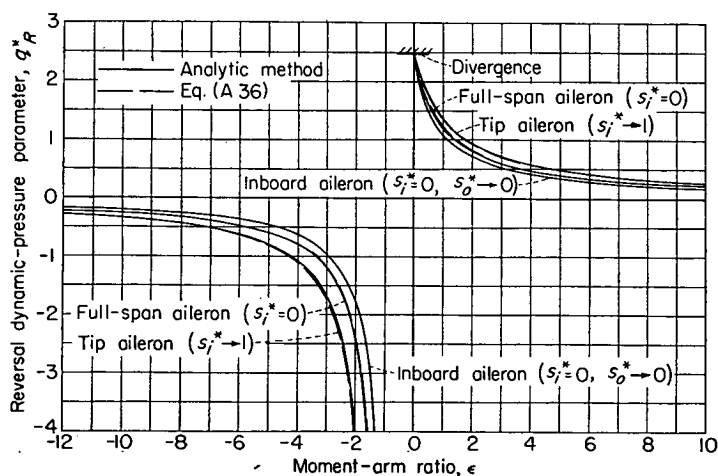


FIGURE 6.—Comparison of dynamic pressures at aileron reversal calculated by the analytic integration method of the appendix with those calculated from equation (A36) for unswept uniform wings.

Nor is there an actual reversal speed if q_R is negative, regardless of whether q_D is positive or negative, although the rolling moment due to aileron deflection will decrease slightly with increasing dynamic pressure if q_R/q_D is greater than 1. Inasmuch as aileron reversal, unlike divergence, is not an instability problem, although it is often convenient to analyze it as such, there is no possibility of encountering aileron reversal in a higher mode when q_R is negative, at least not in the case of ordinary wings with straight leading and trailing edges and with substantially straight elastic axes.

The value of q_R calculated for any given value of q^*_R , $(\epsilon q^*)_R$, or $\bar{q}_R$ depends on the value of the effective lift-curve slope $C_{L_{\alpha_e}}$ and, hence, on the Mach number. As suggested in reference 1, the value of q_R may be plotted against Mach number on log-log coordinates; if the straight lines of the actual dynamic pressure at several altitudes as functions of Mach number are drawn on the same plot, an intersection of the reversal line with one of the lines of actual dynamic pressure designates possible aileron reversal at that value of dynamic pressure, Mach number, and altitude.

Spanwise angle-of-attack distribution.—In the appendix, an approximate expression is determined for the change in angle of attack due to the deflection of ailerons on flexible wings. The ratio of the angle-of-attack distribution due to structural deformation α_s to the effective angle of attack of an aileron α_{δ} is

$$\frac{\alpha_s}{\alpha_{\delta}} = \frac{q/q_D}{1 - \frac{q}{q_D}} \frac{(K_4\epsilon + K_2k)f_a - K_2k\Delta f_a}{1 - K_2k} \quad (29)$$

The functions f_a and Δf_a of the spanwise coordinate s^* are given in figure 7 for wings having half-semispan outboard ailerons, taper ratios of 0.2, 0.5, and 1.0, and the two different types of stiffness distribution. The value of q_D required in equation (29) may be found from the approximate expression for q^*_D or $\bar{q}_D$ given in reference 2 as

$$q^*_D = \frac{K_1}{1 - K_2k} \quad (30)$$

or

$$\bar{q}_D = -\frac{K_1}{K_2 - \frac{1}{k}} \quad (31)$$

When q_D is very large, a more convenient form of equation (29) is as follows:

$$\frac{\alpha_s}{\alpha_{\delta}} = \frac{\epsilon q^*}{1 - \frac{q}{q_D}} \frac{\left(K_4 + K_2 \frac{k}{\epsilon}\right) f_a - K_2 \frac{k}{\epsilon} \Delta f_a}{K_1} \quad (32)$$

Spanwise lift distribution.—Within the limitations of the modified strip theory used in the analysis, the lift per inch of span is proportional to the local effective angle of attack, so that

$$\frac{l}{\alpha_{\delta}} = qcC_{L_{\alpha_e}} \left(\frac{\alpha_s}{\alpha_{\delta}} + I_a \right) \quad (33)$$

where α_s/α_0 is obtained as indicated in the preceding section and I_a is a unit step function of the distance along the span defined by

$$I_a = 0 \quad (\text{when } s < s_t) \\ I_a = 1 \quad (\text{when } s > s_t)$$

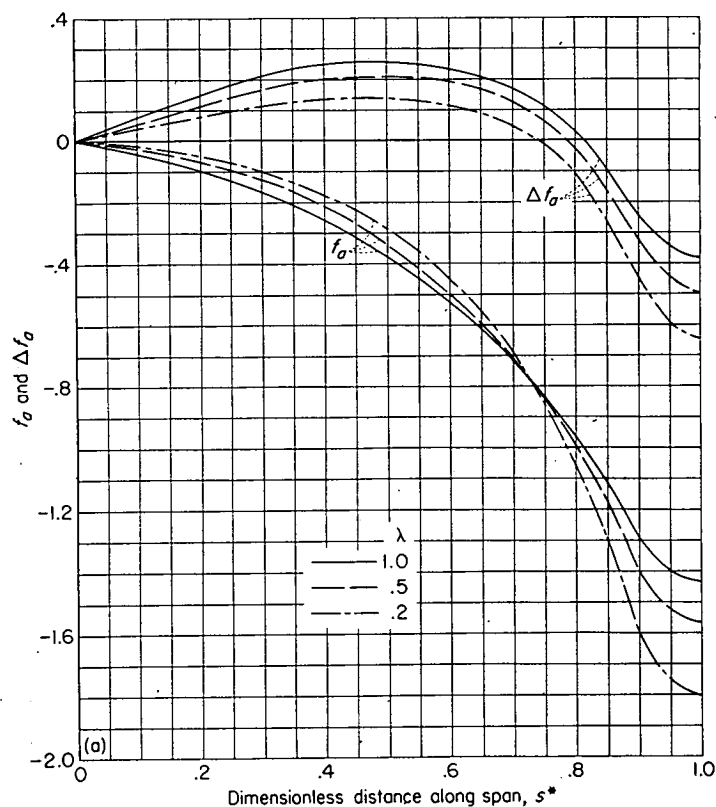
Inasmuch as the functions f_a and Δf_a have been calculated only for half-semispan outboard ailerons or spoilers, the expressions for angle-of-attack distributions and lift distributions (eqs. (29) and (33)) can be used directly only for this case.

Rolling-moment coefficient and rate of roll due to aileron deflection.—The rolling-moment coefficient due to a unit aileron deflection C_{l_δ} may be obtained in terms of its equivalent rigid-wing value from the approximate formula

$$\frac{C_{l_\delta}}{C_{l_\delta 0}} = \frac{1 - \frac{q}{q_R}}{1 - \frac{q}{q_D}} \quad (34)$$

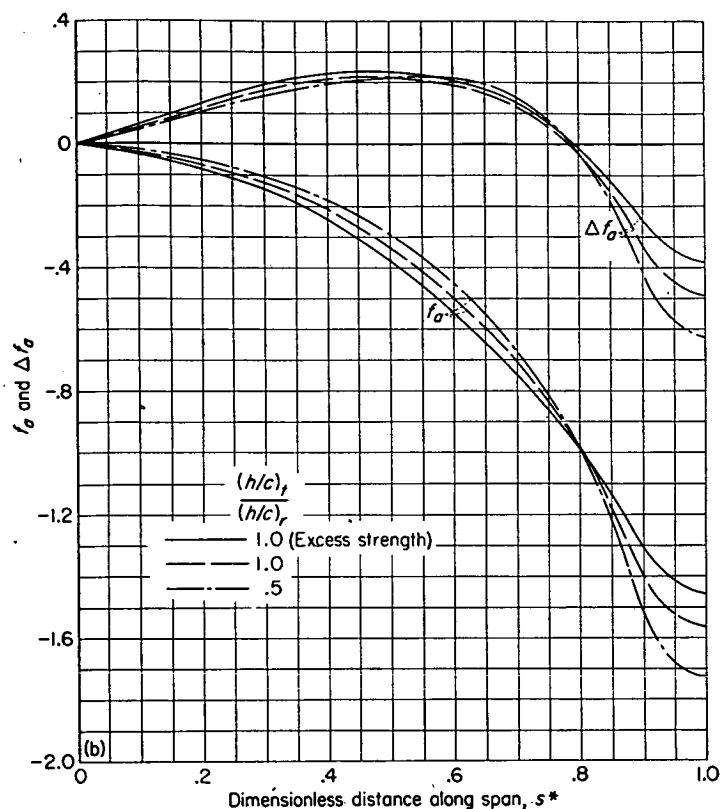
and the wing-tip helix angle due to an aileron deflection $pb/2V$, which is a measure of the rate of roll and the rolling maneuverability, may be obtained from

$$\frac{pb}{2V} = \left(\frac{pb}{2V} \right)_0 \left(1 - \frac{q}{q_R} \right) \quad (35)$$



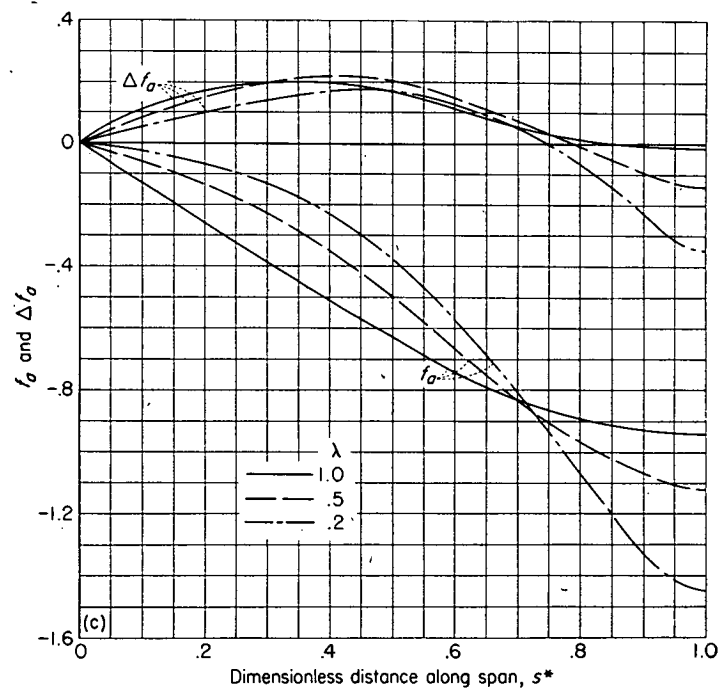
(a) Stiffnesses given by constant-stress criterion for $\frac{(h/c)_t}{(h/c)_r} = 1.0$.

FIGURE 7.—The angle-of-attack distribution functions f_a and Δf_a for aileron deflections. $s_t^* = 0.5$.



(b) Stiffnesses related to those given by constant-stress criterion for wings of taper ratio 0.5.

FIGURE 7.—Continued.



(c) Stiffnesses proportional to c^1 .

FIGURE 7.—Concluded.

The manner in which the aileron-effectiveness parameter $C_{l_s}/C_{l_{s0}}$ varies with dynamic pressure depends on the ratio of the reversal to the divergence dynamic pressure as may be seen from equation (34) and figure 8. When the aerodynamic center is ahead of the elastic axis, as is generally the case at subsonic speeds, q_R/q_D is positive and greater than one for sweptforward wings, positive and less than one for unswept wings, and negative for sweptback wings. Thus, in general, $C_{l_s}/C_{l_{s0}}$ increases with dynamic pressure until divergence is reached for sweptforward wings; it decreases slowly at first and then more rapidly as reversal is approached for unswept wings; and it decreases rapidly at first and then more slowly for sweptback wings. The rapid decrease in rolling effectiveness for sweptback wings can be alleviated by the use of unconventional lateral-control devices which have their centers of pressure ahead of the elastic axis, such as leading-edge ailerons or spoilers. These devices may also serve to make the dynamic pressure at aileron reversal negative, so that there is no reversal of lateral control in the given speed range. The loss or gain of lateral control is then given by the part of figure 8 for negative values of q/q_R .

The analysis summarized by the approximate formulas (34) and (35) is based on rolling moments about the wing root instead of about the fuselage center line, partly to simplify the analysis and partly to avoid the introduction of the fuselage width as another independent parameter. However, these approximate formulas should be valid for obtaining rolling moments and rates of roll about the fuselage center line as well, because equations (34) and (35) are expressed as ratios of flexible-wing to rigid-wing values; that is, the ratio of the rolling moment about the fuselage center line to its rigid-wing value should be nearly the same as the ratio of the equivalent rolling moment about the wing root to its rigid-wing value when the ratio of the fuselage width to the wing span is small.

The rolling-moment coefficient and rates of roll given by equations (34) and (35) are functions of the aileron span inasmuch as q_R is a function of the aileron span. The variation of C_{l_s} or $pb/2V$ with aileron span can therefore be found only for uniform wings; however, since the effect of aileron

span on q_R is not very great, its effect on C_{l_s} or $pb/2V$ is not likely to be great. The rolling moment due to the deflection of an inboard aileron can be found by superposition, because the aeroelastic equations upon which the results of this report are based are linear (that is, the rolling-moment coefficient due to a 30-percent-span inboard aileron, for example, is equal to C_{l_s} for a full-span aileron minus C_{l_s} for a 70-percent-span outboard aileron), but only in the case of the uniform wing is the required information for the full-span aileron presented herein. That the aeroelastic effects on lateral control are likely to be similar for inboard as for outboard ailerons, at least for unswept wings, may be deduced from figure 6.

Inertia effects.—In steady rolling flight no inertia effects are present which can affect the static aeroelastic problem except, possibly, for centrifugal forces on heavy underslung nacelles. The maximum value of $pb/2V$ is therefore usually unaffected by inertia effects. However, in the equally important problem of initial rolling acceleration, which governs the time in which a given rolling velocity can be attained, inertia effects must usually be taken into account. In reference 2 the observation was made that in symmetric flight inertia effects are not as important as other static aeroelastic effects, except for flying wings, because the inertia forces are in about the same ratio to the aerodynamic forces as the wing weight is to the airplane weight. By the same reasoning, however, inertia effects are almost always very important in getting into a roll, because the inertia forces are then in about the same ratio to the aerodynamic forces as the moment of inertia of the wings about the longitudinal axis of the airplane is to the moment of inertia of the entire airplane about its longitudinal axis, a ratio which is usually not much less than 1.

No charts are presented in this report for these inertia effects because the manner in which mass is distributed varies so widely among different airplanes that preparation of a generally applicable set of charts appears to be impractical at present. However, the procedure outlined in reference 2 for taking inertia effects into account in the calculation of quasi-static aeroelastic phenomena by means of the charts presented therein may be applied to the calculation of the inertia effects encountered in starting a roll. This procedure is described in the following paragraphs.

For a given rolling acceleration $\dot{p}$, the linear normal acceleration of an element of mass at a distance y from the center line of the airplane is $\dot{p}y$. From this linear acceleration and the known or estimated mass distribution of the wing, the inertia load l_i per inch of span and the inertia torque t_i per inch of span can be calculated for any given normal, pitching, or rolling acceleration. Substitution of these loads and torques for the terms l and le_1c in equations (A3) or (A36) and equations (A2) or (A35) of reference 2, respectively, yields the values of the accumulated bending moments and torques in equations (A4) and (A5) or in equations (A37) and (A38) of reference 2. Equation (A6) of reference 2, or the matrix equivalent of this equation, then yields the angle-of-attack distribution due to the deformations caused by the inertia effects associated with the given acceleration.

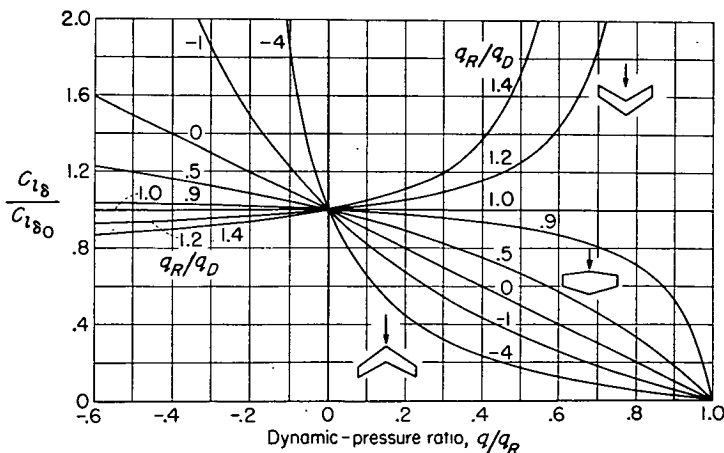


FIGURE 8.—Variation of aileron rolling effectiveness with dynamic-pressure ratio and the parameter q_R/q_D .

This angle-of-attack distribution can be considered as a geometrical angle-of-attack distribution; for the purpose of calculating the increment caused by aeroelastic action, this distribution can be approximated by a linear-twist angle-of-attack distribution with a value at the wing tip which is such that the moment about the effective wing root of the area under the linear-twist distribution equals the moment of the area under the calculated angle-of-attack distribution due to inertia effects. (The moment, rather than the area, is suggested as a basis of correlation because the angles of attack near the wing tip are more important in aeroelastic phenomena than those at the wing root.) The justification for this rather arbitrary approximation to the angle-of-attack distribution is that the correction to be applied as a result of aeroelastic action to the deformations due to inertia loads is usually small compared with these deformations.

The angle of attack due to structural deformation α_s , associated with the linear-twist distribution can then be obtained from equation (21) and figure 7 of reference 2 or, if $\lambda=0$, from figure 8 of reference 2. The lift distribution associated with the total angle-of-attack distribution due to the deformations caused by the inertia effects, including the increment in this angle-of-attack distribution produced by aeroelastic action, can then be found from equation (24b) of reference 2, in which α_s and l_0 pertain to the calculated angle-of-attack distribution due to the inertia effects (not the linear approximation to this distribution). This lift distribution can be integrated to obtain the rolling moment due to inertia effects, as modified by aeroelastic action.

The rolling moment calculated in this manner may then be combined with the rolling moment due to aileron or spoiler deflection, which may be calculated as indicated in the preceding section. If the contributions of the tail and the fuselage to the rolling moment are neglected,

$$\dot{p}(\bar{I}-\bar{I}_w)=\frac{1}{144} C_{l_{\delta_s}} \delta q S b + \left(\frac{\partial L'_w}{\partial \dot{p}} \right)_s \dot{p}$$

where $\dot{p}$ is the angular acceleration in roll, $\bar{I}$ is the mass moment of inertia of the entire airplane about its longitudinal axis, $\bar{I}_w$ is the mass moment of inertia of both wings about the longitudinal axis of the airplane, and $C_{l_{\delta_s}}$ is the flexible-wing value of the rolling-moment coefficient due to aileron deflection (which may be calculated in the manner described in the preceding section). The ratio $(\partial L'_w / \partial \dot{p})_s$ is the rolling moment per unit rolling acceleration due to inertia effects, including aeroelastic effects, and is equal to $-\bar{I}_w$ plus the rolling moment due to the lift which results from the deformations due to the inertia loads per unit angular acceleration in roll as well as from the aeroelastic deformations which accompany these inertia deformations; in other words, $(\partial L'_w / \partial \dot{p})_s$ is equal to $-\bar{I}_w$ plus the rolling moment calculated as described in the preceding paragraphs for $\dot{p}=1$. Then

$$\dot{p} = \frac{1}{144} \frac{1}{1 - \frac{1}{\bar{I} - \bar{I}_w} \left(\frac{\partial L'_w}{\partial \dot{p}} \right)_s} \frac{C_{l_{\delta_s}} \delta q S b}{\bar{I} - \bar{I}_w} \quad (36a)$$

or

$$\dot{p} = \frac{1}{144} \frac{C_{l_{\delta_s}} \delta q S b}{\bar{I} - \bar{I}_w} \quad (36b)$$

where

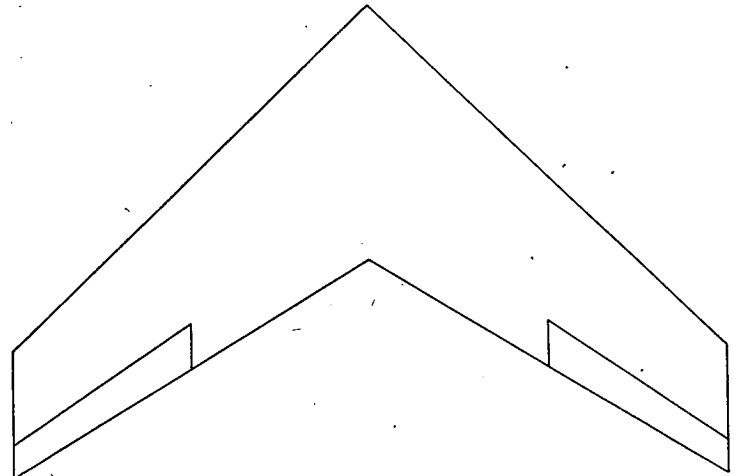
$$C_{l_{\delta_s}, i} = \frac{1}{1 - \frac{1}{\bar{I} - \bar{I}_w} \left(\frac{\partial L'_w}{\partial \dot{p}} \right)_s} C_{l_{\delta_s}}$$

is a rolling-moment coefficient per unit aileron deflection which includes static aeroelastic effects, inertia effects, and the aeroelastic magnification of the inertia effects. This rolling-moment coefficient is a truer index of the rate at which a roll can be initiated than $C_{l_{\delta_s}}$. The initial rolling acceleration (disregarding unsteady-lift effects) can be calculated from equations (36).

ILLUSTRATIVE EXAMPLE

The approximate formulas described in the preceding sections have been used to find the effects of aeroelasticity on some lateral-control properties of the wing considered in the illustrative example of reference 2. The resulting calculations are an extension of those in reference 2, and the additional parameters are presented in table II.

TABLE II.—PARAMETERS OF EXAMPLE WING



Parameter	Subsonic ($M=0$)	Supersonic ($M=1.5$)
s_1^*	0.5	0.5
s_0^*	1.0	1.0
e_2	0.016	0.458
ϵ	0.082	24.1
k	7.76	79.0
d	0.551	0.551
K_1	2.82	2.82
K_2	0.474	0.474
K_3	0.917	0.917
K_4	1.006	1.006
K_5	0.650	0.650
K_6	0.160	0.160
K_7	0.026	0.026
$q^*_{D_0}$	-1.053	-0.0774
$q^*_{R_0}$	1.687	0.0713
q_R , lb/sq ft	10,300	2,500
f_{α}	See fig. 7(c)	See fig. 7(c)
Δf_{α}	See fig. 7(c)	See fig. 7(c)
$\alpha_s / \alpha_{\delta}$	See eq. (29)	See eq. (29)
$C_{l_{\delta}} / C_{l_{\delta_0}}$	$1 + 0.624 \frac{q}{q_D}$	$1 + 1.085 \frac{q}{q_D}$
	$1 - \frac{q}{q_D}$	$1 - \frac{q}{q_D}$
$\frac{(p b / 2 V)}{(p b / 2 V)_0}$	$1 + 0.624 \frac{q}{q_D}$	$1 + 1.085 \frac{q}{q_D}$

The subsonic and supersonic values of the parameters k , ϵ , and d were calculated from equations (9), (11), and (12), respectively. With the appropriate values of the factors K_1 to K_7 interpolated from table I, the values of q_R^* were calculated from equation (21) and are given in table II. From these values of q_R^* , the subsonic and supersonic dynamic pressures at aileron reversal were found by means of equation (6). These values of q_R vary as the reciprocal of the effective lift-curve slope, if the corresponding values of e_1 and e_2 are assumed to remain constant.

In order to find the angle-of-attack distributions due to deflections of the aileron from equation (29), the values of the functions f_a and Δf_a were taken from figure 7(c). The spanwise change in angle of attack is shown in the top plot of figure 9 for different values of the dynamic-pressure ratio. The rolling-moment coefficient and wing-tip helix angle due to deflections of the aileron were calculated from equations (34) and (35) and were plotted in figure 9 as functions of the dynamic-pressure ratio $\frac{q}{-q_D}$.

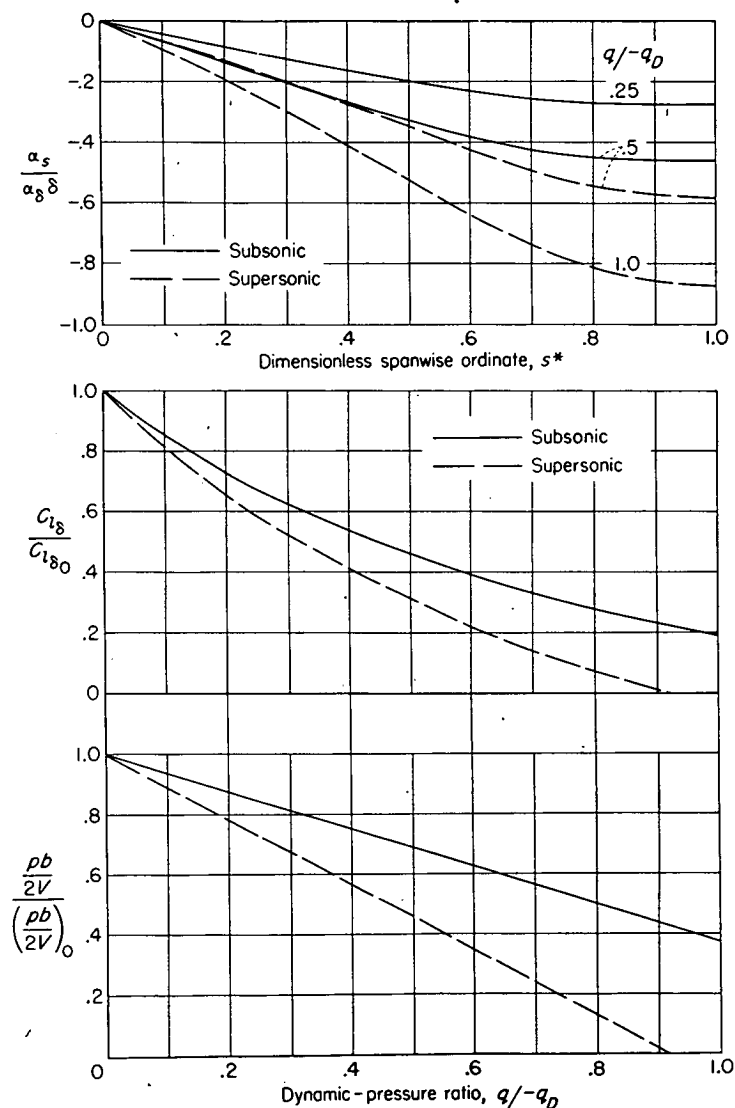


FIGURE 9.—Effect of aeroelastic action on some lateral-control properties of the example wing.

DISCUSSION

LIMITATIONS OF THE CHARTS AND APPROXIMATE FORMULAS

The charts and the approximate formulas presented in this report are subject to certain limitations as a result of the approximations made in the calculations on which they are based. These limitations are discussed fully in reference 2 and can be classified as restrictions on the plan form, on the speed regime, and on the wing structure. The limitations are given very briefly as follows:

(1) The results obtainable by the use of the charts and approximate formulas are likely to be unsatisfactory for wings of very low aspect ratio, very large sweep, or zero taper ratio.

(2) The results are restricted to wings on which the spanwise lift distribution is roughly proportional to the chord and angle of attack and on which the section aerodynamic centers are at an approximately constant fraction of the chord; these restrictions are most likely to be violated by wings flying at transonic speeds and by wings having concentrated sources of lift, such as nacelles and tip tanks.

(3) The results are somewhat restricted to wings with one of the two types of spanwise stiffness distributions used in the analysis, wings with no chordwise bending (relatively thick wings), and wings having an elastic axis at an approximately constant fraction of the chord.

The manner in which the aeroelastic effects of aileron deflection are analyzed in this report imposes certain additional limitations that particularly affect the aileron geometry. In the analysis in the appendix, the spanwise lift distribution due to the deflection of an aileron was approximated by strip theory, an approximation which is probably less valid for aileron deflections than for geometric angles of attack. The assumption was also made that the centers of pressure due to aileron deflections are at a constant fraction of the wing chord; this assumption is also probably less valid than the assumption that the section aerodynamic centers of the wing are at a constant fraction of the chord. Since these assumptions are more nearly true for wings of high than those of low aspect ratio, these limitations serve, in effect, to restrict the applicability of the present report to aspect ratios somewhat higher than those amenable to the analyses of reference 2.

The results of the present report do not take into account explicitly any flexibility of the aileron itself because of the assumption that the angle between the aileron and wing is constant along the span of the aileron. This assumption is almost universally made in analyzing the aeroelastic properties of ailerons and is justifiable because the net effect of the difference between the wing deformations and the aileron deformations on the overall lift and moments appears to be negligible.

As a result of the fact that the static aeroelastic phenomena associated with lateral control involve many more parameters than do those associated with symmetric flight, the coverage of the various parameters is not as complete as in reference 2. Specifically, all the charts and approximate

formulas presented herein are restricted to outboard lateral-control devices (ailerons or spoilers) except for the uniform-wing case, and most of the calculations upon which these results are based were made for wings with half-semispan outboard ailerons ($s_i^*=0.5$, $s_o^*=1$). However, the results of the analysis for the uniform wings with full-span ailerons ($s_i^*=0$, $s_o^*=1$) and tip ailerons ($s_i^*\rightarrow 1$, $s_o^*=1$) indicate that, except for the angle-of-attack distributions, the results based on a half-span aileron are approximately valid for outboard ailerons of spans differing considerably from one-half. The ratio of the aileron chord to the wing chord is assumed in the charts and formulas to be constant over the span of the aileron.

RELATION BETWEEN STRENGTH AND STIFFNESS AS DESIGN CRITERIA

The relation between strength and stiffness as design criteria was discussed in reference 2. The preliminary survey charts in reference 2 indicate the extent to which a wing designed on the basis of strength considerations alone is likely to be affected by aeroelastic phenomena and, consequently, indicate whether the wing has to be stiffened beyond the amount associated with the required strength. The preliminary survey charts of this report (figs. 3 to 5) serve the same purpose in regard to the aeroelastic effects on lateral control; furthermore, even though the charts of reference 2 may indicate that a particular wing is not significantly affected by the aeroelastic phenomena considered in that report, this wing may still have to be stiffened because of an undesirably large loss in lateral control.

As may be concluded from the survey charts of reference 2 and the present report, as well as from the discussion contained in reference 2, the following wings designed on the basis of strength considerations are most likely to be subject to adverse aeroelastic effects on lateral control and rolling maneuverability:

- (1) Wings operating at a high speed or dynamic pressure
- (2) Sweptback wings
- (3) Thin wings
- (4) Wings designed for low wing loading
- (5) Unswept and moderately swept wings with an elastic axis relatively far forward on the chord or with the center of pressure of the lift produced by aileron deflections relatively far back on the chord as a result of the aileron configuration (small aileron chord or wing of low aspect ratio) or flight condition (supersonic speeds)
- (6) Wings with a relatively high lift-curve slope

STRUCTURAL WEIGHT ASSOCIATED WITH THE REQUIRED STIFFNESS

When a given wing has been shown to be subject to undesirably large aeroelastic effects by means of the charts of reference 2 and this report or by any other method, the problem arises how to distribute the additional required stiffness—that is, which spanwise distribution of structural material will alleviate the adverse aeroelastic phenomena to the desired extent with the minimum increase in structural weight.

In order to shed some light on this problem, aeroelastic

and weight calculations have been made for wings with a taper ratio of 0.5 with a family of somewhat arbitrarily selected stiffness distributions which differ from the distribution required by the constant-stress criterion in a manner described in reference 2. These stiffness distributions are designated by the tip stiffness ratio ω , which is the ratio of the stiffness EI or GJ at the wing tip to the corresponding stiffness of a constant-stress wing. The results of the lateral-control calculations for wings with taper ratio 0.5, with constant wing-thickness ratio h/c along the span, and with two of these stiffness distributions are included in table I and figure 7 (b). The designation "excess strength" refers to the stiffness distribution increased over the constant-stress requirement to such an extent that the value of ω is 2.0. The results of the aeroelastic calculations for the stiffness distributions decreased below the constant-stress requirement to a value of $\omega=0.5$ happen to be the same as the results for the constant-stress stiffness distributions for wings with linearly varying wing-thickness ratio and $(h/c)_t/(h/c)_r=0.5$. The structural weight considered in these calculations is that of the primary load-carrying structure; the remaining structure is assumed to be unchanged in the stiffening process.

The results of the weight calculations and aeroelastic calculations in reference 2 indicated that in the case of wings with taper ratio 0.5 stiffening the structure in the outboard region of the wing (ω greater than 1) was more efficient, from weight considerations, in alleviating the aeroelastic effects considered in that report than the addition of stiffness in the inboard region. This conclusion is corroborated, in essence, by the calculations made for the aeroelastic phenomena considered herein. Figure 10, which consists of a plot of the structural weights required for a given loss (20 percent) in lateral control at a given dynamic pressure, indicates that the least weight is associated with values of the tip stiffness ratio ω greater than 1, except for wings with values of the sweep parameters k or k/ϵ equal to -8 . These large negative values of k or k/ϵ , however, pertain to wings that are either (1) sweptforward, or (2) sweptback with the aerodynamic center behind the elastic axis (in the case of negative k), or (3) sweptback with the center of pressure due to aileron deflection ahead of the elastic axis (in the case of negative k/ϵ , as it may be for spoilers or leading-edge ailerons).

For sweptforward wings, the aileron rolling moment usually increases rather than decreases with dynamic pressure, so that lateral control does not impose any structural requirements. For sweptback wings with negative values of e_1 and e_2 , alleviation of the aeroelastic effects in lateral control can be effected at the least cost in weight by adding structural material in the inboard region of the wing; inasmuch as the minimization of the shift of the aerodynamic center of these wings can be effected most efficiently by stiffening the outer region of the wing, a compromise must be made if both types of static aeroelastic phenomena are of concern.

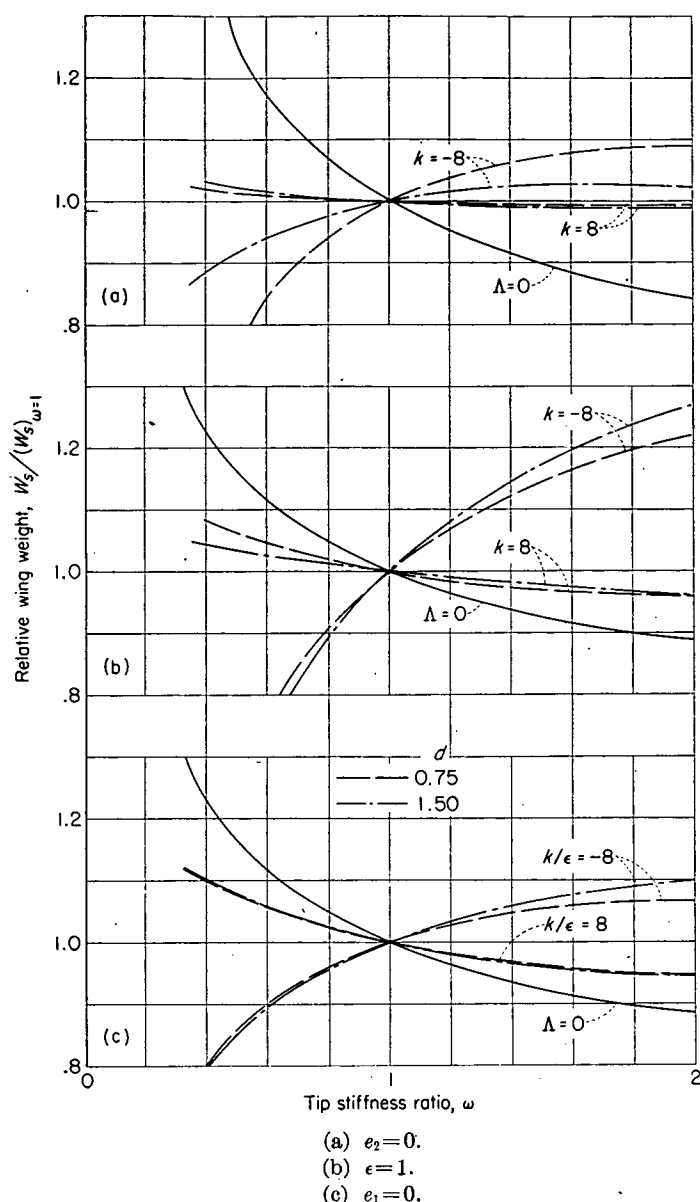


FIGURE 10.—The effect of tip stiffness ratio on the structural weight required to maintain a constant level of lateral-control effectiveness ($C_{ls}=0.8C_{ls0}$) at a given dynamic pressure for wings with a taper ratio of 0.5.

SOME REMARKS CONCERNING THE AEROISOCLINIC WING

As shown in reference 2 an overall type of aeroisoclinicism, in which bending and torsion action tend to cancel for the wing as a whole, can be achieved for the aeroelastic phenomena considered in that report by a choice of a suitable ratio of the bending to the torsion stiffness or by a choice of the elastic-axis location, that is, by satisfying the relation

$$\frac{s_t}{e_1 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda = \frac{1}{K_2} \quad (37)$$

(where K_2 is given in table I). However, also pointed out

in reference 2 was the fact that even if the conditions of equation (37) are achieved there may still be great losses in lateral control, and the wing may still be subject to adverse dynamic phenomena; in fact, the severity of adverse aeroelastic effects in the lateral control and of certain dynamic phenomena may be increased as a result of achieving aeroisoclinicism.

The results of the present report corroborate the conclusion concerning aeroelastic effects on lateral control. However, by suitable additional modifications a wing which has been made aeroisoclinic can also be made to suffer no loss in lateral control due to aeroelastic action. As may be seen from figures 3 to 5 or from equation (34), the condition for no losses in lateral control is that q_R be equal to q_D or, by setting equation (21) equal to equation (30), that

$$K_4 \epsilon + (K_2 + K_7) k + K_5 d + (K_3 + K_6) \frac{\epsilon}{k} d = 0 \quad (38)$$

of which the condition

$$\frac{s_t}{e_2 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda = -\frac{1}{K_2} \quad (39)$$

usually is an approximate solution. Therefore, in order to satisfy both of the conditions specified in equations (37) and (38), the section moment arm e_2 must be nearly equal to $-e_1$ or, in other words, the center-of-pressure parameters a and cp_δ must be nearly equal, a condition which can be satisfied by using full-chord ailerons (all-movable wing tips) or a combination of geared leading- and trailing-edge ailerons, for instance. However, as pointed out in reference 2, attempts at solving static aeroelastic problems by aiming at aeroisoclinicism may tend to aggravate certain dynamic phenomena. The same statement must also be made concerning the foregoing methods of alleviating static aeroelastic effects on lateral control; these methods may, for instance, lead to flutter difficulties which may require excessive mass balancing.

RELATION OF THE CHARTS TO DESIGN PROCEDURE

The conventional procedure of designing a wing on the basis of strength requirements and later checking it for aeroelastic effects can be facilitated at several stages by using the methods described in the present report and in reference 2. As pointed out in reference 2, for instance, the preliminary-survey charts presented therein can be used to establish some static aeroelastic characteristics that would be obtained in symmetric flight if the wing were designed for strength alone.

If these characteristics are deemed satisfactory, the design can proceed on the basis of strength requirements alone. If, on the other hand, they are considered unsatisfactory, the wing must be stiffened. The amount of additional material required can be estimated, as indicated in reference 2, by interpolating between the results presented therein

for the constant-strength case, the "excess strength" case, and the case $\frac{(h/c)_i}{(h/c)_r} = 0.5$. As previously mentioned, the additional structural material is usually most effective if distributed near the wing tip.

Similarly, the preliminary-survey charts of this report can be used to ascertain whether the wing can meet lateral-control requirements if designed for strength alone. If it must be stiffened to meet these requirements, the necessary amount of additional material can be estimated in the same manner as indicated in reference 2 for aeroelastic effects incurred in symmetric flight.

Inasmuch as the charts of reference 2 and of the present report pertain only to static aeroelastic phenomena, the problem remains of ascertaining in the preliminary design stage whether a wing designed for strength alone (or, for that matter, a wing designed both on the basis of strength requirements and of static aeroelastic considerations) is likely to experience flutter difficulties. However, flutter is a much more complicated phenomenon and depends on many more parameters than do static aeroelastic phenomena. Consequently, preparation of a generally applicable set of charts appears impractical. Nonetheless, although phenomenologically flutter is not related to the static aeroelastic phenomena considered in reference 2 and herein, it is mechanically related by virtue of the fact that all these phenomena depend on the wing geometry and the wing stiffness (although the aerodynamic parameters are different and flutter, unlike the static aeroelastic phenomena, involves the mass distribution of the wing and the damping properties of the structure). On the basis of past experience, certain qualitative conclusions can be drawn concerning this relation.

As shown in the charts of this report, the aileron reversal speed, or the speed at which a specified amount of control is retained, is lower for highly sweptback wings than for unswept wings. Similarly, the divergence speed decreases rapidly as the angle of sweepforward increases. For a typical family of wings the values of q^* at divergence and at reversal as obtained by the charts herein are shown as a function of the sweep parameter k in figure 11. For unswept wings the dynamic pressure at flutter is usually within a certain range varying between a value lower than the dynamic pressure at reversal to a value higher than the dynamic pressure at divergence, depending on the geometric, structural, aerodynamic, and mass parameters of the given case, and varying even for a given case and a given speed range with altitude, because a change in air density may change the mode in which the wing flutters. This wide range is indicated in figure 11 by starting three flutter curves (which do not necessarily describe the upper and lower limits) at $k=0$. If the variation of flutter speed with sweep angle is assumed for the purpose of illustration to be similar to that indicated in figure 17 of reference 7,

the flutter curves of figure 11 are obtained.

This figure must not be construed as presenting any quantitative information; to emphasize this point the family of wings is not identified. Even qualitatively the relation between the dynamic pressures at flutter, divergence, and reversal is subject to certain limitations because the flutter tests of reference 7 were performed at subsonic speeds on models without ailerons and concentrated masses, which fluttered in the classical two-degree-of-freedom mode. There is reason to believe that sweptback wings with high aspect ratios flying at high altitudes may experience a possibly mild form of flutter in a single-degree-of-freedom mode, because a vertical motion necessarily implies vertical bending and, hence, in the case of a swept wing, a variation in the angle of attack. In general, the greater the number of degrees of freedom the more difficult it is to relate flutter to the static aeroelastic phenomena.

However, at subsonic speeds and low or moderately high altitudes at least, the trend shown in figure 11 should be valid for wings without very large concentrated masses and with irreversible controls, which tend to minimize the possibility of aileron-coupled flutter. Consequently, if these wings are highly swept back they can be designed to meet lateral-control requirements with the likelihood that they will then be safe against flutter as well, provided conventional lateral-control devices are used. On the other hand, if these wings are unswept or even moderately swept back, they may have to be stiffened beyond the amount required by static aeroelastic considerations, or mass balanced, or both.

In any event the final design must be checked both for static aeroelastic effects and for flutter. In many cases the static aeroelastic effects can probably be calculated with sufficient accuracy by means of the charts and approximate formulas of reference 2 and the present report. In some cases, however, particularly if these effects are in any way critical, a more refined method of analysis, such as that of references 1 and 8, may have to be used.

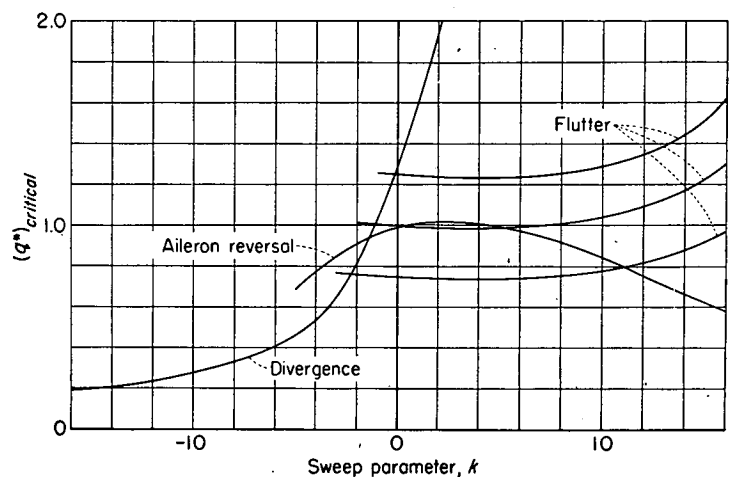


FIGURE 11.—Variation of critical dynamic pressures with the sweep parameter k .

CONCLUDING REMARKS

An approximate method based on charts and approximate formulas has been presented for estimating rapidly the aeroelastic effects on the lateral control of swept and unswept wings at subsonic and supersonic speeds. The charts and approximate formulas presented herein together with those presented in NACA Report 1140 also serve to simplify design procedure in many instances because they can be used at the preliminary design stage to estimate the amount of additional material required to stiffen a wing which is strong enough and because they indicate that the best way of distributing this additional material in most cases is to locate most of it near the wing tip.

For the purpose of making specific calculations, the limitations of the method of this report are that they do not apply directly to wings with very low aspect ratio, with very large angles of sweep, with zero taper ratio, or with large sources of concentrated aerodynamic forces.

The charts and approximate formulas indicate that the control effectiveness of an airplane may be increased by varying some of the design parameters such as the ratio of torsional to bending stiffness and, if necessary, resorting to unconventional lateral-control devices. The charts also indicate that a wing which is strong enough is most likely to be affected by losses in lateral control due to wing flexibility if it is to operate at high dynamic pressures, if it is thin, if it has a large angle of sweepback, if it has an elastic-axis location relatively far forward on the chord or a location of the center of pressure due to aileron deflection far rearward on the chord, or if it is to operate at transonic or high supersonic Mach numbers.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *March 7, 1952.*

APPENDIX

METHODS OF CALCULATIONS ON WHICH THE CHARTS ARE BASED

THE AEROELASTIC EQUATIONS

The assumptions made in the following analysis are the same as those made in reference 2:

(1) Aerodynamic induction is taken into account by applying an overall correction to strip theory.

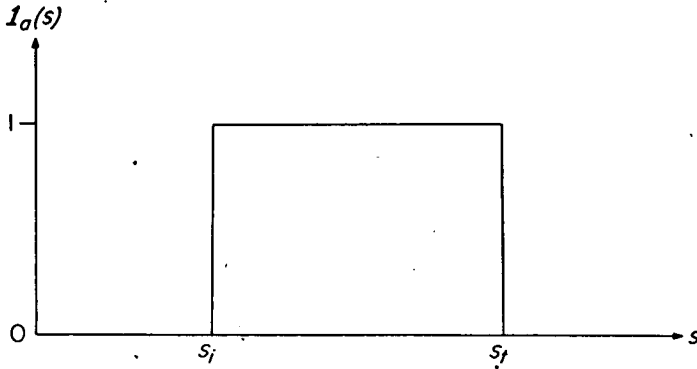
(2) Aerodynamic and elastic forces are based upon the assumption of small deflections.

(3) The wing is clamped at the root perpendicular to a straight elastic axis, and all deformations are considered to be given by the elementary theories of bending and torsion about the elastic axis. In addition it is assumed, as in reference 1 and elsewhere, that the angle between the aileron and the wing is constant along the span of the aileron.

Then, for a wing with an outboard aileron, the force per unit width on sections perpendicular to the elastic axis is

$$l = \frac{qcC_{L_{a_e}}}{144} (\alpha_s + \alpha_s \delta I_a) \quad (A1)$$

where I_a is a unit step function of s defined by



where s_i is the spanwise ordinate of the inboard end of the aileron. The running torque of this force about the elastic axis is

$$\frac{qc^2 e_1 C_{L_{a_e}}}{144} (\alpha_s - \epsilon \alpha_s \delta I_a) \quad (A2)$$

where ϵ is the moment-arm ratio e_2/e_1 .

The integration of these forces yields the accumulated torque and bending moment:

$$T = \frac{qc_r^2 e_1 C_{L_{a_e}}}{144} \int_s^{s_i} \left(\frac{c}{c_r} \right)^2 (\alpha_s - \epsilon \alpha_s \delta I_a) ds \quad (A3)$$

$$M = \frac{qc_r C_{L_{a_e}}}{144} \int_s^{s_i} \int_s^{s_i} \frac{c}{c_r} (\alpha_s + \alpha_s \delta I_a) ds ds \quad (A4)$$

Combining these equations with the equations of elastic deformation presented in appendix A of reference 2 as

$$\varphi = \int_0^s \frac{1}{GJ} T ds$$

$$\Gamma = \int_0^s \frac{1}{EI} M ds$$

results in two simultaneous equations of equilibrium:

$$\frac{d}{ds} \left(GJ \frac{d\varphi}{ds} \right) = - \frac{qc^2 e_1 C_{L_{a_e}}}{144} [(\varphi \cos \Lambda - \Gamma \sin \Lambda) - \epsilon \alpha_s \delta I_a] \quad (A5)$$

$$\frac{d^2}{ds^2} \left(EI \frac{d\Gamma}{ds} \right) = \frac{qc C_{L_{a_e}}}{144} [(\varphi \cos \Lambda - \Gamma \sin \Lambda) + \alpha_s \delta I_a] \quad (A6)$$

These equations are subject to the following boundary conditions:

$$\varphi(0) = 0 \quad (A7a)$$

$$\Gamma(0) = 0 \quad (A7b)$$

$$\left(GJ \frac{d\varphi}{ds} \right)_{s=s_i} = 0 \quad (A7c)$$

$$\left(EI \frac{d\Gamma}{ds} \right)_{s=s_i} = 0 \quad (A7d)$$

$$\left(EI \frac{d^2\Gamma}{ds^2} \right)_{s=s_i} = 0 \quad (A7e)$$

The angle of attack due to structural deformations is related to φ and Γ by the equation

$$\alpha_s = \varphi \cos \Lambda - \Gamma \sin \Lambda \quad (A8)$$

After equations (A5) and (A6) have been solved, the rolling moment about the wing root may be found from the expression

$$\frac{qS \frac{b}{2} C_l}{144} = T_r \sin \Lambda + M_r \cos \Lambda \quad (A9)$$

where the root twisting moment T_r and the root bending moment M_r are given by equations (A3) and (A4) evaluated at s equal to zero. Then the rolling-moment ratio becomes

$$\frac{C_{l_s}}{C_{l_{s_0}}} = 1 + \frac{\frac{d}{k} (B_1 + \epsilon B_2) + B_3 + \epsilon B_4}{\frac{d}{k} \epsilon B_5 + B_6} \quad (A10)$$

where the functions B_1 to B_6 are defined by

$$B_1 + \epsilon B_2 = \int_0^1 \left(\frac{c}{c_r}\right)^2 \frac{\alpha_s}{\alpha_3 \delta} ds^* \quad (A11a)$$

$$B_3 + \epsilon B_4 = \int_0^1 \int_0^{s^*} \frac{c}{c_r} \frac{\alpha_s}{\alpha_3 \delta} ds^* ds^* \quad (A11b)$$

$$B_5 = - \int_0^1 \left(\frac{c}{c_r}\right)^2 I_a ds^* \quad (A11c)$$

$$B_6 = \int_0^1 \int_0^{s^*} \frac{c}{c_r} I_a ds^* ds^* \quad (A11d)$$

and where the parameter d/k is defined by

$$\frac{d}{k} = \frac{e_1 c_r}{s_t} \tan \Lambda \quad (A12)$$

SOLUTION FOR UNIFORM WINGS

If the torsional stiffness, the bending stiffness, and the chord of the wing have constant values of $(GJ)_r$, $(EI)_r$, and c_r , respectively, along the wing span, the equations of equilibrium (A5) and (A6) become

$$\varphi'' \cos \Lambda = -q^*[(\varphi \cos \Lambda - \Gamma \sin \Lambda) - \epsilon \alpha_3 \delta I_a] \quad (A13)$$

$$\Gamma''' \sin \Lambda = -\bar{q}[(\varphi \cos \Lambda - \Gamma \sin \Lambda) + \alpha_3 \delta I_a] \quad (A14)$$

where the differentiation denoted by the prime is with respect to $\xi \equiv 1 - \frac{s}{s_t}$.

Differentiating equation (A13) once with respect to ξ and combining it with equation (A14) yields the single differential equation of equilibrium,

$$\alpha_s''' + q^* \alpha_s' - \bar{q} \alpha_s = \alpha_3 \delta q^* \epsilon I_a' + \bar{q} I_a \quad (A15)$$

subject to the following boundary conditions:

$$\alpha_s(1) = 0 \quad (A16a)$$

$$\alpha_s'(0) = 0 \quad (A16b)$$

$$\alpha_s''(0) = -q^*[\alpha_s(0) - \epsilon \alpha_3 \delta I_a(0)] \quad (A16c)$$

The complete solution of equation (A15) can be readily obtained by means of Laplace transforms as

$$\frac{\alpha_s(\xi)}{\alpha_3 \delta} = \left[f_5(\xi) - f_5(\xi - \xi_i) - \frac{f_5(1) - f_5(1 - \xi_i)}{f_3(1)} f_3(\xi) \right] (1 + \epsilon) q^* + \frac{f_3(1 - \xi_i)}{f_3(1)} f_3(\xi) - f_3(\xi - \xi_i) - I_a(\xi) \quad (A17)$$

where the functions $f_3(\xi)$ and $f_5(\xi)$, as well as $f_4(\xi)$ which will be used subsequently, are defined in appendix A of reference 2 as

$$f_3(\xi) = C_1 e^{-2\beta\xi} + e^{\beta\xi} \left(C_2 \cos \gamma\xi + \frac{C_3}{\gamma} \sin \gamma\xi \right)$$

$$f_4(\xi) = C_4 e^{-2\beta\xi} + e^{\beta\xi} \left(C_5 \cos \gamma\xi + \frac{C_6}{\gamma} \sin \gamma\xi \right)$$

$$f_5(\xi) = C_7 e^{-2\beta\xi} + e^{\beta\xi} \left(C_8 \cos \gamma\xi + \frac{C_9}{\gamma} \sin \gamma\xi \right)$$

where the constants of integration are defined in reference 2 in terms of the roots of the characteristic equation, -2β

and $\beta \pm i\gamma$. These three functions are equal to zero when $\xi < 0$.

The substitution of this solution into equations (All) yields the functions

$$B_1(q^*, k) = \frac{f_3(1 - \xi_i)}{f_3(1)} f_4(1) - f_4(1 - \xi_i) - \xi_i + B_2 \quad (A18a)$$

$$B_2(q^*, k) = \frac{1}{k} [f_3(1) - f_3(1 - \xi_i)] + \left[\frac{1}{k} - \frac{f_4(1)}{f_3(1)} \right] [f_5(1) - f_5(1 - \xi_i)] q^* \quad (A18b)$$

$$B_3(q^*, k) = \frac{f_3(1 - \xi_i)}{f_3(1)} f_5(1) - f_5(1 - \xi_i) - \xi_i + \frac{1}{2} \xi_i^2 + B_4 \quad (A18c)$$

$$B_4(q^*, k) = \frac{1}{k^2} [f_3(1) - f_3(1 - \xi_i)] + \frac{1}{k} [f_4(1) - f_4(1 - \xi_i)] + \left[\frac{1}{k^2} - \frac{f_5(1)}{f_3(1)} \right] [f_5(1) - f_5(1 - \xi_i)] q^* - \frac{1}{k} \xi_i \quad (A18d)$$

$$B_5 = -\xi_i \quad (A18e)$$

$$B_6 = \xi_i - \frac{1}{2} \xi_i^2 \quad (A18f)$$

The value of q^* at reversal is that value which makes $C_{l_s}/C_{l_{s_0}}$ in equation (A10) equal to zero for given values of the parameters k , ϵ , and d .

For a full-span aileron ($\xi_i = 1$, $\xi_0 = 0$), equations (A18) become

$$B_1(q^*, k) = \frac{f_4(1)}{f_3(1)} - 1 + B_2 \quad (A19a)$$

$$B_2(q^*, k) = \frac{1}{k} [f_3(1) - 1] + \left[\frac{1}{k} - \frac{f_4(1)}{f_3(1)} \right] f_5(1) q^* \quad (A19b)$$

$$B_3(q^*, k) = \frac{f_5(1)}{f_3(1)} - \frac{1}{2} + B_4 \quad (A19c)$$

$$B_4(q^*, k) = \frac{1}{k^2} [f_3(1) - 1] + \frac{1}{k} [f_4(1) - 1] + \left[\frac{1}{k^2} - \frac{f_5(1)}{f_3(1)} \right] f_5(1) q^* \quad (A19d)$$

$$B_5 = -1 \quad (A19e)$$

$$B_6 = \frac{1}{2} \quad (A19f)$$

and for a "tip aileron" of very short span ($\xi_i \rightarrow 0$, $\xi_0 = 0$), equations (18) become

$$B_1(q^*, k) = \left\{ f_3(1) + [f_4(1) - k f_5(1)] \frac{f_4(1)}{f_3(1)} q^* - 1 \right\} \xi_i + B_2 \quad (A20a)$$

$$B_2(q^*, k) = \left[q^* f_5(1) - q^* \frac{f_4(1)}{f_3(1)} f_4(1) \right] \xi_i \quad (A20b)$$

$$B_3(q^*, k) = \left\{ f_4(1) + [f_4(1) - k f_5(1)] \frac{f_5(1)}{f_3(1)} q^* - 1 \right\} \xi_i + B_4 \quad (A20c)$$

$$B_4(q^*, k) = \left\{ \frac{1}{k} [f_3(1) - 1] + \left[\frac{1}{k} - \frac{f_4(1)}{f_3(1)} \right] f_5(1) q^* \right\} \xi_i \quad (A20d)$$

$$B_5 = -\xi_i \quad (A20e)$$

$$B_6 = \xi_i \quad (A20f)$$

SOLUTION FOR NONUNIFORM WINGS

By means of strip theory applied at a finite number of points, equation (A1) may be written in matrix notation as

$$\{l\} = \frac{qC_{L\alpha}}{144} [c] \{ \alpha_s \} + \alpha_s \delta \{I_a\} \quad (A21)$$

and the expression for torque (A2) as

$$\frac{qe_1C_{L\alpha}}{144} [c^2] \{ \alpha_s \} - \epsilon \alpha_s \delta \{I_a\} \quad (A22)$$

The matrix notation used in this analysis is the same as in reference 1.

Equations (A3) and (A4), written in matrix notation, are

$$\{T\} = \frac{qc_r^2 e_1 s_i}{144} C_{L\alpha} [I'] \left[\left(\frac{c}{c_r} \right)^2 \right] \{ \alpha_s \} - \epsilon \alpha_s \delta \{I_a\} \quad (A23)$$

$$\{M\} = \frac{qc_r s_i^2}{144} C_{L\alpha} [II'] \left[\frac{c}{c_r} \right] \{ \alpha_s \} + \alpha_s \delta \{I_a\} \quad (A24)$$

where the integrating matrices $[I']$ and $[II']$ are, respectively, for single and double integration from tip to root and are given in reference 1. The value of $\{I_a\}$ at the matrix station nearest the discontinuity of I_a can be modified as in reference 1 in such a manner that premultiplication of $\{I_a\}$ by $[I'_1]$ and $[II'_1]$ yields the same area and moment about the wing root, respectively, as would be obtained by analytic integration. This modification may also be regarded as an attempt to round off the lift distribution near the inboard end of the aileron in a physically reasonable manner.

The combination of equations (A23) and (A24) with matrix expressions for $\{\varphi\}$ and $\{\Gamma\}$ in terms of $\{T\}$ and $\{M\}$ (see appendix A of ref. 2) yields the equilibrium equation

$$[I] - q^*[A] \{ \alpha_s \} = -q^* \alpha_s \delta \{ \beta \} \quad (A25)$$

which can be solved for $\{ \alpha_s \}$. The aeroelastic matrix $[A]$ is defined in reference 1 and the column matrix $\{ \beta \}$ is defined by

$$\{ \beta \} = \left[\epsilon [I']' \left[\frac{(GJ)_r}{GJ} \right] [I'] \left[\left(\frac{c}{c_r} \right)^2 \right] + k [I']' \left[\frac{(EI)_r}{EI} \right] [II'] \left[\frac{c}{c_r} \right] \right] \{I_a\} \quad (A26)$$

After the rolling-moment ratio expressed by equation (A10) is set equal to zero, the condition for aileron reversal is

$$\frac{d}{k} (B_1 + \epsilon B_2) + B_3 + \epsilon B_4 + \frac{d}{k} \epsilon B_5 + B_6 = 0$$

where the matrix equivalents of equations (A11) become

$$B_1 + \epsilon B_2 = \frac{1}{\alpha_s \delta} [I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] \{ \alpha_s \} \quad (A27a)$$

$$B_3 + \epsilon B_4 = \frac{1}{\alpha_s \delta} [II'_1] \left[\frac{c}{c_r} \right] \{ \alpha_s \} \quad (A27b)$$

$$B_5 = -[I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] \{I_a\} \quad (A27c)$$

$$B_6 = [II'_1] \left[\frac{c}{c_r} \right] \{I_a\} \quad (A27d)$$

Therefore, the condition for aileron reversal expressed in matrix notation is

$$\left[\frac{d}{k} [I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] + [II'_1] \left[\frac{c}{c_r} \right] \right] \{ \alpha_s \} = \alpha_s \delta \left[\frac{d}{k} \epsilon [I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] - [II'_1] \left[\frac{c}{c_r} \right] \right] \{I_a\} \quad (A28)$$

Solving equation (A28) for $\alpha_s \delta$ and multiplying the resulting equation by $\{ \beta \}$ yields

$$\alpha_s \delta \{ \beta \} = \frac{\{ \beta \} \left[\frac{d}{k} [I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] + [II'_1] \left[\frac{c}{c_r} \right] \right] \{ \alpha_s \}}{\left[\frac{d}{k} \epsilon [I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] - [II'_1] \left[\frac{c}{c_r} \right] \right] \{I_a\}} \quad (A29)$$

The substitution of this expression for $\alpha_s \delta \{ \beta \}$ into the equilibrium equation (A25) yields

$$\{ \alpha_s \} = q^* [A_R] \{ \alpha_s \} \quad (A30)$$

where the aileron-reversal matrix $[A_R]$ is defined by

$$[A_R] = [A] - \frac{\{ \beta \} \left[\frac{d}{k} [I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] + [II'_1] \left[\frac{c}{c_r} \right] \right]}{\left[\frac{d}{k} \epsilon [I'_1] \left[\left(\frac{c}{c_r} \right)^2 \right] - [II'_1] \left[\frac{c}{c_r} \right] \right] \{I_a\}} \quad (A31)$$

The value of q^* at aileron reversal can be found by the iteration of this aileron-reversal matrix.

REPRESENTATION OF RESULTS BY APPROXIMATE FORMULAS

Approximate formulas, similar to those in reference 2, have been used to combine the results of the many computations indicated in this analysis; as in the case of the formulas presented in reference 2, those presented in this section are based on considerations of a semirigid wing.

The functions B_1 to B_6 in equation (A10) have been found to be given quite accurately by the following approximate expressions:

$$B_1 = -B_6 \frac{q^*/q^*_D}{1 - \frac{q^*}{q^*_D}} \frac{K_5 k}{1 - K_2 k} \quad (A32a)$$

$$B_2 = -B_6 \frac{q^*/q^*_D}{1 - \frac{q^*}{q^*_D}} \frac{K_3 + K_6}{1 - K_2 k} \quad (A32b)$$

$$B_3 = -B_6 \frac{q^*/q^*_D}{1 - \frac{q^*}{q^*_D}} \frac{(K_2 + K_7)k}{1 - K_2 k} \quad (A32c)$$

$$B_4 = -B_6 \frac{q^*/q^*_D}{1 - \frac{q^*}{q^*_D}} \frac{K_4}{1 - K_2 k} \quad (A32d)$$

$$B_5 = -B_6 K_3 \quad (A32e)$$

where the factors K_4 to K_7 depend on the aileron span, the taper ratio, and the spanwise variation of the bending and torsion stiffnesses. The factor K_2 is independent of the

aileron span, and the factor K_3 is independent of the stiffness variation.

The following approximate formula may be obtained by the substitution of equations (A32) into equation (A10):

$$\frac{C_{l_s}}{C_{l_{s0}}} = \frac{1 - \frac{q}{q_R}}{1 - \frac{q}{q_D}} \quad (\text{A33})$$

where the value of q at aileron reversal is

$$q_R = \frac{q_D (1 - K_2 k) \left(1 - K_3 \frac{\epsilon}{k} d\right)}{1 + (K_4 + K_2 K_3 d) \epsilon + K_5 d + K_6 \frac{\epsilon}{k} d + K_7 k} \quad (\text{A34})$$

With the use of the approximate formula presented in reference 2,

$$q_R^* = \frac{K_1}{1 - K_2 k} \quad (\text{A35})$$

equation (A34) becomes

$$q_R^* = \frac{K_1 \left(1 - K_3 \frac{\epsilon}{k} d\right)}{1 + (K_4 + K_2 K_3 d) \epsilon + K_5 d + K_6 \frac{\epsilon}{k} d + K_7 k} \quad (\text{A36})$$

The accuracy of equation (A36) compared with the results calculated directly by the method of the preceding section is illustrated in figures 6 and 12.

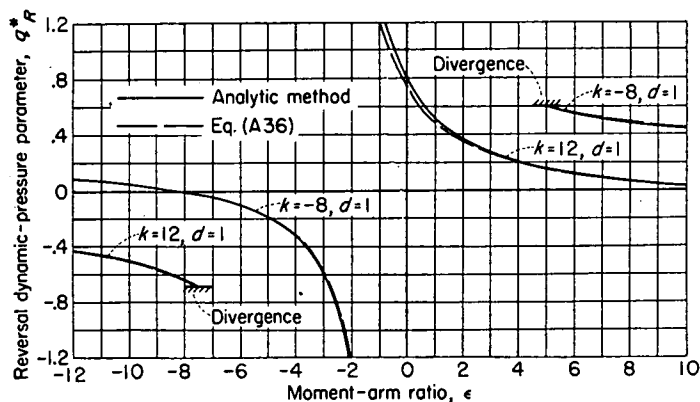


FIGURE 12.—Comparison of dynamic pressures at aileron reversal calculated by the analytic integration method of the appendix with those calculated from equation (A36) for swept uniform wings with tip ailerons. $s_i^* \rightarrow 1$.

The results of reference 2 indicate that the damping-in-roll derivative may be expressed approximately by

$$C_{l_p} = C_{l_{p0}} \frac{1}{1 - \frac{q}{q_D}} \quad (\text{A37})$$

and since the wing-tip helix angle due to roll is

$$\frac{pb}{2V} = -\frac{C_{l_s}}{C_{l_p}} \quad (\text{A38})$$

an approximate formula for rolling maneuverability is

$$\frac{pb}{2V} = \left(\frac{pb}{2V}\right)_0 \left(1 - \frac{q}{q_R}\right) \quad (\text{A39})$$

Figure 13 shows the approximate formulas (A33) and (A39) to be in good agreement with more accurately computed values. (Actually, a slightly more accurate method of estimating C_{l_p} is given in ref. 2 but the simpler expression given here is comparable in accuracy to the expression for

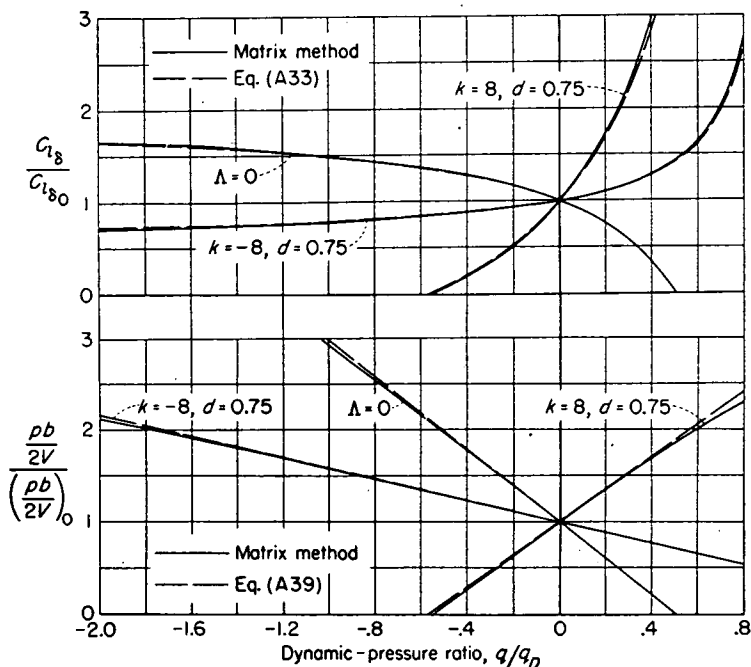


FIGURE 13.—Comparison of rolling power and rolling maneuverability calculated by the matrix method of the appendix with those calculated by equations (A33) and (A39) for uniform wings. $s_i^* = 0.5$; $\epsilon = 1.0$.

C_{l_s} , and its simplicity facilitates its application to the estimation of the rolling maneuverability.)

An approximate expression for the structural twist due to aileron deflection similar to the expressions for the structural twists due to geometrical angles of attack given in reference 2 has been deduced from the results of the analysis in the preceding section:

$$\frac{\alpha_s}{\alpha_0 \delta} = \frac{q/q_D}{1 - \frac{q}{q_D}} \frac{(K_1 \epsilon + K_2 k) f_a - K_2 k \Delta f_a}{1 - K_2 k} \quad (\text{A40})$$

where f_a and Δf_a are functions of the spanwise coordinate s^* , the wing chord and stiffness variations, and the aileron span. The accuracy of equation (A40) is indicated in figure 14.

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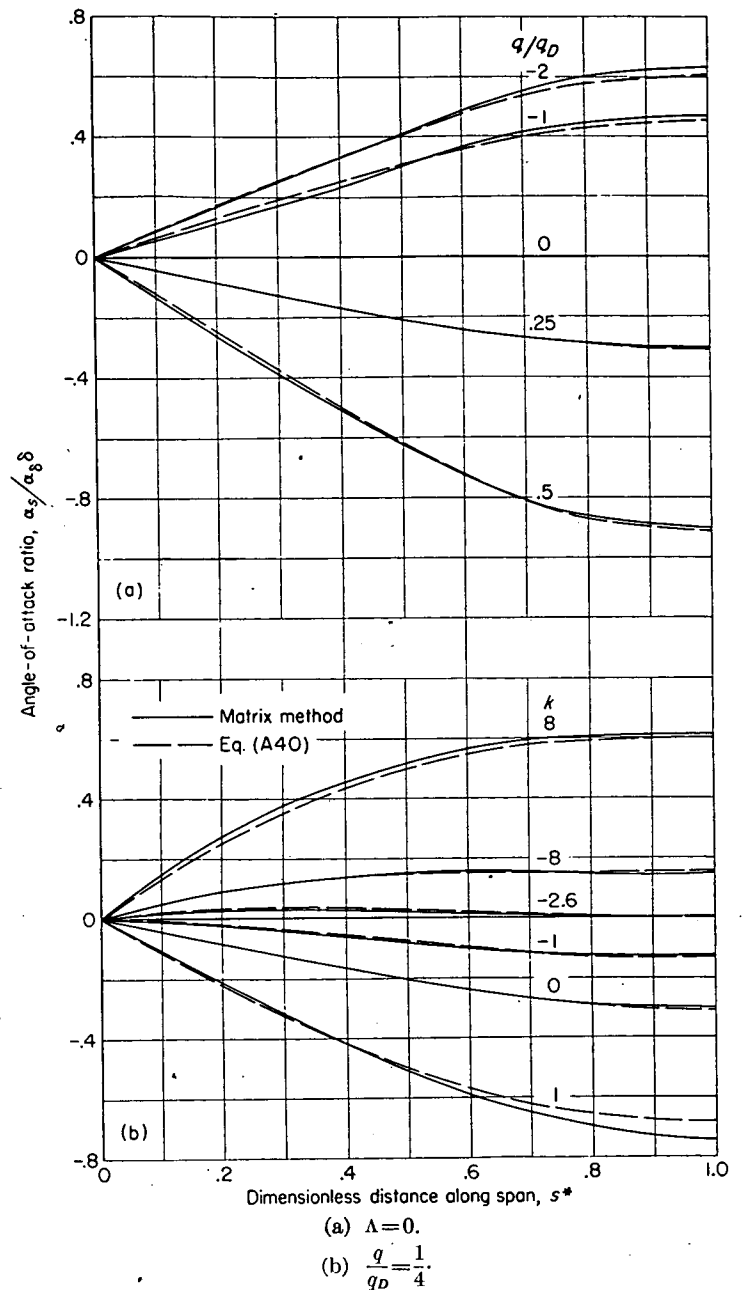


FIGURE 14.—Comparison of angle-of-attack ratios calculated by the matrix method of the appendix with those calculated by equation (A40) for uniform wings. $s_1^* = 0.5$; $\epsilon = 1.0$.

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CHARTS AND APPROXIMATE FORMULAS FOR THE ESTIMATION OF AEROELASTIC EFFECTS ON THE LOADING OF SWEEPED AND UNSWEEPED WINGS¹

By FRANKLIN W. DIEDERICH and KENNETH A. FOSS

SUMMARY

Charts and approximate formulas are presented for the estimation of aeroelastic effects on the spanwise lift distribution, lift-curve slope, aerodynamic center, and damping in roll of swept and unswept wings at subsonic and supersonic speeds. Some design considerations brought out by the results of this report are discussed.

INTRODUCTION

A knowledge of the spanwise lift distribution and of some of the aerodynamic parameters associated with it is required for the design of a wing structure. Under certain conditions, such as high dynamic pressures, thin wings, swept wings, or wings designed for low wing loadings, the spanwise lift distribution may be affected to a significant extent by aeroelastic effects, because a wing which carries a certain lift necessarily deforms under that lift. If the angles of attack along the span are changed as a result of this deformation, the lift carried by the wing is changed as well; in turn, this change in lift causes a change in the deformation of the wing and hence another change in lift, and so on, until an equilibrium condition is reached. The changes in the magnitude and the distribution of the lift are reflected in changes of the wing lift-curve slope, the wing bending and rolling moments, the spanwise center of pressure of the lift, and, on a swept wing, the longitudinal center of pressure.

Inasmuch as the lift produced by a given change in angle of attack is proportional to the dynamic pressure, the various aeroelastic effects tend to increase with dynamic pressure. In fact, for certain wings a sufficiently large dynamic pressure may produce a condition of instability in which the change in lift caused by deformation is greater than the amount of lift required to produce the deformation, so that a given deformation will tend to increase until the structure fails. This phenomenon is aeroelastic divergence; since it involves only torsional deformations in the case of unswept wings, it is often referred to as torsional divergence.

Several methods are available for calculating these effects (ref. 1, for instance), but since these effects depend on the structural characteristics of the wing, which are not ac-

curately known in advance of its design, the relatively large amount of time required for even the most efficient of these methods militates against their use in connection with preliminary design calculations. A need exists, therefore, for means of estimating some of the more important aeroelastic effects on the spanwise lift distribution quickly and with an accuracy that is sufficient for preliminary design purposes.

Charts and approximate formulas are presented in this report for estimating the changes in spanwise lift distribution, lift-curve slope, wing rolling-moment coefficient, spanwise center of pressure, and aerodynamic center occasioned by aeroelastic action of swept and unswept wings at subsonic and supersonic speeds. Also included are summary charts which indicate whether the various aeroelastic phenomena considered are likely to affect any given design. By means of these charts the conventional procedure of designing a wing on the basis of certain strength criteria, checking it for aeroelastic phenomena, and then reinforcing it, when necessary, to meet the stiffness requirements imposed by these phenomena can often be simplified greatly, inasmuch as the effect of some of these phenomena can be estimated in advance of design.

The use of the charts is described in the section headed "Calculation of the Various Aeroelastic Phenomena," and some considerations involved in the selection of the aerodynamic, structural, and geometric parameters are discussed in some detail in the section headed "Selection of Parameters." These two sections, as well as the sections headed "Illustrative Example" and "Preliminary Survey of Aeroelastic Behavior," are likely to prove of greatest interest at a first reading of this report. The various parts of the section headed "Discussion" are concerned with the limitations of the charts, with the light they shed on such practical design problems as the relative significance of strength and stiffness as design criteria, with efficient ways of stiffening a wing that is strong but not stiff enough, and with the achievement of aeroisoclinic conditions.

A brief description of the calculations (based on refs. 1 and 2) used in preparing the charts is contained in the appendixes.

¹ Previously released as NACA TN 2608, "Charts and Approximate Formulas for the Estimation of Aeroelastic Effects on the Loading of Swept and Unswept Wings," by Franklin W. Diederich and Kenneth A. Foss, 1952.

SYMBOLS

A	aspect ratio, b^2/S	M	bending moment about an axis perpendicular to elastic axis, in-lb
A_Δ	swept-span aspect ratio, $A/\cos^2\Lambda$	M_o	free-stream Mach number
$\bar{A}$	cross-sectional area of the (assumed) single torsion cell, sq in.	n	design load factor
a	distance from leading edge to section aerodynamic center, fraction of chord	p	rolling angular velocity, radians/sec
$\bar{a}$	distance from leading edge of mean aerodynamic chord to wing aerodynamic center, fraction of mean aerodynamic chord	q	dynamic pressure, lb/sq ft
b	wing span, in.	q^*	dimensionless dynamic pressure, $\frac{q}{144} \frac{C_{L_{\alpha_e}} e_1 c_r^2 s_i^2 \cos \Lambda}{(GJ)_r}$
b'	wing span less width of fuselage, $b-w$, in.	$\bar{q}$	dimensionless dynamic pressure, $\frac{q}{144} \frac{C_{L_{\alpha_e}} c_r s_i^3 \sin \Lambda}{(EI)_r}$
C_B	wing-root bending-moment coefficient, $4M_r/qSb$	S	total wing area, sq in.
C_{L_w}	lift coefficient of wings alone (exclusive of fuselage), L_w/qS	s	distance along elastic axis measured from wing root, in.
$C_{L_{\alpha}}$	wing lift-curve slope per radian	s^*	dimensionless distance along elastic axis, s/s_i
$C_{L_{\alpha_e}}$	effective lift-curve slope per radian (defined in eqs. 2 and 4)	$\bar{s}$	distance from root to center of pressure of lift along elastic axis, in.
C_{l_w}	rolling-moment coefficient on both wings alone (exclusive of fuselage), Rolling moment/ qSb	$\bar{s}^*$	dimensionless distance from root to center of pressure of lift, $\bar{s}/s_i$
C_T	wing-root twisting-moment coefficient, $2T_r/qSc_r$	T	accumulated torque about elastic axis, in-lb
c	chord (measured perpendicular to elastic axis), in.	t	thickness of most highly stressed element of skin, in.
$\bar{c}$	average chord, $\frac{c_r + c_i}{2}$, in.	t_e	thickness of equivalent skin which includes the material in stringers and spar flanges, in.
$c_{l_{\alpha}}$	section lift-curve slope per radian	t_i	distributed torque due to inertia loading, in-lb/in.
c_{MAC}	mean aerodynamic chord (parallel to plane of symmetry), in.	V	free-stream velocity, ft/sec
E	Young's modulus of elasticity, lb/sq in.	W	design gross weight of airplane, lb
e	distance from leading edge to elastic axis, fraction of chord	W_s	weight of primary structure of both wings, lb
e_1	dimensionless moment arm of the section lift about the elastic axis, $e-a$	W_w	weight of both wings exclusive of fuselage, lb
$\bar{e}_1$	effective or average dimensionless moment arm	w	width of fuselage, in.
F_B	allowable bending stress, lb/sq in.	w_s	weight of primary load-carrying structure per unit distance along span, lb/in.
F_r	root-stiffness function	y	lateral coordinate, in.
F_T	allowable shear stress, lb/sq in.	y^*	dimensionless lateral coordinate, $\frac{y}{b/2}$
F_w	structural weight function	$\bar{y}$	lateral distance to center of pressure
F_1, F_2	dimensionless parameters used in approximate formulas for angle of attack due to aeroelastic deformation	α	angle of attack in a plane parallel to plane of symmetry, radians
f_1, f_2	dimensionless functions of the distance along the span used in approximate formulas for angle of attack due to aeroelastic deformation	Γ	angle of local dihedral, radians; or spanwise slope of normal displacement of elastic axis
G	modulus of rigidity, lb/sq in.	γ_s	density of the material of the primary structure (or an equivalent density in the case of sandwich construction), lb/cu in.
h	wing thickness, in.	η	lateral distance measured from wing root, $y - \frac{w}{2}$, in.
I	section bending moment of inertia, in. ⁴	η^*	dimensionless lateral distance, $\frac{\eta}{b'/2}$
J	section moment of inertia in torsion, in. ⁴	$\eta_{1,2,\dots,21}$	factors defined in table 1
K_1, K_2	dimensionless parameters used in approximate formulas for dimensionless dynamic pressures at divergence	$\eta_{a,b}$	factors defined in equations (15a) and (15b)
k	dimensionless sweep parameter, $\frac{s_i}{e_1 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda$	κ	ratio of lift-curve slopes, $C_{L_{\alpha_e}}/C_{L_{\alpha}}$
L_w	lift of both wings alone (exclusive of fuselage), lb	Λ	angle of sweepback at elastic axis
l	lift per unit distance along span, lb/in.	λ	wing taper ratio, c_i/c_r
		ρ	free-air density, slugs/cu ft

φ	angle of structural twist in planes perpendicular to elastic axis, radians
ω	tip stiffness ratio, $(EI)_t/(EI)_{tcs}$
μ, ν, τ	dimensionless parameters used in approximate formulas for lift, root bending moment, and root twisting moment

Subscripts:

cs	constant stress
D	at divergence
e	effective
g	geometric (due to airplane attitude or built-in twist)
i	inertia
0	rigid wing (for $q=0$)
r	at wing root
s	structural (due to structural or aeroelastic deformation)
t	at wing tip

Superscripts:

M	due to bending moment
T	due to torque
Γ	due to root bending
φ	due to root twist

USE OF THE CHARTS AND APPROXIMATE FORMULAS

SUMMARY OF METHOD AND SCOPE OF CALCULATIONS ON WHICH THE CHARTS AND APPROXIMATE FORMULAS ARE BASED

Although a detailed understanding of the method and scope of the calculations on which the charts of this report are based is not essential to the use of the charts, a brief account of these matters is given, primarily to aid in the appreciation of the limitations of the charts. The method is described more fully in appendix A.

Most of the calculations on which the charts are based were made by the method of reference 1, which consists in solving the differential equations descriptive of an elastically deformed wing under aerodynamic loading by numerical methods employing matrix techniques. Treated by this method were wings with four taper ratios λ (1.0, 0.5, 0.2, and 0), two types of stiffness distributions (one proportional to the fourth power of the chord and one dictated by constant-stress considerations), and four values of a sweep parameter

k at several values of the dynamic-pressure ratio $\frac{q}{q_D}$. Cal-

culated for each case were the dynamic pressure at divergence and the changes due to aeroelastic action in spanwise lift distribution, total wing lift, root bending moment, rolling moment, and spanwise center of pressure of the lift. For the wings of constant chord and constant stiffness, calculations were also performed for six values of k by a method which is an extension of that of reference 2 and consists in solving the differential equations exactly for these relatively simple cases.

Some approximations have been made in the calculations concerning the aerodynamic induction effects, the root rotations, and the stiffness distributions, primarily in order to hold the number of variables considered in the analysis to a

minimum and to make the results more generally applicable.

Aerodynamic induction effects at subsonic speeds are taken into account by an overall reduction of the strip-theory loading and, in the matrix calculations, by rounding off the strip-theory loading at the tip (see refs. 1 and 2); for supersonic speeds, strip theory is used with a small reduction at the tip in the matrix calculations. This approximation has made it unnecessary to consider explicitly the effects of aspect ratio, sweep, and Mach number on the rigid-wing lift distribution; the effects of these parameters on the total lift and on the aeroelastic increment to the lift distribution have been taken into account.

The rigid-body rotations imparted to a swept wing by its triangular root portion vary among different designs in a largely unpredictable manner. They have therefore been taken into account only by the use of an effective root, the selection of which in any given case is discussed briefly in a subsequent section.

The spanwise distributions of the bending and torsional stiffnesses depend on the detailed design of the wing and cannot be generalized easily. The stiffness distributions used in the calculations of aeroelastic effects were obtained from the constant-stress concept outlined in appendix B, which constitutes an effort to relate the stiffness of a wing to its strength on the basis of the following assumptions:

(1) The level of combined bending and torsional stresses is constant along the span.

(2) The structure is designed for combined bending and torsional stresses in such a manner that the sum of the ratio of the actual to the allowable bending stress and the ratio of the actual to the allowable torsion stress is equal to unity when the margin of safety is zero.

(3) The structure is of the thin-skin, stringer-reinforced shell type and its main features do not vary along the span; for instance, the number of spars and their chordwise locations are constant along the span.

(4) At the design condition the spanwise distribution of the applied loading is proportional to the chord.

Also used in the calculations were stiffness distributions which vary as the fourth power of the chord, as do those of solid wings and wings with geometrically similar cross sections; as pointed out in a subsequent section, the results of these calculations can be used to estimate aeroelastic phenomena of some wings which have large cutouts or which for some other reason do not have stiffness distributions represented fairly closely by those of the constant-stress type.

All calculations are based on the assumptions that twisting is resisted primarily by the torsion cells of the wing structure and that the wing deformations can be estimated by means of the elementary theories of bending and torsion about an elastic axis.

SELECTION OF PARAMETERS

Geometric parameters.—The geometric parameters used in the analysis are defined in figure 1. The location of the effective root indicated in figure 1 is discussed in the section concerned with the structural parameters.

for which torsional deformations are predominant; the parameter $\bar{q}$ is primarily useful for highly swept wings, for which bending deformations are predominant. In general, the parameter q^* is used in this report unless e_1 is zero. The ratio of these parameters

$$k = \frac{\bar{q}}{q^*} = \frac{s_t}{e_1 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda \quad (9)$$

is independent of the dynamic pressure and depends only on geometric and structural parameters. This ratio is very useful for analyzing the aeroelastic behavior of swept wings.

Structural parameters.—For the purposes of an aeroelastic analysis, the wing structure is characterized by the location of the elastic axis and the magnitude and distribution of the bending and torsional stiffnesses (EI and GJ), as well as by the magnitude of the rigid-body rotations imparted to the wing by its root.

The elastic axis is usually defined as the locus of points at which normal loads can be applied without causing the wings to twist. Such a locus does not generally exist for practical wings; however, for unswept wings without cutouts an axis can be determined which approximately satisfies this condition. Similarly, for swept wings without cutouts an elastic axis can be defined for the outboard part of the wing if the wing is considered to be clamped along some such line as the effective root shown in figure 1. In most aeroelastic calculations the locus of shear centers for both swept and unswept wings is assumed to be the elastic axis. If the structure has large cutouts which result in sudden changes in the stiffnesses and in the shear center along the span, the charts of this report cannot be used except in a qualitative sense.

The magnitude and the spanwise distributions of the bending and torsional stiffnesses enter aeroelastic calculations by means of the charts and approximate formulas in different ways. The magnitudes, as characterized by the values of the stiffnesses at the effective root, have to be known in order to perform any calculations; the distributions are implicit in the charts. The root stiffnesses, if not known otherwise, can be estimated either from experience with similar designs, from the results of the constant-stress concept outlined in appendix B, or from a combination of the two.

The required bending stiffness at the root $(EI)_r$ is proportional to the design load factor, the wing loading, the wing thickness ratio, the fourth power of the root chord, the square of the swept-span aspect ratio $(A/\cos^2 \Lambda)$, and the ratio of the modulus of elasticity to the allowable bending stress and depends on the taper ratio and on the detailed design of the wing (see appendix B). By means of this relation the bending stiffness of one wing can be estimated from that of a reasonably similar wing. Or, with the constants of proportionality η_{20} and F_w given in table 1 and appendix B, respectively, which take into account some of the detailed design parameters as well as the taper ratio, the stiffness can be estimated directly. However, in view of the fact that these constants have been derived on the basis of a highly idealized structure and loading condition they must be used with caution. The ratio of the root bending

TABLE 1.—DEFINITIONS OF THE FACTORS η_1 TO η_{21}

$\eta_1 = \frac{L_w}{L_{total}}$
$\eta_2 = \frac{W_w}{W}$
$\eta_3 = \eta_1 - \eta_2$
$\eta_4 = \frac{1}{1 + \text{Margin of safety}}$
$\eta_5 = \frac{\text{Ordinate of most highly stressed element}}{\text{One-half of wing thickness}}$
$\eta_6 = \frac{\text{Actual skin thickness of most highly stressed element}}{\text{Equivalent skin thickness of most highly stressed element}}$
$\eta_7 = \frac{\text{Cross-sectional area of (assumed) single torsion cell}}{\text{Chord} \times \text{Wing thickness}}$
$\eta_8 = \frac{\text{Perimeter of torsion cell}}{\text{Twice the chord}}$
$\eta_9 = \frac{\text{Effective perimeter } \bar{p} \text{ of torsion cell}}{\text{Actual perimeter of torsion cell}}$
$\eta_{10} = \frac{\text{Width of equivalent sheet}}{\text{Chord}}$
$\eta_{11} = \frac{\text{Average ordinate of upper skin}}{\text{Maximum ordinate of upper skin}}$
$\eta_{12} = \frac{\text{Equivalent thickness of lower skin}}{\text{Equivalent thickness of upper skin}}$
$\eta_{13} = \frac{\text{Maximum ordinate of lower skin}}{\text{One-half of wing thickness}}$
$\eta_{14} = \frac{\text{Average ordinate of lower skin}}{\text{Maximum ordinate of lower skin}}$
$\eta_{15} = \eta_{10}(\eta_5^2 \eta_{11}^2 + \eta_{12} \eta_{13}^2 \eta_{14}^2)$
$\eta_{16} = \frac{\text{Allowable torsion stress}}{\text{Allowable bending stress}}$
$\eta_{17} = \frac{\bar{c}b'}{S \cos \Lambda}$
$\eta_{18} = \eta_{17} \left(\frac{b}{b'} \right)^2$
$\eta_{19} = \frac{1}{4} \frac{\eta_{15} \eta_{18}}{\eta_5 \eta_6 \eta_7 \eta_{16}}$
$\eta_{20} = \frac{\eta_3 \eta_5}{\eta_4 \eta_{17} \eta_{18}^2}$
$\eta_{21} = \frac{\text{Equivalent perimeter } \bar{p} \text{ of torsion cell}}{\text{Actual perimeter of torsion cell}}$

stiffness to the root torsional stiffness can be estimated by means of equations given in appendix B or, preferably, from experience with structures similar to that under consideration.

The spanwise stiffness distributions need not be known in detail in order to use the charts and approximate formulas. If the wing is solid or nearly solid or if its cross sections are geometrically similar at all points, the charts for stiffness distributions proportional to the fourth power of the chord are used. If the wing does not have large cutouts and is designed for a constant stress level, the charts for the stiffness distributions associated with constant stress are used.

The use of these charts tends to overestimate aeroelastic effects to some extent because, although actual wings are designed for constant stress over most of the span, the portions near their tips are designed on the basis of other considerations, such as handling loads or minimum standard sheet thicknesses; therefore, wings tend to be stiffer near the tip than they would be if designed on the basis of constant stress throughout. This difference in stiffness is particularly large if the taper ratio is zero.

If the wing contains large cutouts or if, for any other reason, the wing stiffness distribution is known to be substantially different from a constant-stress type, the charts can be used to furnish semiquantitative results for the various aeroelastic phenomena by using fictitious stiffnesses, provided the actual stiffness distribution is known at least approximately. The root stiffnesses of these fictitious distributions may be assumed to be the ones that give rise to twist or bending angles at the tip which are the same as those of the actual wing if the bending moments or torques vary as the square of the distance from the tip. For convenience, the spanwise distribution of these fictitious stiffnesses may then be assumed to be proportional to the fourth power of the chord. On the basis of these assumptions,

$$\frac{1}{(EI)_e} = 3\lambda \int_0^1 \frac{1}{EI} (1-s^*)^2 ds^* \quad (10)$$

where the subscript e refers to the fictitious stiffness, and where the integral represents the moment of inertia of the area under the function $\frac{1}{EI}$ plotted against s^* about the point $s^*=1$. The fictitious torsional stiffness at the root can be obtained in the same manner. The aeroelastic phenomena can then be estimated by use of these fictitious root stiffnesses and the charts for stiffness distributions proportional to the fourth power of the chord.

In the derivation of the charts the wing is considered to be clamped at the effective root perpendicular to the elastic axis. From the data and analyses presented in references 1, 2, 4, and 5 a satisfactory location of the effective root appears to be at the intersection of the elastic axis and the side of the fuselage.

If the rotations at this effective root are known as a result of deflection tests or a detailed analysis such as that of reference 5, the root twist due to torque and the root bending due to bending moment may be taken into account by moving the effective root inboard by the distance

$$\Delta s^e = \frac{\varphi_r^T}{T_r} (GJ)_r \quad (11)$$

or

$$\Delta s^r = \frac{\Gamma_r^M}{M_r} (EI)_r \quad (12)$$

where φ_r^T is the angle of twist at the root due to a root torque T_r , and where Γ_r^M is the deflection slope at the effective root due to a bending moment M_r . Since the

distances Δs^e and Δs^r may differ from each other, some compromise between the two must be made; for unswept wings the use of Δs^e appears to be indicated, whereas for highly swept wings the use of Δs^r is more appropriate.

PRELIMINARY SURVEY OF AEROELASTIC BEHAVIOR

The information contained in some of the subsequent sections has been summarized in figure 2 for the purpose of ascertaining in advance of more detailed estimates, if desired, whether the aeroelastic phenomena considered herein are likely to affect the design of the wing structure. This preliminary survey is not essential to any of the further calculations but may show them to be unnecessary in some cases.

The charts of figures 2 (a) to 2 (d) pertain to wings of taper ratios 0, 0.2, 0.5, and 1.0 and constitute plots of the dynamic-pressure parameter q^* defined by equation (7) against the sweep parameter k defined by equation (9). These parameters contain the root stiffnesses $(GJ)_r$ and $(EI)_r$; if, when a preliminary survey of aeroelastic effects is to be made, no information whatever concerning the wing stiffness is available, the following relations for q^* and k may be used:

$$q^* = \frac{C_{L_{\alpha_e}} (1+\lambda)^2 \cos \Lambda}{18432 \frac{G}{F_B} \frac{nW}{S} \frac{h_r}{c_r} \eta_a F_r} q \quad (13)$$

$$k = \frac{1+\lambda}{2} \frac{G}{E} \frac{A_A}{\eta_{19} e_1} \eta_b \tan \Lambda \quad (14)$$

where F_r is a root-stiffness parameter defined in appendix B and plotted in figure 3, and where η_a and η_b are defined by

$$\eta_a = \frac{\eta_3 \eta_5 \eta_6 \eta_7^2}{\eta_4 \eta_8 \eta_9 \eta_{15} \eta_{17}} \quad (15a)$$

and

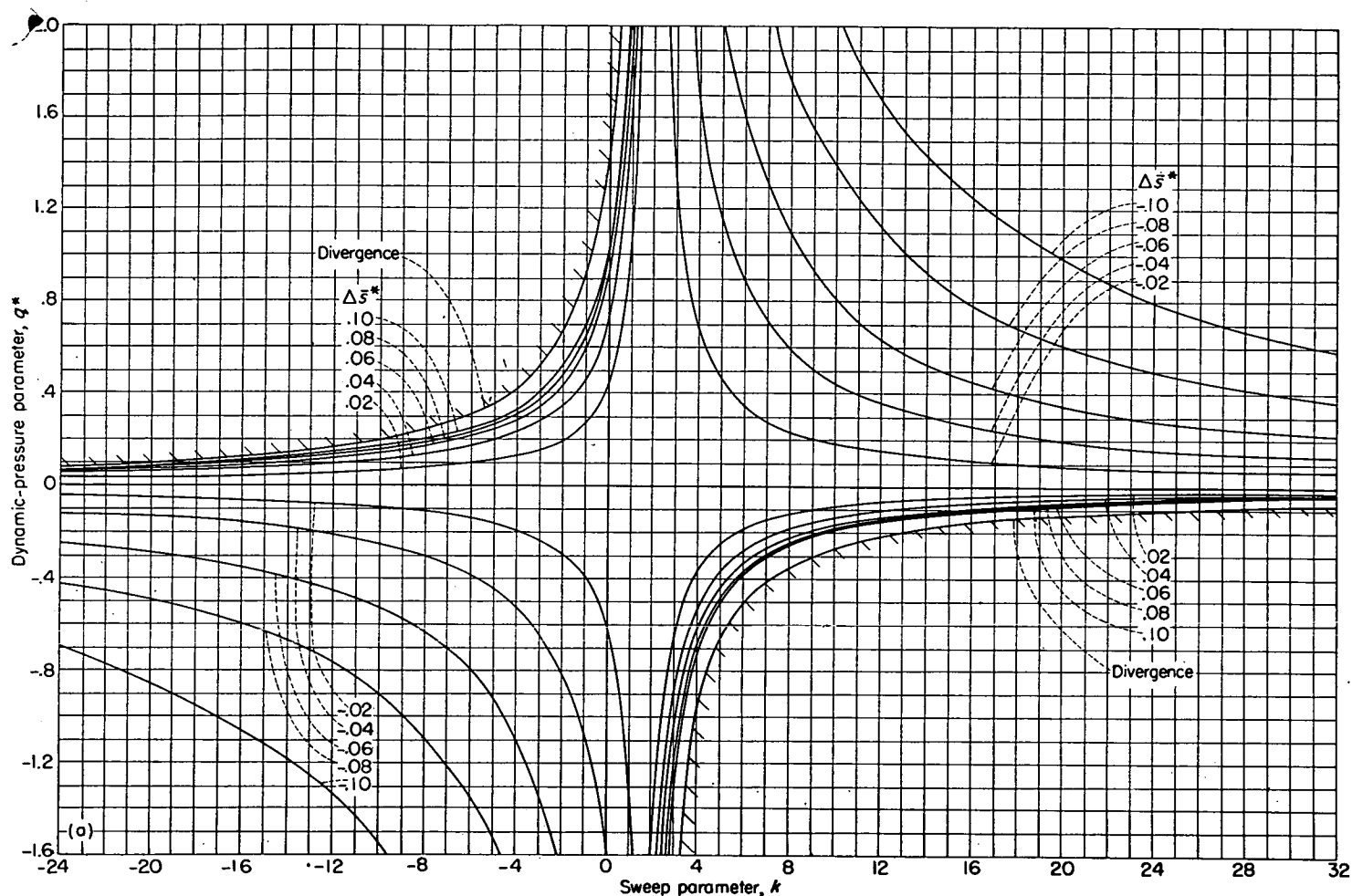
$$\eta_b = \frac{\eta_7}{\eta_5 \eta_8 \eta_9 \eta_{16}} \quad (15b)$$

in terms of some of the factors defined in table 1.

Figure 2 (e) pertains to wings for which the moment arm e_1 is zero and, hence, k is infinite; with the degree of approximation involved in the use of figures 2 (a) to 2 (d), figure 2 (e) can be used for wings with $|k| > 25$. This figure consists in a plot of the dynamic-pressure parameter $\bar{q}$, defined by equation (8), against the taper ratio λ . If no information is available concerning the root bending stiffness $(EI)_r$, contained in $\bar{q}$, the following relation may be used:

$$\bar{q} = \frac{C_{L_{\alpha_e}} (1+\lambda)^3 A_A \sin \Lambda}{36864 \frac{E}{F_B} \frac{nW}{S} \frac{h_r}{c_r} \frac{\eta_a}{\eta_b} \eta_{19} F_r} q \quad (16)$$

The various lines of the charts of figure 2 designate the conditions at which a wing designed on the basis of strength considerations alone is likely to encounter divergence or spanwise shifts of the center of pressure by various amounts; positive shifts are those toward the tip. These spanwise shifts furnish an estimate of the increase in root bending



(a) Wings of taper ratio 0.

FIGURE 2.—Charts for a preliminary survey of aeroelastic phenomena.

moment due to aeroelastic action and an estimate of the shift in wing aerodynamic center, since

$$\Delta \bar{a} = \sin \Lambda \frac{\Delta \bar{s}}{C_{MAC}} \quad (17)$$

Inasmuch as the parameters q^* and $\bar{q}$ contain the dynamic pressure, the negative values of q^* shown in figures 2 (a) to 2 (d) may require some explanation. The four quadrants of each of the charts of figures 2 (a) to 2 (d) may be characterized for practical purposes as follows:

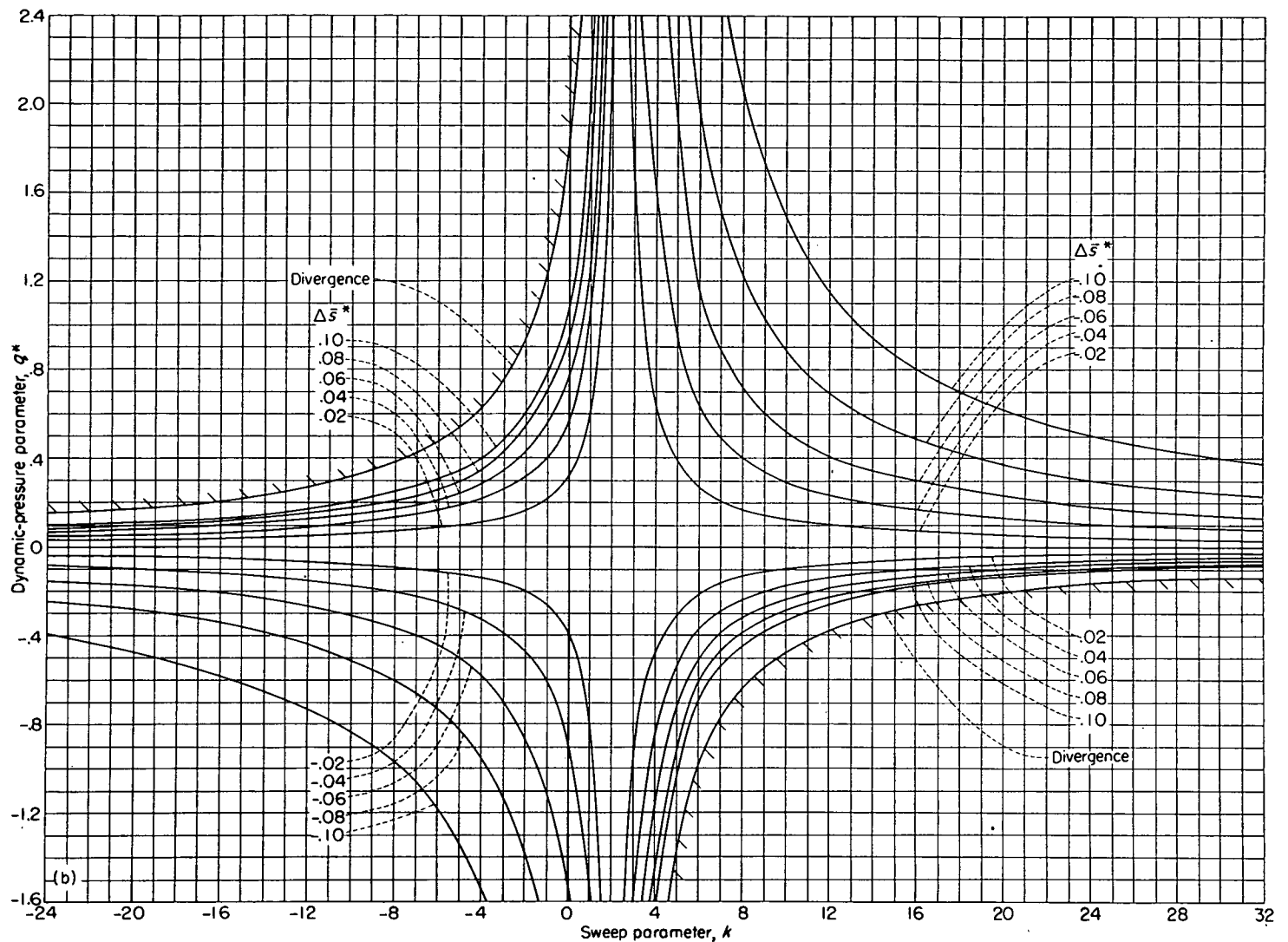
Quadrant	Sweep	e_1	Divergence	Shift in $\bar{y}$
1	Back	Positive	Impossible beyond a certain sweep	Inboard beyond a certain sweep
2	Forward	Positive	Likely	Outboard
3	Back	Negative	Impossible	Inboard
4	Forward	Negative	Possible beyond a certain sweep	Outboard beyond a certain sweep

For unswept wings k is approximately equal to zero, and the aeroelastic phenomena referred to in the charts of figures 2 (a) to 2 (d) are similar to those of swept wings defined by points in quadrant 2 if q^* is positive and by points in quadrant 3 if q^* is negative. In other words, the aeroelastic

phenomena of unswept wings are similar to those of swept-forward wings if e_1 is positive and to those of sweptback wings if e_1 is negative. The aerodynamic-center shift associated with the shift in the lateral center of pressure $\bar{y}$ or in the spanwise center of pressure $\bar{s}$ is always forward, except for small positive values of k (associated with sweep angles smaller than a certain value), in both quadrants 1 and 4.

The significance of negative values of q^* is that e_1 is negative, rather than that q is negative. A negative value of e_1 may be obtained at supersonic speeds; but under most conditions e_1 is likely to be positive. Similarly, in figure 2 (e) a negative value of $\bar{q}$ implies that Λ is negative (that is, that the wing is swept forward), whereas a positive value implies that the wing is swept back.

In using figures 2 (a) to 2 (d), estimates must be made of either the root stiffnesses (in conjunction with eqs. (7) and (9)) or of the effectiveness factors η_a and η_b (for use in eqs. (13) and (14)). The factor F_r is obtained from figure 3 for the largest value of e_1 likely to be encountered at the design load factor and for the given taper ratio λ . The parameter q^* is calculated for the combination of dynamic pressure q , lift-curve slope $C_{L_{\alpha}}$, and moment arm e_1 which is likely to be critical from an aeroelastic point of view. For an unswept



(b) Wings of taper ratio 0.2.

FIGURE 2.—Continued.

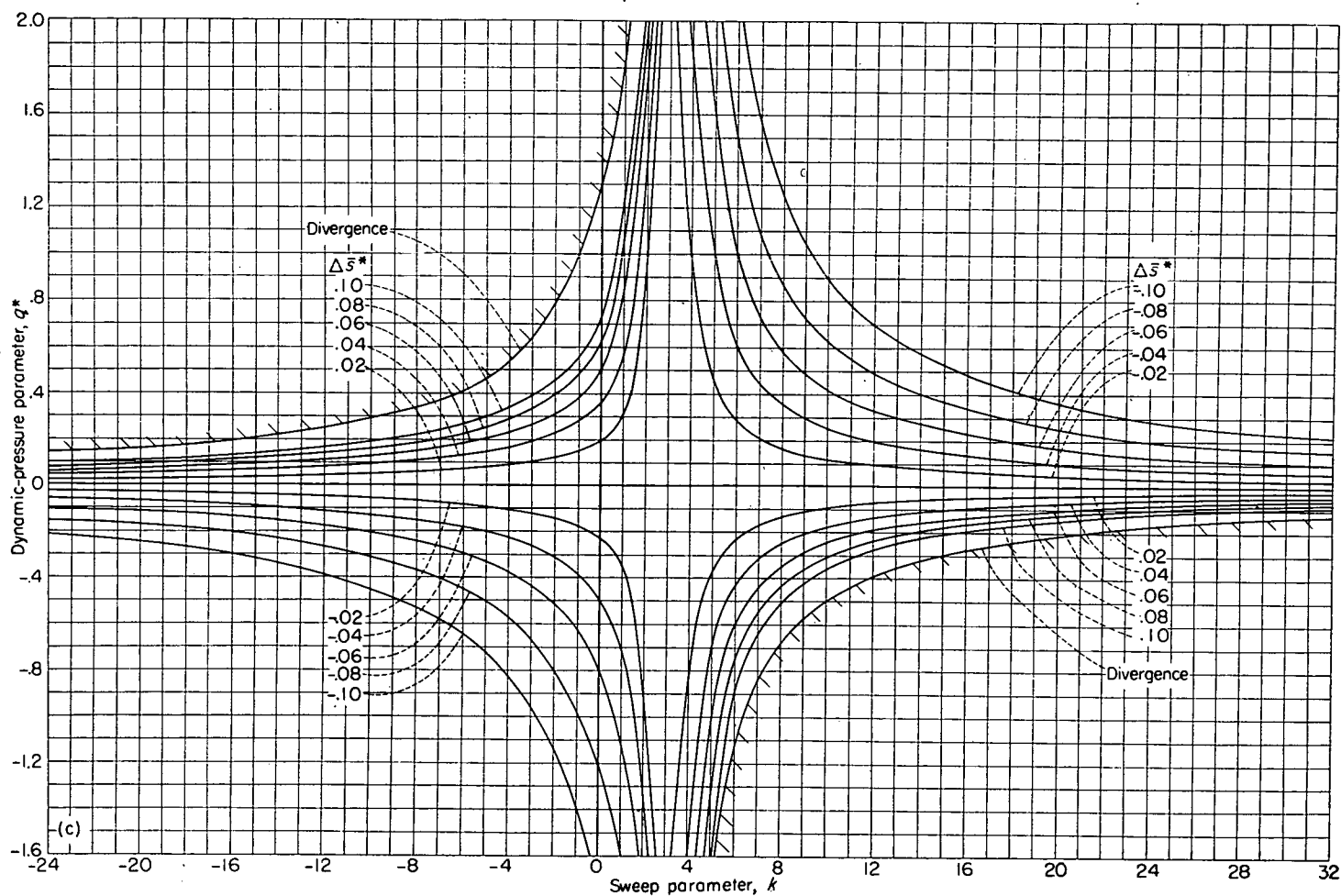
wing the combination for which the product $qC_{L_{\alpha_e}} e_1$ is a maximum is likely to be critical; for a swept wing the combination for which $qC_{L_{\alpha_e}}$ is a maximum is likely to be critical.

The parameter k is then calculated for the same value of e_1 .

The values of q^* and k define a point on one of the charts of figures 2 (a) to 2 (d) (whichever is closest to the actual taper ratio). If the shift in spanwise center of pressure (and any associated shift in the aerodynamic center) at that point is small and, in the case of an unswept or a sweptforward wing, if the absolute value of the ratio of the value of q^* at that point to the value of q^* at divergence for the value of k at that point is small, the static aeroelastic phenomena discussed in this report probably need not be taken into account in designing the wing structure. On the other hand, if the point on the chart indicates the likelihood of significant

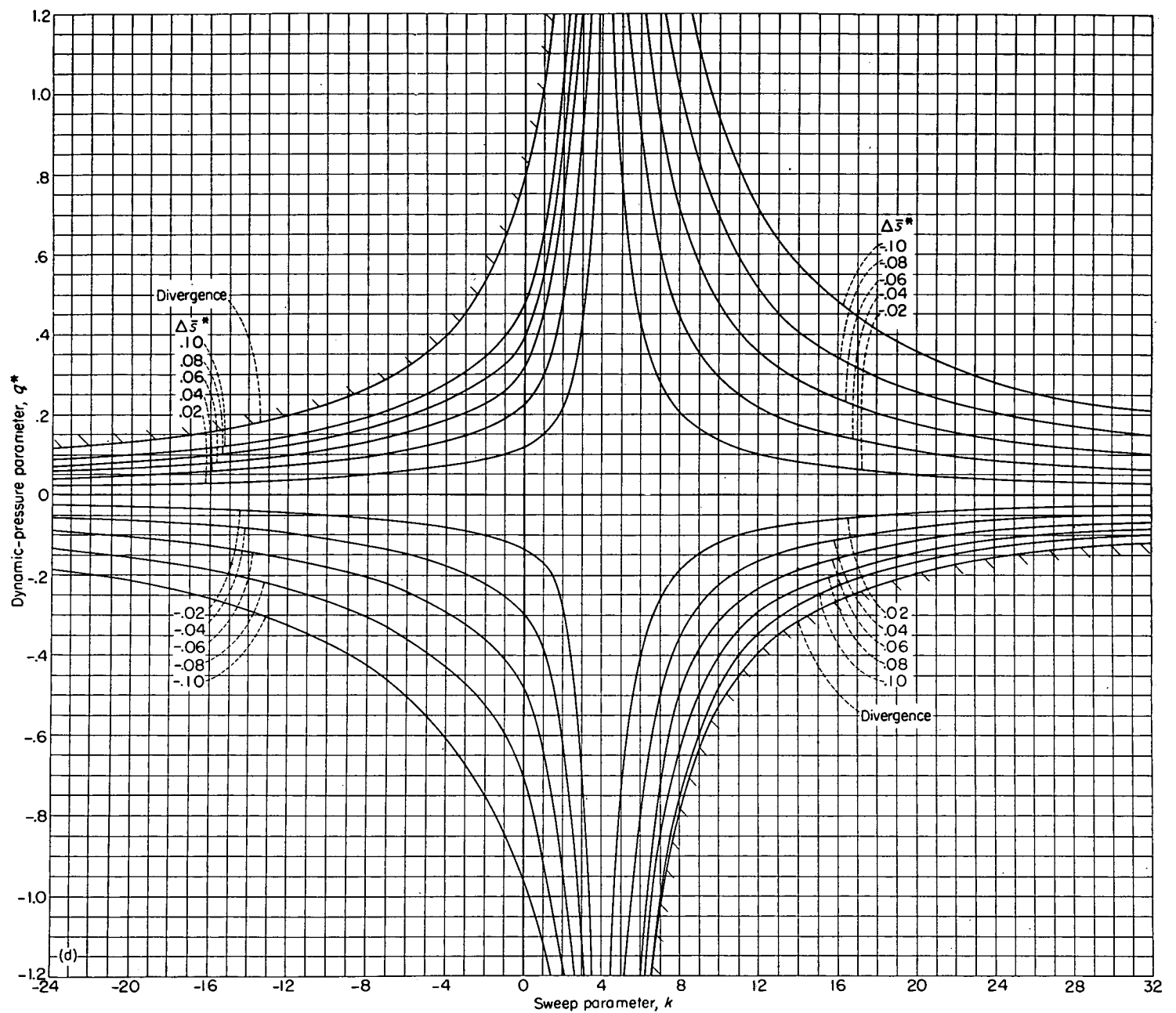
aeroelastic effects on the spanwise center of pressure or the possibility of an approach to the divergence condition, further calculations are desirable. The charts of this report may be used for the preliminary calculations; once the structure has been designed, more refined methods such as that of reference 1 may be used.

If the moment arm e_1 is so small or the angle of sweep so large that the parameter k exceeds the range covered by figures 2 (a) to 2 (d), the chart of figure 2 (e) may be used for the purpose of a preliminary aeroelastic appraisal of the given wing. In this figure only the parameter $\bar{q}$ is required, since k is considered to be infinite. The parameter $\bar{q}$ can be obtained from equation (16). The analysis then proceeds in the same manner as for figures 2 (a) to 2 (d) using, in the parameter $\bar{q}$, the condition for maximum $qC_{L_{\alpha_e}}$.



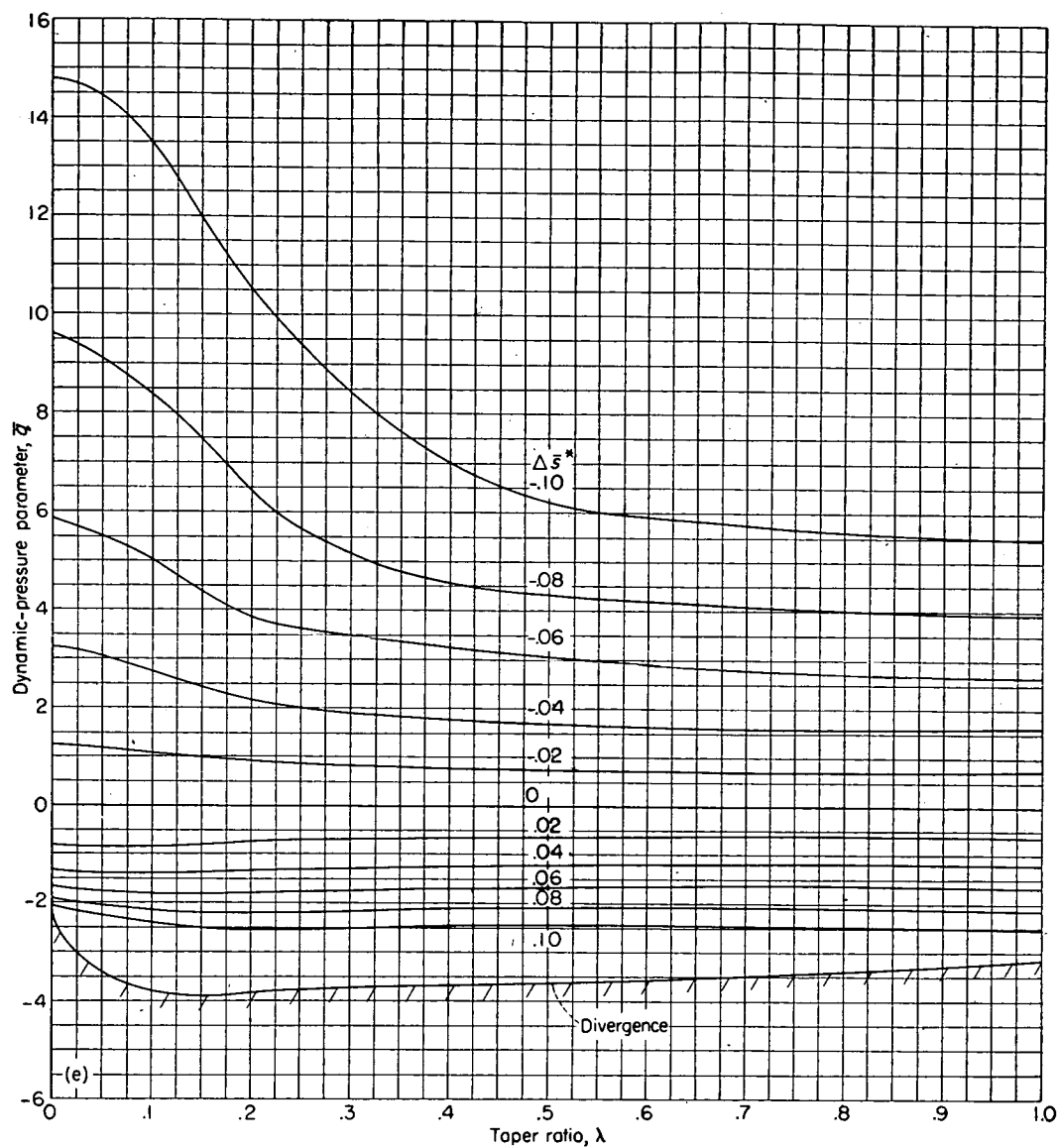
(c) Wings of taper ratio 0.5.

FIGURE 2.—Continued.



(d) Wings of taper ratio 1.0.

FIGURE 2.—Continued.



(e) Wings with moment arm $e_1 = 0$ (or $|k| > 25$).
FIGURE 2.—Concluded.

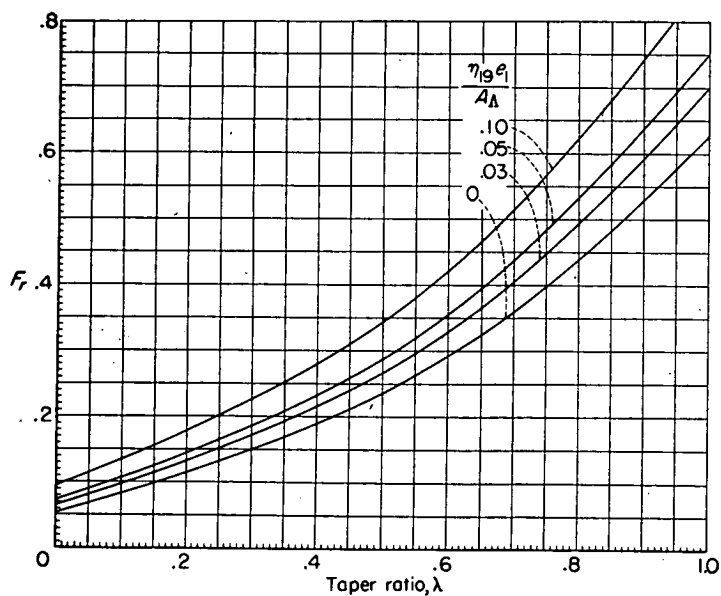


FIGURE 3.—The root-stiffness function F_r .

CALCULATION OF THE VARIOUS AEROELASTIC PHENOMENA

Dynamic pressure at divergence.—The solutions for the divergence speed obtained by the direct method in reference 2 and those obtained by the numerical matrix method given in appendix A can be summarized by approximate formulas which give the dimensionless parameters q^*_D and $\bar{q}_D$ (the values of the parameters defined in eqs. (7) and (8) that correspond to the value of the dynamic pressure q at divergence) in terms of their ratio k defined by equation (9).

These approximate formulas are

$$q^*_D = \frac{K_1}{1 - K_2 k} \quad (18)$$

and

$$\bar{q}_D = -\frac{K_1}{K_2 - \frac{1}{k}} \quad (19)$$

When the angle of sweep is zero, equation (18) reduces to $q^*_D = K_1$, and when the moment arm e_1 is zero, as it may be in supersonic flow, equation (19) reduces to $\bar{q}_D = -\frac{K_1}{K_2}$. The constants K_1 and K_2 are given in table 2 for wings with taper ratios of 0, 0.2, 0.5, and 1.0 for both types of stiffness distributions; the parameter q^*_D for unswept wings and the parameter $\bar{q}_D$ for swept wings with $e_1 = 0$ are plotted against the taper ratio λ in figures 4 (a) and 4 (b), respectively.

With the values of q^*_D or $\bar{q}_D$ given by equations (18) and (19) and the definitions of these two parameters given by equations (7) and (8), the values of q required for divergence may be determined. If desired, the corresponding airspeed may be determined from the relation

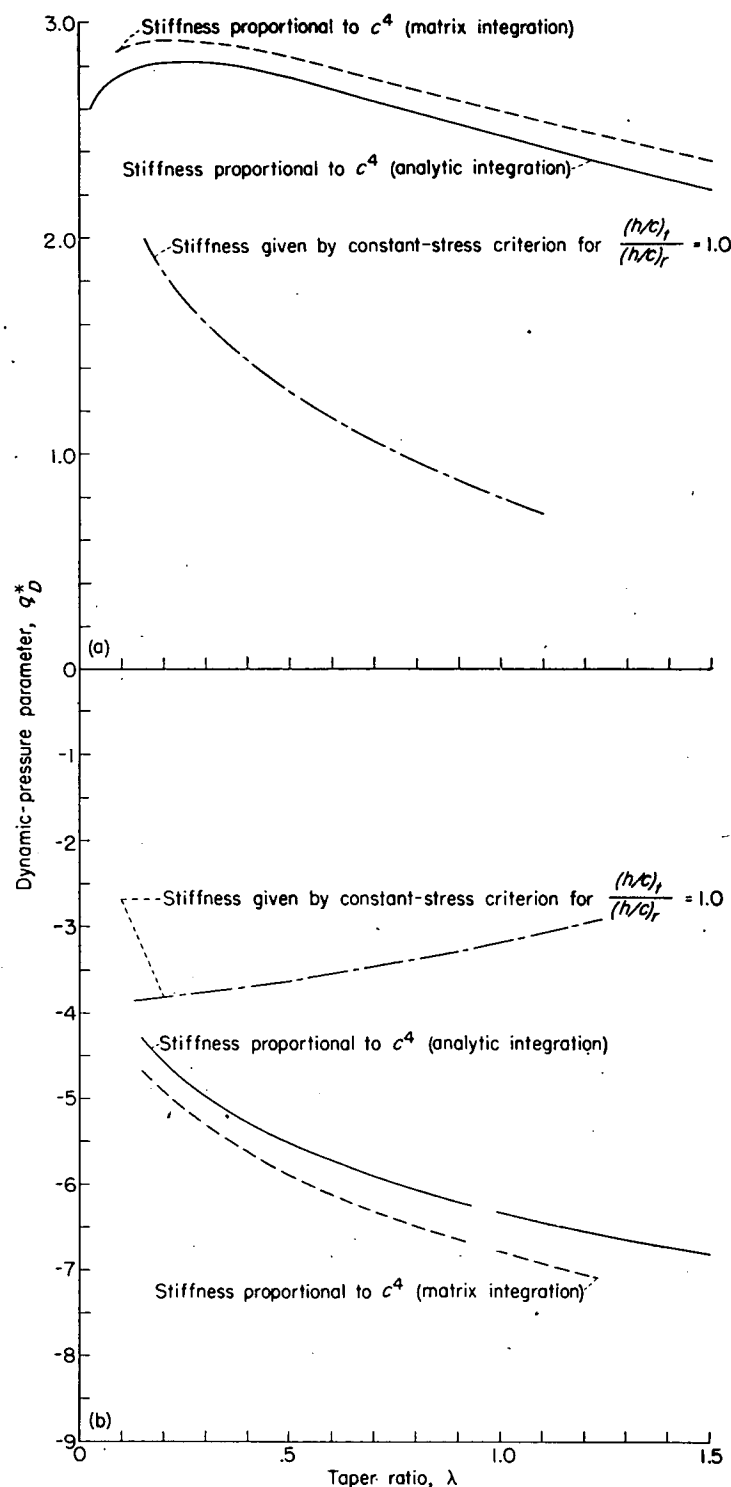
$$V_D = \sqrt{\frac{q_D}{\rho/2}}$$

The value of q_D is often negative for sweptback wings, and, since a negative dynamic pressure does not correspond to any real speed, these wings cannot diverge. These negative values of q_D , nonetheless, are useful as reference values in

TABLE 2.—VALUES OF THE COEFFICIENTS K_1 AND K_2

Taper ratio, λ	GJ and EI	$\frac{(h/c)_t}{(h/c)_r}$	K_1		K_2	
			By matrix integration	By analytic integration (ref. 2)	By matrix integration	By analytic integration (ref. 2)
1.0	Vary as c^4	1.0	2.58	2.47	0.381	0.390
.5		1.0	2.83	2.74	.480	.497
.2		1.0	2.92	2.81	.590	.614
0		1.0	1.45	2.25	.620	-----
1.0	Given by constant-stress criterion	1.0	.795	-----	.252	-----
.5		1.0	1.287	-----	.357	-----
.2		1.0	1.830	-----	.480	-----
0		1.0	1.440	-----	.626	-----
.5	Increased beyond values required by constant-stress criterion	.5	.928	-----	.310	-----
.5		1.0	1.700	-----	.398	-----

other aeroelastic phenomena. (If the negative value of q_D is very low in absolute magnitude, divergence in a higher mode associated with a positive q_D is conceivable. However,



(a) Effect of taper ratio on q^*_D for unswept wings. ($q^*_D = K_1$ in this case.)

(b) Effect of taper ratio on $\bar{q}_D$ for $e_1 = 0$. ($\bar{q}_D = -\frac{K_1}{K_2}$ in this case.)

FIGURE 4.—Effect of taper ratio on the dynamic pressure at divergence.

for the ordinary wings with straight leading and trailing edges and substantially straight elastic axes to which the analysis of this report is applicable, the magnitude of the lowest positive q_D is always much larger than the absolute magnitude of the negative q_D associated with the first mode. Higher-mode divergence of a practical wing for which a negative q_D is indicated by the analysis of this report is therefore very unlikely.)

The values of the constants K_1 and K_2 given in reference 2 differ somewhat from the corresponding values resulting from the matrix solution in appendix A. The matrix results are probably more significant because they are based upon more realistic aerodynamic assumptions; the K_1 and K_2 values in reference 2 tend to give conservative results.

The value of q_D calculated for any given value of q^*_D or $\bar{q}_D$ depends on the value of the effective lift-curve slope $C_{L_{\alpha_e}}$ or $c_{l_{\alpha_e}}$ and, hence, on the Mach number. As suggested in references 1 and 2, the value of q_D calculated at various Mach numbers may be plotted against Mach number. If lines of the actual dynamic pressure at several altitudes as a function of Mach number are drawn on the same plot, an intersection of the divergence line with one of the lines of actual dynamic pressure designates possible divergence at that value of dynamic pressure, Mach number, and altitude. If this plot is on log-log coordinates, the lines of actual dynamic pressure are straight and the ratio of the dynamic pressure at divergence to the actual dynamic pressure at a given Mach number and altitude can be scaled off directly. (See refs. 1 and 2.)

Spanwise angle-of-attack distributions.—In appendix A, an approximate expression is determined for the change in angle of attack due to wing flexibility. For the additional-type angle-of-attack distribution (α_g is constant) the angle of attack due to structural deformation α_s is given by

$$\frac{\alpha_s}{\alpha_g} = \frac{1}{\kappa} \frac{q/q_D}{1 - \frac{q}{q_D}} (f_1 + F_1 \Delta f_1) \quad (20)$$

The functions f_1 and Δf_1 of the spanwise coordinate s^* and the function F_1 of the parameter k are given in figure 5 for swept wings with taper ratios of 0.2, 0.5, and 1.0 and with the two different types of stiffness distributions. For wings with zero taper ratio the structural deformation cannot be obtained from equation (20), as is pointed out in appendix A. However, the ratio $\kappa \frac{\alpha_s}{\alpha_g}$ as a function of the spanwise coordinate s^* is shown in figure 6 for the two different stiffness distributions, several values of q/q_D , and several values of the parameter k .

The spanwise distribution of α_s due to a linear twist ($\alpha_g = s^* \alpha_{g_t}$), which may be either symmetric or antisymmetric, is approximately

$$\frac{\alpha_s}{\alpha_{g_t}} = \frac{q/q_D}{1 - \frac{q}{q_D}} (f_2 + F_2 \Delta f_2) \quad (21)$$

where the functions f_2 , Δf_2 , and F_2 are given in figure 7 for wings of taper ratios 0.2, 0.5, and 1.0. The angle-of-attack ratio is shown in figure 8 as a function of s^* , q/q_D , and k for wings of zero taper ratio.

The results of equations (20) and (21) may be superimposed. For example, if the spanwise distribution of α_s due to rolling is to be found, these equations must be added in such proportion that

$$\alpha_g = y^* \frac{pb}{2V}$$

But

$$y^* = \frac{w}{b} + \frac{b'}{b} s^*$$

so that

$$\alpha_g = \left(1 - \frac{b'}{b} + \frac{b'}{b} s^*\right) \frac{pb}{2V} \quad (22)$$

where $pb/2V$ is the angle of attack at the tip due to roll. The distribution of α_s due to roll is then

$$\frac{\alpha_s}{\frac{pb}{2V}} = \frac{q/q_D}{1 - \frac{q}{q_D}} \left[\frac{1}{\kappa} \left(1 - \frac{b'}{b}\right) (f_1 + F_1 \Delta f_1) + \frac{b'}{b} (f_2 + F_2 \Delta f_2) \right] \quad (23)$$

Spanwise lift distributions.—If desired, the lift distributions can be obtained for the angle-of-attack distributions given in the preceding section by one of the commonly used methods of calculating spanwise lift distributions, such as that of reference 6 or by the approximate method of reference 7. However, the following method is simpler and, within the approximation of the present report, as accurate provided the rigid-wing ($q=0$) loading l_0 is calculated by the methods of reference 6 or 7 or is obtained from the charts of reference 8 or, in general, by an accurate analytical method.

Within the framework of the assumptions made in the analysis the lift per inch of span is proportional to the local angle of attack, so that

$$\frac{l}{l_0} = 1 + \kappa \frac{\alpha_s}{\alpha_g} \quad (24a)$$

for geometrical angles of attack which are constant along the span, and

$$\frac{l}{l_0} = 1 + \frac{\alpha_{g_t}}{\alpha_g} \frac{\alpha_s}{\alpha_{g_t}} \quad (24b)$$

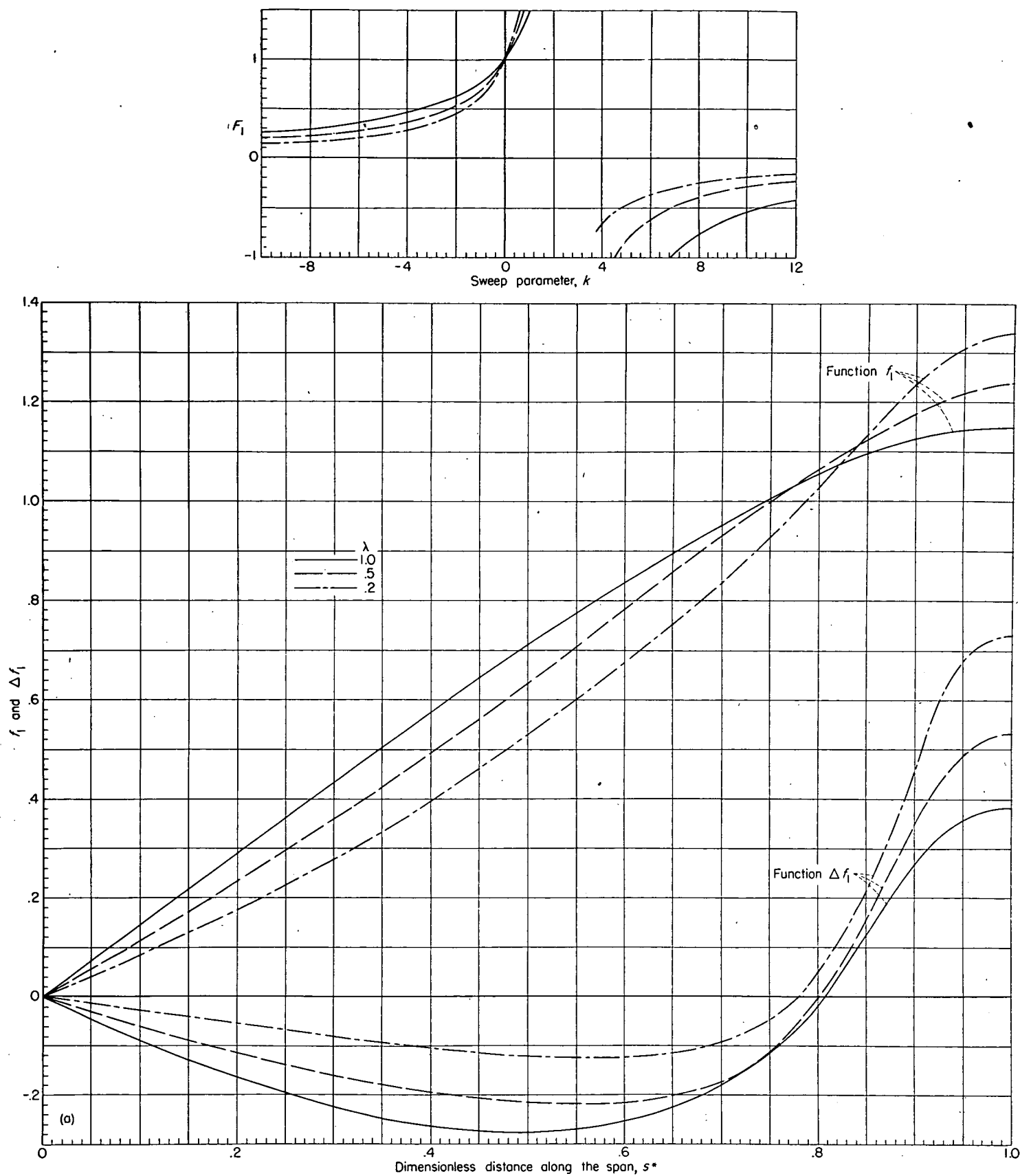
for geometrical angles of attack due to linear twist, where $\kappa \frac{\alpha_s}{\alpha_g}$ and α_s/α_{g_t} are obtained as indicated in the preceding section.

If no better approximation is available for the loading l_0 (which is likely to be the case at transonic speeds), it may be estimated for geometric angles of attack which are constant along the span from the relation

$$l_0 \approx C_{L_{\alpha}} c q \alpha_g \quad (25a)$$

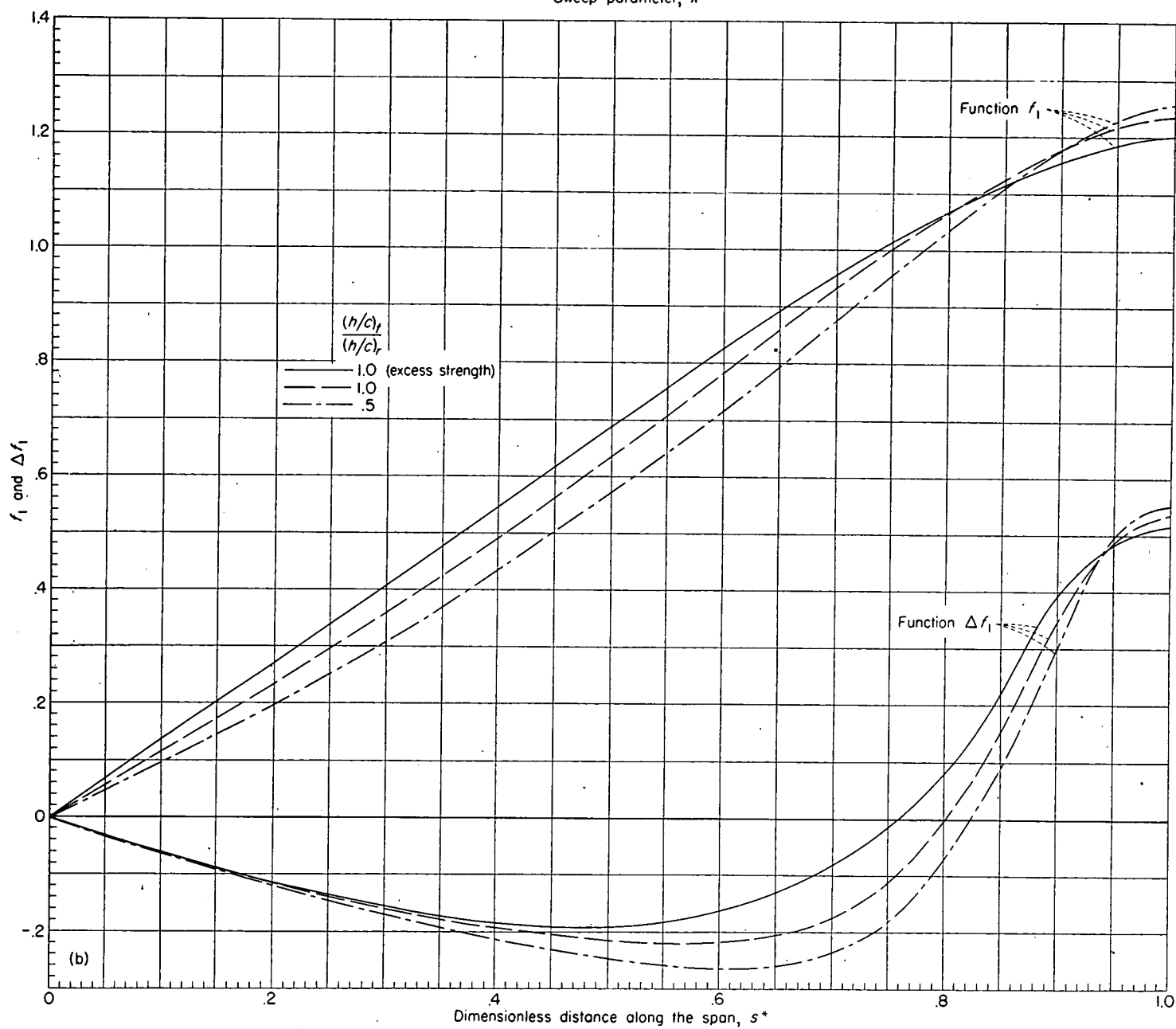
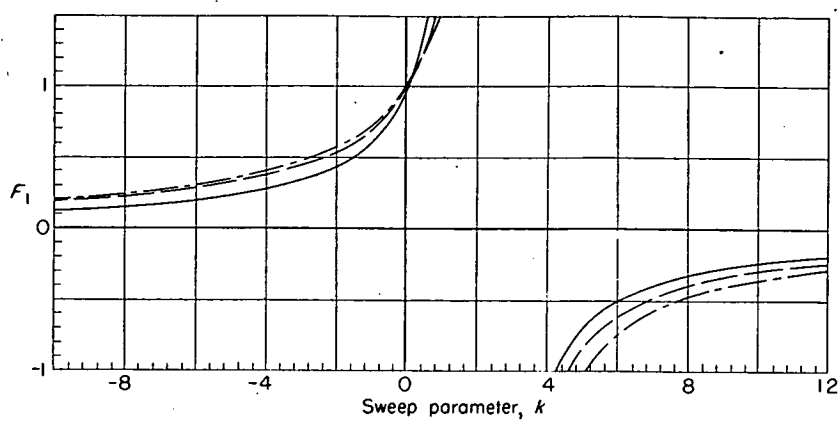
and for all other geometric angles of attack from the relation

$$l_0 \approx C_{L_{\alpha_e}} c q \alpha_g \quad (25b)$$



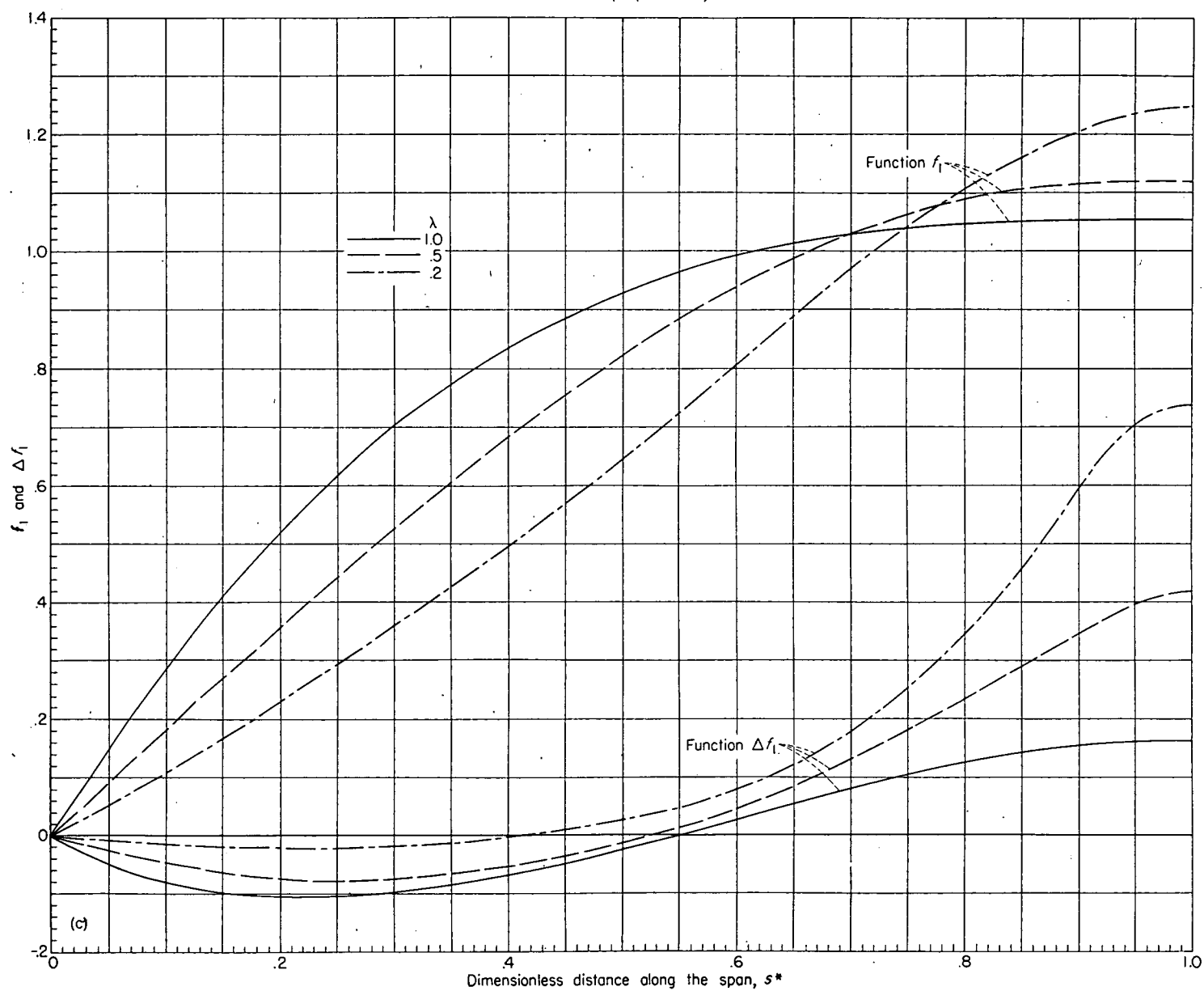
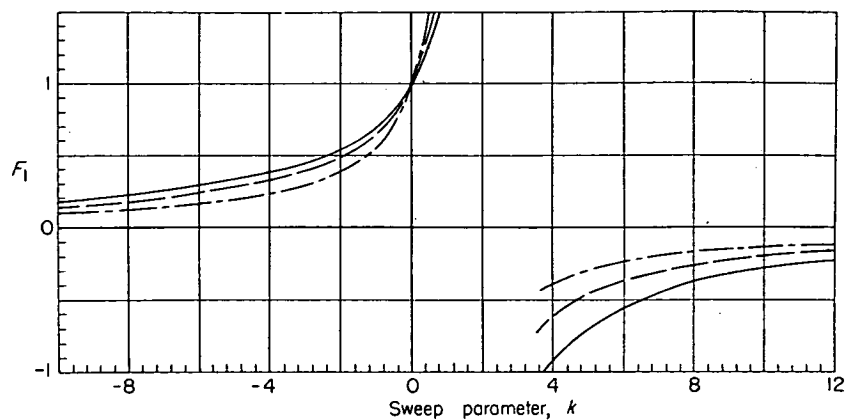
(a) Stiffnesses given by constant-stress criterion for $\frac{(h/c)_t}{(h/c)_r} = 1.0$.

FIGURE 5.--The angle-of-attack distribution functions f_1 , Δf_1 , and F_1 for constant geometric angles of attack.



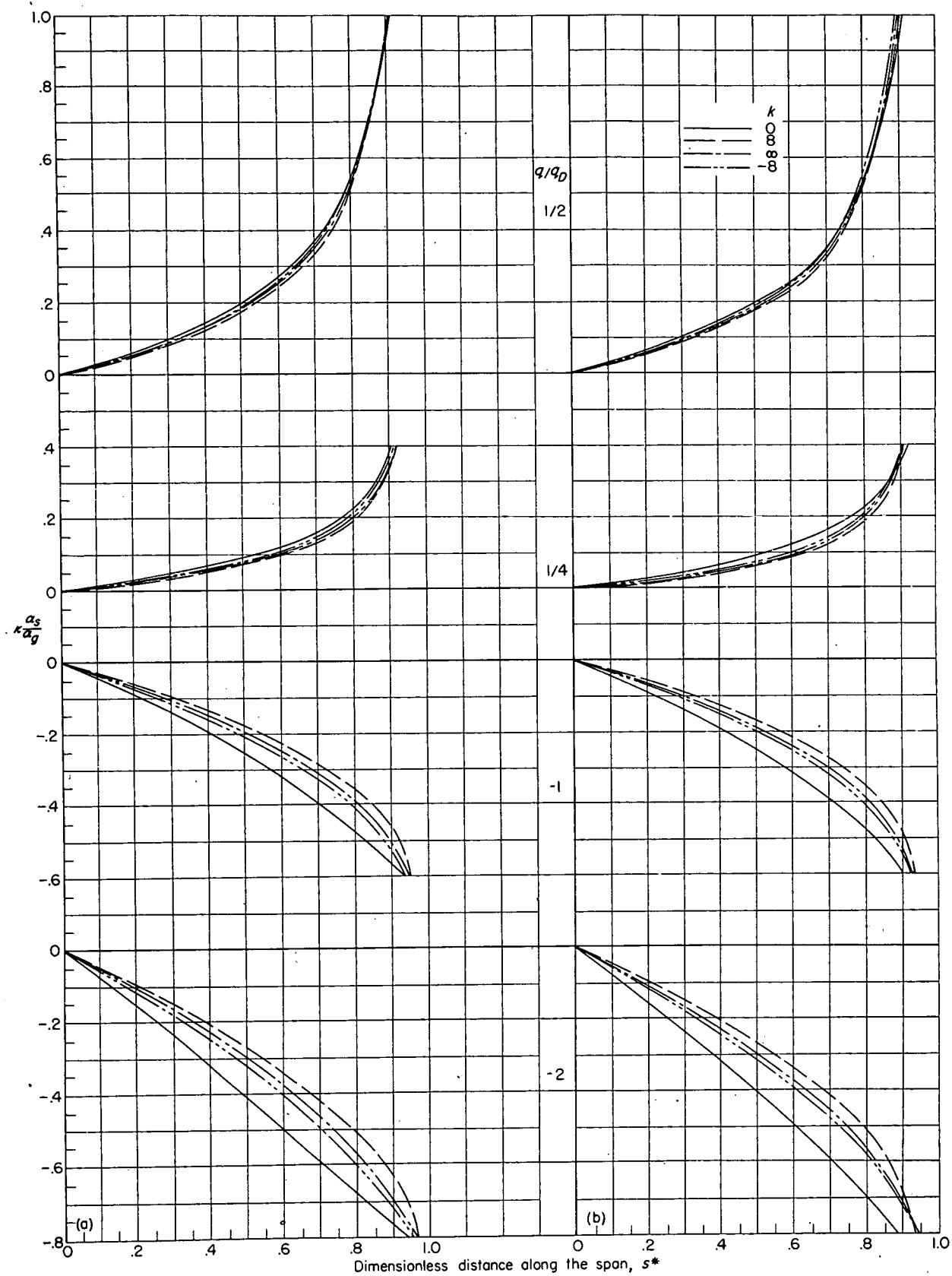
(b) Stiffnesses related to those given by constant-stress criterion for wings with taper ratio 0.5.

FIGURE 5.—Continued.



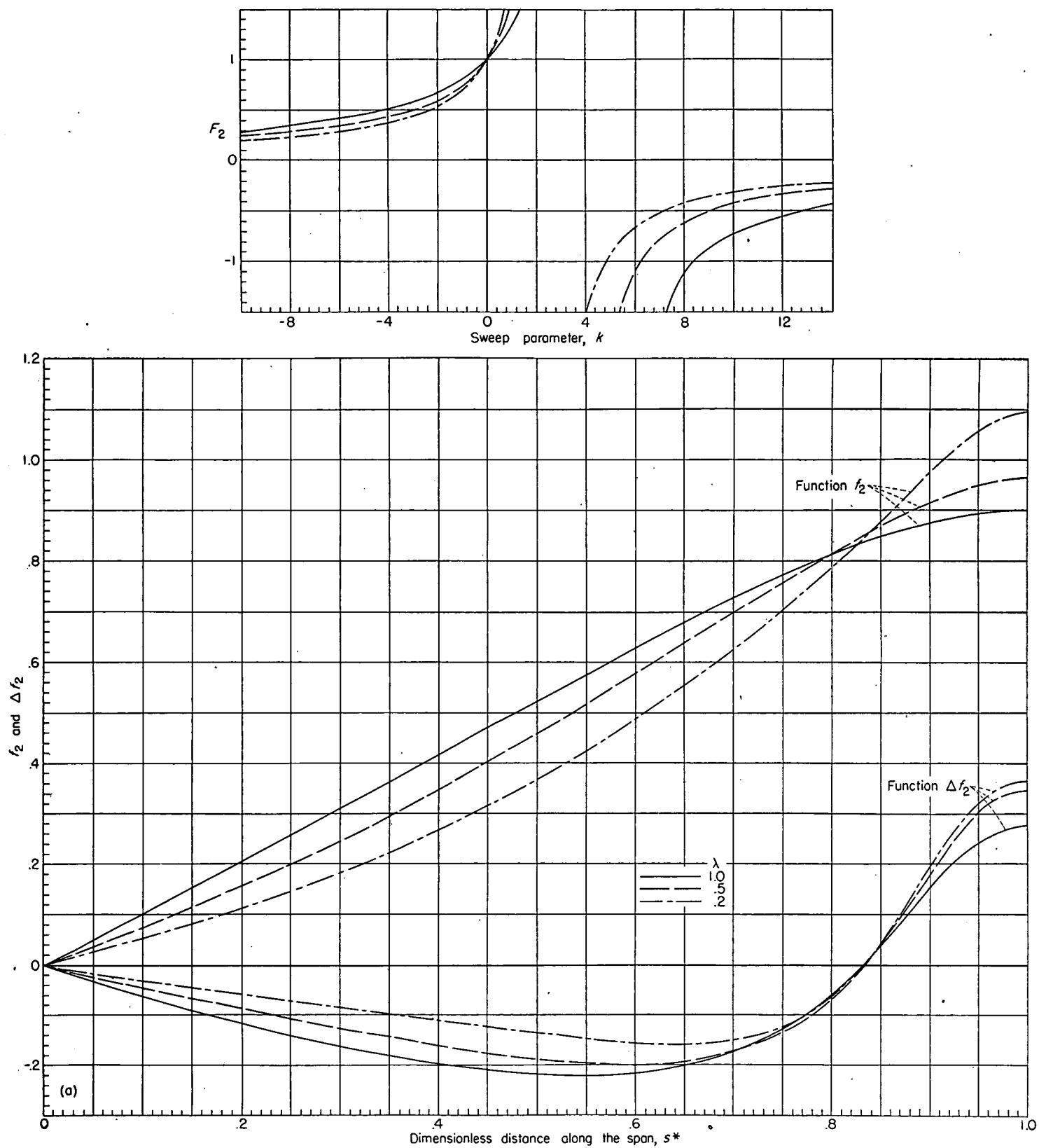
(c) Stiffnesses proportional to c^4 .

FIGURE 5.—Concluded.

(a) Stiffnesses proportional to c^4 .

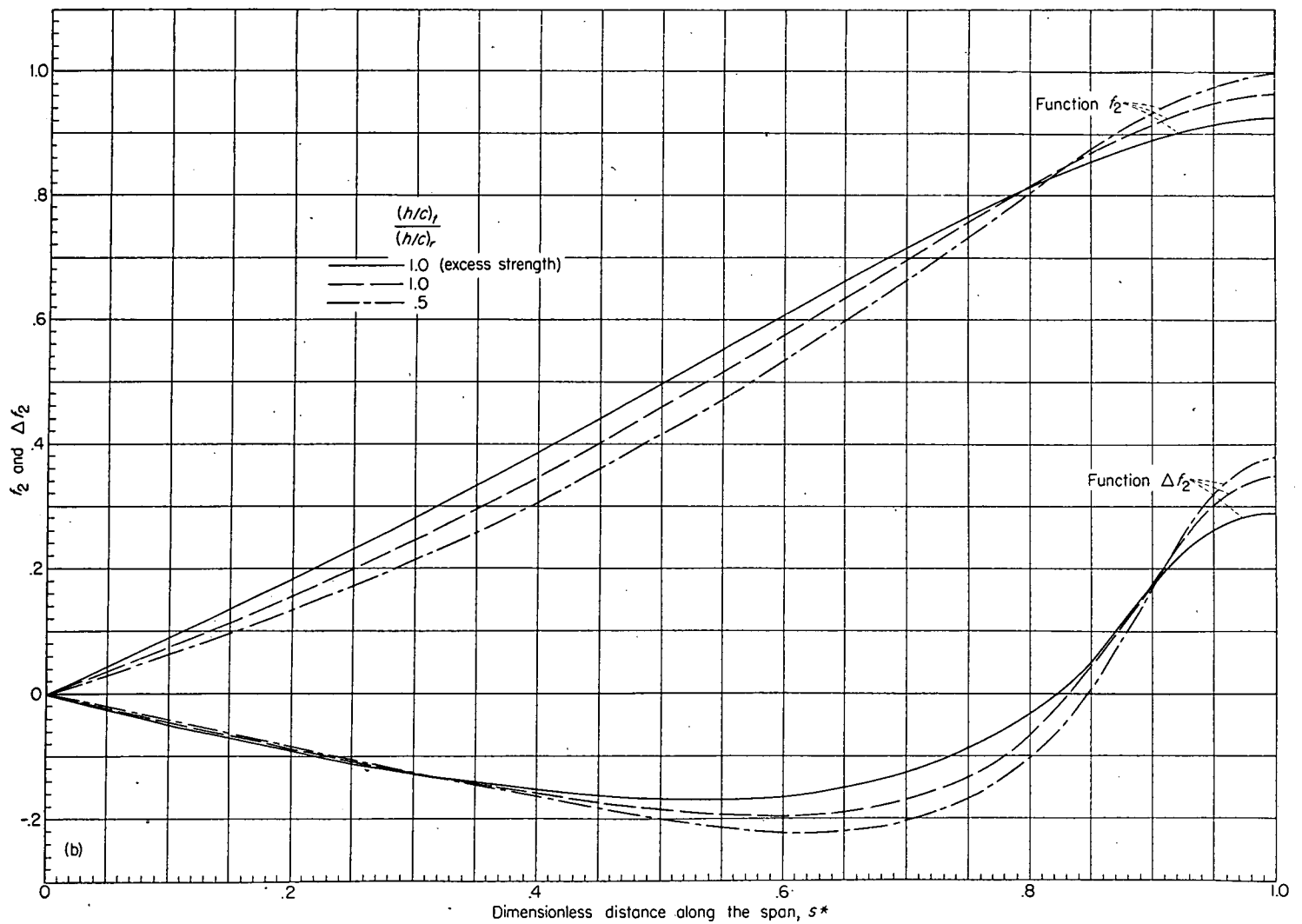
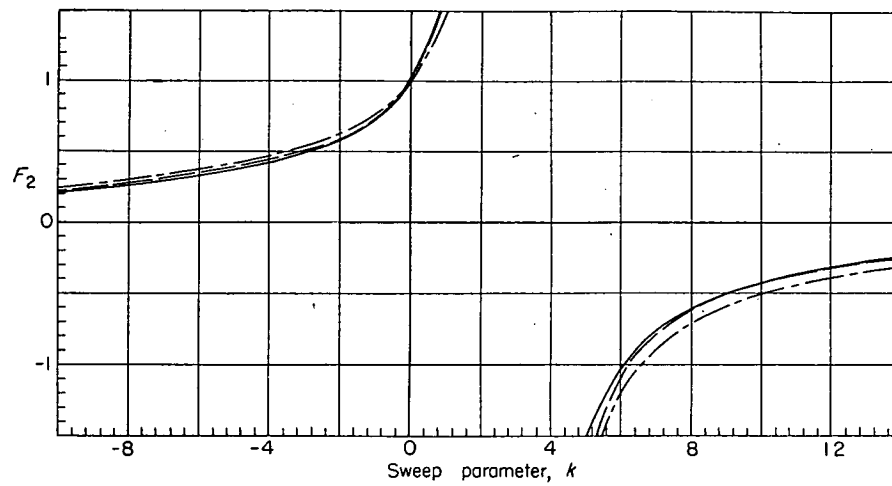
(b) Stiffnesses given by constant-stress criterion.

FIGURE 6.—The angle-of-attack distributions for constant geometric angles of attack for wings of taper ratio 0.



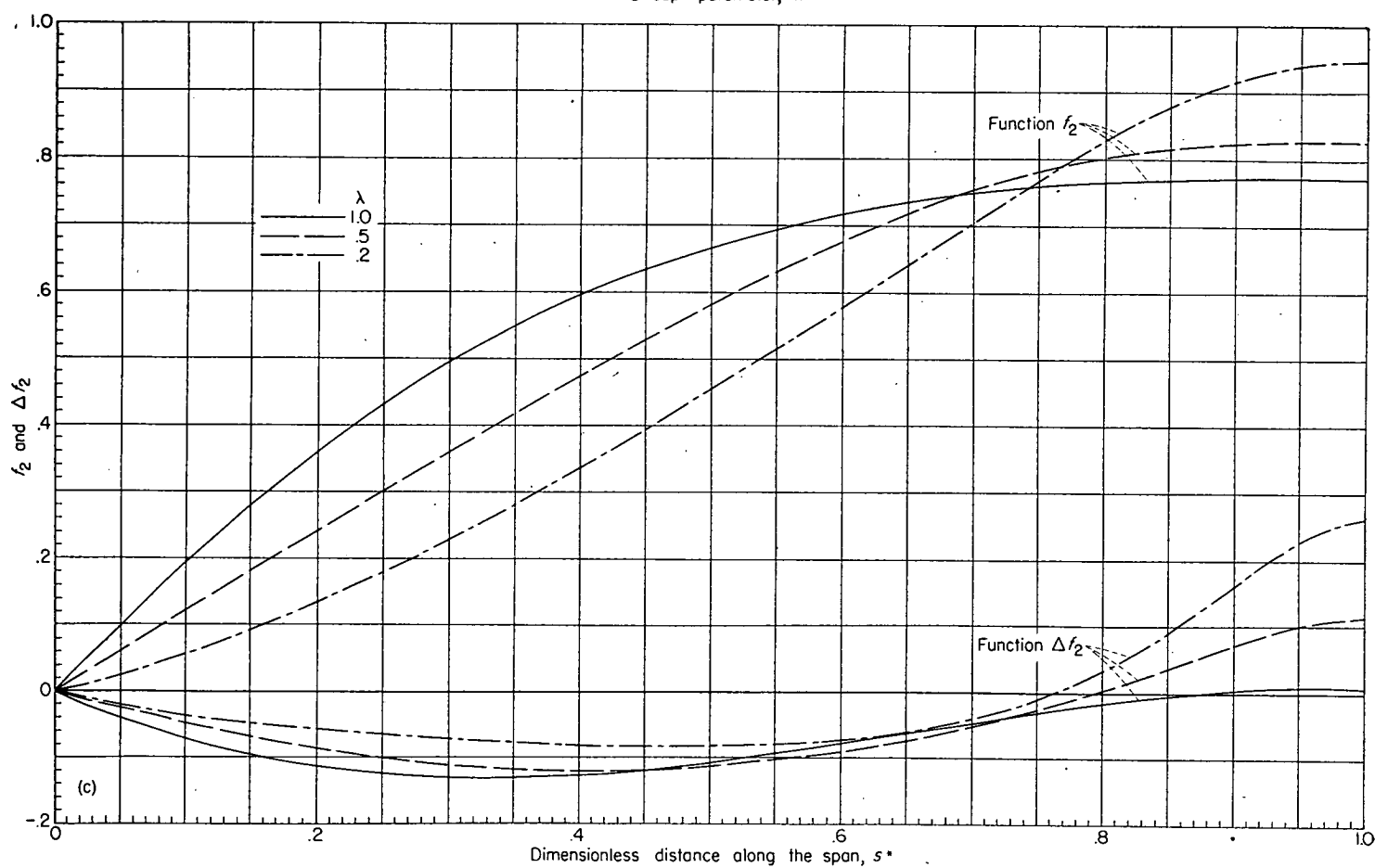
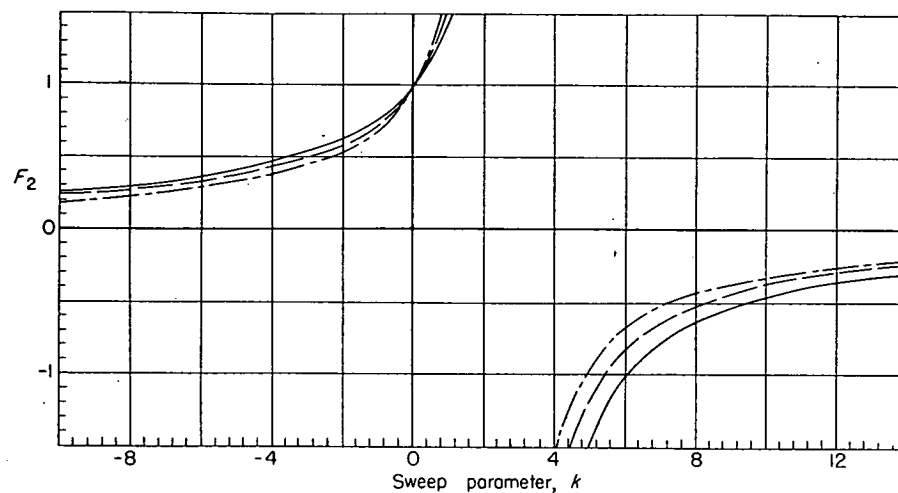
(a) Stiffnesses given by constant-stress criterion for $\frac{(h/c)_t}{(h/c)_r} = 1.0$.

FIGURE 7.—The angle-of-attack distribution functions f_2 , Δf_2 , and F_2 for linearly varying geometric angles of attack.



(b) Stiffnesses related to those given by constant-stress criterion for wings with taper ratio 0.5

FIGURE 7.—Continued.



(c) Stiffnesses proportional to c^4 .

FIGURE 7.—Concluded;

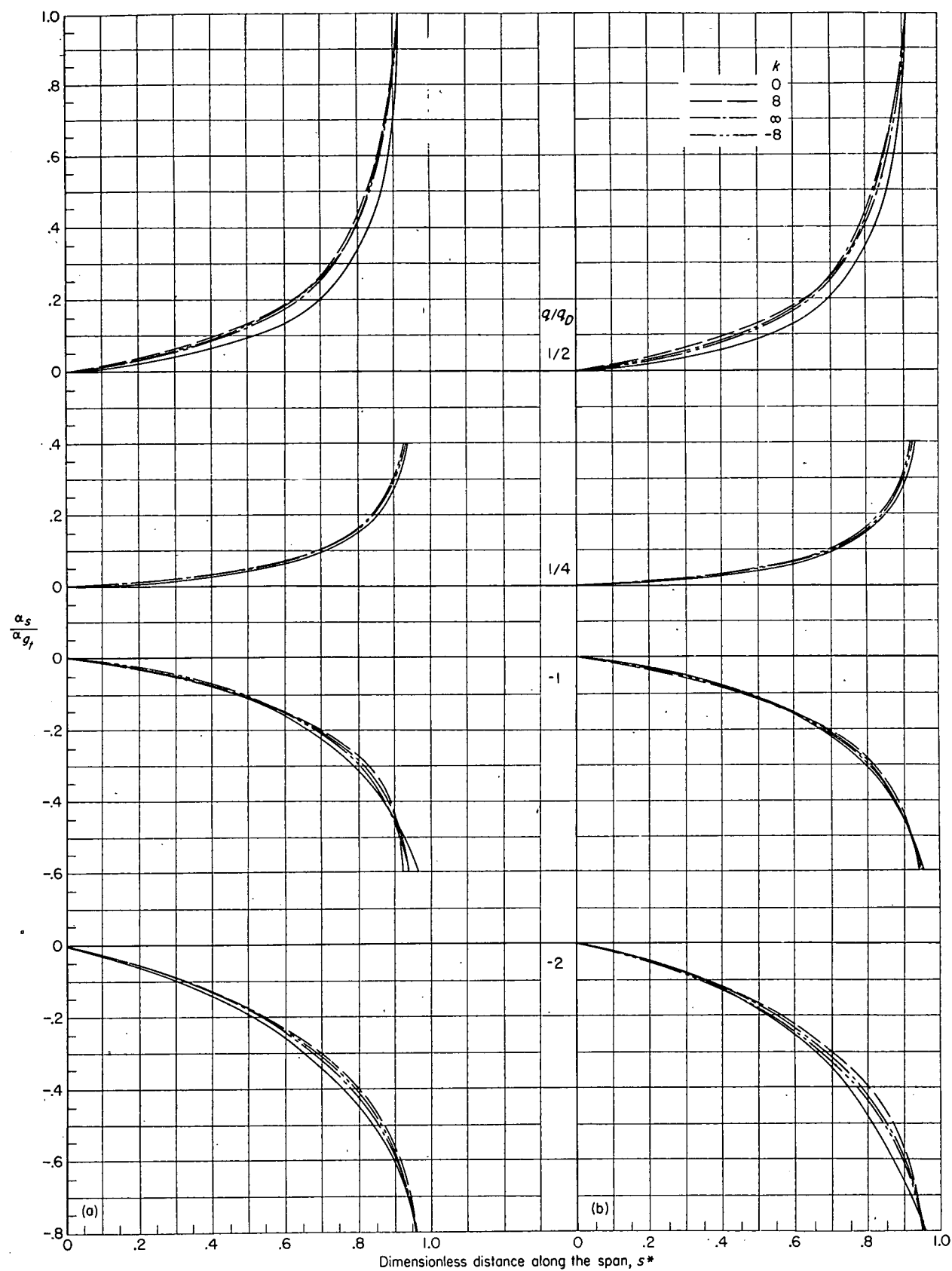


FIGURE 8.—The angle-of-attack distributions for linearly varying angles of attack for wings of taper ratio 0.

Lift and moment coefficients.—The wing lift coefficient C_{L_w} , the wing-root bending-moment coefficient C_B , and the wing-root twisting-moment coefficient C_T may be obtained in terms of their respective rigid-wing values by means of the following approximate expressions:

$$\frac{C_{L_w}}{C_{L_{w0}}} = \frac{1 - \frac{q}{q_D}(1-\nu)}{1 - \frac{q}{q_D}} \quad (26)$$

$$\frac{C_B}{C_{B0}} = \frac{1 - \frac{q}{q_D}(1-\mu\nu)}{1 - \frac{q}{q_D}} \quad (27)$$

$$\frac{C_T}{C_{T0}} = \frac{1 - \frac{q}{q_D}(1-\tau\nu)}{1 - \frac{q}{q_D}} \quad (28)$$

where the coefficients ν , μ , and τ depend on the type of loading. The subscript 1 is used for additional-type angle-of-attack distributions and the subscript 2, for linear-twist-type angle-of-attack distributions. The coefficients ν_1 , μ_1 , and τ_1 are given in figure 9 as functions of the parameter k for wings of taper ratios 0.2, 0.5, and 1.0. For wings with taper ratio zero the ratios of the lift, bending-moment, and twisting-moment coefficients to their respective rigid-wing values are given in figure 10 as a function of q/q_D for several values of the parameter k . The values of ν_2 , μ_2 , and τ_2 are given in figure 11 for wings of taper ratios 0.2, 0.5, and 1.0, and ratios of the lift, bending-moment, and twisting-moment coefficients are given in figure 12 for wings of zero taper ratio. The additional-twist and linear-twist results of equations (26) to (28) may be superimposed in the same way as those of equations (20) and (21).

The wing rolling-moment coefficient C_{l_w} is defined as the rolling moment of the loads on both wings about the fuselage center line divided by qSb . Therefore,

$$C_{l_w} = \frac{2M_r \cos \Lambda + 2T_r \sin \Lambda + \frac{w}{2} L_w}{qSb} \quad (29)$$

The angle-of-attack distribution due to rolling given in equation (22) must be used in finding the values of M_r , T_r , and L_w in equation (29).

Spanwise centers of pressure and aerodynamic centers.—The spanwise location of the center of pressure is given by the distance

$$\bar{s} = \frac{b}{2} \frac{C_B}{C_{L_w}} \quad (30)$$

or the dimensionless distance

$$\bar{s}^* = \frac{C_B}{C_{L_w}} \quad (31)$$

(Inasmuch as η^* is equal to s^* by virtue of the definitions of those dimensionless quantities (see also fig. 1), eq. (31) can be considered to be an expression for $\bar{\eta}^*$ rather than $\bar{s}^*$, if desired.) With the values of the bending-moment and lift coefficients given in the preceding section, the ratio of $\bar{s}$ to its value for the rigid wing may be calculated from either of the equations

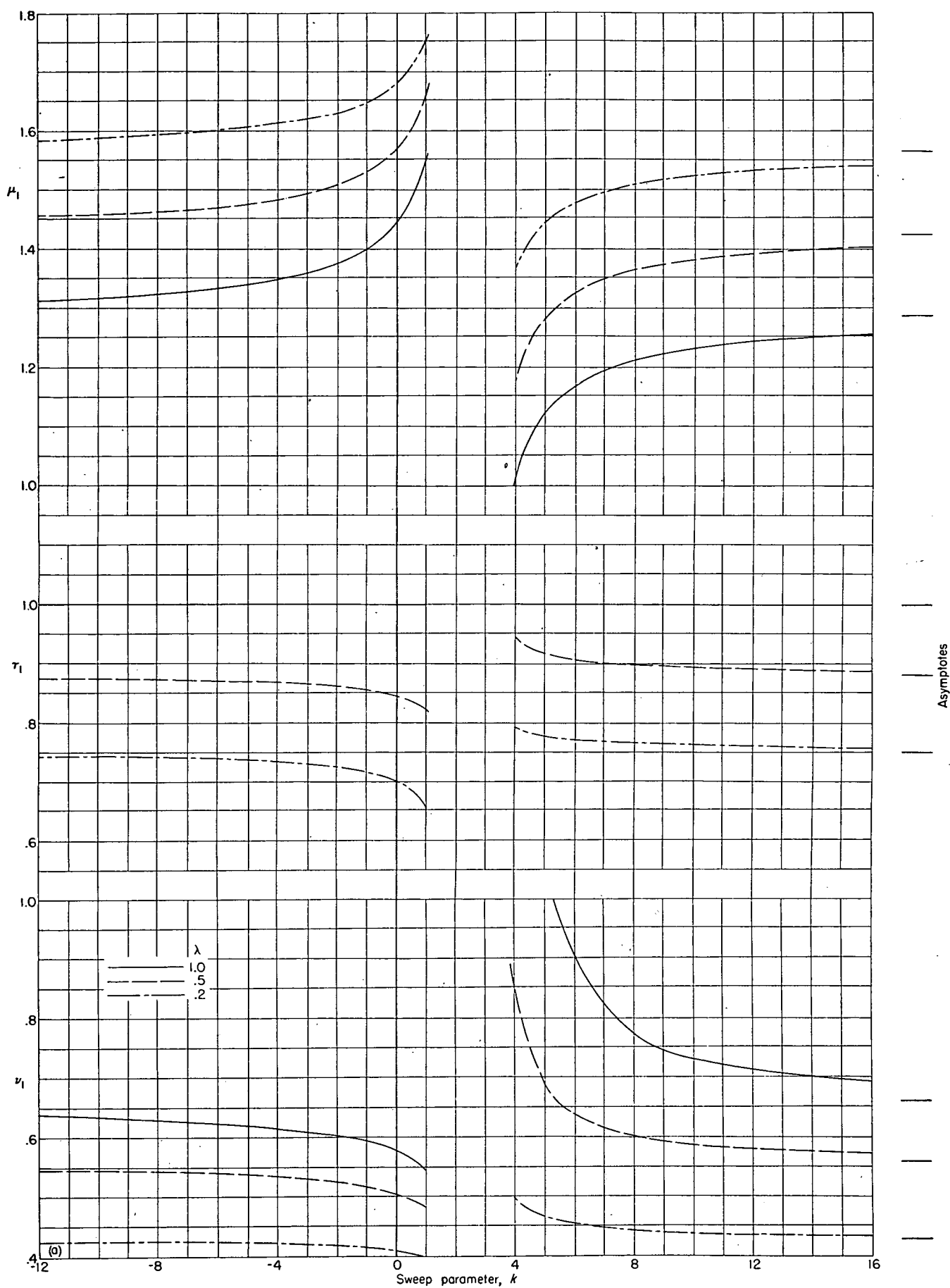
$$\left. \begin{aligned} \frac{\bar{s}}{\bar{s}_0} &= \frac{1 - \frac{q}{q_D}(1-\mu\nu)}{1 - \frac{q}{q_D}(1-\nu)} \\ \frac{\Delta\bar{s}}{\bar{s}_0} &= \frac{\nu \frac{q}{q_D}(\mu-1)}{1 - \frac{q}{q_D}(1-\nu)} \end{aligned} \right\} \quad (32)$$

where the shift in spanwise center of pressure $\Delta\bar{s}$ is defined as $\bar{s} - \bar{s}_0$, and where μ_1 and ν_1 are used for constant geometrical angles of attack and μ_2 and ν_2 , for linearly varying geometrical angles of attack.

The shift due to aeroelastic action of the longitudinal position of the center of pressure associated with a given shift of the spanwise center of pressure is equal to $\sin \Lambda \Delta\bar{s}$. The shift in aerodynamic center (positive when rearward, or stabilizing) can consequently be calculated by substituting into equation (17) the values of $\Delta\bar{s}$ obtained from equation (32) with values of μ_1 and ν_1 .

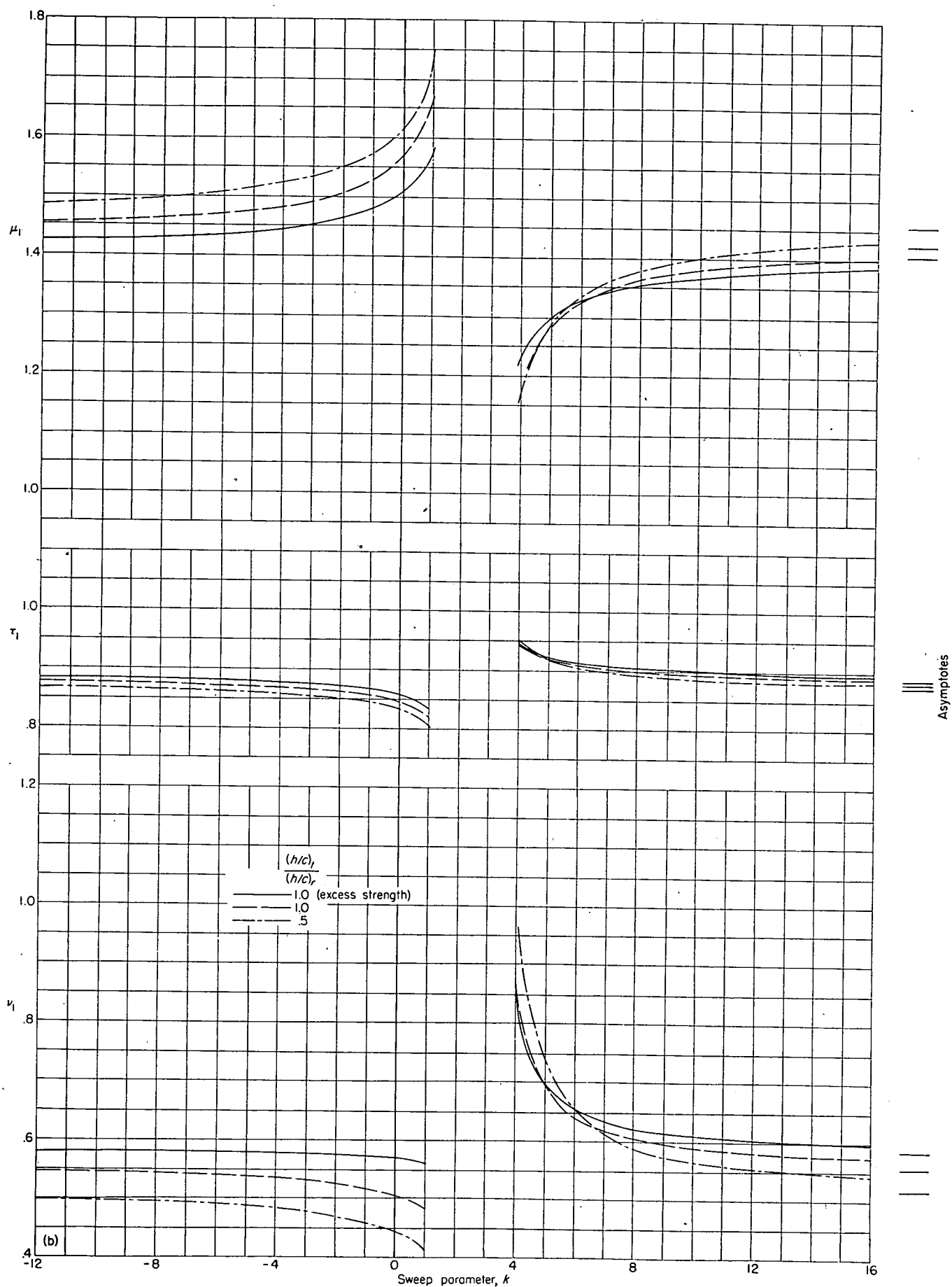
Inertia effects.—No charts are presented in this report for the effects of inertia on quasi-static aeroelastic phenomena—that is, aeroelastic phenomena associated with flight at constant acceleration; the manner in which mass is distributed varies so widely among different wings that preparation of a generally applicable set of charts for inertia effects appears to be impractical at present. Furthermore, except for flying wings, the wing deformations due to inertia loads are small compared with those due to aerodynamic loads, the two types of loads being in about the same ratio as the wing weight to the weight of the entire airplane. If desired, however, inertia effects and the aeroelastic increment in these effects can be estimated in the manner described in the following paragraphs.

From the known or estimated mass distribution of the wing the inertia load l_i per inch of span and the inertia torque t_i per inch of span can be calculated for any given normal, pitching, or rolling acceleration. Substitution of these loads and torques for the terms l and l_{ic} in equations (A3) or (A36) and equations (A2) or (A35), respectively, yields the values of the accumulated bending moment and torque due to the distributed inertia loads and torques. In turn, substitution of these accumulated bending moments and torques in equations (A4), (A5), and (A6), or in equations (A37) and (A38) and the matrix equivalent of equation (A6), yields the angle-of-attack distribution due to the deformations caused by the inertia effects associated with the given acceleration.



(a) Stiffnesses given by constant-stress criterion for $\frac{(h/c)_t}{(h/c)_r} = 1.0$.

FIGURE 9.—The lift- and moment-coefficient parameters μ_1 , ν_1 , and τ_1 for constant geometric angles of attack.



(b) Stiffnesses related to those given by constant-stress criterion for wings with taper ratio 0.5.

FIGURE 9.—Continued.

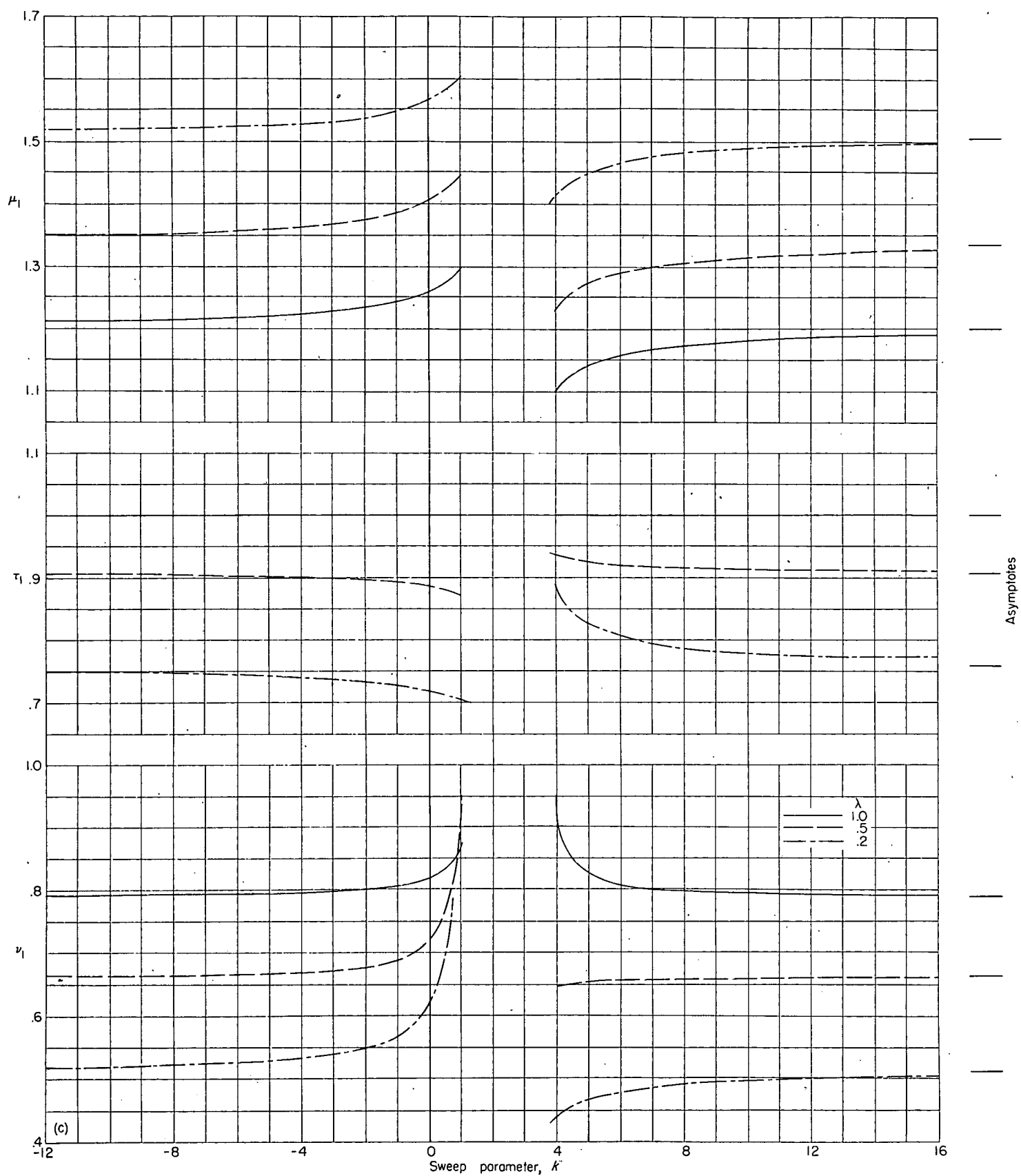
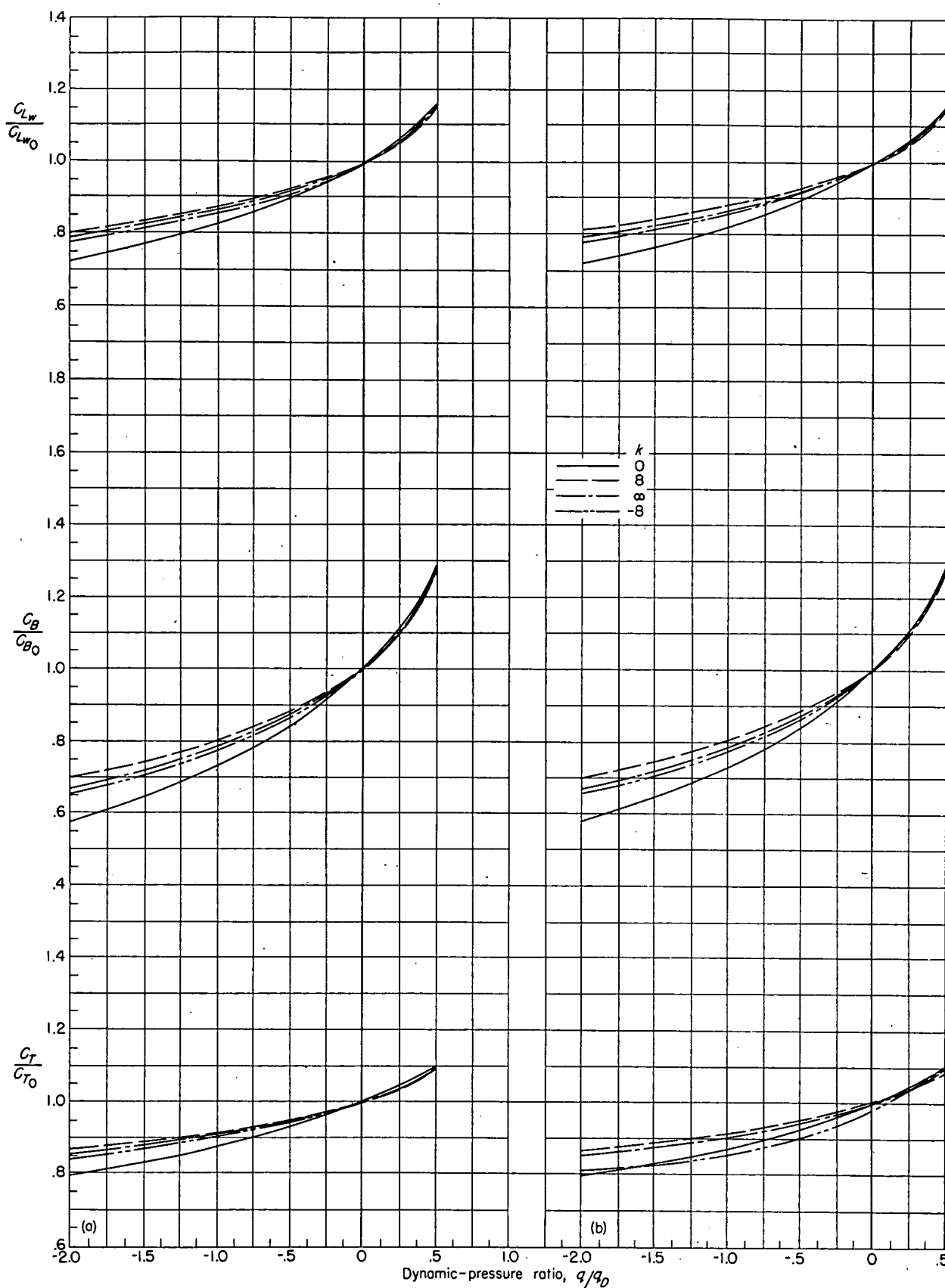
(c) Stiffnesses proportional to c^4 .

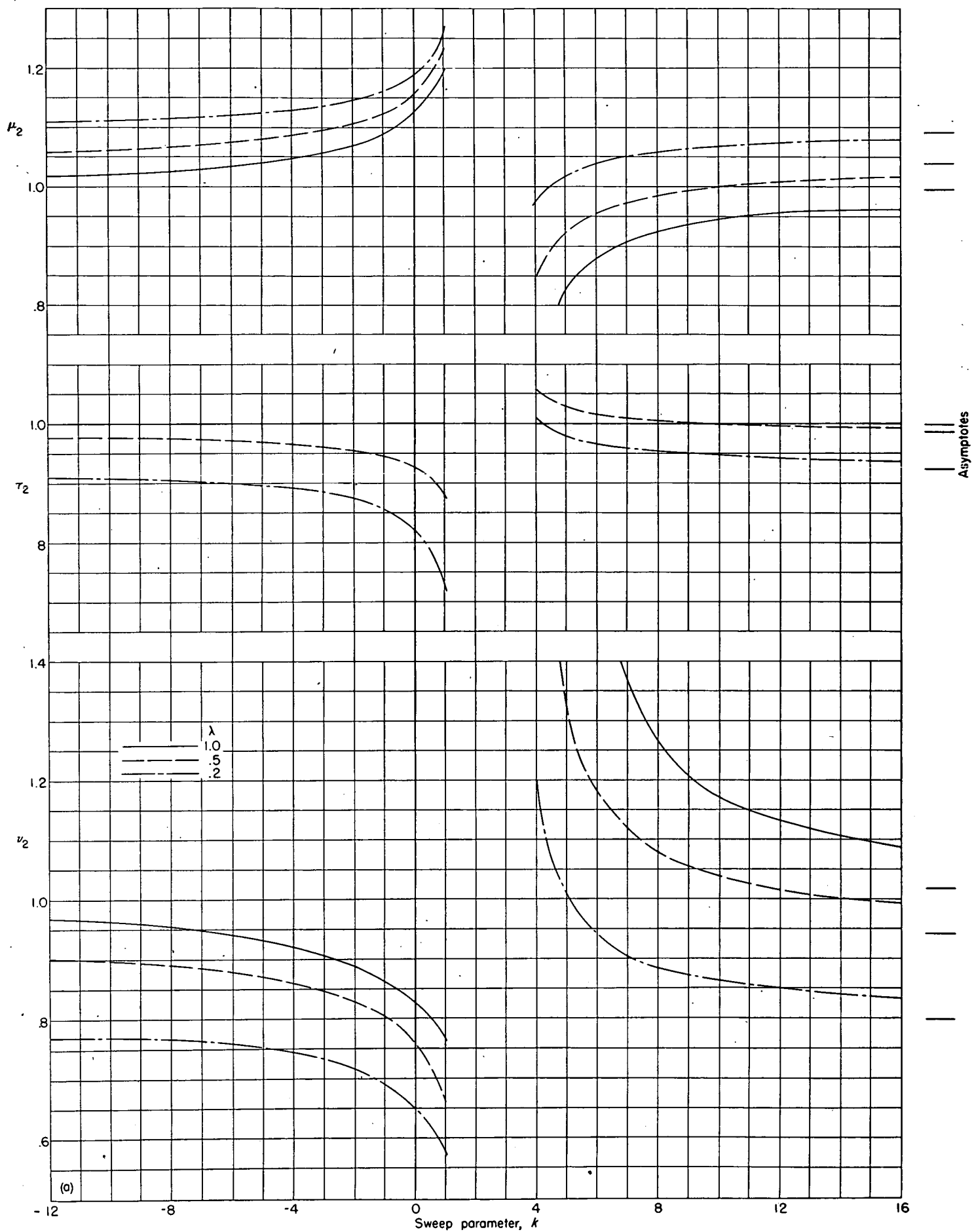
FIGURE 9.—Concluded.



(a) Stiffnesses proportional to c^4 .

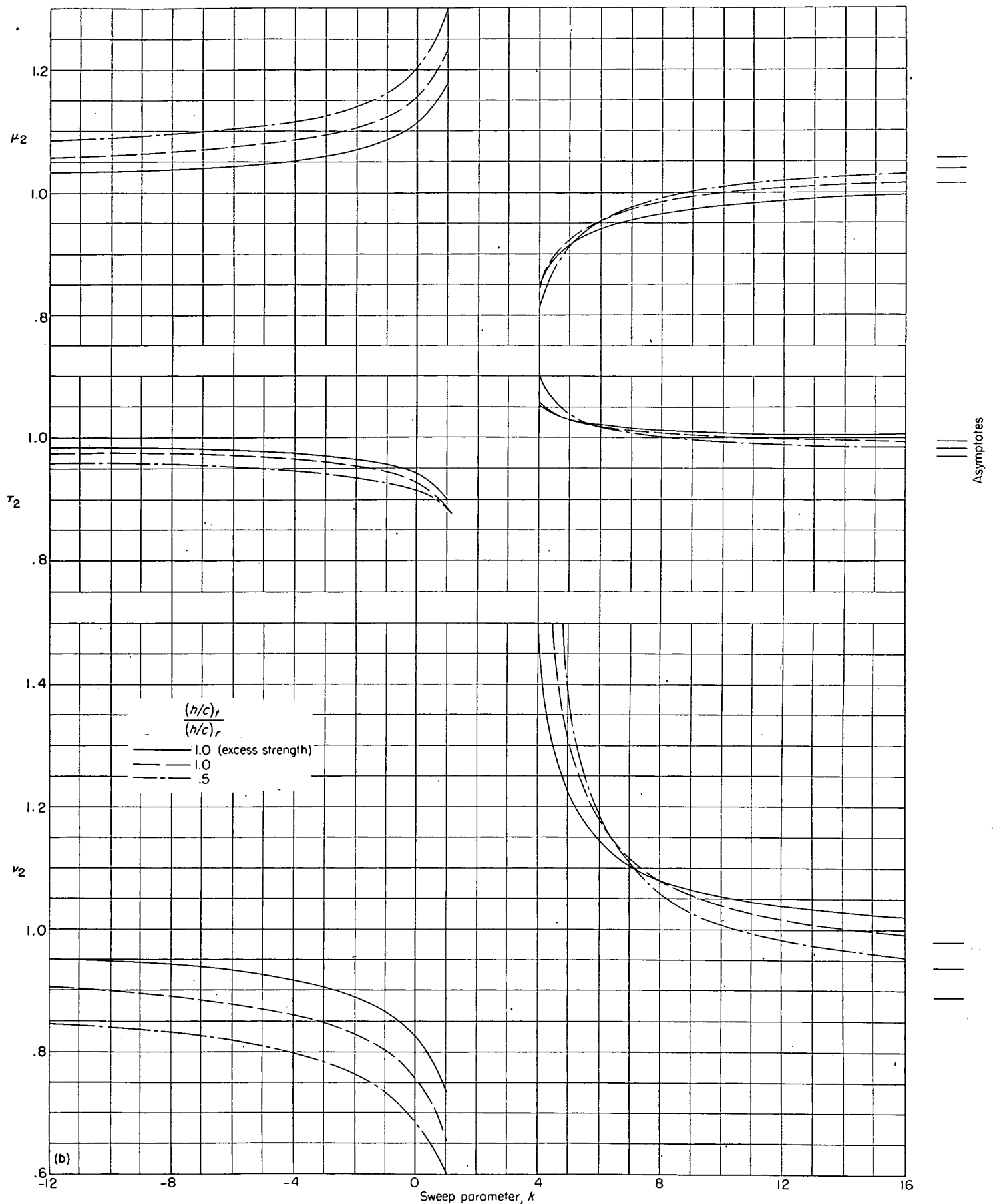
(b) Stiffnesses given by constant-stress criterion.

FIGURE 10.—The lift- and moment-coefficient ratios for constant geometric angles of attack for wings of taper ratio 0.



(a) Stiffnesses given by constant-stress criterion for $\frac{(h/c)_t}{(h/c)_r} = 1.0$.

FIGURE 11.—The lift- and moment-coefficient parameters μ_2 , ν_2 , and τ_2 for linearly varying geometric angles of attack.



(b) Stiffnesses related to those given by constant-stress criterion for wings with taper ratio 0.5.

FIGURE 11.—Continued.

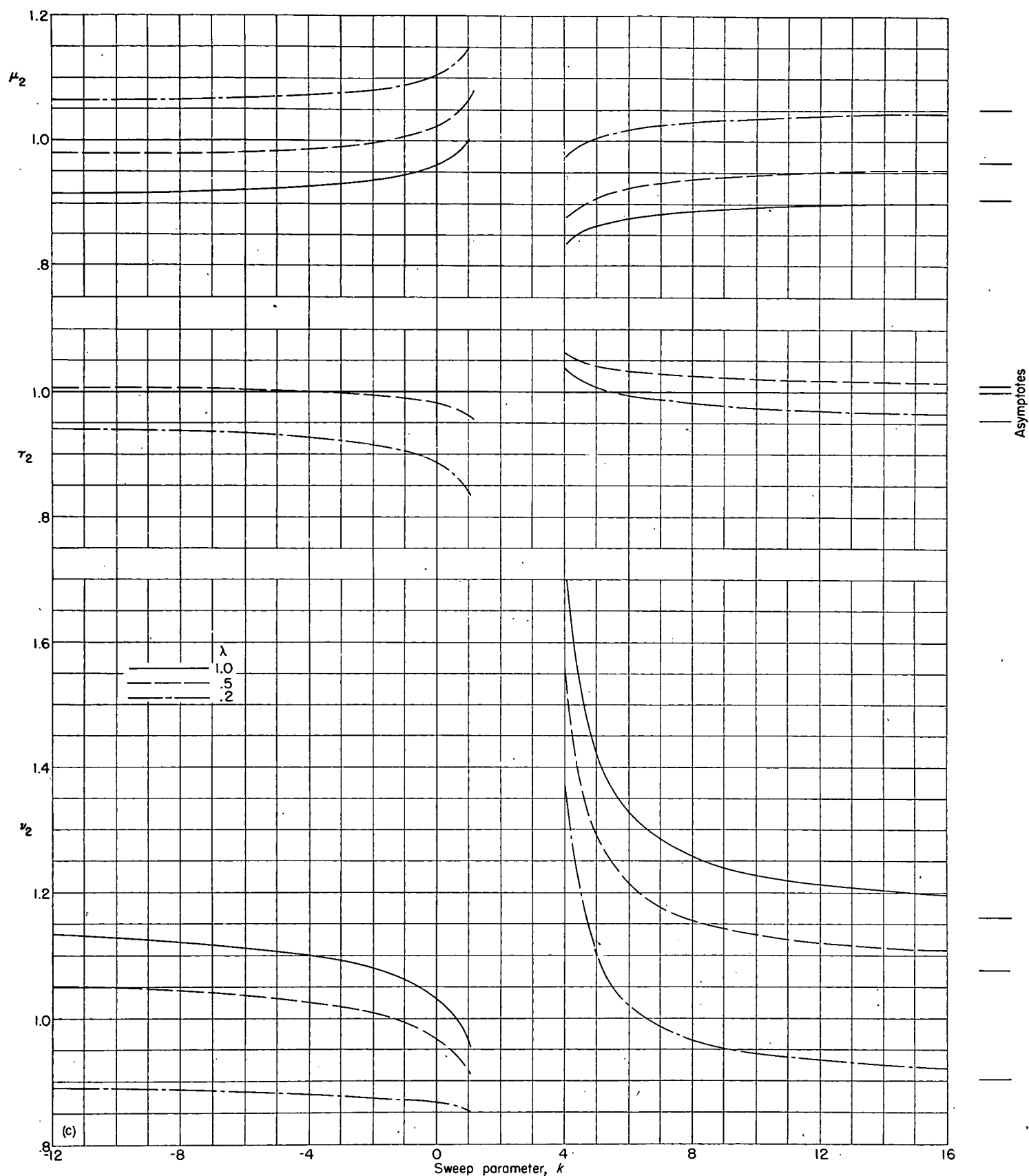
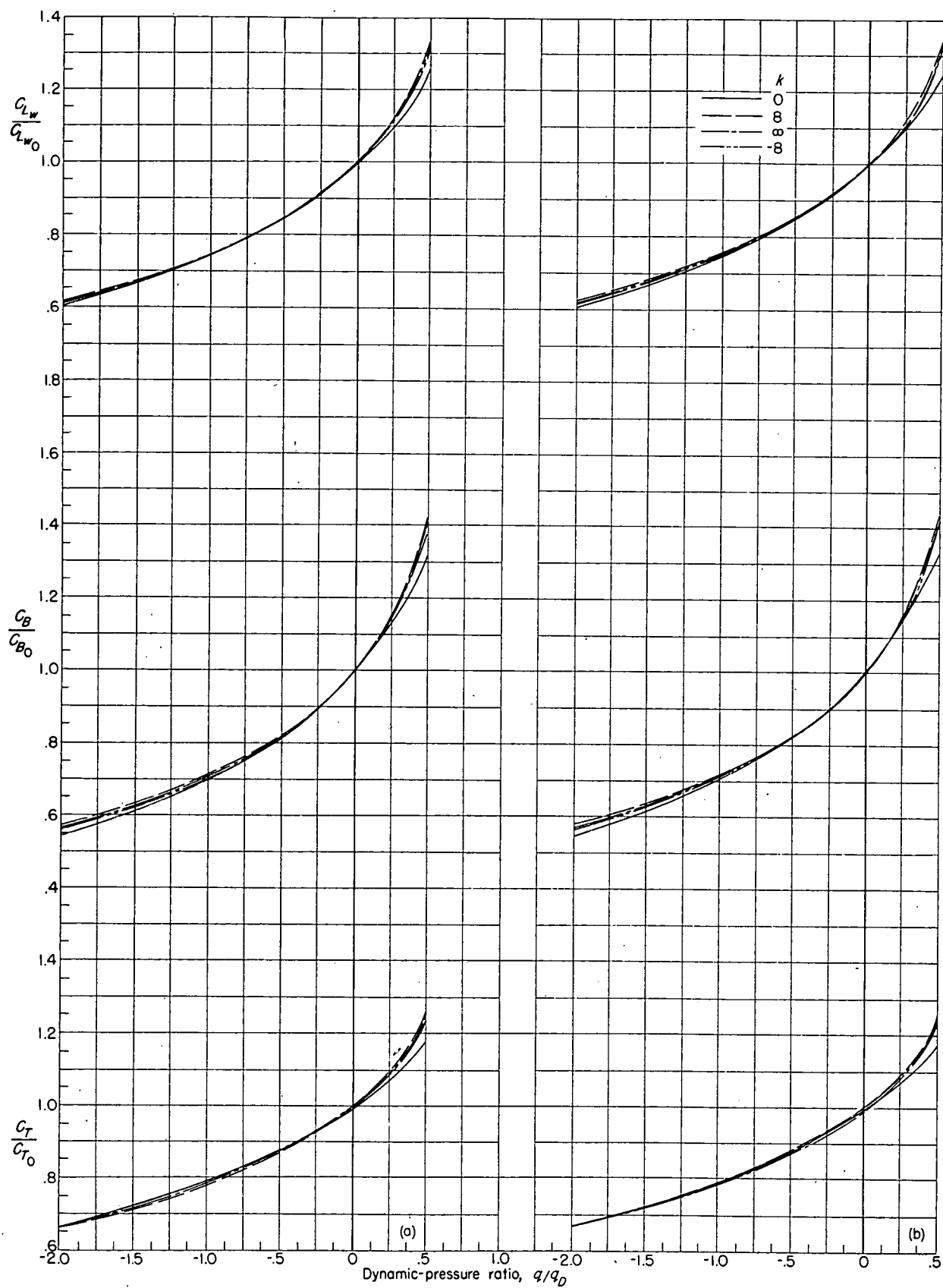
(c) Stiffnesses proportional to c^4 .

FIGURE 11.—Concluded.

(a) Stiffnesses proportional to c^4 .

(b) Stiffnesses given by constant-stress criterion.

FIGURE 12.—The lift- and moment-coefficient ratios for linearly varying geometric angles of attack for wings of taper ratio 0.

This angle-of-attack distribution can be considered as a geometrical angle-of-attack distribution. For the purpose of calculating the increment caused by aeroelastic action, this distribution can be approximated by a linear-twist angle-of-attack distribution with a value at the wing tip which is such that the moment about the effective wing root of the area under the linear-twist distribution equals the moment of the area under the calculated angle-of-attack distribution due to inertia effects. (The moment, rather than the area, is suggested as a basis of correlation because the angles of attack near the wing tip are more important in aeroelastic phenomena than those at the wing root.) The justification for this rather arbitrary approximation to the angle-of-attack distribution is as follows: As previously mentioned, the wing deformations due to inertia loads are likely to be small compared with those due to aerodynamic loads; furthermore, the correction to be applied to these deformations as a result of aeroelastic action is usually small compared with these deformations and, hence, is very small in comparison with the total wing load, so that the correction need not be calculated as accurately as the correction for aeroelastic effects to the rigid-wing lift distribution.

The angle of attack due to structural deformation α_s , associated with the linear-twist distribution can then be obtained from equation (21) and figure 7 or, if $\lambda=0$, from figure 8. The lift distribution associated with the total angle-of-attack distribution due to the deformations caused by the inertia effects, including the increment in this angle-of-attack distribution produced by aeroelastic action, can then be found from equation (24b), in which α_s and l_0 pertain to the calculated angle-of-attack distribution due to the inertia effects (not the linear approximation to this distribution). This lift distribution can be integrated to obtain the lift, bending moment, rolling moment, and aerodynamic-center position due to inertia effects, as modified by aeroelastic action.

The lift and rolling moment calculated in this manner may then be combined with the lift and rolling moment for steady level or rolling flight calculated by the method outlined in the preceding sections. For instance, if the contributions of the tail and the fuselage to the airplane lift can be neglected, the wing lift can be written as

$$n(W - W_w) = \frac{1}{144} C_{L_{\alpha_s}} \alpha q S + \left(\frac{\partial L_w}{\partial n} \right)_s n$$

where $\left(\frac{\partial L_w}{\partial n} \right)_s$ is the total normal force per unit load factor due to inertia effects, including aeroelastic effects; it is equal to $-W_w$ plus the lift on both wings due to inertia effects, as modified by aeroelastic action, per unit load factor and is almost always negative. In the preceding equation $C_{L_{\alpha_s}}$ is a wing lift-curve slope which includes static aeroelastic effects and is equal to $C_{L_{\alpha}}$ multiplied by the factor on the right side of equation (26). Then

$$n = \frac{1}{144} \frac{1}{1 - \frac{1}{W - W_w} \left(\frac{\partial L_w}{\partial n} \right)_s} \frac{C_{L_{\alpha_s}} \alpha q S}{W - W_w}$$

$$= \frac{1}{144} \frac{C_{L_{\alpha_s, t}} \alpha q S}{W - W_w}$$

where

$$C_{L_{\alpha_s, t}} = \frac{1}{1 - \frac{1}{W - W_w} \left(\frac{\partial L_w}{\partial n} \right)_s} C_{L_{\alpha_s}}$$

is a wing lift-curve slope which includes static aeroelastic effects, inertia effects, and aeroelastic modification of the inertia effects.

ILLUSTRATIVE EXAMPLE

The parameters of a swept wing, which differs from the wing of the illustrative example of reference 1 only in the width of the fuselage to which it is attached, are given in table 3. The values of Δs^e and Δs^r were calculated from the dimensionless root-rotation constants used in the example of reference 1, $Q_{\varphi_T} = 0$ and $Q_{r_M} = -0.25$, by means of the relations

$$\Delta s^e = Q_{\varphi_T} w_e$$

$$\Delta s^r = Q_{r_M} w_e$$

where w_e , as defined in reference 1, is the distance along the span between the effective root and the innermost complete section of the torsion box perpendicular to the elastic axis. In the wing of the illustrative examples of the present

TABLE 3.—PARAMETERS OF WING USED IN ILLUSTRATIVE EXAMPLE

Geometrical parameters		Structural parameters	
A , deg	37.5	ϵ	0.444
S , sq in.	37.498	$(GJ)_r$, lb/sq in.	8.94×10^9
b , in.	387.4	$(EI)_r$, lb/sq in.	9.56×10^9
$w/2$, in.	20.0	$GJ/(EI)_r$	$\approx (c/c_r)^4$
$b/2$, in.	173.7	$EI/(EI)_r$	$\approx (c/c_r)^4$
c_r , in.	102.8	Δs^e , in.	0
c_t , in.	54.2	Δs^r , in.	-5.6
λ	-0.527	Δs , in.	-3.0
c_{MAC} , in.	100.4	$s: \left(-\frac{b/2}{\cos A} + \Delta s \right)$, in.	215.9

Aerodynamic parameters		
	Subsonic ($M < 0.65$)	Supersonic ($M = 1.5$)
a	0.25	0.425
ϵ_t	0.194	0.019
$C_{L_{\alpha_s}}$	2.78	4.92
κ	0.78	1.00

Aeroelastic parameters		
	Subsonic ($M < 0.65$)	Supersonic ($M = 1.5$)
k	7.76	79.0
K_1	2.82	2.82
K_2	0.474	0.474
$q^* D$	-1.053	-0.0774
q_D , lb/sq ft	-6,400	-2,700
f_1	Fig. 5 (c)	Fig. 5 (c)
Δf_1	Fig. 5 (c)	Fig. 5 (c)
F_1	-0.27	-0.02
μ_1	0.655	0.662
μ_1	1.303	1.333
C_{L_w}	$1 - 0.345 \frac{q}{q_D}$	$1 - 0.338 \frac{q}{q_D}$
$C_{L_{w_0}}$	$1 - \frac{q}{q_D}$	$1 - \frac{q}{q_D}$
$\frac{s}{s_0}$	$1 - 0.146 \frac{q}{q_D}$	$1 - 0.118 \frac{q}{q_D}$
$\frac{s}{s_0}$	$1 - 0.345 \frac{q}{q_D}$	$1 - 0.338 \frac{q}{q_D}$

report and of reference 1, w_s is 22.4 inches. These relations for Δs^p and Δs^r can be obtained from equations (11) and (12) of the present report in conjunction with the definitions of the root-rotation constants given in equations (15a) and (15d) of reference 1; in the notation of the present report the definitions are

$$Q_{\varphi r} = \frac{\varphi_r^T / T_r}{w_e / (GJ)_r}$$

$$Q_{\Gamma M} = \frac{\Gamma_r^M / M_r}{w_e / (EI)_r}$$

The stiffness is assumed to vary as the fourth power of the chord in the example of the present report.

The subsonic and supersonic values of the parameter k were calculated from equation (9). By means of appropriate values of the constants K_1 and K_2 taken from table 3, the values of q^*_{D} were calculated from equation (18) and included in table 3. From these values of q^*_{D} , the subsonic and supersonic dynamic pressures at divergence were found through the use of equation (7) and are given in table 3. These values of q_D vary as the reciprocal of the effective lift-curve slope, the corresponding values of e_1 being assumed to remain constant.

In order to find the angle-of-attack distribution for additional-type loadings from equation (20), the values of F_1 and of the functions f_1 and Δf_1 were taken from figure 5 (c). The spanwise change in angle of attack is shown in figure 13 for different values of the dynamic-pressure ratio.

The values of ν_1 , τ_1 , and μ_1 were obtained from figure 9 (c) and substituted into equations (26), (27), and (28). The wing lift coefficient, wing rolling-moment coefficient, and spanwise center-of-pressure ratios, as well as the shift in aerodynamic center, were calculated by use of these approximate equations in conjunction with equations (17) and (29) and are shown in figure 13 as functions of the dynamic-pressure ratio $\frac{q}{-q_D}$.

DISCUSSION

LIMITATIONS OF THE CHARTS AND APPROXIMATE FORMULAS

The charts and the approximate formulas presented in this report are subject to certain limitations as a result of the approximations made in the calculations on which they are based. These limitations take the form of restrictions on the plan form, on the speed regime, and on the wing structure. The results obtainable by the use of the charts are likely to be unsatisfactory for wings of very low aspect ratio or very large sweep and relatively unsatisfactory for wings of zero taper ratio.

Wings of low aspect ratio are ruled out on three counts: (1) the extent to which aerodynamic forces are overestimated in replacing the wing by one with an effective root and tip is larger for wings of low aspect ratio than for wings of high aspect ratio, (2) elementary beam theory is unsatisfactory for calculating the deformations of wings of very low aspect ratio (because the effects of end constraint, shear lag, shear deformation, and bending-torsion interaction are more important when the aspect ratio is low), and (3) the assumptions made concerning the lift distribution of the wing are more nearly true for wings of high than for those of low aspect ratio.

For wings with very large angles of sweep, also, the use of an effective root and tip introduces relatively large errors in the aerodynamic forces. Furthermore, the root rotations neglected in the calculations (bending rotation due to torsion and twist due to bending) are likely to be important for wings with large angles of sweep.

The aeroelastic analysis of wings with zero taper ratio entails certain mathematical difficulties which do not arise in the case of wings with nonzero taper ratio. The stiffness of such wings is zero at the tip and very low near the tip, so that the boundary conditions for φ and Γ given by equations (A10a) to (A10c) in appendix A are indeterminate. As a result of the relatively large values of the reciprocal of the stiffness near the tip, the numerical-integration methods used in the matrix calculations are less accurate. These difficulties also occur in other methods of solving the aeroelastic equations, such as energy methods. Furthermore, the structural behavior near the wing tip is not represented adequately by elementary beam theory. Finally, that the aeroelastic results calculated for wings of zero taper ratio are not as reliable as those for other wings is evidenced also by the fact that they do not lend themselves to systematization by means of approximate formulas, as do the aeroelastic results calculated for other wings.

As a result of these considerations delta wings are unsuitable for aeroelastic analysis by means of these charts because

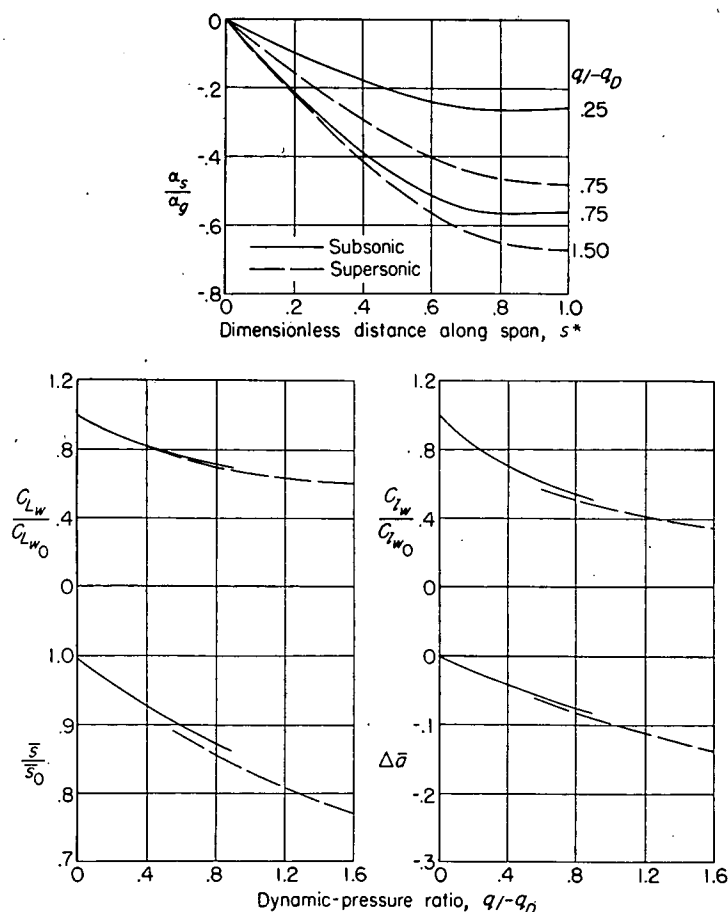


FIGURE 13.—Effect of aeroelastic action on some aerodynamic properties of the wing used in illustrative example.

they have low aspect ratios, large angles of sweep, and zero taper ratio.

In order to use the charts two aerodynamic parameters must be known for any given case: the effective wing lift-curve slope and the section aerodynamic center. From an aerodynamic point of view the charts of this report may be used in almost all cases for which these quantities are known. The exceptions stem from the fact that the spanwise distribution of the lift on the rigid wing is assumed to be proportional to the chord, and the distance from the section aerodynamic center to the elastic axis (as a fraction of the chord) is assumed to be constant along the span. These assumptions are not valid for wings with large angles of sweep and wings of low aspect ratio, as implied previously. They are also invalid to a greater or lesser extent for most wings in the transonic region. Consequently, even when the lift-curve slope and the section aerodynamic center are known, any results calculated for transonic speeds must be used with caution.

Another aerodynamic assumption implied in the charts is that no concentrated aerodynamic forces, such as those due to a tip tank or nacelle, act on the wing. Relatively small nacelles in the inboard half of the span can probably be ignored for the purpose of an aeroelastic analysis at the preliminary design stage. However, large tip tanks cannot usually be ignored even in a preliminary aeroelastic analysis; the aeroelastic phenomena may in such cases be greatly underestimated by calculations made with the charts of this report.

The assumption concerning the applicability of elementary beam theory to the calculation of wing deformations due to aeroelastic action serves to restrict the wings that can be analyzed by means of the charts to those of moderate or high aspect ratio, as stated previously. Neglect of chordwise bending (elastic camber) effects in the calculations on which the charts are based serves to impose a lower limit on the thickness of the wings for which the charts may be used. Whether this limit is within the region of practical thicknesses is questionable, however. The divergence tests of reference 2, which were performed on flat plates of moderately high aspect ratio and with a thickness of 2.5 percent, showed no obvious chordwise bending effects, although the relatively small differences between the measured and calculated divergence speeds may have been due in part to such effects.

As mentioned previously, for wings with taper ratios between 0 and 0.2 the results of aeroelastic calculations are likely to be relatively unreliable. For taper ratios greater than 0.2, the stiffness of actual wings tends to be greater near the tip than that given by the constant-stress criterion; consequently, any given aeroelastic effect is likely to be somewhat less than that calculated on the basis of a constant-stress stiffness distribution but much larger than that calculated on the basis of a c^4 distribution.

If a given structure contains large cutouts which give rise to discontinuities in the stiffness distributions, equation (10) can be used to calculate a fictitious root stiffness to be used in conjunction with charts for c^4 -type stiffness distributions, provided the magnitudes of the discontinuities are known or can be estimated.

Use of the charts of this report is premised on the assumption that the elastic axis is at an approximately constant

fraction of the chord. If the location of the elastic axis varies somewhat along the span, the use of an average value tends to give satisfactory results for the aeroelastic phenomena of swept wings; for unswept wings, however, the results obtained on the basis of this approximation have to be used with caution. If the elastic axis exhibits abrupt shifts along the span as a result of large cutouts or for other reasons, the charts should be used only for moderately or highly swept wings. This restriction is mitigated to a certain extent by the fact that an abrupt shift in the locus of shear centers does not necessarily imply an equally large or equally abrupt shift in the elastic axis.

RELATION BETWEEN STRENGTH AND STIFFNESS AS DESIGN CRITERIA

The strength of a structure is its ability to withstand applied loads without failure; the stiffness of a structure is its ability to deform relatively little under the applied loads. The two terms are related (a fact which forms the basis of the constant-stress type of stiffness distributions used in this report) but are not synonymous. The problem of when to design for strength and when to design for stiffness and the related problem of how to design a wing for stiffness when required to do so have been recognized for a long time. Because of the complexity of these problems no generally satisfactory solution exists at present, but the charts presented herein shed a certain amount of light on the problem insofar as stiffness requirements occasioned by the aeroelastic phenomena considered in this report are concerned.

The charts of figure 2 indicate the extent to which a wing is likely to be affected by aeroelastic phenomena—that is, how far it is from divergence and how much its spanwise center of pressure is likely to shift as a result of aeroelastic action, *provided the wing is designed on the basis of strength considerations alone*. If the margin against divergence is too small, or if the spanwise center of pressure and the associated shift in the aerodynamic center are deemed excessive, the wing has to be stiffened beyond the amount associated with the required strength. The charts of figure 2 therefore serve to delimit the regions in which a wing can be designed on the basis of strength considerations alone and those in which stiffness considerations predominate, at least to the extent of satisfying the stiffness requirements associated with the aeroelastic phenomena considered herein.

The bending moment of inertia required by considerations of strength alone for the root section of a wing is directly proportional to the design load $n(W - W_w)$, to the spanwise coordinate of the center of pressure, and to the wing thickness at the root and is inversely proportional to the allowable bending stress F_B . Alternatively, this bending moment of inertia may be considered to be proportional to the design wing loading $\frac{n(W - W_w)}{S}$, to the square of the wing area, to the wing thickness ratio at the root, and to a function of the taper ratio (which is $\frac{1 + 2\lambda}{(1 + \lambda)^2}$ if strip theory is assumed to apply); it is inversely proportional to F_B and independent of the aspect ratio. These relations for the bending moment of inertia required by considerations of strength alone are implied in the following discussion of structural requirements imposed by considerations of stiffness.

In general, a wing with a high value of q^* (see eqs. (7) and (13)) is most likely to be affected by aeroelasticity (see fig. 2) and, for a given value of q^* , swept wings are much more likely to be affected by aeroelasticity than unswept ones. (See fig. 2 and eqs. (9) and (14).) Consequently, the following wings are most likely to be subject to aeroelastic phenomena, provided they are designed on the basis of strength considerations alone:

- (1) Wings operating at a high flying speed or high dynamic pressure
- (2) Swept wings
- (3) Thin wings
- (4) Wings designed for a low wing loading
- (5) Unswept and moderately swept wings with an elastic axis relatively far back on the chord or likely to fly in a condition in which the section aerodynamic centers are relatively far forward on the chord
- (6) Wings operating at a Mach number at which the lift-curve slope is relatively high

For given wing loadings and given wing areas, some aeroelastic phenomena of wings designed on the basis of strength considerations alone appear to be substantially unaffected by changes in the taper ratio—for instance, the spanwise shift of the center of pressure and the dynamic pressure required for divergence. (In the case of the dynamic pressure required for divergence, the parameter q^*_D (fig. 4), the root stiffness, and the root chord decrease with increasing taper ratio, and the net effect of taper is small.) On the other hand, the change in the lift due to aeroelastic action is more sensitive to the taper ratio; it is more significant for wings with high taper ratio than for wings with low taper ratio.

The effect of aspect ratio on aeroelastic phenomena tends to be small for unswept wings of a given wing area, because these phenomena are determined largely by the magnitude of the parameter q^* , which is independent of the aspect ratio for a given wing area. For the aeroelastic phenomena of highly swept wings, however, the parameter $\bar{q}$ is more significant. This parameter is proportional to the swept-span aspect ratio for wings of a given area. Consequently, with a given wing area, taper ratio, and design wing loading, the aeroelastic effects of swept wings tend to be more pronounced for wings with high aspect ratio than for those with low aspect ratio. This statement is particularly true for the shift of the aerodynamic center, because a given spanwise shift of the center of pressure results in a much greater chordwise shift in the case of a swept wing of high aspect ratio than in the case of a swept wing with low aspect ratio.

STRUCTURAL WEIGHT ASSOCIATED WITH THE REQUIRED STIFFNESS

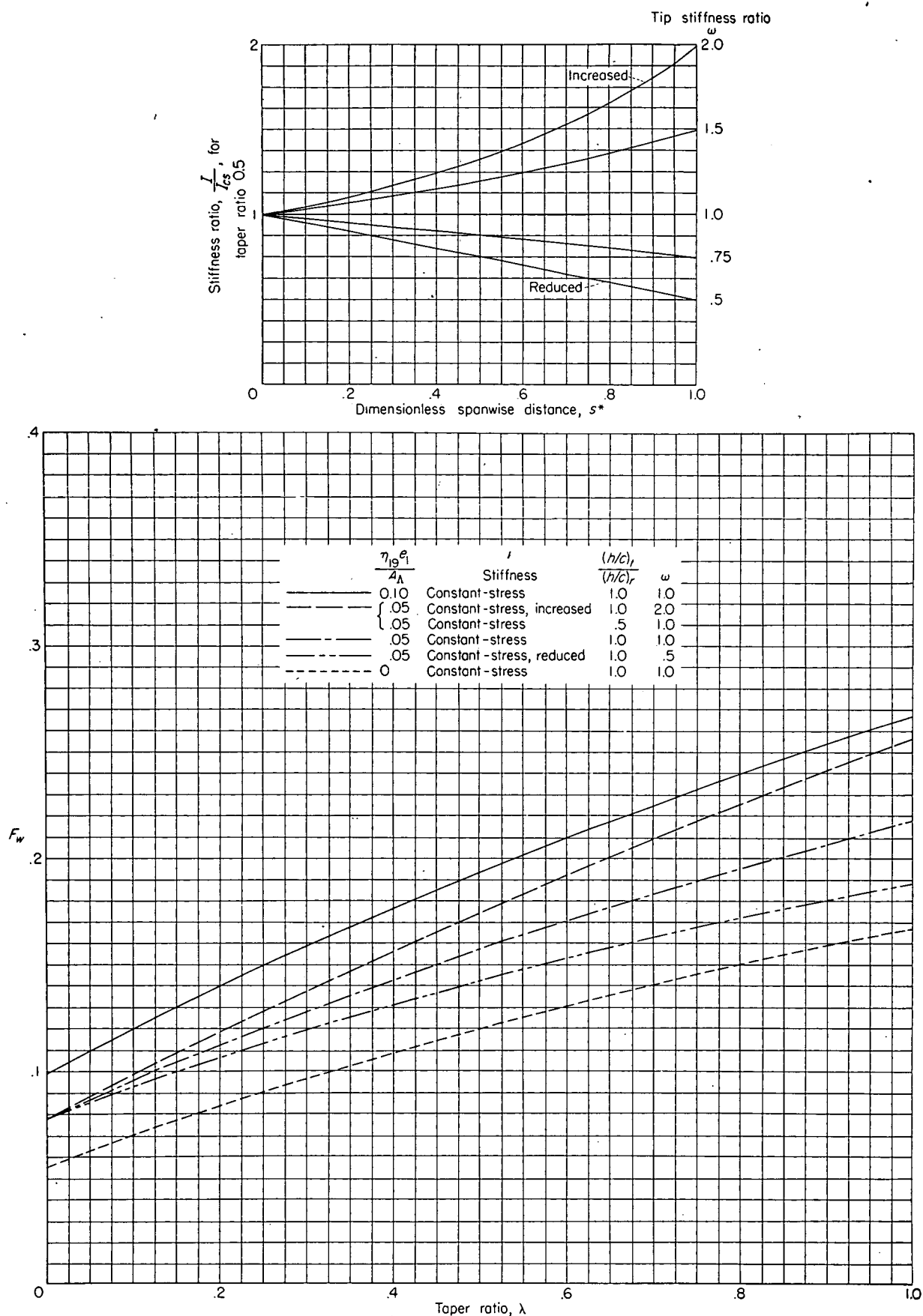
When a given wing has been shown to be subject to undesirably large aeroelastic effects (by means of the charts of this report or by any other method), the problem arises how to distribute the additional required stiffness. If, for instance, the dynamic pressure on an unswept wing is within

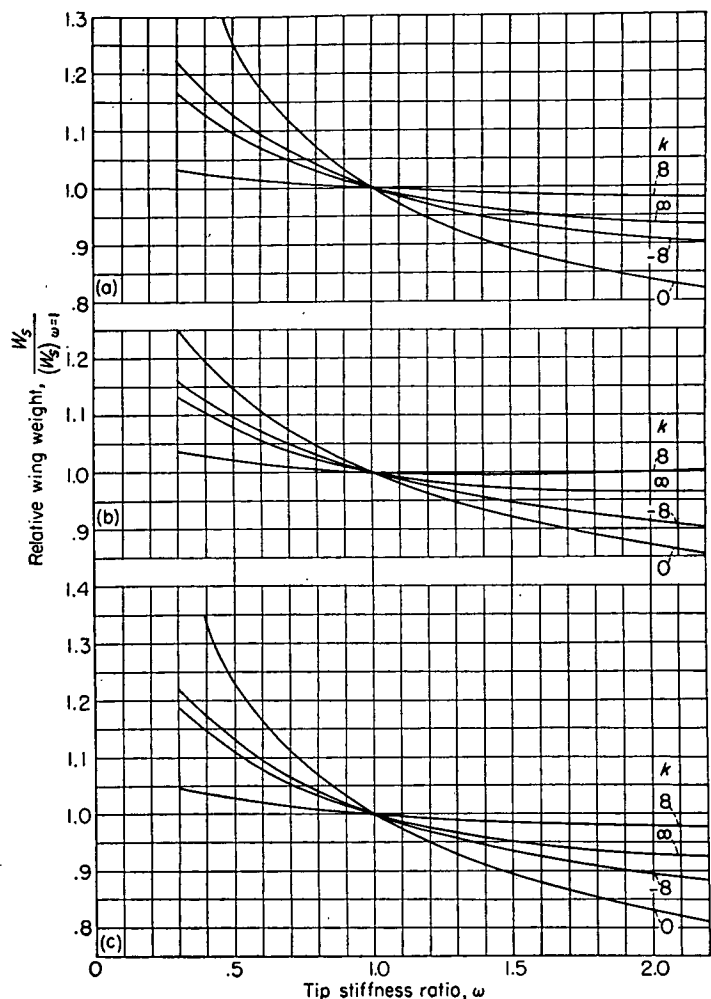
10 percent of the dynamic pressure required for divergence and a margin of 20 percent is desired, an increase of 10 percent in the torsional stiffness along the entire span will produce the desired result. The question remains, however, whether structural weight can be saved by increasing the stiffness more than 10 percent in some places and less in others.

Some insight into this problem may be gained, at least insofar as the aeroelastic phenomena considered herein are concerned, from aeroelastic and weight calculations that have been made for a family of somewhat arbitrarily selected stiffness distributions which differ from the distribution required by the constant-stress criterion. The ratios of the local stiffnesses to those associated with constant stress are shown at the top of figure 14. The structural-weight factor F_w is shown for two of these stiffness distributions as a function of the taper ratio. The function F_w is proportional to the weight W , of the primary load-carrying structure and depends on the manner in which the wing stiffness and thickness are distributed along the span. (See appendix B.)

The results of the aeroelastic calculations for wings with taper ratio 0.5, constant wing thickness ratio h/c along the span, and these two stiffness distributions are included in table 2 and figures 5 (b), 7 (b), 9 (b), and 11 (b). The designation "excess strength" in these figures refers to the stiffness distributions increased over the constant-stress requirement, as shown in figure 14, with a value of $\omega=2.0$. The results of the aeroelastic calculations for the stiffness distributions decreased below the constant-stress requirement to a value of $\omega=0.5$ are the same as those for the constant-stress stiffness distributions for wings with linearly varying wing thickness ratio and $\frac{(h/c)_t}{(h/c)_r}=0.5$.

The results of the weight calculations and the aeroelastic calculations may be combined in several ways. The dynamic pressure at divergence, for instance, can be varied by changing the bending and torsional stiffnesses uniformly along the span, by leaving the stiffnesses at the root unchanged and varying the stiffness distribution in a manner similar to that indicated at the top of figure 14, or by a combination of the processes. A specified dynamic pressure at divergence can therefore be obtained as the result of many combinations of root stiffnesses and stiffness distributions. Figure 15 (a) consists in essence of a plot of the structural weights associated with combinations of this type against the tip stiffness ratio ω for a specified dynamic pressure at divergence. This figure indicates that the least weight is associated with values of the tip stiffness ratio greater than 1. Similarly, figures 15 (b) and 15 (c) consist in essence of plots of the structural weights associated with various combinations of root stiffnesses and stiffness distributions required for shifts of ± 10 percent in the spanwise center of pressure at a specified dynamic pressure. Figures 15 (b) and 15 (c) also indicate that the structural weight is least for values of the stiffness ratio ω greater than 1.

FIGURE 14.—The structural-weight function F_w .



- (a) Relative weight required for a given dynamic pressure at divergence.
 (b) Relative weight required for $\frac{\bar{s}}{s_0} = 0.9$ at a given dynamic pressure.
 (c) Relative weight required for $\frac{\bar{s}}{s_0} = 1.1$ at a given dynamic pressure.

FIGURE 15.—The effect of the tip stiffness ratio on the structural weight required for a given divergence dynamic pressure or given shifts in the spanwise center of pressure of wings with taper ratio 0.5.

The significance of figure 15 is that, if a given wing designed on the basis of strength alone needs to be stiffened for aeroelastic reasons, most of the stiffening material should be added in the outboard regions, provided the weight of the material other than that of the primary load-carrying structure is unaffected by the stiffening process. In fact, on the basis of aeroelastic considerations alone, weight might be saved in some cases by removing material from the root and adding material at the tip; needless to say, however, strength requirements would be violated by this procedure. Just where the material should be added in the outboard regions cannot be said on the basis of the calculations made for figure 15, since these calculations assume any modifications to the constant-stress stiffness distributions to be made as indicated at the top of figure 14. However, it appears unlikely that great weight savings can be had by using modifications which differ substantially from those of figure 14.

SOME REMARKS CONCERNING THE AEROISOCLINIC WING

The term "aeroisoclinic" refers to wings which deform under an aerodynamic load in such a fashion that the angles

of attack of all sections relative to the free stream remain unchanged. Such a wing has the advantage that its aerodynamic loads do not change under aeroelastic action either in magnitude or in distribution; its aerodynamic center, for instance, is unchanged, and the wing cannot diverge. The achievement of such "section aeroisoclinicism" is very difficult and can be realized only by separate variation of the bending and torsion stiffnesses; even so, the aeroisoclinic condition obtains for only one type of aerodynamic loading condition at one Mach number. However, an overall type of aeroisoclinicism in which bending and torsion action tend to cancel for the wing as a whole is relatively easy to achieve. This overall type has, for practical purposes, the same advantages as section aeroisoclinicism, in that the aeroelastic phenomena considered in this report tend to be negligibly small for such a wing.

As may be seen from figure 2, at a small positive value of the parameter k the values of the parameter q^* for divergence as well as those for given shifts in the spanwise center of pressure tend to infinity. This particular value of k represents aeroisoclinic wings in the overall sense; from equations (18) and (19) it may be seen to be the reciprocal of the value of K_2 given in table 2. Hence, from the definition of k (eq. (9)),

$$\frac{s_t}{e_1 c_r} \frac{(GJ)_r}{(EI)_r} \tan \Lambda = \frac{1}{K_2} \quad (33)$$

with the implication that the distributions of the stiffness are of either the c^4 or the constant-stress type and that K_2 pertains to either of these types and to the appropriate taper ratio. Equation (33) indicates that, for a given plan form with assigned values of s_t , c_r , and Λ , the disposable parameters for the achievement of aeroisoclinicism are the elastic-axis location e , which enters into the parameter e_1 , and the root-stiffness ratio $\frac{(GJ)_r}{(EI)_r}$; the aerodynamic center is not under the control of the designer to any appreciable extent.

A decrease in the torsional stiffness can sometimes be effected without decrease in the bending stiffness or impairment of the strength characteristics of the wing, and overall aeroisoclinicism may be achieved in this manner for swept-back wings. Or, if aeroisoclinic conditions are considered at the outset, a wing can be designed with the elastic-axis location relatively far back (in the case of a sweptback wing) or forward (in front of the aerodynamic center, in the case of a sweptforward wing) in order to achieve aeroisoclinicism. However, the fact that only certain types of aeroelastic phenomena are considered in this report must be kept in mind. Locating the elastic axis far back or decreasing the torsional stiffness, for instance, may lead to flutter difficulties, the solution of which may require excessive mass balancing of the wing as a whole.

RELATION OF THE CHARTS TO DESIGN PROCEDURE

The first step in the design of a wing structure, once the wing geometry and the overall airplane characteristics have been decided upon, usually consists of a rough apportioning of structural material along the span in a manner intended to satisfy strength requirements approximately. At a later stage in the design procedure the structure is checked for aeroelastic effects and modified, if necessary.

The modifications are then checked again, and so on, until both stiffness and strength requirements are met with what is believed to be a near-optimum structure from weight considerations. The charts of this report may be used to facilitate the procedure at several stages.

At the very outset, the preliminary-survey charts can be used to establish some overall aeroelastic characteristics of the wing structure that would be obtained by designing the wing for strength alone. If these characteristics are satisfactory, the design of the wing structure can proceed on the basis of strength requirements alone. The final design can then be checked for the aeroelastic effects considered in this report by means of the charts contained herein, and for other aeroelastic effects, such as flutter and loss of lateral control, by equally approximate methods. However, if the preliminary survey indicates that a wing designed on the basis of strength alone would be unsatisfactory from consideration of aeroelasticity, sufficient additional stiffness may be incorporated in the preliminary design stage, provided the taper ratio does not differ greatly from 0.5 and the wing thickness ratio is constant along the span. For instance, the preliminary-survey charts may indicate a shift in the spanwise center of pressure which gives rise to a shift of 4 percent in the aerodynamic center; whereas the desired maximum shift is 2 percent, so that the spanwise shift must be reduced to 50 percent of that indicated on the preliminary-survey chart. The shifts in the spanwise center of pressure for a wing with increased stiffness at the tip (the "excess strength" case, for which $\omega=2.0$) and for a wing with decreased stiffness at the tip (the wing with $\omega=0.5$, for which the results of the case of $\frac{(h/c)_t}{(h/c)_r}=0.5$ may be used) can then be obtained from figure 9 (b) and equation (32), in conjunction with the value of the dynamic pressure at divergence estimated from equation (18) or (19). The fact that the wings with $\omega=2.0$ and $\omega=0.5$ have different dynamic pressures at divergence than does the constant-stress wing must be kept in mind.

From the shifts of the spanwise center of pressure calculated in this manner the value of ω for the desired spanwise shift can be obtained by interpolation and, hence, the approximate magnification factors to be applied to the stiffness distribution for constant stress can be obtained from the chart at the top of figure 14. Estimates for the other aeroelastic characteristics considered in this report can then be obtained for the wing with this modified stiffness distribution by interpolating between the results given for these characteristics for wings with $\omega=0.5$, 1.0, and 2.0; that is, for the cases referred to, respectively, as

$$\frac{(h/c)_t}{(h/c)_r}=0.5$$

$$\frac{(h/c)_t}{(h/c)_r}=1.0$$

and

$$\frac{(h/c)_t}{(h/c)_r}=1.0 \text{ (excess strength)}$$

in table 2 and figures 5 (b), 7 (b), 9 (b), and 11 (b). Once the structure of such a wing has been designed, the various aeroelastic effects considered herein should be checked by a more accurate method, such as that of reference 1, and the loss of lateral control and the flutter characteristics should be calculated.

CONCLUDING REMARKS

Charts have been presented for the estimation of aeroelastic effects on the spanwise lift distribution, lift-curve slope, aerodynamic center, and damping in roll of swept and unswept wings at subsonic and supersonic speeds. Two types of stiffness distributions have been considered: one which consists of a variation of the stiffness with the fourth power of the chord and is appropriate for solid wings, and one which is based on an idealized constant-stress structure and is believed to be more nearly representative of actual structures.

The limitations of these charts are that they do not apply to wings with very low aspect ratio or very large angles of sweep nor to wings with large sources of concentrated aerodynamic forces. The charts are likely to be less reliable for wings with zero taper ratio than for wings with other taper ratios and less reliable when the component of the Mach number perpendicular to the leading edge is transonic than when this component is either subsonic or supersonic. Wings with large discontinuities in the spanwise distribution of the bending or torsional stiffnesses cannot be analyzed directly by use of the charts, but a means of making approximate calculations for such wings has been presented. No charts have been presented for inertia effects but a method of estimating these effects has been outlined.

In addition to facilitating the calculation of various static aeroelastic phenomena, the charts serve to simplify design procedure in many instances, because they can be used at the preliminary design stage to estimate the amount of additional material required to stiffen a wing which is strong enough and because they indicate that the best way of distributing this additional material is to locate most of it near the wing tip.

Also, the charts facilitate the achievement of aeroisoclinic conditions, inasmuch as they serve to define a simple relation between the elastic-axis location and the wing stiffness ratio which is required to obtain this condition for a given plan form. Finally, the charts indicate that a wing which is strong enough is most likely to be affected by aeroelastic phenomena if it is to operate at high dynamic pressures, if it is thin, if it has a large angle of sweep, if it is designed for a low wing loading, if it has an elastic-axis location relatively far back on the chord, and if it is to operate at transonic or high supersonic Mach numbers.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., September 13, 1951.

APPENDIX A

METHODS OF CALCULATIONS ON WHICH THE CHARTS ARE BASED

THE AEROELASTIC EQUATIONS

The methods of calculating aeroelastic phenomena used in preparing the charts of this report are based on the following assumptions:

(1) Aerodynamic induction is taken into account by applying an overall correction to strip theory and, when matrix integrations are used, by rounding off the resulting load distribution at the tip.

(2) Aerodynamic and elastic forces are based upon the assumption of small deflections.

(3) The wing is clamped at the root perpendicular to a straight elastic axis (see fig. 1), and all deformations are considered to be given by the elementary theories of bending and torsion about an elastic axis.

In keeping with assumptions (1) and (2), the force per unit width on a wing section perpendicular to the elastic axis is

$$l = \frac{qc}{144} (C_{L_\alpha} \alpha_s + C_{L_{\alpha_e}} \alpha_s) \quad (A1)$$

where α_s and α_e are, respectively, the angle of attack due to structural deformations and the rigid-wing angle of attack, in planes parallel to the plane of symmetry. (The geometrical angle of attack is considered to be constant along the span in equation (A1); in the case of linear twist the coefficient $C_{L_{\alpha_e}}$ is used instead of C_{L_α} .) The torque of this force about the elastic axis is le_1c for uncambered sections.

The integral equations for the accumulated torque and the bending moment are

$$T = \int_s^{s_t} le_1c \, ds \quad (A2)$$

$$M = \int_s^{s_t} \int_s^{s_t} l \, ds \, ds \quad (A3)$$

and, insofar as assumption (3) holds, the angles of structural twist and bending referred to axes parallel or perpendicular to the elastic axis are

$$\varphi = \int_0^s \frac{1}{GJ} T \, ds \quad (A4)$$

$$\Gamma = \int_0^s \frac{1}{EI} M \, ds \quad (A5)$$

The angle of attack due to structural deformations is related to φ and Γ by the equation

$$\alpha_s = \varphi \cos \Lambda - \Gamma \sin \Lambda \quad (A6)$$

Combining equations (A1) to (A6) gives two simultaneous differential equations:

$$\frac{d}{ds} \left(GJ \frac{d\varphi}{ds} \right) = -\frac{qe_1c^2}{144} [C_{L_\alpha} \alpha_s + C_{L_{\alpha_e}} (\varphi \cos \Lambda - \Gamma \sin \Lambda)] \quad (A7)$$

$$\frac{d^2}{ds^2} \left(EI \frac{d\Gamma}{ds} \right) = \frac{qc}{144} [C_{L_\alpha} \alpha_s + C_{L_{\alpha_e}} (\varphi \cos \Lambda - \Gamma \sin \Lambda)] \quad (A8)$$

These equations are subject to the following boundary conditions:

Zero twist and bending at the root,

$$\varphi(0) = 0 \quad (A9a)$$

$$\Gamma(0) = 0 \quad (A9b)$$

Zero torque, moment, and shear at the tip,

$$\left(GJ \frac{d\varphi}{ds} \right)_{s=s_t} = 0 \quad (A10a)$$

$$\left(EI \frac{d\Gamma}{ds} \right)_{s=s_t} = 0 \quad (A10b)$$

$$\left(EI \frac{d^2\Gamma}{ds^2} \right)_{s=s_t} = 0 \quad (A10c)$$

In the following sections, equations (A7) and (A8) are solved explicitly for an untapered wing with constant stiffness along its span and by matrix integration for a wing with any arbitrary stiffness and chord variation.

SOLUTIONS FOR UNIFORM WINGS

Arbitrary geometric angle of attack.—If the torsional stiffness, the bending stiffness, and the chord of the wing have constant values $(GJ)_r$, $(EI)_r$, and c_r , respectively, along the wing span, equations (A7) and (A8) become

$$\varphi'' \cos \Lambda = -q^* \left[\frac{1}{\kappa} \alpha_s + (\varphi \cos \Lambda - \Gamma \sin \Lambda) \right] \quad (A11)$$

$$\Gamma''' \sin \Lambda = -\bar{q} \left[\frac{1}{\kappa} \alpha_s + (\varphi \cos \Lambda - \Gamma \sin \Lambda) \right] \quad (A12)$$

where the differentiation denoted by the primes is with respect to $\xi \equiv 1 - \frac{s}{s_t}$ and the dimensionless parameters q^* and $\bar{q}$ are defined by

$$q^* = \frac{C_{L_{\alpha_e}} q e_1 c_r^2 s_t^2 \cos \Lambda}{144 (GJ)_r} \quad (A13)$$

$$\bar{q} = \frac{C_{L_{\alpha_e}} q c_r s_t^3 \sin \Lambda}{144 (EI)_r} \quad (A14)$$

Differentiating equation (A11) once with respect to ξ and combining it with equation (A12) yields the single differential equation

$$\alpha_s''' + q^* \alpha_s' - \bar{q} \alpha_s = -q^* \frac{\alpha_g'}{\kappa} + \bar{q} \frac{\alpha_g}{\kappa} \quad (\text{A15})$$

(The factor $\frac{1}{\kappa}$ is used with α_g for the sake of consistency, despite the fact that a geometrical angle of attack which is constant over the span does not have a spanwise derivative.) Equation (A15) is subject to the following boundary conditions:

From equations (A9a) and (A9b)

$$\alpha_s(1) = 0 \quad (\text{A16})$$

From equations (A10a) and (A10b)

$$\alpha_s'(0) = 0 \quad (\text{A17})$$

From equations (A10c) and (A11)

$$\alpha_s''(0) = -q^* \left[\frac{\alpha_g(0)}{\kappa} + \alpha_s(0) \right] \quad (\text{A18})$$

where functional notation is used, so that, for instance, $\alpha_s(1)$ means the value of α_s at $\xi=1$.

The solution of equation (A15) can be effected very readily by means of Laplace transforms. The complete solution of this equation is

$$\alpha_s(\xi) = \frac{H(1)}{f_3(1)} f_3(\xi) - H(\xi) \quad (\text{A19})$$

where the integral $H(\xi)$ is defined as

$$H(\xi) = \int_0^\xi \left[q^* f_4(\xi - \xi_1) - \bar{q} f_5(\xi - \xi_1) \right] \frac{\alpha_g(\xi_1)}{\kappa} d\xi_1 \quad (\text{A20})$$

The functions f_3 , f_4 , and f_5 are defined by

$$f_3(\xi) = C_1 e^{-2\beta\xi} + e^{\beta\xi} \left(C_2 \cos \gamma\xi + \frac{C_3}{\gamma} \sin \gamma\xi \right) \quad (\text{A21a})$$

$$f_4(\xi) = C_4 e^{-2\beta\xi} + e^{\beta\xi} \left(C_5 \cos \gamma\xi + \frac{C_6}{\gamma} \sin \gamma\xi \right) \quad (\text{A21b})$$

$$f_5(\xi) = C_7 e^{-2\beta\xi} + e^{\beta\xi} \left(C_8 \cos \gamma\xi + \frac{C_9}{\gamma} \sin \gamma\xi \right) \quad (\text{A21c})$$

where -2β and $\beta \pm i\gamma$ are the roots of $r^3 + q^*r - \bar{q} = 0$ and

$$C_1 = \frac{4\beta^2}{9\beta^2 + \gamma^2}$$

$$C_2 = 1 - C_1$$

$$C_3 = \frac{3\beta^3 - \beta\gamma^2}{9\beta^2 + \gamma^2}$$

$$C_4 = \frac{-2\beta}{9\beta^2 + \gamma^2}$$

$$C_5 = -C_4$$

$$C_6 = \frac{3\beta^2 + \gamma^2}{9\beta^2 + \gamma^2}$$

$$C_7 = \frac{1}{9\beta^2 + \gamma^2}$$

$$C_8 = -C_7$$

$$C_9 = \frac{3\beta}{9\beta^2 + \gamma^2}$$

The condition for divergence is that α_s be finite when α_g is zero along the entire span. As can be seen from equations (A19) and (A20), divergence can occur only when

$$f_3(1) = 0 \quad (\text{A22})$$

Thus, for a particular value of the parameter $k = \frac{\bar{q}}{q^*}$ the value of q^* (or $\bar{q}$) at divergence is the one which satisfies equation (A22).

Constant geometric angle of attack.—For the additional-type angle of attack, $\alpha_g(\xi) = \text{Constant}$:

$$H(\xi) = [1 - f_3(\xi)] \frac{\alpha_g}{\kappa} \quad (\text{A23})$$

and

$$\frac{\kappa\alpha_s(\xi) + \alpha_g}{\alpha_g} = \frac{f_3(\xi)}{f_3(1)} \quad (\text{A24})$$

For lift distributions based on assumption (1) given at the beginning of this appendix, the lift per unit width of span may then be written as

$$\frac{l}{l_0} = \frac{f_3(\xi)}{f_3(1)}$$

The wing lift coefficient, the wing-root bending-moment coefficient, and the wing-root twisting-moment coefficient are given in general by

$$\begin{aligned} C_{L_w} &= \frac{L_w}{qS} \\ &= C_{L_a} \frac{s_t c_r}{S/2} \int_0^1 \frac{c}{c_r} [\kappa\alpha_s(\xi) + \alpha_g(\xi)] d\xi \end{aligned} \quad (\text{A25})$$

$$C_B = \frac{4M_r}{qSb}$$

$$= C_{L\alpha} \frac{s_t c_r}{S/2} \frac{s_t}{b/2} \int_0^1 \int_0^\xi \frac{c}{c_r} [\kappa \alpha_s(\xi) + \alpha_g(\xi)] d\xi d\xi \quad (\text{A26})$$

$$C_T = \frac{2T_r}{qSc_r}$$

$$= C_{L\alpha} \frac{s_t c_r}{S/2} e_1 \int_0^1 \left(\frac{c}{c_r}\right)^2 [\kappa \alpha_s(\xi) + \alpha_g(\xi)] d\xi \quad (\text{A27})$$

Then, for the uniform wing,

$$\frac{C_{L_w}}{C_{L_{w0}}} = \int_0^1 \frac{\kappa \alpha_s(\xi) + \alpha_g}{\alpha_g} d\xi$$

$$= \frac{f_4(1)}{f_3(1)} \quad (\text{A28})$$

$$\frac{C_B}{C_{B0}} = 2 \int_0^1 \int_0^\xi \frac{\kappa \alpha_s(\xi) + \alpha_g}{\alpha_g} d\xi d\xi$$

$$= 2 \frac{f_5(1)}{f_3(1)} \quad (\text{A29})$$

and

$$\frac{C_T}{C_{T0}} = \frac{C_{L_w}}{C_{L_{w0}}}$$

Linearly varying geometric angle of attack.—For the linear-twist-type angle of attack, $\alpha_g(\xi) = (1-\xi)\alpha_{gt}$, the factor κ is 1, and

$$H(\xi) = \alpha_g(\xi) - [f_3(\xi) - f_4(\xi)] \alpha_{gt} \quad (\text{A30})$$

so that

$$\frac{l}{l_{0t}} = \frac{\alpha_s(\xi) + \alpha_g(\xi)}{\alpha_{gt}}$$

$$= \frac{f_4(1)}{f_3(1)} f_3(\xi) - f_4(\xi) \quad (\text{A31})$$

The ratios of the wing lift, wing-root twisting moment, and wing-root bending-moment coefficients to their rigid-wing values are then, on the basis of assumption (1),

$$\frac{C_{L_w}}{C_{L_{w0}}} = 2 \int_0^1 \frac{\alpha_s(\xi) + \alpha_g(\xi)}{\alpha_{gt}} d\xi$$

$$= 2 \left[\frac{f_4(1)}{f_3(1)} f_4(1) - f_5(1) \right] \quad (\text{A32})$$

$$\frac{C_B}{C_{B0}} = 3 \int_0^1 \int_0^\xi \frac{\alpha_s(\xi) + \alpha_g(\xi)}{\alpha_{gt}} d\xi d\xi$$

$$= 3 \left\{ \frac{f_4(1)}{f_3(1)} f_5(1) - \frac{1}{q} [f_3(1) + q^* f_5(1) - 1] \right\} \quad (\text{A33})$$

and

$$\frac{C_T}{C_{T0}} = \frac{C_{L_w}}{C_{L_{w0}}}$$

as in the preceding section.

SOLUTIONS FOR NONUNIFORM WINGS

Equation (A1) may be written in the matrix notation of reference 1 as

$$\{l\} = \frac{q}{144} C_{L\alpha_s} [c] \left\{ \alpha_s + \frac{\alpha_g}{\kappa} \right\} \quad (\text{A34})$$

and equations (A2) and (A3) as

$$\{T\} = s_t [I'] \{le_1 c\}$$

$$= \frac{q}{144} C_{L\alpha_s} e_1 c_r^2 s_t [I'] \left[\left(\frac{c}{c_r}\right)^2 \right] \left\{ \alpha_s + \frac{1}{\kappa} \alpha_g \right\} \quad (\text{A35})$$

$$\{M\} = s_t^2 [II'] \{l\}$$

$$= \frac{q}{144} C_{L\alpha_s} c_r s_t^2 [II'] \left[\frac{c}{c_r} \right] \left\{ \alpha_s + \frac{1}{\kappa} \alpha_g \right\} \quad (\text{A36})$$

where the matrix $[I']$ performs an integration of the running torque $le_1 c$ from the tip inboard, and the matrix $[II']$ performs a double integration of the running load from the tip inboard. These matrices are derived and given in reference 1. They are based upon Simpson's rule with a modification at the tip, where the load distribution is assumed to go to zero with an infinite slope at the tip.

Equations (A4) and (A5), written in matrix notation, are

$$\{\varphi\} = \frac{s_t}{(GJ)_r} [I''] \left[\frac{(GJ)_r}{GJ} \right] \{T\} \quad (\text{A37})$$

$$\{\Gamma\} = \frac{s_t}{(EI)_r} [I''] \left[\frac{(EI)_r}{EI} \right] \{M\} \quad (\text{A38})$$

where the matrix $[I'']$ serves to integrate the accumulated torque or bending moment outboard from the wing root. This integrating matrix is based upon Simpson's rule without the tip modification and is given in reference 1.

The substitution of equations (A35), (A36), (A37), and (A38) in the matrix equivalent of equation (A6) yields

$$\{\alpha_s\} = q^* [A] \left\{ \alpha_s + \frac{1}{\kappa} \alpha_g \right\} \quad (\text{A39})$$

where the aeroelastic matrix $[A]$ is defined by

$$[A] = [I''] \left[\left[\frac{(GJ)_r}{GJ} \right] [I'] \left[\frac{c}{c_r} \right] - k \left[\frac{(EI)_r}{EI} \right] [II'] \right] \left[\frac{c}{c_r} \right] \quad (\text{A40})$$

The parameters q^* and $\bar{q}$ are defined by equations (A13) and (A14), respectively, and

$$k \equiv \frac{\bar{q}}{q^*}$$

When α_g is zero along the entire span, equation (A39) becomes

$$\{\alpha_s\} = q^* [A] \{\alpha_s\} \quad (\text{A41})$$

Consequently, for a particular value of k , the value of q^* at divergence can be found by the iteration of the aeroelastic matrix $[A]$.

Equation (A39) may be rearranged as follows:

$$[1] - q^*[A]] \{ \kappa \alpha_s + \alpha_g \} = \{ \alpha_g \} \quad (\text{A42})$$

The set of linear simultaneous equations represented by equation (A42) may then be solved for the total angle of attack $\kappa \alpha_s + \alpha_g$ in terms of the values of α_g along the span.

The integrations in equations (A25), (A26), and (A27) may be performed with the first rows of the $[I']$ and $[II']$ matrices. Thus

$$\frac{C_{L_w}}{C_{L_{w_0}}} = \frac{[I']_1 \left[\frac{c}{c_r} \right] \{ \kappa \alpha_s + \alpha_g \}}{[I']_1 \left[\frac{c}{c_r} \right] \{ \alpha_g \}} \quad (\text{A43})$$

$$\frac{C_B}{C_{B_0}} = \frac{[II']_1 \left[\frac{c}{c_r} \right] \{ \kappa \alpha_s + \alpha_g \}}{[II']_1 \left[\frac{c}{c_r} \right] \{ \alpha_g \}} \quad (\text{A44})$$

and

$$\frac{C_T}{C_{T_0}} = \frac{[I']_1 \left[\left(\frac{c}{c_r} \right)^2 \right] \{ \kappa \alpha_s + \alpha_g \}}{[I']_1 \left[\left(\frac{c}{c_r} \right)^2 \right] \{ \alpha_g \}} \quad (\text{A45})$$

The aeroelastic characteristics of uniform wings were calculated by both the direct method of the preceding section and the matrix method given in this section. The values of the divergence parameter q^*_D , calculated by the direct method, were found to be about 5 percent greater than the corresponding values calculated by the matrix method. This discrepancy can be shown to be almost entirely due to the rounding off of the loading of the wing tip in the matrix method. The differences between corresponding values of $\frac{\alpha_s}{\alpha_g}$, $\frac{C_{L_w}}{C_{L_{w_0}}}$, $\frac{C_B}{C_{B_0}}$, and $\frac{C_T}{C_{T_0}}$ are negligible.

COMBINATION OF RESULTS

The forms of the approximate formulas used in combining the results of the many computations indicated in the analysis were obtained by considering a highly idealized semirigid wing; that is, a wing which is rigid along its entire span but can bend and twist at the wing root subject to the restraint of a bending and a torsion spring.

If it is assumed that the two spring constants correspond to $\frac{(GJ)_r}{s_t}$ and $\frac{(EI)_r}{s_t}$, the value of q^* at divergence is given by the simple formula

$$q^*_D = \frac{K_1}{1 - K_2 k} \quad (\text{A46})$$

where the factors K_1 and K_2 depend on the taper ratio and the spanwise variation of the stiffness. As shown in reference 2, this formula serves as a good approximation to the calculated values of q^*_D .

For the semirigid wing, the ratio of α_s to α_g is found to be proportional to $\frac{q/q_D}{1 - \frac{q}{q_D}}$. In order to adapt this expression

to the flexible wings considered in the present analysis, the following approximate expression was found to provide satisfactory correlation:

$$\frac{\alpha_s}{\alpha_g} = \frac{1}{\kappa} \frac{q/q_D}{1 - \frac{q}{q_D}} (f + F \Delta f) \quad (\text{A47})$$

where f and Δf are functions of the spanwise coordinate s^* and the wing-chord and stiffness variations; F is a function of the parameter k and the wing-chord and stiffness variations. The functions f , Δf , and F also depend on the type of spanwise variation of the geometrical angle of attack, the subscripts 1 and 2 being used to distinguish between the two types of interest. The accuracy of equation (A47) is illustrated in figures 16 and 17.

If equation (A25) is used for the wing lift coefficient (with ξ replaced by s^*) and equation (A47) for the angle-of-attack distribution, the wing lift coefficient may be expressed as

$$C_{L_w} = \nu \frac{\frac{q}{q_D}}{1 - \frac{q}{q_D}} C_{L_{w_0}} + C_{L_{w_0}} \quad (\text{A48})$$

or

$$\frac{C_{L_w}}{C_{L_{w_0}}} = \frac{1 - \frac{q}{q_D} (1 - \nu)}{1 - \frac{q}{q_D}} \quad (\text{A49})$$

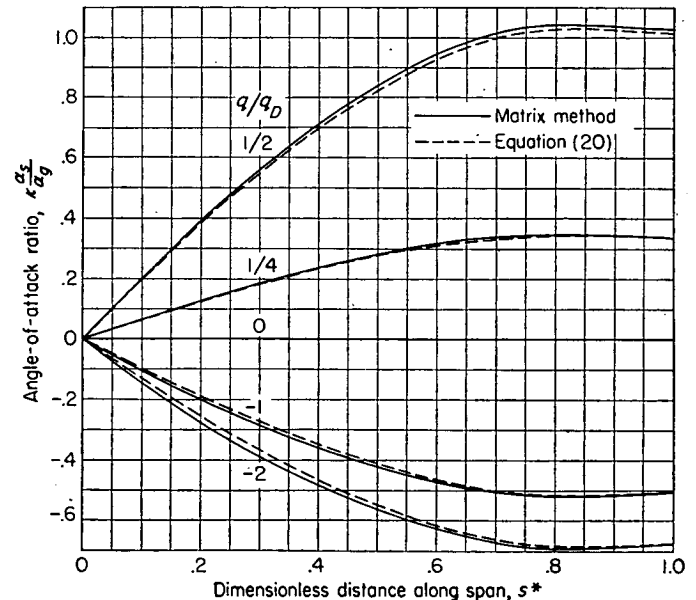


FIGURE 16.—Comparison of angle-of-attack ratios calculated by the matrix method of appendix A with those calculated from equation (20) for constant geometric angles of attack at various dynamic-pressure ratios. $\lambda = 0.5$; $k = 8$; stiffnesses proportional to c^4 .

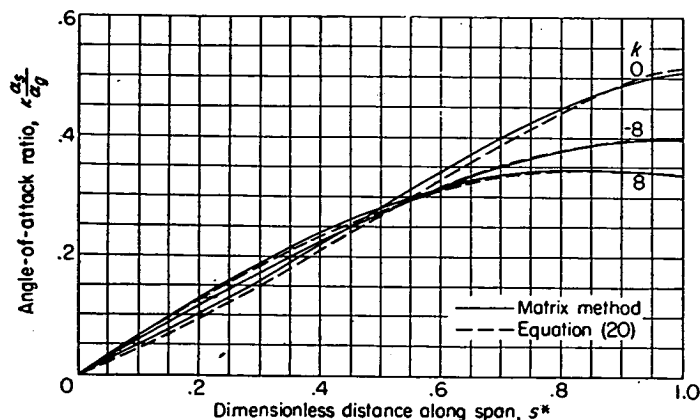


FIGURE 17.—Comparison of angle-of-attack ratios calculated by the matrix method of appendix A with those calculated from equation (20) for constant geometric angles of attack and for various values of the parameter k . $\lambda=0.5$; $\frac{q}{q_D}=\frac{1}{4}$; stiffnesses proportional to c^4 .

where the parameter

$$\nu \equiv \frac{\int_0^{s_t} \frac{c}{c_r} (f + F \Delta f) ds^*}{\int_0^{s_t} \frac{c}{c_r} \frac{\alpha_g}{\alpha_s} ds^*} \quad (A50)$$

so that ν and $\frac{C_{Lw}}{C_{Lw_0}}$ are functions of k and of the wing-chord and

stiffness variations and depend on the type of geometrical angle-of-attack distribution as well.

As indicated by equation (A47), within the approximation inherent in that equation, the shape of the spanwise distribution of α_s does not vary with dynamic pressure. Therefore, to a good approximation, the lateral center of pressure of the lift due to α_s (as well as that due to α_g) does not change its position along the elastic axis when the dynamic pressure changes. The following approximate formula for the wing-root bending-moment coefficient may therefore be deduced from equation (A48):

$$C_B = \bar{s}_s^* \nu \frac{\frac{q}{q_D}}{1 - \frac{q}{q_D}} C_{Lw_0} + \bar{s}_g^* C_{Lw_0} \quad (A51)$$

where $\bar{s}_s^*$ and $\bar{s}_g^*$ are the dimensionless moment arms about the effective wing root of the lifts due to α_s and α_g and are defined by

$$\bar{s}_s^* = \frac{M_{r_s}}{\frac{1}{2} L_{w_s} s_t}$$

and

$$\bar{s}_g^* = \frac{M_{r_g}}{\frac{1}{2} L_{w_g} s_t}$$

Then,

$$\frac{C_B}{C_{B_0}} = \frac{1 - \frac{q}{q_D} (1 - \mu \nu)}{1 - \frac{q}{q_D}} \quad (A52)$$

where μ is defined by

$$\mu \equiv \frac{\bar{s}_s^*}{\bar{s}_g^*} \quad (A53)$$

so that $\frac{C_B}{C_{B_0}}$ is a function of the parameter k , of the taper ratio, and of the stiffness distributions; it also depends on the type of geometrical angle-of-attack distribution.

An approximate formula for the wing-root twisting-moment coefficient may be deduced from equation (A48) as follows:

$$C_T = \bar{e}_{1_s} \nu \frac{\frac{q}{q_D}}{1 - \frac{q}{q_D}} C_{Tw_0} + \bar{e}_{1_g} C_{Tw_0} \quad (A54)$$

where $\bar{e}_{1_s}$ and $\bar{e}_{1_g}$ are the effective dimensionless moment arms about the elastic axis of the lifts due to α_s and α_g and are defined by

$$\bar{e}_{1_s} = \frac{T_{r_s}}{\frac{1}{2} L_{w_s} c_r}$$

and

$$\bar{e}_{1_g} = \frac{T_{r_g}}{\frac{1}{2} L_{w_g} c_r}$$

Then

$$\frac{C_T}{C_{T_0}} = \frac{1 - \frac{q}{q_D} (1 - \nu \tau)}{1 - \frac{q}{q_D}} \quad (A55)$$

where

$$\tau \equiv \frac{\bar{e}_{1_s}}{\bar{e}_{1_g}} \quad (A56)$$

so that $\frac{C_T}{C_{T_0}}$ is a function of k , the taper ratio, and the stiffness distributions and also depends on the type of geometrical angle-of-attack distribution.

The values of ν , μ , and τ are given for the two types of geometrical angle-of-attack distributions in figures 9 and 11.

Figure 18 shows the approximate formulas (A49), (A52), and (A55) to be in good agreement with more accurately computed values.

The foregoing approximate formulas for the structural angle of attack and for the lift, bending-moment, and twisting-moment coefficients are not applicable to wings with zero taper ratio. An attempt was made to combine and to systematize the results calculated for such wings in the manner employed for wings with other taper ratios, but the approximate formulas obtained in this way were found to yield unreliable results. Consequently they are not presented in this report; instead, the results calculated for the wings with zero taper ratio are presented directly in figures 6, 8, 10, and 12.

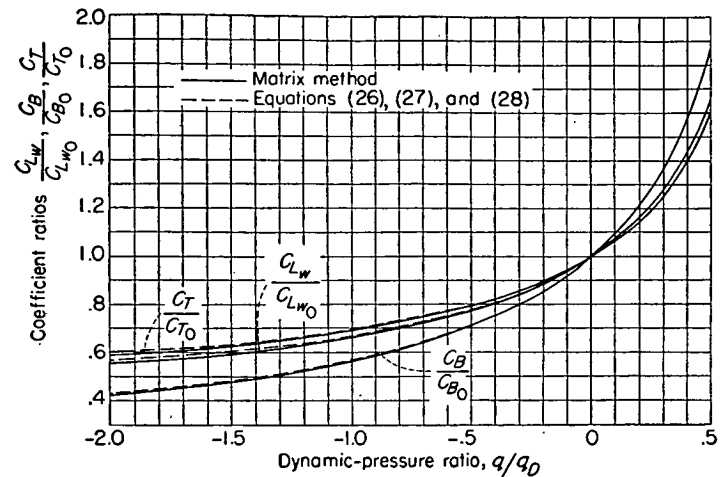


FIGURE 18.—Comparison of lift- and moment-coefficient ratios for constant geometric angles of attack calculated by the matrix method of appendix A with those calculated from equations (26), (27), and (28). $\lambda=0.5$; $k=8$; stiffnesses proportional to c^4 .

APPENDIX B

STIFFNESS DISTRIBUTION OF CONSTANT-STRESS WINGS

OUTLINE OF CONSTANT-STRESS CONCEPT

In order to calculate aeroelastic effects, the bending and torsional stiffnesses of the wing structure EI and GJ have to be known. These stiffnesses enter the calculations in two ways. The root stiffnesses, as indices of the overall bending and torsional stiffnesses, constitute primary parameters which are required in the use of the charts of this report but were not required in the preparation of the charts. On the other hand, the stiffness distributions—that is, the ratios of the local stiffnesses along the span to the root stiffnesses—are secondary parameters which are not required in detail in the use of the charts but did have to be assumed in order to prepare them.

In calculations preliminary to the actual design of the structure, the bending and torsional stiffnesses of the structure are not known; they must be estimated on the basis of either past experience or considerations of an idealized structure. For the purpose of estimating stiffness distributions, past experience with similar structures is likely to be a useful guide in any specific case but does not lend itself to generalization and hence to the preparation of generally applicable charts. The stiffness distributions (other than those which vary as the fourth power of the chord) used to prepare the charts of this report have been obtained from considerations of an idealized structure, as outlined in this appendix.

Basically, the method of this appendix consists in an effort to relate the stiffness of a wing to its strength and to estimate that strength on the basis of certain assumptions. The fundamental assumptions are that the bending and torsional stresses are constant along the span and that the applied loading is proportional to the local chord. The other assumptions concern the bending and torsional stresses caused by this load and their relation to their allowable values. In estimating these stresses the structure is assumed to be essentially of the thin-skin, stringer-reinforced shell type. Certain effectiveness factors are used—for instance, the ratio of the allowable torsional stress to the allowable bending stress, or the ratio of the cross-sectional area of the effective torsion cell to the product of the chord and the wing thickness. The root stiffnesses estimated by the method of this appendix depend directly on the values of these ratios. The stiffness distributions, on the other hand, are largely independent of these ratios but imply the assumption that the ratios are approximately constant along the span. Consequently, the constant-stress concept used in this appendix is more likely to furnish useful results for stiffness distributions than for the root stiffnesses, and, because of the type of structure assumed, the concept is not applicable to very thin wings.

ASSUMED APPLIED LOADS

If the applied normal load is distributed in a manner proportional to the chord, that is, if

$$l = Kc \quad (B1)$$

the bending moment at any point on the span can be obtained by integrating the chord distribution as follows:

$$M = K \left(\frac{b'/2}{\cos \Lambda} \right)^2 \int_{s^*}^1 \int_{s^*}^1 c \, ds^* \, ds^*$$

where s^* is the dimensionless distance along the reference axis measured from the effective root. Similarly, the total normal load on one wing is given by

$$P = K \frac{b'/2}{\cos \Lambda} \int_0^1 c \, ds^*$$

If the wing is linearly tapered, so that

$$c = c_r [1 - (1 - \lambda) s^*] \quad (B2)$$

where the taper ratio λ is defined by

$$\lambda = \frac{c_t}{c_r}$$

then the ratio of the bending moment at any point of the span to the product of the total normal load and the wing semispan less one-half of the fuselage width can be expressed as follows:

$$\frac{M}{P \frac{b'/2}{\cos \Lambda}} = f_6(s^*, \lambda) \quad (B3)$$

where the function f_6 of s^* and λ is defined by

$$f_6 = \frac{1}{3} \left(\frac{1 + 2\lambda}{1 + \lambda} - \frac{1 - \lambda}{1 + \lambda} s^* \right) (1 - s^*)^2 \quad (B4)$$

and shown in figure 19.

Similarly, if the moment arm of the normal load applied to the wing at any station is also proportional to the chord, the constant of proportionality being e_1 , the distributed torque at any station is then $e_1 cl$, and the accumulated torque is

$$T = e_1 K \frac{b'/2}{\cos \Lambda} \int_{s^*}^1 c^2 \, ds^* \quad (B5)$$

which may, in turn, be expressed as

$$\frac{T}{e_1 c_r P} = f_7(s^*, \lambda) \quad (B6)$$

where the function f_7 of s^* and λ is defined by

$$f_7 = \frac{4}{3} \left[\frac{1+\lambda+\lambda^2}{(1+\lambda)^2} - \frac{(2+\lambda)(1-\lambda)}{(1+\lambda)^2} s^* + \frac{(1-\lambda)^2}{(1+\lambda)^2} s^{*2} \right] (1-s^*) \quad (\text{B7})$$

and the average chord $\bar{c}$ is defined by

$$\bar{c} = \frac{c_r + c_t}{2} \quad (\text{B8})$$

The function f_7 is shown in figure 20.

The total normal load on one wing P can be estimated from the design gross weight and the design load factor of the airplane in the following manner:

$$P = \frac{1}{2} (L_w - n W_w) \quad (\text{B9})$$

If the fraction of the wing lift to the total lift carried by the airplane (including that of the fuselage and tail) is designated by η_1 , so that

$$\eta_1 = \frac{L_w}{L_{total}}$$

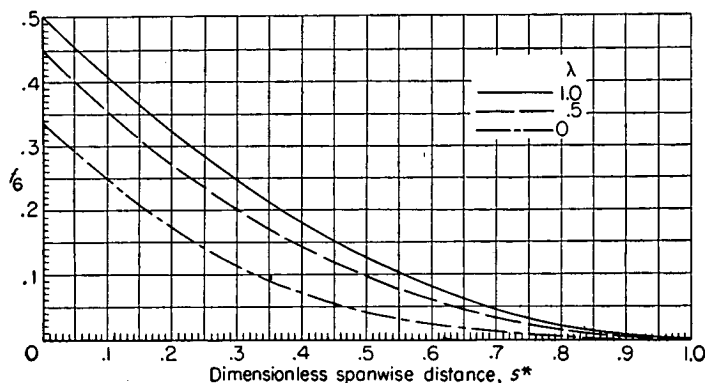


FIGURE 19.—The bending-moment function f_b .

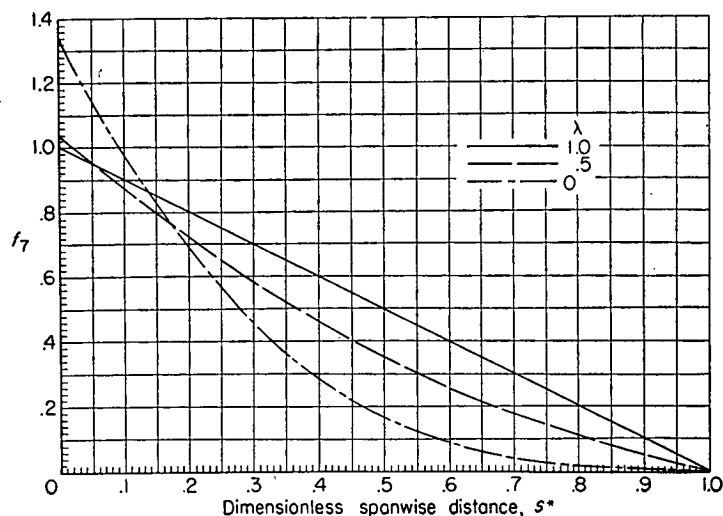


FIGURE 20.—The twisting-moment function f_7 .

and the fraction of the wing weight (including the amount of fuel, external stores, and so on used in the critical design condition) to the total design gross weight is designated by η_2 ,

$$\eta_2 = \frac{W_w}{W}$$

then equation (B9) may be written as

$$P = \frac{1}{2} \eta_3 n W \quad (\text{B10})$$

where

$$\eta_3 = \eta_1 - \eta_2$$

With the value of P given by equation (B10), equations (B3) and (B6) serve to express the local bending and torsional moments in terms of known design parameters.

EFFECTIVE SKIN THICKNESS REQUIRED TO RESIST APPLIED LOADS

The wing structure has to resist both the applied bending moments and the applied torques; in other words, the load-carrying members must resist combined axial and shear stresses. A relation commonly used in the design of wing structures loaded by compressive and shear stresses due to bending and torsion moments is

$$\frac{f_b}{F_B} + \left(\frac{f_t}{F_T} \right)^2 = 1$$

where f_b is the applied bending stress, f_t the applied shear stress, F_B the allowable (compressive) bending stress, and F_T the allowable shear stress. However, a similar relation,

$$\frac{f_b}{F_B} + \frac{f_t}{F_T} = 1 \quad (\text{B11})$$

is somewhat conservative and much more convenient for the present purpose and, consequently, is used as the basis of the following development. If the margin of safety is not zero, equation (B11) can be rewritten as

$$\frac{f_b}{F_B} + \frac{f_t}{F_T} = \eta_4 \quad (\text{B12})$$

where η_4 is an effectiveness factor which can be expressed in terms of the margin of safety (M. S.) as

$$\eta_4 = \frac{1}{1 + \text{M. S.}} \quad (\text{B13})$$

The applied bending stress is

$$f_b = \frac{Mz}{I} \quad (\text{B14})$$

where z is the maximum ordinate on the compression side

measured from and normal to the chordwise principal axis. Similarly, the applied shear stress is

$$f_t = \frac{T}{2\bar{A}t} \quad (\text{B15})$$

where $\bar{A}$ is the cross-sectional area of the (assumed) single torsion cell and t is the skin thickness on the compression side. Substitution of equations (B14) and (B15) into equation (B12) yields

$$\frac{Mz}{IF_B} \left(1 + \frac{T}{M} \frac{F_B}{F_T} \frac{I}{2\bar{A}tz} \right) = \eta_4 \quad (\text{B16})$$

In order to relate the bending and torsion stiffness of the wing to the skin thickness t or to an equivalent thickness t_e which includes the material in the stringers and spar flanges, the bending stress is assumed to be carried by a box covered with sheet of an effective thickness t_e , the webs are assumed to carry no bending stress, and the torsion stress is assumed to be resisted by an equivalent single cell, the two webs of which contain all the material of the actual webs. The torsion and bending moments of inertia may then be written as

$$\begin{aligned} J &= 4 \frac{\bar{A}^2 t}{\bar{p}} \\ &= 2 \frac{\eta_6 \eta_7^2}{\eta_8 \eta_9} c h^2 t_e \end{aligned} \quad (\text{B17})$$

and

$$\begin{aligned} I &= \eta_{10} c t_e \left[\eta_5^2 \left(\frac{h}{2} \right)^2 + \eta_{11}^2 + \eta_{12} \eta_{13}^2 \eta_{14}^2 \left(\frac{h}{2} \right)^2 \right] \\ &= \eta_{15} c \left(\frac{h}{2} \right)^2 t_e \end{aligned} \quad (\text{B18})$$

where the effectiveness factors η_5 to η_{15} are defined in table 1. In the factor η_9 , the effective perimeter $\bar{p}$ of the torsion cell is the sum of the lengths of skin around the perimeter, each weighted by the ratio of the thickness of the critically stressed element to the thickness of the given length of skin.

When the value of I given by equation (B18) is substituted into equation (B16), equation (B16) may be written as

$$t_e = \frac{Mz}{F_B \eta_{15} c \left(\frac{h}{2} \right)^2} \left[1 + \frac{T}{M} \frac{F_B}{F_T} \frac{\eta_{15} c \left(\frac{h}{2} \right)^2}{2\bar{A}z \eta_8} \right] \quad (\text{B19})$$

By making use of equations (B3), (B5), and (B10), as well as the effectiveness factors η_{16} to η_{19} defined in table 1, equation (B19) can be written as

$$t_e = \frac{\eta_3 \eta_5}{\eta_4 \eta_{15}} \frac{nW}{F_B} \frac{b'/2}{c^2 \cos \Lambda} \frac{f_6}{h/c} \left(1 + \frac{\eta_{19} e_1}{A_A} \frac{1}{f_8} \right) \quad (\text{B20})$$

The factor f_8 is defined in terms of the factors f_6 and f_7 given by equations (B4) and (B7) as

$$f_8 = \frac{1}{2} \frac{f_6}{f_7} \quad (\text{B21})$$

The function f_8 is shown in figure 21.

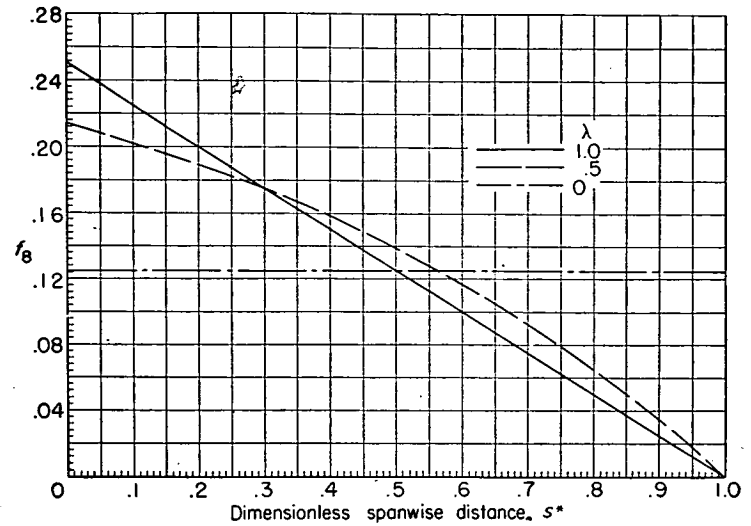


FIGURE 21.—The moment-ratio function f_8 .

BENDING AND TORSIONAL STIFFNESSES

Substitution of the value of t_e given by equation (B20) into equation (B18) yields an expression for the bending moment of inertia I or for the bending stiffness EI at any point along the span. The value of this stiffness at any point on the span may be divided by the stiffness at the wing root $(EI)_r$. This ratio can then be expressed as

$$\begin{aligned} \frac{EI}{(EI)_r} &= \frac{h}{h_r} \frac{f_6}{f_{6r}} \frac{f_8}{f_{8r}} \frac{f_8 + \frac{\eta_{19} e_1}{A_A}}{f_{8r} + \frac{\eta_{19} e_1}{A_A}} \\ &= \frac{h/c}{(h/c)_r} f_8 \left(s^*, \lambda, \frac{\eta_{19} e_1}{A_A} \right) \end{aligned} \quad (\text{B22})$$

where

$$f_8 = \frac{c}{c_r} \frac{f_7}{f_{7r}} \frac{f_8 + \frac{\eta_{19} e_1}{A_A}}{f_{8r} + \frac{\eta_{19} e_1}{A_A}} \quad (\text{B23})$$

The function f_9 is plotted in figure 22. The value $(EI)_r$ may be obtained from equations (B18) and (B20) as

$$(EI)_r = \eta_{20} \frac{E}{F_B} \frac{nW}{S} c_r^4 A_A^2 \left(\frac{h}{c} \right)_r F_r \left(\lambda, \frac{\eta_{19} e_1}{A_A} \right) \quad (\text{B24})$$

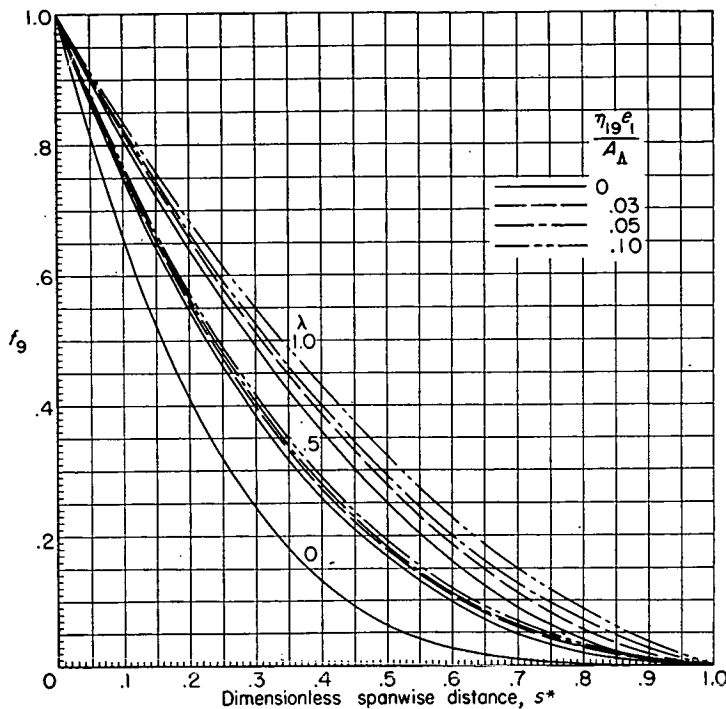
where

$$F_r = \frac{(1+\lambda)^3}{64} \frac{f_{6r}}{f_{8r}} \left(f_{8r} + \frac{\eta_{19} e_1}{A_A} \right) \quad (\text{B25})$$

The function F_r is shown in figure 3 as a function of λ with $\frac{\eta_{19} e_1}{A_A}$ as the parameter.

Similarly the torsional stiffness GJ may be obtained by substituting the value of t_e given by equation (B20) into equation (B17). However, from equations (B17) and (B18) the ratio of the torsional stiffness to the bending stiffness may be obtained in the form

$$\frac{GJ}{EI} = 8 \frac{G}{E} \frac{\eta_6 \eta_7^2}{\eta_8 \eta_9 \eta_{15}} \quad (\text{B26})$$

FIGURE 22.—The stiffness-distribution function f_9 .

This equation shows that the ratio GJ/EI is constant along the span within the framework of the constant-stress concept. Equation (B26) may, therefore, be interpreted as an expression for the value GJ/EI at the wing root, that is, for the value $(GJ)_r/(EI)_r$. The torsional stiffness at any other point on the span can then be obtained from equation (B22), since

$$\frac{GJ}{(GJ)_r} = \frac{EI}{(EI)_r} \quad (\text{B27})$$

because GJ/EI is constant over the span.

The stiffness ratios $GJ/(GJ)_r$ and $EI/(EI)_r$ can be obtained directly from figure 22 when the thickness ratio h/c of the wing is constant along the span; if the thickness ratio is not constant, the factor f_9 obtained from figure 22 must be multiplied by the ratio $\frac{h/c}{(h/c)_r}$ at any station to obtain the stiffness ratio at that station. As may be seen from figure 22, the function f_9 does not vary much with the parameter $\frac{\eta_{19}e_1}{A_\Delta}$; this parameter represents the additional amount of skin thickness required to carry the torque (see eqs. (B19) and (B20)), and this additional thickness is small for most conventional wing structures. Consequently, an average value of $\frac{\eta_{19}e_1}{A_\Delta} = 0.03$ was used to obtain the stiffness distributions used in the aeroelastic calculations on which the charts of this report are based.

Equation (B22) shows that, once a value has been assumed for the term $\frac{\eta_{19}e_1}{A_\Delta}$, the stiffness ratios $EI/(EI)_r$ and $GJ/(GJ)_r$ are independent of the effectiveness factors used in this analysis. Therefore, specific values of these parameters need not be known in order to estimate the stiffness distributions, but one of the assumptions on which equation (B22) is based

is that whatever values the effectiveness factors have are nearly constant along the span. In order to estimate the value of $(EI)_r$, however, these factors must be known, since they enter directly into equation (B24). Estimates of $(EI)_r$ and $(GJ)_r$ obtained in this manner are, therefore, subject to all the limitations imposed by the approximations of the constant-stress concept. Hence, some judgment must be exercised in using these estimates, and, if possible, they should be modified in the light of experience.

STRUCTURAL WEIGHT ASSOCIATED WITH THE STIFFNESS DISTRIBUTION

The increase in structural weight associated with a given increase in stiffness can be estimated on the basis of assumptions similar to those made in relating the stiffness and the strength. For the purpose of this analysis the various components of the wing structure, exclusive of the carry-through structure within the fuselage, are classified in two groups: one which contains the elements that take the bending and torsional loads due to the assumed loading and one which contains all other components. In the first group are

(1) The amount of top and bottom skin that is used in the estimation of the thicknesses required to withstand the bending and torsional loads, including stringers and spar flanges included in the equivalent skin

(2) Webs, including any web stiffeners

In the second group are

(1) Skin, stiffeners, false spars, and so on, which are not considered in the estimation of the equivalent thicknesses

(2) Ribs, bulkheads, and posts designed to raise the buckling strength of the cover sheets

(3) Control surfaces and their supports, attachments, and actuating mechanisms

(4) The supports of internal stores

This analysis is concerned only with the first group and, more specifically, with the relative increase in the weight of this group occasioned by an increase in stiffness of the main structure. Means of estimating the actual magnitude of the weights involved and of estimating the weights of some of the items in the second group as well are given in references 9 and 10.

The weight per unit length of the structural elements of the first group can be written as

$$w_s = 2\eta_8\eta_{21}\gamma_s c t_e \quad (\text{B28})$$

where γ_s is the density of the material of the primary structure (or an equivalent density in the case of sandwich construction), η_{21} is the ratio of an equivalent perimeter $\bar{p}$ to the actual perimeter of the cell, and $\bar{p}$ is the sum of all the lengths which constitute the perimeter, each multiplied by the ratio of its equivalent thickness to the equivalent thickness t_e of the upper cover sheet.

In view of the assumption made concerning the combination of bending and torsional stresses, the thickness t_e required in equation (B28) can be obtained from equation (B18) as

$$t_e = \frac{4I}{\eta_{15}c\bar{h}^2}$$

so that

$$w_s = 8\gamma_s \frac{\eta_s \eta_{21}}{\eta_{15}} \frac{I}{h^2}$$

or

$$\frac{w_s}{w_{s_r}} = \frac{I/I_r}{(h/h_r)^2} \quad (\text{B29})$$

Consequently, the total weight (for both wings) of the structural elements of the first group can be estimated from the relation

$$\frac{W_s/2}{w_{s_r} \cos \Lambda} = \int_0^1 \frac{I/I_r}{(h/h_r)^2} ds^* \quad (\text{B30})$$

Equation (B30) serves to estimate relative changes in the weight of the first group of structural elements. For instance, with a given distribution of I and h , that weight is directly proportional to I_r and inversely proportional to h_r^2 . Similarly, given two different distributions of I and h with the same values at the root, the ratio of the weights is equal to the ratio of the two values obtained by using the respective distributions of I and h in the integral of equation (B30).

Although the actual value of W_s is not relevant to this discussion, it may be estimated by substituting the previously calculated stiffness distributions into equation (B30), and the result is given here as a matter of general interest:

$$W_s = 2 \frac{\eta_3 \eta_5 \eta_8 \eta_{21}}{\eta_4 \eta_{15} \eta_{18}} \frac{\gamma_s}{F_B} \frac{n}{h_r} \frac{W}{c_r} A_\Lambda \frac{b'/2}{\cos \Lambda} F_w \quad (\text{B31})$$

where

$$F_w = \frac{32}{(1+\lambda)^2} F_r \int_0^1 \frac{I/I_r}{(h/h_r)^2} ds^* \quad (\text{B32})$$

According to equation (B31), the structural weight is directly proportional to the design gross weight, load factor,

swept-span aspect ratio, span, and density of the material of the primary structure and inversely proportional to the allowable stress and the wing thickness ratio. The dependence of the weight on the taper ratio (all other parameters, notably the aspect ratio and span, are the same) is illustrated in figure 14 by a plot of the function F_w against taper ratio for several values of the parameter $\frac{\eta_{18} e_1}{A_\Lambda}$ and for several ratios of the wing thickness ratios at the tip and at the root.

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REPORT 1141

METHOD AND GRAPHS FOR THE EVALUATION OF AIR-INDUCTION SYSTEMS¹

By GEORGE B. BRAJNIKOFF

SUMMARY

Graphs have been developed for rapid evaluation of air-induction systems from considerations of their aerodynamic-performance parameters in combination with power-plant characteristics. The graphs cover the range of supersonic Mach numbers up to 3.0. Examples are presented for an air-induction system and engine combination at two Mach numbers and two altitudes in order to illustrate the method and application of the graphs. The examples show that jet-engine characteristics impose restrictions on the use of fixed inlets if the maximum net thrusts are to be realized at all flight conditions.

INTRODUCTION

In order to obtain a true indication of the worth of a given air-induction system as a component of a propulsive unit, it is necessary to employ an evaluation parameter that represents a summation of all the gains and penalties resulting from the use of that particular system. Such a parameter should consider not only the aerodynamics of the entire installation but also such factors as the weight, mechanical complexity, purpose of the aircraft, and many others. Obviously, such a universal parameter is difficult to derive and even more difficult to apply. For this reason, it is convenient to make a partial evaluation based on the aerodynamic considerations before attempting a general evaluation. In such a case, the net thrust or the net thermal efficiency can be used as figures of merit because they provide a measure of the aerodynamic and thermodynamic qualities of the installation. The net thrust represents the force remaining after subtraction of the drag chargeable to the propulsive system from the thrust that it develops. The net thermal efficiency may be obtained from the net thrust, the flight velocity, and the rate of fuel consumption.

The maximum net thrust and thermal efficiency attainable with a jet-engine installation depend greatly on the performance of the air-induction system employed. The characteristics of air-induction systems are usually presented in terms of total-pressure recovery, external drag coefficient, and mass-flow ratio. Unless all three of these parameters for one system excel those for another at supersonic speeds, it is difficult to choose the better system because of the interdependence of the engine and induction-system parameters. Because of this interdependence, it is necessary to combine the induction system and power-plant characteristics so as to obtain a single figure of merit for the complete installation. By comparing the figures of merit, it is possible to establish the relative aerodynamic worth of each of the air-induction systems considered when they are used with a given engine.

The effects of changes in various parameters on the overall performance of propulsive systems have been evaluated in the past (see refs. 1, 2, and 3); however, the scope of each of these investigations was limited because the magnitude of changes due to variations in parameters were determined for specific engines or specific installations. Therefore, the results have quantitative significance for the assumed installations only and cannot be applied directly to propulsive systems having component characteristics different from those used in the analyses.

The purpose of this report is to present a method of evaluation of various air-induction systems when combined with arbitrary jet engines, and to present graphs that were developed to permit rapid determination of the thrust coefficients of a wide variety of jet-propulsion systems (ram jets, turbojets with afterburning, ducted fans, etc.). The evaluation is based on considerations of the air-handling qualities of the induction systems and the component characteristics of engines. The method allows selection of the aerodynamically optimum combination of an induction system and engine for a particular set of flight conditions. Thus, it provides the initial solution in the more general problem that considers the relative merit of systems for a range of flight conditions.

NOTATION

A	cross-sectional area at any point of stream tube containing the air flowing through the propulsive system, sq ft
a	speed of sound, ft/sec
C_{D_p}	external drag coefficient of propulsive system based on maximum frontal area of installation, $\frac{D_p}{q_0 S^*}$, dimensionless
C_{D_p}'	external drag coefficient of propulsive system based on free-stream cross-sectional area of stream tube entering the inlet, $\frac{D_p}{q_0 A_0}$, dimensionless
C_{F_t}	internal thrust coefficient based on maximum frontal area of installation, $\frac{F_t}{q_0 S^*}$, dimensionless
C_{F_t}'	internal thrust coefficient based on the free-stream cross-sectional area of stream tube entering the inlet, $\frac{F_t}{q_0 A_0}$, dimensionless

¹ Supersedes NACA TN 2697, "Method and Graphs for the Evaluation of Air-Induction Systems," by George B. Brajnikoff, 1952.

C_{F_n}'	net thrust coefficient based on the free-stream cross-sectional area of stream tube entering the inlet, $C_{F_i}' - C_{D_p}'$, dimensionless	$\sqrt{\frac{1}{\theta}}$	temperature-correction factor, $\left(\frac{1.4 \times 519}{\gamma T_t}\right)^{\frac{1}{2}}$
C_{F_n}	net thrust coefficient based on maximum frontal area of engine, dimensionless	ρ	mass density, slugs/cu ft
D_p	drag force chargeable to propulsive system if the engine thrust is defined as the total momentum at the exit station less that of the incoming flow in the free stream, $D_{(B+p)} - D_B + D_M$, lb	SUBSCRIPTS	
D_B	pressure and friction drag forces acting on the basic body shape (fuselage) without an air inlet, lb		
D_M	total momentum of the incoming mass of air at the entrance station less the total momentum of the same mass of air in the free stream $[(p_1 A_1 + m_1 V_1) - (p_0 A_0 + m_1 V_0)]$, lb	a	air
$D_{(B+p)}$	pressure and friction drag forces acting on the external surface of the combined basic body and the propulsive system, lb	B	body
F_n	net thrust force, $F_t - D_p$, lb	c	compressor
F_t	internal thrust force (rate of momentum change of internal flow between free-stream and the tail-pipe exit where static pressure is assumed equal to the free-stream static pressure), lb	D	drag
g	acceleration due to gravity, ft/sec ²	F	thrust
h	specific enthalpy, Btu/lb	f	fuel
J	mechanical equivalent of heat, 778 ft-lb/Btu	i	internal (within boundaries of stream tube entering the inlet)
L. H. V.	lower heating value of fuel, Btu/lb	M	total momentum
M	Mach number, dimensionless	M_a	conditions corresponding to flight Mach number
m	mass-flow rate, slugs/sec	n	net
$\frac{m_1}{m_0}$	mass-flow ratio, $\frac{\rho_1 V_1 A_1}{\rho_0 V_0 A_1}$	opt	optimum conditions (conditions of best performance)
n	actual rotational speed of engine, rpm	p	propulsive system
$n_{corr.}$	corrected rotational speed, $n \sqrt{\frac{1.4 \times 519}{\gamma T_t}}$, (where γ and T_t correspond to stagnation conditions at the inlet of the unit under consideration), rpm	P	power plant
p	static pressure, lb/sq ft	sl	at standard sea-level static conditions
p_t	total pressure, lb/sq ft	t	stagnation conditions
q	dynamic pressure, lb/sq ft	0, 1, 2, } 3, ... 11 }	station, as shown in figure 1
S_P	maximum frontal area of power plant, sq ft	SUPERSSCRIPTS	
S^*	maximum frontal area of installation, sq ft		
T	static temperature, °R	'	based on free-stream area A_0
T_t	total temperature, °R	*	reference (such as frontal area, used in drag coefficient)
V	speed, ft/sec	METHOD	
w	weight-flow rate, lb/sec		
γ	ratio of specific heats	The method of evaluation consists of determining the maximum net thrust coefficient based on the frontal area of the engine at various conditions of flight. The present report considers primarily the net thrust coefficient because the net thermal efficiency depends directly on the net thrust coefficient (see eq. (A8) of Appendix A); the evaluation at a given flight condition leads to the same conclusions, regardless of which of the two parameters is used. The net thermal efficiency is useful in evaluating complete flight plans when range and endurance must be considered.	
δ	pressure correction factor, $\frac{p_t}{p_{st}}$	In order to evaluate an induction system, the following information must be available:	
η	net thermal efficiency of propulsive system (net thrust times the flight speed divided by energy input rate)		
$\eta_c \eta_t$	combined adiabatic efficiency of compressor and turbine	1. The induction-system characteristics p_{t_3}/p_{t_0} , m_1/m_0 , and C_{D_p} (which represents the total drag chargeable to the propulsive system) for various Mach numbers and ratios of inlet area to frontal area of the installation	
		2. The engine characteristics, such as the exhaust-to-inlet pressure and temperature ratios and the air-handling capacity (volume of air consumed per second) ²	
		With this information, an inlet size can be selected for the propulsion system such that it operates at any desired mass-flow ratio and provides a constant volume of air as required by the engine. The inlet size and the mass-flow ratio in turn determine the external drag of the system.	

² The ram-jet and turbojet engines operate essentially with constant-volume flow since the velocity past the flame holder and at the compressor intake must remain constant at the maximum allowable values. This condition is required in order to obtain the maximum thrust with an engine of a given frontal area.

By the use of the induction-system and engine-performance data corresponding to the operating condition, it is possible to calculate the net thrust coefficient.

In general, the method entails the following steps:

1. The thrust coefficient C_{F_i}' , based on the free-stream area A_0 of the air required by the engine, is calculated for the given engine as a function of total-pressure recovery for the flight Mach number and altitude. This computation is performed using the induction-system characteristics (p_{t_3}/p_{t_0} as a function of m_1/m_0), the graphs presented in this report, and equation (A3) of Appendix A.

2. The drag coefficient attributable to the propulsive system at various mass-flow ratios, C_{D_p} , is transformed to C_{D_p}' by multiplying C_{D_p} by the ratio S^*/A_0 . (See eq. (A4) of Appendix A.)

3. The net thrust coefficient C_{F_n}' (based on A_0) at various mass-flow ratios is obtained by subtracting the drag coefficient C_{D_p}' (step 2) from the thrust coefficient C_{F_i}' (step 1). (See Appendix A, eq. (A6).)

4. The net thrust coefficient C_{F_n} based on the frontal area of the engine is obtained by transforming C_{F_n}' (step 3) by means of equation (A7) of Appendix A.

After these calculations are performed for the range of operational mass-flow ratios, the maximum net thrust coefficient attainable with the given induction-system and engine combination is found from the plot of C_{F_n} as a function of m_1/m_0 .

DETERMINATION OF THE MAXIMUM NET THRUST COEFFICIENT

Many air-induction systems produce highest total-pressure recoveries at mass-flow ratios less than the maximum attainable at a given supersonic Mach number. As the mass-flow ratio is reduced from its maximum value until the maximum total-pressure recovery is attained, the thrust coefficient C_{F_i}' increases due to rising recovery and so does the drag coefficient C_{D_p}' due to increasing additive drag (see ref. 4). Thus, in this range of mass-flow ratio, the total-pressure recovery and drag have opposite effects on the net thrust coefficient C_{F_n}' . To attain the maximum net thrust-coefficient it is necessary, therefore, to provide the air required by the engine at an optimum mass-flow ratio that provides the best compromise between thrust and drag.

Determination of internal thrust coefficient, C_{F_i}' .—When calculations of the net thrust coefficient are made in order to find the optimum mass-flow ratio, it is convenient to consider the flow field existing about a propulsive system to be divided in two parts, internal and external, the boundary being that of the stream tube surrounding the air which enters the system. The flows within these two regions may be analyzed separately, and the results can be combined to obtain a figure of merit for the complete system. It is shown in Appendix A (eq. (A3)) that when the thrust due to internal flow is expressed in coefficient form (C_{F_i}') based on the free-stream area of the stream tube entering the induction system (A_0), the significance of the quantity of air required by the

engine disappears. Then, the magnitude of C_{F_i}' depends only on the ratio of the exhaust and flight velocities. (The contribution of the mass of fuel burned, w_f/w_a , to the thrust, C_{F_i}' , is usually on the order of only 5 percent at rated conditions and, for purposes of induction system comparison, may be neglected.) This fact makes it possible to isolate the effects of the total-pressure recovery on the thermodynamic cycle which determines the magnitude of the internal thrust coefficient C_{F_i}' attainable with given engine characteristics at a fixed Mach number and altitude. To reduce computational effort and to make the determination of the internal thrust coefficient universal, graphs based on the thermodynamic cycles of jet engines have been developed.

The processes undergone by the internal air flow from one station of a propulsion system to another have been represented graphically in figures 2 to 4 using the tables and methods of reference 5. (See example 16 of ref. 5 for illustration of the method of solution for various states of the gas.) The subscript numerals designate the stations shown in figure 1. The assumptions used are independent of any actual installation. They are as follows:

1. For a turbojet engine, the specific enthalpy change through the turbine is equal to the specific enthalpy rise in the compressor (calculated as if the compression were isentropic) divided by the product of the adiabatic efficiencies (the combined efficiency) of the two units.

2. The air-fuel ratios are those necessary to maintain the assigned combustion temperature by complete combustion of a fuel with the lower heating value (L. H. V.) equal to 19,000 Btu per pound.

3. The flow in the exhaust nozzle is isentropic; it has the properties of air at the exhaust temperatures and leaves the tail pipe at free-stream static pressure at all flight conditions. (This assumption requires an adjustable exhaust nozzle with variable throat and exit areas A_{10} and A_{11} .)

Flight and exhaust velocities can be determined from figure 2 for ram jets and also for turbojets if the engine performance is available in the generalized form (p_{t_3}/p_{t_0} and T_{t_0}/T_{t_3}) suggested in reference 6. If not, figures 3 to 5 must be used to find the effects of engine operation on the fluid conditions at the engine outlet or at the entrance of the exhaust nozzle. Quadrant I of figure 2 presents the variation of the free-stream Mach number with the speed of flight at various altitudes. Quadrant II yields the ratio of recovered total pressure to free-stream static pressure (p_{t_3}/p_0) as a function of total-pressure recovery (p_{t_3}/p_{t_0}) of the air-induction system. Quadrant III shows the highest exhaust Mach number obtainable with the available pressure ratio (p_{t_3}/p_0), which includes mechanical compression due to the power plant, as a function of the total temperature of the exhaust. The last quadrant provides the exhaust velocity corresponding to the exhaust Mach number and the total temperature T_{t_0} when $p_{t_1}=p_0$. A correction for the exhaust-nozzle losses can be applied to the exhaust velocity, which was calculated on the assumption of isentropic flow, if the actual-to-theoretical jet-speed ratio is known. (See ref. 4.)

The temperature graph, from which the total temperature at various stations throughout the propulsive system can be determined, is shown in figure 3. The effects of altitude, flight Mach number, mechanical compression, and burning of the fuel are included in this figure. Quadrant IV can be used to find the temperatures or the amounts of fuel consumed in afterburning as well as in the main combustion chambers. The effects of incomplete combustion or of a different fuel heating value on fuel consumption can be taken into account by direct ratios of combustion efficiencies or of the heating values. The temperature graph is used in conjunction with the compressor-turbine graph shown in figure 4 from which the turbine expansion ratio necessary to drive the compressor can be determined. This figure also provides the resulting total temperature at the turbine outlet (T_{t_0}) for a given inlet temperature.

Figure 5 presents the variation of the temperature correction factor with temperature at the compressor or turbine inlet. This figure is used to calculate the engine operational conditions from the performance parameters corrected to standard sea-level conditions. The effect of temperature on the ratio of specific heats for air has been included in the temperature correction factor for the compressor because the stagnation temperature of the free stream varies sufficiently to cause error if it were neglected.

The graphs described above allow determination of the velocities and air-fuel rates that must be known in order to calculate the internal thrust coefficient C_{F_i} . The coefficient C_{F_i} is computed by substituting these quantities in equation (A3) of Appendix A.

Determination of the external drag coefficient C_{D_p} .—The total drag chargeable to a propulsive unit consists of drag attributable to the induction system, to the modification of the airframe necessary to house the engine, and to the interference of the pressure field of the propulsive system with other components of the aircraft. The magnitude of drag chargeable to the induction system depends on the amount of diffusion ahead of the inlet, which is a function of mass-flow ratio (see ref. 7), and on the geometric proportions of the induction system which may be described, in general, by the length of the induction system and the ratio of inlet area to the frontal area of the installation (A_1/S^*). The inlet area necessary to provide the air required by the engine depends on the inlet mass-flow ratio; if the optimum mass-flow ratio is unknown, the value of the ratio A_1/S^* necessary for optimum operation is also unknown. Thus, when the optimum mass-flow ratio is being calculated, it is necessary to compute the variation of the net thrust coefficient C_{F_n} with mass-flow ratio for several values of A_1/S^* which are between the value for the inlet operating at the maximum mass-flow ratio and that required at the mass-flow ratio for maximum total-pressure recovery.

In practice, induction-system characteristics are usually obtained from tests with models having a fixed A_1/S^* ratio. If data for various ratios are unavailable, it is necessary to estimate the effects of A_1/S^* ratio on the total-pressure

recovery and drag. The total-pressure recovery usually is not affected appreciably by small changes in the inlet size. The changes in drag, however, must be taken into consideration at a given mass-flow ratio, the additive drag varies directly with the inlet area, while the surface pressure drag, which is roughly proportional to the square of the angle of inclination of the external surface of the induction system with respect to the flow direction decreases slightly as the inlet area is increased.

DETERMINATION OF INLET SIZE FOR OPTIMUM OPERATION

To develop maximum thrust at a given flight condition with an engine operating with a fixed volumetric capacity, it is necessary to match the engine and the induction system so that the latter operates at the optimum mass-flow ratio. This condition is attained when the induction-system inlet area (A_1) is such that the cross-sectional area of the free-stream tube entering the inlet (A_0) is equal to that required by the engine. For a fixed volumetric capacity, the area A_0 required by the engine varies directly with the total-pressure recovery at the exit of the induction system. This area (A_0) can be found using relations given in Appendix B (see eqs. (B1) and (B8)) and the engine characteristics. The inlet area (A_1) can be found from the equation defining the mass-flow ratio, that is, $m_1/m_0 = A_0/A_1$.

The engine air requirements change with Mach number and altitude, and the optimum mass-flow ratio usually changes with Mach number. Thus, it is unlikely that any one propulsive system will develop the maximum net thrust at all flight conditions unless the inlet area is adjustable. If the engine air requirements are adjusted to the characteristics of the induction system having a constant inlet area by means of engine-speed control, the propulsive system will not develop the attainable maximum net thrust at off-design flight conditions.³ Appendix C presents the relations between the mass-flow ratio, the total-pressure recovery, and the engine air requirements at various engine speeds; these relations provide the information necessary for computation of engine performance at part-throttle operation.

ILLUSTRATIVE EXAMPLES

APPLICATION OF GRAPHS

The use of the graphs can be demonstrated best by illustrative examples. For this purpose a ram jet and a turbojet with and without afterburning have been selected. When the charts are used for determining the internal thrust coefficients of actual installations, the actual induction system and engine characteristics are used; in the present report, these characteristics have been assumed. The computations were performed for various total-pressure recoveries (p_{t_3}/p_{t_0} from 1.0 to 0.4) and Mach numbers (M_0 from 1.0 to 3.0) in order to obtain data for the example induction-system calculations presented later in this report.

Ram jet.—The steps of solution, presented in tabular form, for a ram jet flying at a Mach number 2.0 in the isothermal

³ Since the engine ordinarily runs at the maximum speed allowable for continuous operation (rated rpm), the speed control can only reduce the speed. Thus, the weight of air handled per second and the operating temperature decrease with the result that thrust decreases.

region of the atmosphere are as follows (the assumed values are indicated by asterisks):

Item	Source	Units	Quantity
V_0	Fig. 2, quadrant I.....	$\frac{\text{ft}}{\text{sec}}$	1940
$\frac{p_{t_3}}{p_{t_0}}$	Induction-system characteristics.....	None.....	*0.80
$\frac{p_{t_3}}{p_0}$	Fig. 2, quadrant II.....	None.....	6.25
T_{t_0-3}	Fig. 3, quadrants I and II.....	° R.....	708
T_{t_0}	Operating temperature.....	° R.....	*3000
$\left(\frac{w_a}{w_f}\right)_s$	Fig. 3, for T_{t_0} and T_{t_0-3}	lb air lb fuel	29.6
$\frac{p_{t_3}}{p_0}$	$\left(\frac{p_{t_3}}{p_0}\right) \left(\frac{p_{t_0}}{p_0}\right)$	None.....	6.25
V_{t_1}	Fig. 2, quadrants III and IV for p_{t_3}/p_0 and T_{t_0}	None.....	3913
C_{F_i}'	$2 \left[\left(1 + \frac{w_f}{w_a}\right) \frac{V_{t_1}}{V_0} - 1 \right]$	None.....	2.170

The loss of total pressure between stations 3 and 9 was neglected. However, such loss can be taken into account by reducing the value of p_{t_0}/p_{t_3} by the amount of loss.

Figure 6 shows the variation of the internal thrust coefficient C_{F_i}' with the total-pressure recovery and Mach number computed using the procedure outlined in the sample calculation. The temperature of combustion was assumed to be 3000° R throughout the range of Mach numbers. It must be remembered in the use of this figure that, since a fixed temperature was used, either the size of the inlet or the mass-flow ratio must reduce with decreasing total-pressure recovery or the critical area of the exhaust nozzle must be increased to compensate for the greater specific volume of the air handled.

Turbojet.—In order to find the variation of the internal thrust coefficient for the turbojet without and with afterburning, the generalized characteristics of the engine shown in figure 7 were used. Constant actual speed n of 12,500 rpm (rated speed at standard intake conditions) was assumed for the entire range of operation in both cases. The compressor characteristics were selected so that the pressure ratio at Mach number 1.4 would be 6.25. (The reasons for selecting this pressure ratio are discussed in Appendix D.) The effects of Reynolds number index (the ratio of Reynolds number to Mach number, see ref. 6) on the compressor characteristics have been neglected. The turbine characteristics necessary to satisfy the engine operational requirements were determined using figures 3, 4, and 5 upon assumption of the variation of the adiabatic efficiency η_t with corrected turbine speed for sea-level static conditions. The operational limits were fixed by the assumptions of: (1) the combustion temperature (2000° R) at rated engine speed; (2) the equality of actual rotational speeds of compressor and turbine; and (3) the equality of compressor and turbine pressure ratios at the idling speed.

The internal thrust coefficients for the turbojet with afterburning were computed on the assumption that the

total fuel consumption (engine plus afterburner) per pound of air is equal to that of the ram jet at the same Mach number. The steps of solution for the turbojet without and with afterburning are given in chronological order in table I which contains the computations for flight at Mach number 2.0 in the isothermal region of the atmosphere.

As in the case of the ram jet, the losses of total pressure due to combustion and friction $\left(\frac{p_{t_5}}{p_{t_4}}, \frac{p_{t_7}}{p_{t_6}}, \frac{p_{t_8}}{p_{t_7}}, \text{ and } \frac{p_{t_9}}{p_{t_8}}\right)$ were neglected. Such losses, however, can be readily accounted for when p_{t_0}/p_0 is computed.

The variation of the internal-thrust coefficient C_{F_i}' with total-pressure recovery computed for a range of Mach numbers is shown in figure 8. The variations of the internal thrust coefficients with Mach number and total-pressure recovery were calculated for the engine characteristics pertaining to a fixed actual engine speed. Thus, as the total-pressure recovery decreases at a given Mach number and altitude, the size of the air inlet or the mass-flow ratio must be reduced for a fixed-size engine.

CALCULATION OF OPTIMUM MASS-FLOW RATIO AND INLET AREA

To illustrate the calculation of the maximum net thrust coefficient and of the required inlet area, the assumed characteristics of an air-induction system will be combined with the assumed engine characteristics shown in figure 7. In addition, the flight conditions of the example calculations will be selected so as to demonstrate the effects of Mach number and altitude on the maximum net thrust coefficient, the optimum mass-flow ratio, and the inlet area necessary for optimum performance.

Example 1.—Flight is in the isothermal region of the atmosphere at Mach number 2.5. The engine operates at the rated actual speed and uses afterburning to the extent that the total fuel consumption is equal to that of a ram jet having a 3000° R combustion temperature T_{t_0-9} . (See figs. 1 and 6.)

Example 2.—Flight is in the isothermal region of the atmosphere at Mach number 1.2. The engine operates at the rated actual speed and uses no afterburning.

Example 3.—Flight is at sea level at Mach number 1.2. The engine operates at the rated actual speed and uses no afterburning.

To simplify presentation, the effects of the ratio of inlet area to the maximum frontal area of the installation (A_1/S^*) on the drag coefficient and total-pressure recovery will be neglected; that is, the characteristics for a fixed (A_1/S^*) ratio will be used to find the optimum mass-flow ratio. The engine frontal area S_F will be assumed equal to 70 percent of S^* , and $A_1/S^*=0.20$ will be used in all examples; it will also be assumed that the engine operates at a constant speed (12,500 rpm) at all times, and that its size is such that it requires $A_0=1$ square foot when $p_{t_3}/p_{t_0}=1.0$ and $M_0=1.0$ at sea level, that is $(w_{a_1})=54.6$ lb/sec.

The assumed characteristics of the induction system to be evaluated are shown in figures 9 and 10. This system, designed for $M_0=2.5$, uses one oblique shock wave ahead of the entrance and a normal shock wave just inside the inlet

at the maximum mass-flow ratio. The A_1/S^* ratio is typical of induction systems of the side-scoop type. At $M_0=1.2$, the maximum mass-flow ratio is less than at $M_0=2.5$ because the normal shock wave is well ahead of the entrance and a considerable portion of flow is deflected past the inlet. For the same reason, the minimum external drag coefficient C_{D_p} is higher as a result of a greater additive drag coefficient (see ref. 7). The total-pressure recovery is higher at $M_0=1.2$ because the normal shock wave occurs at a lower Mach number.

In all cases the solution follows the outline given in the section entitled "Method" of this report.

Step 1: The variation of the internal thrust coefficient C_{F_i}' with p_{t_2}/p_{t_0} at flight Mach number and altitude is calculated. Figure 11 shows the internal thrust coefficient C_{F_i}' for example 1 obtained from the cross plot of data of figure 8 for $M_0=2.5$.

Step 2: The external drag coefficients shown in figure 9, when converted to A_0 using equation (A4) of Appendix A and $A_1/S^*=0.20$, assume the values shown in figure 11.

Step 3: The net thrust coefficient C_{F_n}' is obtained by subtracting C_{D_p}' from C_{F_i}' of figure 11.

Step 4: The net thrust coefficient C_{F_n} , obtained using the relation

$$C_{F_n} = C_{F_n}' \left(\frac{A_1}{S_P} \right) \left(\frac{m_1}{m_0} \right) = C_{F_n}' \left(\frac{A_1}{S^*} \right) \left(\frac{m_1}{m_0} \right) \left(\frac{S^*}{S_P} \right) \\ = 0.286 C_{F_n}' \left(\frac{m_1}{m_0} \right) .$$

and the values found in step 3, are shown in figure 12 for example 1.

Similar calculations for examples 2 and 3 yield the results shown in figure 13. In the case of example 3, curves similar to those of figure 8 for sea-level flight must be used.

The free-stream-tube area necessary to supply the air required by the turbojet engine of the illustrative examples is shown in figure 14 for various Mach numbers and total-pressure recoveries at two altitudes. The curves shown were obtained using equation (B8) of Appendix B. The inlet areas required at each of the three flight conditions to produce maximum net thrust coefficients C_{F_n} were obtained by dividing the free-stream areas A_0 , required by the engine when the total-pressure recoveries are equal to those at $\left(\frac{m_1}{m_0} \right)_{opt}$, by the respective $\left(\frac{m_1}{m_0} \right)_{opt}$.

The results of the calculations are summarized in the following table:

Example	$\left(\frac{m_1}{m_0} \right)_{opt}$	Maximum C_{F_n}	Optimum inlet area $(A_1)_{opt}$ sq ft
1	1.65	0.750	0.794
2	.94	.450	1.075
3	.94	.149	.915

These results indicate that the inlet area required for optimum performance must change with altitude at a fixed Mach number, as well as with Mach number at a fixed altitude. Although the results depend entirely on the variation of the induction system and engine characteristics with Mach number and altitude, it is unlikely that any one propulsive system will develop the attainable maximum net thrust throughout the range of Mach numbers and altitudes unless some means of varying the inlet area are employed.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., February 19, 1952.

APPENDIX A

RELATIONS DESCRIBING NET THRUST COEFFICIENT AND NET THERMAL EFFICIENCY

The internal thrust force and the internal thrust coefficient are given by the following relations when the exhaust pressure is equal to free-stream static pressure:

$$F_i = \frac{w_a}{g} \left[\left(1 + \frac{w_f}{w_a} \right) V_{11} - V_0 \right] \\ = \rho_0 A_0 V_0^2 \left[\left(1 + \frac{w_f}{w_a} \right) \frac{V_{11}}{V_0} - 1 \right] = 2 q_0 A_0 \left[\left(1 + \frac{w_f}{w_a} \right) \frac{V_{11}}{V_0} - 1 \right] \quad (A1)$$

$$C_{F_i} = \frac{F_i}{q_0 S^*} = 2 \frac{A_0}{S^*} \left[\left(1 + \frac{w_f}{w_a} \right) \frac{V_{11}}{V_0} - 1 \right] \quad (A2)$$

where w_f/w_a represents the total amount of fuel consumed per pound of air handled. When based on the free-stream area of the stream tube entering the duct, the internal thrust coefficient is given by

$$C_{F_i}' = C_{F_i} \frac{S^*}{A_0} = 2 \left[\left(1 + \frac{w_f}{w_a} \right) \frac{V_{11}}{V_0} - 1 \right] \quad (A3)$$

Similarly,

$$C_{D_p}' = C_{D_p} \frac{S^*}{A_0} = C_{D_p} \frac{S^*}{A_1 \frac{m_1}{m_0}} \quad (A4)$$

where $\frac{m_1}{m_0} = \frac{\rho_1 V_1 A_1}{\rho_0 V_0 A_0}$ by definition. By continuity, $\rho_1 V_1 A_1 = \rho_0 V_0 A_0$, and thus $\frac{m_1}{m_0} = \frac{\rho_0 V_0 A_0}{\rho_0 V_0 A_1} = \frac{A_0}{A_1}$.

The net thrust coefficient based on A_0 is given by

$$C_{F_n}' = C_{F_i}' - C_{D_p}' \quad (A5)$$

thus,

$$C_{F_n}' = 2 \left[\left(1 + \frac{w_f}{w_a} \right) \frac{V_{11}}{V_0} - 1 \right] - C_{D_p} \frac{S^*}{A_1 \frac{m_1}{m_0}} \quad (A6)$$

The net thrust coefficient based on the engine frontal area is related to C_{F_n}' by the following relation:

$$C_{F_n} = C_{F_n}' \left(\frac{A_1}{S_P} \right) \left(\frac{m_1}{m_0} \right) = C_{F_n}' \left(\frac{A_1}{S^*} \right) \left(\frac{S^*}{S_P} \right) \left(\frac{m_1}{m_0} \right) \quad (A7)$$

The net thermal efficiency of a propulsive system is given as

$$\eta = \frac{F_n V_0}{[(L. H. V.) w_f] J}$$

where

$$F_n = C_{F_n}' \frac{1}{2} \rho_0 V_0^2 A_0 = \frac{1}{2} C_{F_n}' (\rho_0 V_0 A_0) V_0 = \frac{1}{2} C_{F_n}' \frac{w_a}{g} V_0$$

thus

$$\eta = \frac{1}{2} \frac{C_{F_n}' V_0^2}{(L. H. V.) g J} \frac{w_a}{w_f} \quad (A8)$$

where w_a/w_f is the weight ratio of air to fuel for the complete installation.

If fuel is added at stations A and B and the local w_a/w_f ratios are known, it can be shown that

$$\left(\frac{w_a}{w_f} \right)_{total} = \frac{\left[\left(\frac{w_a}{w_f} \right)_A + 1 \right] \left[\left(\frac{w_a}{w_f} \right)_B + 1 \right]}{\left(\frac{w_a}{w_f} \right)_A + \left(\frac{w_a}{w_f} \right)_B + 2} - 1$$

since for one pound of mixture (see ref. 5)

$$\left(\frac{w_f}{w_a + w_f} \right)_{total} = \left(\frac{w_f}{w_a + w_f} \right)_A + \left(\frac{w_f}{w_a + w_f} \right)_B$$

Similarly, solving for the air-to-fuel ratio necessary to make up a total ratio,

$$\left(\frac{w_a}{w_f} \right)_B = \frac{\left[\left(\frac{w_a}{w_f} \right)_A + 1 \right] \left[\left(\frac{w_a}{w_f} \right)_{total} + 1 \right]}{\left(\frac{w_a}{w_f} \right)_A - \left(\frac{w_a}{w_f} \right)_{total}} - 1 \quad (A9)$$

APPENDIX B

RELATION BETWEEN ENGINE REQUIREMENTS AND THE OPTIMUM INLET AREAS AT VARIOUS FLIGHT CONDITIONS

The corrected air flow given by the relation

$$(w_a)_{corr. c} = \left(w_a \frac{\sqrt{\theta_3}}{\delta_3} \right)_{M_a}$$

can be plotted in the form of a ratio to the rated corrected value

$$(w_a)_{st} = \left(w_a \frac{\sqrt{\theta_3}}{\delta_3} \right)_{st}$$

Thus,

$$\frac{(w_a)_{corr. c}}{(w_a)_{st}} = \frac{\left(w_a \frac{\sqrt{\theta_3}}{\delta_3} \right)_{M_a}}{\left(w_a \frac{\sqrt{\theta_3}}{\delta_3} \right)_{st}}$$

or

$$(w_a)_{M_a} = \left(w_a \frac{\sqrt{\theta_3}}{\delta_3} \right)_{st} \frac{(w_a)_{corr. c}}{(w_a)_{st}} \left(\frac{\delta_3}{\sqrt{\theta_3}} \right)_{M_a} \quad (B1)$$

where $(w_a)_{M_a} = g \rho_0 V_0 A_0$ is the actual weight rate of air flow through the engine at a given Mach number. For any two Mach numbers M_{a1} and M_{a2} the weight rates are thus related by the ratio

$$\frac{(w_a)_{M_{a1}}}{(w_a)_{M_{a2}}} = \frac{\left[\frac{(w_a)_{corr. c}}{(w_a)_{st}} \left(\frac{\delta_3}{\sqrt{\theta_3}} \right) \right]_{M_{a1}}}{\left[\frac{(w_a)_{corr. c}}{(w_a)_{st}} \left(\frac{\delta_3}{\sqrt{\theta_3}} \right) \right]_{M_{a2}}} \quad (B2)$$

for a given engine.

The weight flow in the free stream through an area equal to that of inlet passage is given by the relation $g \rho_0 V_0 A_1 = g \rho_0 M_0 A_1 a_0$ and the weight entering the inlet is given by

$$(w_a)_{M_a} = \left[(g \rho_0 M_0 A_1 a_0) \frac{m_1}{m_0} \right]_{M_a} \quad (B3)$$

At any two Mach numbers M_{a1} and M_{a2}

$$\frac{(w_a)_{M_{a1}}}{(w_a)_{M_{a2}}} = \frac{\left(g \rho_0 M_0 A_1 a_0 \frac{m_1}{m_0} \right)_{M_{a1}}}{\left(g \rho_0 M_0 A_1 a_0 \frac{m_1}{m_0} \right)_{M_{a2}}} \quad (B4)$$

If the two static pressures and temperatures are equal (fixed standard altitude)

$$\frac{(w_a)_{M_{a_1}}}{(w_a)_{M_{a_2}}} = \frac{\left(A_1 M_0 \frac{m_1}{m_0}\right)_{M_{a_1}}}{\left(A_1 M_0 \frac{m_1}{m_0}\right)_{M_{a_2}}} \quad (B5)$$

The flow through the inlet must be equal to the flow through the engine for a matched condition. Thus,

$$\frac{\left(A_1 M_0 \frac{m_1}{m_0}\right)_{M_{a_1}}}{\left(A_1 M_0 \frac{m_1}{m_0}\right)_{M_{a_2}}} = \frac{\left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{\delta_3}{\sqrt{\theta_3}}\right]_{M_{a_1}}}{\left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{\delta_3}{\sqrt{\theta_3}}\right]_{M_{a_2}}}$$

since $\delta_3 = \frac{p_{t_3}}{p_0} \frac{p_0}{p_{sl}}$ and $\frac{p_0}{p_{sl}} = \text{constant}$ for a fixed altitude,

$$\frac{(A_1)_{M_{a_2}}}{(A_1)_{M_{a_1}}} = \frac{\left[M_0 \left(\frac{m_1}{m_0}\right)_{opt}\right]_{M_{a_1}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_0} \sqrt{\frac{1}{\theta_3}}\right]_{M_{a_2}}}{\left[M_0 \left(\frac{m_1}{m_0}\right)_{opt}\right]_{M_{a_2}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_0} \sqrt{\frac{1}{\theta_3}}\right]_{M_{a_1}}} \quad (B6)$$

or

$$\frac{(A_1)_{M_2}}{(A_1)_{M_1}} = \frac{\left[M_0 \left(\frac{m_1}{m_0}\right)_{opt}\right]_{M_{a_1}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_0} \frac{p_{t_0}}{p_0} \sqrt{\frac{1}{\theta_3}}\right]_{M_{a_2}}}{\left[M_0 \left(\frac{m_1}{m_0}\right)_{opt}\right]_{M_{a_2}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_0} \frac{p_{t_0}}{p_0} \sqrt{\frac{1}{\theta_3}}\right]_{M_{a_1}}}$$

The values of $\frac{(w_a)_{corr. c}}{(w_a)_{sl}}$ are given by the engine (compressor) characteristics for corrected compressor speed $n_{corr. c}$, which may be found using figures 3 and 5. The rest of the terms are determined from the inlet characteristics and flight conditions with the help of figures 2, 3, and 5.

For flight at different altitudes and Mach numbers it can be shown, similarly to equation (B6), that

$$\frac{(A_1)_{M_{a_2}}}{(A_1)_{M_{a_1}}} = \frac{\left[M_0 \left(\frac{m_1}{m_0}\right)_{opt} \frac{p_0}{p_{t_0}}\right]_{M_{a_1}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_{t_0}} \frac{a_0}{a_{t_0}}\right]_{M_{a_2}}}{\left[M_0 \left(\frac{m_1}{m_0}\right)_{opt} \frac{p_0}{p_{t_0}}\right]_{M_{a_2}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_{t_0}} \frac{a_0}{a_{t_0}}\right]_{M_{a_1}}} \quad (B7)$$

where $a_t = \sqrt{\gamma g R T_t}$ is the speed of sound at stagnation conditions.

The free-stream area A_0 required by the engine at various flight conditions can be obtained using equation (B7) by

letting $\left(\frac{m_1}{m_0}\right)_{opt} = 1$, and $A_1 = A_0$. Thus

$$\frac{(A_0)_{M_2}}{(A_0)_{M_1}} = \frac{\left(M_0 \frac{p_0}{p_{t_0}}\right)_{M_{a_1}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_{t_0}} \frac{a_0}{a_{t_0}}\right]_{M_{a_2}}}{\left(M_0 \frac{p_0}{p_{t_0}}\right)_{M_{a_2}} \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \frac{p_{t_3}}{p_{t_0}} \frac{a_0}{a_{t_0}}\right]_{M_{a_1}}} \quad (B8)$$

where $\frac{p_0}{p_{t_0}} = \frac{1}{p_{t_3}/p_0}$ when $\frac{p_{t_3}}{p_0} = 1.0$. Quadrant II of figure 2 gives $\frac{p_{t_3}}{p_0}$ for M_0 .

$$\frac{a_0}{a_t} = \frac{a_0}{a_{sl}} \frac{a_{sl}}{a_t} = \frac{a_0}{a_{sl}} \sqrt{\frac{1.4 \times 519}{\gamma T_t}} = \frac{a_0}{a_{sl}} \sqrt{\frac{1}{\theta}}$$

where

$\frac{a_0}{a_{sl}}$ is the ratio of speed of sound at static temperature for flight altitude to that at standard sea-level conditions.

This ratio can be found from quadrant I of figure 2.

$\sqrt{\frac{1}{\theta}}$ is given in figure 5 for $T_t \equiv T_{t_0-3}$ which can be found from quadrants I and II of figure 3.

APPENDIX C

RELATION BETWEEN INLET MASS-FLOW RATIO AND THE ENGINE AIR REQUIREMENTS AT A FIXED MACH NUMBER AND ALTITUDE

The mass-flow ratio, by definition, is given by

$$\frac{m_1}{m_0} = \frac{\rho_1 V_1 A_1}{\rho_0 V_0 A_1}$$

The mass of air flowing through the inlet is then

$$\rho_1 V_1 A_1 = \frac{m_1}{m_0} \rho_0 V_0 A_1$$

The weights of air flowing through the inlet and the engine are equal and are given by

$$w_a = \frac{m_1}{m_0} g \rho_0 V_0 A_1 = (w_a)_{corr. c} \left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}$$

and

$$\left(\frac{m_1}{m_0}\right)_{opt} g \rho_0 V_0 A_1 = \left[(w_a)_{corr. c} \left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}\right]_{opt}$$

Thus, the ratio of mass flows at different m_1/m_0 is given by

$$\frac{\frac{m_1}{m_0}}{\left(\frac{m_1}{m_0}\right)_{opt}} = \frac{\left[(w_a)_{corr. c} \left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}\right]_{opt}}{\left[(w_a)_{corr. c} \left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}\right]_{opt}}$$

or

$$\frac{\frac{m_1}{m_0}}{\left(\frac{m_1}{m_0}\right)_{opt}} = \frac{\left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}\right]_{opt}}{\left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}\right]_{opt}} \quad (C1)$$

Solving for $\frac{(w_a)_{corr. c}}{(w_a)_{sl}}$ and rearranging terms, one obtains

$$\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \bigg|_{\frac{m_1}{m_0}} = \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}}\right]_{opt} \frac{\frac{m_1}{m_0}}{\left(\frac{m_1}{m_0}\right)_{opt}} \frac{\left[\left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}\right]_{opt}}{\left(\frac{\delta_3}{\sqrt{\theta_3}}\right)_{M_a}} \quad (C2)$$

For a given Mach number and altitude the total temperature T_{t_0-3} is constant, hence

$$[(\sqrt{\theta_3})_{M_a}]_{opt} = (\sqrt{\theta_3})_{M_a}$$

By definition,

$$\delta_3 = p_{t_3} / p_{sl}$$

or

$$\delta_3 = \frac{p_{t_3} p_{t_0}}{p_{t_0} p_{sl}} = \frac{p_{t_3}}{p_{t_0}} \frac{p_{t_0}}{p_{sl}}$$

where $\frac{p_{t_0}}{p_{sl}} = \text{constant}$ for a given M_0 and altitude. Thus,

$$\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \left| \frac{m_1}{m_0} \right| = \left[\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \right]_{opt} \frac{\frac{m_1}{m_0}}{\left(\frac{m_1}{m_0} \right)_{opt}} \frac{\left(\frac{p_{t_3}}{p_{t_0}} \right)}{\left(\frac{p_{t_3}}{p_{t_0}} \right)_{opt}} \quad (C3)$$

Using $\frac{(w_a)_{corr. c}}{(w_a)_{sl}} \left| \frac{m_1}{m_0} \right|$, the compressor corrected speed $n_{corr. c}$ can

be found from the engine characteristics. The actual engine speed n can be found, once $\sqrt{\theta_3}$ is known, since $n = n_{corr. c} \sqrt{\theta_3}$.

APPENDIX D

SELECTION OF COMPRESSOR PRESSURE RATIO

The compressor pressure ratio p_{t_4}/p_{t_3} , as used in this report, has been selected using the maximum obtainable internal thrust coefficient as a criterion. Figures 2, 3, and 4 were used together with the equation for the internal thrust coefficient C_{F_i} (Appendix A, eq. (A3)). The calculations were based on the following principles:

1. In order to segregate the effects of variation of engine parameters from the effects of altitude and pressure recovery, the conditions of the isothermal region of the atmosphere were used and isentropic recovery was assumed.

2. The combined efficiency of the compressor and turbine $\eta_c \eta_t$ was assumed to be 100 percent in all computations, and a combustion temperature of 2000° R was used.

3. The internal thrust coefficient C_{F_i}' was determined for assumed Mach numbers of flight. The exhaust nozzle was considered 100 percent efficient.

The results of these calculations are shown in figure 15 for the turbojet without afterburning. The points of maximum internal thrust coefficients have been joined by a curve which indicates the compressor pressure ratios necessary to obtain optimum operation. The data of figure 15 are idealized since the variation in compressor efficiency with temperature was not included ($\eta_c \eta_t$ was assumed to be 100 percent).

The optimum pressure ratios, however, are not affected appreciably by the combined efficiency; the thrust coefficients, on the contrary, are largely dependent on efficiency of every component of an installation. Pressure ratios of the compressor of figure 7 at various Mach numbers also are shown in figure 15. The value of $p_{t_4}/p_{t_3} = 6.25$ at $M_0 = 1.4$ was selected as one that would produce approximately maximum internal thrust coefficient without afterburning at Mach numbers less than 2.0. Figure 16, which presents data similar to those of figure 15 but with afterburning, shows that the same engine using afterburning is capable of producing nearly maximum internal thrust coefficients in the range of Mach numbers between 2.0 and 3.0. Again, the amount of afterburning was controlled so that total fuel consumption would be equal to that of the ram jet of figure 6.

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TABLE I.—SAMPLE CALCULATIONS FOR A TURBOJET ENGINE WITH AND WITHOUT AFTERBURNING.

Item	Source	Units	Turbojet without afterburning	Turbojet with afterburning
V_0	Fig. 2, quadrant I.....	$\frac{\text{ft}}{\text{sec}}$	1940	1940
p_{t_3}/p_{t_0}	Induction-system characteristics.....	None.....	* 0.80	* 0.80
p_{t_3}/p_0	Fig. 2, quadrant II.....	None.....	6.25	6.25
T_{t_0-3}	Fig. 3, quadrants I and II.....	° R.....	708	708
$\sqrt{\frac{1}{\theta_3}}$	Fig. 5, for T_{t_3}	None.....	0.858	0.858
n	Actual engine speed.....	rpm.....	* 12,500	* 12,500
$n_{corr. c}$	$n \sqrt{\frac{1}{\theta_3}}$	rpm.....	* 10,720	* 10,720
η_c	Compressor characteristics for $n_{corr. c}$, fig. 7.....	None.....	* 0.900	* 0.900
p_{t_4}/p_{t_3}	Compressor characteristics for $n_{corr. c}$, fig. 7.....	None.....	* 4.56	* 4.56
T_{t_4}	Fig. 3 for p_{t_4}/p_{t_3} and T_{t_0-3}	° R.....	1060	1060
T_{t_5}	Engine characteristics for $n_{corr. c}$, fig. 7.....	° R.....	* 2000	* 2000
$\sqrt{\frac{1}{\theta_5}}$	Fig. 5 for T_{t_5}	None.....	0.522	0.522
$n_{corr. t}$	$n \sqrt{\frac{1}{\theta_5}}$	rpm.....	6530	6530
η_t	Turbine characteristics for $n_{corr. t}$, fig. 7.....	None.....	* 0.900	* 0.900
p_{t_6}/p_{t_5}	Fig. 4 for T_{t_5} , $\eta_c \eta_t$, and T_{t_5}	None.....	2.50	2.50
T_{t_6}	Fig. 4 for p_{t_6}/p_{t_5} and T_{t_5}	° R.....	1584	1584
$\left(\frac{w_a}{w_f} \right)_s$	Fig. 3 for T_{t_4} and T_{t_5}	lb air lb fuel	77.06	77.06
$\left(\frac{w_a}{w_f} \right)$ total at station 8.	Assumed equal to that of ram jet operating at 3000° R.	lb air lb fuel	-----	* 29.6
$\left(\frac{w_a}{w_f} \right)$ added at station 8.	Equation (A8) of appendix A.....	lb air lb fuel	-----	49.29
T_{t_0}	Fig. 3, quadrant IV, for T_{t_6} and $\left(\frac{w_a}{w_f} \right)$ added at station 8.	° R.....	1584	2926
p_{t_0}/p_0	$\left(\frac{p_{t_6}}{p_0} \right) \left(\frac{p_{t_4}}{p_{t_3}} \right) \left(\frac{p_{t_5}}{p_{t_4}} \right) \left(\frac{p_{t_6}}{p_{t_5}} \right) \left(\frac{p_{t_7}}{p_{t_6}} \right)$ $\left(\frac{p_{t_3}}{p_{t_7}} \right) \left(\frac{p_{t_9}}{p_{t_8}} \right)$	None.....	11.30	11.30
V_{t1}	Fig. 2, quadrants III and IV, and $\left(\frac{p_{t_0}}{p_0} \right)$ and T_{t_0}	$\frac{\text{ft}}{\text{sec}}$	3118	4297
C_{F_i}'	$2 \left[\left(1 + \frac{w_f}{w_a} \right) \frac{V_{t1}}{V_0} - 1 \right]$	None.....	1.256	2.520

* Values for assumed condition of operation.

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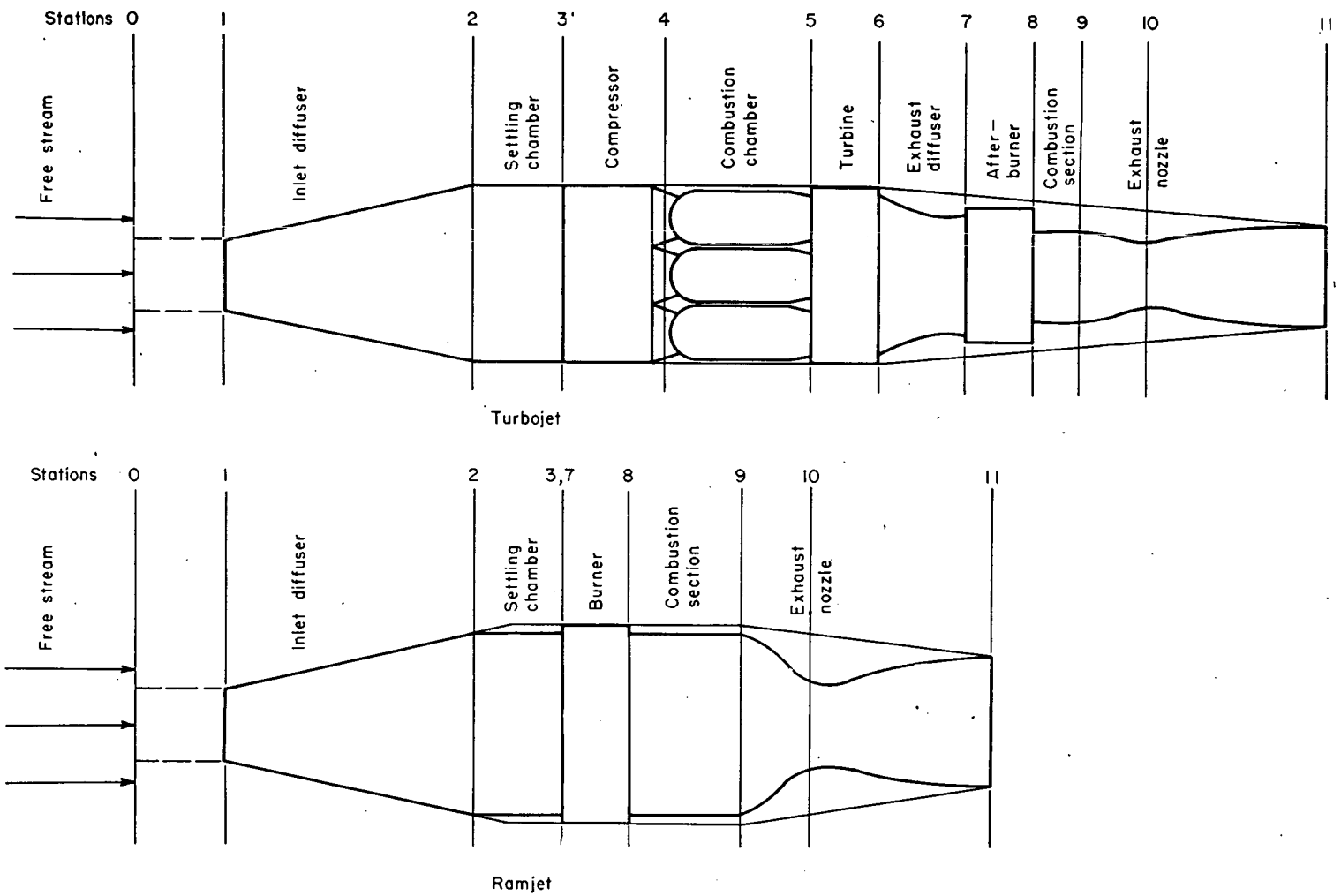


FIGURE 1.—Schematic representation of propulsive systems.

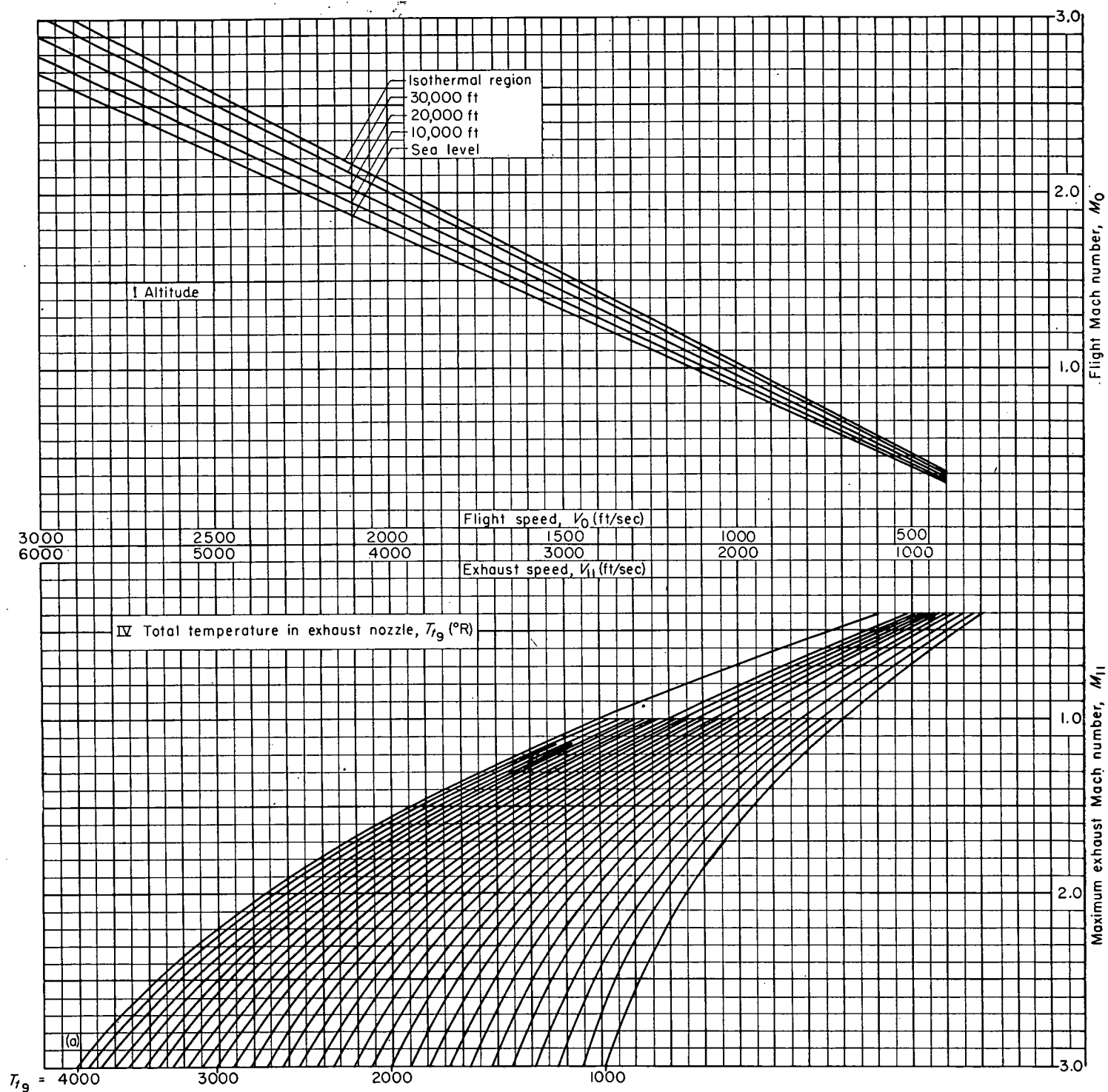


FIGURE 2.—Velocity graph.

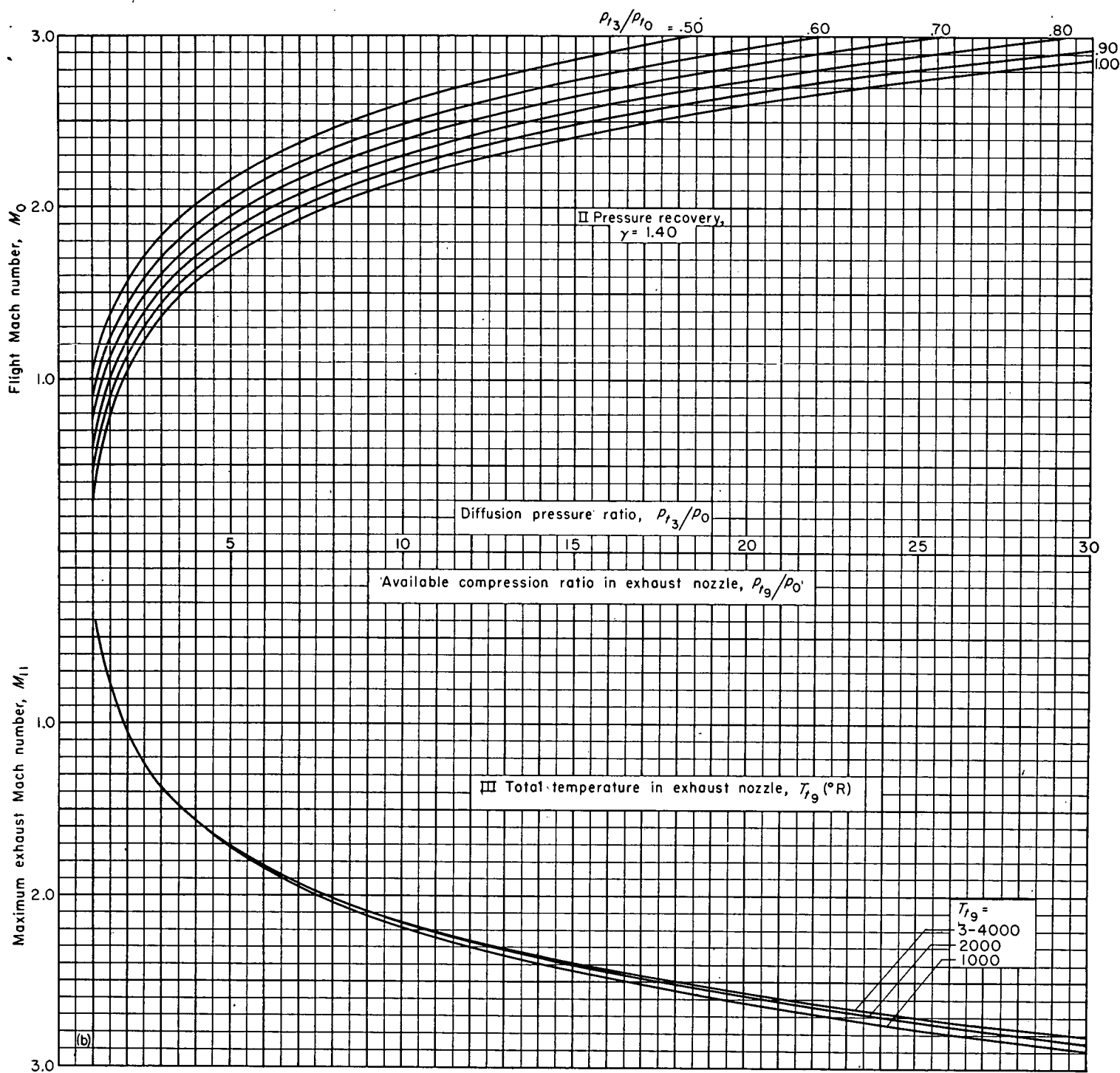


FIGURE 2.—Concluded.

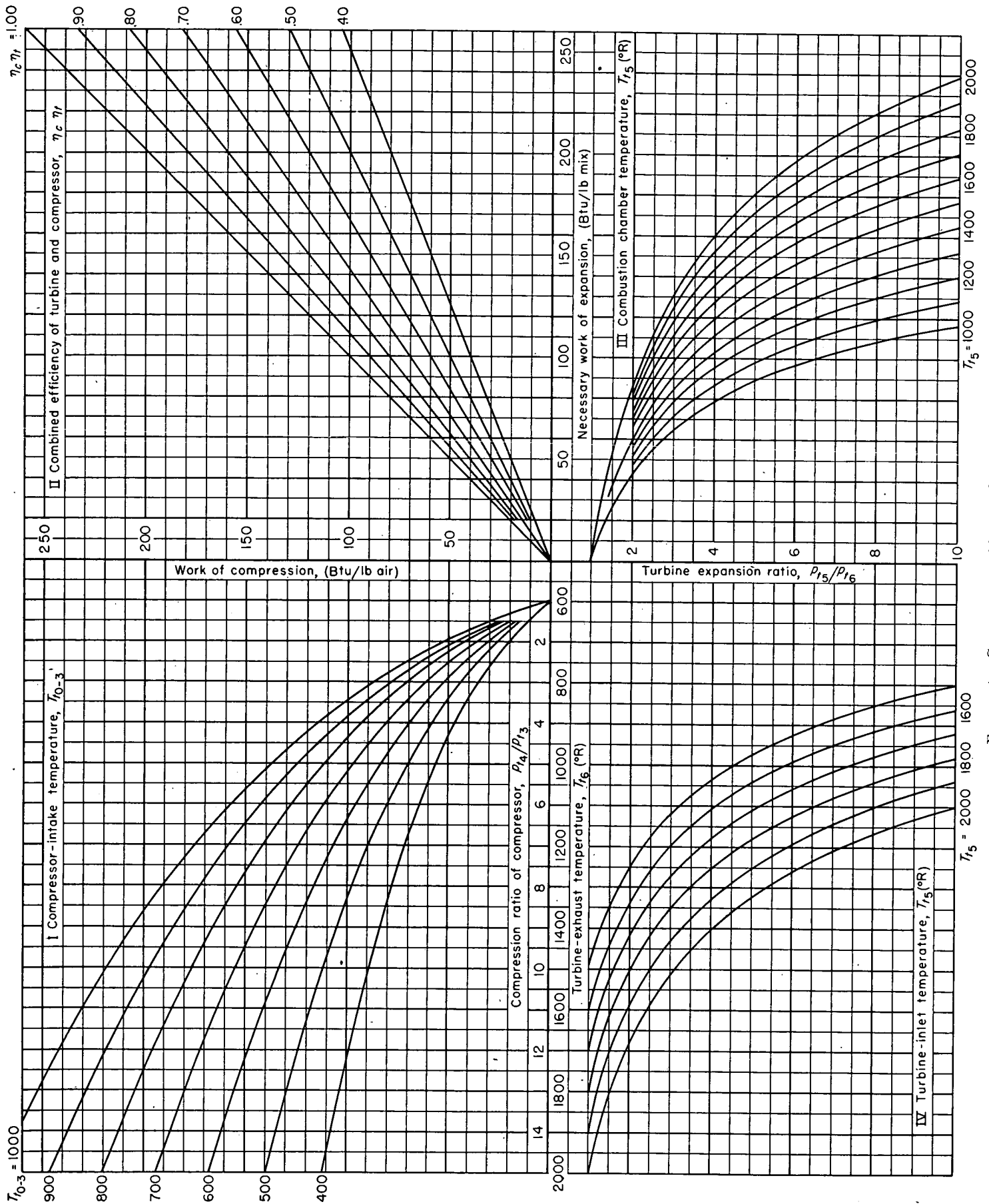


FIGURE 4.—Compressor-turbine graph.

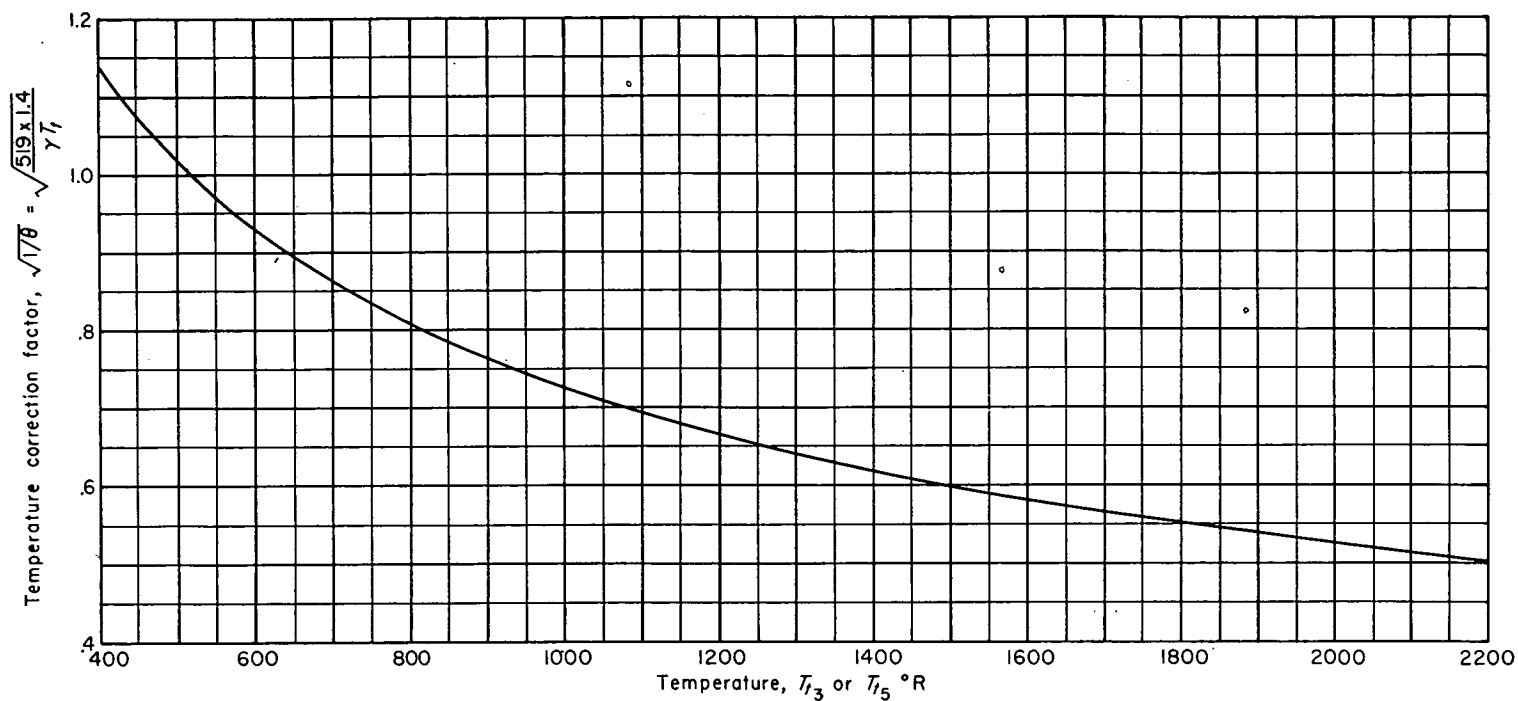


FIGURE 5.—Temperature correction graph.

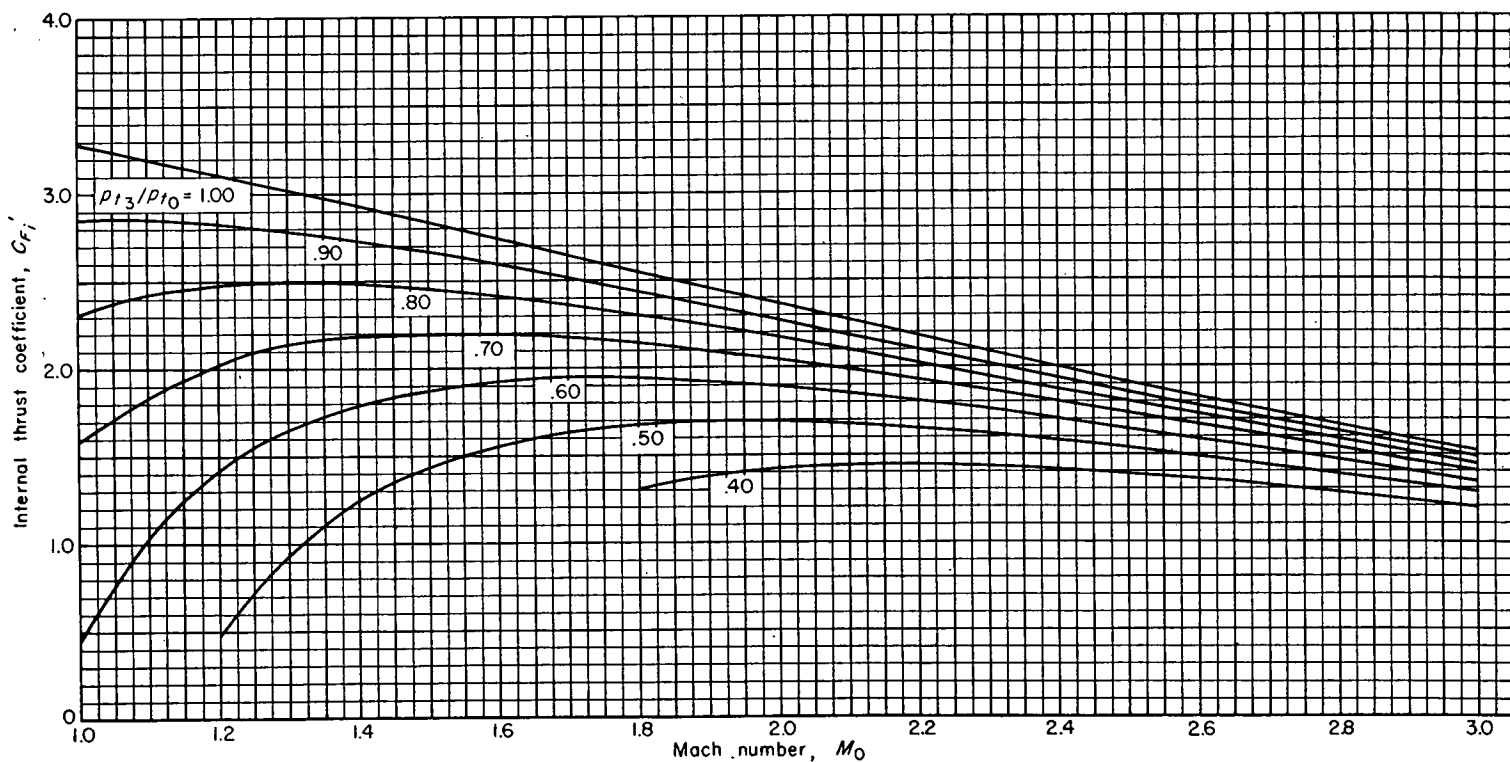


FIGURE 6.—Effects of total-pressure recovery on internal thrust coefficient of a ram jet operating at 3000° Rankine in the isothermal region of atmosphere.

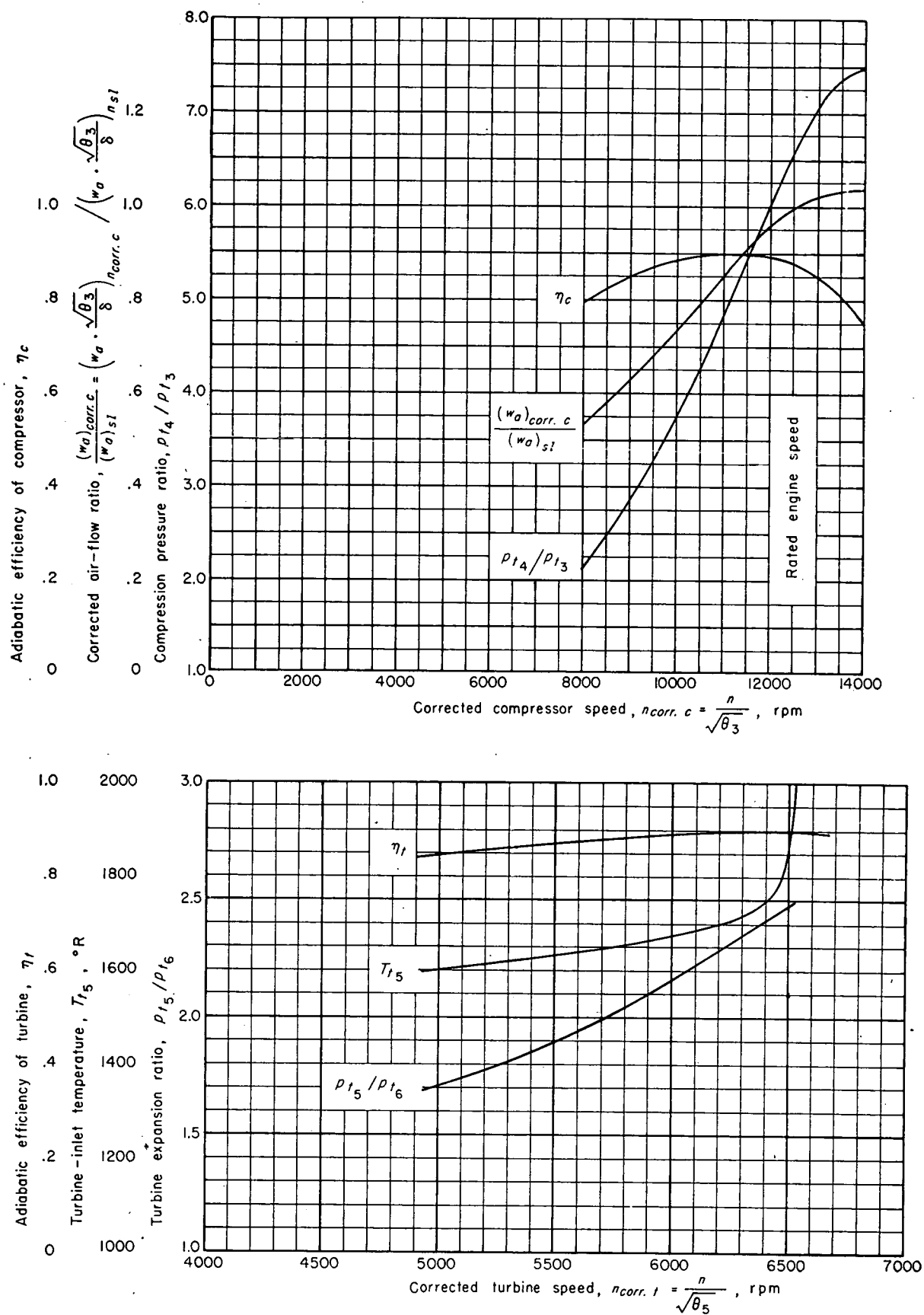


FIGURE 7.—Turbojet-engine characteristics corrected to standard sea-level static atmospheric conditions.

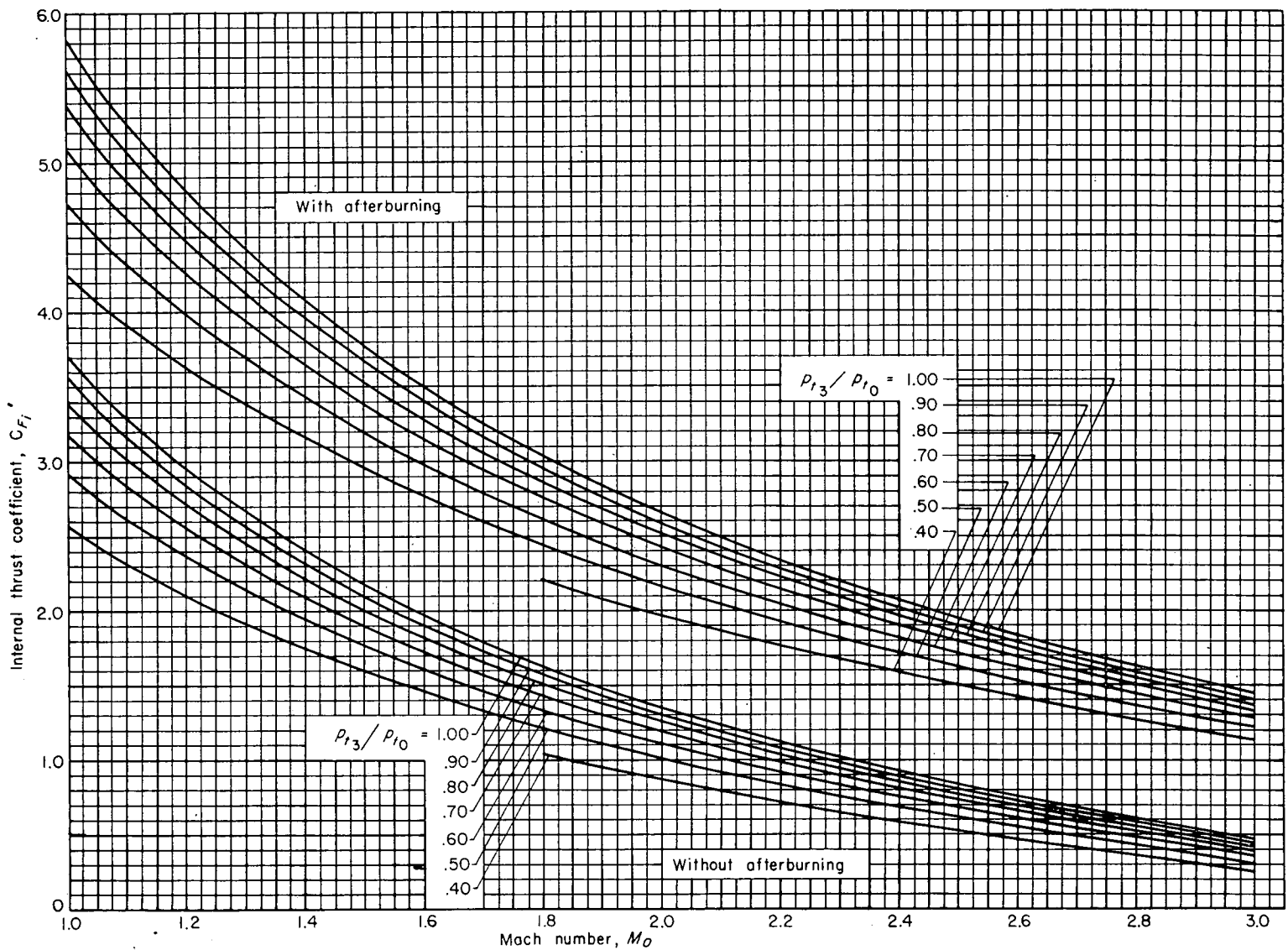


FIGURE 8.—Effects of total-pressure recovery on internal thrust coefficient of turbojet of figure 7 operating in the isothermal region of the atmosphere at 2000° Rankine without afterburning and the same turbojet with afterburning having a total air-to-fuel ratio equal to that of the ram jet of figure 6.

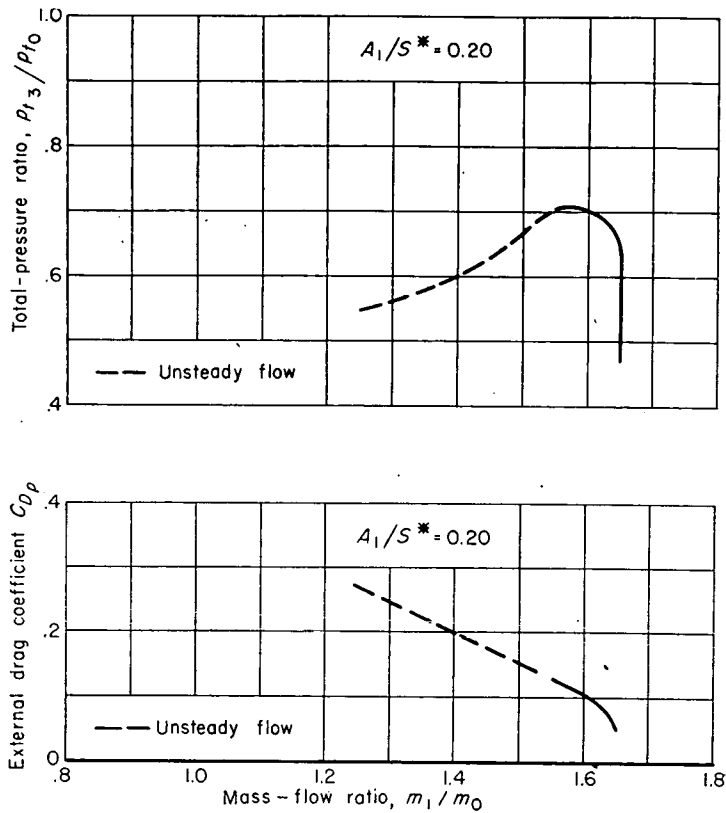


FIGURE 9.—Induction-system characteristics at Mach number 2.5.

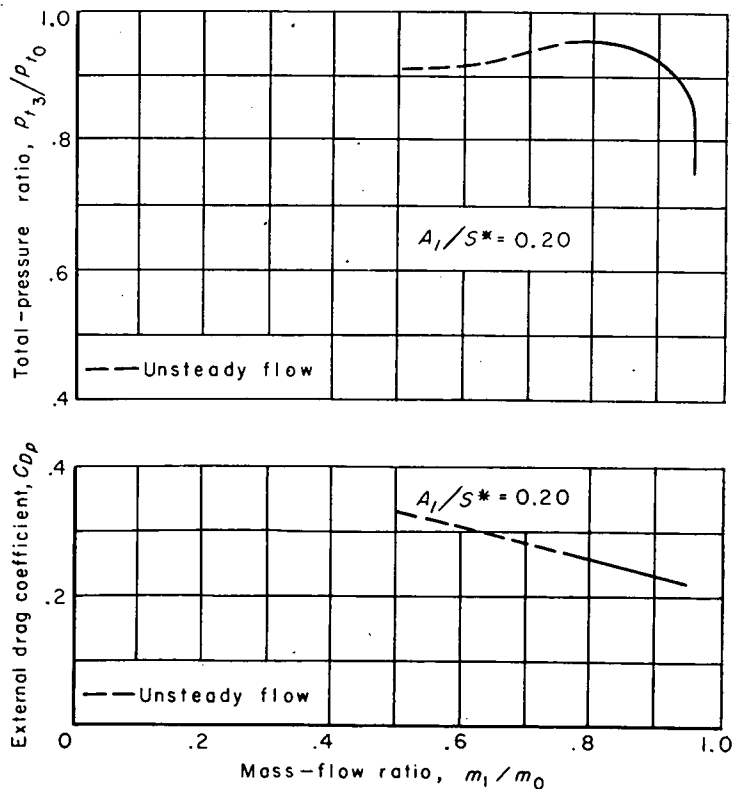


FIGURE 10.—Induction-system characteristics at Mach number 1.2.

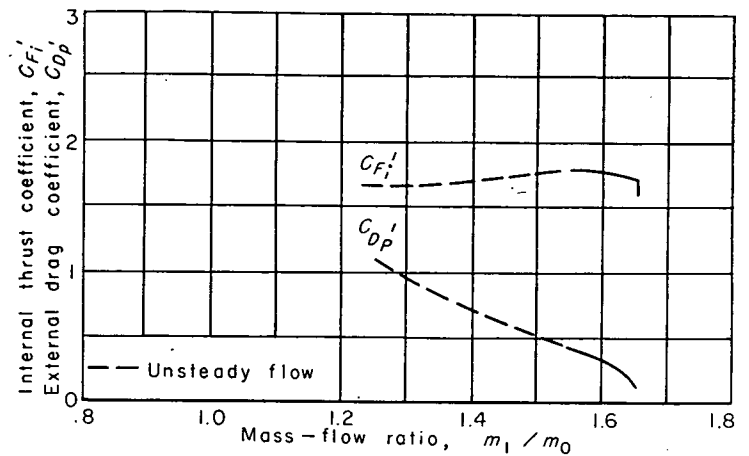


FIGURE 11.—Effects of mass-flow ratio on thrust and drag coefficients.

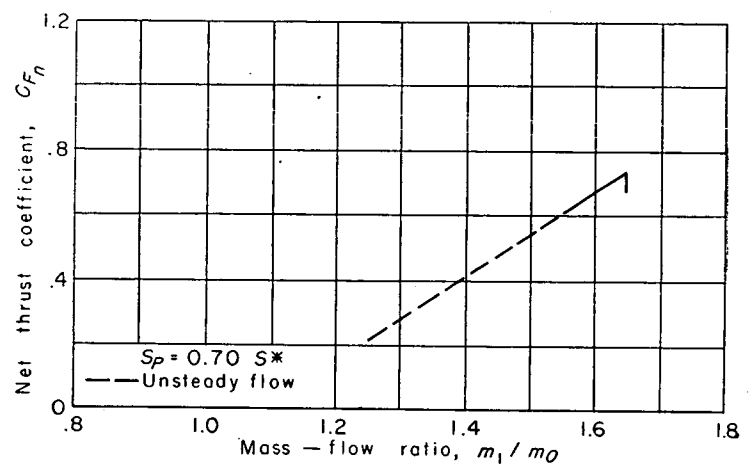


FIGURE 12.—Effects of mass-flow ratio on net thrust coefficient at Mach number 2.5 in the isothermal region of the atmosphere.

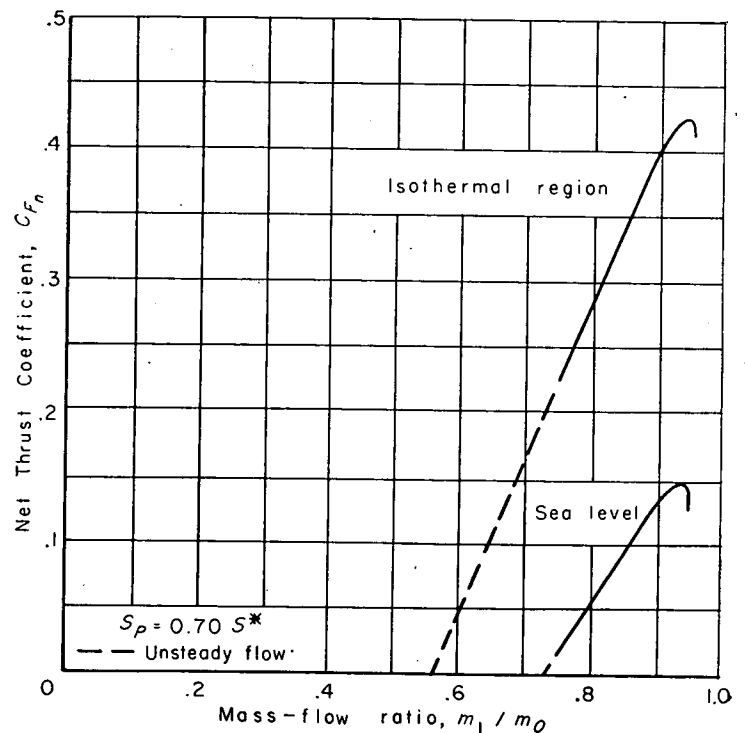
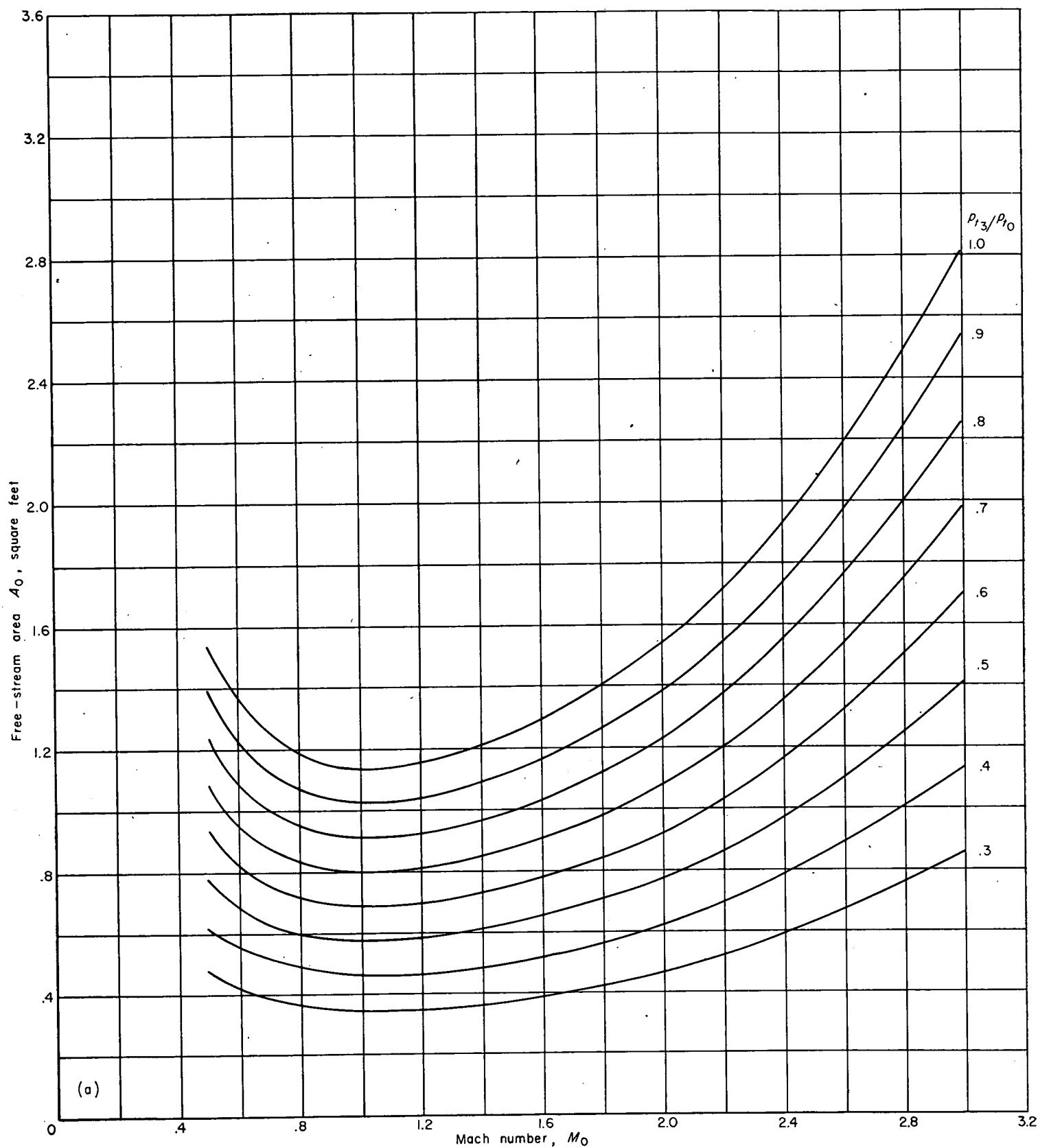
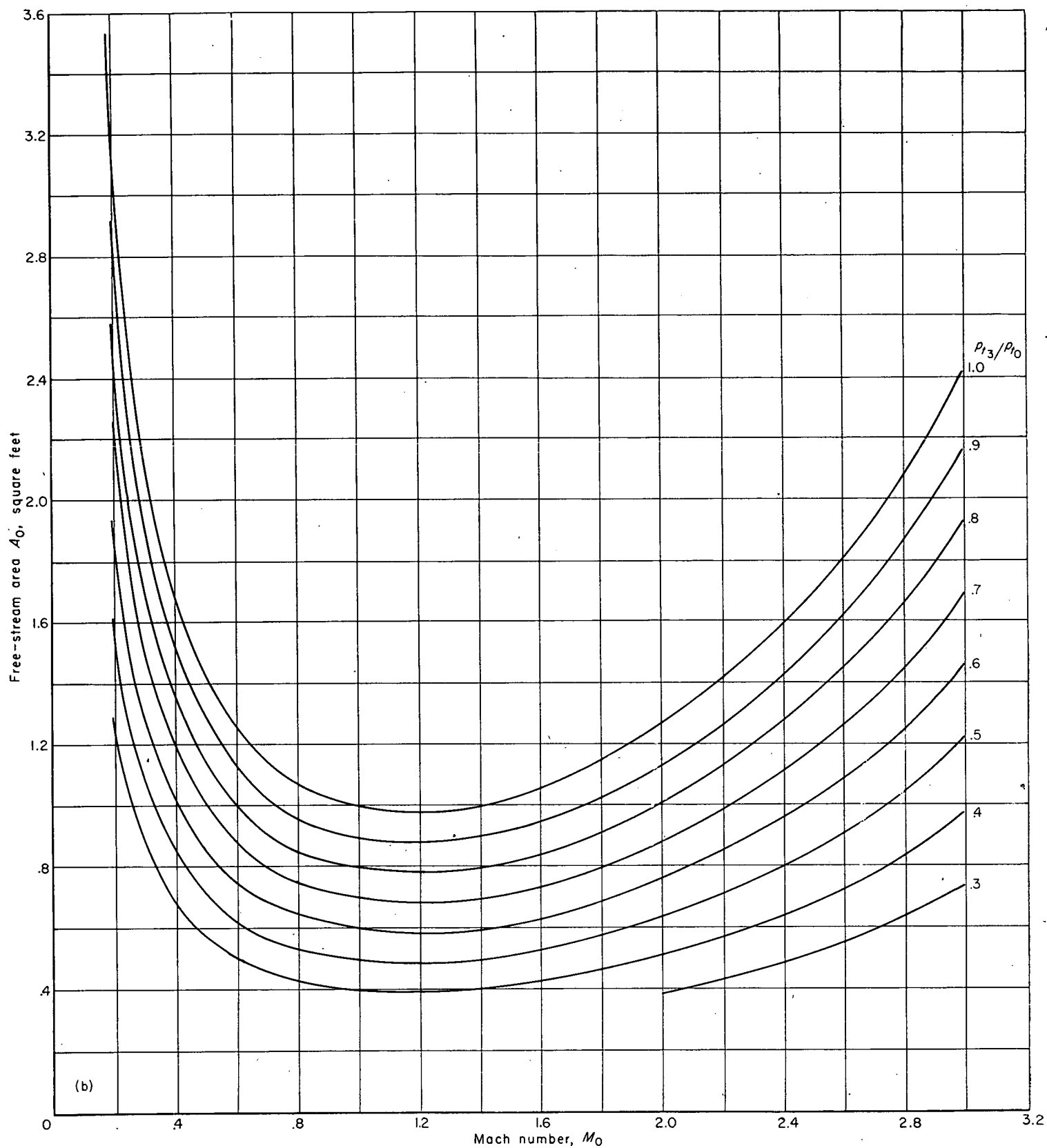


FIGURE 13.—Effects of mass-flow ratio on net thrust coefficient at Mach number 1.2.



(a) Isothermal region.

FIGURE 14.—Free-stream area required by the turbojet engine of the illustrative examples.



(b) Sea level, standard atmosphere.

FIGURE 14.—Concluded.

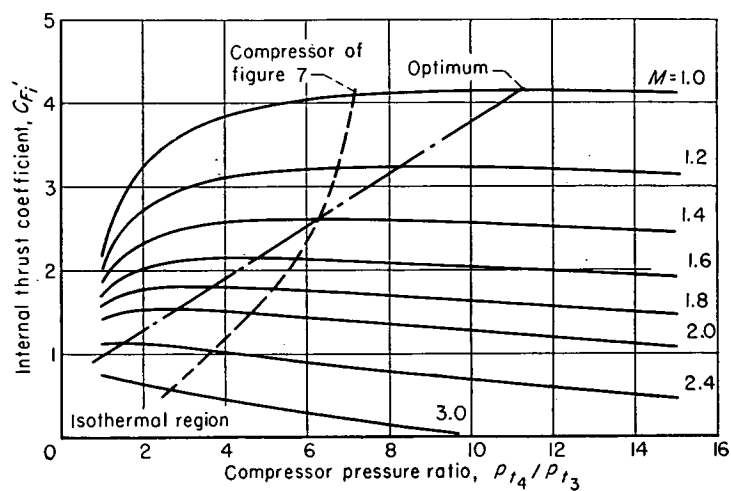


FIGURE 15.—Effects of compressor pressure ratio on internal thrust coefficient of turbojet without afterburning.

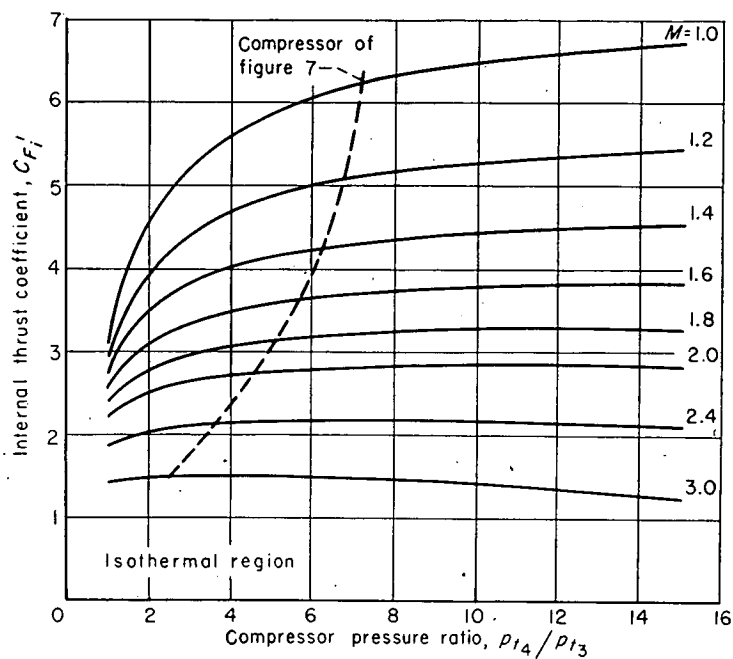


FIGURE 16.—Effects of compressor pressure ratio on internal thrust coefficient of turbojet with afterburning.

REPORT 1142

DIFFUSION OF HEAT FROM A LINE SOURCE IN ISOTROPIC TURBULENCE¹

By MAHINDER S. UBEROI and STANLEY CORRISIN

SUMMARY

An experimental and analytical study has been made of some features of the turbulent heat diffusion behind a line heated wire stretched perpendicular to a flowing isotropic turbulence. The mean temperature distributions have been measured with systematic variations in wind speed, size of turbulence-producing grid, and downstream location of heat source. The nature of the temperature fluctuation field has been studied.

A comparison of Lagrangian and Eulerian analyses for diffusion in a nondecaying turbulence yields an expression for turbulent-heat-transfer coefficient in terms of turbulence velocity and a Lagrangian "scale."

The ratio of Eulerian to Lagrangian microscale has been determined theoretically by generalization of a result of Heisenberg and, with arbitrary constants taken from independent sources, shows rough agreement with experimental results.

A convenient form has been deduced for the criterion of interchangeability of instantaneous space and time derivatives in a flowing turbulence.

INTRODUCTION

One of the most striking aspects of turbulent motion in fluids is its dispersive property. This "convective diffusion," illustrated by the general statistical tendency of (noncontiguous) fluid elements to get farther apart with increasing time, was probably first observed long before the era of analytical fluid mechanics. An analytical start on this problem was not made, however, until the now-classic work by Taylor in 1921 on diffusion by continuous movements (reference 1). Not only did this paper lay a groundwork for the study of turbulent diffusion but it also represented a forward step in the ideas essential to development of a general statistical theory of turbulence, a field which had scarcely progressed since Reynolds' original formulation of the equations of motion for a flow in which mean and fluctuating parts could be distinguished.

The diffusive action of a turbulent flow may manifest itself in various ways, depending upon the initial and/or boundary conditions and upon the interests of the observer. The following possible measures of the diffusive powers are neither exhaustive nor mutually independent:

- (1) The average rate of dispersion of particles from a fixed source
- (2) The average rate of increase of spacing between different particles

- (3) The average rate of transport of particle concentration under a given mean concentration gradient
- (4) The average rate of increase of the length of a fluid line
- (5) The average rate of increase of the area of a fluid surface

The word "particles" means simply indelibly tagged fluid elements, much smaller than the smallest length associated with the turbulence.

The present report is concerned primarily with measure (1). The measurements have all been made in the thermal wake of a long thin heated wire mounted perpendicular to an isotropic turbulent air flow and producing no turbulent wake. Here the tagging is thermal, and the degree of indelibility (negligibility of molecular diffusion) is one of the matters to be investigated.

The diffusive property (for scalars) of a turbulent flow is apparently a secondary characteristic at least in the sense that it need not explicitly enter the dynamical problem. The diffusion may be regarded as a kinematic phenomenon, to be deduced from the dynamical solution to the problem if and when the latter is obtained. Thus the objective of research on turbulent diffusion may be to seek a connection between the diffusive and the dynamical statistical variables, even before the complete dynamical theory is available.

Measure (3) is usually termed the "turbulent transport" or "transfer" problem. Although of extensive practical importance, it has not yet been subjected to genuine theoretical study.

Most of the semiempirical "theories" of turbulent transport, for both scalar and vector properties, employ an Eulerian formulation of the basic equations, and up to now they have been unable to relate the turbulent transport correlation to other statistical functions describing the flow. Taylor (reference 1) showed that in the simple case of a homogeneous field of isotropic turbulence, and even in a decaying isotropic turbulence (reference 2), a Lagrangian formulation of the transport (i. e., diffusion; the terms will be used interchangeably) problem leads to some important results.²

Up to the present time little theoretical or experimental work has been done to find relations, if any, between the Lagrangian statistical measures of a turbulent field with its Eulerian statistical measures. Since turbulence dynamics seems best handled in the latter terms and turbulent diffusion in the former, it is evident that such a connection is impor-

¹Supersedes NACA TN 2710, "Diffusion of Heat From a Line Source in Isotropic Turbulence" by Mahinder S. Uberoi and Stanley Corrsin, 1952.

²In his tensorial generalization of Taylor's work on case (1), Batchelor (reference 3) has chosen to call this an "Eulerian" analysis, describing only case (2) as "Lagrangian." In keeping with previously accepted nomenclature, both cases are Lagrangian, (1) involving a single particle and (2) involving a pair. In fact, case (2) might be termed a mixed (Eulerian and Lagrangian) problem.

tant. Hence, one of the purposes of the present experiments has been to compare the magnitudes of some of these quantities under variations in the turbulent field. For example, the postulates of Taylor and Heisenberg on a relation between Lagrangian and Eulerian microscales can be examined and, in corrected form, compared with experiment.

The turbulent diffusion from a fixed line source can be set up analytically as an ordinary (Eulerian) "heat-transfer" problem, permitting a start to be made in relating measures (1) and (3) of the diffusive power of a turbulent flow, under certain simplifying assumptions.

Measures (4) and (especially) (5) may well be classed as characteristic of the "turbulent-mixing" problem rather than diffusion in the common connotation.

Experimental work on diffusion from a fixed local source in a turbulent flow has been meager. In isotropic turbulence, there have been the measurements of Schubauer (reference 4), Simmons (reported by Taylor in reference 2), Dupuis (reported by Kampé de Fériet in reference 5), Frenkiel (reference 6), and Collis (reference 7). Of these, only the data of Simmons and Collis are extensive enough to permit confident computation of the Lagrangian correlation function. In turbulent shear flow, Skramstad and Schubauer (reference 8), Dryden (reference 9), and the present authors (reference 10) have measured distributions close to a source; Kalinske and Pien (reference 11) and Van Driest (reference 12) have made measurements somewhat farther downstream.

None of these studies was repeated with a systematic variation of the properties of the turbulence. In spite of the poor precision inherent in this type of measurement, it was hoped that such an approach would at least show up some general trends in the relations between Eulerian and Lagrangian variables.

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SYMBOLS

c	root-mean-square molecular velocity
c_p	specific heat per unit volume at constant pressure
$f(r)$	Eulerian velocity correlation coefficient (notation of Von Kármán and Howarth)
$H \equiv 2\rho c_p \bar{U} \int_0^\infty \bar{\theta} dy$	
$H^* \equiv \frac{H}{2\rho c_p \int_0^\infty f(\xi) d\xi}$	
j	width of rectangular heat pulse
k	thermal conductivity
k_T	turbulent-heat-transfer coefficient
L	Eulerian scale ($L \equiv \int_0^\infty f(r) dr$)

L_L	Lagrangian scale ($L_L \equiv \int_0^\infty R_r(\tau) d\tau$)
L_η	Lagrangian scale for nondecaying and decaying turbulence ($L_\eta \equiv \int_0^\infty R_\eta(\eta) d\eta$)
l	mixing length
M	grid mesh size
$M_1 \equiv \int_0^\infty \eta R_\eta(\eta) d\eta$	
$P(\vartheta)$	probability density of fluctuating temperature
p	static pressure
R_L	turbulence Reynolds number based on Eulerian scale ($R_L \equiv \frac{u' L}{\nu}$)
$R_{t-\tau} \equiv R_r(t, \tau) \equiv \frac{\overline{v(t)v(t-\tau)}}{\overline{v'(t)v'(t-\tau)}}$	
$R_r(\tau)$	Lagrangian correlation coefficient for nondecaying turbulence
$R_\eta(\eta)$	Lagrangian correlation as a function of η for nondecaying and decaying turbulence
$R_{\vartheta} \equiv \frac{\overline{\partial v}}{\overline{\partial U}}$	
R_λ	turbulence Reynolds number based on Eulerian microscale ($R_\lambda \equiv \frac{v' \lambda}{\nu}$)
r	scalar distance between two points
$S \equiv -\left(\frac{\partial v}{\partial y}\right)^3 \left/ \left[\left(\frac{\partial v}{\partial y}\right)^2 \right]^{3/2} \right.$	
s	average on-center spacing of pulses
$T \equiv \left[\left(\frac{dv}{dt}\right)^2 / \bar{U}^2 \left(\frac{\partial v}{\partial x}\right)^2 \right]^{1/2}$	
t	time
$\bar{U}$	mean velocity in x -direction
u	instantaneous velocity fluctuation in x -direction
$u' \equiv \sqrt{\overline{u^2}}$	
$\bar{V}$	mean velocity in y -direction
v	instantaneous velocity fluctuation in y -direction
$v' \equiv \sqrt{\overline{v^2}}$	
w	instantaneous velocity fluctuation in z -direction
X	distance traveled in x -direction by a fluid particle
x	distance downstream from grid
x_0	location of heating wire
$\Delta x = x - x_0$	
Y	distance traveled in y -direction by a fluid particle
Y'	root-mean-square displacement of a fluid particle in y -direction ($Y' \equiv \sqrt{\overline{Y^2}}$)
Y_m	root-mean-square displacement of a molecule
y	distance in direction of measured diffusion
z	distance in direction of heating wire
$\epsilon \equiv \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x}$	

$$\eta(t_o, t) \equiv \int_{t_o}^t v'(\xi) d\xi$$

θ instantaneous temperature (measured above ambient room temperature)

$\bar{\theta}$ mean temperature

$\bar{\theta}_o$ maximum mean temperature at a cross section, a function of Δx

ϑ instantaneous temperature difference

$$\vartheta' \equiv \sqrt{\vartheta^2}$$

ϑ_o temperature difference of rectangular heat pulse

κ dimensionless empirical constant

Λ mean free path of a molecule

λ Eulerian microscale of turbulence

$$\left(\lambda \equiv \left(\frac{1}{-f''(0)} \right)^{1/2} \right)$$

λ_o Lagrangian microscale of turbulence for non-decaying turbulence $\left(\lambda_o \equiv \left(\frac{2}{-R_o''(0)} \right)^{1/2} \right)$

λ_η Lagrangian microscale of turbulence for nondecaying and decaying turbulence $\left(\lambda_\eta \equiv \left(\frac{2}{-R_\eta''(0)} \right)^{1/2} \right)$

ν kinematic viscosity

$$\xi \equiv \frac{y}{Y'(x)}$$

ρ density

τ time difference

(-) mean value or ensemble average

Subscripts:

max maximum

min minimum

EQUIPMENT AND PROCEDURE

AERODYNAMIC EQUIPMENT

The wind tunnel (fig. 1) is an open-return NPL type tunnel with a 2- by 2-foot working section and a free-stream turbulence level of $v'/\bar{U}=0.06$ percent and $u'/\bar{U}=0.05$ percent at a mean velocity of 26 feet per second. The turbulence-producing grids were as follows:

Designation	Type	Mesh size (in.)	Rod diameter (in.)	Solidity
1-in. grid	Biplane, wood	1.00	0.25	0.437
1/2-in. grid	Woven, metal	.50 by 0.53	.120	.41
1/4-in. grid	Woven, metal	.25	.060	.42

* The 0.50-in. mesh was set in direction of measured diffusion.

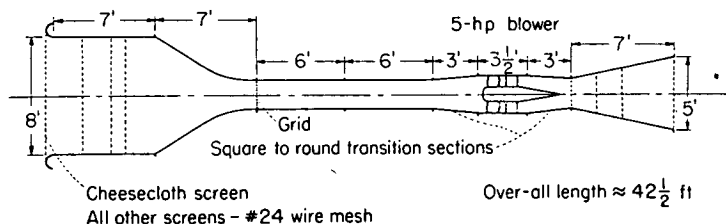


FIGURE 1.—Sketch of open-return wind tunnel.

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They were mounted in turn at the upstream end of the working section.

The heat source was an 0.008-inch-diameter platinum wire stretched vertically across the tunnel at various distances from the grid. It was heated by direct current to temperatures between 500° and 700° C, with the latter figure only at the highest operating velocity of 38.0 feet per second. The wire Reynolds numbers at this condition and at the two other velocities were as follows:

Mean speed (fps)	Wire temperature (°C)	Reynolds number based on—	
		Air temperature	Wire temperature
8.5	500	35	6.4
25.6	500	105	19
38.0	700	157	19

A preliminary investigation was made without grids to insure that these operating conditions did not generate a vortex street downstream of the heated wire.

With the grids in place, the mean momentum wake became practically undetectable with total-head tube and manometer at distances greater than 1 or 2 inches downstream of the heating wire.

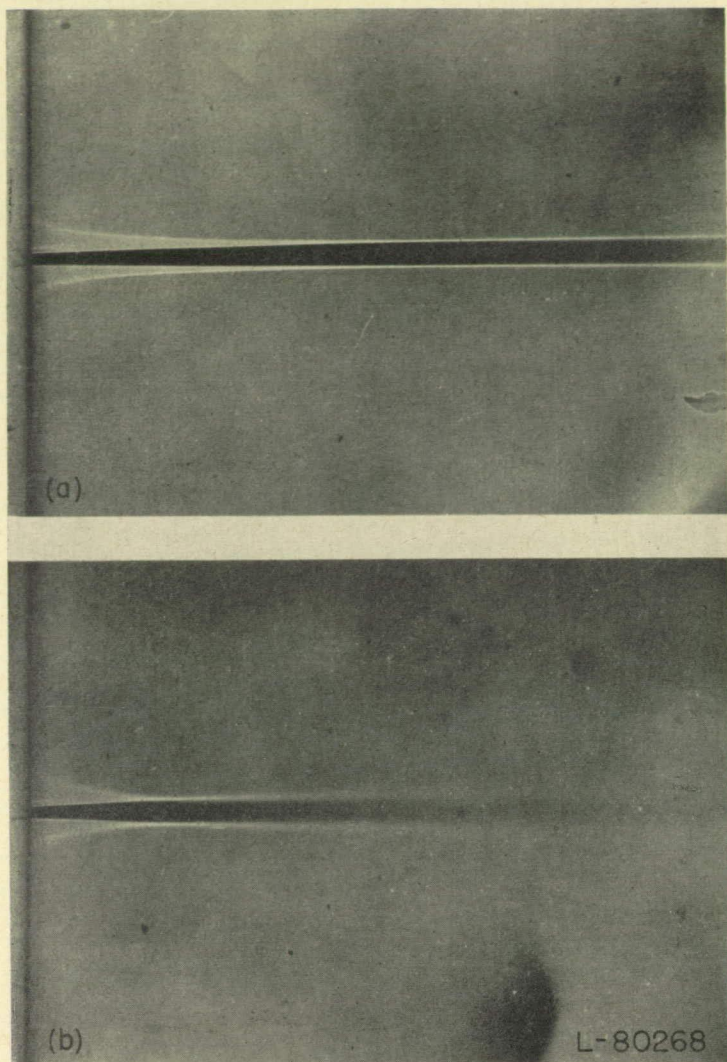
MEASURING EQUIPMENT AND PROCEDURES

The mean-temperature distributions were measured with a Chromel-Alumel thermocouple and a Leeds and Northrup type K-2 potentiometer. The cold junction was kept outside of the wind tunnel.

The shadowgraph technique was used to photograph the laminar thermal wake close to the source with no grid in the wind tunnel. This information was applied to the problem of "correcting" the thermal wake in turbulent flow for the effects of molecular diffusion and of finite source size. Figure 2 (a) is a shadowgraph of the wire wake with no grid in the tunnel; figure 2 (b) is a typical time exposure with grid-produced turbulence. A resistance-thermometer traverse of the laminar wake in a flow of very small turbulence showed that the temperature profile had already become very nearly Gaussian at a distance of 1 inch (125 wire diameters) downstream. The white lines on the sides of the dark wake shadow in figure 2 (a) correspond to the minimums in the second derivative of the density profile. For small temperature differences these coincide with the maximums in the second derivative of the temperature profile. Although the temperature differences are not small in the immediate vicinity of the wire, this condition is reasonably well satisfied at relatively small values of Δx as evidenced by the parabolic spread of this laminar wake.

The standard deviation of the wake could then be computed from the spacing of these two bright lines, with the assumption of a Gaussian distribution. Since the closest points of traverse in the turbulent cases were 1/2 inch (63 wire diameters) from the heat source, this was probably a reasonable assumption.

As pointed out by Taylor (reference 2), the molecular and



(a) No grid in tunnel.
(b) Grid-produced turbulence.

FIGURE 2.—Shadowgraph time exposure of wire wake.

the turbulent diffusive phenomena are statistically independent, so that the squares of the standard deviations due to these two effects are additive. Hence the wake spread due to turbulence alone, from a true line source, was obtained by subtracting the square of the standard deviation of the laminar wake (computed from the shadowgraph) from the square of the standard deviation of the total wake (from thermocouple traverses with grid turbulence present) at all stations. This difference was the square of the standard deviation of the desired phenomena. All wake-spread data presented in the next section have been corrected in this fashion.

Parenthetically, it should be remarked that the laminar

wake in figure 2 (a) spreads parabolically within the limits of precision, from at least 1 inch on, so that the effects of density differences on the flow phenomena must have been negligible for this investigation.

The transverse turbulence levels $v'/\bar{u}$ behind the grids were obtained from the initial rate of spread of the mean thermal wake (method due to Schubauer (reference 4)) after the effects of molecular spread and finite source had been removed. The resulting levels were somewhat higher than those obtained with a hot-wire anemometer but were used because of their consistency with the rest of the measured diffusion curve.

Free-stream velocity fluctuations (without grids) and the wake temperature fluctuations (with grids) were measured with the hot-wire anemometry equipment described in reference 10. The wires were 0.00025-inch platinum etched from Wollaston; the compensated response of the system was flat within ± 2 percent over a frequency range from 3 to 12,000 cycles per second.

Oscillograms of the temperature fluctuations were recorded by photographing a blue oscilloscope tube with fast 35-millimeter film in a General Radio typé 651-AE camera.

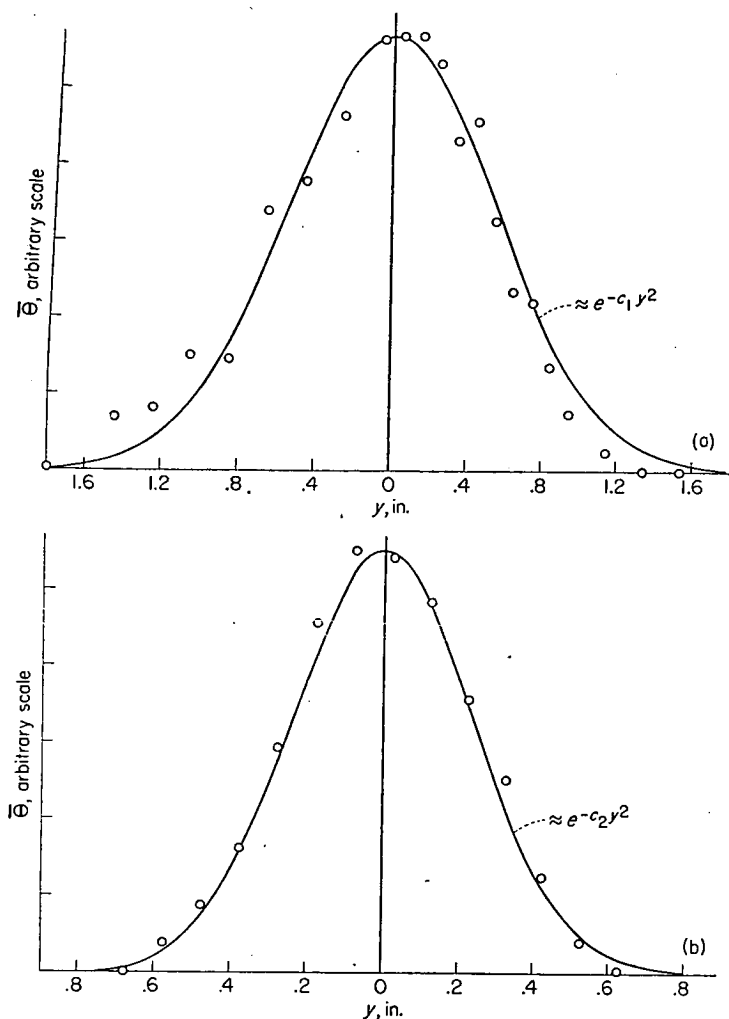
Probability densities of the temperature fluctuations at fixed points in the mean thermal wake were determined from photodensitometer traverses of time-exposure photographs of a short-persistence (0.001 sec) blue oscilloscope tube with the temperature fluctuations on one pair of plates and a 30,000-cycle-per-second sweep on the opposite plates. The technique is essentially that used by Simmons and Salter (reference 13).

EXPERIMENTAL RESULTS

MEAN THERMAL WAKE

Complete mean-temperature wakes behind a line source of heat were measured for 10 different conditions. Arranged to indicate the systematic variation of one parameter at a time, these conditions were as follows:

1-in. grid; $\frac{x_o}{M} = 43.4$; $\bar{U}$, fps.....	8.5 25.6 38.0
$\bar{U} = 25.6$ fps; $\frac{x_o}{M} = 43.4$; grid, in.....	1 $\frac{1}{2}$ $\frac{1}{4}$
$\bar{U} = 25.6$ fps; $\frac{x_o}{M} = 86.1$; grid, in.....	1 $\frac{1}{2}$ $\frac{1}{4}$
$\bar{U} = 25.6$ fps; $\frac{1}{2}$ -in. grid; x_o/M	43.4 86.1 172.3
$\bar{U} = 25.6$ fps; $\frac{1}{4}$ -in. grid; x_o/M	43.4 86.1 172.3



(a) $\frac{\Delta x}{M} = 15$, $\frac{x_o}{M} = 43.4$, $M = 1$ inch, and $\bar{U} = 25.6$ feet per second.
 (b) $\frac{\Delta x}{M} = 39$, $\frac{x_o}{M} = 86.1$, $M = \frac{1}{4}$ inch, and $\bar{U} = 25.6$ feet per second.

FIGURE 3.—Experimental scatter; temperature behind line source of heat.

Here $\bar{U}$ is the mean velocity, M is the grid mesh size, and x_o is the heat-source location measured from the grid. Since some individual cases enter as elements in two sequences, the total number of elements is more than 10.

Two of the many mean-temperature traverses in the y -direction (perpendicular to mean flow and to source line) are shown in figure 3 to give an idea of the amount of experimental scatter. The upper traverse was the worst of the lot, even showing an apparent skewness which was not borne out by the investigation as a whole. The lower traverse is more nearly typical of the measured temperature distributions from which the standard deviations of the mean

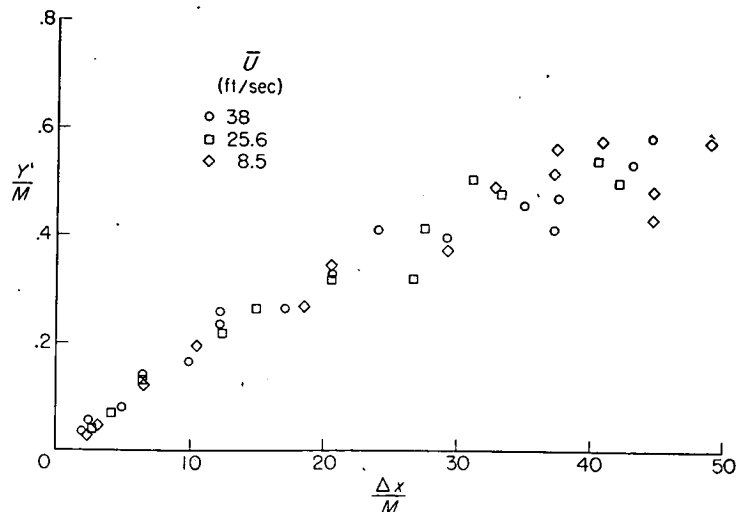


FIGURE 4.—Spread of heat from a line source. $\frac{x_o}{M} = 43.4$, $M = 1$ inch.

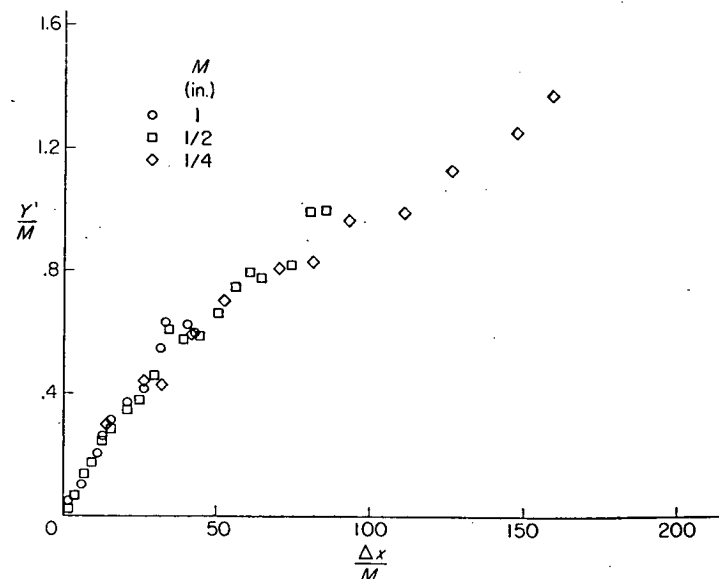


FIGURE 5.—Spread of heat from a line source. $\frac{x_o}{M} = 43.4$, $\bar{U} = 25.6$ feet per second.

thermal wake were computed. By comparison with the reference curve, it is seen to be essentially Gaussian. This was the case for temperature profiles at all stations. Since the virtually Gaussian character of such a wake has already been established by several of the earlier publications, there seemed to be no point in reproducing here all of the large number of traverses measured.

The mean-thermal-wake spread for the 10 different configurations studied is given in figures 4 to 7 as plots of corrected standard deviation Y' against distance from the heat

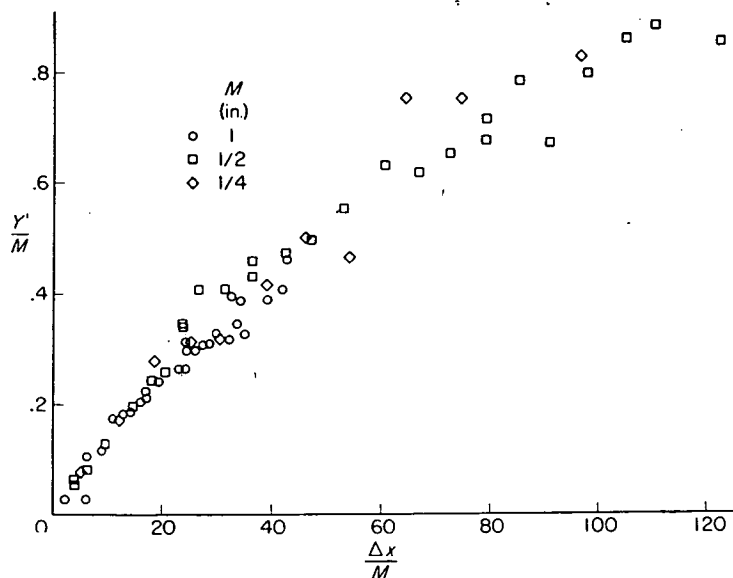


FIGURE 6.—Spread of heat from a line source. $\frac{x_o}{M}=86.1$, $\bar{U}=25.6$ feet per second.

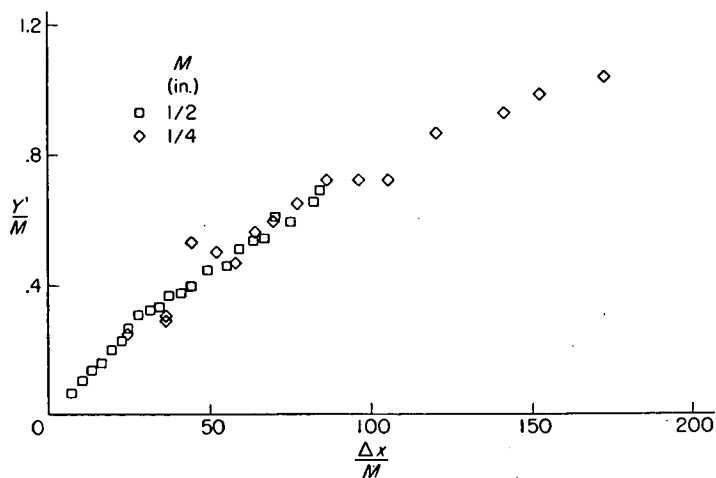
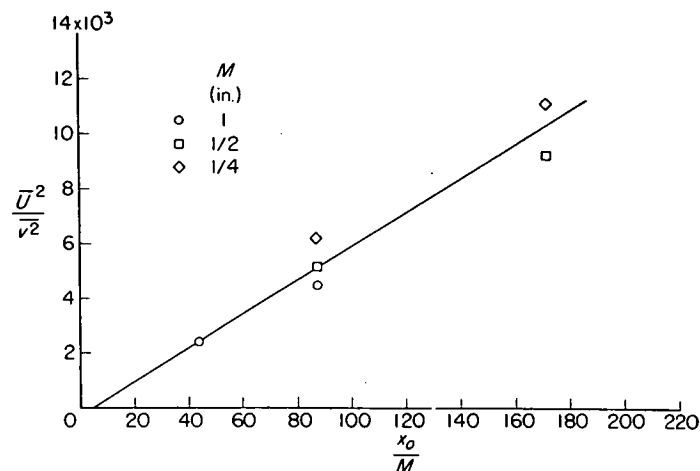


FIGURE 7.—Spread of heat from a line source. $\frac{x_o}{M}=172.25$, $\bar{U}=25.6$ feet per second.

source Δx . Each point in these figures corresponds to a complete transverse temperature traverse.

In order to have values of transverse turbulence level $v'/\bar{U}$ consistent with the thermal-wake behavior, these values were determined from the initial angle of spread of the corrected wake standard deviation (references 2 and 4) instead of from direct hot-wire anemometer measurements. The results are plotted in figure 8. Since these represent an insufficient number of points per grid to permit the drawing of reliable curves, some simplifying assumptions were made based upon the results of several more-detailed turbulence-decay investigations. (See references 14 to 20. The pertinent results of references 14 to 16 are summarized in refer-



M (in.)	x_o/M	$\bar{U}$ (fps)	$v'/\bar{U}$ (percent)
1	43.4	8.5	2.0
1	43.4	25.6	2.0
1	43.4	38.0	2.0
1/2	43.4	25.6	2.0
1/4	43.4	25.6	2.0
1	86.1	25.6	1.50
1/2	86.1	25.6	1.40
1/4	86.1	25.6	1.30
1/2	172.3	25.6	1.05
1/4	172.3	25.6	.95

FIGURE 8.—Decay of turbulence as determined by thermal wake of a line source.

ence 17.) In the light of these papers, it was assumed that the decay curves had a common apparent origin and that this was obtainable by drawing the best straight line for all the available points, independent of the differing wind speeds and mesh sizes. In computing any individual Lagrangian correlation function from the corresponding wake history, the turbulence decay rate was assumed to be given by the line drawn through this common origin and the specific turbulence value giving the measured initial spread angle for this wake. This is, of course, a very rough procedure, but the experimental scatter in this whole method of determining Lagrangian correlation functions is so great that a more extensive study of decay (including the resolution of inconsistencies between wake method and hot-wire method) seemed unwarranted at this time.

TEMPERATURE FLUCTUATIONS

Distributions of temperature-fluctuation level $\vartheta'/\bar{\theta}$ in the thermal wake have been measured by using the hot-wire anemometer as a resistance thermometer, that is, at a current low enough to render the sensitivity to velocity fluctuations negligible compared with the sensitivity to temperature fluctuations (reference 21). A representative distribution of $\vartheta'/\bar{\theta}$ in the x -direction is given in figure 9. Typical transverse distributions are given in figures 10 (a) and 10 (b). It is clear that the temperature-fluctuation inten-

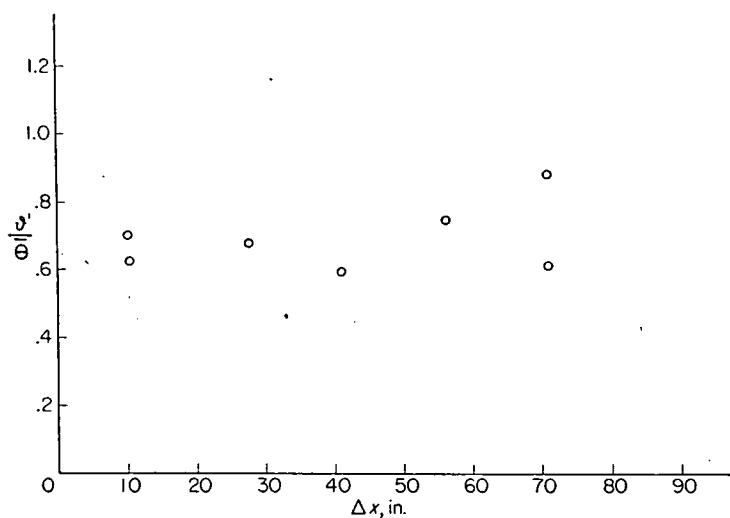
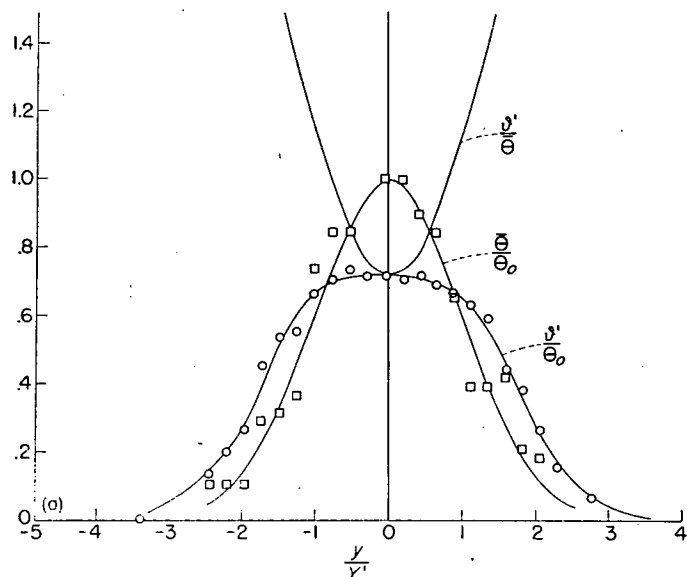


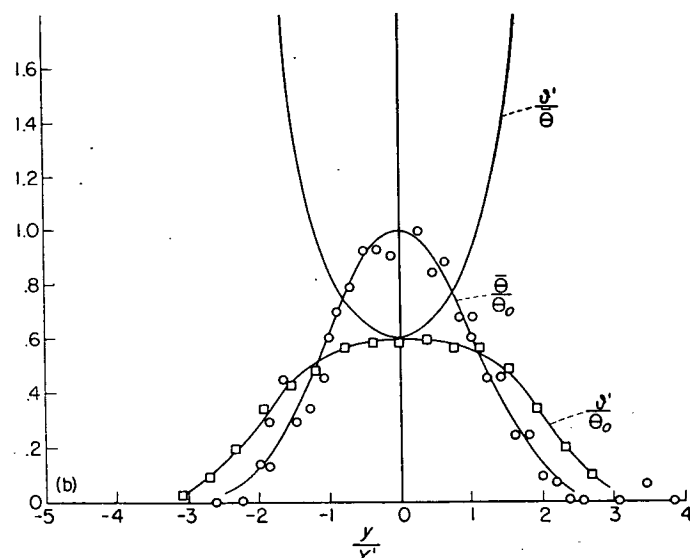
FIGURE 9.—Temperature fluctuations at $x_0=43$ inches, $M=1$ inch, and $\bar{U}=25.6$ feet per second along wake axis.



(a) $\Delta x=10$ inches.

FIGURE 10.—Temperature fluctuations behind a line heat source at $\frac{x_0}{M}=43$, $M=1$ inch, and $\bar{U}=25.6$ feet per second.

sity changes very little with increasing values of Δx . A rough explanation of the very high values of θ'/θ (compared with the concomitant turbulence level, for example) in terms of the highly intermittent structure of the thermal wake has been given in reference 10 and will be discussed in more detail later in this report. This intermittency is shown very clearly in figure 11, a series of temperature oscillograms recorded at two different positions across the thermal wake for a fixed value of Δx and at two different



(b) $\Delta x=70$ inches.

FIGURE 10.—Concluded.

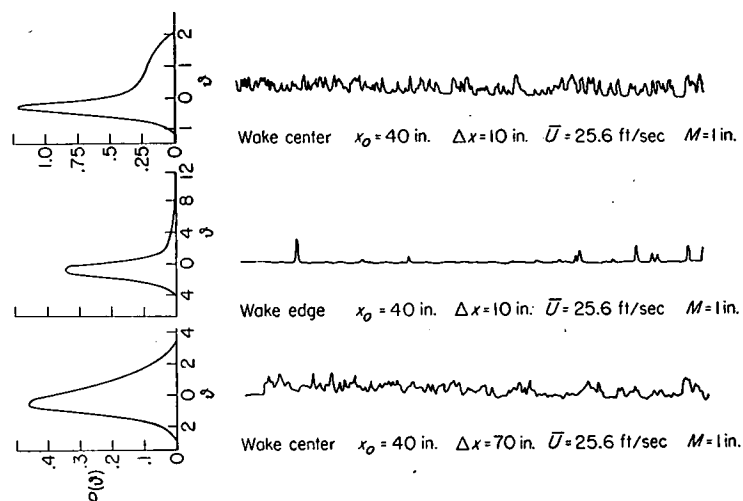


FIGURE 11.—Oscillogram records and normalized probability density of temperature fluctuations. Film speed, 1 foot per second.

values of Δx with $y=0$. The one-sided and pulse character of the instantaneous temperature at a fixed point in space is also demonstrated by its probability density.

THEORETICAL CONSIDERATIONS

HOMOGENEOUS STEADY TURBULENCE AT REST

For a nondecaying incompressible turbulence with no mean motion, Taylor (reference 1) was followed in getting an expression for the mean time rate of diffusion in the y -direction (say) from a fixed source as measured by the second moment of the probability density of the diffusion, that is, the mean-square particle displacement $\bar{Y}^2(t)$:

$$\begin{aligned}\frac{d}{dt}\overline{Y^2} &= 2\overline{Y\frac{dY}{dt}} \\ &= 2\overline{v(t)\int_0^t v(t_1)dt_1}\end{aligned}$$

The bar denotes ensemble average.

Taking $v(t)$ inside the integral, interchanging the processes of integration and averaging, and introducing the Lagrangian (auto) correlation coefficient (following the fluid particle), where $\tau = t - t_1$,

$$R_v(\tau) \equiv \frac{\overline{v(t)v(t-\tau)}}{\overline{v^2}}$$

there results

$$\overline{Y^2} = 2\overline{v^2} \int_0^t \int_0^\tau R_v(\tau) d\tau dT \quad (1)$$

This is Taylor's form. Integration by parts yields a form like that in the work of Kampé de Fériet (reference 5):

$$\overline{Y^2} = 2\overline{v^2} \int_0^t (t-\tau) R_v(\tau) d\tau \quad (2)$$

In this Lagrangian analysis, $v(t)$ is the velocity of a fluid particle in the y -direction at time t ; $v(t-\tau)$ is the velocity of the same particle at time $t-\tau$. Corresponding expressions can be written for the rate of diffusion in any direction.

Diffusion from an infinite line source, the case to be discussed here, is a two-dimensional problem in the mean, and, in addition to equation (2),

$$\overline{X^2} = 2\overline{u^2} \int_0^t (t-\tau) R_u(\tau) d\tau \quad (3)$$

A tensorial generalization of these concepts has been given by Batchelor (reference 3) dealing with the behavior of $\overline{X_i X_j(t)}$ where X_i and X_j are any two of the orthogonal displacements of the particle at time t .

It should be noted that this analysis gives no information on the shape of the probability density of $Y(t)$ or of $X(t)$. In fact there still exists no theory for these. However, experiments in flowing turbulence (the case to be considered next) show Gaussian density, within the experimental precision, for $Y(t)$ at all values of t .

For this stationary random process, Taylor (reference 1) introduced the concept of the Lagrangian "scale,"

$$L_L = \int_0^\infty R_v(\tau) d\tau \quad (4)$$

These have the dimensions of time and are characteristic constants of the system.

In his later work on the (Eulerian) dynamics (reference 2), Taylor had occasion to introduce another measure of the

correlation function, which he called the "microscale." Applying the same geometrical concept to the present function, the Lagrangian microscale

$$\lambda_v^2 = -\frac{2}{R_v''(0)} \quad (5)$$

is simply the τ -intercept of the vertex-osculating parabola of the even function $R_v(\tau)$. The kinematic significance is clearly shown by a series expansion of $v(t+\tau)$ in $R_v(\tau)$:

$$\begin{aligned}R_v(\tau) &= \frac{\overline{v(t)v(t+\tau)}}{\overline{v^2}} \\ &= \frac{1}{\overline{v^2}} \left(\overline{v(t) \left\{ v(t) + \left[\left(\frac{dv}{dt} \right)_t \tau + \left[\left(\frac{d^2v}{dt^2} \right)_t \frac{\tau^2}{2} + \dots \right] \right\}} \right) \\ &= \frac{1}{\overline{v^2}} + \left\{ \overline{v^2} \left[\frac{1}{2} \left(\frac{d^2v^2}{dt^2} \right)_t \right] \tau + \left[\frac{1}{2} \left(\frac{d^2v^2}{dt^2} \right)_t - \left(\frac{dv}{dt} \right)_t^2 \right] \frac{\tau^2}{2} + \dots \right\}\end{aligned}$$

But $\overline{v^2} = \text{Constant}$, so that

$$R_v(\tau) = 1 - \frac{1}{2\overline{v^2}} \left(\frac{dv}{dt} \right)_t^2 \tau^2 + \dots \quad (6)$$

From equations (6) and (5), the Lagrangian microscale for $v(t)$ is

$$\lambda_v^2 = \frac{2\overline{v^2}}{\left(\frac{dv}{dt} \right)_t^2} \quad (7)$$

or

$$\left(\frac{dv}{dt} \right)_t^2 = 2 \frac{\overline{v^2}}{\lambda_v^2} \quad (7a)$$

TURBULENCE IN A FLOWING MEDIUM

The dictates of both practical interest and experimental feasibility require analysis of the diffusion when there is a mean velocity $\overline{U}$ relative to the source. Since the diffusion phenomenon is linear, the probability density (mean-concentration distribution of tagged particles in the wake) is simply proportional to the superimposed probability densities of a continuous line of sources moving with the mean velocity $\overline{U}$ with their time (and space) origin at the actual fixed source.

This is illustrated in figure 12(a) for $\frac{v', u'}{\overline{U}} < 1$. The circles

(corresponding to isotropy) are the standard deviations of the dispersions that would occur from moving sources. The envelope of these circles gives a measure of the mean wake. It is obvious that in general the functional form of the mean-concentration distribution along a line $\Delta x = \text{Constant}$ will not be the same as the functional form of the same quantity for the individual source at time t , that is, at position $x = \overline{U}t$.

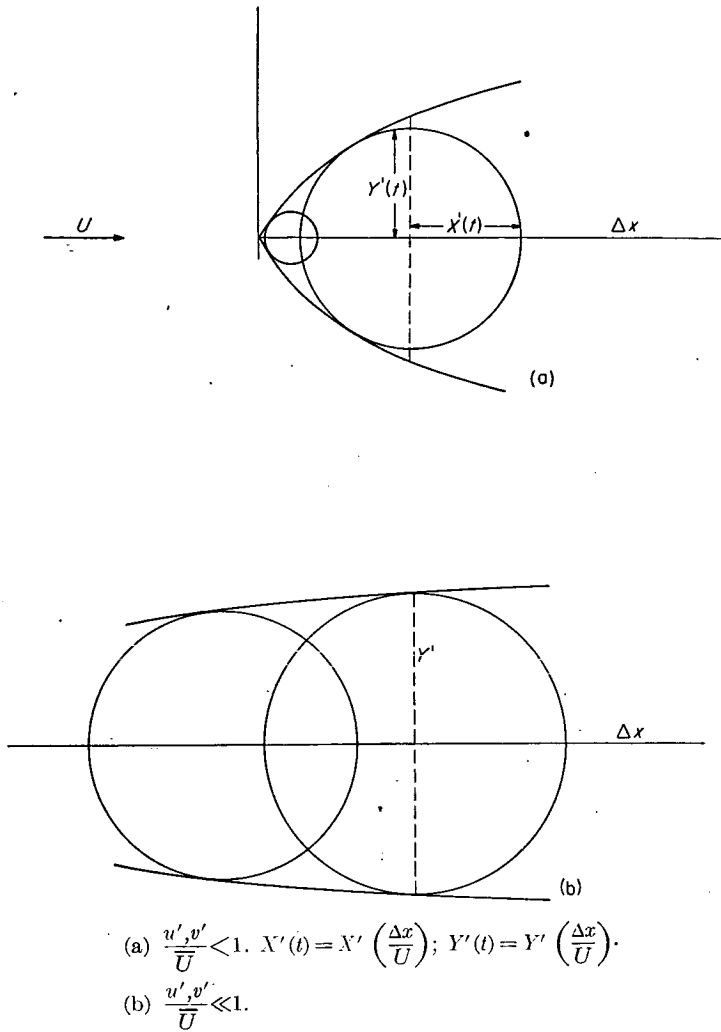


FIGURE 12.—Propagation of turbulence from source moving with mean velocity $\bar{U}$.

However, it is extremely likely that if $\frac{dY'}{dx} \ll 1$ (fig. 12(b)) the mean concentration along a line $\Delta x = \text{Constant}$ becomes very nearly of the same functional form as that for the source at $x = \bar{U}t$. For a zero-correlation Gaussian density, the equivalence is easily demonstrable.³

The condition $\frac{dY'}{dx} \ll 1$ will always occur at large enough values of Δx when $\frac{v'}{\bar{U}} < 1$. This follows from the asymptotic

³ The most general mathematical restrictions under which the superposition of a line of identical densities will yield a "cross-section" density of the same form have not been studied here. It is obvious that the condition of statistical independence is sufficient. The fact that the density of each of the velocity components in isotropic turbulence has been found to be Gaussian within the experimental precision seems to show that the equivalence under discussion is at least a good approximation.

parabolic behavior, $Y'(t) \sim \sqrt{t}$ (reference 1), of the diffusive process. It will occur for all values of Δx when $\frac{v'}{\bar{U}} \ll 1$. Therefore, in this case a simple approximate space-time transformation *in the mean* is permissible, and the t -variation in Taylor's theory of diffusion by continuous movements becomes a variation of $x/\bar{U}$. (It must be emphasized that the foregoing discussion does not apply directly to the possibility of applying a space-time transformation to the *instantaneous* turbulence variables. This latter question will be discussed later.)

In the present measurements $\frac{v'}{\bar{U}} \ll 1$, and, therefore, the Δx -variation of diffusion gives an approximate measure of the Lagrangian correlation coefficient (in time). Equation (2) can be rewritten as:

$$\bar{Y}^2 \left(\frac{x}{\bar{U}} \right) = 2 \frac{\bar{v}^2}{\bar{U}^2} \int_0^x (x - \xi) R_{\tau} \left(\frac{\xi}{\bar{U}} \right) d\xi \quad (8)$$

DECAYING ISOTROPIC TURBULENCE

When the turbulence is decaying in time (similar to space in the flowing turbulence) $\bar{v}^2$ is no longer constant, and the analysis cannot be carried out as far as equation (1). The same approach stops with

$$\frac{d}{dt} \bar{Y}^2 = v'(t) \int_0^t v'(t - \tau) R_{t-\tau} d\tau \quad (9)$$

where the prime denotes root-mean-square value, and

$$R_{t-\tau} \equiv \frac{\overline{v(t)v(t-\tau)}}{v'(t)v'(t-\tau)}$$

There is no a priori reason to believe that $R_{t-\tau}$ is a function of τ alone, as in the nondecaying turbulent flow.

At this point Taylor (reference 2) invokes the empirical fact that over a wide range of mean velocities (all of which give essentially the same distribution of $v'/\bar{U}$ in x behind a grid) the thermal wake behind a line heat source at fixed x appears to be unchanged in form, within the experimental error. This is consistent with dependence of the diffusive process upon a variable of type

$$\begin{aligned} \eta &= \int_{t_0}^t v'(t_1) dt_1 \\ &= \frac{1}{\bar{U}} \int_{x_0}^x v'(x_1) dx_1 \end{aligned} \quad (10)$$

Therefore, Taylor has postulated the unique dependence of $R_{t-\tau}$ on the variable η . With this postulate and the space-time transformation valid for small turbulence level, he

arrives at

$$\frac{d}{dx} \overline{Y^2} = 2 \frac{v'}{\overline{U}} \int_0^\eta R_\eta(\eta_1) d\eta_1 \quad (11)$$

or

$$\frac{d}{d\eta} \overline{Y^2} = 2 \int_0^\eta R_\eta(\eta_1) d\eta_1 \quad (12)$$

where

$$d\eta = \frac{v'}{\overline{U}} dx \\ = v' dt$$

Equations (11) and (12) look like the equation for non-decaying turbulence. They also give

$$\overline{Y^2} = 2 \int_0^\eta (\eta - \eta_1) R_\eta(\eta_1) d\eta_1 \quad (13)$$

A physical significance of the length η is underscored by the limiting form of equation (13) as $\eta \rightarrow 0$ and $R_\eta \rightarrow 1$. Then,

$$Y' = \eta$$

so that η is a measure of the lateral diffusion that would occur if the lateral velocity fluctuation following a particle $v(t)$ remained perfectly correlated but decreased in magnitude according to the decay rate of the turbulence level.

In postulating $R_{t-\tau} = R_\eta(\eta)$, Taylor was apparently comparing only diffusive processes in turbulence fields with identical $\frac{v'}{\overline{U}}(x)$. Of more general interest is the comparison

of diffusion in fields with differing turbulence-level distributions. Although such a generalized application of his postulate is doubtless not too well applicable, it is conceivable it might have approximate success in the more general comparison. For both convenience and lack of any obviously superior alternative, his suggestion is therefore applied in computing the results of the measurements reported here.⁴

With the η -postulate, the R_η Lagrangian correlation function can be obtained from measurements of $\overline{Y^2}$ as a function of Δx :

$$R_\eta(\eta) = \frac{1}{2} \frac{d^2}{d\eta^2} \overline{Y^2} \\ = \frac{1}{2} \frac{\overline{U}^2}{v'^2} \frac{d^2}{dx^2} \overline{Y^2} \quad (14)$$

⁴ After this work was completed, Dr. Batchelor suggested an alternative approximate approach: In order to construct a stationary random function out of the nonstationary $v(t)$ one first normalizes the dependent variable with its root-mean-square value. (This has been automatically accomplished by use of the correlation coefficient.) If the decaying quantity is assumed to maintain complete similarity during decay, all characteristic times (e. g., Lagrangian time scale and microscale) vary in the same way with t and the new independent variable is constructed by dividing τ by this t -variation. Unfortunately, this variation is unknown a priori, so it would be necessary to assume further that the Lagrangian scales are directly proportional to the Eulerian scales, about which there is previous experimental information. His work has now been published; see reference 22.

A scale and a microscale can also be defined for R_η :

$$L_\eta = \int_0^\infty R_\eta(\eta_1) d\eta_1 \quad (15)$$

$$\lambda_\eta^2 = -\frac{2}{R_\eta(0)''} \quad (16)$$

With nondecaying turbulence, no one-parameter true (time) Lagrangian correlation function exists, and the η -formulation is much more convenient. A further significance of this variable will appear in the comparison of Eulerian and Lagrangian treatments of diffusion from a line source in flowing turbulence.

ACCELERATIONS IN DECAYING TURBULENCE

A series expansion of $v(t+\tau)$ for decaying turbulence will show something about the initial behavior of the true (time) Lagrangian correlation function and will indicate an experimental method for examining a hypothesis of Taylor on the interchangeability of instantaneous time and space derivatives when the turbulence level is low (reference 23).

Write the Lagrangian correlation coefficient

$$R_v(t, \tau) = \frac{\overline{v(t)v(t-\tau)}}{v'(t)v'(t-\tau)} \quad (17)$$

Substitute

$$v(t-\tau) = v(t) - \left(\frac{dv}{dt}\right)_t \tau + \left(\frac{d^2v}{dt^2}\right)_t \frac{\tau^2}{2!} + \dots$$

into numerator and denominator, and restrict the analysis to small values of τ :

$$R_v(t, \tau) \approx \frac{\overline{v^2(t)} - \frac{1}{2} \left(\frac{d\overline{v^2}}{dt}\right)_t \tau + \left[\frac{1}{2} \left(\frac{d^2\overline{v^2}}{dt^2}\right)_t - \frac{\left(\frac{d\overline{v}}{dt}\right)_t^2}{2}\right] \frac{\tau^2}{2}}{v'(t) \sqrt{\overline{v^2(t)} - \left(\frac{d\overline{v^2}}{dt}\right)_t \tau + \frac{1}{2} \left(\frac{d^2\overline{v^2}}{dt^2}\right)_t \tau^2}} \quad (18)$$

Divide numerator and denominator by $\overline{v^2}$, expand the square root in the numerator, and keep terms in τ^2 :

$$R_v(t, \tau) \approx 1 - \left[-\left(\frac{dv'}{dt}\right)_t^2 + \left(\frac{d\overline{v}}{dt}\right)_t^2 \right] \frac{\tau^2}{2\overline{v^2(t)}} \quad (19)$$

For negligible decay rate this reduces to equation (6).

Equation (19) shows that a Lagrangian microscale defined by

$$\frac{1}{\lambda_v^2(t)} = \lim_{\tau \rightarrow 0} \left[\frac{1 - R_v(t, \tau)}{\tau^2} \right] \\ = - \left[\frac{\partial^2 R_v(t, \tau)}{\partial \tau^2} \right]_{\tau=0} \quad (20)$$

is expressible as

$$\frac{1}{\lambda_v^2(t)} = \frac{1}{2\overline{v^2}} \left[\left(\frac{d\overline{v}}{dt}\right)_t^2 - \left(\frac{dv'}{dt}\right)_t^2 \right] \quad (21)$$

Introduction of Taylor's η -postulate transforms equation (19) to

$$R_\eta(\eta) \approx 1 - \left[\left(\frac{dv}{d\eta} \right)_{\eta=0}^2 - \left(\frac{dv'}{d\eta} \right)_{\eta=0}^2 \right] \frac{\eta^2}{2v_{\eta=0}^2} \quad (22)$$

after the additional approximation that $v'(\eta) \approx v'(0)$ when η is very small.

This gives a new expression for Lagrangian microscale λ_η :

$$\frac{1}{\lambda_\eta^2} = \frac{1}{2v^2} \left[\left(\frac{dv}{d\eta} \right)^2 - \left(\frac{dv'}{d\eta} \right)^2 \right] \quad (23)$$

Equation (23) is in contradiction to equation (17) of part IV of reference 2. In that work Taylor has apparently assumed that v' is a nondecaying function of η . However it certainly is decaying, even in terms of this distorted coordinate, and the $dv'/d\eta$ -term must be included.

Since λ_η and $v'(x)$ can be determined experimentally from the mean thermal wake behind a line heat source, equation

(23) permits determination of $\left(\frac{dv}{d\eta} \right)^2$ which is simply related

to the mean-square "Stokes" acceleration $\left(\frac{dv}{dt} \right)^2 = (v')^2 \left(\frac{dv}{d\eta} \right)^2$.

This quantity is of particular interest for the possibility of an instantaneous space-time transformation at low turbulence levels. This was first proposed by Taylor (reference 23) and has since been used very widely, especially to get approximate values of partial derivatives with respect to x (the mean-flow direction) by measurements of time partial derivatives.

The total (or Stokes) derivative of $v(x, y, z, t)$ in a turbulent flow with mean velocity $\bar{U}$ along the x -direction is

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + (\bar{U} + u) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \quad (24)$$

Taylor's hypothesis amounts to the statement that

$$\frac{\partial v}{\partial t} \approx -\bar{U} \frac{\partial v}{\partial x}$$

or

$$\frac{\partial v}{\partial t} = -\bar{U} \frac{\partial v}{\partial x} + \epsilon \quad (24a)$$

with

$$\left[\frac{\epsilon^2}{\bar{U}^2 \left(\frac{\partial v}{\partial x} \right)^2} \right]^{1/2} \ll 1 \quad (25)$$

In detail, equation (25) is

$$\left[\frac{\left(\frac{dv}{dt} \right)^2 + u_i u_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - 2u_i \frac{dv}{dt} \frac{\partial v}{\partial x_i}}{\bar{U}^2 \left(\frac{\partial v}{\partial x} \right)^2} \right]^{1/2} \ll 1 \quad (25a)$$

In the absence of information on the algebraic sign of the triple correlation term, it is sufficient to require the two conditions.⁵

$$\left. \begin{aligned} \left[\frac{\left(\frac{dv}{dt} \right)^2}{\bar{U}^2 \left(\frac{\partial v}{\partial x} \right)^2} \right]^{1/2} &\ll 1 \\ \left[\frac{u_i u_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}}{\bar{U}^2 \left(\frac{\partial v}{\partial x} \right)^2} \right]^{1/2} &\ll 1 \end{aligned} \right\} \quad (26)$$

But $\left(\frac{dv}{dt} \right)^2$ can be determined from measurements of $Y'(x)$;

$\left(\frac{\partial v}{\partial x} \right)^2 = \frac{v^2}{\lambda^2}$, where λ is the Eulerian microscale (reference 2);

and upper bounds, in terms of measurable functions, can be

set on $u_i u_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}$ with the use of Schwarz inequalities. Thus,

an experimental check of the requirements in equation (26) is to be made in the section entitled "Computation of Results."

RELATION BETWEEN EULERIAN AND LAGRANGIAN MICROSCALES

Taylor (reference 2) inferred an approximate relation between λ_η and λ by neglecting the effect of viscosity on

pressure-gradient fluctuations and estimating $\left[\left(\frac{\partial p}{\partial y} \right)^2 \right]^{1/2}$ as

being approximately $3\rho \left[v^2 \left(\frac{\partial v}{\partial y} \right)^2 \right]^{1/2}$. This led to a constant

ratio λ_η/λ for all turbulence. The rough nature of this

analysis induced Heisenberg (reference 25) to conduct a

more detailed study of the static-pressure fluctuations and to reestimate the λ_η/λ ratio. However, he followed Taylor

in ignoring the $dv'/d\eta$ -term in the relation between λ_η and $\left(\frac{dv}{d\eta} \right)^2$ (see equation (23)) and in neglecting viscous terms in

the relation between λ_η and $\left(\frac{\partial p}{\partial y} \right)^2$.

Although these omissions are probably not serious except in the low Reynolds number range, it seems interesting, if

only for the sake of completeness, to use a Heisenberg type of approximation for $\left(\frac{\partial p}{\partial y} \right)^2$ and to repeat his treatment with the omissions rectified.

⁵ Lin has discussed the validity of Taylor's hypothesis using a slightly different formulation in reference 24. He points out there that if $A = \sum_1^n a_\bullet$, then (from the Schwarz inequality) $\overline{(A^2)}^{1/2} \leq \sum_1^n \overline{(a_\bullet^2)}^{1/2}$ where $a_\bullet$ is a set of numbers.

From the complete Navier-Stokes equations in the wave-number space, Heisenberg deduced an approximate expression for $(\nabla p)^2$ in terms of mean quadruple products of the "harmonics" of the velocity field. His principal simplifying assumptions were:

(a) Different Fourier components of the velocity field are uncorrelated

(b) The turbulent energy spectrum is given by the solution to his equilibrium-energy-transfer equation, above a lower cut-off wave number k_0 .

Following these, but using Chandrasekhar's (reference 26) solution to the Heisenberg equation instead of the interpolation formula used by Heisenberg, there results

$$\overline{\left(\frac{\partial p}{\partial y}\right)^2} = \frac{25.2}{\kappa} \rho^2 \nu \frac{(v')^3}{\lambda^3} \quad (27)$$

where κ is a dimensionless empirical constant.

The numerical constant in equation (27) would perhaps have been given more accurately by the use of a "self-preserving" spectrum (calculated by Chandrasekhar from Heisenberg's equation) instead of the stationary spectrum with low cut-off wave number. Time was not taken to make the requisite additional calculations because: (a) The value of $\overline{\left(\frac{\partial p}{\partial y}\right)^2}$ depends principally upon the high-wave-number region of the velocity spectrum rather than the low-wave-number region, where the difference would be greatest, and (b) the experimental results and (especially) the value of κ both have a considerable range of uncertainty.

The mean square of the y -component of the Navier-Stokes equation will lead to a relation between λ_η and λ :

$$\begin{aligned} \frac{dv}{dt} &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \end{aligned}$$

therefore,

$$\overline{\left(\frac{dv}{dt}\right)^2} = \frac{1}{\rho^2} \overline{\left(\frac{\partial p}{\partial y}\right)^2} + \nu^2 \overline{(\nabla^2 v)^2} \quad (28)$$

where the correlation between pressure gradient and velocity Laplacian function is zero because of isotropy.

Equations (23) and (27) give the first two terms in terms of the microscales, and the mean-square Laplacian function is expressible in terms of the fourth derivative of the Von Kármán-Howarth $f(r)$ correlation coefficient at $r=0$ (reference 27):

$$\overline{(\nabla^2 v)^2} = \frac{35}{3} (v')^2 f^{iv}(0) \quad (29)$$

Consequently, equation (28) becomes

$$2 \frac{(v')^4}{\lambda_\eta^2} + \left(\frac{dv'}{dt}\right)^2 = \frac{25.2}{\kappa} \nu \frac{(v')^3}{\lambda^3} + \frac{35}{3} \nu^2 (v')^2 f^{iv}(0)$$

The second term in this equation can be replaced by the

turbulence-decay equation

$$\frac{d(v')^2}{dt} = -10\nu \frac{(v')^2}{\lambda^2} \quad (30)$$

whence

$$\frac{dv'}{dt} = -5\nu \frac{v'}{\lambda^2}$$

so that

$$2 \frac{(v')^4}{\lambda_\eta^2} + 25\nu^2 \frac{(v')^2}{\lambda^4} = \frac{25.2}{\kappa} \nu \frac{(v')^3}{\lambda^3} + \frac{35}{3} \nu^2 (v')^2 f^{iv}(0)$$

therefore,

$$\frac{\lambda^2}{\lambda_\eta^2} = \frac{12.6}{\kappa} \frac{1}{R_\lambda} - \frac{12.5}{R_\lambda^2} + \frac{35}{6} f^{iv}(0) \frac{\nu^2 \lambda^2}{(v')^2} \quad (31)$$

where $R_\lambda = \frac{v' \lambda}{\nu}$.

Batchelor and Townsend (reference 19) have deduced an expression for $f^{iv}(0)$ which is valid in the region of decay where both $1/(v')^2$ and λ^2 increase linearly with t (corresponding to large values of R_λ):

$$\lambda^4 f^{iv}(0) = \frac{30}{7} + \frac{1}{2} R_\lambda S \quad (32)$$

where $S = -\overline{\left(\frac{\partial v}{\partial y}\right)^3} / \left[\overline{\left(\frac{\partial v}{\partial y}\right)^2}\right]^{3/2}$, the skewness factor. Their experimental results showed $S=0.39$ approximately constant for isotropic turbulence. Then the estimate for λ/λ_η becomes

$$\left(\frac{\lambda}{\lambda_\eta}\right)^2 = \frac{12.5}{R_\lambda^2} + \left(\frac{12.6}{\kappa} + 1.14\right) \frac{1}{R_\lambda} \quad (33)$$

The value of κ was first estimated by Heisenberg (reference 25), from measurements of turbulence decay, as 0.85. This method may be regarded as emphasizing the (relatively low wave number) energy-bearing range of the spectrum. Lee (reference 28) worked out an estimate based upon skewness factor ($\kappa=0.13$), which gives heavy weight to the high-wave-number range. Proudman (reference 29) has reestimated κ by comparison with measured curves of the double and triple velocity correlations. The value $\kappa=0.45$ leads to reasonably good agreement for the moderately high-wave-number region, over a wide range of values of R_λ .

It may be remarked that the supposed constancy of κ is merely a postulate of the Heisenberg dimensional formulation of the spectral transfer function. In fact it is by no means obvious that this turbulent part of the transfer is quantitatively independent of the amount of spectrally local dissipation to heat. In any case, Proudman's estimate of $\kappa=0.45$ has been used here. Therefore,

$$\left(\frac{\lambda}{\lambda_\eta}\right)^2 = \frac{12.5}{R_\lambda^2} + \frac{29}{R_\lambda} \quad (34)$$

In the limit of $R_\lambda \rightarrow 0$, equation (34) does not apply since equation (32) does not apply. However, the appropriate

limiting relation can be obtained directly. In this limiting condition the pressure term in equation (28) is negligible compared with the viscous term (the former goes with $1/R_\lambda$ and the latter with $1/R_\lambda^2$), and the Eulerian velocity correlation coefficient is (reference 27) $f(r)=e^{-r^2/2\lambda^2}$. This gives $f''(0)=3/\lambda^4$ and

$$\left(\frac{\lambda}{\lambda_\eta}\right)^2 = \frac{3.5}{R_\lambda^2} \quad (35)$$

EULERIAN ANALYSIS OF HEAT DIFFUSION FROM A LINE SOURCE

The two-dimensional turbulent-heat-transfer equation is

$$\bar{U} \frac{\partial \bar{\theta}}{\partial x} + \bar{V} \frac{\partial \bar{\theta}}{\partial y} = \frac{k}{\rho c_p} \left(\frac{\partial^2 \bar{\theta}}{\partial x^2} + \frac{\partial^2 \bar{\theta}}{\partial y^2} \right) - \frac{\partial}{\partial x} (\bar{\theta} \bar{u}) - \frac{\partial}{\partial y} (\bar{\theta} \bar{v}) \quad (36)$$

where $\bar{\theta}$ is mean temperature, $\bar{\theta}$ is temperature fluctuation about the mean, k is thermal conductivity, and c_p is specific heat at constant pressure.

For the thermal wake behind a line source in isotropic turbulence with constant mean velocity $\bar{V}=0$. With restriction to low turbulence level, a "boundary-layer" type of approximation can be applied to the mean wake, so that

$$\frac{\partial^2 \bar{\theta}}{\partial x^2} \ll \frac{\partial^2 \bar{\theta}}{\partial y^2}$$

and

$$\frac{\partial}{\partial x} (\bar{\theta} \bar{u}) \ll \bar{U} \frac{\partial \bar{\theta}}{\partial x}$$

so that equation (36) takes the approximate form

$$\bar{U} \frac{\partial \bar{\theta}}{\partial x} = \frac{k}{\rho c_p} \frac{\partial^2 \bar{\theta}}{\partial y^2} - \frac{\partial}{\partial y} (\bar{\theta} \bar{v}) \quad (37)$$

It must be emphasized that for this particular initial condition on the temperature (effectively a "point source"), the restriction to small turbulence level $v'/\bar{U} \ll 1$ does not imply that $\bar{v}'/\bar{\theta}$ is small. In fact, for this problem $\bar{v}'/\bar{\theta}$ is often greater than unity, especially at the "edge" of the mean wake, as has been discussed in reference 10 and will be brought out again later in the present report.

When the molecular transport can be neglected relative to turbulent transport

$$\bar{U} \frac{\partial \bar{\theta}}{\partial x} = - \frac{\partial}{\partial y} (\bar{\theta} \bar{v}) \quad (38)$$

an equation given in reference 10; a slightly more general treatment follows:

With a constant rate of heat generation (similar to steady state in the average), the application of a Von Kármán integral-relation treatment to equation (38) yields an integral condition:

$$2\rho c_p \bar{U} \int_0^\infty \bar{\theta} dy = \text{Constant} = H \quad (39)$$

where H is the average time rate at which heat crosses all planes perpendicular to $\bar{U}$ per unit length of heat source

(z -direction). Of course, $\bar{\theta} \bar{u}$ has been neglected relative to $\bar{\theta} \bar{U}$ in equation (39).

Equation (38) has two unknowns, and the first objective is to express $\bar{\theta} \bar{v}$ as a function of the (more easily measurable) $\bar{\theta}(x, y)$. After integration with respect to y ,

$$\bar{\theta} \bar{v} = -\bar{U} \int_0^y \frac{\partial \bar{\theta}}{\partial x} dy + F(x)$$

But, by symmetry, $\bar{\theta} \bar{v}=0$ for $y=0$. Hence $F(x)=0$ and

$$\bar{\theta} \bar{v} = -\bar{U} \int_0^y \frac{\partial \bar{\theta}}{\partial x} dy \quad (40)$$

This relation is sufficient for the computation of $\bar{\theta} \bar{v}(x, y)$ from the measured $\bar{\theta}(x, y)$ but the empirical fact of simple geometrical similarity in $\bar{\theta}(x, y)$ suggests exploitation of the consequent simplification.

Assume

$$\bar{\theta}(x, y) = \bar{\theta}_o(x) f(\xi) \quad (41)$$

where $\xi = y/Y'(x)$. This transforms equation (39) to

$$\bar{\theta}_o(x) Y'(x) = \frac{H^*}{\bar{U}} \quad (42)$$

where $H^* = \frac{H}{2\rho c_p \int_0^\infty f(\xi) d\xi} = \text{Constant}$. It transforms equation (40) to

$$\bar{\theta} \bar{v} = \left(-\bar{U} \int_0^\xi f(\xi) d\xi \right) Y'(x) \frac{d\bar{\theta}_o}{dx} + \left(\bar{U} \int_0^\xi \xi \frac{df}{d\xi} d\xi \right) \bar{\theta}_o(x) \frac{dY'}{dx} \quad (43)$$

With equation (42), $\bar{\theta}_o(x)$ can be eliminated from equation (43), and after integration of the dimensionless integrals this leads to the final form for the turbulent-heat-transfer correlation,

$$\bar{\theta} \bar{v}(x, y) = H^* \xi f(\xi) \frac{1}{Y'(x)} \frac{dY'}{dx} \quad (44)$$

where

$$\xi = \frac{y}{Y'(x)}$$

The same sort of analysis can be made on equation (37) which includes the molecular conduction, but the rather large experimental scatter in the present measurements seems to make such a refinement inappropriate.

An "exchange" coefficient or "diffusion" coefficient for turbulent heat transfer k_T is simply expressible in terms of $\bar{\theta}(x, y)$. A conventional procedure for semiempirical analyses is to write for the turbulent transport an expression just like that for the molecular transport:

$$-\rho c_p \bar{\theta} \bar{v} = k_T \frac{\partial \bar{\theta}}{\partial y} \quad (45)$$

which serves as the definition of k_T .

For the simple case of equation (44), it turns out that

$$k_T = \rho c_p \bar{U} \left[\frac{\xi f(\xi)}{\frac{df}{d\xi}} \right] Y'(x) \frac{dY'}{dx} \quad (46)$$

This has the particularly interesting property that in a nondecaying turbulence at very large values of x , where the mean thermal wake spreads parabolically (reference 1), k_T becomes independent of explicit dependence on x .

A more startling simplification follows for a particular mean temperature distribution across the wake: All dependence of k_T on y disappears if

$$\frac{df}{d\xi} \propto \xi f$$

that is, if

$$f(\xi) = e^{-\frac{1}{2}\xi^2} \quad (47)$$

But this is the Gaussian function, which is found empirically to fit all the measurements within the experimental scatter. Hence one arrives at the empirical result that in both nondecaying and decaying turbulence k_T is independent of y in the thermal wake behind a line source of heat. From equation (47), $\frac{df}{d\xi} = -\xi f(\xi)$ and

$$k_T = \rho c_p \bar{U} Y'(x) \frac{dY'}{dx} \quad (48)$$

It can be seen that in the nondecaying case at very large values of x , k_T is constant and independent of both y and x .

RELATION BETWEEN SOME LAGRANGIAN AND EULERIAN PARAMETERS IN TRANSPORT

There has apparently been little effort to relate the Eulerian and Lagrangian formulations of turbulent diffusion up to the present time. Exceedingly simple boundary conditions permit some connection to be made in restricted ranges of the present problem.

For nondecaying or (with much less accuracy) decaying turbulence, equation (13) applies:

$$\bar{Y}^2 = 2 \int_0^\eta (\eta - \eta_1) R_\eta(\eta_1) d\eta_1$$

or, equation (12),

$$\begin{aligned} \frac{d\bar{Y}^2}{d\eta} &= 2Y' \frac{dY'}{d\eta} \\ &= 2 \int_0^\eta R_\eta(\eta_1) d\eta_1 \end{aligned}$$

For small values of η , $R_\eta = 1 - \frac{\eta^2}{\lambda_\eta^2}$ whence

$$\bar{Y}^2 = \eta^2 \left(1 - \frac{1}{6} \frac{\eta^2}{\lambda_\eta^2} \right) \quad (49)$$

$$\frac{d\bar{Y}^2}{d\eta} = 2\eta \left(1 - \frac{1}{3} \frac{\eta^2}{\lambda_\eta^2} \right) \quad (50)$$

Substituted into equation (44) these give

$$\bar{\vartheta} v = H^* f(\xi) \frac{y}{\eta^2} \left(1 - \frac{1}{12} \frac{\eta^2}{\lambda_\eta^2} \right) \frac{v'}{\bar{U}} \quad (51)$$

while equation (46) becomes

$$k_T = \rho c_p v' y \frac{f(\xi)}{df/d\xi} \left(1 - \frac{1}{4} \frac{\eta^2}{\lambda_\eta^2} \right) \quad (52)$$

where $\eta = \frac{1}{\bar{U}} \int_0^x v' dx$ and $\xi = \frac{y}{Y'(x)} \approx \frac{y}{\eta} \left(1 + \frac{1}{12} \frac{\eta^2}{\lambda_\eta^2} \right)$. In fact, for small values of η , $\eta \approx \frac{v'}{\bar{U}} x$.

For low-level nondecaying turbulence, $\eta = \frac{v'}{\bar{U}} x$, and equation (51) becomes

$$\bar{\vartheta} v = H^* f(\xi) \frac{y}{x^2} \left(1 - \frac{1}{12} \frac{\bar{v}^2}{\bar{U}^2} \frac{x^2}{\lambda_\eta^2} \right) \frac{\bar{U}}{v'} \quad (53)$$

while equation (52) becomes

$$k_T = \rho c_p v' y \frac{f(\xi)}{df/d\xi} \left(1 - \frac{1}{4} \frac{\bar{v}^2}{\bar{U}^2} \frac{x^2}{\lambda_\eta^2} \right) \quad (54)$$

As $\eta \rightarrow 0$ both equations (51) and (53) reduce to

$$\bar{\vartheta} v = H^* f(\xi) \frac{\bar{U}}{v'} \frac{y}{x^2} \quad (55)$$

while equations (52) and (54) reduce to

$$k_T = \rho c_p v' \frac{f(\xi)}{df/d\xi} y \quad (56)$$

At the other extreme, when η is very large,

$$\bar{Y}^2 = 2\eta L_\eta - 2M_1 \quad (57)$$

where $M_1 = \int_0^\infty \eta R_\eta(\eta) d\eta = \text{Constant}$ and

$$\frac{d\bar{Y}^2}{d\eta} = 2L_\eta \quad (58)$$

In this case,

$$\bar{\vartheta} v = H^* f(\xi) \frac{y L_\eta \frac{v'}{\bar{U}}}{[2(\eta L_\eta - M_1)]^{3/2}} \quad (59)$$

and

$$k_T = \rho c_p v' \frac{f(\xi)}{df/d\xi} \frac{y L_\eta}{[2(\eta L_\eta - M_1)]^{1/2}} \quad (60)$$

As indicated previously, the η -variation can be expressed

in terms of x or t .

If η is allowed to become large enough to make $yL_\eta \gg M_1$

$$\overline{\partial v} = H^* f(\xi) \frac{y \overline{v'}}{2^{3/2} \eta^{3/2} L_\eta^{1/2}} \quad (61)$$

$$k_T = \rho c_p v' \frac{f(\xi)}{df/d\xi} y \left(\frac{L_\eta}{2\eta} \right)^{1/2} \quad (62)$$

The above formulas take on particularly simple forms if the empirical result of a Gaussian $f(\xi)$ is utilized:

$$f(\xi) = e^{-\frac{1}{2}\xi^2}$$

Then the general expressions for $\overline{\partial v}$ and k_T (equations (44) and (46)) become

$$\overline{\partial v}(x, y) = \frac{H^*}{(Y')^2(x)} \frac{dY'}{dx} y e^{-\frac{1}{2}(\frac{y}{Y'})^2} \quad (63)$$

and

$$k_T(x, y) = \rho c_p Y'(x) \frac{dY'}{dx} \overline{U} \quad (64)$$

the latter having been deduced in the previous section. The particular forms for small values of η would follow from substitution of equations (49) and (50) into these two.

However, the most interesting form occurs for very large values of η . There $Y'(x)$ is given by equation (57) and

$$\overline{\partial v}(x, y) = \frac{H^* y L_\eta \overline{v'}}{[2(\eta L_\eta - M_1)]^{3/2}} \exp\left(\frac{-\frac{1}{4}y^2}{\eta L_\eta - M_1}\right) \quad (65)$$

$$k_T(x, y) = \rho c_p v' L_\eta = \text{Constant} \quad (66)$$

For still larger values of η , such that $M_1 \ll \eta L_\eta$,

$$\overline{\partial v}(x, y) = \frac{H^*}{2\sqrt{2}} \frac{y \overline{v'}}{\eta^{3/2} L_\eta^{1/2}} \exp\left(-\frac{1}{4} \frac{y^2}{\eta L_\eta}\right) \quad (67)$$

and

$$k_T(x, y) = \rho c_p v' L_\eta = \text{Constant}$$

The constancy of k_T for large values of η (large values of t or x) is to be expected; a treatment of molecular diffusion by this method must certainly yield a constant coefficient for times much larger than the mean free time (of flight) of the molecules—that is, for all “macroscopic times.” Put another way, the simple parabolic behavior of Y' for large values of η is a sure indication that $\bar{\theta}$ obeys the simple classical diffusion equation with constant coefficient, when viewed extremely “coarsely.”

Perhaps the chief interest of equation (66) is its identification of L_η as a significant Lagrangian length for diffusion at a large distance from the source. It enters the expression

for turbulent-diffusion coefficient in much the same way as mean free path enters the expressions for the molecular-diffusion coefficients. Furthermore, its role appears to be much like that attributed to Prandtl's “mixing length,” which was brought into the turbulent-transport problem in a more or less intuitive fashion.

Of course, the possible crude nature of Taylor's original η -postulate may render the significance of L_η more qualitative than quantitative in the case of decaying turbulence.

COMPUTATION OF RESULTS

Although Taylor's assumption of the unique dependence of $R_{\tau-\tau}$ upon η is not likely to be accurate for collapsing together cases with widely differing turbulence decay rates, it does provide a relatively simple relation between $Y'(x)$ and $R_\eta(\eta)$. Therefore all of the mean-thermal-wake data were reduced on the η -basis.

In principle the complete $R_\eta(\eta)$ curve can be obtained from $\overline{Y^2}(x)$ by double differentiation (equation (12)):

$$R_\eta = \frac{1}{2} \frac{d^2}{d\eta^2} (\overline{Y^2}) \quad (68)$$

or

$$R_\eta = \frac{1}{2} \left(\frac{\overline{U}}{v'} \right)^2 \frac{d^2}{dx^2} (\overline{Y^2})$$

However, simple double differentiation of the squares of a curve as uncertain as $Y'(x)$ seems almost hopelessly indeterminate—although Taylor (reference 2) and Collis (reference 7) have apparently followed this procedure. A somewhat more circumspect technique has been tried here: The values of λ_η and L_η were determined first, through certain limit relations (to be described). Then the R_η curve was determined by double differentiation, subject to the restrictions of agreement with the previously determined scales. Thanks to rather poor determinacy of values of λ_η and L_η , this method is not so much of an improvement as it might first appear.

LAGRANGIAN MICROSCALE λ_η

If equation (13) is restricted to very small values of η the parabolic approximation for R_η can be introduced:

$$\overline{Y^2} = 2 \int_0^\eta (\eta - \eta_1) \left(1 - \frac{\eta_1^2}{\lambda_\eta^2} \right) d\eta_1 \quad (69)$$

therefore

$$\frac{\overline{Y^2}}{\eta^2} = 1 - \frac{1}{6} \frac{\eta^2}{\lambda_\eta^2}$$

whence

$$\lambda_\eta^2 = -\frac{1}{6} \left[\frac{d}{d(\eta^2)} \left(\frac{\overline{Y^2}}{\eta^2} \right) \right]_{\eta \rightarrow 0} \quad (70)$$

The computational procedure was to plot $\overline{Y^2}/\eta^2$ against η^2 and to estimate the slope of the faired curve at $\eta=0$ where the curve must pass through unity. The abscissa intercept of the 0-tangent is $6\lambda_\eta^2$. The actual points, faired curves,

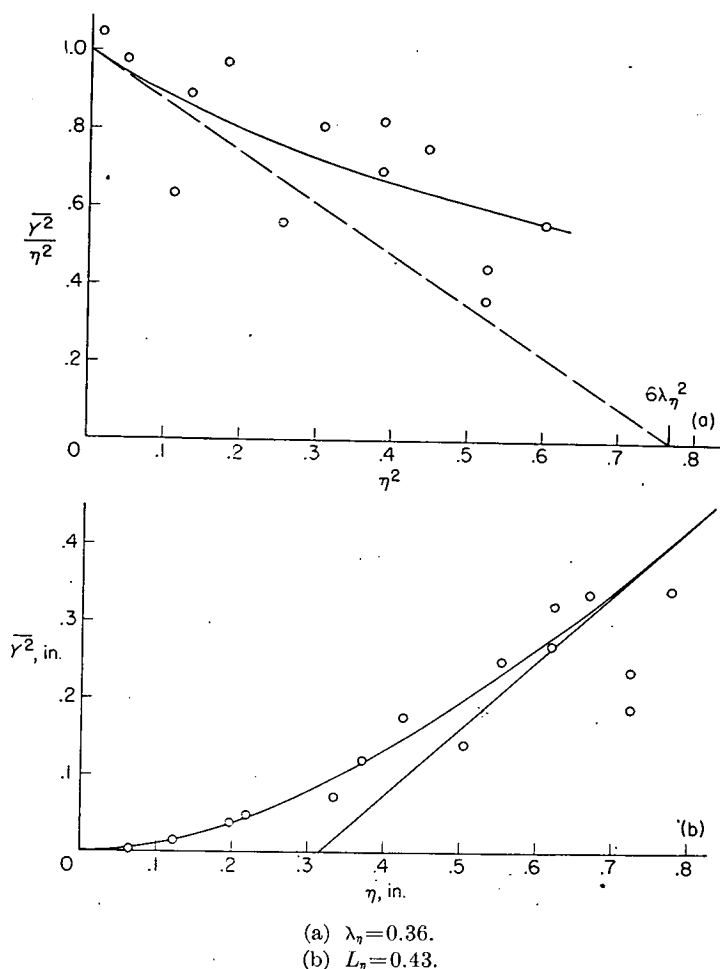


FIGURE 13.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 43.4$, $M = 1$ inch, and $\bar{U} = 8.5$ feet per second.

and tangents for all cases are presented in figures 13 to 22. Clearly the precision is poor.

LAGRANGIAN SCALE L_η

Consider equation (12) in the limit as $\eta \rightarrow \infty$. It immediately gives

$$L_\eta = \frac{1}{2} \lim_{\eta \rightarrow \infty} \left(\frac{d\bar{Y}^2}{d\eta} \right) \quad (71)$$

and the graphical procedure based on this is also presented in figures 13 to 22. Some of the asymptotic slopes drawn are not the best representation of the experimental points. This is due to the auxiliary (assumed) restriction that R_η cannot increase with increasing values of η as long as R_η has not previously dropped below zero. The graphical precision attainable is perhaps a little better here than that for λ_η^2 , but

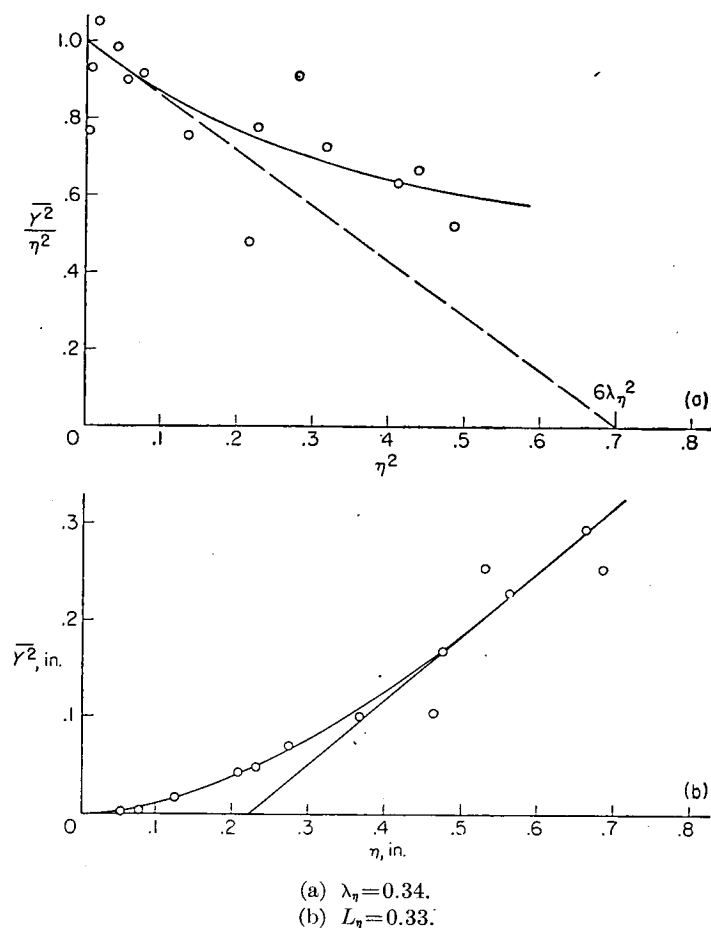


FIGURE 14.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 43.4$, $M = 1$ inch, and $\bar{U} = 25.6$ feet per second.

the square root necessary to get λ_η means that λ_η is determined about as well as is L_η .

LAGRANGIAN CORRELATION FUNCTION $R_\eta(\eta)$

With λ_η and L_η determined, the initial (small Δx) and final (large Δx) behavior of the curve $Y'(\Delta x)$ is prescribed. These parts of the curve were drawn on a graph with the experimental points. Then the fairing in of a reasonable central portion to this mean $Y'(\Delta x)$ curve was a relatively simple matter. The R_η curve was then obtained by double differentiation.

The curves drawn for $Y'(\Delta x)$ in figures 23 to 32 were determined in the fashion described above, as were the curves for R_η in the same figures.

EULERIAN MICROSCALE λ

In view of the approximate nature of the determination of λ_η , no new direct measurements were made of the Eulerian microscale λ . Instead, λ was computed with the energy

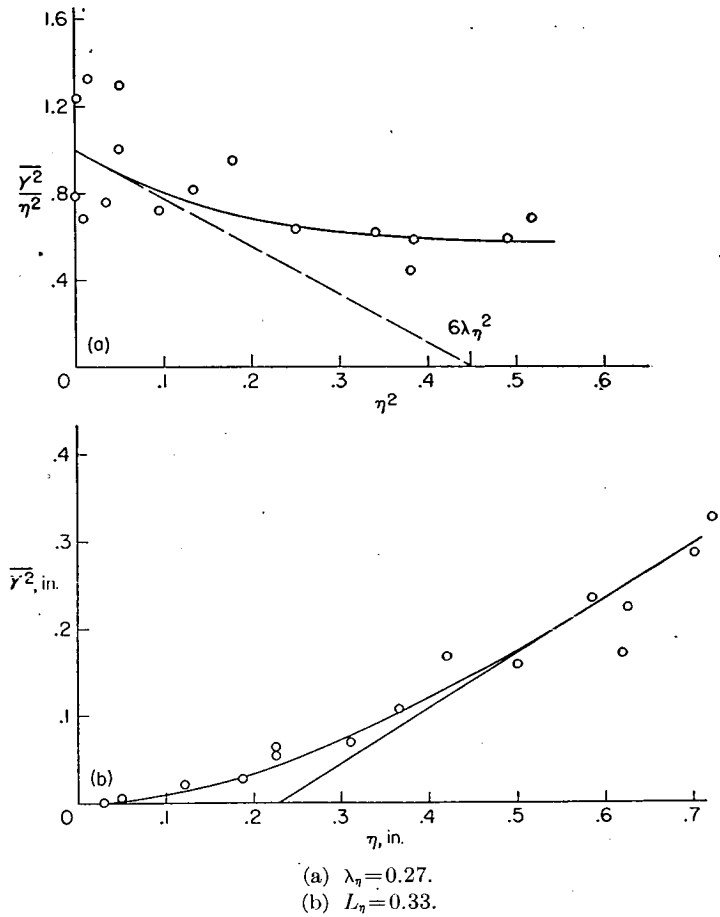


FIGURE 15.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 43.4$, $M = 1$ inch, and $\overline{U} = 38$ feet per second.

equation for isotropic turbulence, from the measurements of turbulence decay:

$$\lambda^2 = -10\nu \frac{\overline{v^2}}{d\overline{v^2}/dt} \quad (72)$$

or, with the space-time transformation,

$$\lambda^2 = -10\nu \frac{\overline{v^2}}{\overline{U} \frac{d\overline{v^2}}{dx}} \quad (73)$$

EULERIAN SCALE L

Earlier investigations have shown that the Eulerian scale in a grid-produced turbulence is closely a linear function of the mesh size of the grid producing the turbulence (for a given value of x and grid geometry) and is not significantly dependent upon the mean velocity (or grid Reynolds number, provided it is sufficient to cause turbulence). Therefore the values of L have been deduced from earlier measurements

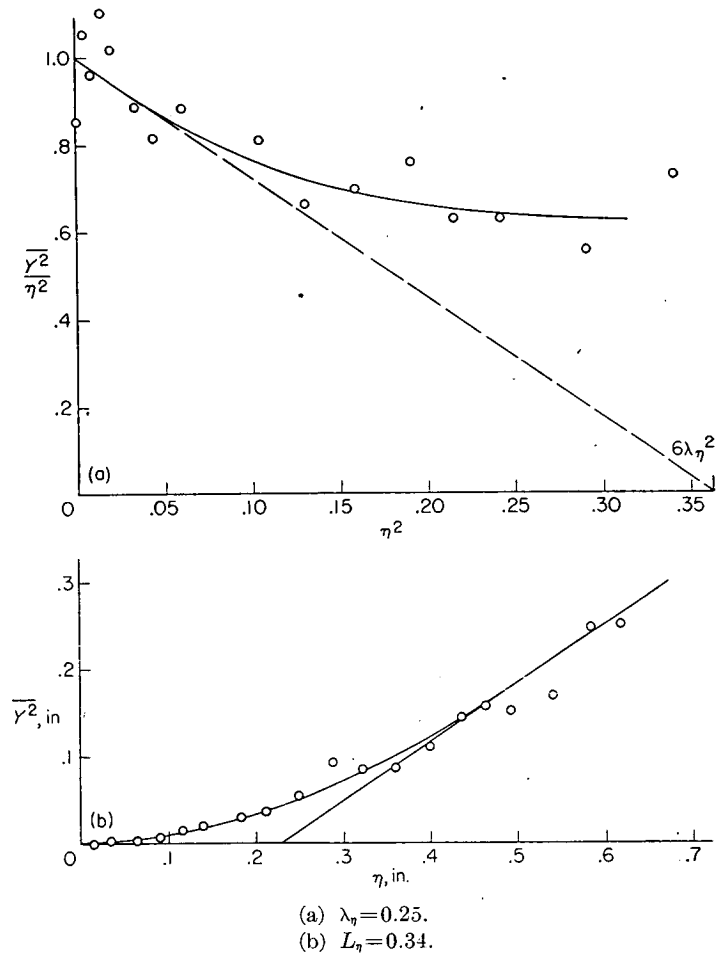


FIGURE 16.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 43.4$, $M = \frac{1}{2}$ inch, and $\overline{U} = 25.6$ feet per second.

at the California Institute of Technology (reference 18) on grids of essentially the same geometry.

Table I summarizes the results for Lagrangian and Eulerian scales and microscales. The results have been grouped to show the effect of systematic variation of one parameter at a time. Some of the results are presented in figures 33, 34, and 35.

INSTANTANEOUS SPACE-TIME TRANSFORMATION

The permissibility of an instantaneous space-time transformation in flowing turbulence,

$$\frac{\partial v}{\partial x} \approx -\frac{1}{\overline{U}} \frac{\partial v}{\partial t} \quad (74)$$

can be estimated in accordance with equations (26). For equation (74) to be valid, the sufficient requirements are those given in equations (26) that:

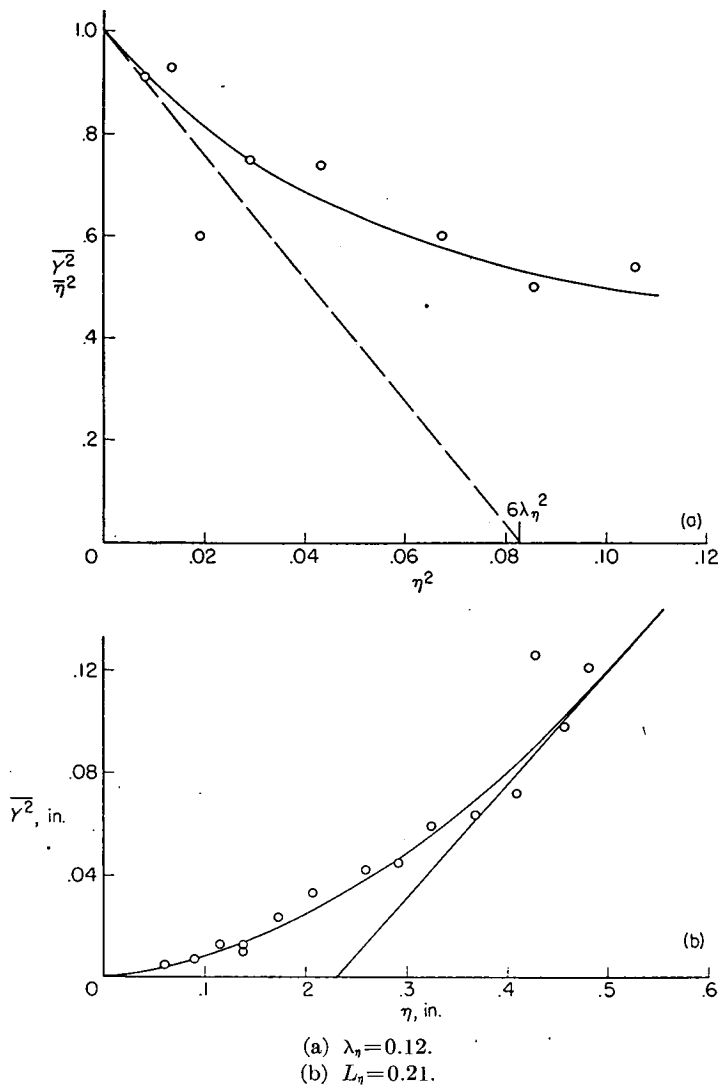


FIGURE 17.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 43.4$, $M = \frac{1}{4}$ inch, and $\bar{U} = 25.6$ feet per second.

$$T \equiv \left[\frac{\left(\frac{dv}{dt} \right)^2}{\bar{U}^2 \left(\frac{\partial v}{\partial x} \right)^2} \right]^{1/2} \ll 1$$

$$\left[\frac{u_i u_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}}{\bar{U}^2 \left(\frac{\partial v}{\partial x} \right)^2} \right]^{1/2} \ll 1$$

With the aid of equation (23), the turbulence decay equation, and the Taylor relation $\left(\frac{\partial v}{\partial x} \right)^2 = 2 \frac{\bar{v}^2}{\lambda^2}$, the first of these

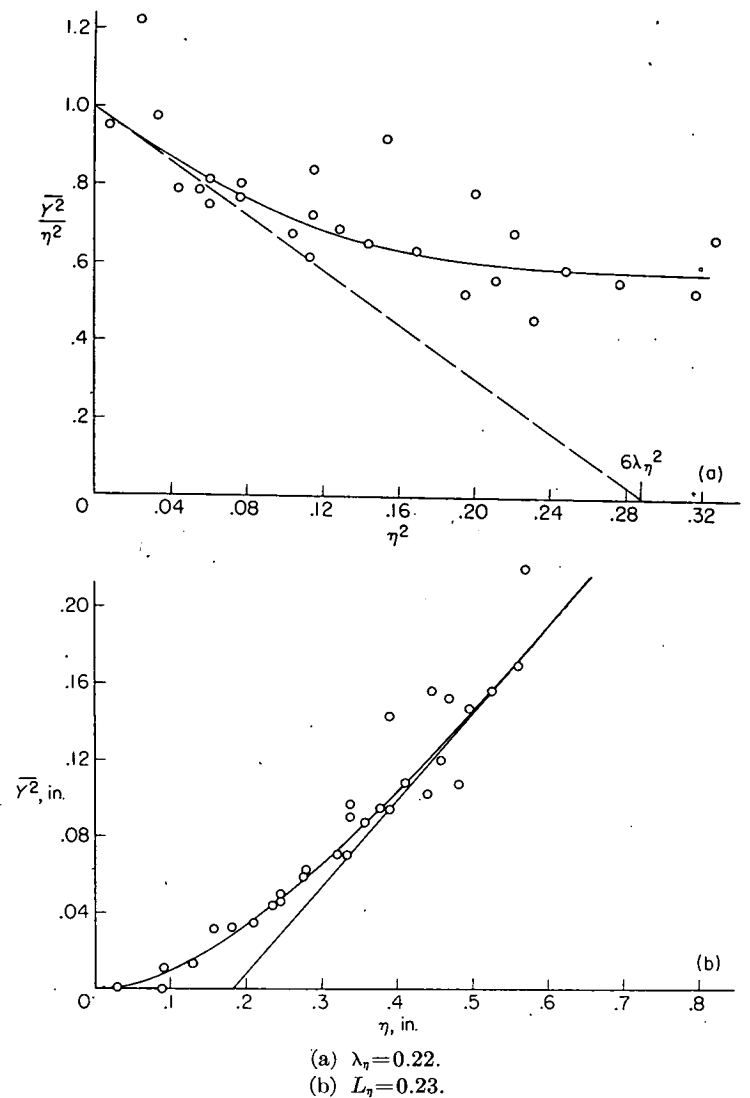


FIGURE 18.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 86.1$, $M = 1$ inch, and $\bar{U} = 25.6$ feet per second.

conditions can be written in the form

$$T \equiv \left(\frac{v'}{\bar{U}} \right) \left[\frac{12.5}{R_\lambda^2} + \left(\frac{\lambda}{\lambda_\eta} \right)^2 \right]^{1/2} \ll 1, \quad (75)$$

From the Schwarz inequality, essentially the necessity that the magnitude of any correlation coefficient be less than or equal to unity,

$$u_i u_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \leq u_i' u_j' \left(\frac{\partial v}{\partial x_i} \right)' \left(\frac{\partial v}{\partial x_j} \right)' \quad (76)$$

where the prime in this expression denotes root-mean-square

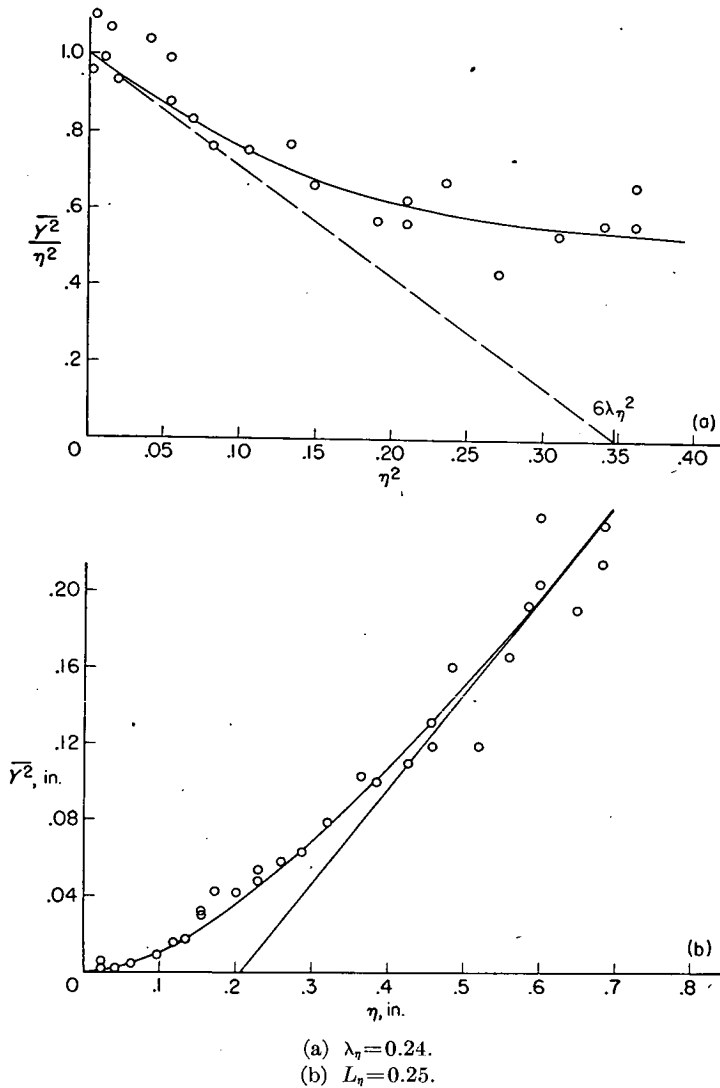


FIGURE 19.—Determination of λ_η and L_η ; $\frac{x_0}{M} = 86.1$, $M = \frac{1}{2}$ inch, and $\bar{U} = 25.6$ feet per second.

value. For isotropic turbulence, equation (76) can be written

$$u_i u_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \leq (9 + 4\sqrt{2}) \frac{(v')^4}{\lambda^2} \quad (77)$$

Thus, the second condition in equation (26) will be satisfied if

$$2.7 \frac{v'}{\bar{U}} \ll 1 \quad (78)$$

Both T and $v'/\bar{U}$ for the flows studied are presented in table I. It is clear that for these flows instantaneous x and t partial derivatives may be taken proportional with reasonable confidence.

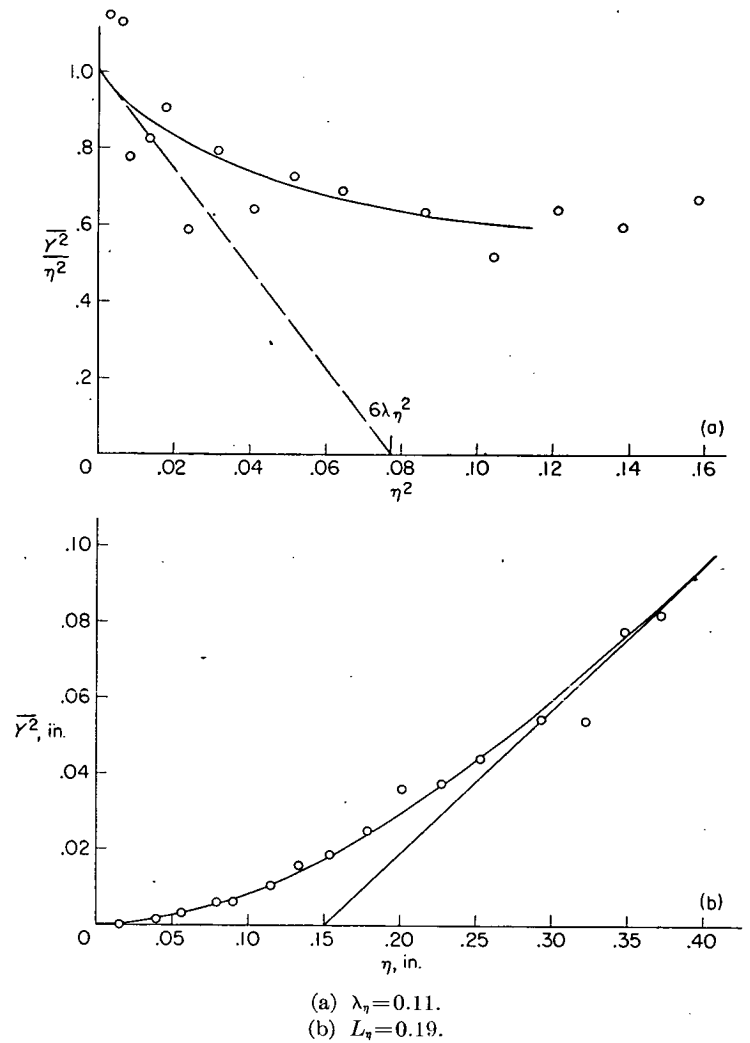


FIGURE 20.—Determination of λ_η and L_η ; $\frac{x_0}{M} = 86.1$, $M = \frac{1}{4}$ inch, and $\bar{U} = 25.6$ feet per second.

$\bar{\partial v}$ -CORRELATION

The Eulerian measure of transverse turbulent heat transport is computed from the mean temperature distribution. The dimensionless form, $\bar{\partial v}/\bar{\theta}_0 \bar{U}$, is given for two typical cross sections in figures 36 and 37.

The measurements of $v'/\bar{U}$ and of $\partial'/\bar{\theta}_0$ permit calculation of the correlation coefficient $R_{\partial v} = \frac{\bar{\partial v}}{\partial' v'}$, and this is also given in figures 36 and 37.

For the convenient and reasonably accurate assumption of Gaussian mean temperature distribution, the corresponding turbulent-heat-transfer coefficient k_T follows from equation (64). It was found to be independent of y , and typical

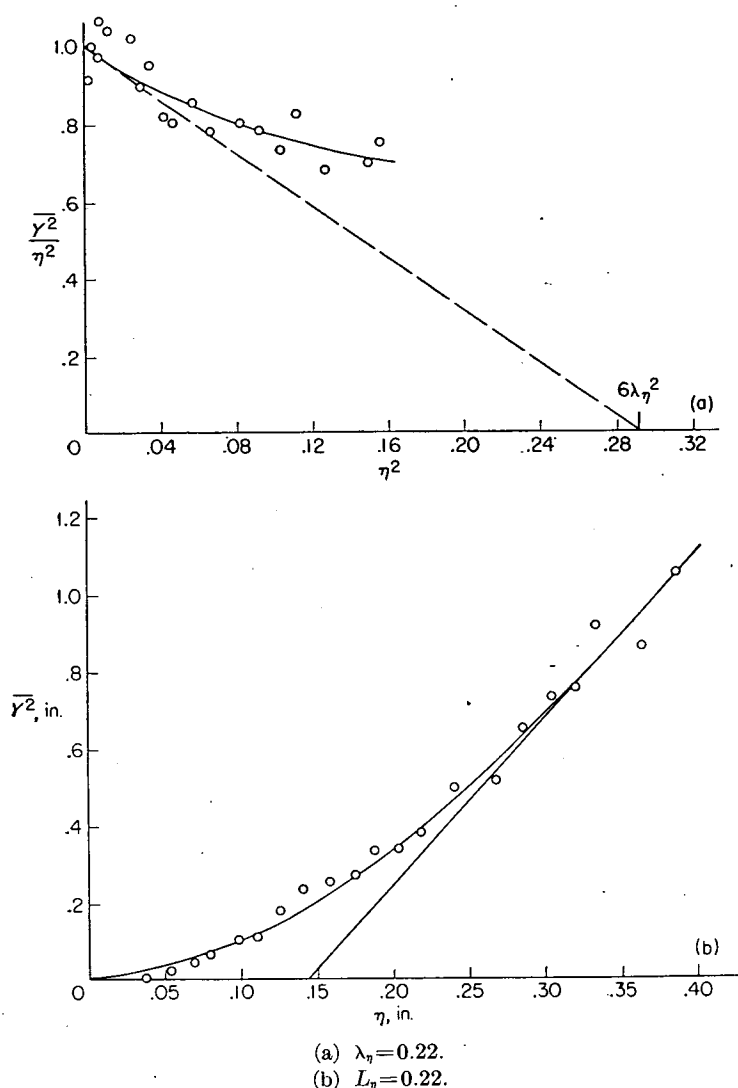


FIGURE 21.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 172.3$, $M = \frac{1}{2}$ inch, and $\bar{U} = 25.6$ feet per second.

curves of k_T/k are given in figure 38. The data for k_T at three different speeds behind the 1-inch grid are roughly collapsed together through division of k_T by $\rho c_p v' L_\eta$, as suggested by equation (66a), an asymptotic result for nondecaying turbulence (fig. 39).

DISCUSSION

LAGRANGIAN VARIABLES

Even a cursory examination of the technique used in this investigation for the determination of Lagrangian correlation shows that, as physical measurements go, this method is a "bad" one, largely because of the inherent double

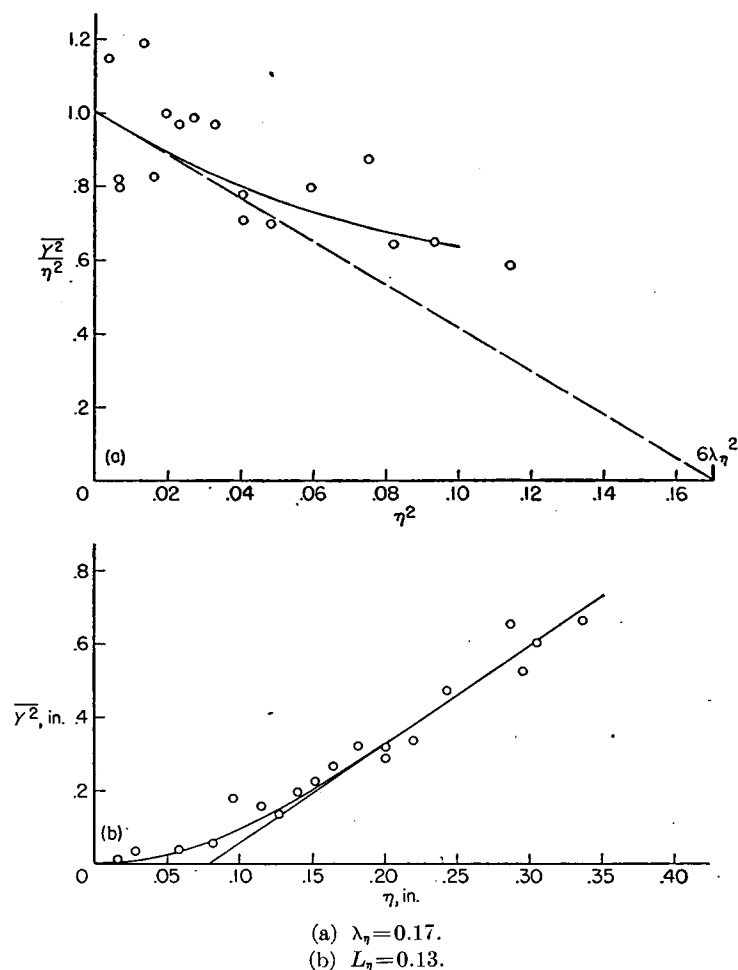
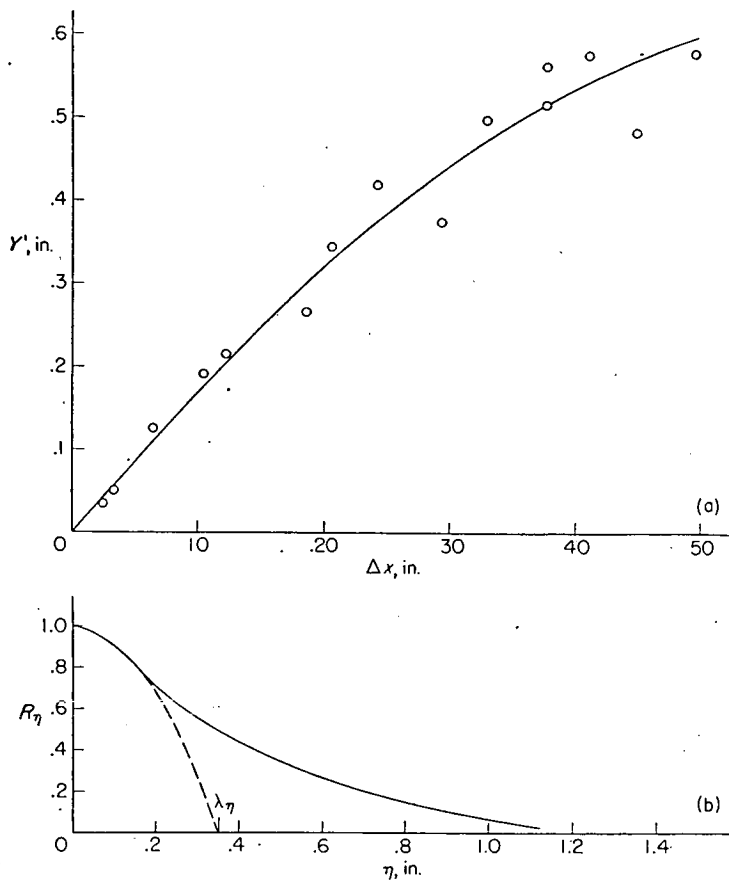


FIGURE 22.—Determination of λ_η and L_η ; $\frac{x_o}{M} = 172.3$, $M = \frac{1}{4}$ inch, and $\bar{U} = 25.6$ feet per second.

differentiation between measured variable and desired information.

Figures 13 to 22 suggest an uncertainty in values of λ_η and L_η as large as ± 20 percent, in spite of moderately good precision in the measurement of individual temperature distributions such as the lower curve in figure 3.

As mentioned earlier, the values of $v'/\bar{U}$ computed from initial wake spread are consistently higher than those measured with the hot-wire anemometer (reference 18). The same relative result was encountered during a brief investigation following that reported in reference 18. Up to the present time there has been no satisfactory explanation of the discrepancy. A tentative hypothesis which would at least account for its direction may be based upon a human weakness in the visual averaging of the reading of a fluctuation pointer; there seems to be a tendency to choose an



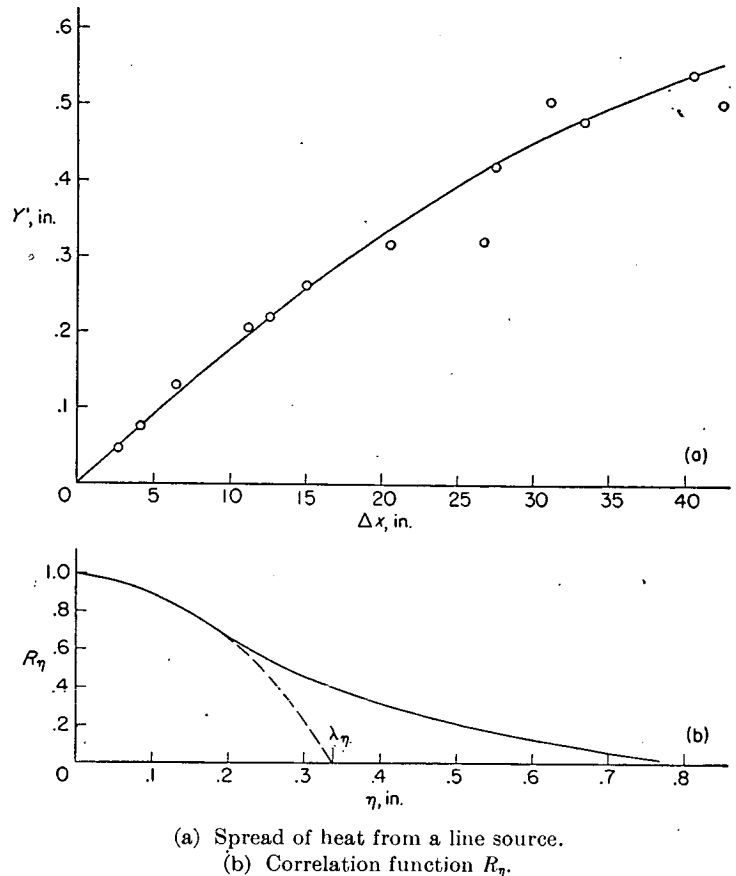
(a) Spread of heat from a line source.
(b) Correlation function R_η .

FIGURE 23.—Spread of heat from a line source and correlation function R_η for $M=1$ inch, $\bar{U}=8.5$ feet per second, and $\frac{x_0}{M}=43.4$.

“average” more or less halfway between the extremes of the needle travel. Thus a pointer motion with very skew probability density (greater than 0) would tend to be “averaged” at too high a value. The thermocouple voltage in one of these thermal wake traverses has just this character (fig. 11). Hence a visual averaging might yield too high a wake width. If this effect is nonnegligible, it is advisable to employ some electrical means of averaging for skew signals, for example, the fluxmeter and bucking circuit described in reference 10.

In view of the considerable uncertainty in λ_η as well as that in λ and v' the poor degree of agreement between experiment and theory shown in figure 33 is understandable. Since the two undetermined constants in the theoretical result have been evaluated from sets of experiments completely independent of the present ones, this agreement can be viewed as an affirmative result.

Since some sort of Lagrangian scale should be a significant



(a) Spread of heat from a line source.
(b) Correlation function R_η .

FIGURE 24.—Spread of heat from a line source and correlation function R_η for $M=1$ inch, $\bar{U}=25.6$ feet per second, and $\frac{x_0}{M}=43.4$.

length in turbulent heat and mass transport, as demonstrated in the analytical section of this report, an effort has been made to find some systematic variation in the values of L_η . Figure 34 might be construed to indicate a monotonic decrease of L_η/L with increasing values of R_L . It is interesting to note that a decrease was also observed for the ratio of mixing length to tube radius by Nikuradse (reference 30) in fully developed turbulent tube flows. In order to determine whether these two rates of decrease with increasing Reynolds numbers are of the same order of magnitude, an estimate has been made of the magnitudes of R_L corresponding to Nikuradse's results given in figures 28 and 29 of reference 30. Both scale-to-diameter ratio and average turbulent levels for various Reynolds numbers were estimated with the help of Laufer's data on turbulent channel flow (reference 31) at various Reynolds numbers. The absolute level (i. e., the ordinate scale) of the resulting l_{max}/L against R_L curve was adjusted to give the most reasonable-looking fit with the L_η/L data. This is the dashed line in

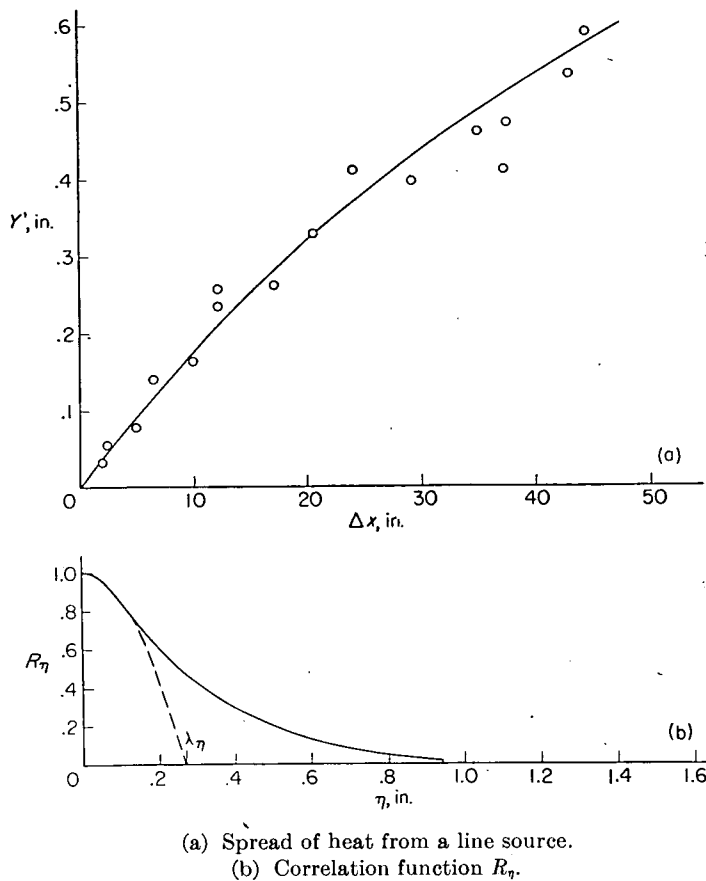


FIGURE 25.—Spread of heat from a line source and correlation function R_η for $M=1$ inch, $\bar{U}=38$ feet per second, and $\frac{x_o}{M}=43.4$.

figure 34 and it shows at least a qualitative resemblance. It is likely, however, that L_η/L is not a unique function of R_L .

Of course not all of the scatter in figures 34 and 35 (involving L_η) can be attributed to simple lack of experimental precision. Some is evidence of the fact that the Taylor postulate of Lagrangian correlation function being uniquely a function of η is certainly not very closely true. Furthermore, table I does show rather systematic variations of L_η in some of the three-point groups. Most noticeably, there is a regular decrease in L_η with increasing x_o/M (or perhaps with decreasing $v'/\bar{U}$) for each of the three grids.

TEMPERATURE-FLUCTUATION FIELD

Fairly close behind a line heat source in turbulent flow, the random pulse nature of the temperature fluctuations at a fixed point has been established by the oscillogram in reference 10. This is confirmed by the first two oscillograms in figure 11, with their highly skew probability densities at

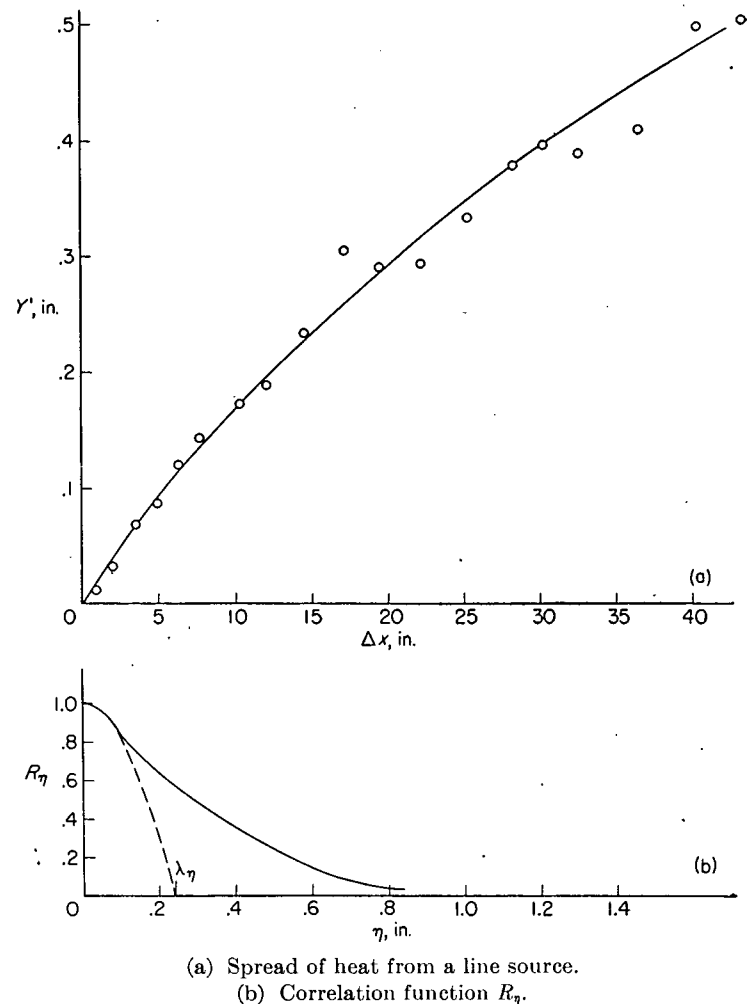
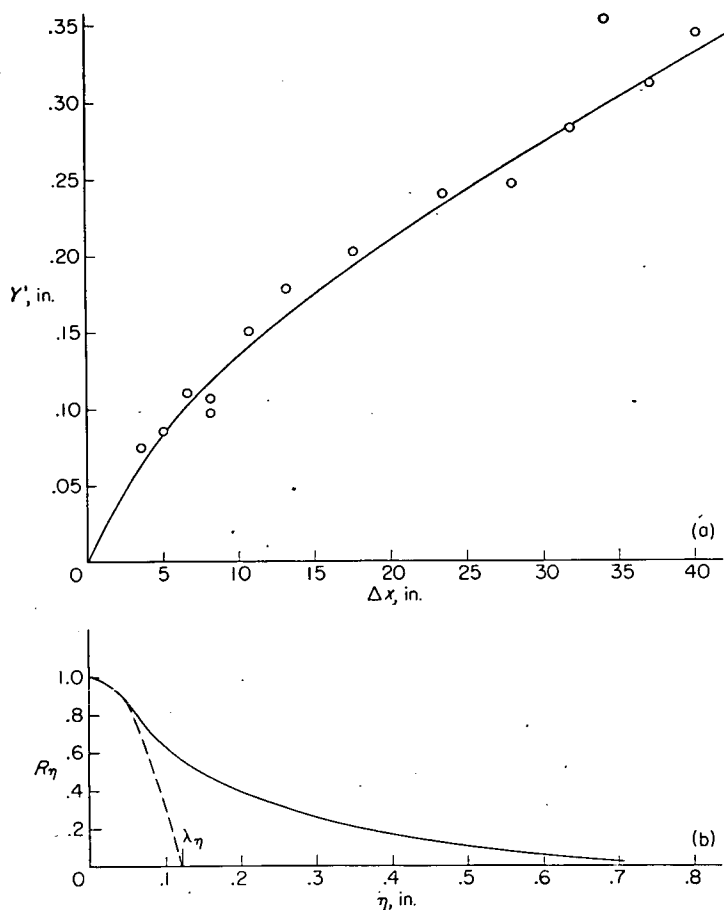


FIGURE 26.—Spread of heat from a line source and correlation function R_η for $M=\frac{1}{2}$ inch, $\bar{U}=25.6$ feet per second, and $\frac{x_o}{M}=43.4$.

$\Delta x/M=10$. One of the objectives of the present investigation was to find out whether this distinctly pulsed character persisted far downstream or whether molecular heat conduction becomes increasingly effective in smearing out the pulses, until they are no longer distinguishable as such. The third oscillogram and probability density in figure 11 ($\Delta x/M=70$) does show a decided trend away from the pulse-type signal. The molecular broadening of the laminar wake (corresponding to the pulses) decreases the relative spacing of the pulses in $\vartheta(t)$ at any point in the "turbulent wake" region. This is a reduction in relative length of the flat ($\vartheta=0$) base lines between pulses, giving greater statistical symmetry in $\vartheta(t)$ about its mean, that is, reducing the skewness of $P(\vartheta)$.

A simple analysis will show the existence of an asymptotic



(a) Spread of heat from a line source.
(b) Correlation function R_η .

FIGURE 27.—Spread of heat from a line source and correlation function R_η for $M = \frac{1}{4}$ inch, $\bar{U} = 25.6$ feet per second, and $\frac{x_0}{M} = 43.4$.

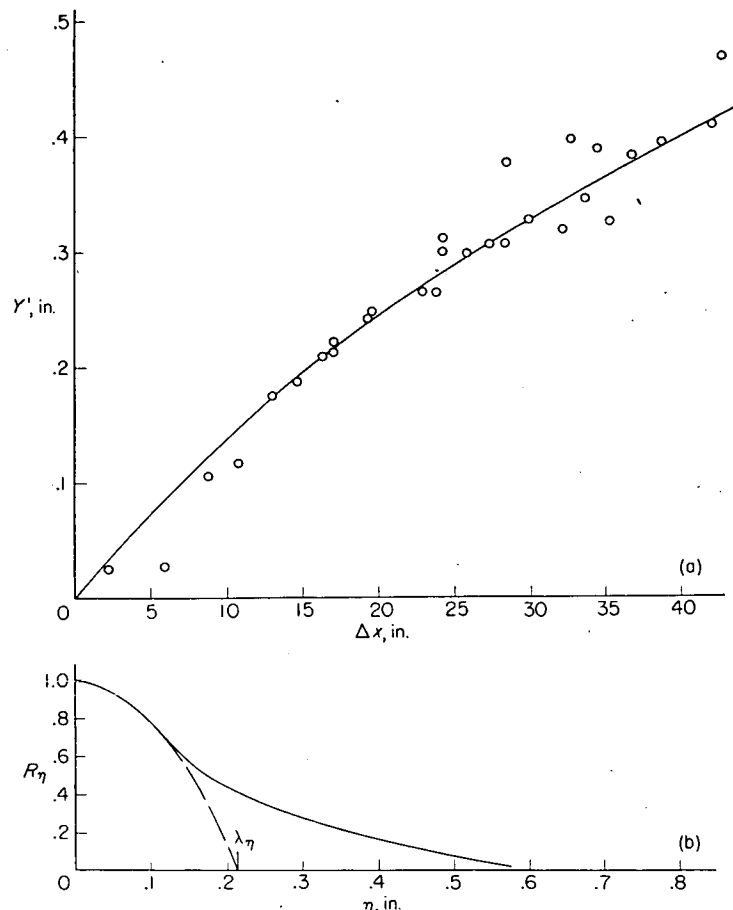
behavior of molecular-conduction effects in a nondecaying turbulence. For a nondecaying flow turbulence and very large values of $t(\approx x/\bar{U})$ the mean-square wake spread due to turbulent motion is

$$\bar{Y}^2 \approx 2L_\eta \frac{v'}{\bar{U}} x \quad (79)$$

On the other hand, an approach to molecular diffusion through Taylor's concept of "continuous movements" gives, for any macroscopic distance downstream, the mean-square thermal wake width,

$$\bar{Y}_m^2 \approx 2\Lambda \frac{c}{\bar{U}} x \quad (80)$$

where Λ is the mean free path and c is the root-mean-square



(a) Spread of heat from a line source.
(b) Correlation function R_η .

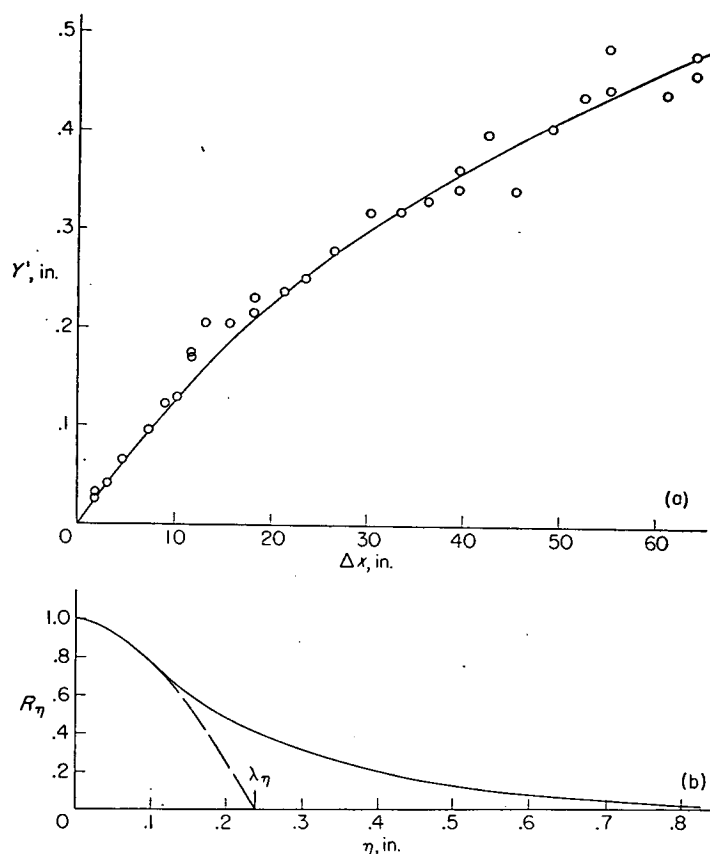
FIGURE 28.—Spread of heat from a line source and correlation function R_η for $M = 1$ inch, $\bar{U} = 25.6$ feet per second, and $\frac{x_0}{M} = 86.1$.

molecular velocity.

From equations (79) and (80)

$$\frac{Y_m'}{Y'} \approx \left(\frac{\Lambda}{L_\eta} \frac{c}{v'} \right)^{1/2} \quad (81)$$

For a typical case, take $L = 1$ centimeter, $v' = 10$ centimeters per second, $\Lambda = 6 \times 10^{-6}$ centimeter, and $c = 5 \times 10^4$ centimeters per second. Then $\frac{Y_m'}{Y'} \approx 0.17$. For people accustomed to thinking of molecular transport as negligibly small in turbulent flow (e. g., in shear flow), this ratio will appear quite large. The values of k_T/k plotted in figure 38 also show that at these low turbulence levels the molecular thermal conductivity is not necessarily negligible compared with the turbulent transport.



(a) Spread of heat from a line source.

(b) Correlation function R_η .

FIGURE 29.—Spread of heat from a line source and correlation function

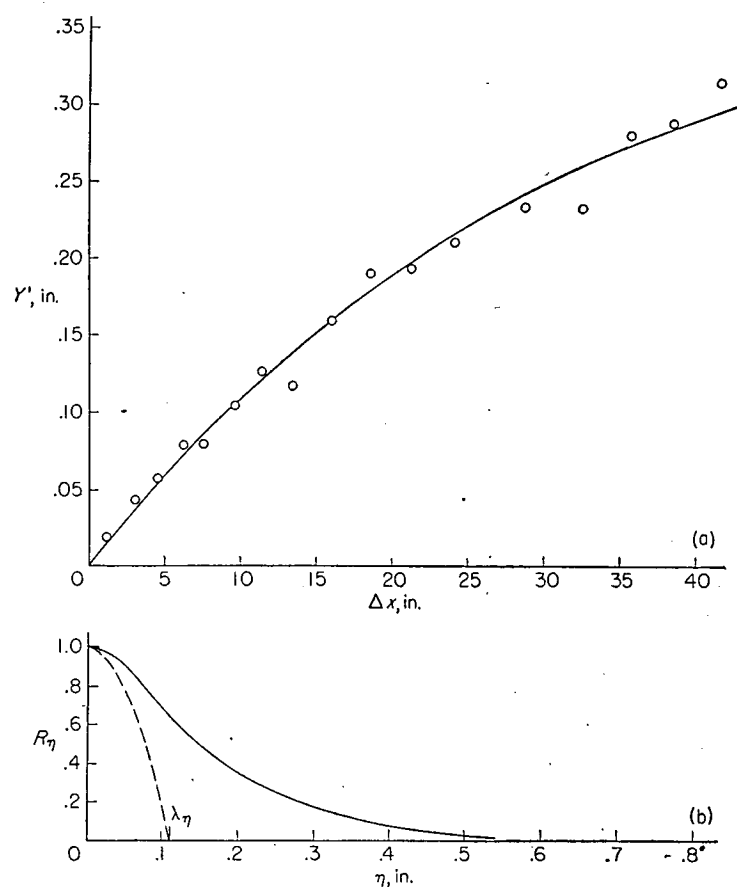
R_η for $M = \frac{1}{2}$ inch, $\bar{U} = 25.6$ feet per second, and $\frac{x_o}{M} = 86.1$.

The temperature-fluctuation-level distributions $\vartheta'/\bar{\theta}$ across the wake (figs. 10 (a) and 10 (b)) show the same character as that measured at much higher turbulence level in a jet (reference 10), with somewhat lower minimum values, which are attributable to the lower turbulence level. A rough evaluation of the behavior of the statistical variables in this turbulent thermal wake is obtainable by recalling that it consists of a randomly "waving" laminar thermal wake. If $\vartheta(t)$ is crudely represented by a randomly spaced sequence of identical rectangular pulses with height ϑ_o , width j , and average spacing s , it is easily seen that

$$\bar{\theta} = \frac{\vartheta_o j}{s} \quad (82)$$

$$\frac{\vartheta'}{\bar{\theta}} = \left(\frac{s}{j} - 1 \right)^{1/2} \quad (83)$$

This permits $\vartheta'/\bar{\theta}$ to vary between 0 and ∞ as s/j travels the permissible range from 1 to ∞ . Since points nearer the edge



(a) Spread of heat from a line source.

(b) Correlation function R_η .

FIGURE 30.—Spread of heat from a line source and correlation function

R_η for $M = \frac{1}{4}$ inch, $\bar{U} = 25.6$ feet per second, and $\frac{x_o}{M} = 86.1$.

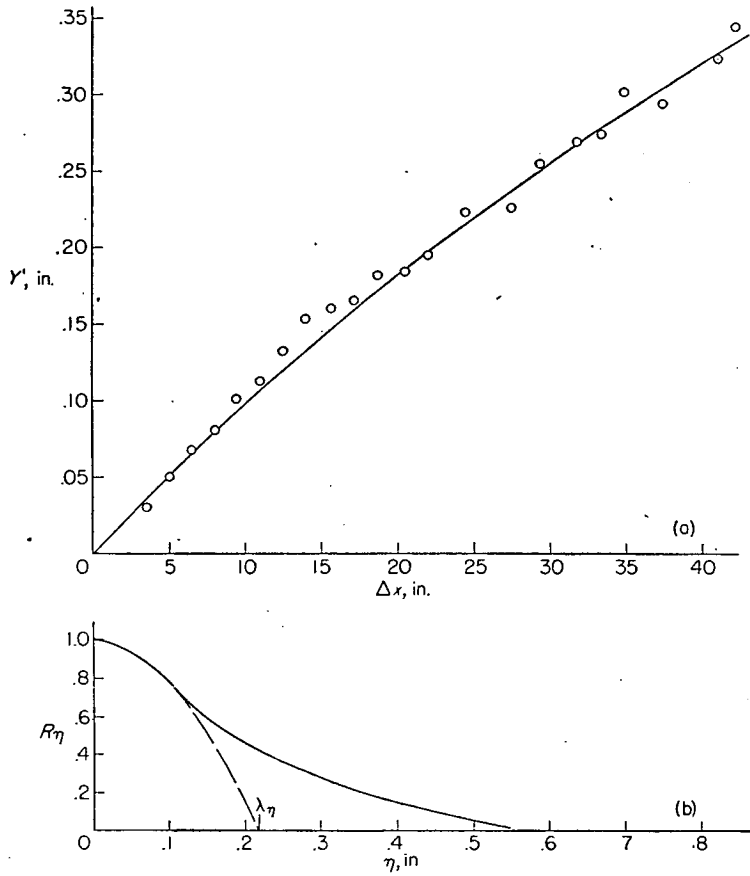
of the turbulent thermal wake have higher values of s/j , the behavior of equation (83) is consistent with the experimental distribution. If the analysis were repeated with triangular pulses, for example, the quantitative estimate would doubtless be more realistic. The higher values of $(\vartheta'/\bar{\theta})_{min}$ encountered at higher values of $v'/\bar{U}$ are an indication that for a given width of laminar thermal wake, the higher value of $v'/\bar{U}$ leads to a higher minimum value of s/j .

The closely Gaussian shape of $\bar{\theta}/\bar{\theta}_o$ against y has already been pointed out. If

$$\frac{\bar{\theta}}{\bar{\theta}_o} = \exp \left[-\frac{1}{2} \frac{y^2}{(Y')^2} \right]$$

be introduced, there results

$$\frac{\vartheta'}{\bar{\theta}} = \left\{ \frac{\vartheta_o}{\bar{\theta}_o} \exp \left[\frac{1}{2} \frac{y^2}{(Y')^2} \right] - 1 \right\}^{1/2} \quad (84)$$



(a) Spread of heat from a line source.
(b) Correlation function R_η .

FIGURE 31.—Spread of heat from a line source and correlation function R_η for $M = \frac{1}{2}$ inch, $\bar{U} = 25.6$ feet per second, and $\frac{x_o}{M} = 172.3$.

and

$$\frac{\vartheta'}{\bar{\theta}_o} = \exp \left[-\frac{1}{2} \frac{y^2}{(\bar{Y}')^2} \right] \left\{ \frac{\vartheta_o}{\bar{\theta}_o} \exp \left[\frac{1}{2} \frac{y^2}{(\bar{Y}')^2} \right] - 1 \right\}^{1/2} \quad (85)$$

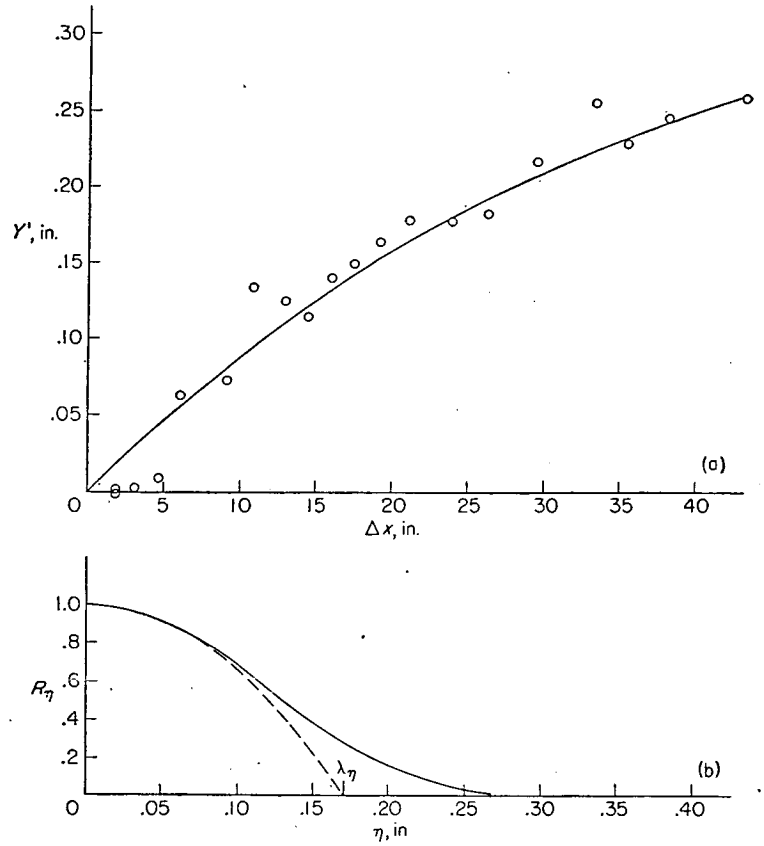
Both of these expressions have behavior consistent with the experiments.

The form of dimensionless transverse turbulent-heat-transfer rate $\frac{\partial v}{\partial \theta_o \bar{U}}$ can be deduced for small values of Δx (such that $R_\eta \approx 1$) with this pulse representation of $\vartheta(t)$. In this picture, ϑv is the correlation between a continuous random variable v and a random pulse signal ϑ which "fires" every time the continuous variable passes through a specific value

$$v_1 = \frac{y}{\Delta x} \bar{U}$$

Therefore

$$\frac{\partial v}{\partial \theta_o} = \frac{\vartheta_o}{s} \frac{y}{\Delta x} \bar{U}$$



(a) Spread of heat from a line source.
(b) Correlation function R_η .

FIGURE 32.—Spread of heat from a line source and correlation function R_η for $M = \frac{1}{4}$ inch, $\bar{U} = 25.6$ feet per second, and $\frac{x_o}{M} = 172.3$.

$$\frac{\partial v}{\partial \theta_o} = \frac{\bar{U}}{\Delta x} \bar{\theta} y \quad (86)$$

where s and $\bar{\theta}$ are functions of y . With Gaussian $\bar{\theta}(y)$,

$$\frac{\partial v}{\partial \theta_o \bar{U}} = \frac{y}{\Delta x} \exp \left[-\frac{1}{2} \frac{y^2}{(\bar{Y}')^2} \right] \quad (87)$$

The direct comparison between this crude picture and the experimental results will be confined to the correlation coefficient $R_{\vartheta v} = \frac{\partial v}{\partial \theta_o} \frac{\vartheta_o}{\bar{\theta}_o}$. This is of particular interest in view of the surprisingly high experimental values. With equation (85) and the fact that $v'/\bar{U} = Y'/\Delta x$, there results

$$R_{\vartheta v} = \frac{y}{\bar{Y}'} \left\{ \frac{\vartheta_o}{\bar{\theta}_o} \exp \left[\frac{1}{2} \frac{y^2}{(\bar{Y}')^2} \right] - 1 \right\}^{-1/2} \quad (88)$$

This contains the undetermined constant $\vartheta_o/\bar{\theta}_o$, which can be obtained from any one of several experimental results. Figure 36 includes one plot of equation (88) with $\vartheta_o/\bar{\theta}_o$ de-

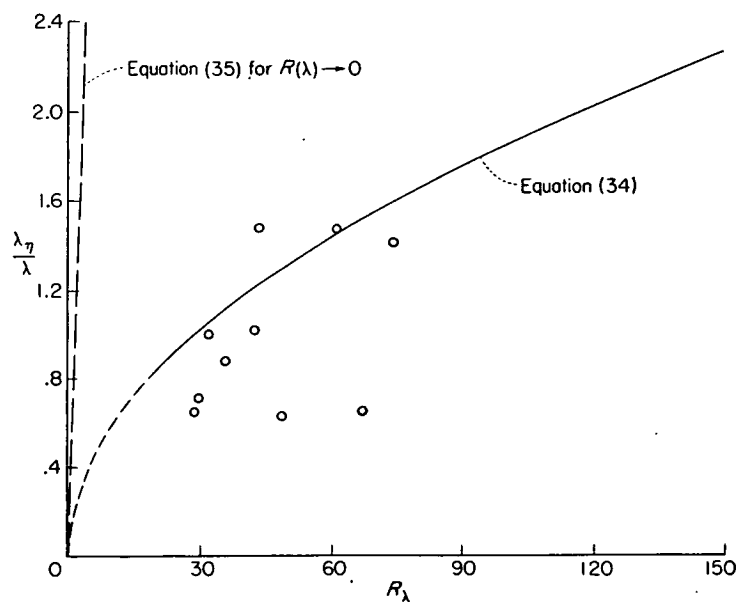


FIGURE 33.—Ratio of Lagrangian and Eulerian microscales.

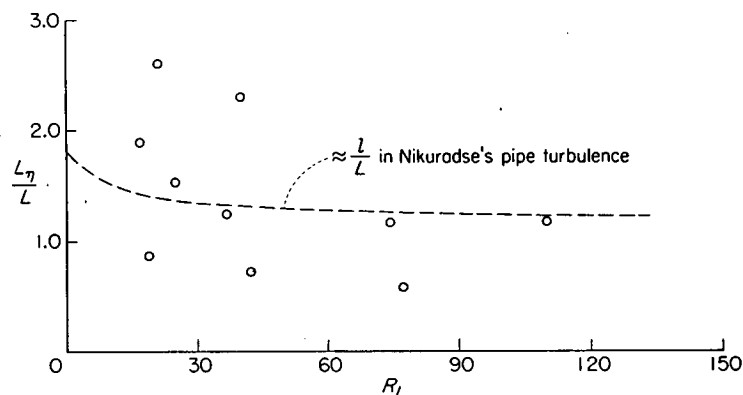


FIGURE 34.—Ratio of Lagrangian and Eulerian scales.

terminated from equation (84) and the experimental value of $(\vartheta'/\bar{\theta})_{min}$ and one plot with $\vartheta_o/\bar{\theta}$ determined by matching equation (88) with the experimental result at $y/Y' = 1.4$.

It is also surprising to find that the experimental $(R_{\vartheta v})_{max}$ at large values of Δx is even larger than that at small values of Δx . This may be due to a considerable experimental error; the resistance-thermometer voltage signal is much lower here. Unfortunately no relation corresponding to equation (88) has been deduced for large values of Δx , where R_η is essentially zero.

The criteria for Taylor's hypothesis of the interchangeability of instantaneous space $\bar{U} \frac{\partial}{\partial x}$ and time $\frac{\partial}{\partial t}$ derivatives (assumed by him to depend only upon turbulence level) have been expressed in equations (75) and (78) as functions of turbulence level R_λ and λ/λ_η . If λ/λ_η is replaced by its theoretical expression (equation (34)) in terms of R_η , equation (75) becomes

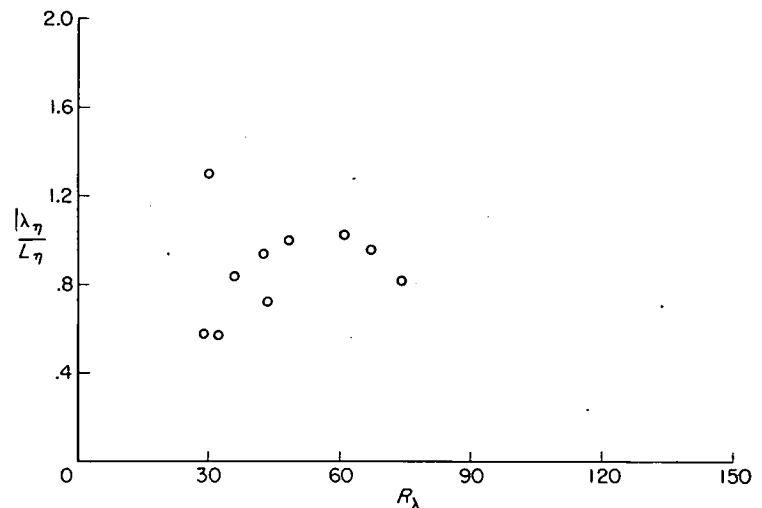
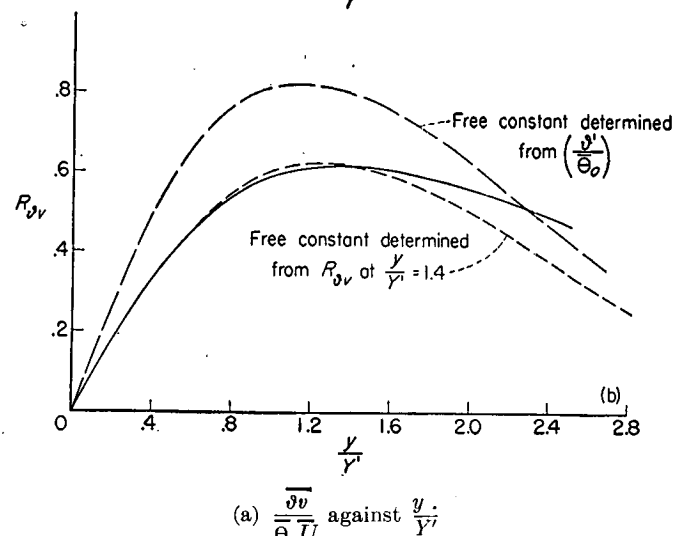
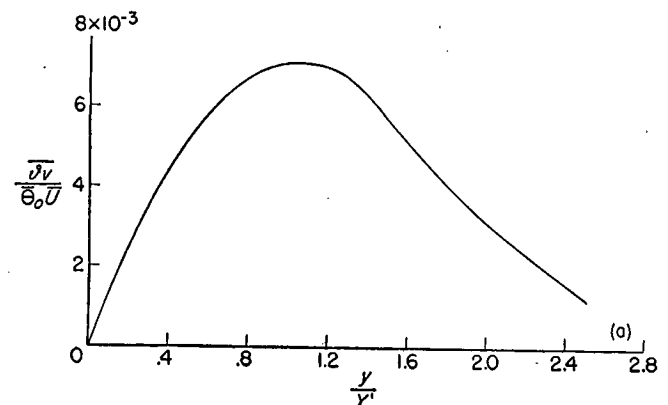


FIGURE 35.—Ratio of Lagrangian microscale and scale.



(a) $\frac{\overline{\vartheta v}}{\bar{\theta}_o \bar{U}}$ against $\frac{y}{Y'}$.
 (b) $R_{\vartheta v}$ against y/Y' . Dashed curves are result of analysis based on rectangular temperature pulses.

FIGURE 36.—Heat-transfer correlation across thermal wake, computed from measured mean temperature distribution. $\bar{U} = 25.6$ feet per second, $M = 1$ inch, $x_o = 43.4$ inches, and $\Delta x = 10.5$ inches.

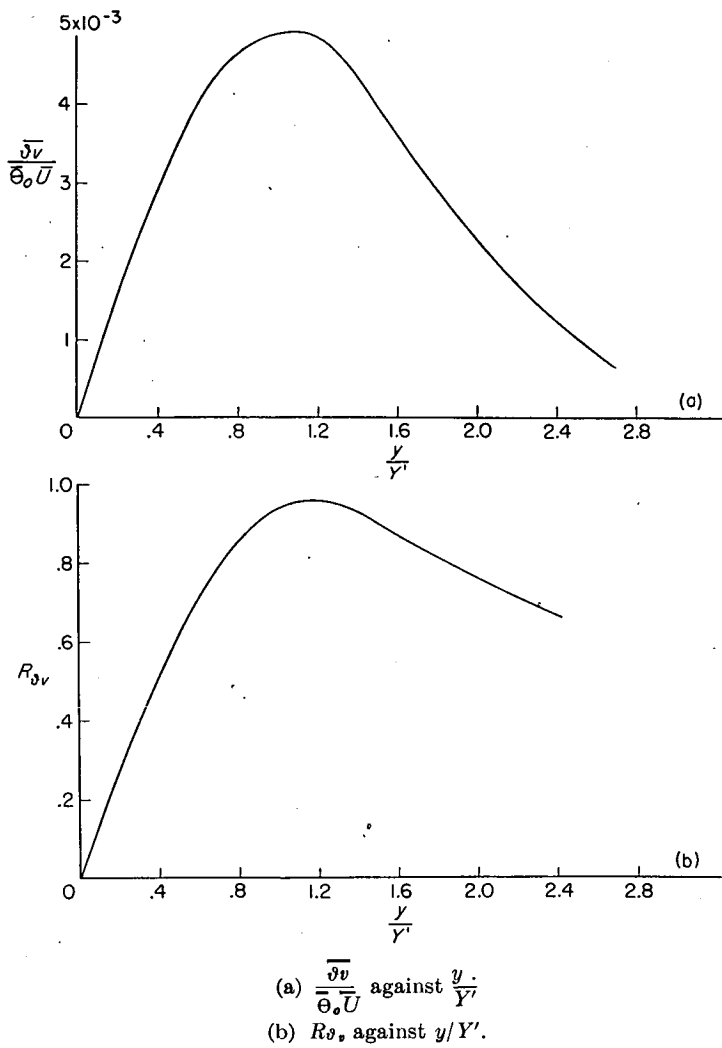


FIGURE 37.—Heat-transfer correlation across thermal wake, computed from measured mean temperature distribution. $\bar{U}=25.6$ feet per second, $M=1$ inch, $x_0=43.4$ inches, and $\Delta x=70$ inches.

$$\frac{v'}{\bar{U}} \left(\frac{25}{R_\lambda^2} + \frac{29}{R_\lambda} \right)^{1/2} \ll 1 \quad (89)$$

For estimates of most flows the first term in the parentheses can be neglected; values of R_λ less than 5 or 10 are rare.

Since equation (34) has now been roughly verified by experiment, equations (89) and (78) may serve as approximate criteria for the validity of Taylor's hypothesis.

In the limit of $R_\lambda \rightarrow 0$ when equation (35) replaces equation (34), there follows a simpler criterion to replace equation (75):

$$\frac{4}{R_\lambda} \left(\frac{v'}{\bar{U}} \right) \ll 1 \quad (90)$$

SUMMARY OF RESULTS

The following results were obtained from the investigation of the diffusion of heat from a line source in isotropic turbulence.

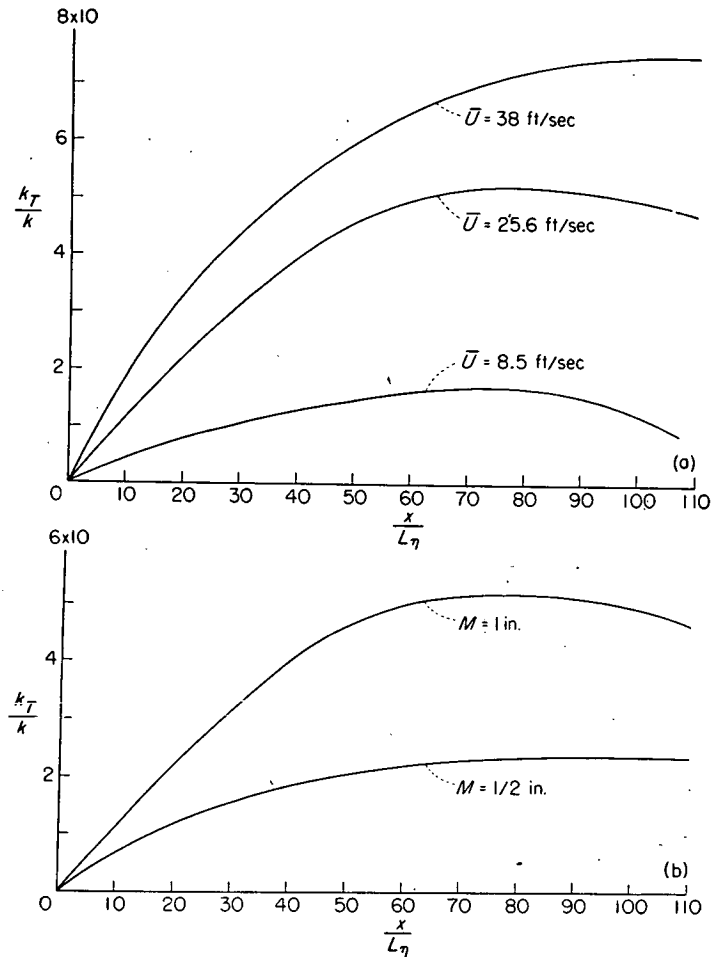


FIGURE 38.—Turbulent-heat-transfer coefficient for three airspeeds and two different grids.

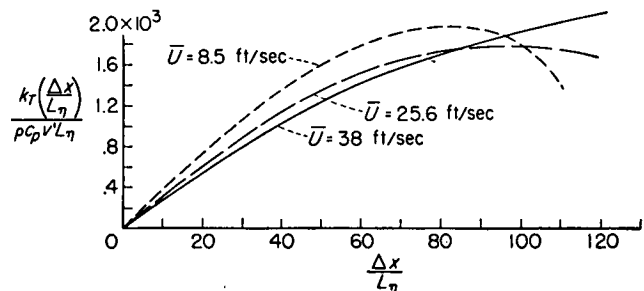


FIGURE 39.—Dimensionless turbulent-heat-transfer coefficient.

$$\frac{x_0}{M}=43.4 \text{ and } M=1 \text{ inch.}$$

1. The thermal wake behind a heated wire set perpendicular to a flowing isotropic turbulence (at sufficiently low wire Reynolds number) consists of a randomly "waving" thin, laminar, thermal wake whose variations in lateral position give what may be called the turbulent thermal wake. At a fixed point not too far behind the wire the instantaneous temperature difference $\vartheta(t)$ is a random pulse function, and the nature of the turbulent heat transfer can be deduced on this basis. Farther downstream the distinct pulse nature tends to disappear.

2. The mean transverse temperature distribution $\frac{\bar{\theta}}{\theta_0}(y)$ appears to be Gaussian within the experimental precision for all distances behind the wire.

3. An Eulerian analysis of this turbulent-heat-transfer problem permits computation of the turbulent-heat-transfer coefficient k_T which is essentially constant with respect to the distance in the direction of the measured diffusion y for these boundary conditions. It is found that at low turbulence levels (approximately equal to 1 to 2 percent) the molecular heat transport is not vanishingly small compared with the turbulent heat transport.

4. Although Taylor's postulate that Lagrangian correlations in decaying turbulence can be made similar by introduction of an independent variable $\eta = \int_0^t v'(t) dt$ (where t is time and v' is the root-mean-square instantaneous velocity fluctuation in the y -direction) seems to be an oversimplification, it has been applied here for convenience in the reduction of data. A simple comparison of Eulerian and Lagrangian analyses for diffusion in nondecaying turbulence shows that for large values of the distance from the heat source Δx the Lagrangian scale L_η enters the expression for k_T , the turbulent-heat-transfer coefficient, much like the empirical mixing length in the old turbulent transport theories. Therefore some properly modified generalization of Taylor's η -postulate should prove useful.

5. A correction and generalization of Heisenberg's theoretical expression for the ratio of Eulerian to Lagrangian microscale λ/λ_η as a function only of the turbulence Reynolds number based on microscale R_λ has been made and seems to agree roughly with experiment. It must be noted that since λ_η depends only upon a transformation $d\eta = v' dt$, and not upon the integral postulate stated above, its validity is not impaired by any failure of the integral postulate.

6. Taylor's hypothesis for the interchangeability of space and time derivatives at low turbulence levels has been expressed in terms of criteria which depend upon turbulence level, Reynolds number, and λ/λ_η . Applied to the flows studied here it shows that in these cases such a transformation is permissible. By substitution of the theoretical expression for $\frac{\lambda}{\lambda_\eta}(R_\lambda)$, a slightly simpler and rougher criterion is derived, depending only upon turbulence level and R_λ .

THE JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MD., June 5, 1951.

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TABLE I
RESULTS FOR LAGRANGIAN AND EULERIAN
SCALES AND MICROSCALES

$\bar{U}$ (fps)	M (in.)	x_0/M	$v'/\bar{U}$ (percent)	λ (in.)	λ_g (in.)	L (in.)	L_g (in.)	λ_g/λ	L_g/L	R_λ	R_{L_g}	T
8.5	1	43.4	2.0	0.41	0.36	0.28	0.43	0.88	1.54	36	25	0.0228
25.6	1	43.4	2.0	.23	.34	.28	.33	1.48	1.18	61	74	.0136
38.0	1	43.4	2.0	.19	.27	.28	.33	1.42	1.18	74	110	.0150
25.6	1	43.4	2.0	0.23	0.34	0.28	0.33	1.48	1.18	61	74	0.0136
	$\frac{1}{2}$	43.4	2.0	.165	.245	.15	.34	1.48	2.3	43.5	40	.0136
	$\frac{1}{4}$	43.4	2.0	.12	.12	.08	.21	1.00	2.6	32	21	.0202
25.6	1	86.1	1.5	0.34	0.22	0.39	0.23	0.65	0.59	67	77	0.0225
	$\frac{1}{2}$	86.1	1.4	.23	.235	.20	.25	1.02	1.25	42.5	37	.0138
	$\frac{1}{4}$	86.1	1.3	.17	.11	.10	.19	.65	1.9	29	17	.0203
25.6	$\frac{1}{4}$	43.4	2.0	0.165	0.245	0.15	0.34	1.48	2.3	43.5	40	0.0136
		86.1	1.4	.23	.235	.20	.25	1.02	1.25	42.5	37	.0138
		172.3	1.05	.35	.22	.30	.22	.63	.73	49	42	.0164
25.6	$\frac{1}{4}$	43.4	2.0	0.12	0.12	0.08	0.21	1.00	2.6	32	21	0.0202
		86.1	1.3	.17	.11	.10	.19	.65	1.9	29	17	.0203
		172.3	.95	.24	.17	.15	.13	.71	.87	30	19	.0137

REPORT 1143

A VECTOR STUDY OF LINEARIZED SUPERSONIC FLOW APPLICATIONS TO NONPLANAR PROBLEMS¹

By JOHN C. MARTIN

SUMMARY

A vector study of the partial-differential equation of steady linearized supersonic flow is presented. General expressions, which relate the velocity potential in the stream to the conditions on the disturbing surfaces, are derived. In connection with these general expressions the concept of the finite part of an integral is discussed.

A discussion of problems dealing with planar bodies is given and the conditions for the solution to be unique are investigated.

Problems concerning nonplanar systems are investigated, and methods are derived for the solution of some simple nonplanar bodies. The surface pressure distribution and the damping in roll are found for rolling tails consisting of four, six, and eight rectangular fins for the Mach number range where the region of interference between adjacent fins does not affect the fin tips.

INTRODUCTION

In the presentation of the theory of the flow of an idealized incompressible fluid, vector methods can be used to reduce greatly the mathematical manipulations involved. The study of steady linearized supersonic flow may also be aided by the use of vector methods. Two types of approaches, however, can be used. Perhaps the more obvious is to make use of common vector methods as was done in reference 1. The other vector method, which was introduced by Robinson in reference 2 and is used in this report, appears to be more suited to the study of the linearized partial-differential equation of steady supersonic flow. This method allows a derivation of a hyperbolic scalar potential and a hyperbolic vector potential along lines analogous to the derivation sometimes used (ref. 3, ch. VIII) in dealing with common scalar and vector potentials.

The present report presents a vector derivation of many general results which have been found by various methods and are given in the published literature on the linearized partial-differential equation of supersonic flow and also presents some results which are not found in the literature. The general results of Hadamard (ref. 4, p. 207), Puckett (ref. 5), and Heaslet and Lomax (ref. 6) are found as special cases of a general expression for a scalar potential, and the results found by Robinson (ref. 2) are obtained by the use of a vector potential. The derivation of the scalar potential doubtlessly helps to clarify the concept of the finite part of an integral.

A discussion of problems dealing with planar bodies immersed in a supersonic flow is given, and the conditions necessary for the solution to be unique are investigated.

Problems dealing with nonplanar systems are also discussed, and methods are derived for the solution of some simple problems dealing with nonplanar bodies. The surface pressure distribution, the spanwise loading, and the damping in roll are found for rolling tails consisting of four, six, and eight rectangular fins for the Mach number range where the region of interference between adjacent fins does not affect the fin tips.

SYMBOLS

A	hyperbolic vector potential
A	aspect ratio of tail fin
a	positive constant
C_1, C_2, C_3	arbitrary constants
c	chord
E, F	arbitrary vector functions
f	scalar function defined by equation (19)
G	vector function associated with vector function F
H	vorticity vector
$b/2$	span of tail fin
i, j, k	unit vectors in x -, y -, and z -directions, respectively
M	Mach number
n	unit vector normal to element of area da
$n_h = -i\beta^2 v_1 + jv_2 + kv_3$	
$n_h' = -i\beta^2 v_1' + jv_2' + kv_3'$	
$n_h^* = -i\beta^2 v_1^* + jv_2^* + kv_3^*$	
ΔC_p	pressure-difference coefficient
p	rate of roll
Q	function used in equation of surface of discontinuity
q	part of velocity vector which is made up of hyperbolic curl of vector potential
q'	total perturbation velocity
$R = \sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2(z-\zeta)^2}$	
R'	small constant
$r = \sqrt{(x-\xi)^2 + \beta^4(y-\eta)^2 + \beta^4(z-\zeta)^2}$	
S	area of tail fin
S_0	surface of discontinuity

¹Supersedes NACA TN 2641, "A Vector Study of Linearized Supersonic Flow Applications to Nonplanar Problems" by John C. Martin, 1952.

$S_1, S_2, S_3, S_4, S_5,$ S_6, T, T'	surfaces of integration
V	free-stream velocity
v_0, v_1, v_2	volumes of integration
$W = \frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R}$	
x, y, z	Cartesian coordinates (x -axis parallel to free-stream direction)
$\beta = \sqrt{M^2 - 1}$	
Γ	spanwise circulation
	$\left(2 \int_{L.E.}^{T.E.} \phi_z dx = \frac{V}{2} \int_{L.E.}^{T.E.} \Delta C_p dx \right)$
ϵ	small positive quantity
ξ, η, ζ	Cartesian coordinates (ξ -axis parallel to free-stream direction)
θ, ρ	polar coordinates
Λ, ψ	scalar functions
λ	given volume
ν_1, ν_2, ν_3	direction cosines of outward normal to element of area da
ν_1', ν_2', ν_3'	direction cosines of normal (directed away from point (x, y, z)) to surface S_6
$\nu_1^*, \nu_2^*, \nu_3^*$	direction cosines of normal to element of area da used in equation (45)
σ	slope of deflected area
τ	area of integration
$\phi, \phi_0, \phi_1, \phi_2, \psi_0$	scalar potentials
C_l	rolling-moment coefficient per fin, Rolling moment per fin $\frac{1}{2} \rho V^2 S \frac{b}{2}$
$C_{l_p} = \left[\frac{\partial C_l}{\partial \frac{p(b/2)}{V}} \right]_{p=0}$	
$\oint$	indicates integration over closed line or surface
$\int$	denotes finite part of integral

THEORY

This report deals with the linearized partial-differential equation of steady supersonic flow. This equation is given by

$$-\beta^2 \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad (1)$$

The potential is assumed to be continuous in the stream direction, and the potential is assumed to be always finite. Assuming the potential to be finite and continuous in the stream direction has the effect of requiring the aerodynamic lift and moment (calculated by use of the linearized pressure) of finite bodies to be finite since the linearized pressure is related to the derivative of the potential in the stream direction. The expression "linearized pressure" refers to the pressure obtained by neglecting all powers of the perturbation-velocity components above the first.

VECTOR OPERATORS AND IDENTITIES

Certain operators, which are closely associated with the linearized hyperbolic partial-differential equation of supersonic flow (the two-dimensional wave equation), are added to the vector operators commonly used. The basic operators have been used previously in references 2 and 7.

The gradient operator is defined by

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

The analogous hyperbolic gradient operator defined by Robinson in reference 2 may be expressed as

$$\nabla h = -i\beta^2 \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

The hyperbolic divergence of an arbitrary vector $\mathbf{E}$ is given by

$$\nabla h \cdot \mathbf{E}$$

Similarly, the hyperbolic curl of the vector $\mathbf{E}$ is given by

$$\nabla h \times \mathbf{E}$$

The divergence of the gradient operator is sometimes denoted by

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The analogous divergence of the hyperbolic gradient operator is denoted by

$$\nabla^2 h \equiv \nabla \cdot \nabla h = -\beta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The following identities are needed. Let $\mathbf{E}$ be a vector and ψ and Λ be scalar functions of x, y , and z . Then,

$$\nabla \psi \cdot \nabla \Lambda = \nabla \Lambda \cdot \nabla \psi \quad (2a)$$

$$\nabla \cdot \psi \mathbf{E} = \psi \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \psi \quad (2b)$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (2c)$$

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0 \quad (2d)$$

$$\nabla h \psi \cdot \nabla \Lambda = \nabla h \Lambda \cdot \nabla \psi \quad (2e)$$

$$\nabla h \cdot \psi \mathbf{E} = \psi \nabla h \cdot \mathbf{E} + \mathbf{E} \cdot \nabla h \psi \quad (2f)$$

$$\nabla \times (\nabla h \times \mathbf{E}) = \nabla h (\nabla \cdot \mathbf{E}) - \nabla^2 h \mathbf{E} \quad (2g)$$

$$\nabla h \times (\nabla \times \mathbf{E}) = \nabla (\nabla h \cdot \mathbf{E}) - \nabla^2 h \mathbf{E} \quad (2h)$$

$$\nabla h \cdot (\nabla h \times \mathbf{E}) = 0 \quad (2i)$$

These identities can be proved by direct expansion.

The divergence theorem may be expressed as

$$\oint \mathbf{E} \cdot \mathbf{n} da = \int \nabla \cdot \mathbf{E} dv \quad (3)$$

where $\mathbf{n}$ is the normal unit vector to the element of area da . The vector $\mathbf{n}$ is expressed mathematically as

$$\mathbf{n} = i\nu_1 + j\nu_2 + k\nu_3$$

where ν_1 , ν_2 , and ν_3 are the direction cosines of the outward drawn normal to the element of area da .

A theorem more general than the divergence theorem is given by (this theorem follows from the results of ref. 8, p. 87)

$$\oint (C_1\nu_1 E_x + C_2\nu_2 E_y + C_3\nu_3 E_z) da = \int \left(C_1 \frac{\partial E_x}{\partial x} + C_2 \frac{\partial E_y}{\partial y} + C_3 \frac{\partial E_z}{\partial z} \right) dv$$

where the subscripts x , y , and z refer to components of the vector $\mathbf{E}$, and C_1 , C_2 , and C_3 are arbitrary constants. Note that if $C_1 = C_2 = C_3 = 1$ the preceding equation reduces to equation (3). If

$$C_1 = -\beta^2$$

$$C_2 = C_3 = 1$$

the preceding equation reduces to

$$\oint (-\beta^2\nu_1 E_x + \nu_2 E_y + \nu_3 E_z) da = \int \left(-\beta^2 \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) dv$$

or

$$\oint \mathbf{E} \cdot \mathbf{n}_h da = \int \nabla h \cdot \mathbf{E} dv \quad (4)$$

where

$$\mathbf{n}_h = -i\beta^2\nu_1 + j\nu_2 + k\nu_3$$

If the divergence theorem as expressed by equation (3) is applied to a volume throughout which

$$\nabla \cdot \mathbf{E} = 0$$

then the surface integral over the bounding surface is

$$\oint \mathbf{E} \cdot \mathbf{n} da = 0$$

provided that no surfaces exist inside the volume of integration across which the normal component of $\mathbf{E}$ is discontinuous. Similarly, if equation (4) is applied to a volume throughout which

$$\nabla h \cdot \mathbf{E} = 0$$

then the surface integral over the bounding surface is

$$\oint \mathbf{E} \cdot \mathbf{n}_h da = 0$$

provided that there are no surfaces inside the volume of integration across which $\mathbf{E} \cdot \mathbf{n}_h$ is discontinuous. It is in-

teresting to note, however, that surfaces exist inside the volume of integration across which $\mathbf{E} \cdot \mathbf{n}$ can be discontinuous while at the same time $\mathbf{E} \cdot \mathbf{n}_h$ remains continuous. It follows that for such a surface $\mathbf{n}$ and $\mathbf{n}_h$ must satisfy the relation

$$\mathbf{n} \cdot \mathbf{n}_h = 0 \quad (5)$$

Let $Q(x, y, z) = 0$ be the equation of such a surface. Then,

$$\mathbf{n} = \frac{1}{\sqrt{Q_x^2 + Q_y^2 + Q_z^2}} \nabla Q$$

and

$$\mathbf{n}_h = \frac{1}{\sqrt{Q_x^2 + Q_y^2 + Q_z^2}} \nabla h Q$$

where the subscripts indicate differentiation. Substituting the preceding expressions for $\mathbf{n}$ and $\mathbf{n}_h$ into equation (5) yields

$$-\beta^2 \left(\frac{\partial Q}{\partial x} \right)^2 + \left(\frac{\partial Q}{\partial y} \right)^2 + \left(\frac{\partial Q}{\partial z} \right)^2 = 0 \quad (6)$$

Any solution of equation (6) set equal to zero is the equation of a surface across which $\nabla \cdot \mathbf{E}$ may be discontinuous while $\nabla h \cdot \mathbf{E}$ remains continuous. The fact that the Mach cone from any arbitrary point satisfies equation (6) can be easily verified. The equation of the envelope of the Mach cones from an arbitrary line also satisfies equation (6) (ref. 9, p. 106).

FINITE PART OF INTEGRALS WHICH ARISE IN STEADY SUPERSONIC FLOW

In the following sections use is made of the concept of the finite part of an infinite integral. This concept was introduced by Hadamard (ref. 4) and has been used by a number of other investigators. The concept of the finite part is, however, sometimes confusing. This section was therefore included in an attempt to give a realistic picture of the finite-part concept and also to present the first steps of the derivation of the scalar and vector potentials.

The concept of the finite part of double integrals as defined by Hadamard and used in this report is different from the concept of the finite part of double integrals as defined in reference 10. The essential difference between these two definitions lies in the manner in which the singular points along the Mach cone are treated.

In reference 3, page 183, a vector function is used in the derivation of the common scalar and vector potentials. The analogous vector function based on equation (1) is

$$\mathbf{W} = \frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R}$$

where

$$R = \sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2(z-\zeta)^2}$$

The hyperbolic divergence of vector $\mathbf{W}$ with respect to variables ξ , η , and ζ is given by

$$\nabla h \cdot \mathbf{W} = \frac{1}{R} \nabla^2 h \phi = \frac{1}{R} \left(-\beta^2 \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial^2 \phi}{\partial \zeta^2} \right)$$

The preceding equation indicates that the hyperbolic divergence of the vector W set equal to zero yields the partial-differential equation of linearized supersonic flow. A mathematical derivation of W can be obtained; however, for the purposes of this report such a derivation is not needed.

The result of applying equation (4) to the vector W is

$$\oint \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da = \int \frac{1}{R} \nabla^2 h \phi dv \quad (7)$$

When ϕ satisfies equation (1) throughout the volume of integration, the right-hand side of equation (7) is zero; thus,

$$\oint \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da = 0 \quad (8)$$

when

$$\nabla^2 h \phi = 0$$

Equation (7) is applied to a volume (denoted by v_0) enclosed in the forward Mach cone from the point (x, y, z) . This volume is bounded by the surface given by $R=R'$, where R' is a small constant, and an arbitrary surface S_1 enclosed in the forward Mach cone from the point (x, y, z) . A cross section of the region of integration is shown in figure 1. Note that this region is analogous to the region that is sometimes used in calculating the potential function satisfying Laplace's equation (ref. 3, pp. 151-153). For regions such as the one shown in figure 1, equation (7) may be written as

$$\begin{aligned} \int_T \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da + \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da \\ = \int_{v_0} \frac{1}{R} \nabla^2 h \phi dv \quad (9) \end{aligned}$$

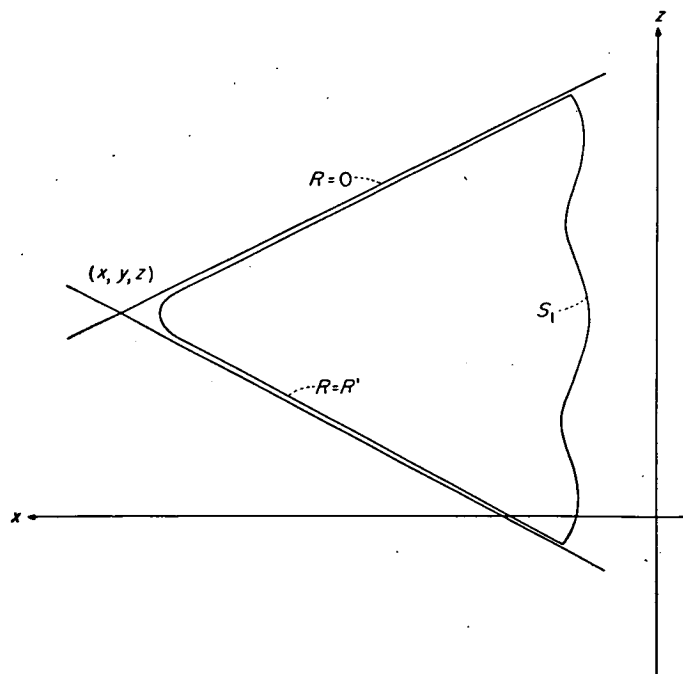


FIGURE 1.—Cross section of the region of integration used in connection with equation (9).

where T represents the area of integration when $R=R'$.

The integral over the area T may be reduced to

$$\int_T \frac{1}{R'} \left(\nabla \phi \cdot \mathbf{n}_h + \frac{\beta^2 \phi}{r} \right) da \quad (10)$$

where r is given by

$$r = \sqrt{(x-\xi)^2 + \beta^4(y-\eta)^2 + \beta^4(z-\zeta)^2}$$

Since R' is a constant, equation (10) can be written as

$$\frac{1}{R'} \int_T \left(\nabla \phi \cdot \mathbf{n}_h + \frac{\beta^2 \phi}{r} \right) da \quad (11)$$

Equation (9) can now be written as

$$\begin{aligned} -\frac{1}{R'} \int_T \left(\nabla \phi \cdot \mathbf{n}_h + \frac{\beta^2 \phi}{r} \right) da + \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da \\ = \int_{v_0} \frac{1}{R} \nabla^2 h \phi dv \quad (12) \end{aligned}$$

If ϕ is required to satisfy the linearized partial-differential equation of steady supersonic flow, then

$$\nabla^2 h \phi = 0$$

and equation (12) reduces to

$$\frac{1}{R'} \int_T \left(\nabla \phi \cdot \mathbf{n}_h + \frac{\beta^2 \phi}{r} \right) da + \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da = 0 \quad (13)$$

If R' is made smaller and smaller the integrand of the integral over the area T in equation (13) remains finite except on the small area close to the point (x, y, z) . In anticipation of taking the limit of equation (13) as R' approaches zero, the small area close to the point (x, y, z) is removed from the area T . The area T is divided into two parts. One part is the area of T which is downstream of the surface given by

$$\xi = x - \epsilon$$

where ϵ is small but larger than R' . This area is denoted by τ . The remaining part of T (denoted by T') is the area of T which is upstream of the surface

$$\xi = x - \epsilon$$

A cross section of the region of integration with T divided into τ and T' is shown in figure 2. Equation (13) can now be expressed as

$$\begin{aligned} \frac{1}{R'} \int_{\tau} \left(\nabla \phi \cdot \mathbf{n}_h + \frac{\beta^2 \phi}{r} \right) da + \frac{1}{R'} \int_{T'} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da + \\ \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da = 0 \quad (14) \end{aligned}$$

where R' is smaller than ϵ .

Since ϕ is continuous and therefore its values over τ are approximately constant for small values of ϵ , the integral over the area τ can be written as

$$\frac{1}{R'} \int_{\tau} \nabla \phi \cdot \mathbf{n}_h da + \frac{\beta^2 \phi(x, y, z)}{R'} \int_{\tau} \frac{da}{r} \quad (15)$$

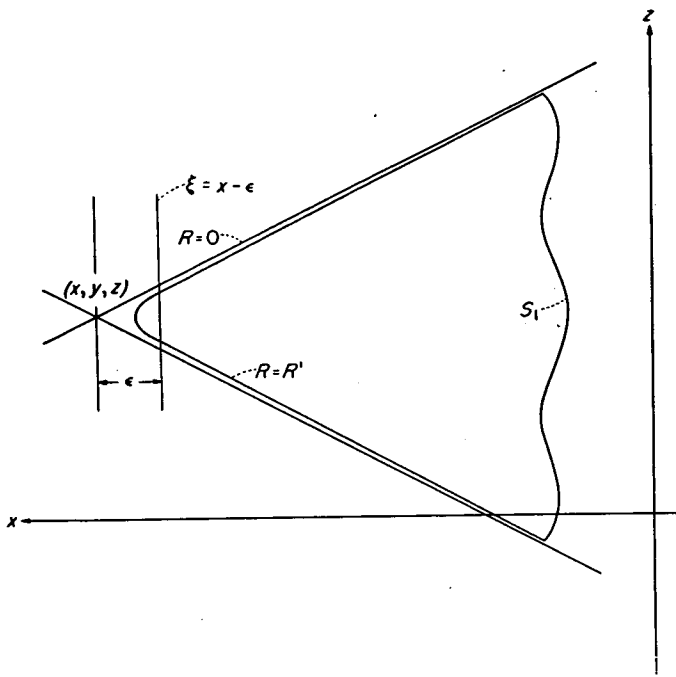


FIGURE 2.—Cross section of the region of integration used in connection with equation (14).

When the second integral of expression (15) is integrated, equation (14) becomes

$$\frac{1}{R'} \int_{\tau} \nabla \phi \cdot \mathbf{n}_h da + \frac{2\pi \epsilon \phi(x, y, z)}{R'} - 2\pi \phi(x, y, z) + \frac{1}{R'} \int_{T'} \left(\nabla \phi \cdot \mathbf{n}_h + \frac{\beta^2 \phi}{r} \right) da + \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da = 0 \quad (16)$$

If R' is made to approach zero, equation (16) applies even to the limit where R' is zero.

The limit of equation (16) as R' approaches zero may be written as

$$\lim_{R' \rightarrow 0} \left[\frac{1}{R'} \int_{\tau} \nabla \phi \cdot \mathbf{n}_h da + \frac{2\pi \epsilon \phi(x, y, z)}{R'} - 2\pi \phi(x, y, z) + \frac{\beta^2}{R'} \int_{T'} \frac{\phi}{r} da + \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da \right] = 0 \quad (17)$$

The integrands of the integrals over the areas τ and T' are always finite and it can be shown that their first derivatives with respect to R' approach zero as R' approaches zero; therefore, the product of $1/R'$ and these integrals either approaches zero in at least the order of R' or approaches infinity as R' approaches zero. Thus it follows that the integrals over the areas τ and T' have no finite terms remaining after the limit ($R' \rightarrow 0$) has been taken. The sum of the terms of equation (17) must be zero; thus the singularities resulting from the integrals over the areas τ and T' must cancel the singularities which arise from the integral over the area S_1 .

From the preceding considerations it follows that one method of evaluating the finite part of infinite integrals of the type appearing in equation (17) is to evaluate the integral when R' is small but not zero and neglect the terms multi-

plied by powers of $1/R'$. Other infinite integrals sometimes arise, however, for which the finite part cannot be obtained by neglecting powers of $1/R'$. For example, if equation (17) is differentiated with respect to one of the variables (x , y , or z) an equation containing the velocity component is obtained. In some cases, when the point (x, y, z) lies on the surface S_1 the infinite terms are of the order $(\ln R')/R'$ and of the orders $(1/R')^n$. In these cases, the finite part of the infinite integrals can be obtained by evaluating the integrals when R' is small and neglecting the terms multiplied by powers of $1/R'$ and $(\ln R')/R'$.

The process of removing the infinite parts of an integral, however, has been derived by Hadamard (ref. 4, book III, ch. I). Hadamard used his methods of evaluating the finite part of integrals in finding solutions to certain hyperbolic equations including the linearized equation of steady supersonic flow. Perhaps a fact worth noting is that the integrals of equation (17) are double integrals and when the methods given by Hadamard are used the methods given for multiple integrals should be used. In the past, the singular points (points on the Mach cone where the derivative of $(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2(z-\zeta)^2$ with respect to the variable of integration is zero) have caused some confusion; as Hadamard points out (ref. 4, p. 147), these singular points must be removed from the area of integration before the finite part is taken. Particular attention should be given to paragraph 92 of reference 4 since the special type of integrals discussed therein sometimes arises in dealing with planar problems.

Robinson (ref. 2) has shown that when using Hadamard's methods the order of integration may be changed without affecting the finite part and that it is permissible to differentiate under the integral sign of a multiple integral without considering the variable limits which lie along the boundary where the integrand is singular, provided that only the finite part is taken. Both Hadamard and Robinson have shown that in differentiating an improper integral which has an integrand that has a one-half power singularity along variable limits the variable limits may be neglected provided the finite part of the resulting integral is taken.

The term "finite part" is somewhat misleading since the finite part of an integral can be infinite. In certain cases the integral is infinite even after the terms which approach infinity as R' approaches zero have been neglected.

SCALAR POTENTIAL

The preceding arguments show that the finite parts of equation (17) can be equated to zero; thus,

$$-2\pi \phi(x, y, z) + \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da = 0$$

where the symbol f before the integral denotes that only the finite part is to be taken. The preceding equation may be solved for the value of the potential at the point (x, y, z) ; the result of this operation is given by

$$\phi(x, y, z) = \frac{1}{2\pi} \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da \quad (18)$$

It should be remembered that surfaces can exist inside the

forward Mach cone from the point (x, y, z) across which $\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R}$ can be discontinuous and across which $\left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R}\right) \cdot \mathbf{n}_h$ remains continuous.

Equation (18) is an expression for the scalar potential at the point (x, y, z) in terms of the potential and its derivatives with respect to $\mathbf{n}_h$ on the surface S_1 . A more general expression for the scalar potential than that given by equation (18) can be obtained. If within the volume enclosed by the forward Mach cone from the point (x, y, z) and the surface S_1 equation (1) is not satisfied and $\nabla^2 h$ operating upon ϕ yields

$$\nabla^2 h \phi(\xi, \eta, \zeta) = f(\xi, \eta, \zeta) \quad (19)$$

then equation (10) becomes

$$\begin{aligned} \frac{1}{R'} \int_v \left(\nabla \phi \cdot \mathbf{n}_h + \frac{\beta^2 \phi}{r} \right) da + \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da \\ = \int_{v_0} \frac{f(\xi, \eta, \zeta)}{R} dv \end{aligned} \quad (20)$$

Provided that $f(\xi, \eta, \zeta)$ is always finite, the right-hand side of equation (20) is finite; furthermore, the right-hand side of equation (20) remains finite as R' approaches zero. If in equation (20) R' is made to approach zero and only the finite parts of the integrals are retained, then the resulting expression is

$$\phi(x, y, z) = \frac{1}{2\pi} \int_{S_1} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da - \frac{1}{2\pi} \int_{v_1} \frac{f(\xi, \eta, \zeta)}{R} dv \quad (21)$$

where v_1 represents the volume v_0 when R' is equal to zero. Equation (21) is equation (58) of reference 4 where β^2 has been set equal to one. Note that the volume integral in equation (21) has the appearance of the integral for the potential resulting from a volume distribution of sources in an incompressible flow.

The assumption has been made that ϕ is continuous throughout the volume v_1 . It is also assumed that no surfaces exist inside v_1 across which $\partial \phi / \partial n_h$ is discontinuous. If equation (21) is applied to a volume v_1 , which has surfaces across which ϕ and/or the derivative of ϕ in the direction of $\mathbf{n}_h$ is discontinuous, these surfaces of discontinuity can be removed from the volume of integration by allowing the arbitrary surface S_1 to envelop them (see fig. 3). For volumes of integration where the surfaces of discontinuity have been removed in this manner, the scalar potential can be written as

$$\begin{aligned} \phi(x, y, z) = -\frac{1}{2\pi} \int_{v_1} \frac{f(\xi, \eta, \zeta)}{R} dv + \frac{1}{2\pi} \int_{S_0} \left(\frac{1}{R} \Delta \frac{\partial \phi}{\partial n_h} - \right. \\ \left. \Delta \phi \frac{\partial}{\partial n_h} \frac{1}{R} \right) da + \frac{1}{2\pi} \int_{S_1-S_0} \left(\frac{1}{R} \nabla \phi - \phi \nabla \frac{1}{R} \right) \cdot \mathbf{n}_h da \end{aligned} \quad (22)$$

where S_0 denotes the surface of discontinuity, and $\Delta \phi$ is the potential difference across the surface S_0 . The notation $\partial / \partial n_h$ is used to denote the operator

$$-\beta^2 \nu_1' \frac{\partial}{\partial \xi} + \nu_2' \frac{\partial}{\partial \eta} + \nu_3' \frac{\partial}{\partial \zeta}$$

For the cases where no surface of discontinuity exists inside the volume v_1 and ϕ and $\nabla \phi$ are zero on the surface S_1-S_0 , equation (22) reduces to

$$\phi(x, y, z) = -\frac{1}{2\pi} \int_{v_1} \frac{f(\xi, \eta, \zeta)}{R} dv \quad (23)$$

From equation (19)

$$\nabla^2 h \phi(x, y, z) = f(x, y, z) \quad (24)$$

Note that equation (23) is a solution of the partial-differential equation (24).

For most problems in linearized supersonic flow, $f(\xi, \eta, \zeta)$ is zero and ϕ is zero upstream of the disturbing body. For such problems, the surface S_1-S_0 can be taken to be located upstream of the disturbing body where ϕ and $\nabla \phi$ are zero. In this case, equation (22) reduces to

$$\phi(x, y, z) = \frac{1}{2\pi} \int_{S_0} \left(\frac{1}{R} \Delta \frac{\partial}{\partial n_h} \phi - \Delta \phi \frac{\partial}{\partial n_h} \frac{1}{R} \right) da \quad (25)$$

If the surface S_0 is confined to the $\zeta=0$ plane, equation (25) reduces to equation (10) of reference 6. In this reference the boundary conditions for airfoils are discussed.

COMPONENTS OF VECTOR FIELD

Let $\mathbf{F}$ be a vector which is finite and integrable in a given volume (denoted by λ) and is zero outside the volume λ . To each point in the volume associate the vector

$$\mathbf{G}(x, y, z) \equiv \int_{v_2} \frac{\mathbf{F}(\xi, \eta, \zeta)}{R} dv \quad (26)$$

where v_2 denotes the part of the volume λ enclosed in the forward Mach cone from the point (x, y, z) .

From equations (24) and (26), it follows that each component of $\mathbf{G}$ satisfies the relation

$$\nabla^2 h G_i(x, y, z) = -2\pi F_i(x, y, z) \quad (27)$$

where the subscript i refers to any component of the vector field.

Let $\psi_0(x, y, z)$ be a scalar and $\mathbf{A}(x, y, z)$ be a vector defined by the equations

$$2\pi \psi_0 \equiv \nabla h \cdot \mathbf{G} = \int_{v_2} \mathbf{F}(\xi, \eta, \zeta) \cdot \nabla h \frac{1}{R} dv \quad (28)$$

and

$$2\pi \mathbf{A} \equiv \nabla \times \mathbf{G} = \int_{v_2} \mathbf{F}(\xi, \eta, \zeta) \times \nabla \frac{1}{R} dv \quad (29)$$

Equation (2h) indicates that

$$\nabla h \times (\nabla \times \mathbf{G}) = \nabla (\nabla h \cdot \mathbf{G}) - \nabla^2 h \mathbf{G} \quad (30)$$

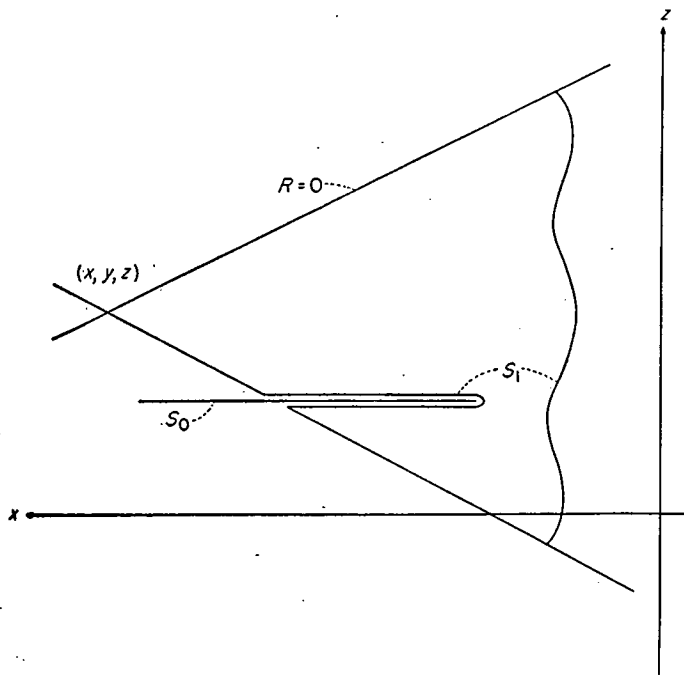


FIGURE 3.—A cross section of the region in the forward Mach cone from the point (x, y, z) showing the surface S_1 enveloping a surface of discontinuity S_0 .

Substituting the expressions for $\nabla \times \mathbf{G}$, $\nabla h \cdot \mathbf{G}$, and $\nabla^2 h \mathbf{G}$ as given by equations (27), (28), and (29), respectively, into equation (30) and solving for $\mathbf{F}$ yields

$$\mathbf{F}(x, y, z) = -\nabla \psi_0(x, y, z) + \nabla h \times \mathbf{A}(x, y, z) \quad (31)$$

Since $\mathbf{F}$ is an arbitrary vector, equation (31) indicates that any finite integrable vector field can be expressed in terms of the gradient of a scalar and the hyperbolic curl of a vector. Equation (31) has the appearance of the Helmholtz theorem (ref. 3, p. 187); however, since ψ_0 and $\mathbf{A}$ are found by integration only in the forward Mach cone from the point (x, y, z) , equation (31) hardly seems to be a statement of the Helmholtz theorem as is commonly given. The result given by equation (31) was obtained by Robinson in reference 2.

HYPERBOLIC VECTOR POTENTIAL

Equation (31) indicates that the perturbation velocity vector can be divided into two parts. One part is the gradient of a scalar function, and the other is the hyperbolic curl of a vector function. The vector function is analogous to the common vector potential (ref. 3, pp. 104 and 188); therefore, the vector function is referred to as the hyperbolic vector potential. Thus, if $\mathbf{q}'$ is the total perturbation velocity vector, then

$$\mathbf{q}' = \nabla \phi + \nabla h \times \mathbf{A} \quad (32)$$

where ϕ is the scalar potential and $\mathbf{A}$ is the hyperbolic vector potential. The part of the velocity vector which is made up of the hyperbolic curl of the vector potential is denoted by $\mathbf{q}$.

By direct expansion it can be shown that

$$\nabla h \cdot \mathbf{q}' = \nabla h \cdot \nabla \phi + \nabla h \cdot (\nabla h \times \mathbf{A}) = \nabla^2 h \phi + \nabla h \cdot (\nabla h \times \mathbf{A}) = 0 \quad (33)$$

Equation (33) indicates that the hyperbolic divergence of the perturbation velocity vector is zero.

The vorticity vector is given by

$$\mathbf{H} = \nabla \times \mathbf{q}' \quad (34)$$

Therefore, from equation (32),

$$\mathbf{H} = \nabla \times (\nabla h \times \mathbf{A})$$

or

$$\mathbf{H} = \nabla h (\nabla \cdot \mathbf{A}) - \nabla^2 h \mathbf{A}$$

From equations (2d) and (29), the divergence of the hyperbolic vector potential is zero; thus,

$$\mathbf{H} = -\nabla^2 h \mathbf{A} \quad (35)$$

Each component of equation (35) is a partial-differential equation of the form of equation (24); thus, from equation (23) each component of equation (35) has a solution given by

$$A_i(x, y, z) = \frac{1}{2\pi} \int_{v_1} \frac{H_i(\xi, \eta, \zeta)}{R} dv \quad (36)$$

where the subscript i refers to any component of the vector $\mathbf{H}$. Since each component of $\mathbf{A}$ is given by equation (36), then

$$\mathbf{A}(x, y, z) = \frac{1}{2\pi} \int_{v_1} \frac{\mathbf{H}(\xi, \eta, \zeta)}{R} dv \quad (37)$$

The velocity vector resulting from the hyperbolic vector potential is therefore given by

$$\mathbf{q} = \nabla h \times \mathbf{A} = \frac{1}{2\pi} \int_{v_1} \nabla h \times \frac{\mathbf{H}}{R} dv \quad (38)$$

or

$$u = \frac{\beta^2}{2\pi} \int_{v_1} \frac{(y-\eta)H_z - (z-\zeta)H_y}{R^3} dv \quad (39a)$$

$$v = \frac{\beta^2}{2\pi} \int_{v_1} \frac{(z-\zeta)H_x - (x-\xi)H_z}{R^3} dv \quad (39b)$$

$$w = \frac{\beta^2}{2\pi} \int_{v_1} \frac{(x-\xi)H_y - (y-\eta)H_x}{R^3} dv \quad (39c)$$

where the subscripts refer to the components of the vector $\mathbf{H}$. The results given by equations (39) were obtained by Robinson in reference 2.

VORTEX SHEETS

If the vorticity is confined to a surface S_2 , equation (37) becomes

$$\mathbf{A}(x, y, z) = \frac{1}{2\pi} \int_{S_2} \frac{\mathbf{H}(\xi, \eta, \zeta)}{R} da \quad (40)$$

Equation (40) is an expression for the hyperbolic vector potential resulting from a surface of vorticity. Note that if the vorticity is zero except on the surface S_2 , then equation (35) reduces to

$$\nabla^2 h \mathbf{A} = 0$$

By removing the surface S_2 from the volume of integration

each component of A can be expressed as (from eq. (25))

$$A_i(x, y, z) = \frac{1}{2\pi} \int_{S_2} \left(\frac{1}{R} \Delta \frac{\partial A_i}{\partial n_h} - \Delta A_i \frac{\partial}{\partial n_h} \frac{1}{R} \right) da \quad (41)$$

where the subscript i refers to any component of the vector A . Since each component of A is given by equation (41), then

$$A(x, y, z) = \frac{1}{2\pi} \int_{S_2} \left(\frac{1}{R} \Delta \frac{\partial A}{\partial n_h} - \Delta A \frac{\partial}{\partial n_h} \frac{1}{R} \right) da \quad (42)$$

Note that if ΔA is zero, equation (42) reduces to

$$A(x, y, z) = \frac{1}{2\pi} \int_{S_2} \frac{1}{R} \Delta \frac{\partial A}{\partial n_h} da \quad (43)$$

By comparing equations (40) and (43) it follows that, on the surface S_2 ,

$$H = \Delta \frac{\partial A}{\partial n_h} \quad (44)$$

Equation (44) indicates that across a surface of vorticity the derivative of the hyperbolic vector potential in the direction of n_h is discontinuous. Thus, a lifting surface can be represented by a continuous hyperbolic vector potential, while it can be shown that a thickness effect can be represented by a discontinuous hyperbolic vector potential. Note the contrast with the scalar potential, which uses a continuous potential to represent a thickness effect and a discontinuous potential to represent a lifting surface.

FURTHER DEVELOPMENT OF SCALAR POTENTIAL

The scalar potential can be expressed in forms other than those already presented. Equation (8) is applied to the region bounded by the arbitrary surface S_1 , the forward Mach cone from the point (x, y, z) , and a second arbitrary surface S_3 enclosed in the forward Mach cone from the point (x, y, z) and upstream of the surface S_1 . A cross section of such a region is shown in figure 4. The result of applying equation (8) to this region is

$$\int_{S_1} \left(\frac{1}{R} \frac{\partial \phi'}{\partial n_h} - \phi' \frac{\partial}{\partial n_h} \frac{1}{R} \right) da + \int_{S_3} \left(\frac{1}{R} \frac{\partial \phi'}{\partial n_h} - \phi' \frac{\partial}{\partial n_h} \frac{1}{R} \right) da = 0 \quad (45)$$

provided that ϕ' is a solution of equation (1). Note that the scalar potential as given by equation (18) is independent of ϕ' so that ϕ' is arbitrary so long as it satisfies equation (1) throughout the proper volume.

If for a finite distance upstream ϕ' is zero and remains zero for greater distances upstream, the surface S_3 may be chosen in this region so that the integral over S_3 in equation (45) is zero; thus,

$$-\frac{1}{2\pi} \int_{S_1} \left(\frac{1}{R} \frac{\partial \phi'}{\partial n_h} - \phi' \frac{\partial}{\partial n_h} \frac{1}{R} \right) da = 0 \quad (46)$$

Equations (18) and (46) can be combined to yield

$$\phi(x, y, z) = \frac{1}{2\pi} \int_{S_1} \left[\frac{1}{R} \left(\frac{\partial \phi}{\partial n_h} - \frac{\partial \phi'}{\partial n_h} \right) - (\phi - \phi') \frac{\partial}{\partial n_h} \frac{1}{R} \right] da \quad (47)$$

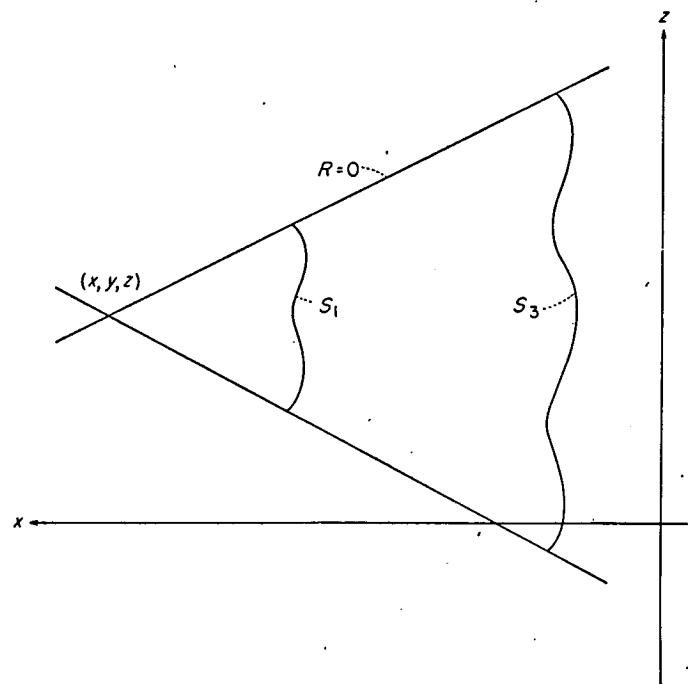


FIGURE 4.—Cross section of the region of integration used in connection with the potential function ϕ' .

The only restrictions placed on ϕ' at this point are that it satisfy equation (1) and be zero at a finite distance upstream.

In many cases ϕ' may be chosen so that $\frac{\partial \phi}{\partial n_h} - \frac{\partial \phi'}{\partial n_h}$ or $\phi - \phi'$ is zero; therefore, in these cases, ϕ can be expressed as

$$\phi(x, y, z) = \frac{1}{2\pi} \int_{S_1} \frac{1}{R} \left(\frac{\partial \phi}{\partial n_h} - \frac{\partial \phi'}{\partial n_h} \right) da \quad (48a)$$

or

$$\phi(x, y, z) = -\frac{1}{2\pi} \int_{S_1} (\phi - \phi') \frac{\partial}{\partial n_h} \frac{1}{R} da \quad (48b)$$

Equations (48) are quite useful; however, remember that they apply only when ϕ' can be chosen so that ϕ' does not violate any of its restrictions.

Note that equations (48) can be applied to problems where either ϕ or $\partial \phi / \partial n_h$ is given on the surface S_1 . The application of these equations to most nonplanar problems of either type, however, lead to quite unwieldy integral equations.

APPLICATIONS

PLANAR PROBLEMS

Many problems in linearized supersonic flow deal with the surface of discontinuity confined to a plane surface parallel to the x -axis. In this section a general discussion of this type of problem is given. The coordinates are located so that the surface of discontinuity is in the $\xi=0$ plane.

The scalar potential at an arbitrary point (x, y, z) above the $\xi=0$ plane is (from eq. (47))

$$\phi(x, y, z) = -\frac{1}{2\pi} \int_{S_1} \left[(\phi_z - \phi'_z) \frac{1}{R} + (\phi - \phi') \frac{\beta^2 z}{R^3} \right] d\xi d\eta \quad (49)$$

In this case, the surface S_1 is the $\xi=0$ plane.

If $\phi'(x,y,0)$ is chosen equal to $\phi(x,y,0)$ the potential becomes symmetric with respect to the $\zeta=0$ plane. Thus, for $z=a$

$$\phi(x,y,a) = \phi'(x,y,-a)$$

and

$$\phi_z(x,y,a) = -\phi'_z(x,y,-a)$$

For this case, equation (49) reduces to

$$\phi(x,y,z) = -\frac{1}{\pi} \int_{S_1} \frac{\phi_z(\xi,\eta,0)}{R} d\xi d\eta \quad (50)$$

Equation (50) was given by Puckett in reference 5.

If $\phi'(x,y,0)$ is chosen equal to $-\phi(x,y,0)$ the potential becomes antisymmetric with respect to the $\zeta=0$ plane. In this case,

$$\phi(x,y,a) = -\phi'(x,y,-a)$$

and

$$\phi_z(x,y,a) = \phi'_z(x,y,-a)$$

Thus, equation (49) is reduced to

$$\phi(x,y,z) = -\frac{\beta^2 z}{\pi} \int_{S_1} \frac{\phi(\xi,\eta,0)}{R^3} d\xi d\eta \quad (51)$$

Note that for surface S_1 not confined to a plane parallel to the x -axis, a choice of $\phi'(x,y,z)$ at the surface S_1 to equal $\phi(x,y,z)$ at S_1 does not cause $\frac{\partial \phi(x,y,z)}{\partial n_h}$ at the surface S_1 to equal

$-\frac{\partial \phi'(x,y,z)}{\partial n_h^*}$ at S_1 . Similarly, choosing $\phi'(x,y,z)$ at the surface S_1 equal to $-\phi(x,y,z)$ at S_1 does not cause $\frac{\partial \phi'(x,y,z)}{\partial n_h^*}$ at S_1 to equal $\frac{\partial \phi(x,y,z)}{\partial n}$ at S_1 .

Provided the discontinuities are restricted to the $\zeta=0$ plane, the scalar potential can also be expressed as follows (from eq. (18)):

$$\phi(x,y,z) = -\frac{1}{2\pi} \int_{S_1} \left[\frac{\phi_z(\xi,\eta,0)}{R} + \phi(\xi,\eta,0) \frac{\beta^2 z}{R^3} \right] d\xi d\eta \quad (52)$$

for positive z . A comparison of equations (50) and (51) with equation (52) shows that the two terms of the integrand of equation (52) contribute equal amounts to the potential at any point (x,y,z) .

Since the terms of the integrand of equation (52) contribute equal amounts to the potential at the point (x,y,z) as z approaches zero, equation (52) must reduce to

$$\phi(x,y,0) = -\frac{1}{2\pi} \int_{S_1} \frac{\phi_z(\xi,\eta,0)}{R} d\xi d\eta + \frac{\phi(x,y,0)}{2}$$

The preceding equation can also be obtained by examining the limit of equation (52) as z approaches zero. If this procedure is done the entire contribution of the second term of the integrand of equation (52) is found to come from the point at the apex of the hyperbola formed by the intersection of the Mach cone from the point (x,y,z) and the $\zeta=0$ plane. Note that if the integration is performed first with respect to

η then, when the methods of Hadamard are used, the point at the apex of the hyperbola is a singular point and must be removed from the area of integration by a process such as is given in reference 4, page 147.

If ϕ_z is prescribed over the $\zeta=0$ plane, then the potential is given uniquely by equation (50). Similarly, if the potential is prescribed over the $\zeta=0$ plane, then the derivative of ϕ with respect to z is determined over the $z=0$ plane. This result follows from equation (51) since prescribing ϕ over the $z=0$ plane determines the potentials in the space above the $z=0$ plane; therefore, it also determines ϕ_z in the space above the $z=0$ plane and the limit of ϕ_z as z approaches zero from the positive direction.

The question that arises is whether $\phi(x,y,z)$ is uniquely determined in the space above the $z=0$ plane if ϕ is prescribed over certain areas of the $z=0$ plane and ϕ_z is prescribed over the remaining areas. If the assumption is made that ϕ is not determined uniquely, then at least two potential functions satisfy the condition that either ϕ or ϕ_z is prescribed in all regions on the $z=0$ plane and that ϕ is identically zero upstream of a given point. Let ϕ_1 and ϕ_2 denote two potential functions which satisfy the same boundary conditions, and let ϕ_0 denote the potential function formed by taking the difference between ϕ_1 and ϕ_2 . Mathematically, the potential function ϕ_0 is given by

$$\phi_0(x,y,z) = \phi_1(x,y,z) - \phi_2(x,y,z) \quad (53)$$

Since ϕ_1 and ϕ_2 have the same values in certain regions in the $z=0$ plane then ϕ_0 is zero in these regions. Similarly, since $\partial \phi_1 / \partial z$ and $\partial \phi_2 / \partial z$ have the same values in the remaining regions of the $z=0$ plane, then $\partial \phi_0 / \partial z$ is zero in these remaining regions. The potential function ϕ_0 has the boundary conditions that either ϕ_0 or $\partial \phi_0 / \partial z$ is zero in all regions of the $z=0$ plane and that ϕ_0 is identically zero upstream of a given point.

Consider the case where all the boundaries between the regions are supersonic. (The slopes of the boundaries are such that the component of the free stream perpendicular to the boundary is always supersonic.) The potential function ϕ_0 can be evaluated by use of equation (50) or (51) for points in areas which are far enough upstream to be affected only by a region where ϕ or ϕ_z is prescribed. For all points in these areas ϕ_0 is zero as indicated by equation (50) or (51). It follows from equations (50) and (51) that ϕ_0 is also zero inside the volume above the $z=0$ plane, which is affected by these areas alone. Thus, the volume where ϕ_0 is identically zero has been moved downstream. The same argument can be repeated until the complete $z=0$ plane has been covered. The preceding arguments cannot be applied to cases where the regions have subsonic boundaries; however, if it is permissible to distort the boundaries within a strip of infinitesimal width these subsonic edges can be converted into supersonic edges by replacing every element of the subsonic boundaries by a broken line made up of supersonic segments. Such a procedure is illustrated in figure 5. If the assumption is made that the subsonic boundaries may be distorted an infinitesimal amount, then ϕ_0 is zero over the $z=0$ plane and also in the space above the $z=0$ plane.

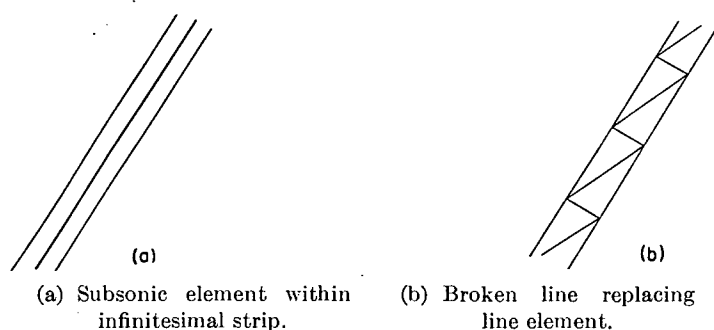


FIGURE 5.—An illustration of a method of replacing a subsonic line element with a broken line made up of supersonic segments.

Equation (53) now reduces to

$$\phi_1(x,y,z) = \phi_2(x,y,z)$$

Since ϕ_1 and ϕ_2 are any two potential functions with the same values in certain regions of the $z=0$ plane and with the same partial derivatives with respect to z in the remaining regions, proof has been given that only one potential function exists for which the potential is prescribed over certain areas in the $z=0$ plane and the partial derivative with respect to z is prescribed over the remaining areas.

The boundary conditions for a zero-thickness lifting airfoil with a given local angle-of-attack distribution are not of the type discussed in the preceding paragraph. The conditions prescribed in the $z=0$ plane for this type of problem are: The potential function ϕ is identically zero upstream of the airfoil; $\phi(x,y,0)$ is zero except on the plan form or in the wake; the partial derivative of the potential with respect to z , ϕ_z , is given on the plan form; and $\phi_x(x,y,0)$ is zero in the wake. The preceding boundary conditions do not specify that ϕ or ϕ_z be prescribed in all regions on the $z=0$ plane since not ϕ but ϕ'_x is given in the wake. For airfoils which have trailing edges which are always supersonic, the requirement that ϕ be continuous in the stream direction necessitates the potential in the wake to have the value of the potential at the trailing edge of the airfoil. In this case, the potential function is uniquely determined. For airfoils which have subsonic trailing edges the Kutta-Joukowski condition is generally applied to the trailing edges to determine ϕ uniquely. If the assumption is made that the trailing edge can be distorted within a strip of infinitesimal width, then the requirement that ϕ be continuous in the stream direction can be used to determine ϕ uniquely. If the assumption is made that the subsonic trailing edge is distorted within the infinitesimal strip so that each segment of each line element of the trailing edge is always supersonic (see fig. 6), then ϕ is determined uniquely. It is well-known that for airfoils with subsonic trailing edges there are an infinite number of solutions which satisfy the boundary conditions as stated at the beginning of this paragraph. The preceding arguments however prove that there is but one solution for an airfoil which has had its subsonic edge replaced by broken lines which are always supersonic. Note that it has not been proved that the solution obtained by distorting the subsonic trailing edges corresponds to the

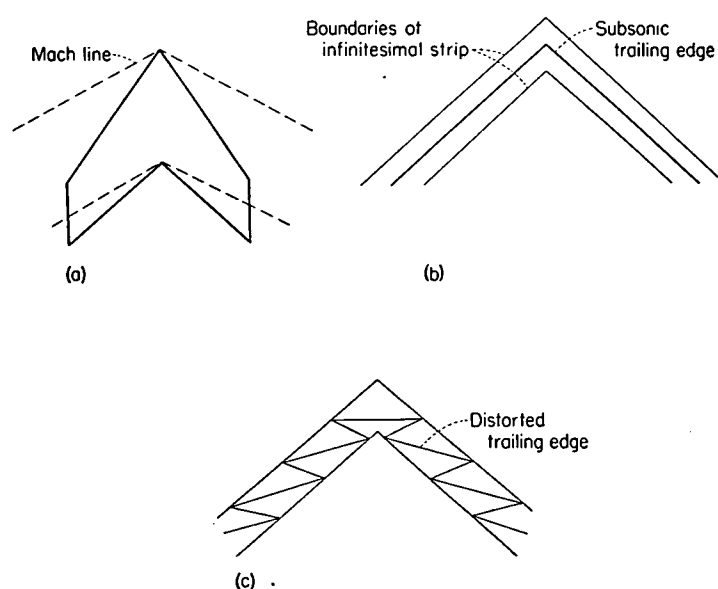


FIGURE 6.—Methods of distorting a subsonic trailing edge to determine the potential function uniquely.

solution satisfying the Kutta-Joukowski condition, nor has it been proved that the solution of the distorted trailing edge is independent of the manner of distortion.

NONPLANAR PROBLEMS

The scalar potential resulting from the disturbances caused by a nonplanar body can be found from equation (18) provided that both ϕ and $\partial\phi/\partial n_n$ are known on some surface S_1 . Unfortunately, ϕ and $\partial\phi/\partial n_n$ are not generally known on a surface which fills the requirement of the surface S_1 ; therefore, equation (18) appears to have little value in the calculation of the potential functions for nonplanar systems in general. Certain properties of equation (18) are, however, worth investigating.

The problem of evaluating the potential on the upper surfaces of a long rectangular body is discussed. The assumption is made that the body extends upstream to infinity and that the sides are parallel to the free-stream direction except for small local variations which cause small disturbances in the stream. Figure 7 (a) shows the forward Mach cone from a point on the upper surface of such a rectangular body. This figure also shows that there is a certain part of the surface of the rectangular body in the forward Mach cone from the point (x,y,z) that cannot possibly affect the potential at the point (x,y,z) . If the surface S_1 in equation (18) is taken to be the surface of the rectangular body, then equation (18) indicates that the values of ϕ and $\partial\phi/\partial n_n$ in the region which cannot possibly affect the potential at the point (x,y,z) should be used in evaluating the potential at the point (x,y,z) . The only possible explanation of this consideration is that the integral of ϕ and $\partial\phi/\partial n_n$ caused by the disturbances in the "blind spot" add to zero. This consideration can be shown mathematically as follows. Let ψ_0

denote the potential function resulting from the disturbances inside the blind spot. From equation (46), it follows that

$$\int_{S_1} \left(\frac{1}{R} \frac{\partial \psi_0}{\partial n_h^*} - \psi_0 \frac{\partial}{\partial n_h^*} \frac{1}{R} \right) da = 0 \quad (54)$$

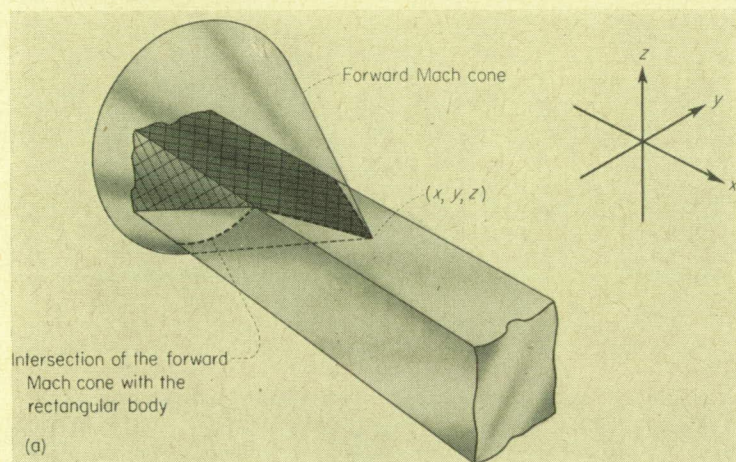
Equation (54) indicates that the potential at the point (x, y, z) can be evaluated by applying equation (18) to the surface of the rectangular body regardless of the blind spots. The same argument holds for other bodies with blind spots.

The preceding arguments can be clarified by a simple illustration of the effect of a blind spot. Consider an infinite rectangular body such as shown in figure 7 where the only disturbances are caused by a small deflected area with a constant slope σ with respect to the free-stream direction located on the lower surface of the body. The leading edge of the deflected area is chosen perpendicular to the free-stream direction so that the potential in the region not affected by the vertical sides is of a two-dimensional nature. Figure 7 (b) illustrates such a disturbing surface.

The disturbance potential in the two-dimensional region is given by

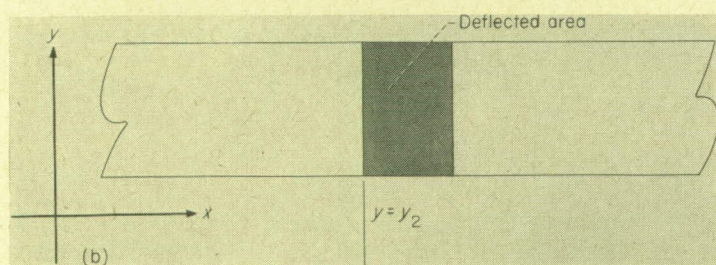
$$\phi(x, z) = \frac{\sigma V [x - x_2 + \beta(z - z_2)]}{\beta}$$

where the lower surface of the body lies in the $z = z_2$ plane and the leading edge of the deflected area is in the $x = x_2$ plane.



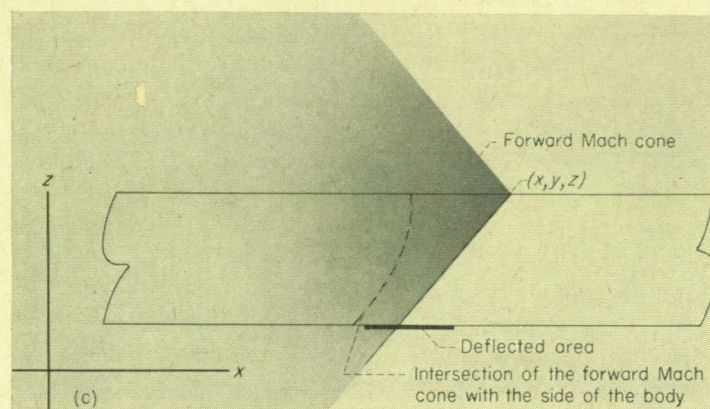
(a) The forward Mach cone from a point on the upper surface of a rectangular body.

FIGURE 7.—Rectangular body parallel to free-stream direction.



(b) Bottom view of rectangular body with deflected area on the lower surface.

FIGURE 7.—Continued.



(c) Side view of rectangular body and the forward Mach cone from the point (x, y, z) .

FIGURE 7.—Concluded.

A point on the upper surface which has only the two-dimensional flow in its forward Mach cone is illustrated in figure 7 (c). The disturbance potential for this point is (from eq. (18)) given by

$$\phi(x, y, z) = \frac{1}{2\pi} \int \left[\frac{\sigma V}{R} + \frac{\beta^2 \sigma V (\xi - x_2)(z - z_2)}{R^3} \right] d\xi d\eta$$

Upon performing the indicated integrations the preceding expression becomes

$$\phi(x, y, z) = -\frac{\sigma V [x - x_2 - \beta(z - z_2)]}{2\beta} + \frac{\sigma V [x - x_2 - \beta(z - z_2)]}{2\beta}$$

which reduces to

$$\phi(x, y, z) = 0$$

This result is a demonstration that the disturbances in blind spots do not contribute to the potential.

The scalar potential resulting from the disturbance produced by a nonplanar body can also be obtained by use of equations (47) and (48) provided that the necessary values of ϕ , $\partial\phi/\partial n_h$, ϕ' , and $\partial\phi'/\partial n_h$ are known. In dealing with planar bodies ϕ' could be chosen so that

$$\phi - \phi' = 0$$

and thus equation (47) is reduced to equation (50). Similarly, ϕ' could also be chosen so that

$$\frac{\partial\phi}{\partial n_h} = \frac{\partial\phi'}{\partial n_h^*}$$

and thus equation (47) is reduced to equation (51). Unfortunately, for nonplanar bodies, choosing ϕ' equal to ϕ does not make $\partial\phi'/\partial n_h^*$ known as was the case for planar problems and, similarly, choosing $\partial\phi'/\partial n_h^*$ equal to $-\partial\phi/\partial n_h$ does not make ϕ' known. Certain problems exist in which ϕ' can be chosen so that ϕ can be written as a simple integral.

INTERSECTING PLANES

Many problems concerning nonplanar bodies deal with disturbances produced by two intersecting planes parallel to the free-stream direction. In this section, methods of

solutions for two planes intersecting at various angles are given. The component of velocity normal to the surface is assumed to be known.

Perhaps the simplest case of two intersecting planes occurs when the planes intersect at right angles. It is desired to find the potential in space resulting from the disturbances produced by the two intersecting planes. This type of problem could represent an isolated cruciform tail with supersonic leading edges undergoing various motions. Problems of this type have been solved in references 11 and 12. The axes are chosen so that $y=0$ and $z=0$ are the disturbing planes (see fig. 8). When y and z are positive, equation (18) becomes

$$\phi(x,y,z) = -\frac{1}{2\pi} \int_{S_3} \frac{1}{R} \frac{\partial \phi(\xi, 0^+, \zeta)}{\partial \eta} da - \frac{\beta^2 y}{2\pi} \int_{S_3} \frac{\phi(\xi, 0^+, \zeta)}{R^3} da - \frac{1}{2\pi} \int_{S_4} \frac{1}{R} \frac{\partial \phi(\xi, \eta, 0^+)}{\partial \zeta} da - \frac{\beta^2 z}{2\pi} \int_{S_4} \frac{\phi(\xi, \eta, 0^+)}{R^3} da \quad (55)$$

The surface S_1 has been taken to be the disturbing surface; thus, S_3 is the part of the $y=0$ plane (z positive) bounded by the $z=0$ line in the $y=0$ plane and the forward Mach cone from the point (x, y, z) . Similarly, S_4 is the part of the $z=0$ plane (y positive) bounded by the $y=0$ line in the $z=0$ plane and the trace of the forward Mach cone from the point (x, y, z) (see fig. 9).

The assumption is made that $\frac{\partial \phi(\xi, 0^+, \zeta)}{\partial \eta}$ and $\frac{\partial \phi(\xi, \eta, 0^+)}{\partial \zeta}$ are known and that $\phi(\xi, 0^+, \zeta)$ and $\phi(\xi, \eta, 0^+)$ are unknown. The integrals containing $\phi(\xi, 0^+, \zeta)$ and $\phi(\xi, \eta, 0^+)$ can be eliminated by several applications of equation (46). Equation (46) is applied to the volume on the left-hand side of the $y=0$ plane

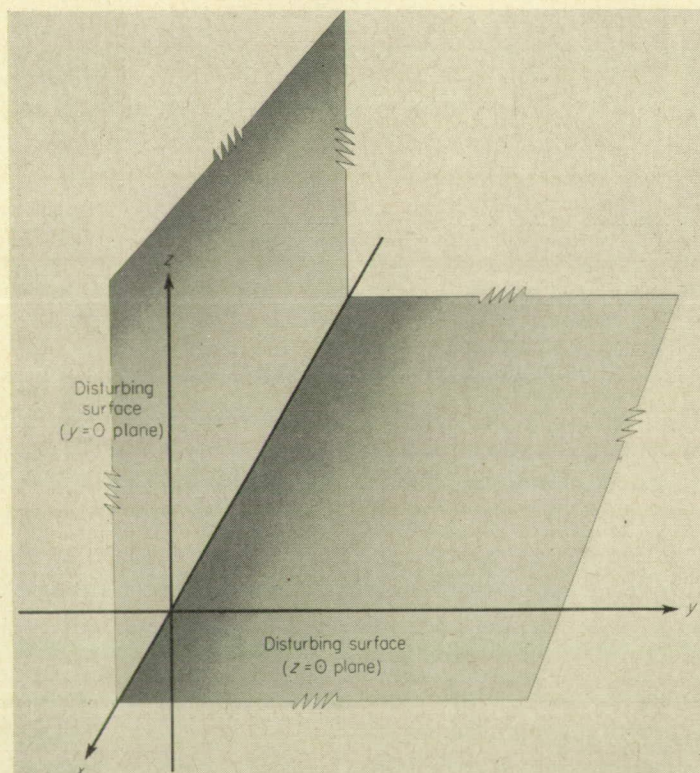


FIGURE 8.—Two disturbing surfaces intersecting at right angles.

enclosed by the forward Mach cone from the point (x, y, z) , the $y=0$ plane, the $z=0$ plane, and an arbitrary surface upstream of the disturbance (see fig. 10). The result of applying equation (46) to this volume is

$$-\frac{1}{2\pi} \int_{S_5} \frac{1}{R} \frac{\partial \phi'(\xi, \eta, 0^+)}{\partial \zeta} da - \frac{\beta^2 z}{2\pi} \int_{S_5} \frac{\phi'(\xi, 0^+, \zeta)}{R^3} da + \frac{1}{2\pi} \int_{S_3} \frac{1}{R} \frac{\partial \phi'(\xi, 0^-, \zeta)}{\partial \eta} da + \frac{\beta^2 y}{2\pi} \int_{S_3} \frac{\phi'(\xi, 0^-, \zeta)}{R^3} da = 0 \quad (56)$$

The surface S_1 has been taken to be the $y=0$ plane (z positive) and the $z=0$ plane (y negative); thus, S_5 is the part of the $z=0$ plane in the forward Mach cone from the point (x, y, z) (see fig. 10). Adding equations (55) and (56) yields

$$\begin{aligned} \phi(x,y,z) = & -\frac{1}{2\pi} \int_{S_3} \frac{1}{R} \left[\frac{\partial \phi(\xi, 0^+, \zeta)}{\partial \eta} - \frac{\partial \phi'(\xi, 0^-, \zeta)}{\partial \eta} \right] da - \\ & \frac{\beta^2 y}{2\pi} \int_{S_3} \frac{1}{R^3} [\phi(\xi, 0^+, \zeta) - \phi'(\xi, 0^-, \zeta)] da - \\ & \frac{1}{2\pi} \int_{S_4} \frac{\partial \phi(\xi, \eta, 0^+)}{\partial \zeta} da - \\ & \frac{\beta^2 z}{2\pi} \int_{S_4} \frac{\phi(\xi, \eta, 0^+)}{R^3} da - \frac{1}{2\pi} \int_{S_5} \frac{\partial \phi'(\xi, \eta, 0^+)}{\partial \zeta} da - \\ & \frac{\beta^2 z}{2\pi} \int_{S_5} \frac{\phi'(\xi, 0^+, \zeta)}{R^3} da \end{aligned} \quad (57)$$

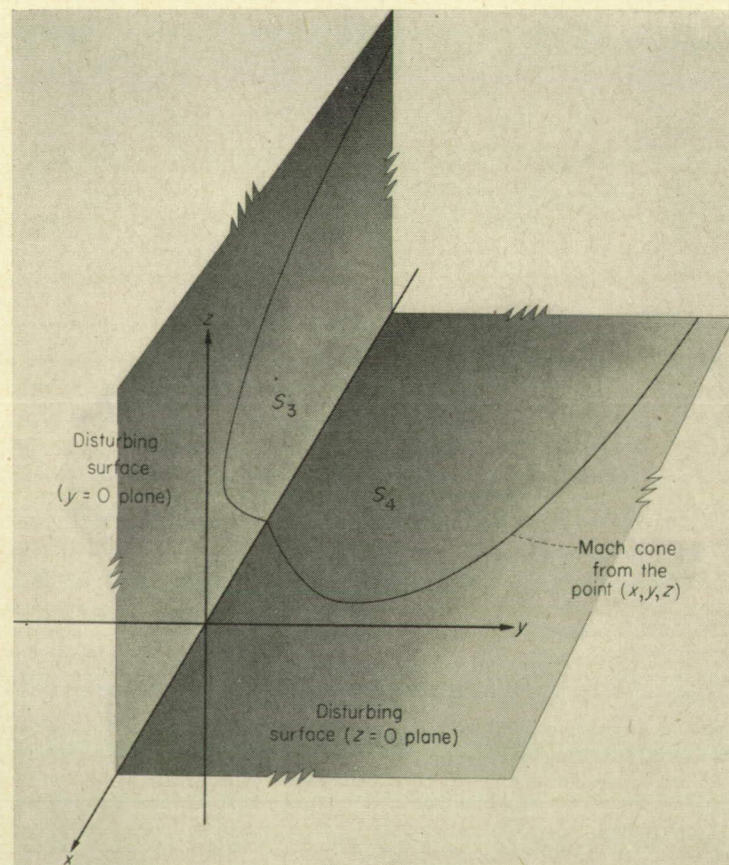


FIGURE 9.—Regions of integration for equation (55).

Equation (61) contains only integrals of known expressions, and it is, therefore, the solution to the problem of two planes intersecting at right angles parallel to the x -axis where the normal derivative of the potential function is known on both planes. Figure 11 shows the cross section of the distribution of velocity normal to the surfaces for a problem as represented by equation (55) and its solution as given by equation (61). Note that $\frac{\partial \phi(\xi, \eta, 0^+)}{\partial \xi}$ (η negative) in S_5 is the reflection of $\frac{\partial \phi(\xi, \eta, 0^+)}{\partial \xi}$ in S_4 (η positive) across the $\eta=0$ plane and that $\frac{\partial \phi(\xi, 0^+, \zeta)}{\partial \eta}$ in S_6 (ζ negative) is the reflection of $\frac{\partial \phi(\xi, 0^+, \zeta)}{\partial \eta}$ in S_3 (ζ positive) across the $\zeta=0$ plane. This condition suggests that the result given by equation (61) could also be obtained by utilizing the concept of reflecting surfaces.

The mathematical derivation required for finding solutions to problems consisting of two planes parallel to the x -axis intersecting at various angles can be reduced by making use of the concept of the reflecting surfaces. For this reason, the result given by equation (61) is obtained by use of reflecting

surfaces. The potential function can be separated into two parts, ϕ_1 and ϕ_2 , satisfying the following boundary conditions, on the disturbing surfaces:

$$\frac{\partial \phi_1(\xi, \eta, 0^+)}{\partial \xi} = \frac{\partial \phi(\xi, \eta, 0^+)}{\partial \xi}$$

$$\frac{\partial \phi_1(\xi, 0^+, \zeta)}{\partial \eta} = 0$$

$$\frac{\partial \phi_2(\xi, \eta, 0^+)}{\partial \xi} = 0$$

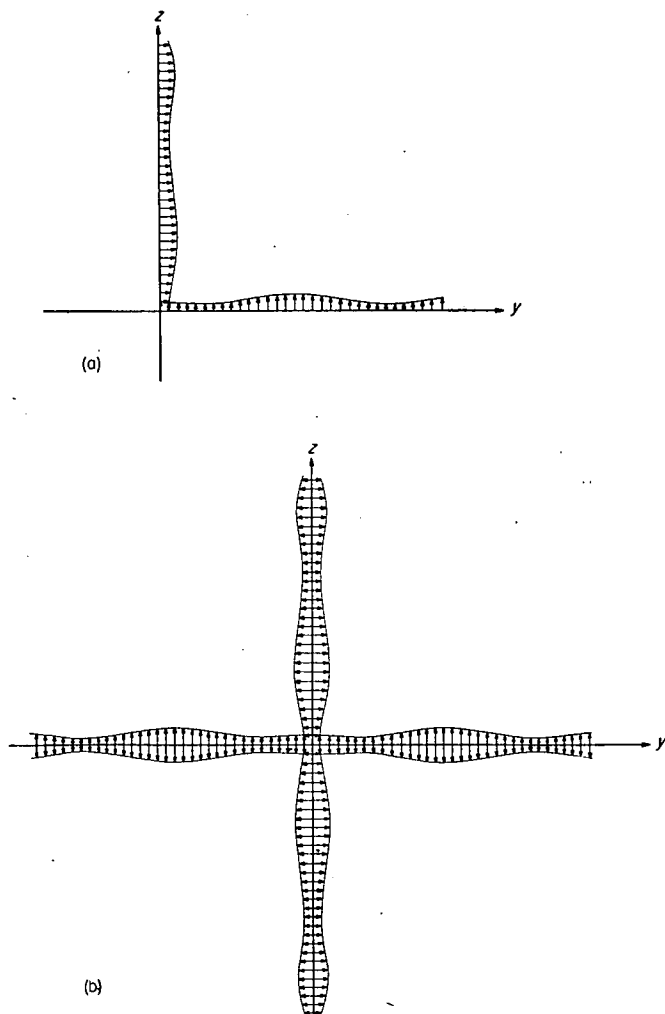
$$\frac{\partial \phi_2(\xi, 0^+, \zeta)}{\partial \eta} = \frac{\partial \phi(\xi, 0^+, \zeta)}{\partial \eta}$$

A cross section of these boundary conditions is shown in figure 12. Only the potential function ϕ_1 is treated in detail since the boundary conditions for ϕ_1 and ϕ_2 are of the same type. The normal derivative of ϕ_1 is zero on the $\eta=0$ plane; thus, the $\eta=0$ plane can be considered as a reflecting plane. The potential function ϕ_1 is, therefore, the potential resulting from a distribution of $\frac{\partial \phi(\xi, \eta, 0^+)}{\partial \xi}$ which is symmetric with respect to the $\eta=0$ line and has the value of $\frac{\partial \phi_1(\xi, \eta, 0^+)}{\partial \xi}$

when η is positive. Figure 13 illustrates such a distribution. The problem of finding ϕ_1 has been reduced to a planar problem which can be solved by use of equation (50).

Equation (50) was obtained by defining the potential below the $z=0$ plane so that the total potential function was symmetric with respect to the $z=0$ plane. This result caused the derivative of the potential function with respect to z to be antisymmetric with respect to the $z=0$ plane. Figure 14 illustrates the distribution of the normal derivative of the function across the $z=0$ plane. The problem of evaluating the potential function ϕ_1 has been reduced to a planar problem. Similarly, the problem of evaluating the potential function ϕ_2 can be reduced to a planar problem. Figure 15 illustrates such a procedure. The original potential function is the sum of ϕ_1 and ϕ_2 . Equation (61) follows from the preceding results for ϕ_1 and ϕ_2 . The addition of ϕ_1 and ϕ_2 is illustrated by figure 16.

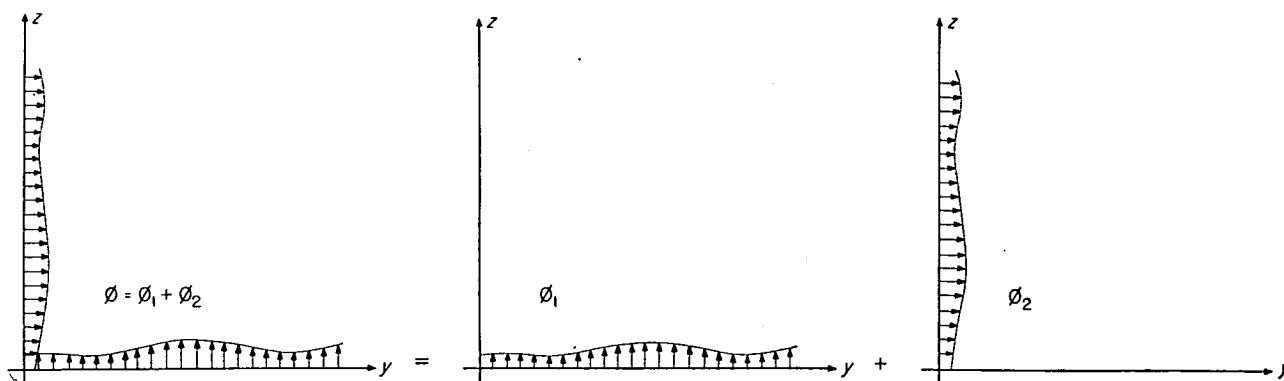
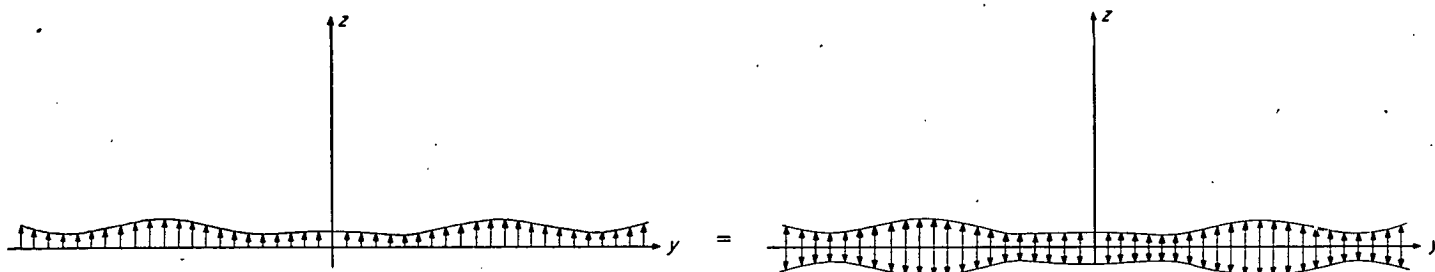
The concept of reflecting surfaces is now utilized to find the potential resulting from two disturbing surfaces parallel to the x -axis and intersecting at an angle of 45° . The axis is chosen so that the x -axis lies along the intersection of the disturbing surfaces and one of the disturbing surfaces lies in the $z=0$ plane (see fig. 17). The potential function ϕ is divided into two parts, ϕ_1 and ϕ_2 . The boundary conditions on ϕ_1 and ϕ_2 are similar to the corresponding potential functions used for the disturbing surfaces intersecting at 90° . Figure 18 illustrates the boundary conditions for ϕ_1 and ϕ_2 . The surfaces on which the normal derivative of ϕ_1 is zero can be considered as a reflecting surface. This consideration leads to the same distribution of the normal derivative of ϕ_1 on the $\eta=0$ plane as is given on the $\zeta=0$ plane. Figure 19 illustrates such a distribution. The problem of finding ϕ_1 for two disturbing surfaces intersecting at 45° has been reduced to a problem of two surfaces intersecting at 90° .



(a) Original problem (equation (55)).

(b) Solution to problem as given by equation (61).

FIGURE 11.—A cross section of the distribution of the velocity component normal to the $z=0$ and the $y=0$ planes represented by equations (55) and (61).

FIGURE 12.—A cross section of the distribution of velocity normal to the disturbing surfaces for the potential functions ϕ , ϕ_1 , and ϕ_2 .FIGURE 13.—The reduction of ϕ_1 to a planar problem.FIGURE 14.—The normal derivative of the potential function across the $z=0$ plane obtained by applying equation (50) to a planar problem.

The solution of ϕ_1 can be obtained from equation (61). Figure 20 shows the surfaces across which the normal derivative of ϕ_1 is discontinuous. Since ϕ_1 and ϕ_2 have the same type of boundary conditions, then ϕ_2 has a solution as illustrated in figure 21. The original potential function ϕ is the sum of ϕ_1 and ϕ_2 ; therefore, ϕ can be found by considering surfaces of discontinuity as illustrated in figure 22. The potential function ϕ can be evaluated by use of equation (22), because no surfaces across which ϕ is discontinuous exist and the values of $\Delta(\partial\phi/\partial n_h)$ are known across all surfaces of discontinuity.

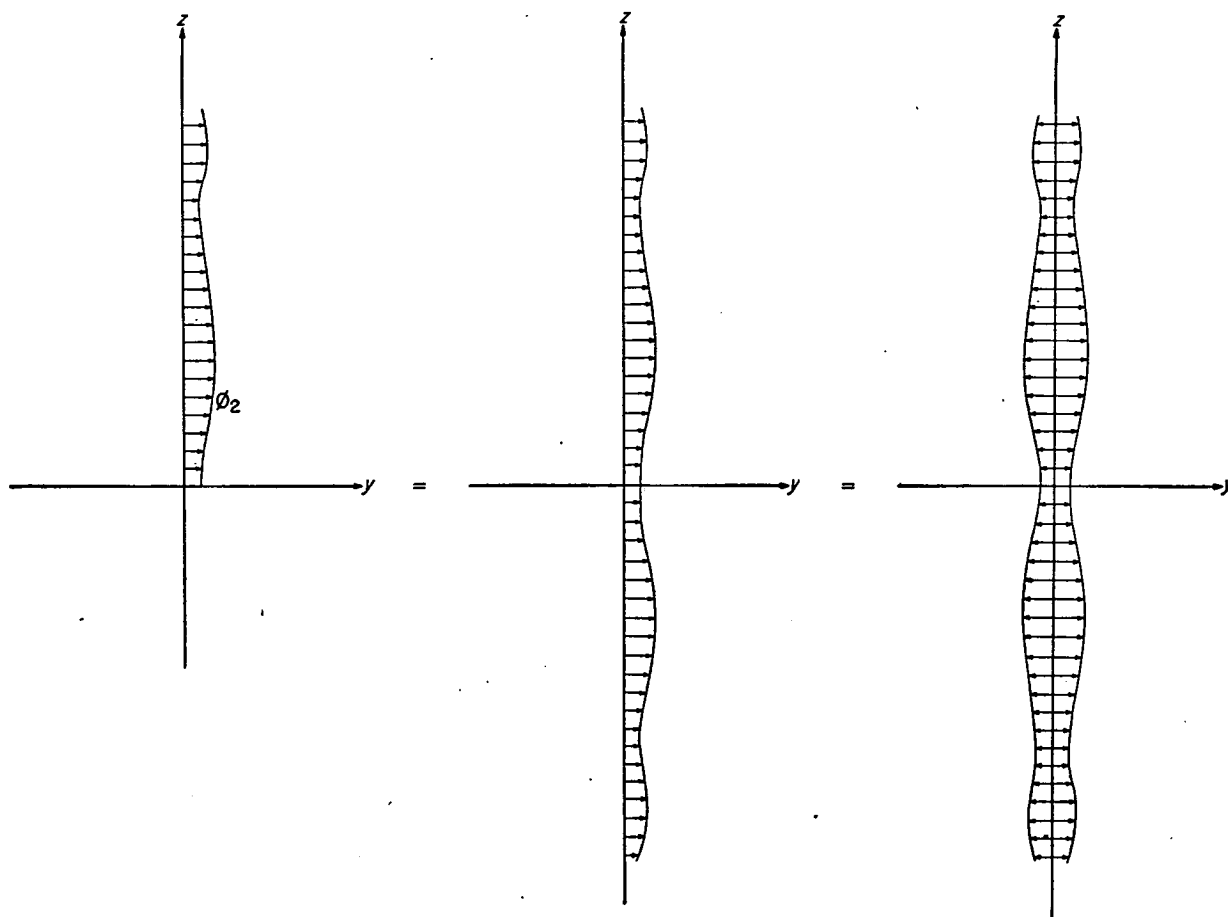
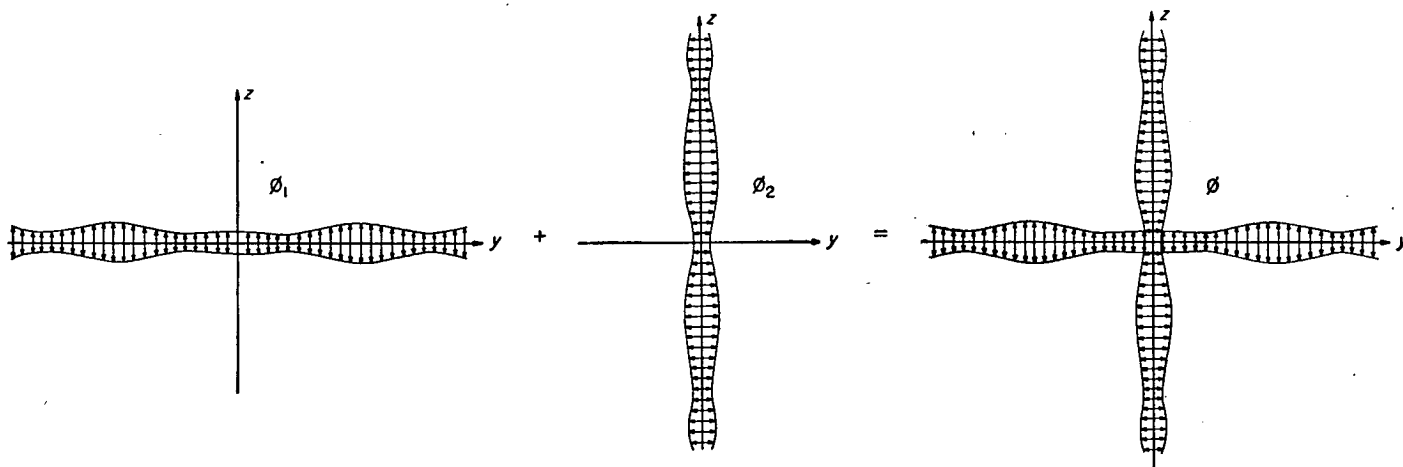
Another simple case of two disturbing surfaces parallel to the x -axis occurs when the surfaces intersect at an angle of 60° . The potential function ϕ is divided into two parts, ϕ_1 and ϕ_2 . The boundary conditions on ϕ_1 and ϕ_2 are similar to the corresponding functions used previously. Figure 23 shows a cross section of these boundary conditions. By use of a reflecting surface, the function ϕ_1 can be represented by the boundary conditions as shown in figure 24. The function ϕ_1 is undefined for 240° of the total angle around the x -axis. The function ϕ_1 is defined as shown in figure 25.

Since no surfaces exist across which ϕ_1 is discontinuous, the function ϕ_1 can be evaluated by using equation (22). Similarly, ϕ_2 can be defined as shown in figure 26. The sum of ϕ_1 and ϕ_2 is illustrated by figure 27. The potential function ϕ can be found by using equation (22).

In the preceding paragraphs, methods have been found for determining the potential resulting from two plane disturbing surfaces parallel to the stream direction intersecting at certain angles. The same method can be used to find methods for determining the potential resulting from two plane disturbing surfaces intersecting at various other angles.

ROLLING TAILS WITH MULTIPLE RECTANGULAR FINS

The methods derived in the preceding section are used to find the surface velocity potential, the pressure distribution, and the damping in roll of rolling tails consisting of four, six, and eight rectangular fins. For comparison, these same quantities are also presented for the planar tail configurations consisting of one and two rectangular fins. An illustration of the tails treated is shown in figure 28. The analysis is limited to tail configurations having surfaces of

FIGURE 15.—An illustration of the reduction of ϕ_2 to a planar problem.FIGURE 16.—An illustration of the addition of ϕ_1 and ϕ_2 .

vanishingly small thickness and of zero camber. The investigation is also limited to the range of Mach numbers for which the region of interference between the adjacent fins does not affect the fin tips (see fig. 29).

TAIL CONSISTING OF ONE FIN

The pressure distribution and the velocity potential on the surface of rolling tails made up of one and two rectangular fins can be obtained from the results of reference 13. The pressure and potential for the tail consisting of only one fin can be found by transforming the axis of roll of the tail consisting of two rectangular fins.

The tail consisting of one fin is divided into regions as shown in figure 29 (a). The velocity potential on the sur-

face facing the negative y -direction is given for the various regions as follows:

For region I,

$$\phi(x, z) = \frac{2p}{\pi} \left[\frac{xz}{\beta} \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} - \frac{1}{3} \left(\frac{2x}{3} - z - b \right) \sqrt{\left(\frac{b}{2} - z \right) \left(z + \frac{x}{\beta} - \frac{b}{2} \right)} \right] \quad (62a)$$

For region II,

$$\phi(x, z) = \frac{pxz}{\beta} \quad (62b)$$

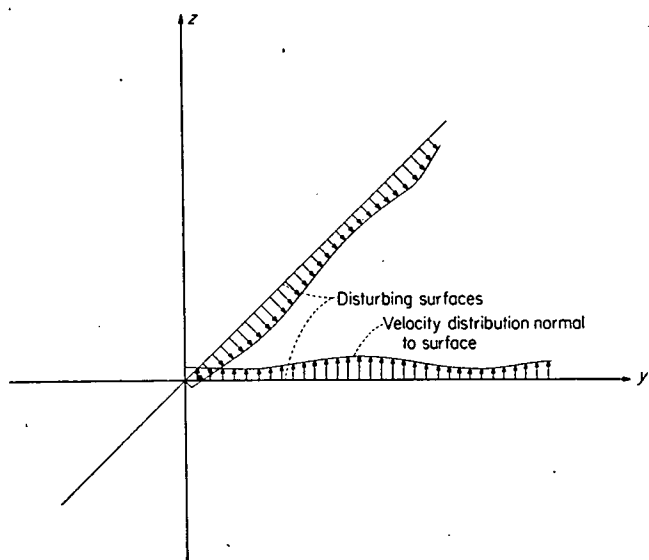


FIGURE 17.—Position of coordinate axes for disturbing surfaces intersecting at 45°.

For region III,

$$\phi(x, z) = \frac{2p}{\pi} \left[\frac{xz}{\beta} \sin^{-1} \sqrt{\frac{\beta z}{x}} + \frac{1}{3} \left(\frac{2x}{\beta} + z \right) \sqrt{z \left(\frac{x}{\beta} - z \right)} \right] \quad (62c)$$

For region IV (note that the potential in region IV is the potential in region I plus the potential in region III minus the potential in region II),

$$\begin{aligned} \phi(x, z) = \frac{2p}{\pi} & \left[\frac{xz}{\beta} \sin^{-1} \sqrt{\frac{\beta z}{x}} + \frac{xz}{\beta} \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} + \right. \\ & \frac{1}{3} \left(\frac{2x}{\beta} + z \right) \sqrt{z \left(\frac{x}{\beta} - z \right)} - \\ & \left. \frac{1}{3} \left(\frac{2x}{\beta} - z - b \right) \sqrt{\left(\frac{b}{2} - z \right) \left(z - \frac{b}{2} + \frac{x}{\beta} \right)} - \frac{\pi x z}{2\beta} \right] \quad (62d) \end{aligned}$$

The pressure-difference coefficient is given for the various regions as follows:

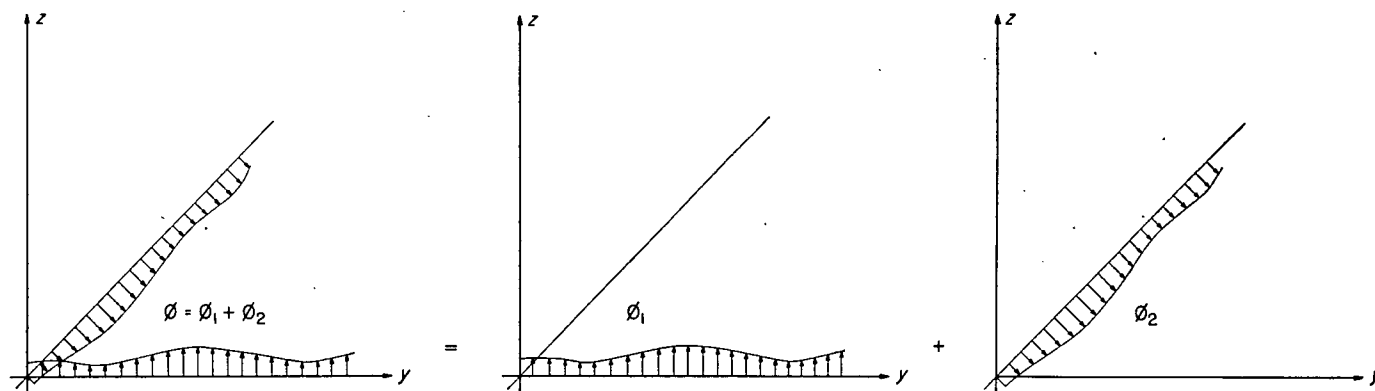


FIGURE 18.—A cross section of the velocity distribution on the disturbing surfaces for the functions ϕ_1 and ϕ_2 for the disturbing surfaces intersecting at 45°.

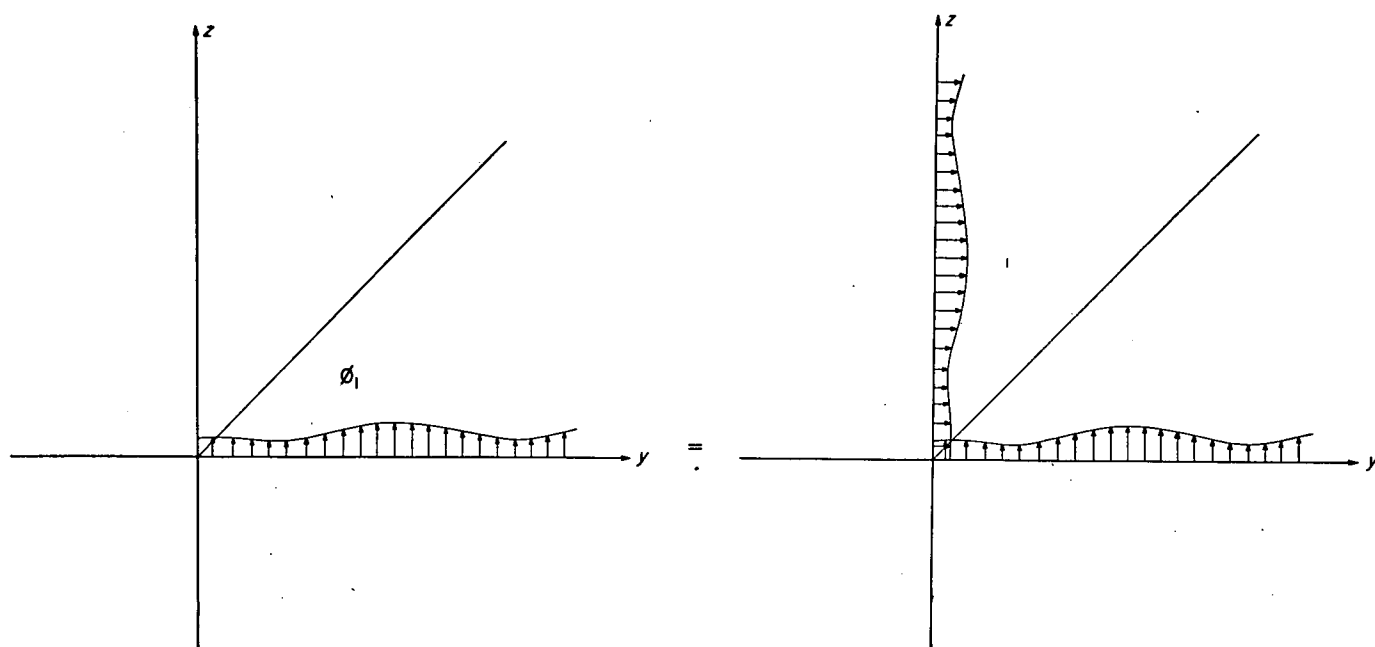


FIGURE 19.—Reflection of the normal derivative of ϕ_1 on the $y=0$ plane.

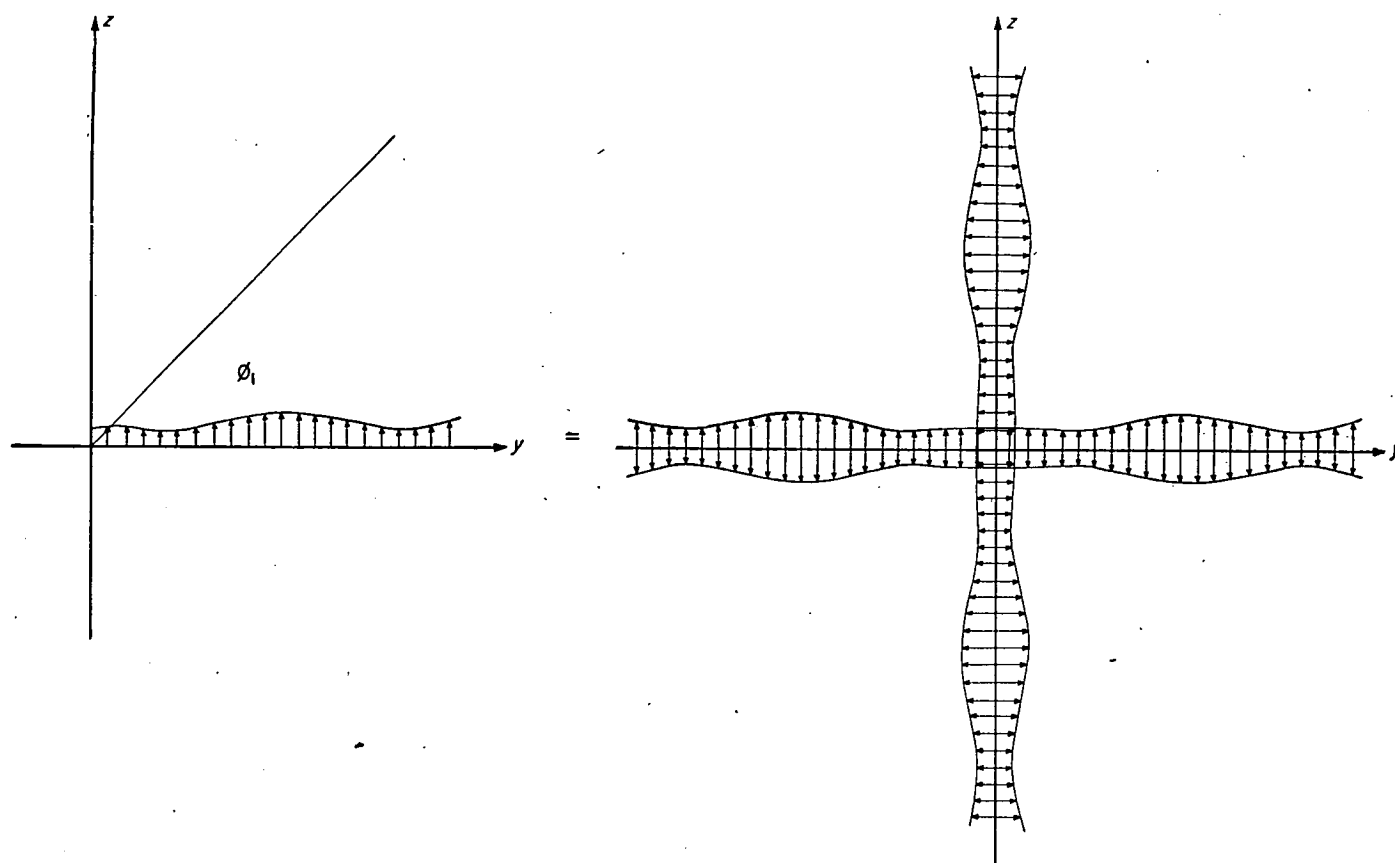


FIGURE 20.—A cross section of the surfaces of discontinuity which can be used to evaluate the potential function ϕ_1 for two surfaces intersecting at 45° .

For region I,

$$\Delta C_p = \frac{8p}{\pi\beta V} \left[z \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z\right)}{x}} - \sqrt{\left(\frac{b}{2} - z\right) \left(z - \frac{b}{2} + \frac{x}{\beta}\right)} \right] \quad (63a)$$

For region II,

$$\Delta C_p = \frac{4pz}{\beta V} \quad (63b)$$

For region III,

$$\Delta C_p = \frac{8p}{\pi\beta V} \left[z \sin^{-1} \sqrt{\frac{\beta z}{x}} + \sqrt{z \left(\frac{x}{\beta} - z\right)} \right] \quad (63c)$$

For region IV,

$$\Delta C_p = \frac{8p}{\pi\beta V} \left[z \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z\right)}{x}} + z \sin^{-1} \sqrt{\frac{\beta z}{x}} + \sqrt{z \left(\frac{x}{\beta} - z\right)} - \sqrt{\left(\frac{b}{2} - z\right) \left(z - \frac{b}{2} + \frac{x}{\beta}\right)} - \frac{\pi z}{2} \right] \quad (63d)$$

TAIL CONSISTING OF TWO FINS

The tail consisting of two fins (fig. 29 (b)) has the same potential and pressure distribution as a rectangular rolling wing and can, therefore, be obtained from the results of reference 13. For each tail consisting of two fins divided into regions as shown in figure 29 (b) the pressure and potential in regions I and II are the same as the pressure and potentials in the corresponding regions for tails consisting of one fin.

TAIL CONSISTING OF FOUR FINS

Each fin of the tail consisting of four fins is divided into regions as shown in figure 29 (c). The pressure and potentials in regions I and II are the same as the pressure and potentials in the corresponding regions for tails consisting of either one or two fins. The regions III and IV are affected by the interaction between adjacent fins. The potential in region IV is made up of a combination of the potentials of regions I, II, and III. Thus, the only real problem is the determination of the potential in region III.

The potential in region III is not affected by the tip and is, therefore, the same potential as would be obtained if the fins were infinitely long. With the coordinate axes chosen

as shown in figure 28 (c) the point (x,y,z) is restricted to values of y which are negative while the values of z are restricted to positive values. Note that for a tail with finitely long fins, the potential at a point (x,y,z) in the region of interaction is independent of the disturbances produced at points located so that their projection on the yz -plane does not lie in the second quadrant. The general method previously derived for finding the potential resulting from two plane surfaces intersecting at right angles can thus

be used to find the potential in the part of the region of interaction which is not affected by the tip.

The velocity component normal to the fin in the $z=0$ plane is given by

$$\phi_z(x,y,0^+) = -py$$

and the velocity component normal to the fin in the $y=0$ plane is given by

$$\phi_y(x,0^-,z) = pz$$

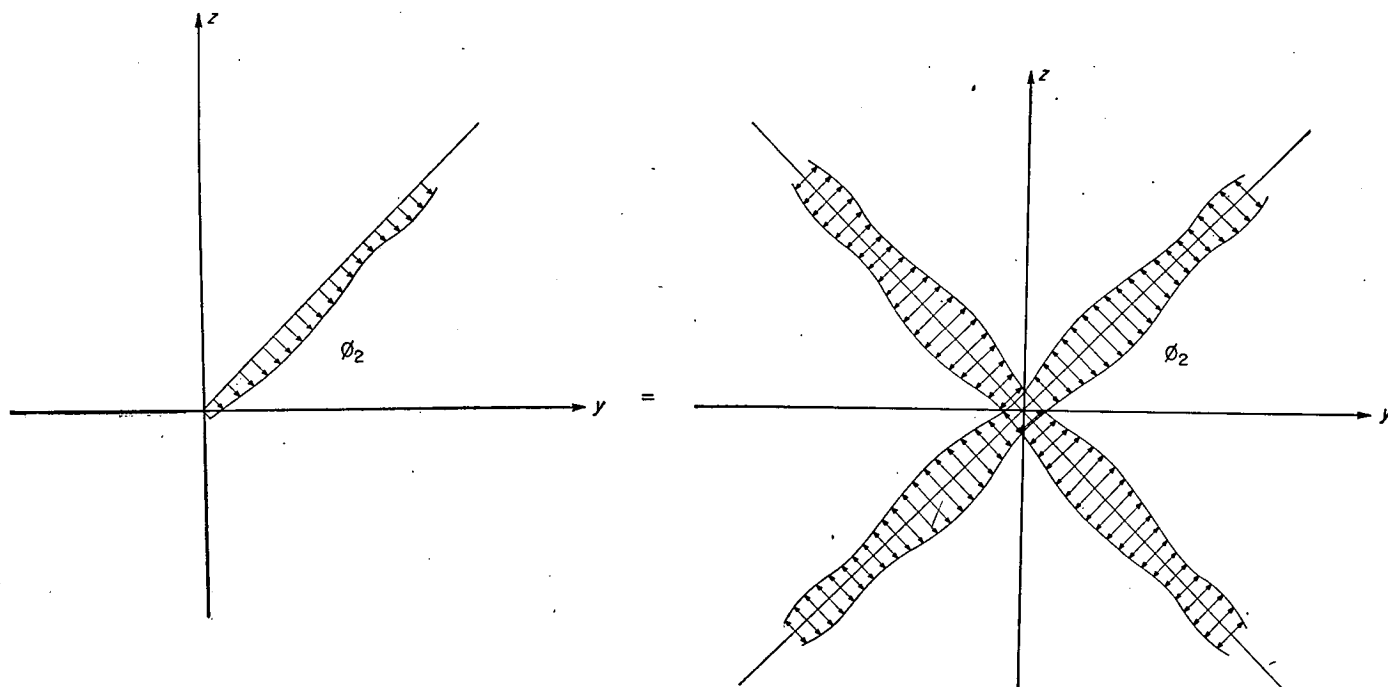


FIGURE 21.—A cross section of the surfaces of discontinuity which can be used to evaluate the potential function ϕ_2 for two planes intersecting at 45° .

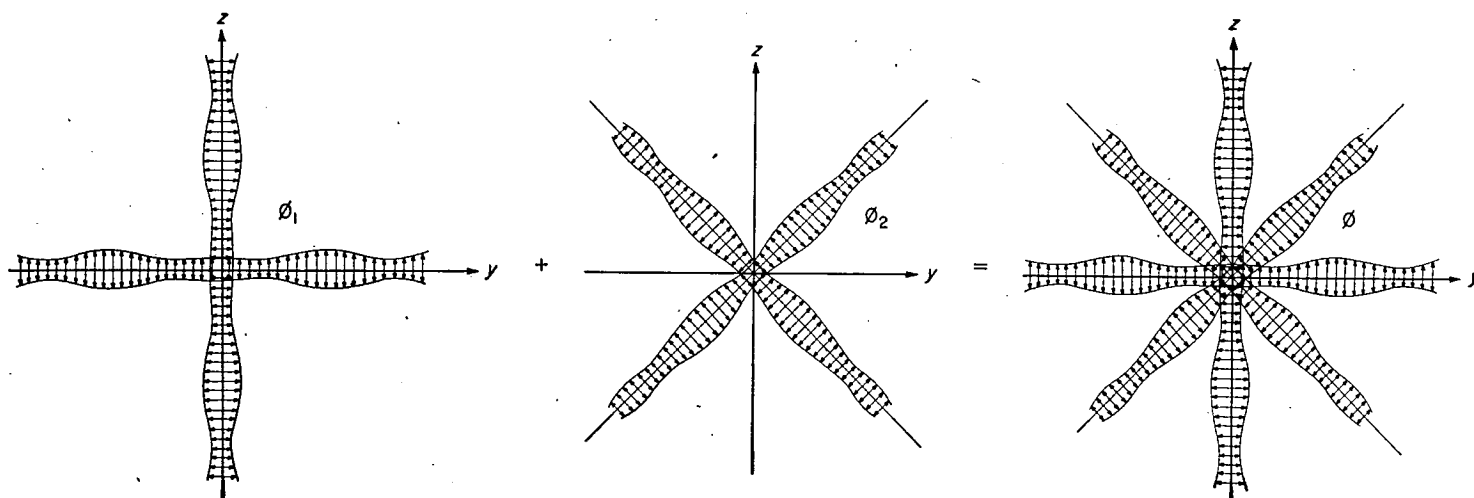


FIGURE 22.—The addition of ϕ_1 and ϕ_2 to obtain the potential function ϕ for two surfaces intersecting at 45° .

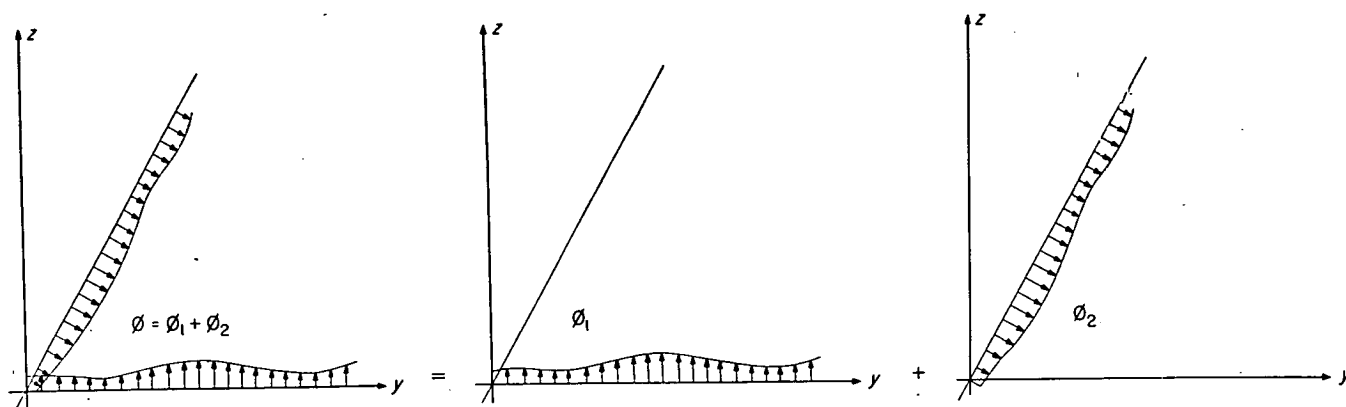


FIGURE 23.—A cross section of the boundary conditions of the functions ϕ_1 and ϕ_2 for two disturbing surfaces intersecting at 60° .

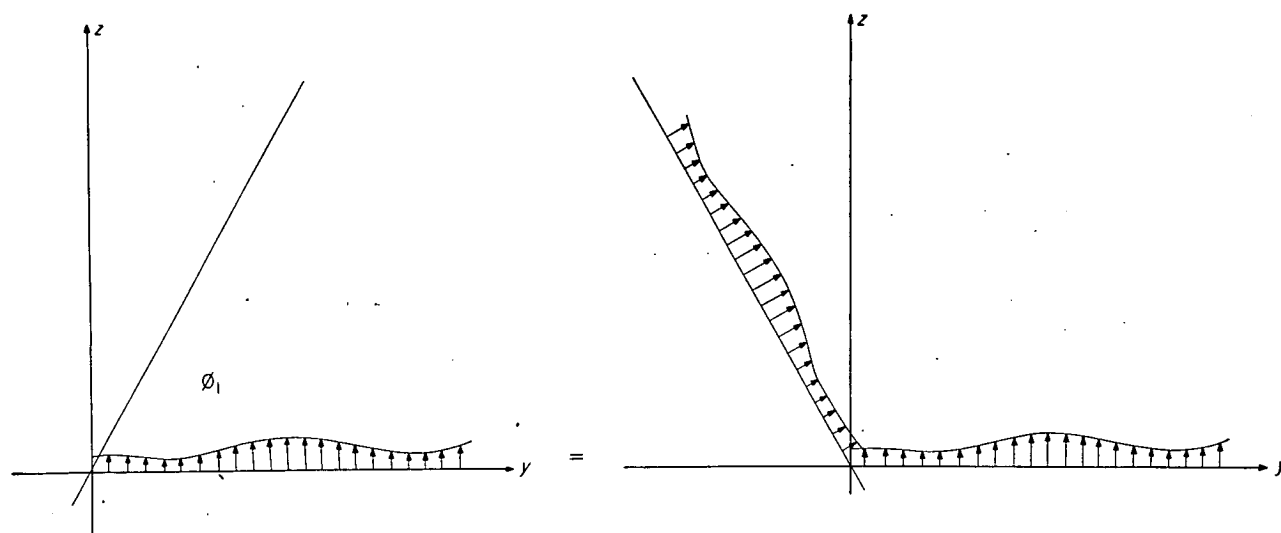


FIGURE 24.—The reflection of ϕ_1 through one surface.

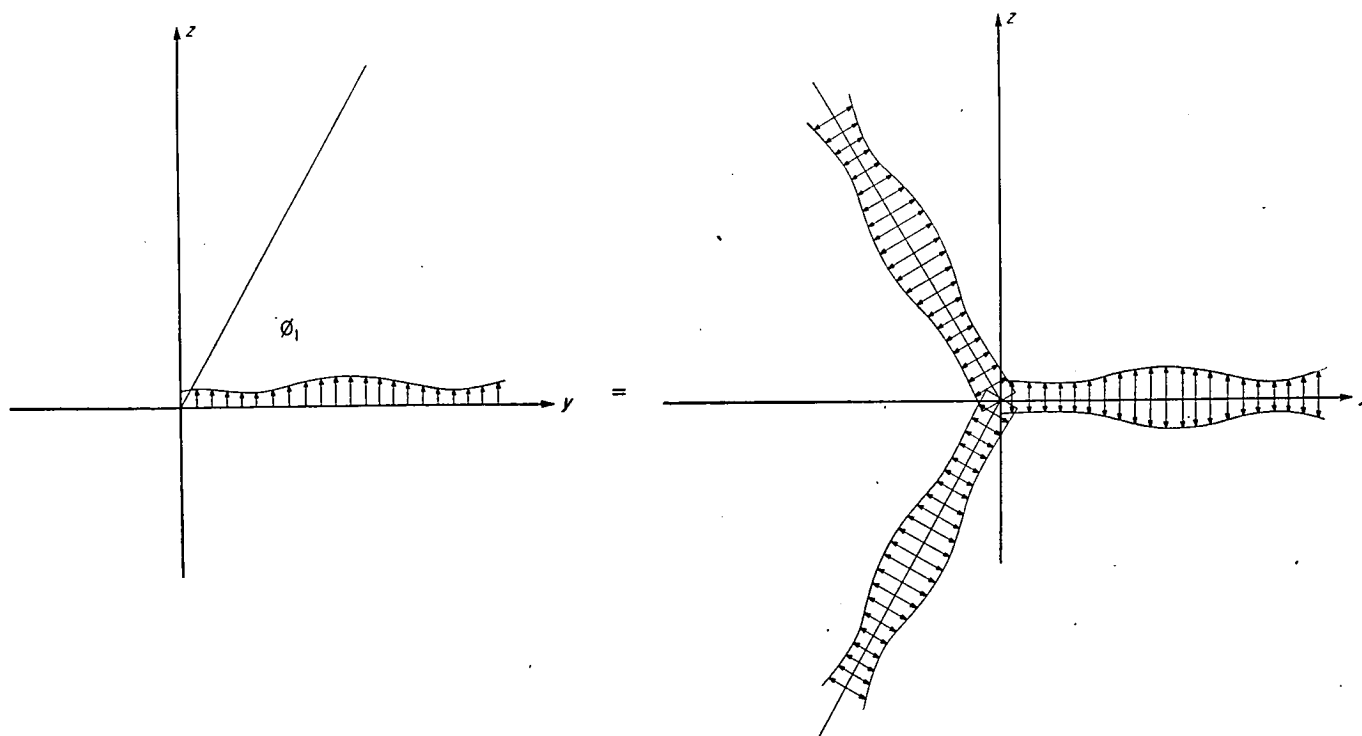


FIGURE 25.—A method of defining ϕ_1 so as to eliminate discontinuities in the potential function.

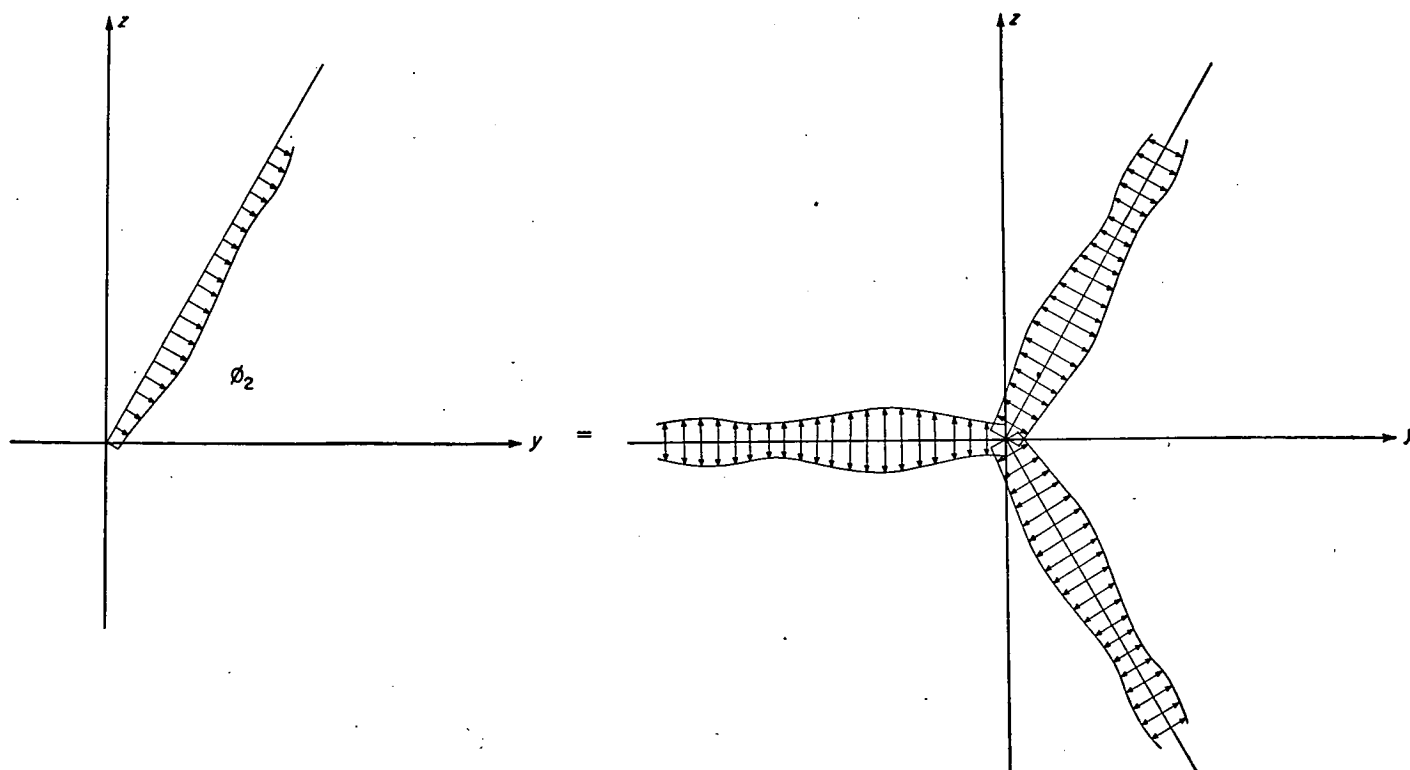


FIGURE 26.—A method of defining ϕ_2 so as to eliminate discontinuities in the potential function.

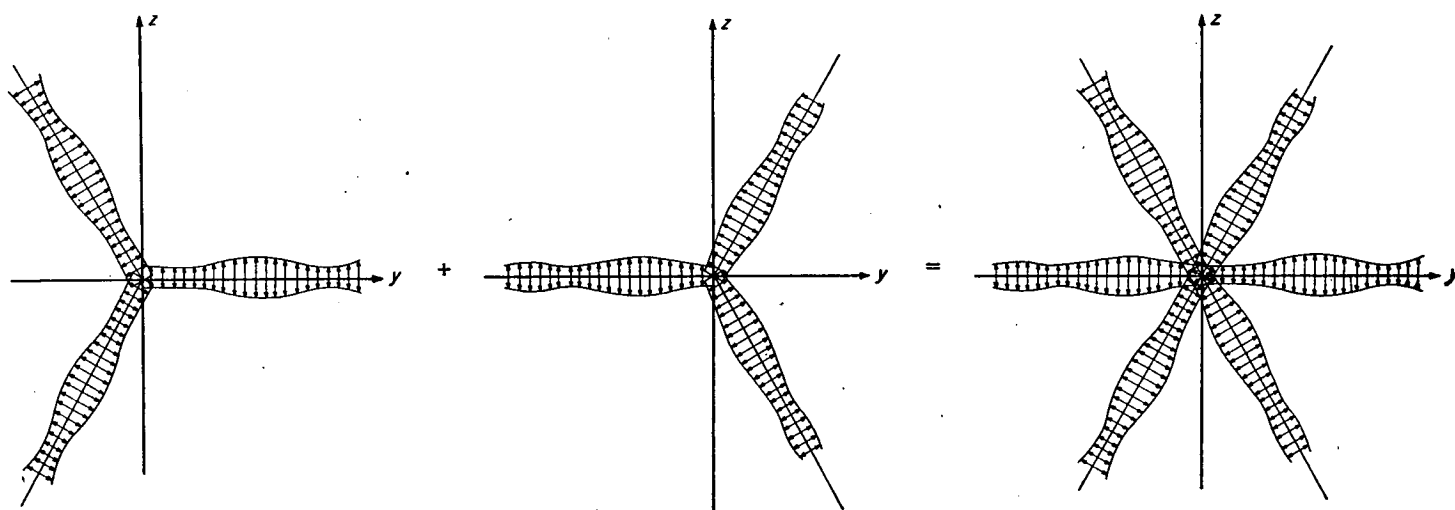
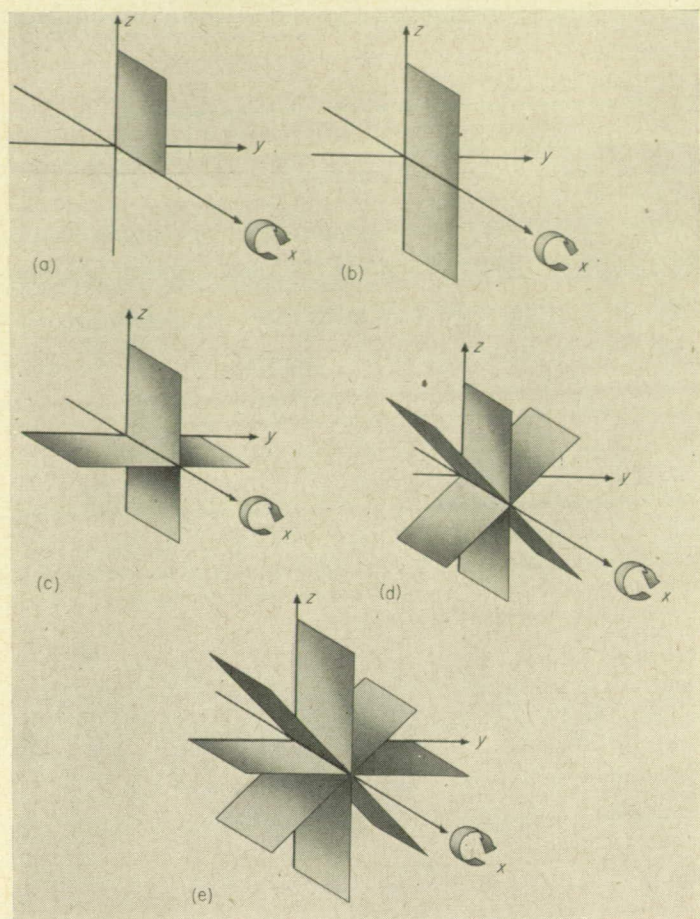


FIGURE 27.—The addition of ϕ_1 and ϕ_2 .



(a) One fin. (b) Two fins.
(c) Four fins. (d) Six fins.
(e) Eight fins.

FIGURE 28.—Types of tails treated.

Figure 30 illustrates this type of normal-velocity distribution, and figure 31 illustrates a cross section of the surfaces of normal-velocity discontinuity, which previous results show can be used to obtain the potential in the part of the region of interaction which is not affected by the tips.

Note that in figure 32 the discontinuity in the normal velocity across the $z=0$ plane is the same type as the discontinuity in the $y=0$ plane. Thus, if the potential resulting from this type of discontinuity (see fig. 32) is known, then the potential resulting from any combination of discontinuities of this type can be found. The potential for this type of distribution is denoted by ϕ_0 . By the use of cylindrical coordinates, as shown in figure 32, the potential at the point (x, ρ, θ) can be expressed in terms of ϕ_0 by

$$\phi(x, \rho, \theta) = \phi_0(x, \rho, \theta) - \phi_0\left(x, \rho, \theta - \frac{\pi}{2}\right) \quad (64)$$

Equation (64) follows from figures 31 and 32.

The potential function ϕ_0 was evaluated by use of equation

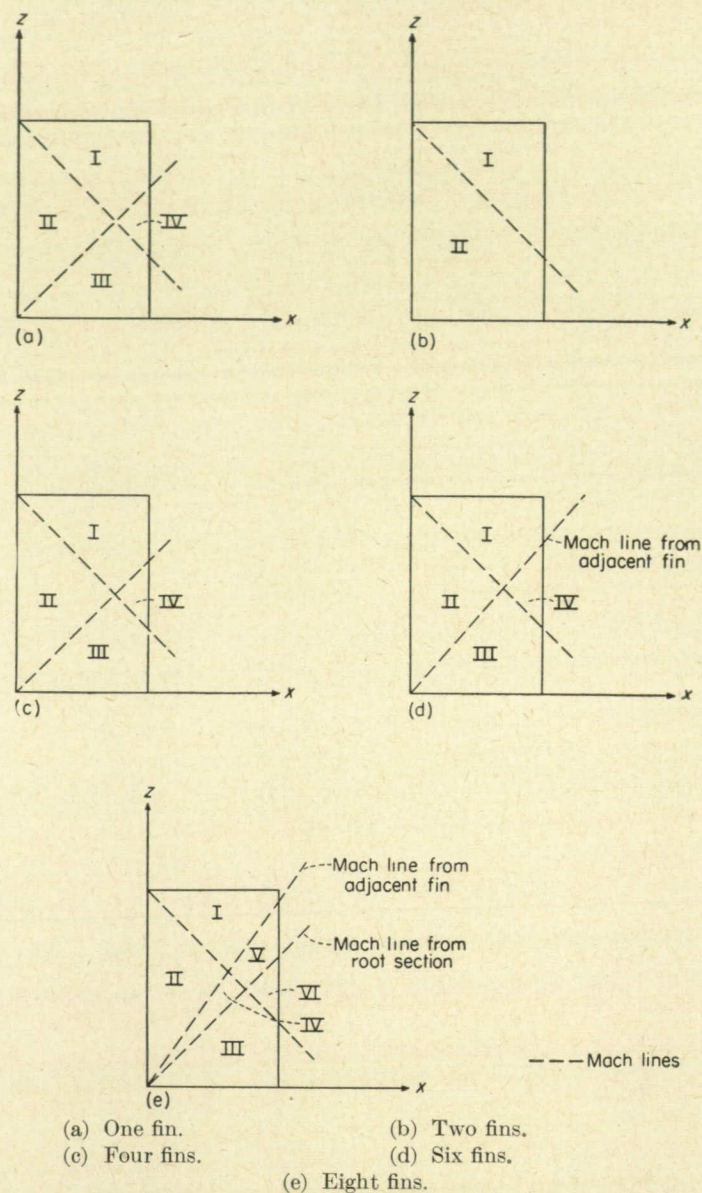


FIGURE 29.—Regions of similar disturbances for tails consisting of rectangular fins.

(50) and is given as follows:

For $0 \leq \theta \leq \frac{\pi}{2}$,

$$\begin{aligned} \phi_0(x, \rho, \theta) = & -\frac{p}{\pi\beta^2} \left[x\sqrt{x^2 - \beta^2\rho^2} + \right. \\ & \beta^2\rho^2(1 - 2\cos^2\theta) \ln \frac{\beta\rho}{x + \sqrt{x^2 - \beta^2\rho^2}} + \\ & 2x\beta\rho \cos\theta \sin^{-1} \left(\frac{\beta\rho \cos\theta}{\sqrt{x^2 - \beta^2\rho^2} \sin^2\theta} \right) - \\ & \left. 2\beta^2\rho^2 \cos\theta \sin\theta \tan^{-1} \left(\frac{x \cot\theta}{\sqrt{x^2 - \beta^2\rho^2}} \right) \right] \quad (65a) \end{aligned}$$

For $\frac{\pi}{2} \leq \theta \leq \pi$,

$$\begin{aligned} \phi_0(x, \rho, \theta) = & -\frac{p}{\pi\beta^2} \left[x\sqrt{x^2 - \beta^2\rho^2} + \right. \\ & \left. \beta^2\rho^2(1 - 2\cos^2\theta) \ln \frac{\beta\rho}{x + \sqrt{x^2 - \beta^2\rho^2}} + \right. \\ & \left. 2x\beta\rho \cos\theta \sin^{-1} \left(\frac{\beta\rho \cos\theta}{\sqrt{x^2 - \beta^2\rho^2} \sin^2\theta} \right) + \right. \\ & \left. 2\beta^2\rho^2 \sin\theta \cos\theta \tan^{-1} \left(\frac{x \cot\theta}{\sqrt{x^2 - \beta^2\rho^2}} \right) \right] \quad (65b) \end{aligned}$$

From equations (64) and (65), the potential function in the region of interaction, which is not affected by the tips, is given by the following equation for $\frac{\pi}{2} \leq \theta \leq \pi$:

$$\begin{aligned} \phi(x, \rho, \theta) = & -\frac{p}{\pi\beta^2} \left\{ 2\beta^2\rho^2(1 - 2\cos^2\theta) \ln \frac{\beta\rho}{x + \sqrt{x^2 - \beta^2\rho^2}} + \right. \\ & \left. 2x\beta\rho \left[\cos\theta \sin^{-1} \left(\frac{\beta\rho \cos\theta}{\sqrt{x^2 - \beta^2\rho^2} \sin^2\theta} \right) - \right. \right. \\ & \left. \left. \sin\theta \sin^{-1} \left(\frac{\beta\rho \sin\theta}{\sqrt{x^2 - \beta^2\rho^2} \cos^2\theta} \right) \right] + \right. \\ & \left. 2\beta^2\rho^2 \cos\theta \sin\theta \left(\tan^{-1} \frac{x|\cot\theta|}{\sqrt{x^2 - \beta^2\rho^2}} - \right. \right. \\ & \left. \left. \tan^{-1} \frac{x|\tan\theta|}{\sqrt{x^2 - \beta^2\rho^2}} \right) \right\} \quad (66) \end{aligned}$$

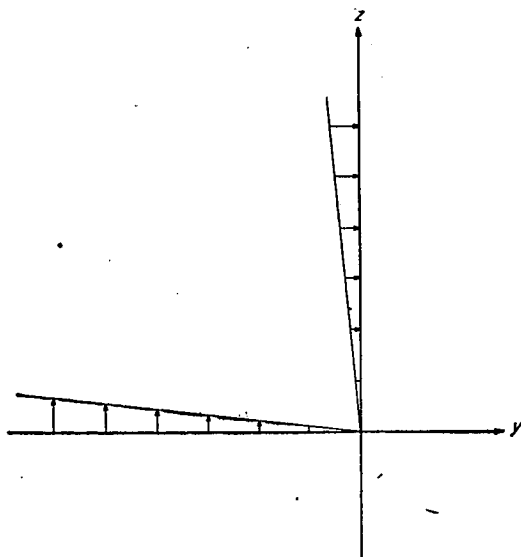


FIGURE 30.—A cross section of the normal-velocity distribution on two plane surfaces representing the region of interaction for a rolling tail with four fins.

The potential in region III of the fin surface is a special case ($\theta = \frac{\pi}{2}$) of the preceding equation. Thus, the potential in region III is given (in Cartesian coordinates) by

$$\phi(x, 0^-, z) = \frac{2pz}{\pi\beta} \left(x \sin^{-1} \frac{\beta z}{x} - \beta z \ln \frac{\beta z}{x + \sqrt{x^2 - \beta^2 z^2}} \right) \quad (67)$$

From equation (67) the pressure-difference coefficient is found to be

$$\Delta C_p = \frac{8pz}{\pi V\beta} \sin^{-1} \frac{\beta z}{x} \quad (68)$$

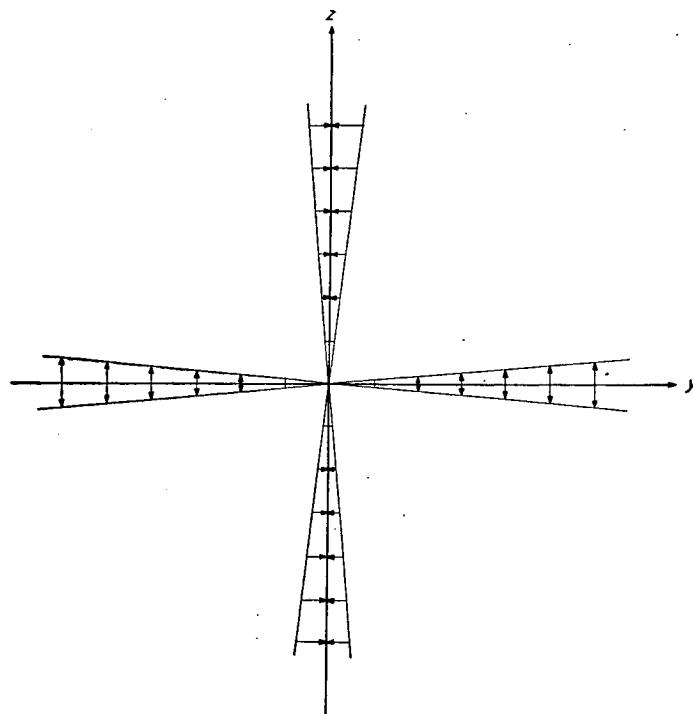


FIGURE 31.—A cross section of the velocity discontinuity distribution used to find the potential in part of the region of interaction for a tail of four fins.

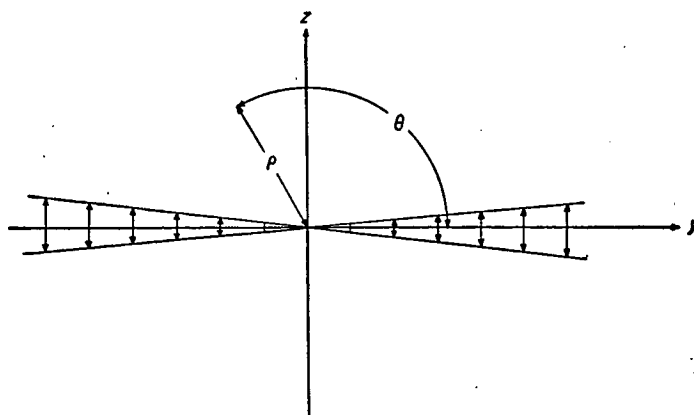


FIGURE 32.—A cross section of the velocity discontinuity distribution associated with the function ϕ_0 for a tail with four fins.

As previously stated, the potential in region IV (see fig. 29) is a combination of the potentials in regions I, II, and III. Assume that the fins are infinitely long. In this case only two regions (II and III) exist since regions I and IV are affected by the tip. The effect of the tip can be taken into account by adding a potential which has zero normal velocity on the fin and the negative of the pressure of the infinite fin in the plane of the fin outboard of the tip. The value of such a potential on the fin is given by the difference between the potential of region I and the potential of region II. (This potential is only the effect of the tip on a semi-infinite rolling wing.) Thus, the potential in region IV is the potential in region III plus the difference between the potential of region I and the potential of region II. Mathematically, the potential in region IV is given by

$$\phi(x, 0^-, z) = \frac{2p}{\pi} \left[-\frac{xz}{\beta} \cos^{-1} \frac{\beta z}{x} + \frac{xz}{\beta} \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z\right)}{x}} - \frac{1}{3} \left(\frac{2x}{\beta} - z - b \right) \sqrt{\left(\frac{b}{2} - z \right) \left(z + \frac{x}{\beta} - \frac{b}{2} \right)} - z^2 \ln \frac{\beta z}{x + \sqrt{x^2 - \beta^2 z^2}} \right] \quad (69)$$

From equation (69) the pressure-difference coefficient is given by

$$\Delta C_p = \frac{8p}{\pi V \beta} \left[-z \cos^{-1} \frac{\beta z}{x} + z \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z\right)}{x}} - \sqrt{\left(\frac{b}{2} - z \right) \left(z - \frac{b}{2} + \frac{x}{\beta} \right)} \right] \quad (70)$$

TAIL CONSISTING OF SIX FINS

The pressure and potentials on the surface of the tail consisting of six fins can be obtained in a manner similar to that used for the tail consisting of four fins. The pressure and potentials in regions I and II (fig. 29 (d)) are the same as the pressure and potentials in the corresponding regions for tails consisting of one, two, or four fins. Regions III and IV are affected by the interaction between adjacent fins. The potential in region IV is made up of a combination of the potential in regions I, II, and III; therefore, the main problem, as for the case of four fins, is the determination of the potential in region III.

The potential in region III is the same as the potential for a tail consisting of six infinitely long fins. The induced velocities normal to two of the planes of the fins are illustrated in figure 33. For two plane surfaces parallel to the stream direction and intersecting at an angle of 60° , the potential in region III can be obtained by a distribution of discontinuities in velocity as illustrated in figure 34. Note that the potential in region III can be made up of a combination of the potentials from a velocity discontinuity as shown in figure 35. The potential from this type of discontinuity is denoted by ϕ_0 . By use of cylindrical coordinates as shown in figure 35, the potential at the point (x, ρ, θ) can be expressed in terms of ϕ_0 by

$$\phi(x, \rho, \theta) = \phi_0 \left(x, \rho, \theta - \frac{\pi}{6} \right) - \phi_0 \left(x, \rho, \theta - \frac{\pi}{2} \right) - \phi_0 \left(x, \rho, \theta + \frac{\pi}{6} \right) \quad (71)$$

Equation (71) follows from figures 34 and 35.

The potential function ϕ_0 was evaluated by use of equation (50) and is given by the following equation for $0 \leq \theta \leq \pi$:

$$\phi_0(x, \rho, \theta) = -\frac{p\rho \cos \theta}{\beta} (x - \beta \rho \sin \theta) \quad (72)$$

From equations (71) and (72) the potential function in the region of interaction, which is not affected by the tips, is given by the following equation for $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$:

$$\phi(x, \rho, \theta) = \frac{p\rho^2}{2} [\sqrt{3}(1 - 2 \cos^2 \theta) + 2 \cos \theta \sin \theta] \quad (73)$$

The potential in region III of the fin surface is a special case ($\theta = \frac{\pi}{2}$) of equation (73). The potential in this region is given (in Cartesian coordinates) by

$$\phi(x, 0^-, z) = \frac{pz^2\sqrt{3}}{2} \quad (74)$$

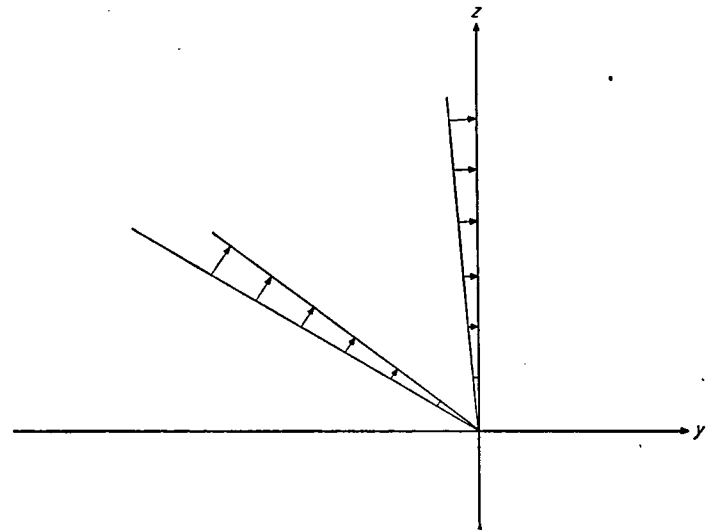


FIGURE 33.—A cross section of the velocity distribution normal to the planes of two fins of a rolling tail consisting of six fins.

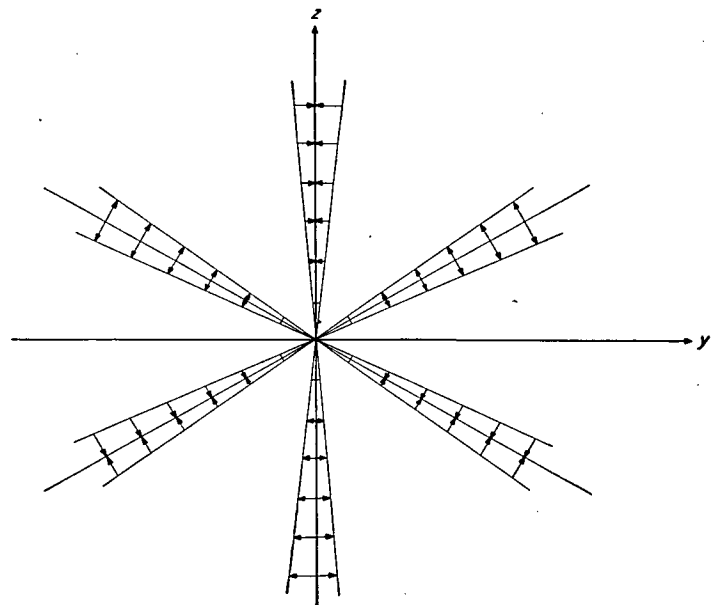


FIGURE 34.—A cross section of the velocity discontinuity distribution used to find the potential in part of the region of interaction for a tail of six fins.

From equation (74) the pressure-difference coefficient in region III is found to be zero.

The potential in region IV is a combination of the potentials in regions I, II, and III; this can be shown in the same way as the potential in region IV of the tail consisting of four fins was shown to be a combination of potentials from other regions. Specifically, the potential in region IV for the tail consisting of six fins is the potential in region III plus the difference between the potentials of region I and of region II. Mathematically, the potential in region IV is given by

$$\phi(x, 0^-, z) = \frac{2p}{\pi} \left[\frac{\pi z^2 \sqrt{3}}{4} - \frac{xz}{\beta} \cos^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} - \frac{1}{3} \left(\frac{2x}{\beta} - z - b \right) \sqrt{\left(\frac{b}{2} - z \right) \left(z + \frac{x}{\beta} - \frac{b}{2} \right)} \right] \quad (75)$$

From equation (75) the pressure-difference coefficient is given by

$$\Delta C_p = -\frac{8p}{\pi \beta V} \left[z \cos^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} + \sqrt{\left(\frac{b}{2} - z \right) \left(z - \frac{b}{2} + \frac{x}{\beta} \right)} \right] \quad (76)$$

TAIL CONSISTING OF EIGHT FINS

The pressure and potential on the surface of the tail consisting of eight fins can be found by utilizing the potential functions ϕ_0 used in finding the pressure and potentials on the surface of the tail consisting of four and six fins. The pressure and potentials in regions I and II are the same as the pressure and potentials for the corresponding regions of the other tails. The potentials in regions III, IV, V, and VI (see fig. 29(e)) are affected by the interaction between adjacent fins. Since the potentials in regions V and VI are combinations of the potentials in the remaining regions, the main problem is to find the potentials in regions III and IV.

The potentials in regions III and IV are the same as the potential for a tail of eight infinitely long fins. The induced velocity normal to two of the planes of the fins is illustrated in figure 36. From the results for two plane surfaces intersecting at an angle of 45° , the potentials in regions III and IV can be obtained by a distribution of discontinuities in velocity as illustrated in figure 37. The potential resulting from the distribution of discontinuities in velocity as illustrated in figure 37 can be obtained from a distribution of

discontinuities as possessed by the potential function ϕ_0 used in connection with the four-finned tail. The potential function ϕ_0 used in connection with the four-finned tail was evaluated only in the region affected by the root sections of the fins. For the case of the eight-finned tail, interaction occurs between adjacent fins in regions which are not affected by the root sections of the fins. The potential function ϕ_0 of the four-finned tail in the region not affected by the root section must be known. In this region, the potential functions ϕ_0 for the four- and six-finned tails are the same.

From figures 32 and 37 the potential function in the region of fin interaction which is affected by the root sections of the fins is, for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$,

$$\phi(x, \rho, \theta) = -\phi_0(x, \rho, \theta) + \phi_0\left(x, \rho, \theta - \frac{\pi}{4}\right) - \phi_0\left(x, \rho, \theta - \frac{\pi}{2}\right) + \phi_0\left(x, \rho, \theta + \frac{\pi}{4}\right) \quad (77)$$

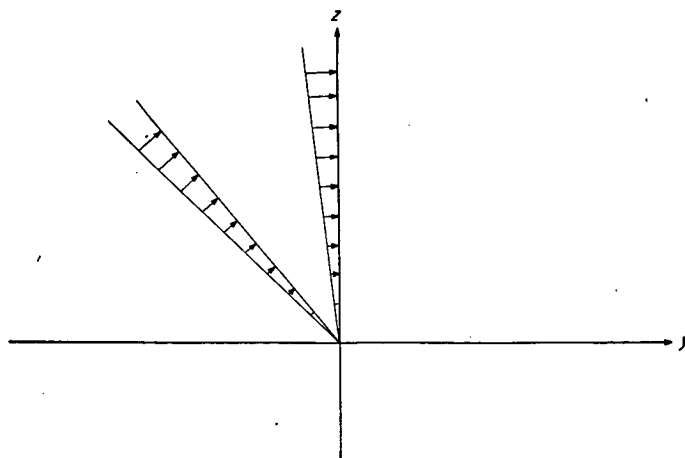


FIGURE 36.—A cross section of the normal velocity induced on the planes of two fins of a rolling tail consisting of eight fins.

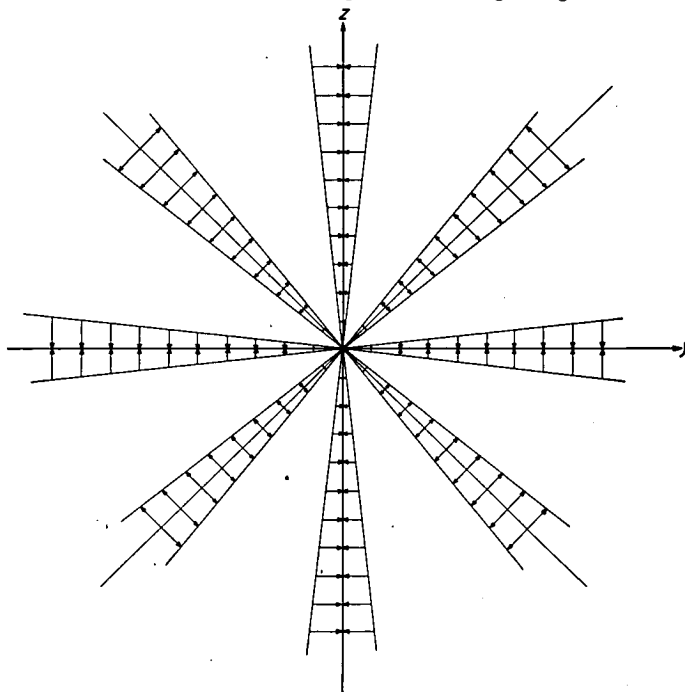


FIGURE 37.—A cross section of the velocity discontinuities which can be used to obtain the potential in the region of interaction between adjacent fins for a tail consisting of eight fins.

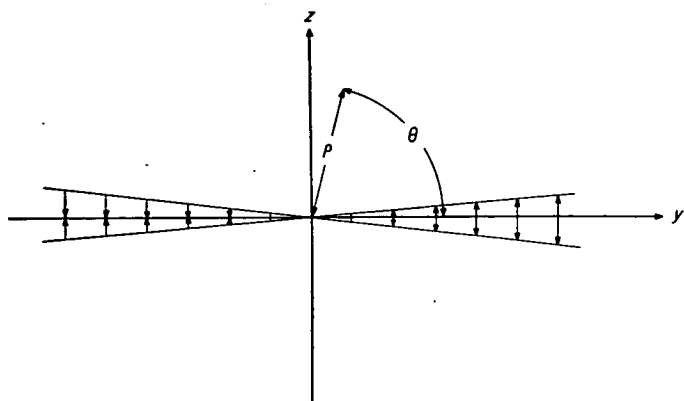


FIGURE 35.—A cross section of the velocity discontinuity distribution associated with the function ϕ_0 for a tail of six fins.

where ϕ_0 is given by equations (65).

The potential in the part of the region of interaction which is affected by the root section is (from eqs. (65) and (77)), for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$,

$$\begin{aligned} \phi(x, \rho, \theta) = & -\frac{2p\rho}{\pi\beta} \left\{ -x \cos \theta \sin^{-1} \frac{\beta\rho \cos \theta}{\sqrt{x^2 - \beta^2 \rho^2 \sin^2 \theta}} + x \cos \left(\theta - \frac{\pi}{4} \right) \sin^{-1} \frac{\beta\rho \cos \left(\theta - \frac{\pi}{4} \right)}{\sqrt{x^2 - \beta^2 \rho^2 \sin^2 \left(\theta - \frac{\pi}{4} \right)}} - x \sin \theta \sin^{-1} \frac{\beta\rho \sin \theta}{\sqrt{x^2 - \beta^2 \rho^2 \cos^2 \theta}} + \right. \\ & x \cos \left(\theta + \frac{\pi}{4} \right) \sin^{-1} \frac{\beta\rho \cos \left(\theta + \frac{\pi}{4} \right)}{\sqrt{x^2 - \beta^2 \rho^2 \sin^2 \left(\theta + \frac{\pi}{4} \right)}} - \beta\rho \sin \theta \cos \theta \left[\tan^{-1} \frac{x |\cot \theta|}{\sqrt{x^2 - \beta^2 \rho^2}} + \tan^{-1} \frac{x |\cot \left(\theta + \frac{\pi}{4} \right)|}{\sqrt{x^2 - \beta^2 \rho^2}} \right] - \\ & \left. \frac{\beta\rho}{2} (1 - 2 \cos^2 \theta) \left[\tan^{-1} \frac{x |\cot \left(\theta - \frac{\pi}{4} \right)|}{\sqrt{x^2 - \beta^2 \rho^2}} + \tan^{-1} \frac{x |\cot \left(\theta + \frac{\pi}{4} \right)|}{\sqrt{x^2 - \beta^2 \rho^2}} \right] \right\} \end{aligned} \quad (78)$$

The potential in region III is a special case ($\theta = \frac{\pi}{2}$) of equation (78). Setting $\theta = \frac{\pi}{2}$ in equation (78) yields (in Cartesian coordinates)

$$\phi(x, 0^-, z) = \frac{2pz}{\pi\beta} \left(x \sin^{-1} \frac{\beta z}{x} - x \sqrt{2} \sin^{-1} \frac{\beta z}{\sqrt{2x^2 - \beta^2 z^2}} + \beta z \tan^{-1} \frac{x}{\sqrt{x^2 - \beta^2 z^2}} \right) \quad (79)$$

From equation (79) the pressure-difference coefficient is found to be given by

$$\Delta C_p = \frac{8pz}{\pi\beta V} \left(\sin^{-1} \frac{\beta z}{x} - \sqrt{2} \sin^{-1} \frac{\beta z}{\sqrt{2x^2 - \beta^2 z^2}} \right) \quad (80)$$

By inspection of figures 35 and 37 the potential function in the region of fin interaction, which is not affected by the root sections of the fins, is, for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$,

$$\phi(x, \rho, \theta) = \phi_0(x, \rho, \theta) + \phi_0 \left(x, \rho, \theta - \frac{\pi}{4} \right) - \phi_0 \left(x, \rho, \theta - \frac{\pi}{2} \right) - \phi_0 \left(x, \rho, \theta - \frac{3\pi}{4} \right) \quad (81)$$

where ϕ_0 is given by equation (72). Substituting equation (72) into equation (81) yields (remember that ϕ_0 is zero upstream of the Mach cone from the y -axis), for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$,

$$\begin{aligned} \phi(x, \rho, \theta) = & \frac{p\rho x}{\beta} \left[\left(1 - \frac{1}{\sqrt{2}} \right) \sin \theta + \frac{\cos \theta}{\sqrt{2}} \right] + p\rho^2 \cos \theta \sin \theta - \frac{p\rho^2}{2} (1 - 2 \sin^2 \theta) - \\ & \frac{p\rho}{\beta \sqrt{2}} (\cos \theta + \sin \theta) \left\{ \begin{array}{l} 0; \text{ if } x < \frac{\beta\rho}{\sqrt{2}} (\sin \theta - \cos \theta) \\ x - \frac{\beta\rho}{\sqrt{2}} (\sin \theta - \cos \theta); \text{ if } x > \frac{\beta\rho}{\sqrt{2}} (\sin \theta - \cos \theta) \end{array} \right\} - \frac{p\rho}{\beta} \cos \theta \left\{ \begin{array}{l} 0; \text{ if } x < \beta\rho \sin \theta \\ x - \beta\rho \sin \theta; \text{ if } x > \beta\rho \sin \theta \end{array} \right\} \end{aligned} \quad (82)$$

The potential in region IV is a special case ($\theta = \frac{\pi}{2}$) of equation (82). Setting $\theta = \frac{\pi}{2}$ in equation (82) yields (in Cartesian coordinates)

$$\phi(x, 0^-, z) = \frac{pz}{\beta} [(1 - \sqrt{2})x + \beta z] \quad (83)$$

From equation (83) the pressure-difference coefficient is found to be given by

$$\Delta C_p = \frac{4pz}{\beta x} (1 - \sqrt{2}) \quad (84)$$

The potential in region V is the potential in region IV plus the difference between the potential in region I and the potential in region II; thus, from equations (62a), (62b), and (83), the potential in region V is found to be given by

$$\phi(x, 0^-, z) = \frac{2p}{\pi} \left[\frac{\pi z}{2} \left(z - \frac{x\sqrt{2}}{\beta} \right) + \frac{xz}{\beta} \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} - \frac{1}{3} \left(\frac{2x}{3} - z - b \right) \sqrt{\left(\frac{b}{2} - z \right) \left(z + \frac{x}{\beta} - \frac{b}{2} \right)} \right] \quad (85)$$

From equation (85) the pressure-difference coefficient is found to be given by

$$\Delta C_p = -\frac{8p}{\pi\beta V} \left[\frac{\pi z}{\sqrt{2}} - z \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} + \sqrt{\left(\frac{b}{2} - z \right) \left(z - \frac{b}{2} + \frac{x}{\beta} \right)} \right] \quad (86)$$

The potential in region VI is the potential in region III plus the difference between the potential in region I and the potential in region II; thus, from equations (62a), (62b), and (79), the potential in region VI is found to be given by

$$\phi(x, 0^-, z) = \frac{2pz}{\pi\beta} \left[-x \cos^{-1} \frac{\beta z}{x} - x\sqrt{2} \sin^{-1} \frac{\beta z}{\sqrt{2x^2 - \beta^2 z^2}} + \beta z \tan^{-1} \frac{x}{\sqrt{x^2 - \beta^2 z^2}} + xz \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} - \frac{\beta \left(\frac{2x}{\beta} - z - \frac{b}{2} \right) \sqrt{\left(\frac{b}{2} - z \right) \left(z + \frac{x}{\beta} - \frac{b}{2} \right)}}{3} \right] \quad (87)$$

From equation (87), the pressure-difference coefficient in region VI is found to be given by

$$\Delta C_p = -\frac{8p}{\pi\beta V} \left[z \cos^{-1} \frac{\beta z}{x} + z\sqrt{2} \sin^{-1} \frac{\beta z}{\sqrt{2x^2 - \beta^2 z^2}} - z \sin^{-1} \sqrt{\frac{\beta \left(\frac{b}{2} - z \right)}{x}} + \sqrt{\left(\frac{b}{2} - z \right) \left(z - \frac{b}{2} + \frac{x}{\beta} \right)} \right] \quad (88)$$

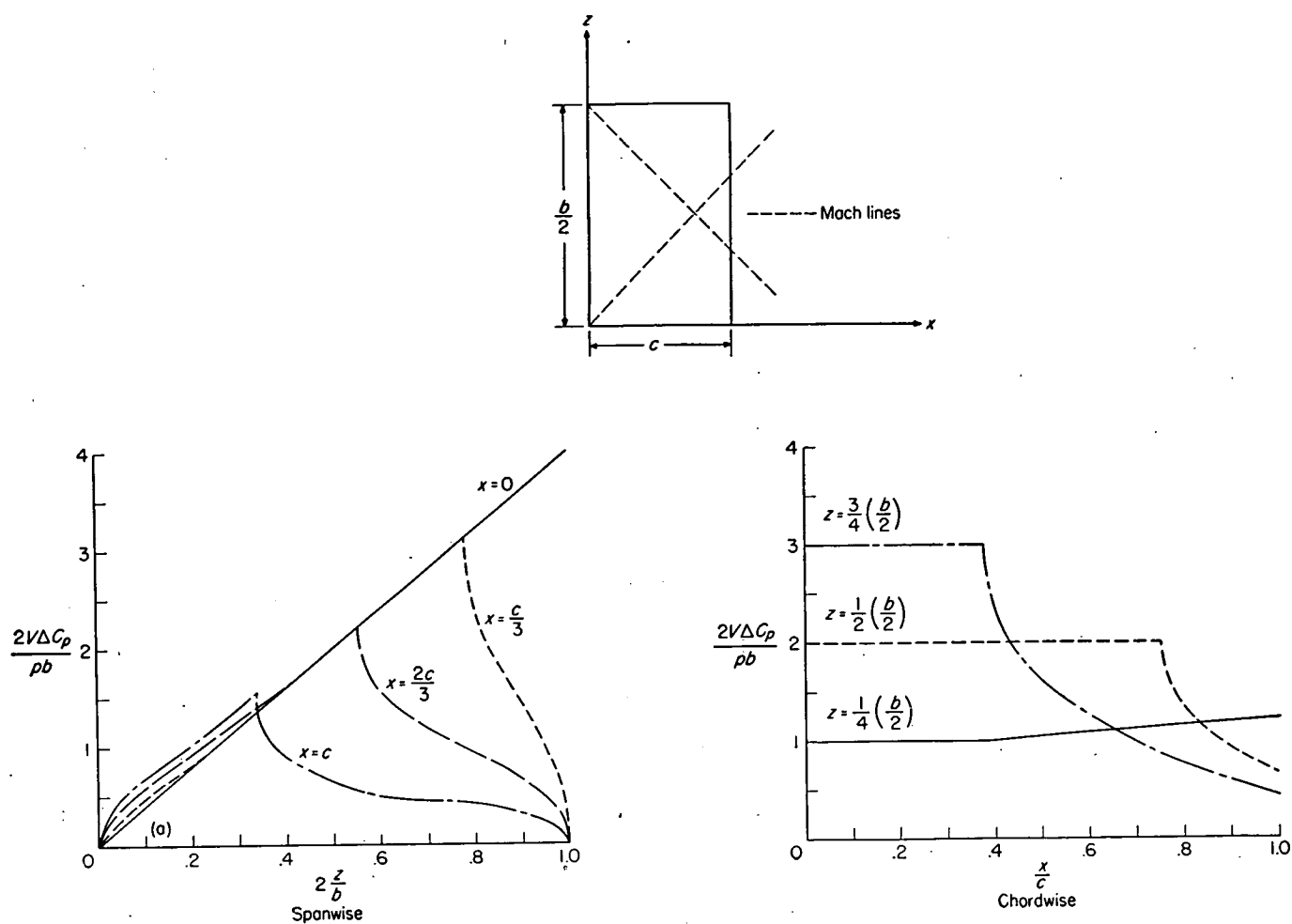
DISCUSSION OF RESULTS FOR ROLLING TAILS

Illustrative plots of the chordwise and spanwise pressure distributions across one fin for tails with various numbers of fins are shown in figure 38. Figure 39 shows illustrative plots of the spanwise loadings on one fin for tails made up of various numbers of fins.

The potential function ϕ_0 used in finding the pressures and potentials for the tails consisting of four, six, and eight fins could be used in finding pressures and potentials for tails consisting of any even number of fins provided that the region of interaction between adjacent fins does not affect the tip. The restriction on the region of interaction causes the range of validity to decrease as the number of fins is increased. The range of validity could be extended, however, by use of a pressure or potential cancellation method such as given in references 14 and 15.

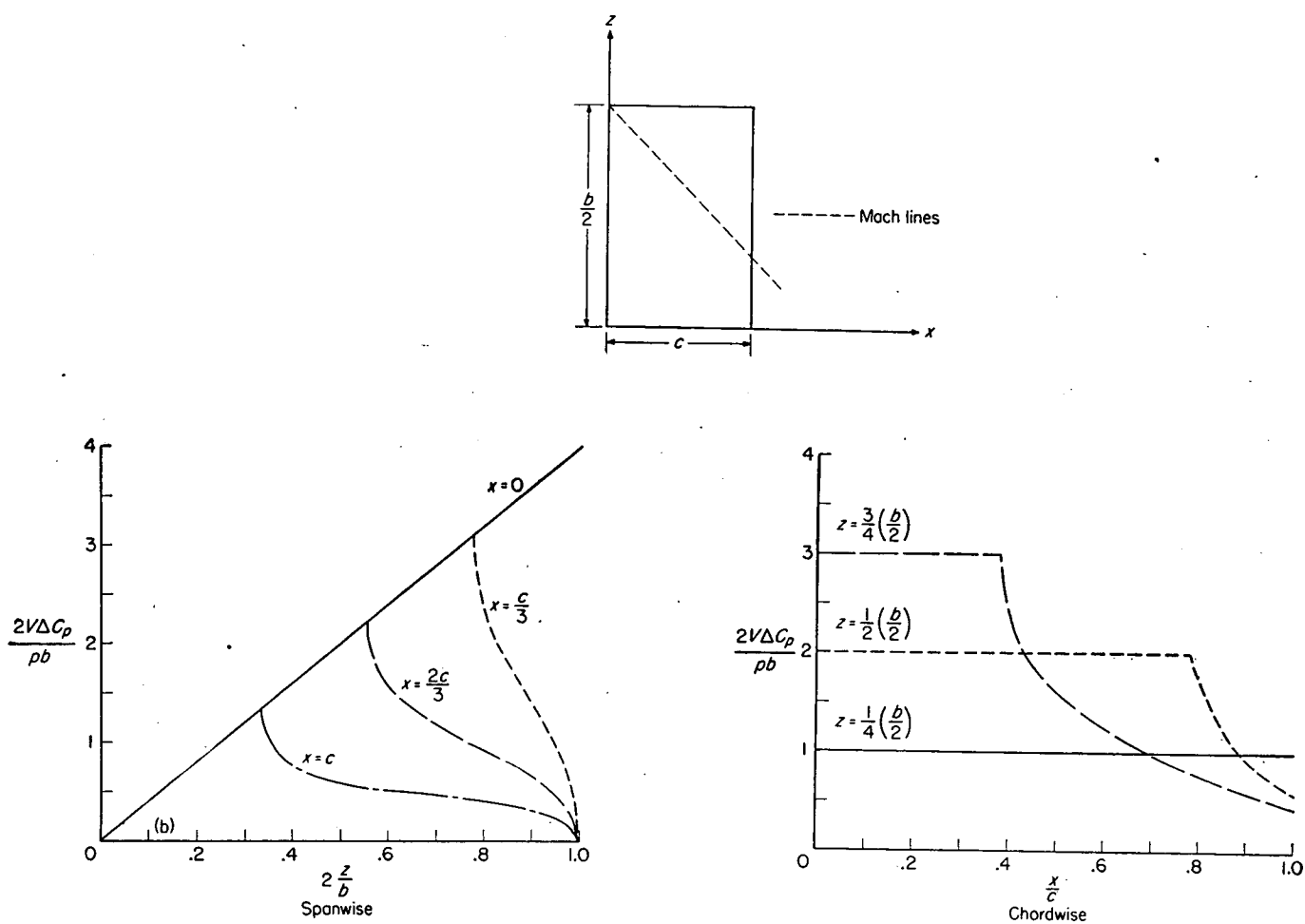
From the potential, the damping in roll per fin was calculated. Table I presents the results of these calculations. Figure 40 presents the variation of the damping in roll per fin with $A\beta$ for tails made up of various numbers of fins. For a given Mach number (β constant), figure 40 shows the variation of the damping with aspect ratio. Figure 41 presents the variation of the damping in roll per fin with Mach number for tails consisting of various numbers of fins with a fin aspect ratio of 1.5.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., October 25, 1951.

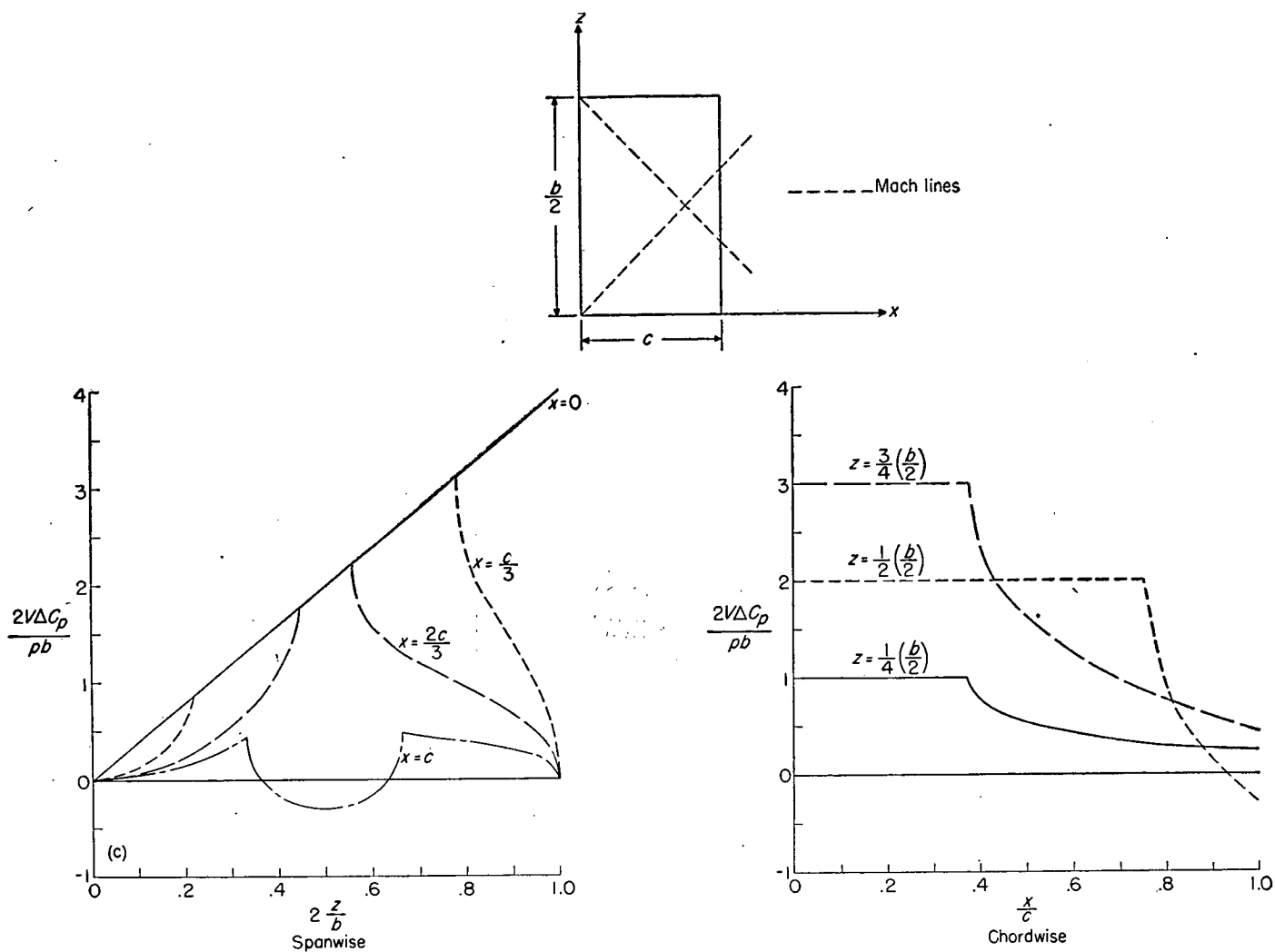


(a) Tail of one fin.

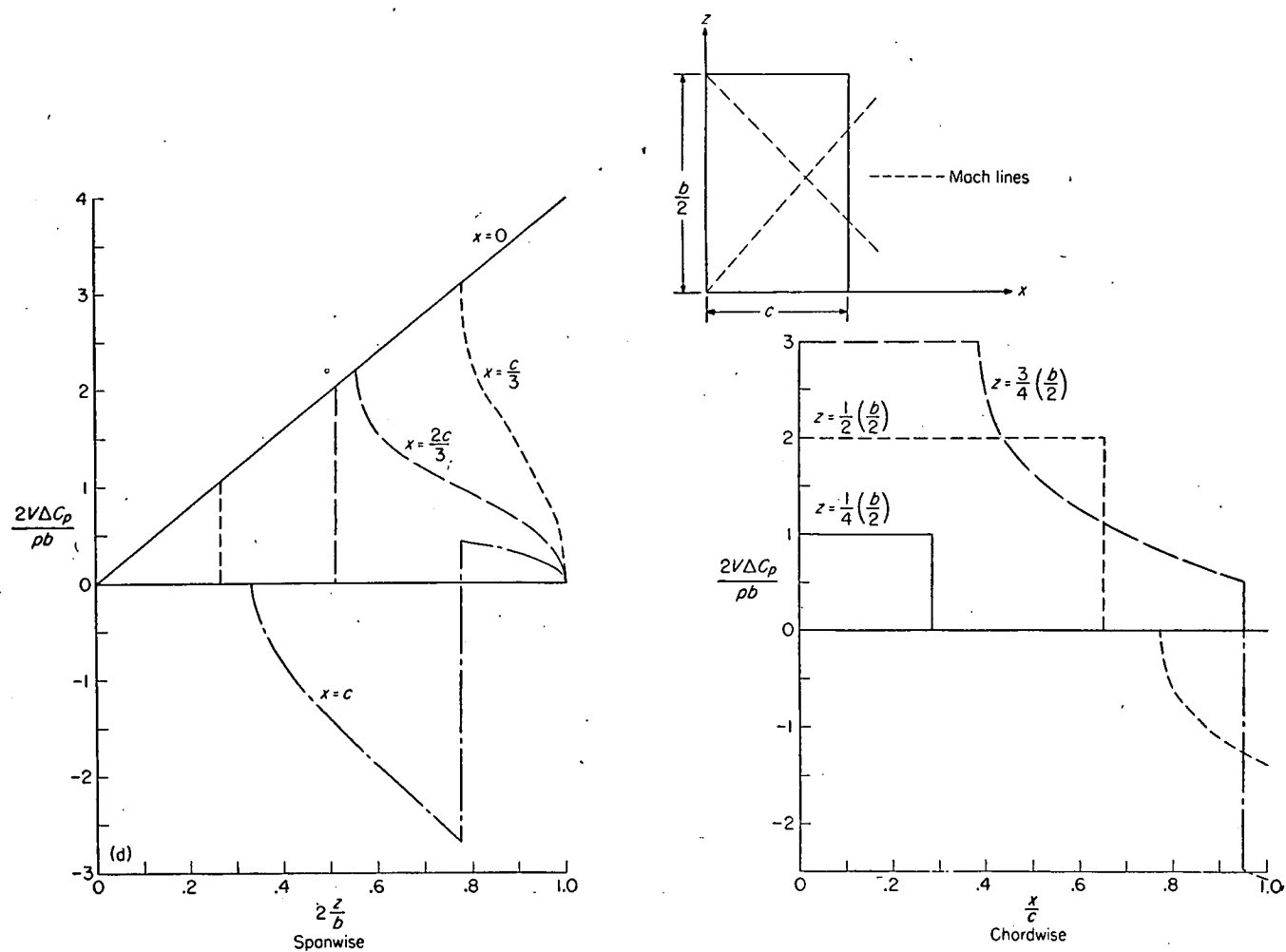
FIGURE 38.—Chordwise and spanwise pressure distributions on a fin of aspect ratio 1.5 at a Mach number of $\sqrt{2}$ for tails consisting of rectangular fins.



(b) Tail of two fins.
FIGURE 38.—Continued.

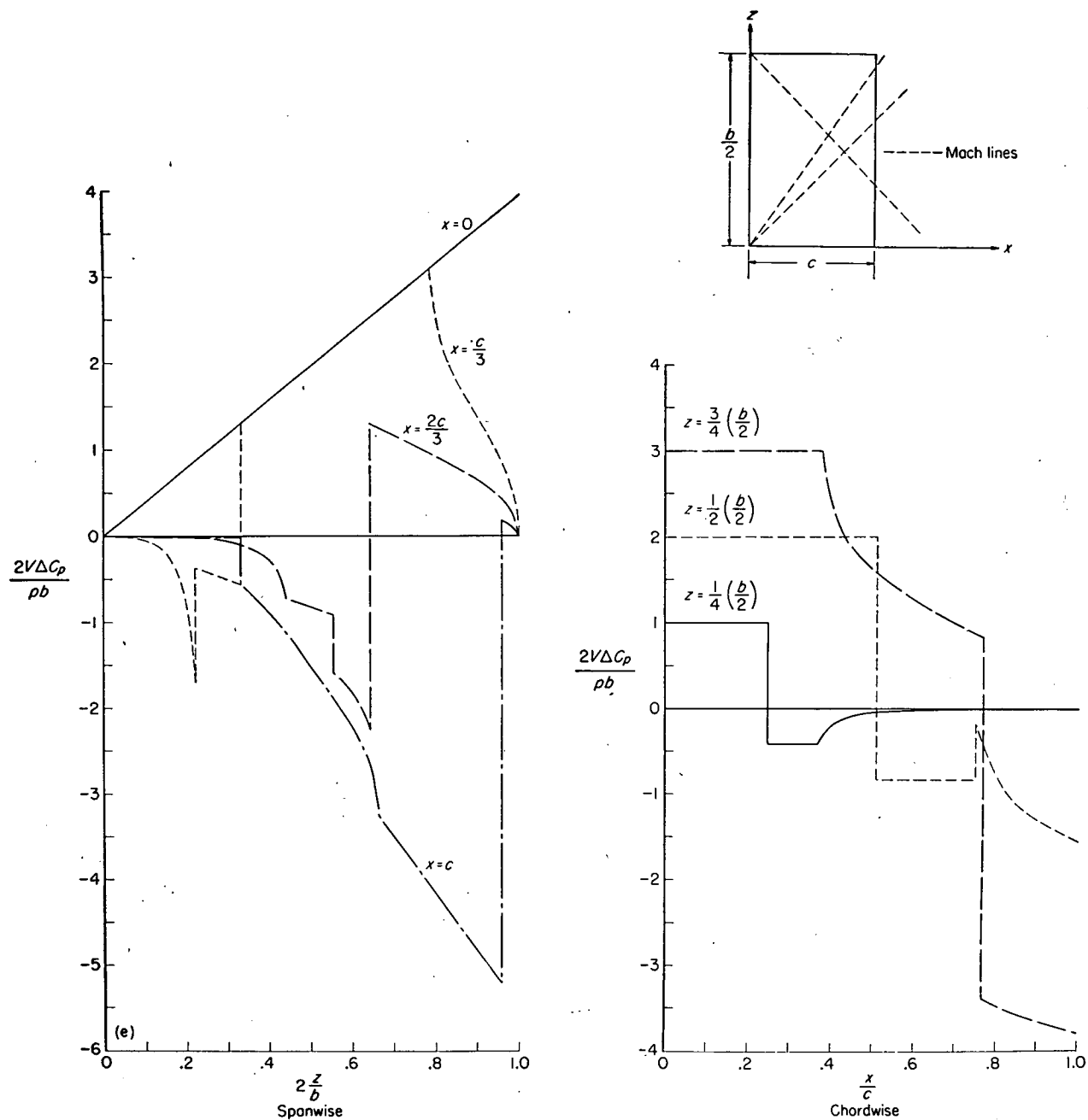


(c) Tail of four fins.
FIGURE 38.—Continued.



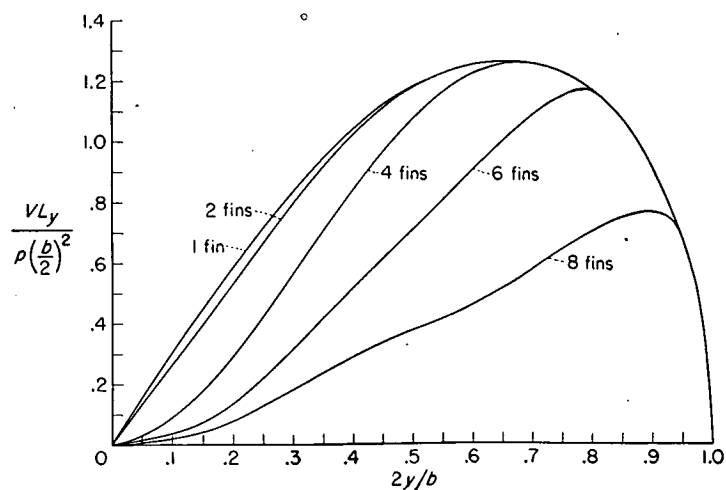
(d) Tail of six fins.

FIGURE 38.—Continued.



(e) Tail of eight fins.

FIGURE 38.—Concluded.

FIGURE 39.—Spanwise loading on a fin of aspect ratio 1.5 at a Mach number of $\sqrt{2}$ for tails consisting of rectangular fins.

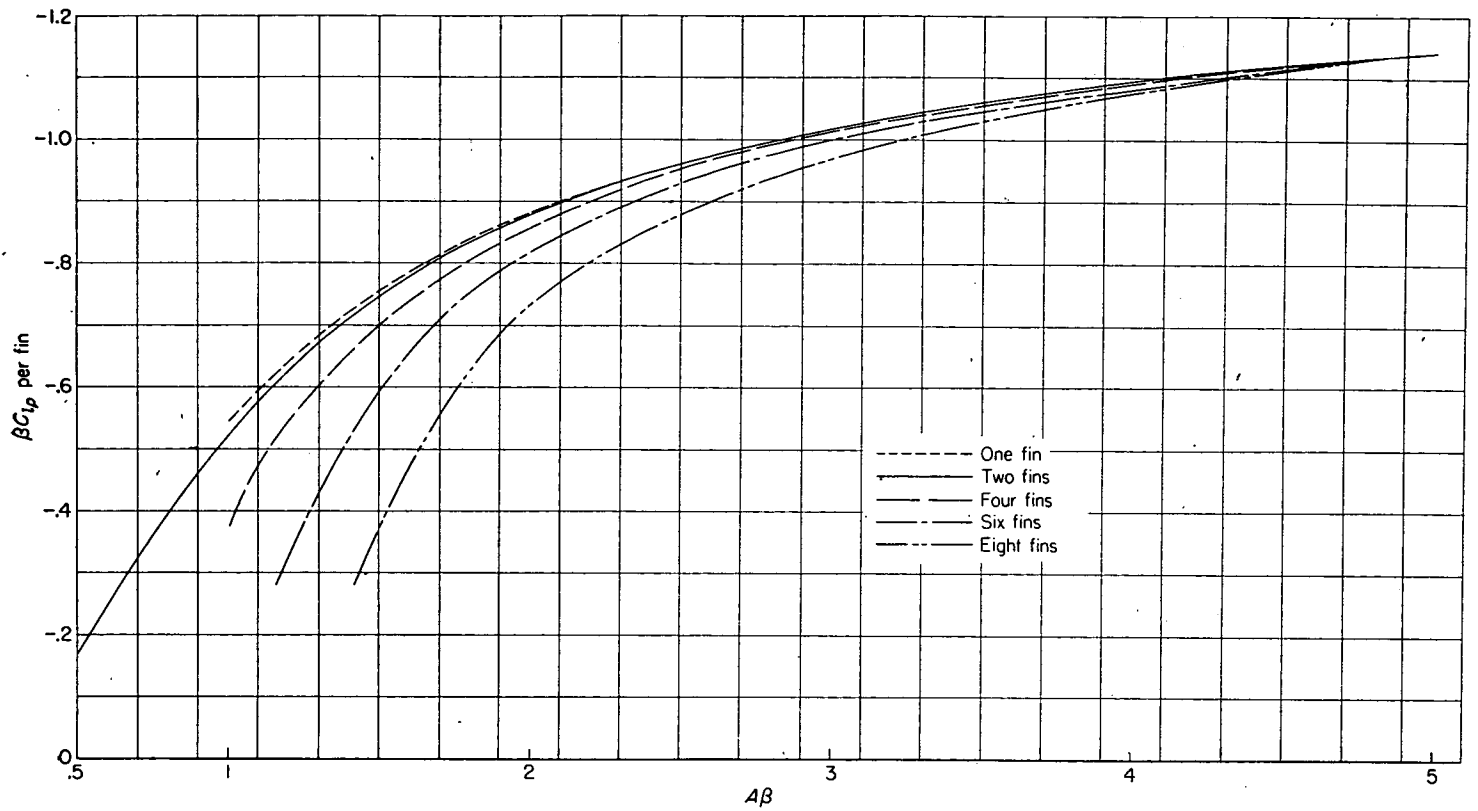


FIGURE 40.—The damping in roll per fin for tails consisting of rectangular fins.

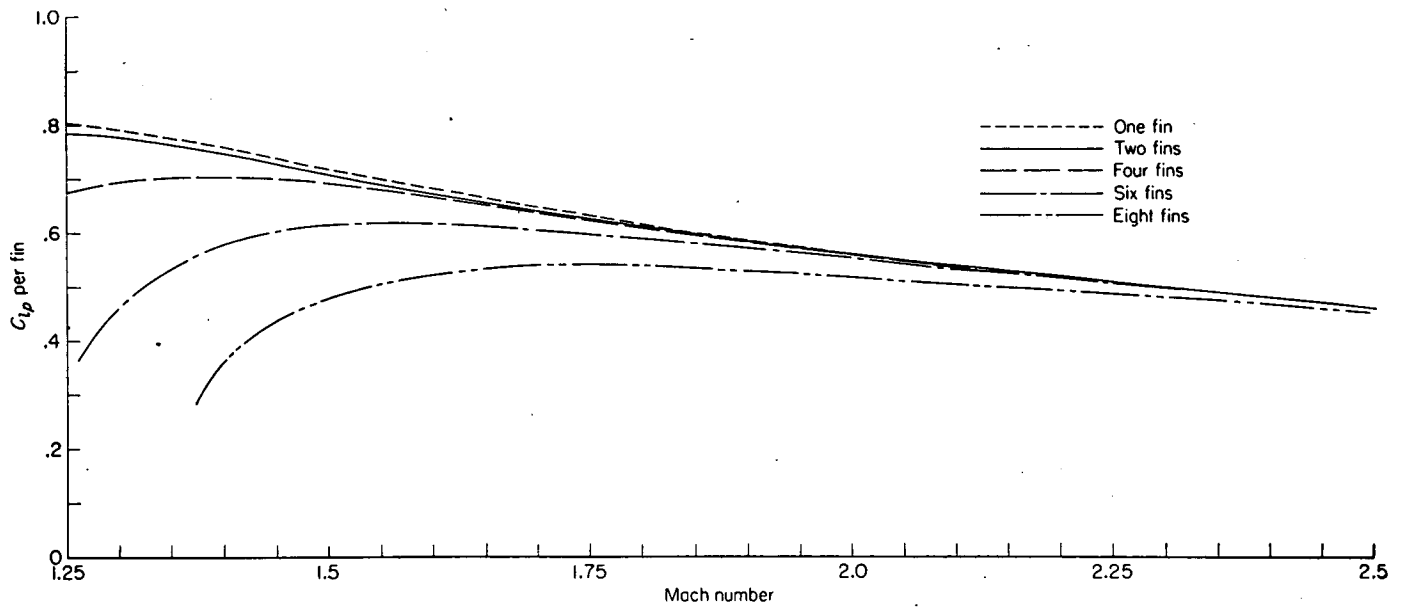
FIGURE 41.—An illustrative variation of C_{l_p} per fin with Mach number for tails consisting of rectangular fins with a fin aspect ratio of 1.5.

TABLE I
DAMPING-IN-ROLL COEFFICIENT PER FIN

Number of fins	βC_{i_p} per fin	Valid for
1	$\frac{1 + 4A\beta - 24A^2\beta^2 + 32A^3\beta^3}{24A^3\beta^3}$	$A\beta \geq 1$
2	$\frac{1 + 8A\beta - 48A^2\beta^2 + 64A^3\beta^3}{48A^3\beta^3}$	$A\beta \geq \frac{1}{2}$
4	$-\frac{4}{\pi A^3\beta^3} \left[-\frac{1}{9} + \frac{\pi}{192} (1 + 8A\beta - 48A^2\beta^2 + 64A^3\beta^3) \right]$	$A\beta \geq 1$
6	$-\frac{4}{A^3\beta^3} \left[-\frac{2}{9\sqrt{3}} + \frac{1}{192} (1 + 8A\beta - 48A^2\beta^2 + 64A^3\beta^3) \right]$	$A\beta \geq \frac{2}{\sqrt{3}}$
8	$-\frac{4}{A^3\beta^3} \left[-\frac{4}{9\pi} - \frac{1}{6} + \frac{1}{192} (1 + 8A\beta - 48A^2\beta^2 + 64A^3\beta^3) \right]$	$A\beta \geq \sqrt{2}$

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REPORT 1144

AERODYNAMIC CHARACTERISTICS OF A REFINED DEEP-STEP PLANING-TAIL FLYING-BOAT HULL WITH VARIOUS FOREBODY AND AFTERBODY SHAPES¹

By JOHN M. RIEBE and RODGER L. NAESETH

SUMMARY

An investigation was made in the Langley 300 MPH 7- by 10-foot tunnel to determine the aerodynamic characteristics of a refined deep-step planing-tail hull with various forebody and afterbody shapes. For comparison, tests were made on a streamline body simulating the fuselage of a modern transport airplane.

The results of the tests, which include the interference effects of a 21-percent-thick support wing, indicated that for corresponding configurations the hull models incorporating a forebody with a length-beam ratio of 7 had lower minimum drag coefficients than the hull models incorporating a forebody with a length-beam ratio of 5. The lowest minimum drag coefficients, 0.0024 and 0.0023, which were considerably less than that of a comparable conventional hull of length-beam ratio 9, were obtained on the length-beam-ratio-7 forebody, alone and with round center boom, respectively. The streamline body had a minimum drag coefficient of 0.0025; flying-boat hulls can, therefore, have drag values comparable to landplane fuselages. The hull angle of attack for minimum drag varied from 2° to 4°.

Longitudinal and lateral stability was generally about the same for all hull models tested and about the same as that of a conventional hull.

INTRODUCTION

Because of the requirements for increased range and speed in flying boats, an investigation of the aerodynamic characteristics of flying-boat hulls as affected by hull dimensions and hull shape is being conducted at the Langley Aeronautical Laboratory. The results of one phase of this investigation, presented in reference 1, have indicated that hull drag can be reduced without causing large changes in aerodynamic stability and hydrodynamic performance by the use of high length-beam ratios. Another phase of the investigation, reference 2, indicated that hulls of the deep-step planing-tail type have much lower air drag than the conventional type of hull and about the same aerodynamic stability; tank tests, reference 3, have indicated that this type of hull also has hydrodynamic performance equal to and in some respects superior to the conventional type of hull.

In an attempt to improve the aerodynamic performance of hulls still further without causing excessive penalties in hydrodynamic performance, several refined deep-step planing-tail hulls were designed jointly by the Hydrodynamics

Division and the Stability Research Division of the Langley Laboratory. It was believed that improved aerodynamic performance could be facilitated mainly by refinement of the forebody plan form and by a reduction in the volume and surface area of the afterbody. This report presents the results of the tests of these hulls.

In order to make a preliminary study of overall flying-boat configurations, tests were also made on models incorporating a typical engine nacelle and an engine nacelle extended into a boom which is to function as the afterbody and reduce the size of and possibly eliminate wing-tip floats; the nacelle and nacelle boom were also tested without the hull models. For comparing the drag and stability, tests were made on a streamline body simulating the fuselage of a modern transport airplane.

Tank tests (ref. 4) have indicated that the hull models presented in the present report (with the possible exception of the forebody alone for which data are not available) will have acceptable hydrodynamic performance.

COEFFICIENTS AND SYMBOLS

The results of the tests are presented as standard NACA coefficients of forces and moments. Rolling-, yawing-, and pitching-moment coefficients are given about the locations (wing 30-percent-chord point) shown in figures 1, 2, and 3. The wing area, mean aerodynamic chord, and span used in determining the coefficients and Reynolds numbers are those of a hypothetical flying boat (ref. 1). The hull, fuselage, and nacelle coefficients were derived by subtraction of data for the wing alone from data for the wing plus hull, fuselage, or nacelle. The wing-alone data were determined by including in the tests that part of the wing which is enclosed in the hull, fuselage, or nacelle. The hull, fuselage, and nacelle coefficients therefore include the wing interference resulting from the interaction of the velocity fields of the wing and the bodies and also the negative wing interference caused by shielding from the airstream that part of the wing enclosed within the hull, fuselage, or nacelle. The data are referred to the stability axes, which are a system of axes having their origin at the center of moments shown in figures 1, 2, and 3 and in which the Z-axis is in the plane of symmetry and perpendicular to the relative wind, the X-axis is in the plane of symmetry and perpendicular to the Z-axis, and the Y-axis

¹ Supersedes NACA TN 2489, "Aerodynamic Characteristics of a Refined Deep-Step Planing-Tail Flying-Boat Hull With Various Forebody and Afterbody Shapes" by John M. Riebe and Rodger L. Naeseth, 1952.

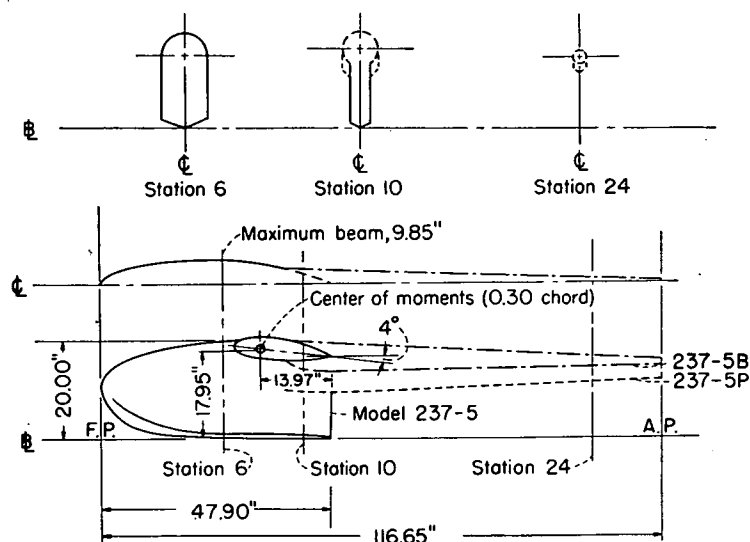


FIGURE 1.—Lines of Langley tank models 237-5, 237-5B, and 237-5P.

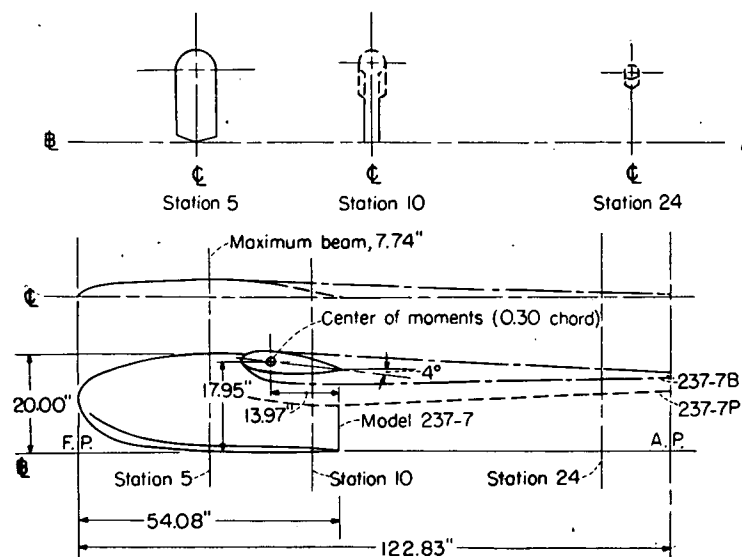
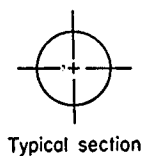


FIGURE 2.—Lines of Langley tank models 237-7, 237-7B, and 237-7P.



Typical section

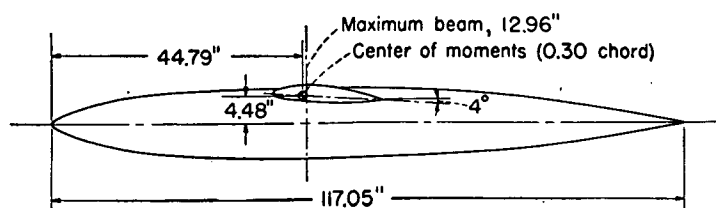


FIGURE 3.—Lines of the streamline fuselage.

is perpendicular to the plane of symmetry. The positive directions of forces and moments about the stability axes are shown in figure 4.

The coefficients and symbols are defined as follows:

C_L	lift coefficient, $Lift/qS$ where $Lift = -Z$
C_D	drag coefficient, D/qS
C_Y	lateral-force coefficient, Y/qS
C_l	rolling-moment coefficient, L/qSb
C_m	pitching-moment coefficient, $M/qS\bar{c}$
C_n	yawing-moment coefficient, N/qSb
D	drag, $-X$ when $\beta = 0$
X	force along X -axis, lb
Y	force along Y -axis, lb
Z	force along Z -axis, lb
L	rolling moment, ft-lb
M	pitching moment, ft-lb
N	yawing moment, ft-lb
q	free-stream dynamic pressure, $\rho V^2/2$, lb/sq ft
S	wing area of $\frac{1}{10}$ -scale model of hypothetical flying boat, 18.264 sq ft
$\bar{c}$	wing mean aerodynamic chord of $\frac{1}{10}$ -scale model of hypothetical flying boat, 1.377 ft
b	wing span of $\frac{1}{10}$ -scale model of hypothetical flying boat, 13.971 ft
V	air velocity, fps
ρ	mass density of air, slugs/cu ft
α	angle of attack of hull base line, deg
β	angle of sideslip, deg
R	Reynolds number, based on wing mean aerodynamic chord of $\frac{1}{10}$ -scale model of hypothetical flying boat

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

MODELS AND APPARATUS

The hull lines were determined through the joint cooperation of the Hydrodynamics Division and the Stability Research Division of the Langley Laboratory. The hull forebodies were derived in plan form from modified NACA 16-series symmetrical airfoil sections of thickness ratios 20 and 14.3 percent airfoil chord, resulting in forebody length-beam ratios of approximately 5 and 7, respectively. The forebody length-beam ratio is equal to the distance from the forward perpendicular (F. P.) to the step divided by the maximum beam of the forebody (figs. 1 and 2 show maximum beam of forebody). Dimensions of the hulls are given in figures 1 and 2 and tables I to IV. The lines of a tail float used for several of the tests are given in figure 5; offsets are given in table V. The streamline body, fineness ratio of

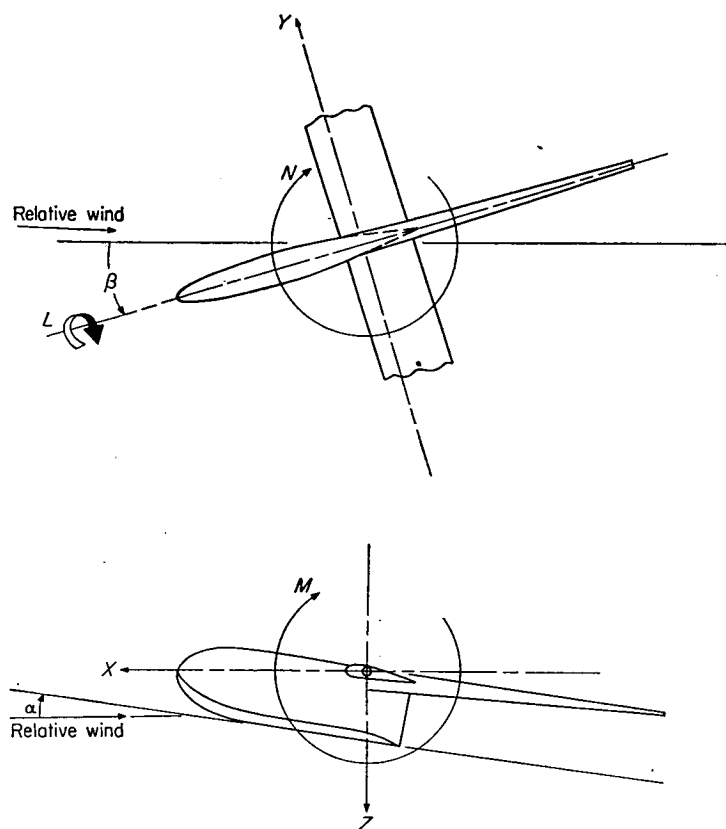


FIGURE 4.—System of stability axes. Positive values of forces, moments, and angles are indicated by arrows.

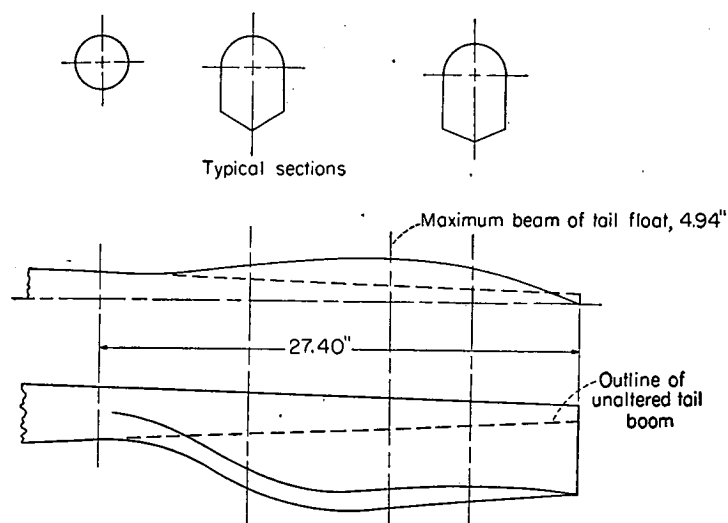


FIGURE 5.—Lines of tail float incorporated on hulls 237-5F1 and 237-7F1.

about 9, represents the fuselage of a typical high-speed land-plane; dimensions are given in figure 3 and table VI. The engine nacelle (fig. 6) was a scale model of the engine nacelle of the XPBB-1 flying boat (ref. 1). The manner in which the engine-nacelle boom was derived is also shown in figure 6. Photographs of the hulls with the corresponding Langley

tank designation numbers are given in figure 7. All models and interchangeable parts were constructed of laminated mahogany and finished with pigmented varnish. The volumes, surface areas, maximum cross-sectional areas, and side areas for the hulls and fuselage are given in table VII.

The hull was attached to a wing which was mounted horizontally in the tunnel as shown in figure 8. The wing was the one used in the investigations of reference 1. It was set at an incidence of 4° with respect to the base line on all models and had a 20-inch chord, a 94.2-inch span, and an NACA 4321 airfoil section.

TESTS

TEST CONDITIONS

The tests were made in the Langley 300 MPH 7-by 10-foot tunnel at dynamic pressures of approximately 25, 100, and 170 pounds per square foot, corresponding to airspeeds of 100, 201, and 274 miles per hour. Reynolds numbers for these airspeeds, based on the mean aerodynamic chord of the hypothetical flying boat, were approximately 1.30×10^6 , 2.50×10^6 , and 3.10×10^6 , respectively. Corresponding Mach numbers were 0.13, 0.26, and 0.35.

CORRECTIONS

Blocking corrections have been applied to the wing and wing-plus-hull data. The drag coefficients of the hulls and fuselage have been corrected for longitudinal buoyancy effects caused by a tunnel static-pressure gradient. Angles of attack have been corrected for structural deflections caused by aerodynamic forces.

TEST PROCEDURE

The aerodynamic characteristics of the hulls with interference of the support wing were determined by testing the wing alone and the wing-and-hull combinations under identical conditions. The hull aerodynamic coefficients

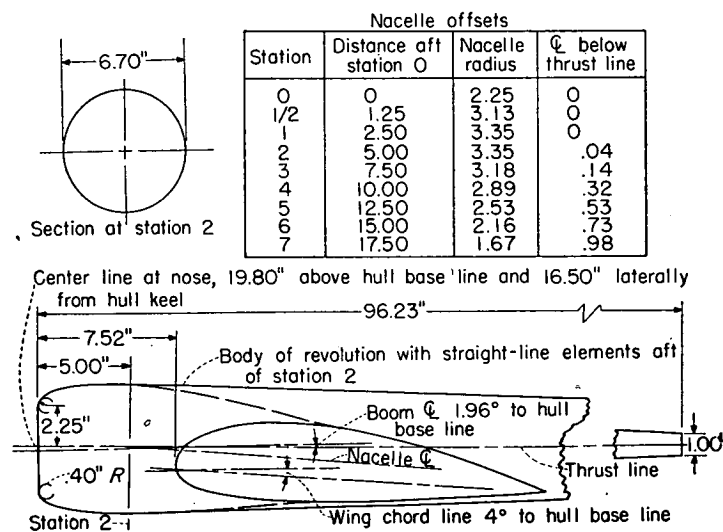


FIGURE 6.—Lines of engine nacelle and engine-nacelle boom.

were determined by subtraction of wing-alone coefficients from wing and hull coefficients after the data were plotted in order to account for structural deflections.

Tests were made at three Reynolds numbers. Because of structural limitations of the support wing, it was necessary to limit the data at the higher Reynolds numbers to the angle-of-attack range shown.

In order to minimize possible errors resulting from transition shift on the wing, the wing transition was fixed at the leading edge by means of roughness strips of carborundum particles of approximately 0.008-inch diameter. The particles were applied for a length of 8 percent airfoil chord measured along the airfoil contour from the leading edge on both upper and lower surfaces.

Hull transition for all tests was fixed by a $\frac{1}{2}$ -inch strip of 0.008-inch-diameter carborundum particles located approximately 5 percent of the hull length aft of the bow. All tests were made with the support setup shown in figure 8.

RESULTS AND DISCUSSION

The aerodynamic characteristics in pitch of the refined deep-step planing-tail hulls with various afterbody configurations are presented in figures 9 and 10 and the aerodynamic characteristics in sideslip, in figures 11 and 12. The aerodynamic characteristics of the streamline fuselage are included in figures 9 and 11. The aerodynamic characteristics in pitch of models incorporating the engine nacelle and the engine-nacelle boom are presented in figures 13 and 14 and the aerodynamic characteristics in sideslip, in figures 11 and 12. The aerodynamic characteristics of the engine nacelle and the engine-nacelle boom without the hull are included in figure 13 (a); the coefficients are plotted against hull angle of attack and therefore correspond to the increments that result from the nacelle or the nacelle boom when the hull is at a given attitude. Minimum drag coefficients and stability parameters, as determined from the figures, are presented in table VIII for comparison.

The following discussion of the longitudinal characteristics is based on the results for Reynolds number 2.5×10^6 . A comparison of figures 9 and 10 indicates that for corresponding configurations the hull models incorporating a forebody with a length-beam ratio of 7 had lower minimum drag coefficients than the hull models incorporating a forebody with a length-beam ratio of 5. The incremental difference in minimum drag coefficient between corresponding configurations varied from 0.0008 for the hull forebodies alone ($C_{D_{min}} = 0.0032$ for model 237-5 and 0.0024 for model 237-7) to 0.0003 for the deep-center-boom configuration ($C_{D_{min}} = 0.0030$ for model 237-5P and 0.0027 for model 237-7P).

According to reference 5, the difference in minimum profile-drag coefficients between airfoil sections of thickness ratios 0.20 and 0.143 is about 20 percent; the difference in

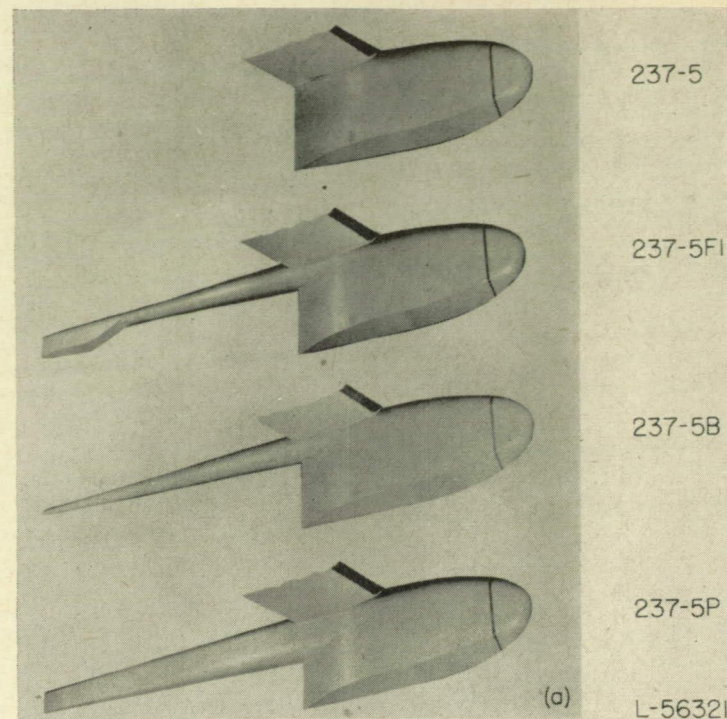


FIGURE 7.—Hull models tested in the Langley 300 MPH 7-by 10-foot tunnel.

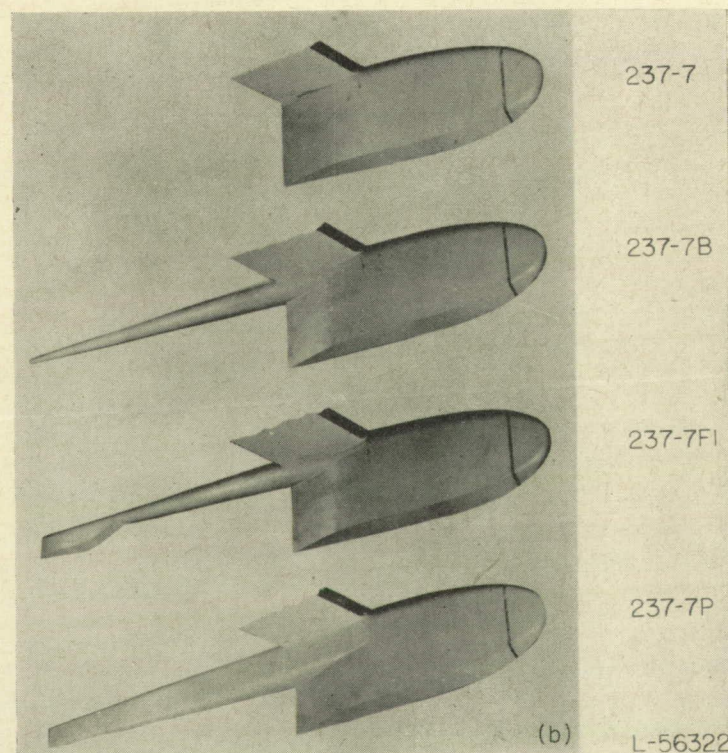


FIGURE 7.—Continued.

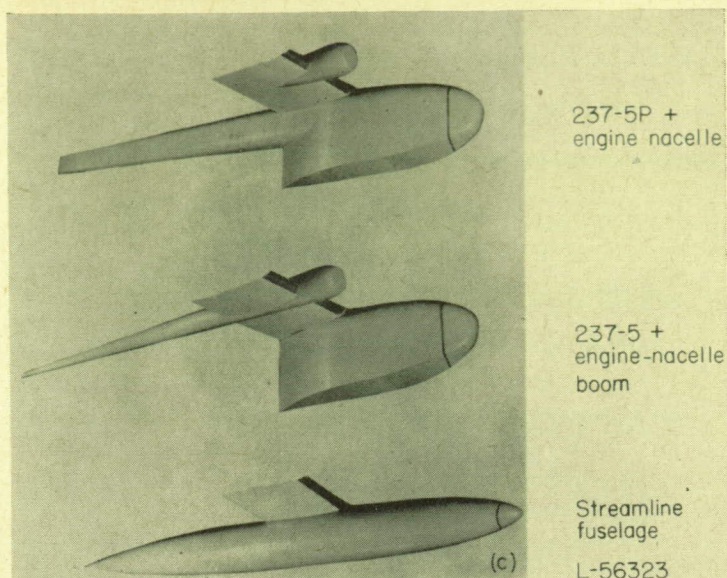


FIGURE 7.—Concluded.

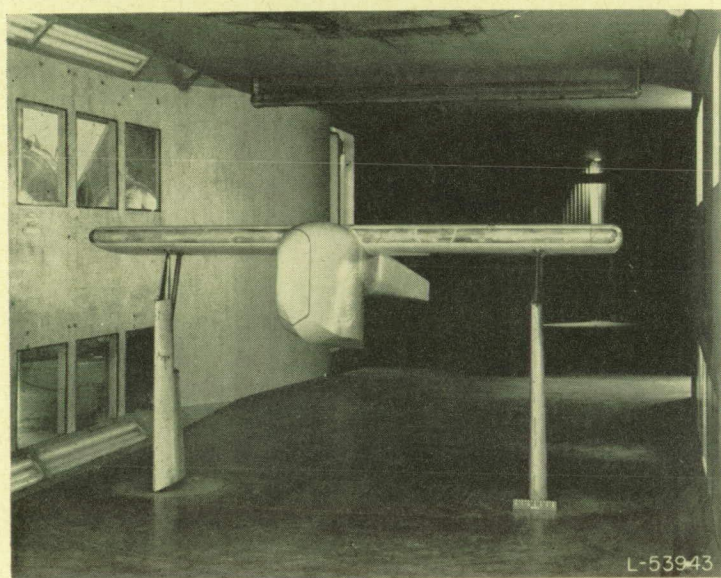


FIGURE 8.—Langley tank model 237-5P mounted in the Langley 300 MPH 7-by 10-foot tunnel.

minimum drag coefficients between hull models 237-7 and 237-5 which were derived from airfoils of these same corresponding thickness ratios agreed favorably with this value.

At negative angles of attack the drag coefficients for hulls with forebody length-beam ratios of 5 were much larger than those for hulls with length-beam ratios of 7 (figs. 9 and 10). The steep drag rise at negative angles can be explained by an examination of the tuft studies of hull models 237-5B, 237-5, 237-7B, and 237-7 presented in figures 15, 16, 17, and 18, respectively. For the length-beam-ratio-5 forebody

alone (fig. 16) a large amount of separation occurred on the upper rear of the forebody and rear of the wing. Fairing the juncture with the boom (fig. 15) reduced the separation somewhat and consequently the hull drag coefficient. Little or no separation occurred for the length-beam-ratio-7 forebody configurations throughout the angle-of-attack range tested (figs. 17 and 18). Unpublished tests of the hulls alone have indicated that the separation was caused primarily by the interference effect of the support wing; tuft studies of the hulls alone at angles of attack corresponding to those of the present report showed no occurrence of separation.

The lowest minimum drag coefficients, 0.0024 and 0.0023, were obtained on hull models 237-7 and 237-7B, respectively. Although the skin area of model 237-7B was larger than that of model 237-7 (table VII) because of the addition of the boom, the drag increase corresponding to the added skin friction was probably offset by the boom's causing a better flow condition at the wing-hull juncture.

As indicated by figures 9 and 10, the hull angle of attack for minimum drag varied from 2° to 4° .

A comparison of the lowest minimum drag coefficient, 0.0023 for hull 237-7B, with that of a conventional hull, 0.0066 for hull model 203 of length-beam ratio 9 (ref. 1), indicated a minimum-drag-coefficient reduction of 0.0043 or 65 percent.

The minimum drag coefficient for the streamline body was 0.0025 (fig. 9); flying-boat hulls can, therefore, have drag values comparable to that of a fuselage of a landplane approximately similar in size and gross weight to a hypothetical flying boat incorporating hull model 237-7B. Tank tests (ref. 4) have shown that a flying boat incorporating hull 237-7B and a gross weight similar to a landplane incorporating the streamline fuselage will take off from and land on water if a small vertical chine strip is added to the hull. There are several disadvantages to this type of hull, however. The hull volume is less than the fuselage volume (table VII) and, because of the location of the major portion of hull volume ahead of the wing where the pay load would be carried, a balance problem would probably be encountered on large flying-boat designs. These disadvantages are much less serious on model 237-7P because of the deep tail boom; the increase in minimum drag coefficient, 0.0004, may be worth the alleviation of the volume and balance problem.

Hydrodynamic considerations have indicated that improved hydrodynamic performance on the deep-step hulls might be facilitated by incorporating a tail float on the hulls such as shown in figure 5. If tank tests indicate that a tail float is much desired, a more refined float than that shown in figure 5 should be used. The minimum drag coefficients of the hull models with tail float, models 237-5F1 and 237-7F1, were 0.0043 and 0.0038, respectively. These

drag-coefficient values were about 0.0015 larger than similar configurations without the tail float.

Figures 9 and 10 show negative values of hull lift coefficient throughout most of the angle-of-attack range tested; the values are especially more negative than those of conventional hulls (ref. 1) in the minimum drag range. In order to compensate for these negative values, the wing lift coefficient of flying boats would have to be increased; this increase would result in an increase in induced-drag coefficient. However, the increase in induced drag for the wing of the hypothetical flying boat, used as a basis in the present investigation, would be small and would not seriously alter the relative merits in performance of the hulls of the present investigation over conventional hulls.

In order to make a preliminary study of overall flying-boat configurations, tests were also made on a typical engine nacelle and an engine nacelle extended into a boom (fig. 6) which is to function as the afterbody and reduce the size of, or possibly eliminate, wing-tip floats. The drag coefficients for one engine nacelle and one engine-nacelle boom near the angle of attack for minimum drag of the hulls without nacelles were about equal, with a value of 0.0022 (fig. 13 (a)). This drag coefficient agreed favorably with the increment of drag coefficient resulting from the addition of the engine nacelle or the engine-nacelle boom to the hull models as determined by a comparison of figures 13 and 14 with figures 9 and 10. The drag coefficient for the nacelle alone and nacelle boom alone decreased as the hull angle of attack became less positive. A more rapid decrease occurred for the nacelle alone; this effect probably accounts for the negative shift in angle of attack for minimum drag of the forebody alone plus the engine nacelle.

The minimum drag coefficients for both combinations were about equal so that a flying-boat configuration with twin engine-nacelle booms probably has an advantage in aerodynamic performance over a flying boat with a single round boom and conventional nacelles resulting from the reduction in size of, or possible elimination of, wing-tip floats. As noted previously, the length-beam-ratio-5 forebody alone had a greater drag than the forebody with a round center boom, mainly because of an adverse wing interference effect. However, the configuration with nacelle booms still might be better aerodynamically, especially if the wing-hull juncture had a suitable fairing. These results show the need for investigation of overall flying-boat hull configurations if further progress is to be made in improving the aerodynamic performance of flying boats.

The longitudinal stability for the various hulls, as indicated by the parameter C_{m_α} , is given in table VIII. The hull models incorporating a forebody with a length-beam ratio of 7 were generally less unstable longitudinally than those with a length-beam ratio of 5. This increase in longitudinal stability with length-beam ratio is similar to that reported

in reference 1. As expected, because of the large part of the hull ahead of the center of moments, the most longitudinally unstable hull models were forebody-alone configurations 237-5 and 237-7 which had C_{m_α} values of 0.0028 and 0.0026, respectively. The addition of afterbodies had only a small effect on the stability which corresponds to a rearward aerodynamic-center shift of less than 1 percent mean aerodynamic chord on a flying boat. Of the models tested, the choice of hulls probably should be determined mainly from hull drag, hull volume, and balance considerations; the increase in horizontal-tail area necessary to compensate for the hulls with less stability would give only a small drag increase which would be blanketed by the reduction obtained by using the lower drag hulls. These factors should also be considered when comparison is made with the conventional-type hulls of reference 1. The deep-step hulls were slightly less unstable longitudinally for the present wing and center-of-gravity positions, which were located from hydrodynamic considerations.

The directional stability as determined by C_{n_β} (table VIII) was -0.0008 for hull model 237-5 and -0.0009 for model 237-7. As expected, the addition of the afterbodies reduced the directional instability slightly, the amount depending upon the amount of side area added and its location aft of the center of moments. The least directionally unstable configurations tested were models 237-5P and 237-5F1 which both had a C_{n_β} value of -0.0006 . The increase in directional instability with length-beam ratio is also similar to that reported in reference 1 and probably resulted from the increase in side area ahead of the center of moments with length-beam ratio.

The addition of the engine nacelle to models 237-5 and 237-7B increased C_{m_α} slightly but showed no change in C_{n_β} . The directional stability of the flying-boat hulls of the present investigation was generally about the same as that of conventional hulls. This result can largely be explained by the fact that the different center-of-gravity positions compensated for the difference in body shape.

CONCLUSIONS

The results of tests in the Langley 300 MPH 7- by 10-foot tunnel to determine the aerodynamic characteristics of refined deep-step planing-tail flying-boat hulls with various forebody and afterbody shapes and a streamline fuselage indicate the following conclusions:

1. For corresponding configurations the hull models incorporating a forebody with a length-beam ratio of 7 had lower minimum drag coefficients than the hull models incorporating a forebody with a length-beam ratio of 5.
2. The lowest minimum drag coefficients, 0.0024 and 0.0023, which were about 65 percent less than that of a comparable conventional hull of a previous investigation,

were obtained on the length-beam-ratio-7 forebody, alone and with round center boom, respectively.

3. The minimum drag coefficient obtained for the streamline body was 0.0025; flying-boat hulls can, therefore, have drag coefficients comparable to landplane fuselages.

4. The hull angle of attack for minimum drag varied from 2° to about 4° .

5. Longitudinal and lateral stability was generally about the same for all hull models tested and about the same as a conventional hull of a previous aerodynamic investigation.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., June 30, 1948.

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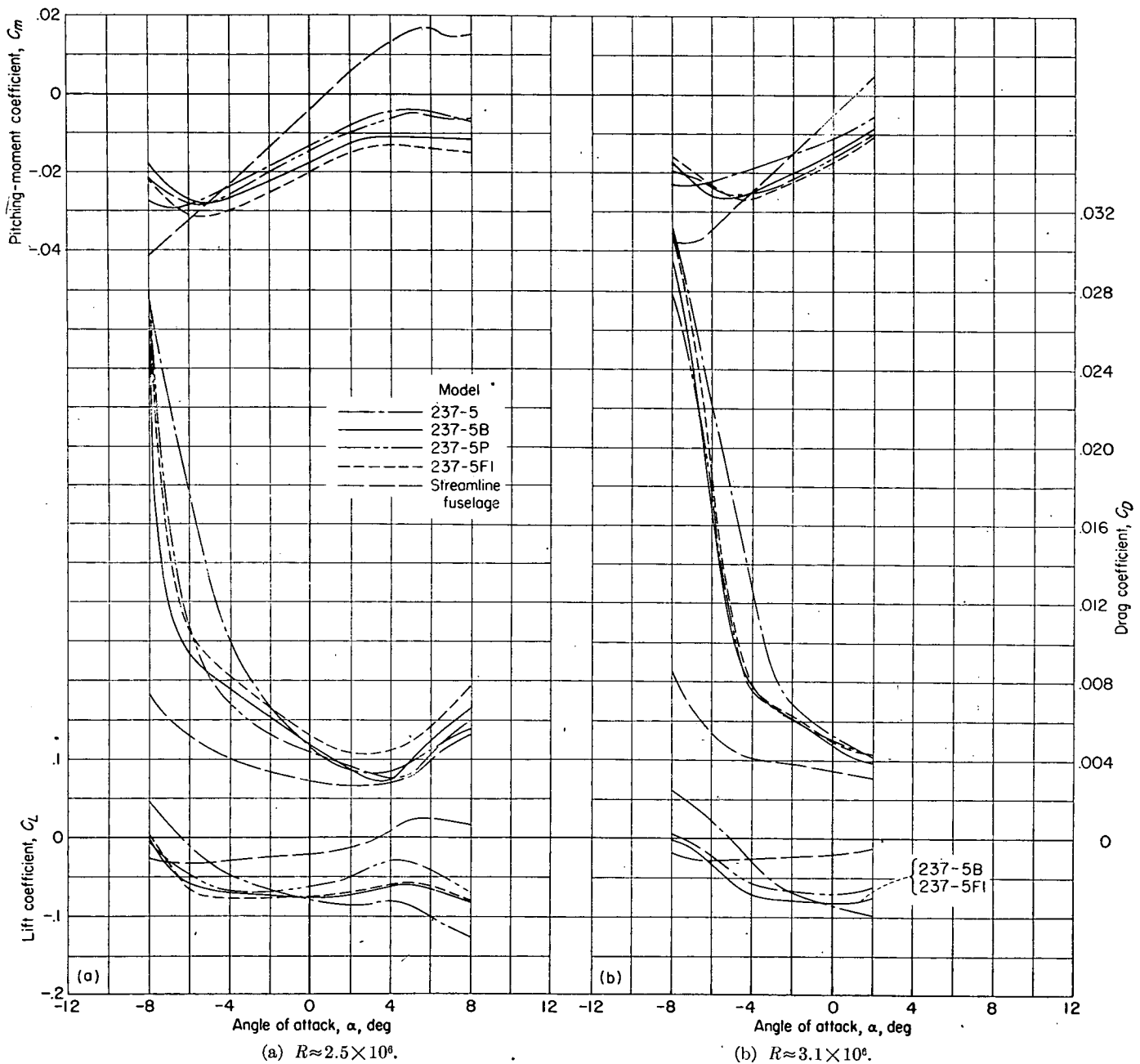


FIGURE 9.—Aerodynamic characteristics in pitch of Langley tank model 237-5 with various afterbody configurations and streamline fuselage.

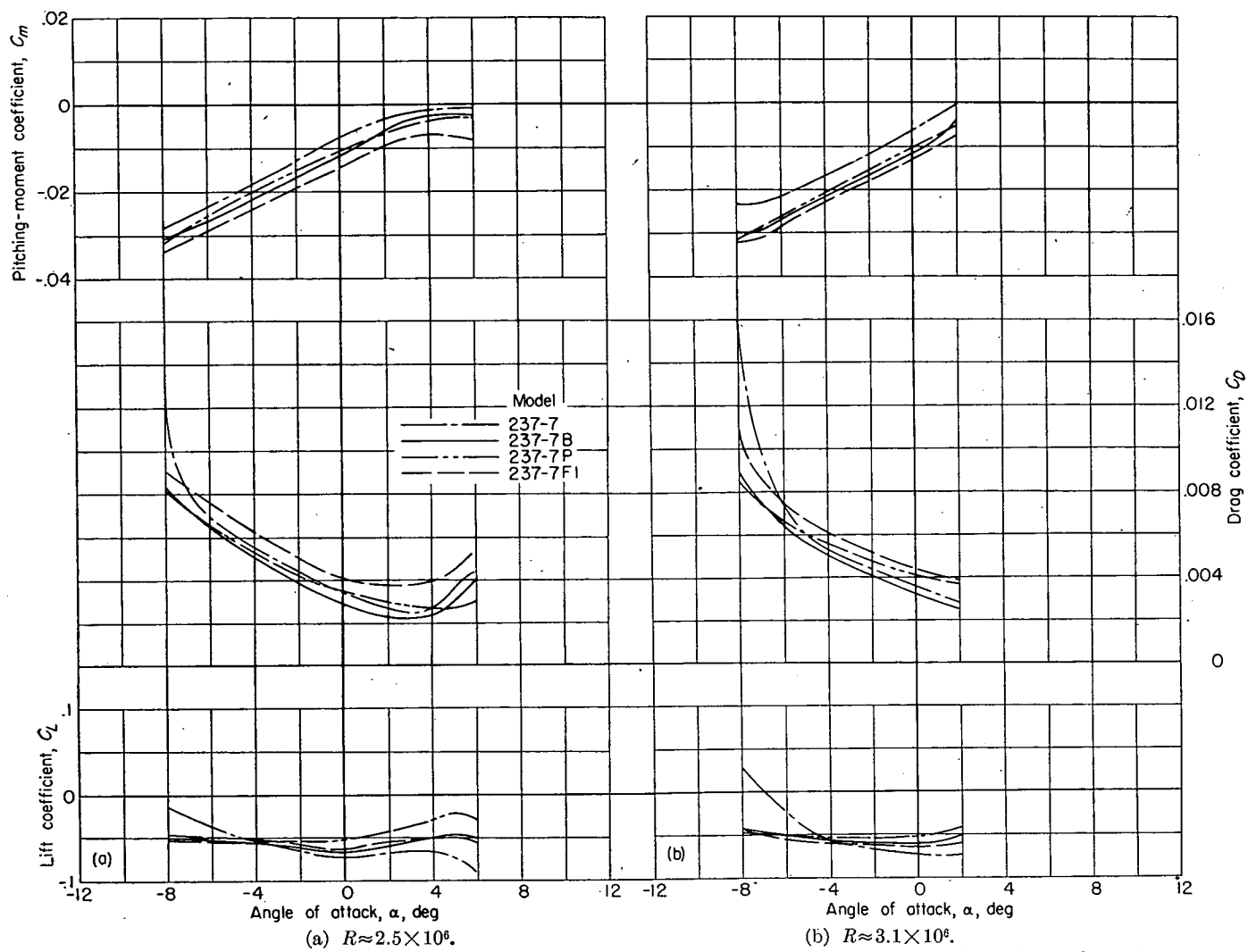


FIGURE 10.—Aerodynamic characteristics in pitch of Langley tank model 237-7 with various afterbody configurations.

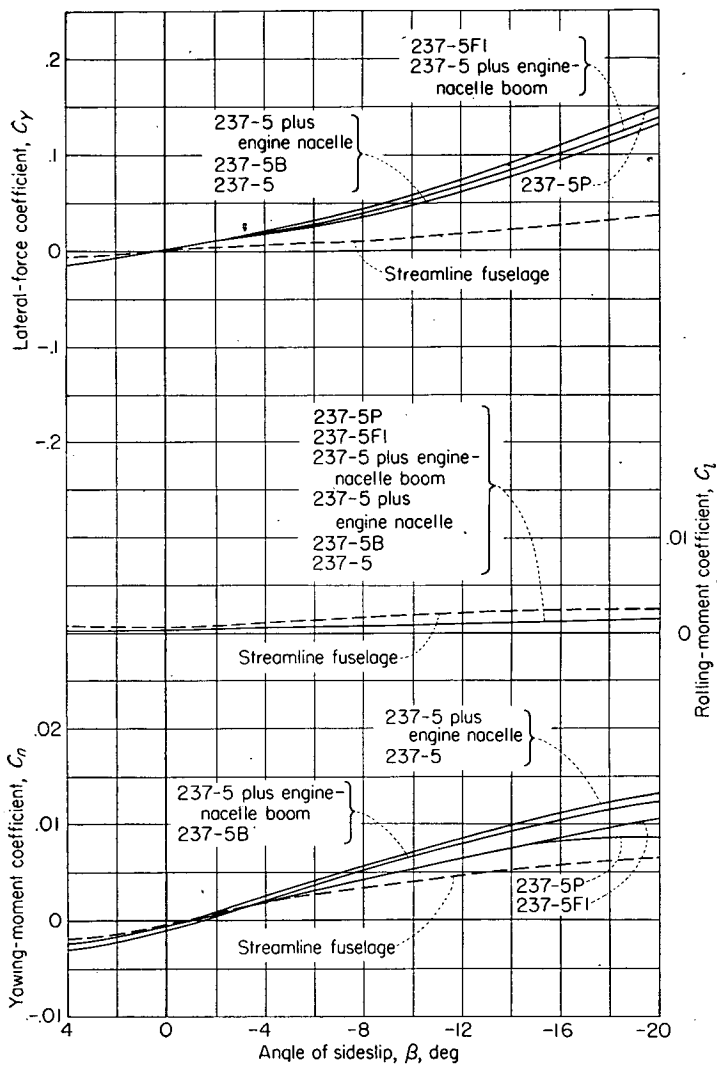


FIGURE 11.—Aerodynamic characteristics in sideslip of Langley tank model 237-5 with various afterbody configurations. $R \approx 1.3 \times 10^6$; $\alpha = 2^\circ$.

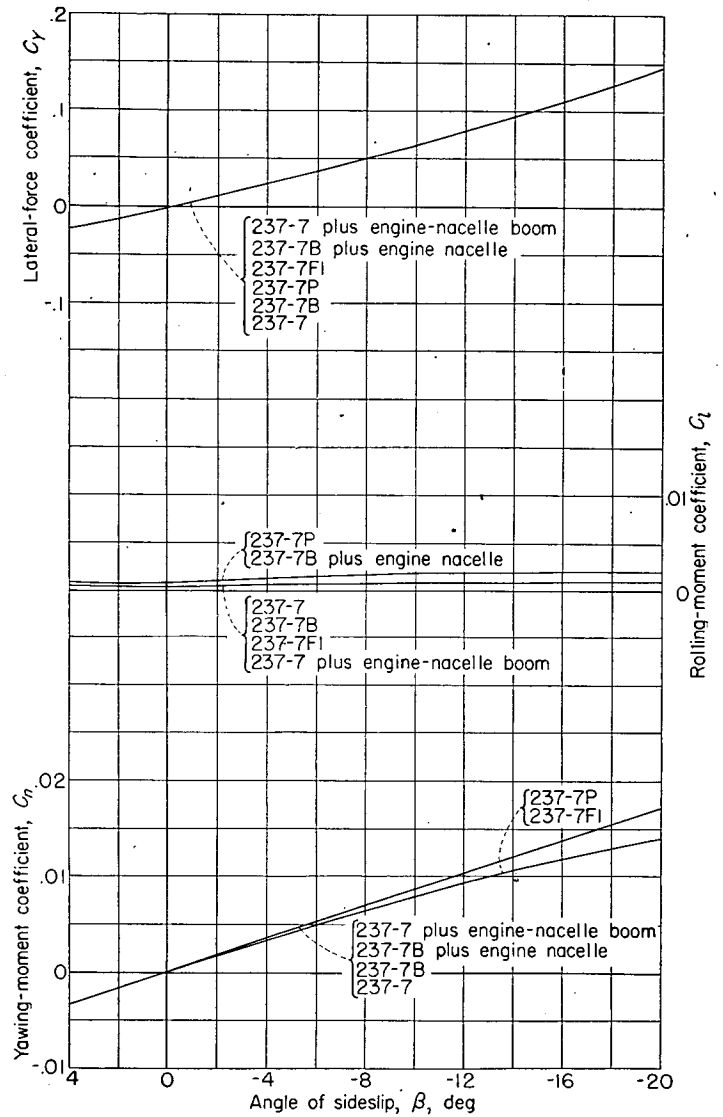


FIGURE 12.—Aerodynamic characteristics in sideslip of Langley tank model 237-7 with various afterbody configurations. $R \approx 1.3 \times 10^6$; $\alpha = 2^\circ$.

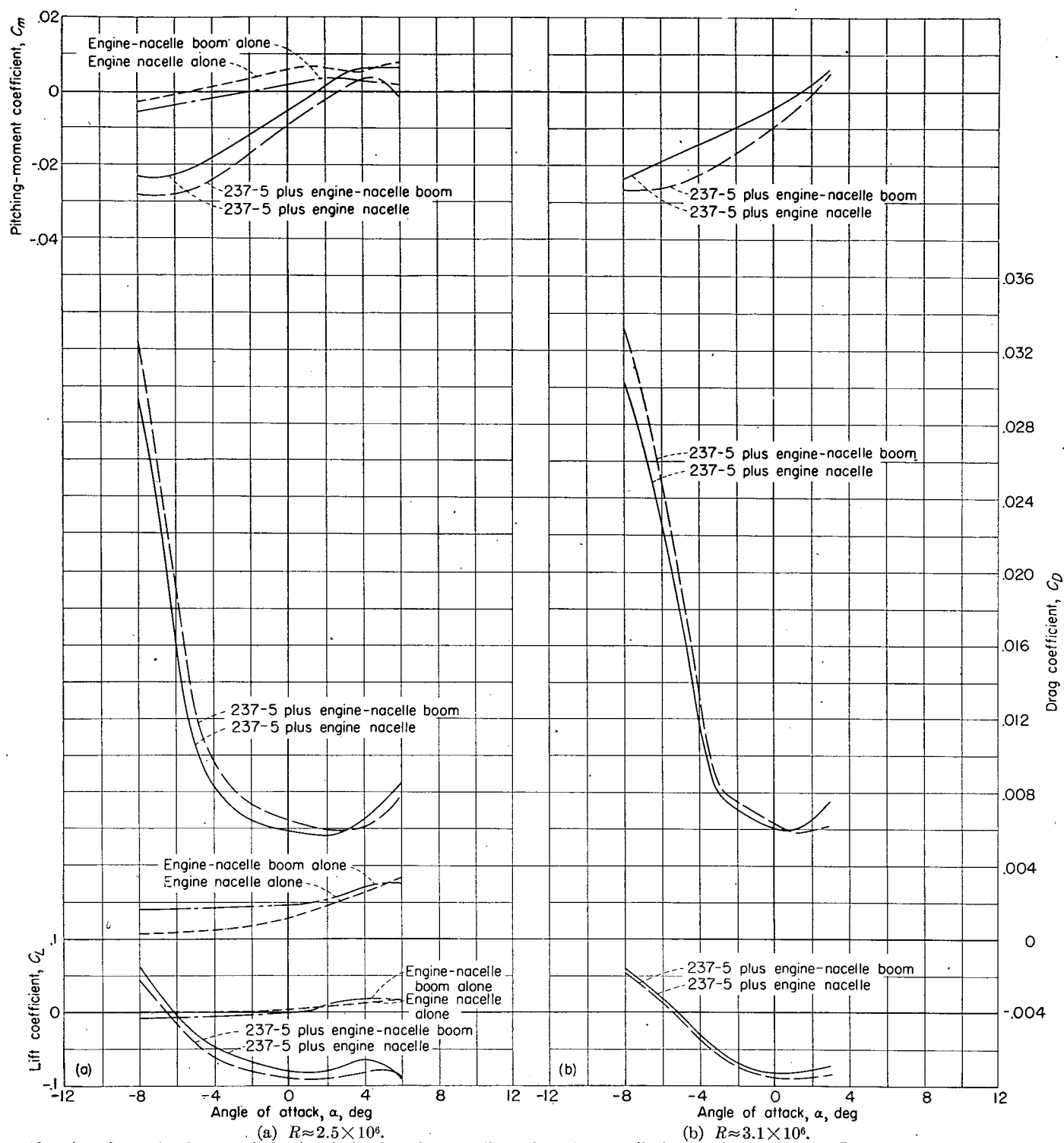


FIGURE 13.—Aerodynamic characteristics in pitch of engine nacelle and engine-nacelle boom alone and with Langley tank model 237-5. The coefficients for the nacelle alone and the nacelle boom alone are given for corresponding hull angles of attack.

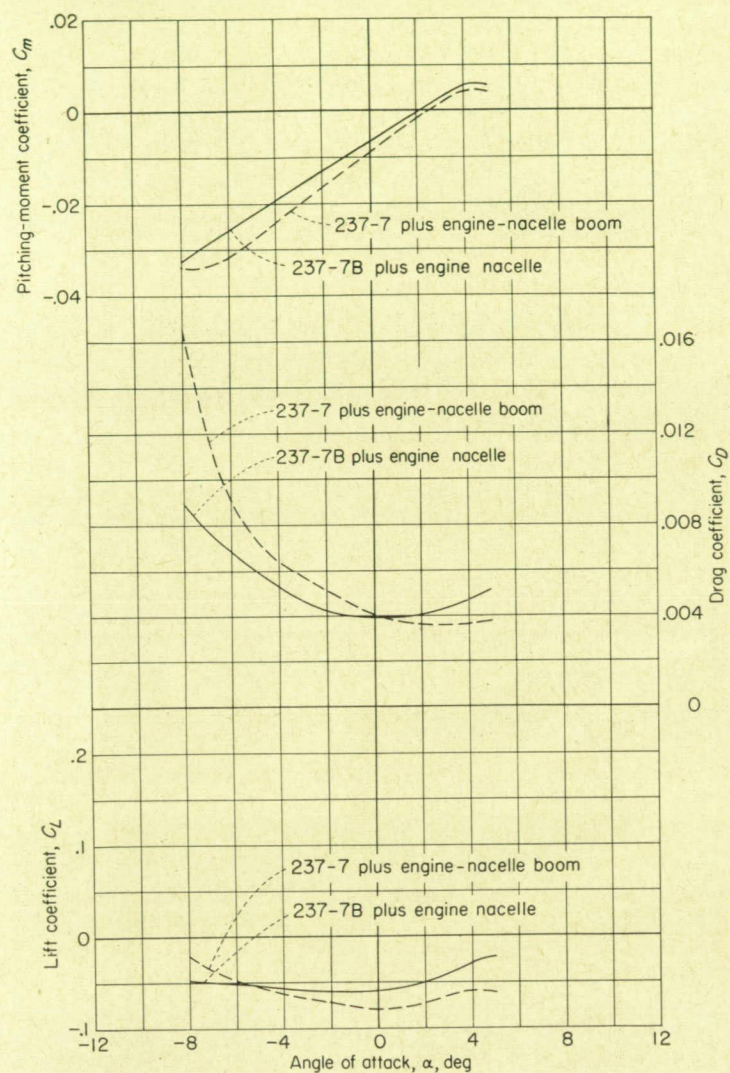


FIGURE 14.—Aerodynamic characteristics in pitch of Langley tank model 237-7 with engine nacelle and engine-nacelle boom, $R \approx 2.5 \times 10^6$.

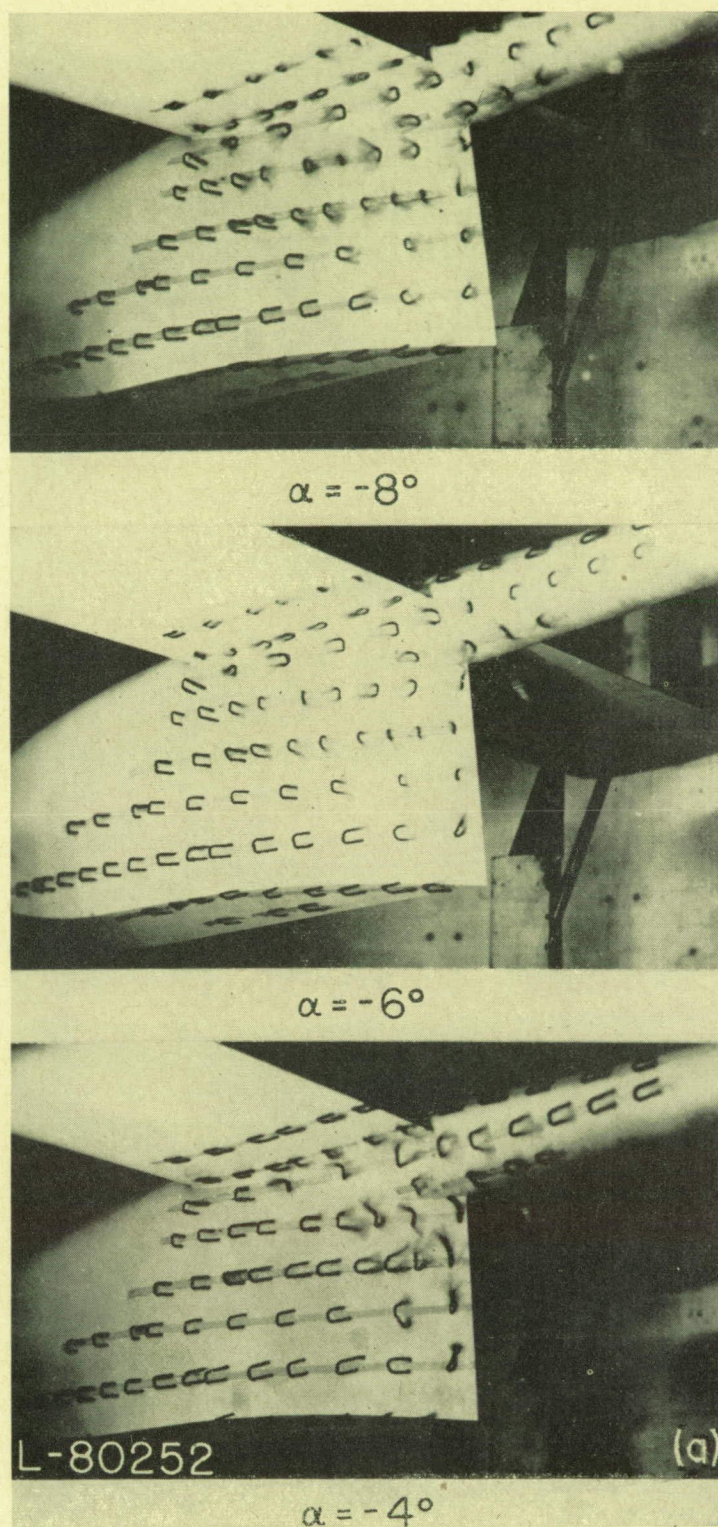


FIGURE 15.—Tuft studies of Langley tank model 237-5B.

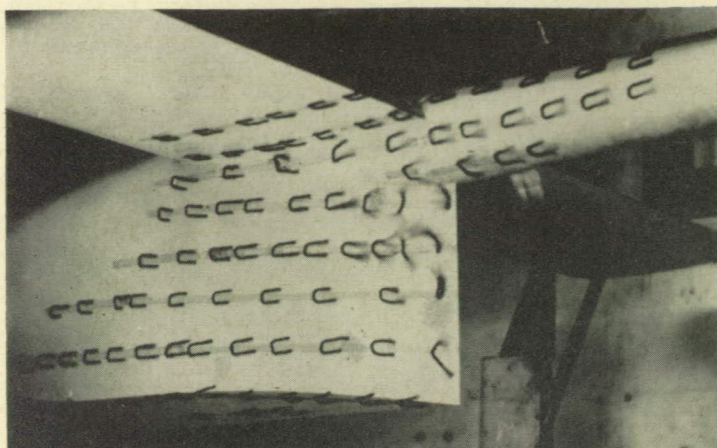
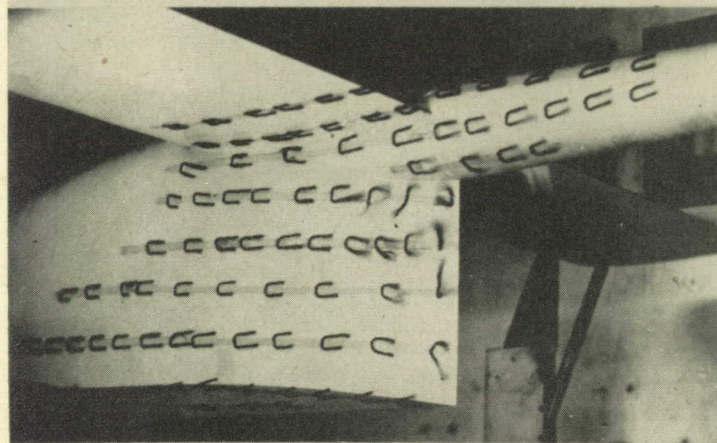
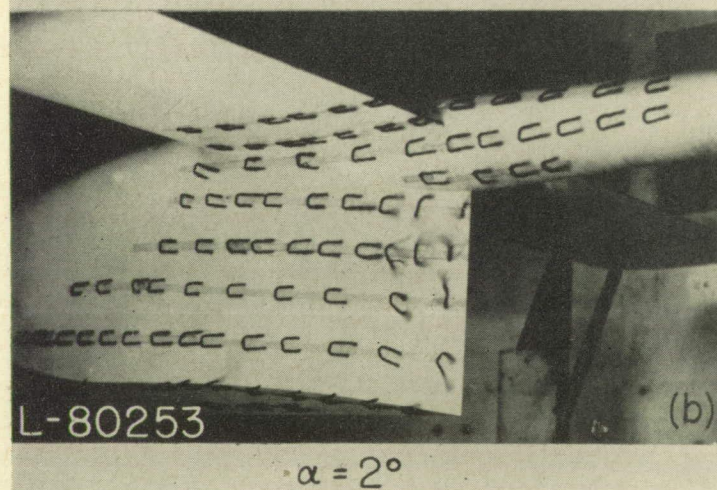

 $\alpha = -2^\circ$

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 $\alpha = 2^\circ$

FIGURE 15.—Continued.

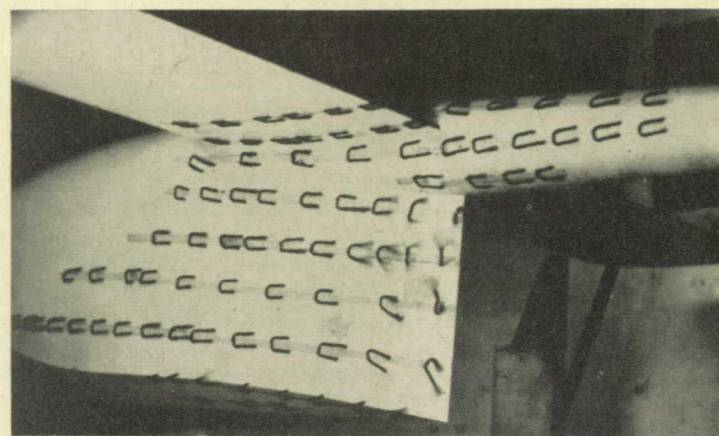
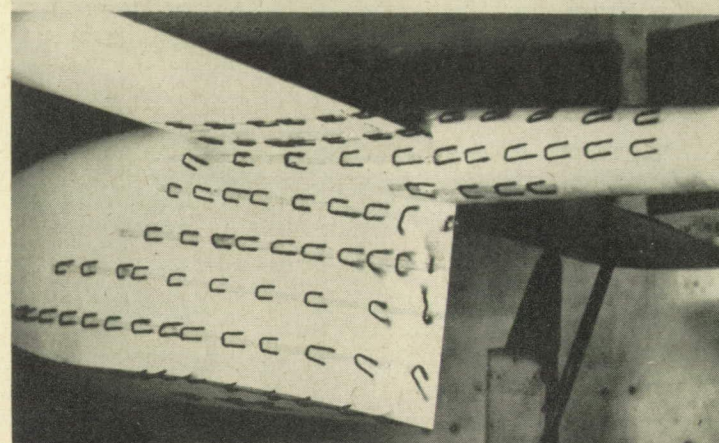
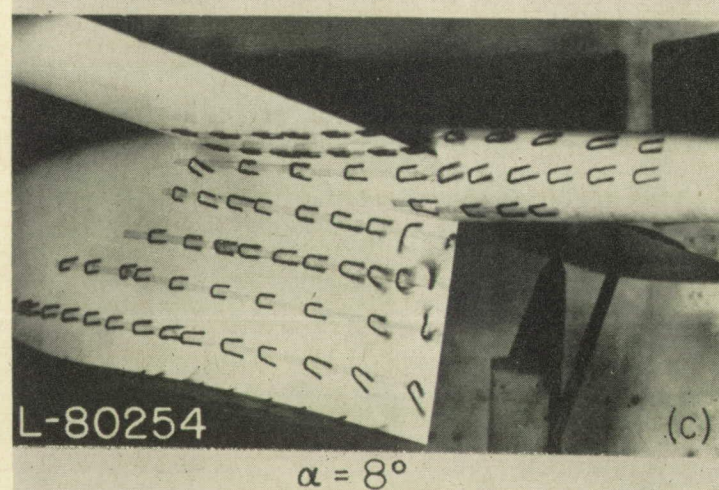

 $\alpha = 4^\circ$

 $\alpha = 6^\circ$

 $\alpha = 8^\circ$

FIGURE 15.—Concluded.

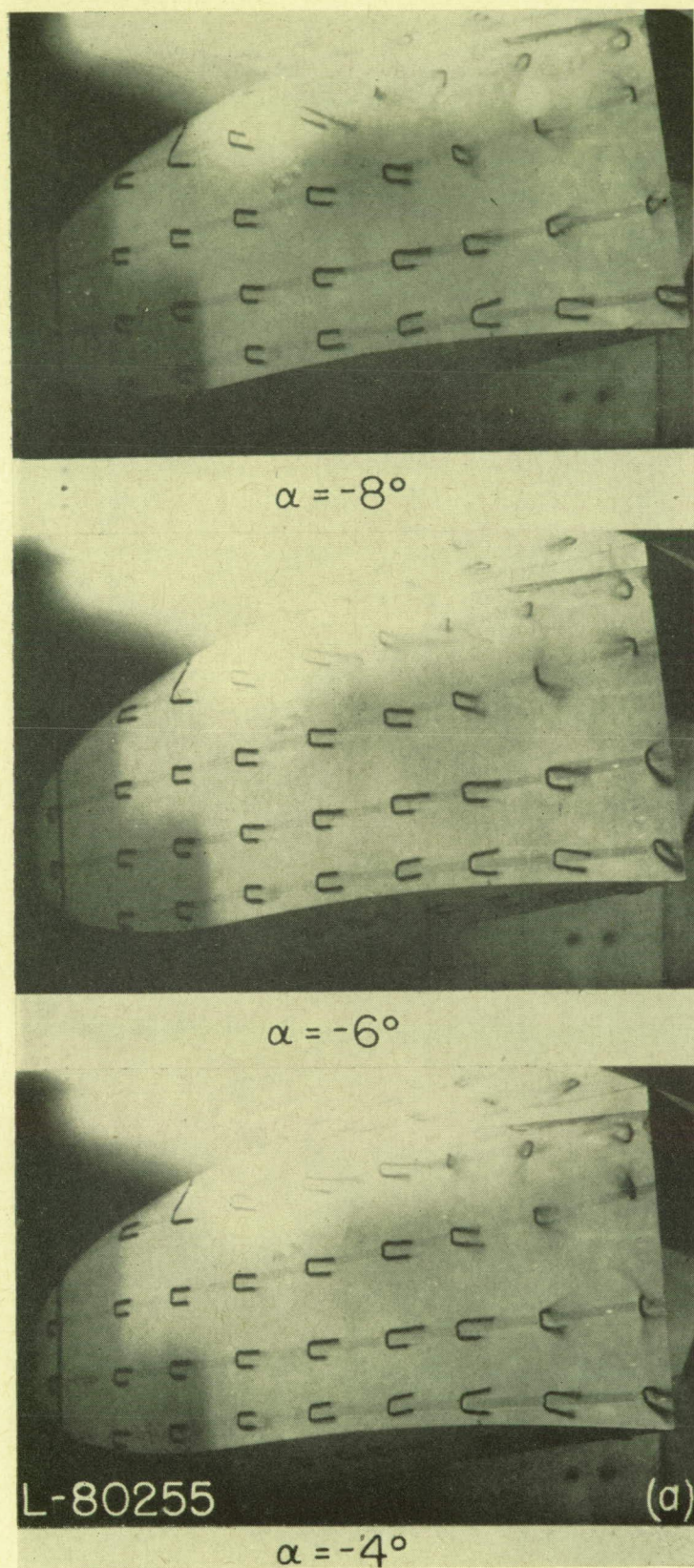


FIGURE 16.—Tuft studies of Langley tank model 237-5.

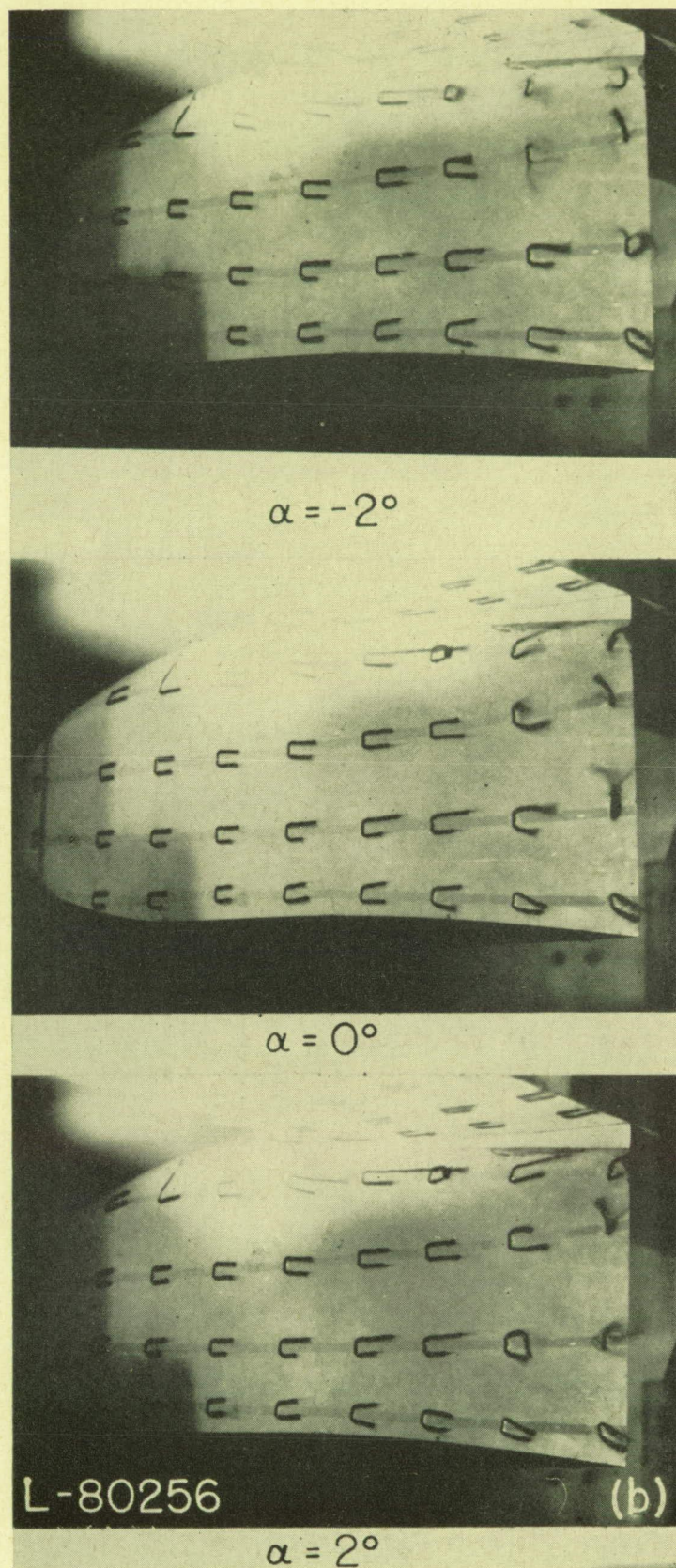


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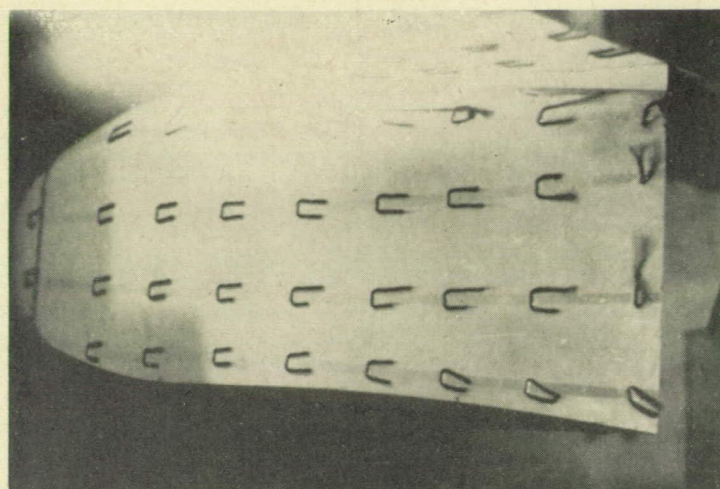
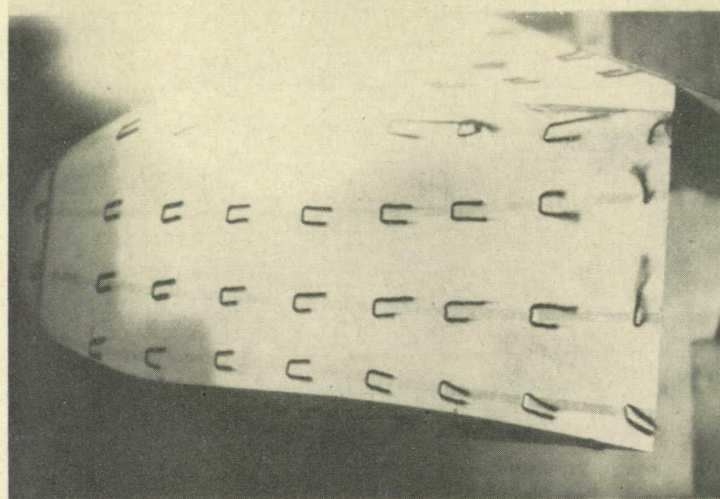
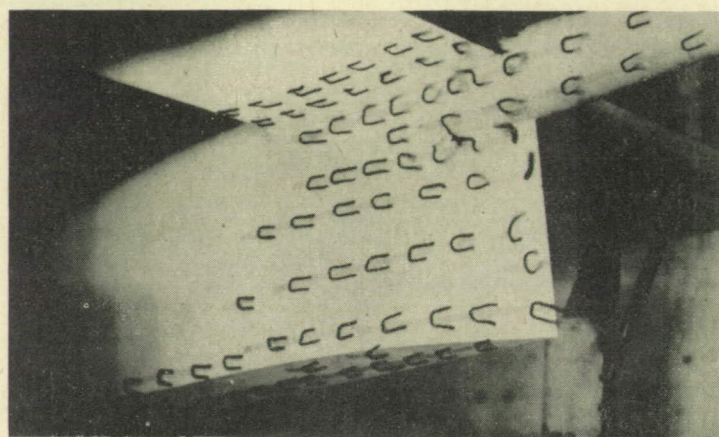
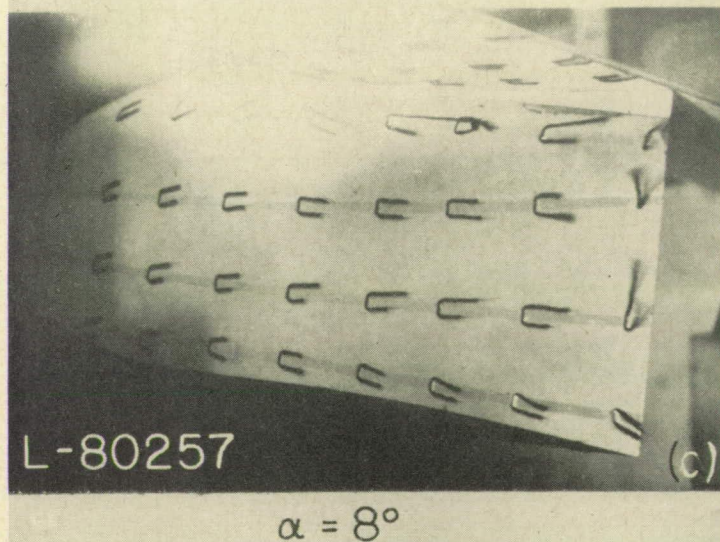
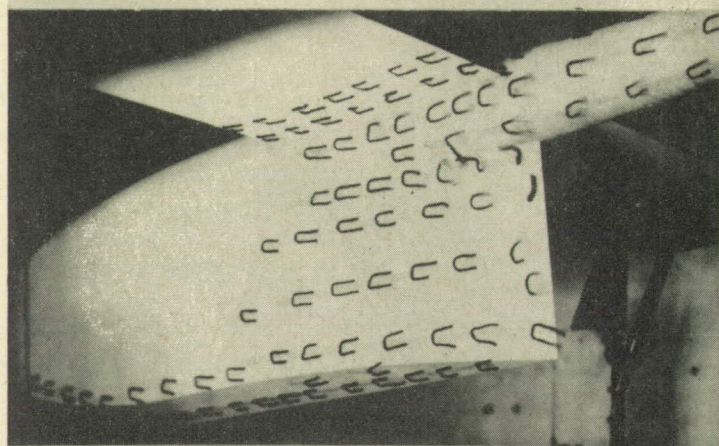
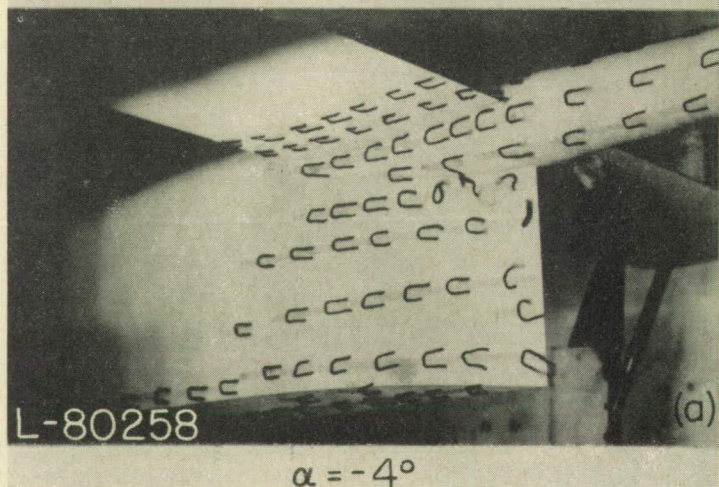

 $\alpha = 4^\circ$

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 $\alpha = -6^\circ$

 $\alpha = -4^\circ$

FIGURE 17.—Tuft studies of Langley tank model 237-7B.

FIGURE 16.—Concluded.

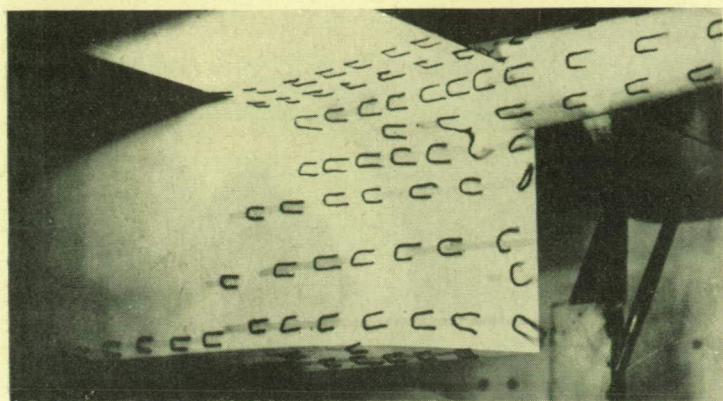
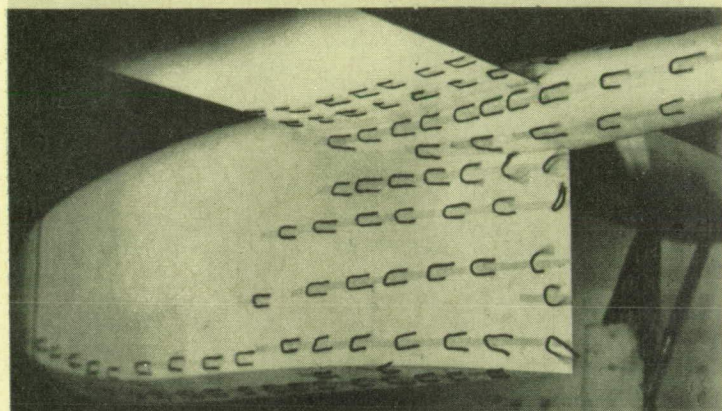
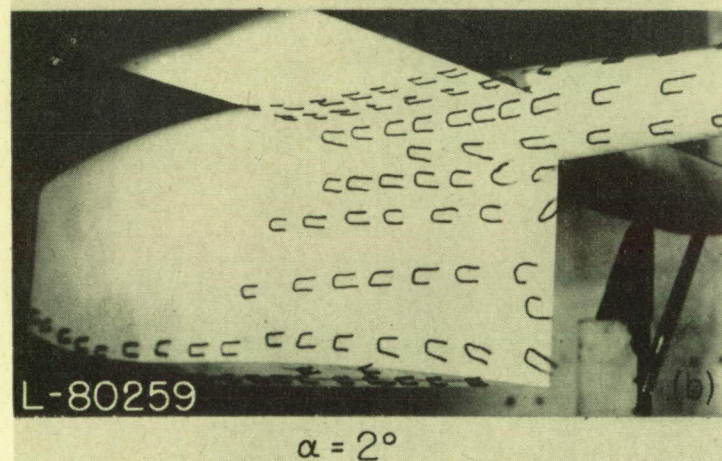
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FIGURE 17.—Continued.

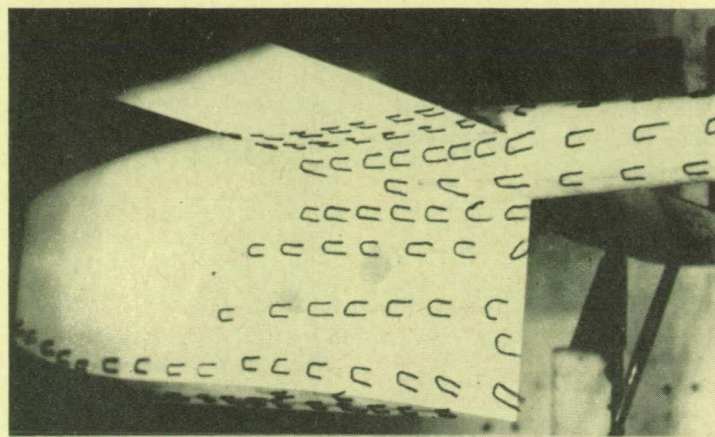
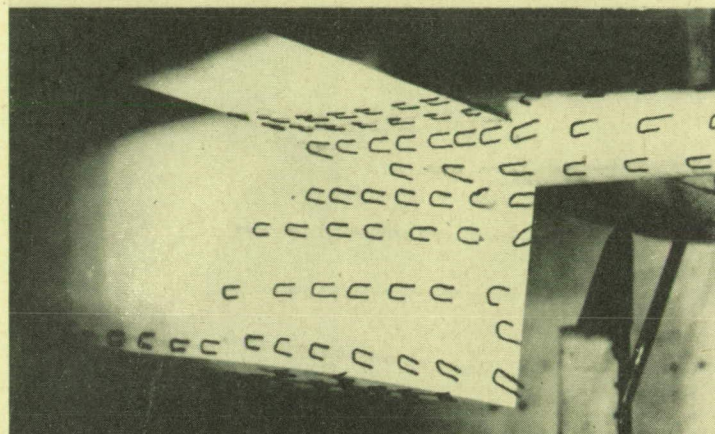
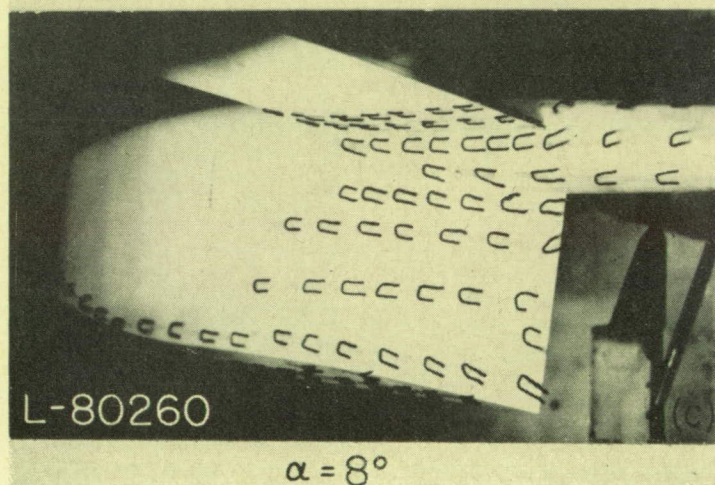
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FIGURE 17.—Concluded.

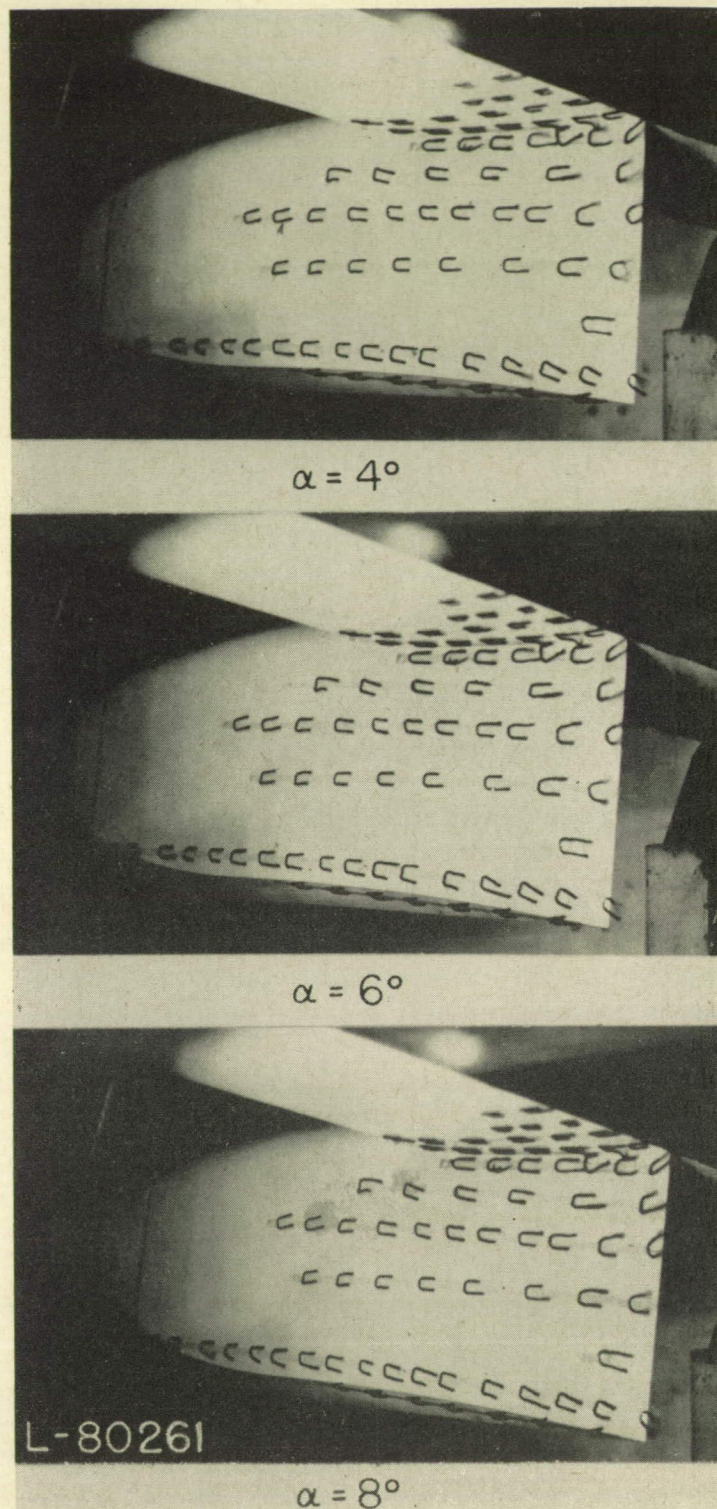


FIGURE 18.—Tuft studies of Langley tank model 237-7.

TABLE I
OFFSETS FOR LANGLEY TANK MODEL 237-5

[All dimensions are in inches]

[illegible]

TABLE II
OFFSETS FOR LANGLEY TANK MODEL 237-7

[All dimensions are in inches]

[illegible]

TABLE III
OFFSETS FOR LANGLEY MODELS 237-5B AND 237-7B

Offsets for hull ahead of stations 9 and 7 are given in tables I and II, respectively. All dimensions are in inches.]

Station	Distance to F. P., table I, or distance to station 0, table II	Keel above base line	Chine above base line	Half beam at chine	Radius and half maximum beam	Height of hull at center line	Line of centers above base line
237-5B							
9	38.25	0	1.19	3.28	3.32	19.85	16.53
10	42.50	0	.72	1.98	3.17	19.70	16.53
11	46.75	0	.15	.43	3.00	19.53	16.53
11½	47.90	{ 13.55 }	0	0	2.96	19.49	16.53
237-7B							
7	29.75	0	1.30	3.57	3.62	20.00	16.38
7½	31.87	0	1.25	3.40	3.54	19.97	16.43
8	34.00	0	1.18	3.18	3.46	19.95	16.49
9	38.25	0	.93	2.47	3.32	19.85	16.53
10	42.50	0	.55	1.45	3.17	19.70	16.53
11	46.75	0	.12	.32	3.00	19.53	16.53
11½	47.90	{ 13.55 }	0	0	2.96	19.49	16.53
237-5B and 237-7B							
12	51.00	13.67	-----	-----	2.86	19.39	16.53
13	55.25	13.83	-----	-----	2.70	19.23	16.53
14	59.50	13.98	-----	-----	2.55	19.08	16.53
15	63.75	14.13	-----	-----	2.40	18.93	16.53
16	68.00	14.28	-----	-----	2.25	18.78	16.53
17	72.25	14.44	-----	-----	2.09	18.62	16.53
18	76.50	14.58	-----	-----	1.95	18.48	16.53
19	80.75	14.73	-----	-----	1.80	18.33	16.53
20	85.00	14.90	-----	-----	1.63	18.16	16.53
21	89.25	15.04	-----	-----	1.49	18.02	16.53
22	93.50	15.20	-----	-----	1.33	17.86	16.53
23	97.75	15.36	-----	-----	1.17	17.70	16.53
24	102.00	15.51	-----	-----	1.02	17.55	16.53
25	106.25	15.65	-----	-----	.88	17.41	16.53
26	110.50	15.80	-----	-----	.73	17.26	16.53
27	114.75	15.96	-----	-----	.57	17.10	16.53
A. P	116.65	16.03	-----	-----	.50	17.03	16.53

TABLE IV
OFFSETS FOR LANGLEY TANK MODELS 237-5P AND 237-7P

[Offsets for hull ahead of stations 9 and 7 are given in tables I and II, respectively. All dimensions are in inches.]

Station	Distance to F. P., table I, or distance to station 0, table II	Keel above base line	Chine above base line	Half beam at chine	Maximum half beam	Height of cove above base line	Height of hull at center line	Line of centers top of hull	Line of centers bottom of hull	1-in. buttock	2-in. buttock	3-in. buttock	10-in. water line	12-in. water line
237-5P														
9	38.25	0	1.19	3.28	3.32	12.37	19.85	16.53	12.82					3.28
10	42.50	0	.72	1.98	3.17	10.33	19.70	16.53	12.80					3.05
11	46.75	0	.15	.43	3.00	9.80	19.53	16.53	12.79	9.97	10.36	11.80	1.11	2.89
11½	47.90	9.65	0	0	2.96	9.65	19.49	16.53	12.79	9.99	10.59	12.79	1.00	2.85
237-7P														
7	29.75	0	1.30	3.57	3.62	12.24	20.00	16.38	12.84					3.57
7½	31.87	0	1.25	3.40	3.54	11.83	19.97	16.43	12.83					3.45
8	34.00	0	1.18	3.18	3.46	11.43	19.95	16.49	12.83					3.36
9	38.25	0	.93	2.47	3.32	10.62	19.85	16.53	12.82					3.21
10	42.50	0	.55	1.45	3.17	10.02	19.70	16.53	12.80					3.05
11	46.75	0	.12	.32	3.00	9.72	19.53	16.53	12.79	9.97	10.36	11.80	1.11	2.89
11½	47.90	9.65	0	0	2.96	9.65	19.49	16.53	12.79	9.99	10.59	12.79	1.00	2.85
237-5P and 237-7P														
13	55.25	9.91	-----	-----	2.70		19.23	16.53	12.77	10.27	10.96		0.25	2.57
15	63.75	10.21	-----	-----	2.40		18.93	16.53	12.75	10.57	11.43			2.27
17	72.25	10.51	-----	-----	2.09		18.62	16.53	12.72	10.91	12.14			1.95
18	76.50	10.67	-----	-----	1.95		18.48	16.53	12.71	11.07				1.82
19	80.75	10.82	-----	-----	1.80		18.33	16.53		11.20				1.70
20	85.00	10.97	-----	-----	1.63		18.16	16.53		11.32				1.60
21	89.25	11.12	-----	-----	1.48		18.01	16.53		11.46				1.48
22	93.50	11.27	11.75	-----	1.33		17.86	16.53		11.63				1.33
24	102.00	11.58	11.95	-----	1.02		17.55	16.53		11.90				1.02
26	110.50	11.88	12.15	-----	0.73		17.26	16.53						.29
A. P	116.65	12.10	12.29	-----	0.50		17.03	16.53						

TABLE V
OFFSETS FOR TAIL FLOAT INCORPORATED WITH LANGLEY TANK MODELS 237-5F1 AND 237-7F1

[All dimensions are in inches.]

Station	Distance to F. P., table I, or distance to station 0, table II	Keel above base line	Chine above base line	Radius of tail boom	Half maximum beam	Height of hull at center line	Line of centers above base line	½-in. buttock	1-in. buttock	1½-in. buttock	2-in. buttock	12-in. water line	13-in. water line	14-in. water line	15-in. water line	16-in. water line	18-in. water line
21	89.25	15.05	16.53	1.48	1.48	18.01	16.53	15.14	15.43							1.39	0.17
21¼	90.31	15.04	16.50	1.44	1.45	17.96	16.51	15.17	15.49							1.33	
21½	91.38	14.94	16.35	1.40	1.46	17.93	16.47	15.21	15.54						0.08	1.30	
21¾	92.44	14.70	16.05	1.36	1.50	17.90	16.40	15.14	15.57	16.03					.33	1.45	
22	93.50	14.33	15.59		1.56	17.86	16.30	14.73	15.12	15.53					.82		
22¼	94.56	13.82	15.04		1.64	17.81	16.17	14.20	14.55	14.93				0.22	1.58		
22½	95.63	13.28	14.46		1.74	17.78	16.04	13.62	13.95	14.30				1.06	1.74		
22¾	96.69	12.74	13.88		1.86	17.74	15.88	13.04	13.36	13.66			0.42	1.86	1.86		
23	97.75	12.26	13.35		1.98	17.70	15.72	12.54	12.82	13.09			1.29	1.98	1.98		
23½	99.88	11.56	12.56		2.24	17.62	15.38	11.80	12.01	12.24	12.46	0.95	2.24	2.24	2.24		
24	102.00	11.24	12.16		2.41	17.55	15.14	11.43	11.61	11.81	12.00	2.00	2.41	2.41	2.41		
24¼	103.06	11.21	12.10		2.44	17.51	15.07	11.39	11.57	11.76	11.94	2.17	2.44	2.44	2.44		
24½	104.13	11.24	12.13		2.47	17.48	15.01	11.41	11.60	11.78	11.96	2.10	2.46	2.46	2.46		
25	106.25	11.38	12.26		2.43	17.41	14.98	11.56	11.74	11.92	12.10	1.70	2.43	2.43	2.43		
26	110.50	11.68	12.39		1.94	17.26	15.32	11.86				.87	1.93	1.93	1.93		
27	114.75	11.98	12.23		.69	17.10	16.41	12.16				.04	.69	.69	.69		
A. P.	116.65	12.12	12.12		0	17.03	17.03						0	0	0		

TABLE VI
ORDINATES FOR LANDPLANE FUSELAGE

[All dimensions are given in inches.]

Station	Radius	Station	Radius
0.158	0.408	50.989	6.440
.527	.838	54.309	6.420
1.054	1.263	58.143	6.354
2.108	1.887	62.267	6.254
3.373	2.462	66.378	6.121
5.059	3.071	69.896	5.980
7.906	3.864	72.557	5.854
8.432	3.989	76.404	5.642
10.804	4.496	79.843	5.420
14.124	5.064	84.033	5.103
17.457	5.492	87.538	4.797
20.580	5.790	91.015	4.451
23.584	6.003	94.494	4.058
26.483	6.156	97.973	3.616
29.513	6.274	101.451	3.118
33.031	6.369	104.837	2.573
36.918	6.436	108.144	1.978
40.185	6.487	111.543	1.293
43.716	6.481	114.521	.624
45.166	6.482	117.050	0
47.524	6.479		

TABLE VII
VOLUMES, SURFACE AREAS, AND MAXIMUM CROSS-SECTIONAL AREAS OF LANGLEY TANK MODELS 237 AND OF STREAMLINE FUSELAGE

Configuration	Volume, cu in.	Surface area, sq in.	Side area, sq in.	Maximum cross-sectional area, sq in.
237-5.....	5,649	2,095	841	176
237-7.....	5,228	2,303	964	142
237-5B.....	6,519	2,884	1,090	176
237-7B.....	6,174	3,100	1,213	142
237-5P.....	7,574	3,427	1,359	176
237-7P.....	7,276	3,645	1,482	142
237-5F1.....	6,869	3,106	1,177	176
237-7F1.....	6,524	3,321	1,300	142
Streamline body.....	10,270	3,630	1,162	132
Engine nacelle.....	471	406	108	39
Engine-nacelle boom.....	1,419	1,220	363	39

TABLE VIII
MINIMUM DRAG COEFFICIENTS AND STABILITY PARAMETERS FOR LANGLEY TANK MODELS 237 AND STREAMLINE BODY

[The drag coefficients are given for a Reynolds number of about 2.5×10^6 based on wing M. A. C.]

Model	$C_{D_{min}}$	$C_{m_{\alpha}}$	$C_{n_{\beta}}$	$C_{r_{\beta}}$
237-5.....	0.0032	0.0028	-0.0008	-0.0042
237-5P.....	.0030	.0026	-.0006	-.0042
237-5B.....	.0028	.0025	-.0008	-.0042
237-5F1.....	.0043	.0026	-.0006	-.0042
237-5 + engine-nacelle boom.....	.0059	.0037	-.0008	-.0042
237-5 + engine nacelle.....	.0056	.0034	-.0008	-.0042
237-7.....	.0024	.0026	-.0009	-.0060
237-7P.....	.0027	.0024	-.0008	-.0060
237-7B.....	.0023	.0025	-.0009	-.0060
237-7F1.....	.0038	.0024	-.0008	-.0060
237-7 + engine-nacelle boom.....	.0036	.0037	-.0009	-.0060
237-7B + engine nacelle.....	.0039	.0032	-.0009	-.0060
Streamline body.....	.0025	.0049	-.0005	-.0015
Engine nacelle.....	.0021	.0011		
Engine-nacelle boom.....	.0022	.0009		

* At $\alpha = 3^\circ$ (not minimum drag coefficient).

REPORT 1145

A METHOD OF CALIBRATING AIRSPEED INSTALLATIONS ON AIRPLANES AT TRANSONIC AND SUPERSONIC SPEEDS BY THE USE OF ACCELEROMETER AND ATTITUDE-ANGLE MEASUREMENTS¹

By JOHN A. ZALOVCIK, LINDSAY J. LINA, and JAMES P. TRANT, Jr.

SUMMARY

A method is described for calibrating airspeed installations on airplanes at transonic and supersonic speeds in vertical-plane maneuvers in which use is made of measurements of normal and longitudinal accelerations and attitude angle. In this method, all the required instrumentation is carried within the airplane.

An analytical study of the effects of various sources of error on the accuracy of an airspeed calibration by the accelerometer method indicated that the required measurements can be made accurately enough to insure a satisfactory calibration. This conclusion was verified experimentally by a calibration of the airspeed installation of a jet fighter airplane. The tests included shallow dives up to a Mach number of about 0.80 with pull-ups of about 4, 3, and 2g normal acceleration. A calibration by the radar-phototheodolite method (NACA Rep. 985) was also made for comparison.

INTRODUCTION

A method of calibrating the static-pressure source of a pitot-static installation on an airplane at high speed and high altitude by the use of radar-phototheodolite tracking equipment was described in reference 1. In this method, the radar-phototheodolite equipment is used to establish the geometric altitude of the airplane in surveys of atmospheric pressure made at speeds for which the airspeed calibration is known and in maneuvers under conditions for which the calibration is desired. Although this method is precise, its use is limited since radar-phototheodolite equipment is not generally available. In order that airspeed calibrations may be performed universally with the added convenience of having instrumentation all within the airplane, a method, herein referred to as the accelerometer method, was devised at the Langley Aeronautical Laboratory of the National Advisory Committee for Aeronautics. The method makes use of measurements of total pressure, static pressure, temperature, acceleration, and attitude angle in vertical-plane maneuvers.

This report presents an analysis of the accelerometer method including a discussion of the required instrumentation, calibration procedures, and the effect of errors in measurement on the accuracy of a calibration. Also included

is a comparison of airspeed calibrations of a jet fighter airplane evaluated by both the accelerometer and radar-phototheodolite methods. These calibrations were made simultaneously in shallow dives up to a Mach number of about 0.8.

SYMBOLS

p	free-stream static or atmospheric pressure
p_m	indicated static pressure
p_a	free-stream static pressure computed from equation (7) by using values of temperature and altitude computed from accelerometer measurements
p_r	free-stream static pressure computed from equation (7) by using values of temperature and altitude determined by radar
ρ	density
e_p	static-pressure error ratio, $\frac{p_m - p}{p}$
p_T	free-stream total pressure for subsonic flow and total pressure behind normal shock for supersonic flow
q_c'	indicated impact pressure, $p_T - p_m$
t	elapsed time
t_m	measured elapsed time
e_t	error in elapsed time, $\frac{t_m - t}{t}$
h	altitude
M	free-stream Mach number
M'	indicated free-stream Mach number
T	free-stream temperature, absolute units
T_m	measured temperature, absolute units
T'	temperature defined by equation (5)
e_T	error in free-stream temperature, $\frac{T' - T}{T}$
K	temperature recovery factor, $\frac{T_m - T}{0.2M^2T}$
a_x	longitudinal acceleration, positive forward along x -axis
a_z	normal acceleration, positive upward along z -axis
a_v	vertical acceleration, positive upward along vertical of earth (defined by eq. (10))
a_y	lateral acceleration, positive to right of y -axis
α	angle between rays of sun and longitudinal axis as measured by sun camera

¹Supersedes NACA TN 2099, "A Method of Calibrating Airspeed Installations on Airplanes at Transonic and Supersonic Speeds by Use of Accelerometer and Attitude-Angle Measurements" by John A. Zalovecik, 1950, and NACA TN 2570, "Comparison of Airspeed Calibrations Evaluated by the Accelerometer and Radar Methods" by Lindsay J. Lina and James P. Trant, Jr., 1952.

- β longitude of airplane
 γ ratio of specific heats, 1.4
 ϵ declination of sun
 λ elevation angle of sun
 θ attitude angle, positive below horizon
 τ latitude of airplane
 ϕ angle of bank, positive right wing down
 ψ angle of yaw, positive to right
 ω Greenwich hour angle
 v vertical velocity
 R gas constant
 s_β error in estimating latitude of airplane (distance along longitude β)
 s_τ error in estimating longitude of airplane (distance along latitude τ)
 g acceleration due to gravity
 Δ prefix denoting error

Subscripts:

- 0 beginning of level-flight run or shallow dive preceding calibration maneuver or end of level-flight run or shallow dive following calibration maneuver
 1 beginning of evaluation of maneuver

ANALYSIS OF METHOD

THEORY

The principal feature of most procedures for calibration of airspeed installations on airplanes is the determination of ambient or free-stream static pressure. In the method described herein, free-stream static pressure is obtained over a range of altitude with the aid of the relation that the rate of change of static pressure with altitude is equal to the density, or

$$\frac{dp}{dh} = -\rho \quad (1)$$

Since

$$\rho = \frac{p}{RT}$$

then

$$\frac{dp}{dh} = -\frac{p}{RT}$$

or

$$dp = -\frac{p}{RT} dh \quad (2)$$

Equation (2) may be integrated as

$$p = p_1 e^{-\int_{h_1}^h \frac{dh}{RT}} \quad (3)$$

Therefore, if the pressure at one altitude is known, the pressure at other altitudes may be determined provided the ambient temperature and the change in altitude are determined. The ambient temperature T may be determined from the measured temperature T_m with the use of the relation

$$T = \frac{T_m}{1 + \frac{\gamma-1}{2} KM^2} \quad (4)$$

Since only the indicated Mach number M' is known for

flight conditions where no airspeed calibration exists, the ambient temperature is determined only approximately as

$$T' = \frac{T_m}{1 + \frac{\gamma-1}{2} KM'^2} \quad (5)$$

The use of T' in equation (3) would result in a small error in p and, hence, two or more approximations may be necessary.

An alternative integral form of equation (2) is

$$\left(\frac{p}{p_1}\right)^n = 1 - n \int_{h_1}^h \left(\frac{p}{p_1}\right)^n \frac{dh}{RT} \quad (6)$$

After substitution of the measured pressure p_m and the temperature T' in the right side of equation (6), the equation becomes

$$\left(\frac{p}{p_1}\right)^n = 1 - n \int_{h_1}^h \left(\frac{p_m}{p_1}\right)^n \frac{dh}{RT'}$$

or

$$\left(\frac{p}{p_1}\right)^n = 1 - n \int_{h_1}^h \left(\frac{p_m}{p_1}\right)^n \frac{\left(1 + \frac{\gamma-1}{2} KM'^2\right)}{RT_m} dh \quad (7)$$

The value of n may be selected for the flight conditions encountered so that only one approximation is required in the determination of p . (See appendix A.) For values of temperature recovery factor K of the thermometer near unity and for subsonic and low supersonic airspeeds, a value of n of $\frac{\gamma-1}{\gamma}$ or 0.286 gives satisfactory results.

The change in altitude dh in equation (7) may be determined from vertical velocity computed from measurements of pressure and temperature at an instant in the calibration run when the airspeed calibration or the static pressure is known and from vertical acceleration computed from measurements of longitudinal and normal acceleration and attitude angle, as

$$dh = \left(v_1 + \int_{t_1}^t a_r dt\right) dt \quad (8)$$

where

$$v_1 = \frac{-RT_1}{p_1} \left(\frac{dp}{dt}\right)_1 \quad (9)$$

and

$$a_r = a_z \cos \theta - a_x \sin \theta - g \quad (10)$$

Vertical velocity v_1 may also be determined as described in appendix B.

The relation given by equation (7) may also be used to advantage in the radar method of reference 1, particularly for tests which would not permit a survey of static pressure to be made over the desired range of altitude at flight conditions (Mach number and lift coefficient) at which the airspeed calibration was known. For such a flight condition, however, the airspeed calibration at least at one instant during the test must be known and the change in altitude from the altitude at that instant would, of course, be determined from radar measurements.

REQUIRED EQUIPMENT

The airplane on which the pitot-static installation is to be calibrated should be equipped with instruments to record the following: static pressure measured by the static-pressure source; impact pressure, or the difference between total and static pressure, measured by the pitot-static installation; temperature; normal and longitudinal components of acceleration; attitude angle; and time. If the static-pressure recorder does not have the required sensitivity for accurately measuring changes in static pressure for the determination of vertical velocity, a statoscope with a sensitive differential-pressure recorder should also be included. The impact-pressure recorder, the static-pressure recorder, and the statoscope, if one is used, should be the only instruments connected to the static-pressure source and should be located as near as possible to it in order to minimize the lag of the pressure system. Provision must also be made so that the large volume of the statoscope is not continuously open to the static-pressure source because of lag considerations and so that the sensitive pressure cell is not subjected to pressures beyond its limit in order to prevent damage. The magnitude of the pressure lag of the airspeed installation may be determined by methods described in reference 2. Where the lag is appreciable, corrections must be made to the measured static pressure. The thermometer is used to determine free-stream temperature. A properly shielded thermometer with a high temperature recovery factor (approaching 1.0) is recommended since it should be least affected by position on the airplane. The thermometer should also have a low lag; corrections should be made if the lag is appreciable. The attitude recorder is used to measure the attitude of the airplane and may consist of a horizon camera, a sun camera, or an attitude gyroscope. A horizon camera shooting either forward or laterally is probably the most desirable attitude recorder in localities where the horizon is not obscured by haze. The attitude gyroscope and the sun camera, however, may be more generally used. The attitude gyroscope measures the change in attitude angle. The attitude angle at some instant during the calibration must therefore be determined from other measurements, perhaps from the statoscope measurements and from the estimated angle of attack. When a sun camera is used, the attitude angle is determined by subtracting the elevation angle of the sun from the angle recorded by the camera

$$\theta = \alpha - \lambda \quad (11)$$

The elevation angle of the sun may be found in navigational tables by use of the date of calibration, the time at the start or end of the calibration run, and the longitude and latitude of the airplane. The elevation angle may also be found from the expression

$$\sin \lambda = \sin \epsilon \sin \tau + \cos \epsilon \cos \tau \cos (\omega - \beta) \quad (12)$$

When the sun camera is used, the airplane should be equipped with an indicating device to enable the pilot to fly the airplane in a vertical plane with the lateral axis normal to the rays of the sun. One such device is a sundial.

CALIBRATION PROCEDURE

The calibration procedure described herein may be used for level flight, climb, dive, push-down, pull-out and any combination of these maneuvers in a vertical plane. A calibration may be obtained in a single run for some conditions and does not require any additional survey of static pressure and temperature as is necessary for the method of reference 1.

The calibration maneuver consists essentially of two parts. In one part the airplane is flown at a condition (lift coefficient and Mach number) for which the airspeed calibration is known. This part of the maneuver should preferably be made in or near level flight over an interval of time consistent with an accurate evaluation of vertical velocity. In the other part of the maneuver the airplane is flown at conditions for which the calibration is desired. This part of the maneuver should cover as short a time interval as practical in order to avoid the accumulation of errors in the evaluation. Either part of the maneuver may precede the other. Measurements, continuous throughout the maneuver, are made of impact pressure, static pressure, temperature, longitudinal acceleration, normal acceleration, attitude angle, and change in static pressure if a statoscope is used.

When a sun camera is used to obtain the attitude angle, the airplane must be flown with the lateral axis normal or very nearly normal to the rays of the sun. A simple sundial mounted ahead of the canopy may be used by the pilot as an indicator for keeping the lateral axis normal to the rays of the sun. The local time when the instruments are turned on should be determined accurately. The clock that is used for this purpose may be checked against radio time signal (National Bureau of Standards radio station WWV).

When a horizon camera or an attitude gyroscope is used, the airplane must be flown in a vertical plane. The airplane, however, is not restricted to any particular vertical plane as in the case with the sun camera.

EFFECT OF ERRORS IN MEASUREMENT

The accuracy of the method depends principally on the accuracy of determining the attitude angle and the accuracy of the accelerometer measurements. The equations for the various errors are derived in appendix C, and the errors in the computed quantities due to assumed errors in measured quantities are presented in figures 1 to 11. When a sun camera is used to determine the attitude angle, errors in attitude angle may arise from an error in time of the calibration, an error in longitude and latitude of the airplane, and an error in measurement of angle with the sun camera. By measuring the time of the start or end of the calibration run to within a few seconds, the error in attitude angle due to an error in the measurement of the time of the calibration may be almost eliminated (see fig. 1). The clock should be checked against an accurate source of time, perhaps against radio time signal, before or after the calibration. Because of the speed of the airplane, and hence the distance covered in a calibration run, the latitude and longitude of the airplane may vary appreciably during the run. If, however, the pilot can estimate his position to within 10 miles, the error in attitude angle can be kept to within $\pm 0.1^\circ$ (fig. 2). The

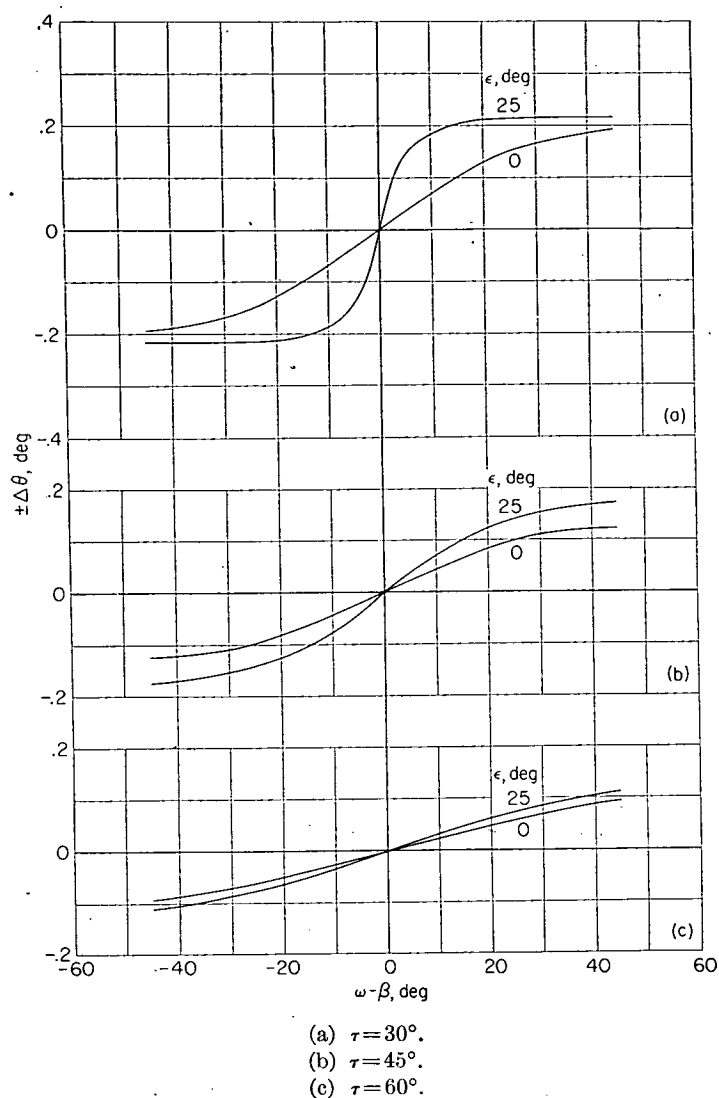


FIGURE 1.—Error in attitude angle due to error of ± 60 seconds in determining local time of calibration when use is made of a sun camera.

error in the angle of sun rays relative to the longitudinal axis as measured with the sun camera need not be more than about 0.1° for a properly designed sun camera. The probable maximum error in attitude angle should be of the order of 0.1° .

A properly designed horizon camera can probably measure attitude angle to within 0.1° . When the horizon camera is shooting forward, the correction for dip of the horizon can be estimated to within $\pm 0.1^\circ$.

When the attitude angle is obtained from the flight-path angle and the angle of attack at one instant in the calibration run, together with the change in attitude angle as measured with an attitude gyroscope for other times during the calibration run, the error in attitude angle may be of a larger magnitude than that for the method with which the sun camera is used. The error in the flight-path angle arises from an error in determining vertical velocity. If the error in vertical velocity is 1 foot per second, the corresponding error in flight-path angle is about 0.1° at a flight Mach number of 0.6. The error in estimating the angle of attack at the instant when the flight-path angle is determined depends on the applicability and the accuracy of the information on

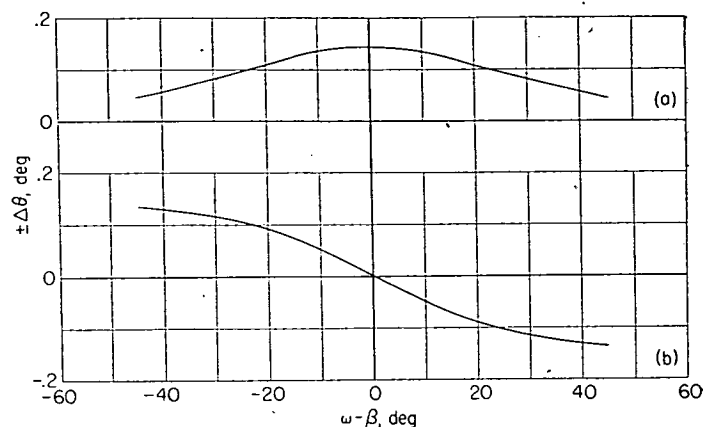


FIGURE 2.—Error in attitude angle due to error in estimating position of airplane. $\epsilon = 15^\circ$; $\tau = 40^\circ$.

which the estimate is based. The NACA attitude gyroscopes have a drift of about 3° per minute. For a calibration run lasting, for example, 20 seconds, the error in attitude angle due to the drift of the gyroscope would be of the order of 1° at the end of the calibration run. If the error in estimating the angle of attack is 0.3° and the error in determining attitude angle is 0.1° , the maximum error in attitude angle for a calibration run lasting 20 seconds would vary from 0.4° at the start to about 1.4° at the end.

The error in the vertical component of acceleration due to an error of $\pm 1^\circ$ in attitude angle as shown in figure 3 increases with increase in normal acceleration and with increase in attitude angle. In dives, normal acceleration would vary from about $1g$ near level flight to 0 in a vertical dive. The error in vertical acceleration in a dive would probably be of the order of $0.01g$ for an error of $\pm 1^\circ$ in attitude angle. In a pull-out, the error in vertical acceleration would be larger but would probably occur only near the end of the calibration run.

The error in the vertical component of acceleration due to an error of $\pm 0.01g$ in the normal and longitudinal components of acceleration is of the order of $0.01g$ (fig. 4). A standard NACA recording accelerometer has an accuracy of $\frac{1}{4}$ percent of full range. A longitudinal accelerometer of $\pm \frac{1}{2}g$ range and a normal accelerometer of 0 to $1g$ range would have an accuracy of $0.0025g$. A normal accelerometer of 0 to $4g$ range would have an accuracy of $0.01g$. In order to get improved accuracy for normal accelerations, a combination low-range and high-range normal accelerometer can be used. For example, in a calibration run involving a dive and pull-out, a low-range normal accelerometer can be evaluated for the dive and the high-range accelerometer for the pull-out. Errors in normal acceleration due to zero shift of the instrument may be largely eliminated by the method of appendix B.

The error in the vertical component of acceleration due to neglecting the angle of bank varies with attitude angle and normal acceleration (fig. 5). For $1g$ normal acceleration, neglecting an angle of bank of 10° results in an error in vertical acceleration of $-0.015g$ at zero attitude angle and of $0g$ in a vertical dive.

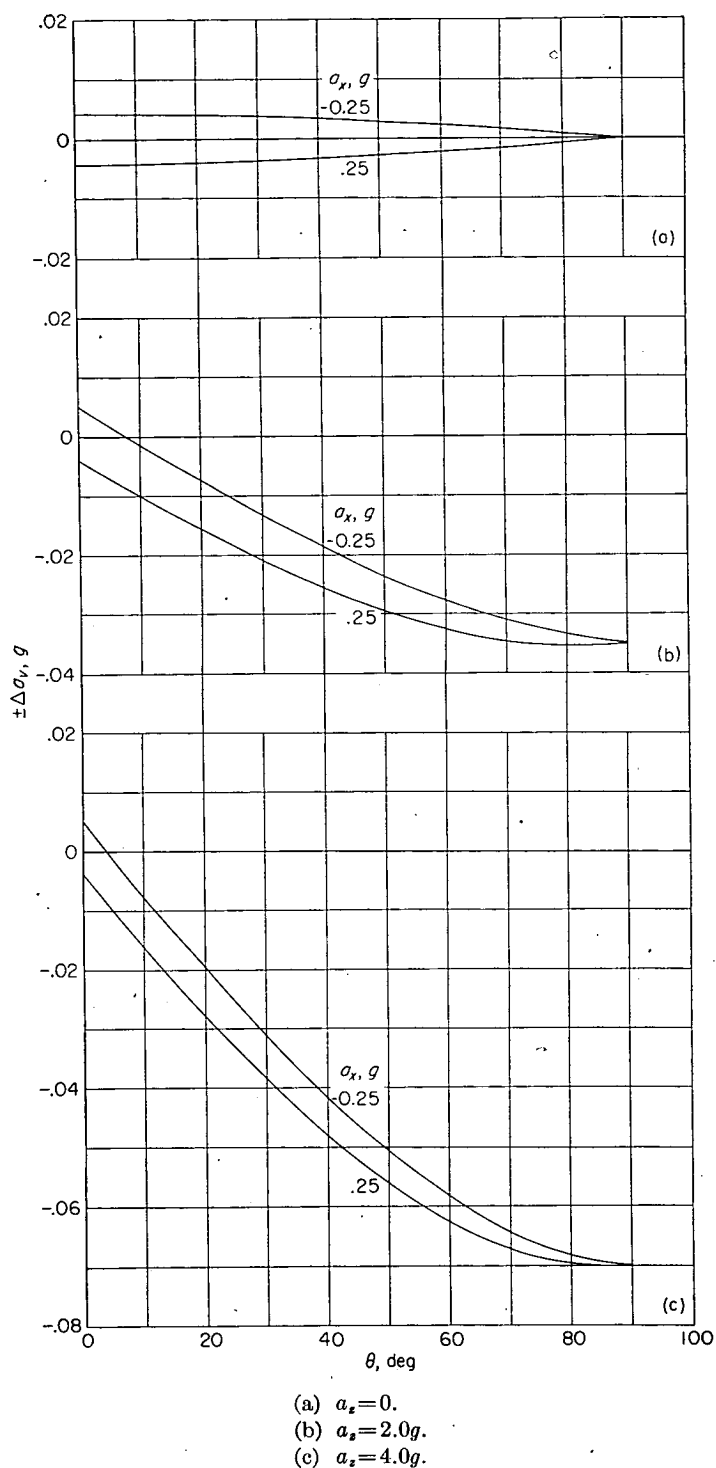
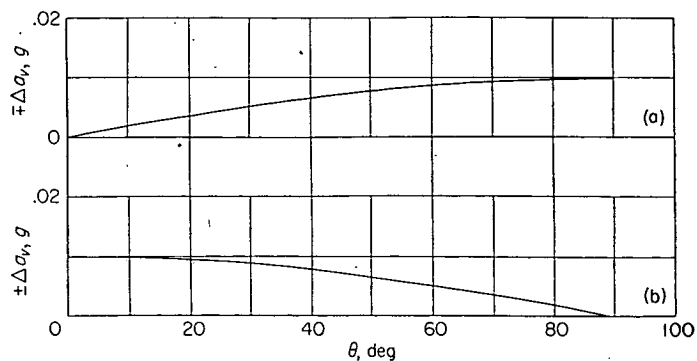


FIGURE 3.—Error in vertical component of acceleration due to $\pm 1^\circ$ error in attitude angle.

The error in the vertical component of acceleration due to neglecting the angle of yaw is shown in figure 6. For a 2° angle of yaw, the error in vertical acceleration is less than $0.01g$. For transonic and supersonic speeds, an angle of yaw of 2° probably would not be unintentionally exceeded and certainly would not be maintained.

The error in static pressure due to a constant error of $\pm 0.01g$ in vertical acceleration varies as the square of the time for the calibration run (fig. 7). For a calibration run lasting 20 seconds, the error, as percent of free-stream static pressure, is 0.3, and for 40 seconds, the error is 1.1.



(a) Due to error in longitudinal acceleration.

(b) Due to error in normal acceleration.

FIGURE 4.—Error in vertical component of acceleration due to an error of $\pm 0.01g$ in longitudinal and normal accelerations.

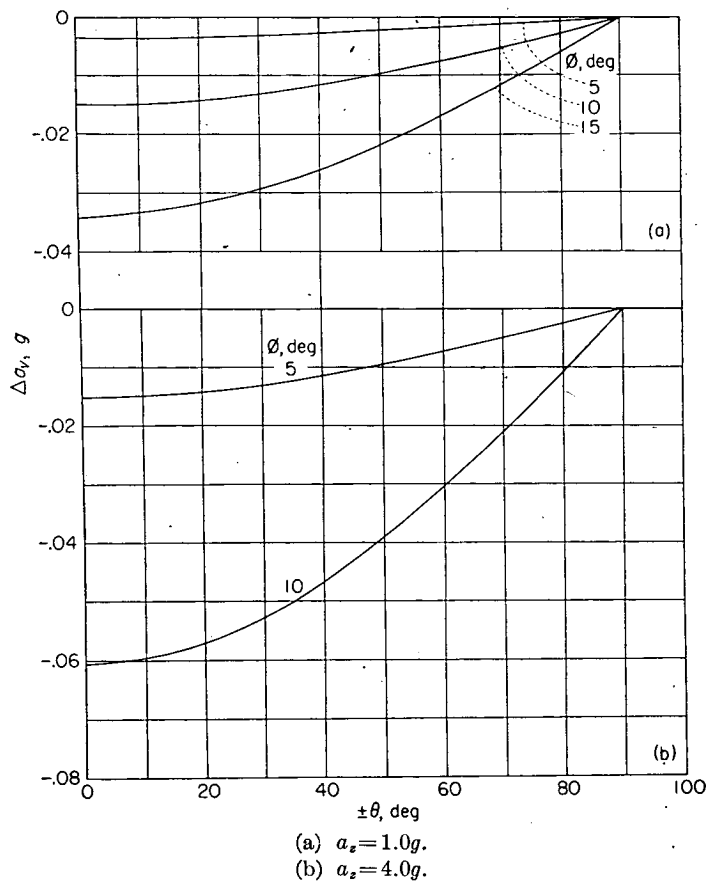


FIGURE 5.—Error in vertical component of acceleration due to neglecting angle of bank.

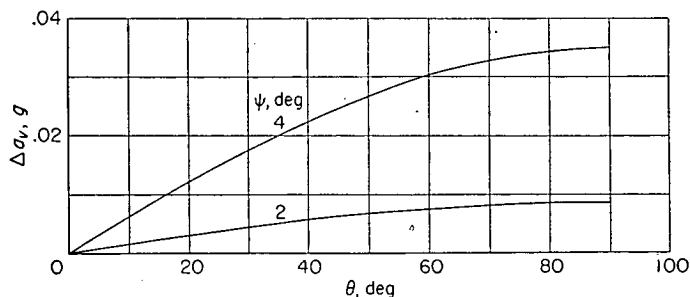


FIGURE 6.—Error in vertical component of acceleration due to neglect of angle of yaw. $a_y = 0.25g$ and $0.50g$ for $\psi = 2^\circ$ and 4° , respectively. $a_z = 0$.

Errors in the time rate of change of static pressure may arise not only from the recording instruments but also from the time rate of change of Mach number and of angle of attack when Mach number and angle of attack affect the airspeed calibration. These Mach number and angle-of-attack effects may be avoided by making the measurements for the determination of vertical velocity v_1 in approximately constant-speed level flight. This type of maneuver will also minimize errors in vertical velocity due to uncertainties in the static-pressure measurements if the method of appendix B (eq. (20)) is used.

The error in vertical velocity due to an error of ± 0.01 inch of water in the time rate of change of static pressure at 40,000 feet is 2.8 feet per second (fig. 8). This error in vertical velocity results in a static-pressure error which is shown as a function of time in figure 9. For calibration runs lasting 20 and 40 seconds the errors in static pressure are about 0.3 and 0.5, respectively.

The error in static pressure due to an error of ± 1 percent in measured temperature (or about $\pm 5^\circ \text{F}$) varies with the range of pressures covered in the calibration (fig. 10). For a range of static pressures from 0.7 to 3 times the initial static pressure, the error is within 1 percent of free-stream static pressure.

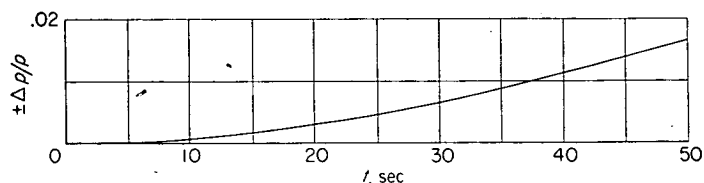


FIGURE 7.—Error in free-stream static pressure due to consistent error of $\pm 0.01g$ in vertical component of acceleration.

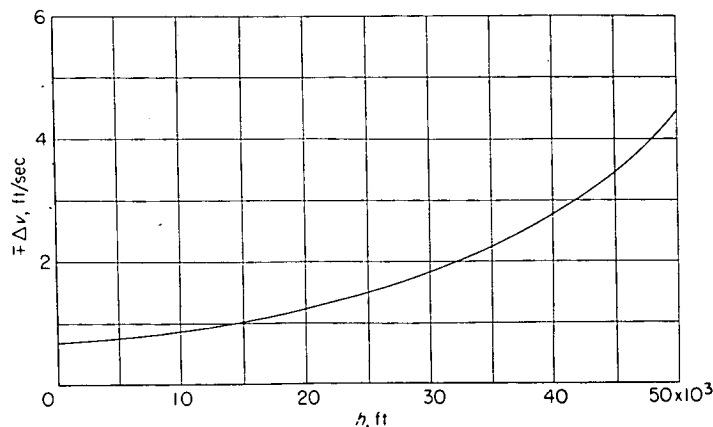


FIGURE 8.—Error in vertical velocity due to error of ± 0.01 inch of water per second in time rate of change of static pressure.

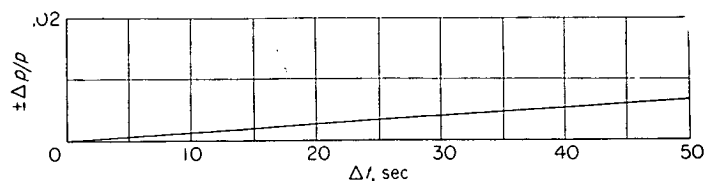


FIGURE 9.—Error in static pressure due to error of ± 0.01 inch of water per second in time rate of change of static pressure. $h \approx 40,000$ feet.

An error in total pressure results in an error in Mach number and, hence, in static pressure as determined from equation (7). Errors in total pressure due, for instance, to angularity of flow may be avoided by the use of properly designed pitot tubes. The error in free-stream static pressure due to an error of ± 1 percent of total pressure is seen in figure 10 to be within 0.3 percent of free-stream static pressure for a range of static pressures from 0.7 to 3 times the initial static pressure.

In evaluating the free-stream static pressure with the use of equation (6), the value of the gas constant may be taken to be 53.3, the value for dry air. The resulting error due to neglecting moisture content may be shown to be negligible. For example, if the air were saturated, the use of the gas constant for dry air would introduce an error of about 0.2 percent of free-stream static pressure in a dive from an altitude of 10,000 feet to sea level and of 0.01 percent in a dive from 30,000 feet to 20,000 feet.

The error in static pressure due to error in a standard NACA static-pressure recorder is $\pm \frac{1}{4}$ percent of the full-scale reading. For a static-pressure recorder covering a range from sea level to 50,000 feet, the error in static pressure is ± 1 inch of water or ± 1.3 percent of static pressure at an altitude of 40,000 feet. For a static-pressure recorder covering a range of altitudes above 30,000 feet, the error is ± 0.2 inch of water or ± 0.3 percent of static pressure at 40,000 feet. Further improvement in accuracy of the static-pressure measurements may be obtained with the use of a statoscope equipped with a differential-pressure recorder having a range to cover the change in static pressure over the desired range of altitudes. The error in Mach number due to an error of ± 1 percent of static pressure is shown in figure 11.

The elapsed time t may be measured to within 0.01 percent with the use of a tuning-fork timer. The static-pressure error corresponding to this error in time is (according to eq. (50)) within 0.05 percent of free-stream static pressure for a range of altitudes ($h-h_1$) of 50,000 feet. Since all instruments can be of the continuous-recording type, no

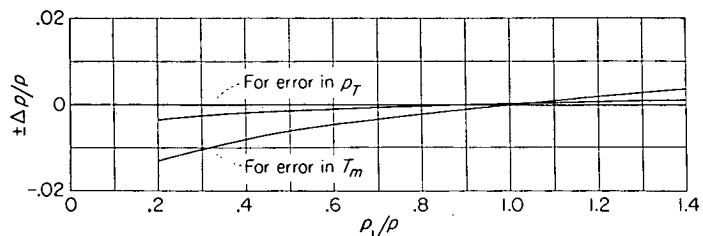


FIGURE 10.—Error in free-stream static pressure due to ± 1 -percent error in measured temperature and ± 1 -percent error in total pressure.

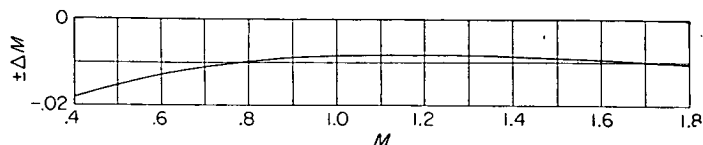


FIGURE 11.—Error in Mach number corresponding to ± 1 -percent error in static pressure.

consistent error should result from the correlation of these records.

When a calibration run begins and ends near level flight at a speed for which the calibration is known, a check on the constant errors in calibration (due to errors in acceleration or attitude angle) is obtained in that the vertical velocity at the end of the calibration run should be equal to vertical velocity at the start plus the time integral of the vertical acceleration. A consistent error in the static-pressure recorder does not affect the determination of the static-pressure error since the consistent error would be included in the computed free-stream static pressure as well as in the measured pressure.

The effect of the various errors on a calibration was evaluated for a hypothetical maneuver involving a level-flight run of 4 seconds followed by a 30° dive lasting 30 seconds on the assumption that standard NACA instruments were used. This study indicated that the various quantities could be measured accurately enough with standard instruments to insure a satisfactory calibration.

FLIGHT EVALUATION OF METHOD

An experimental verification of the conclusions reached in the analytical study of the effect of errors on the calibrations was considered desirable. Flight tests were therefore made to obtain a calibration of an airspeed installation on a jet fighter airplane by the accelerometer method. A calibration was also made at the same time by the radar-phototheodolite method for comparison. These flight tests are described in the following sections.

EQUIPMENT

The jet-powered fighter airplane used for the calibration tests was equipped with a pitot-static tube mounted on a boom about 1 fuselage maximum diameter ahead of the fuselage nose. A resistance-type free-stream thermometer equipped with two radiation shields was mounted about $\frac{2}{3}$ fuselage maximum diameter ahead of the nose on the airspeed boom. The thermometer had a recovery factor very nearly 1.0 and a time lag of about $\frac{1}{10}$ second for the altitude and speed range at which the tests were made. The thermometer and pitot-static head are shown mounted on the boom in figure 12.

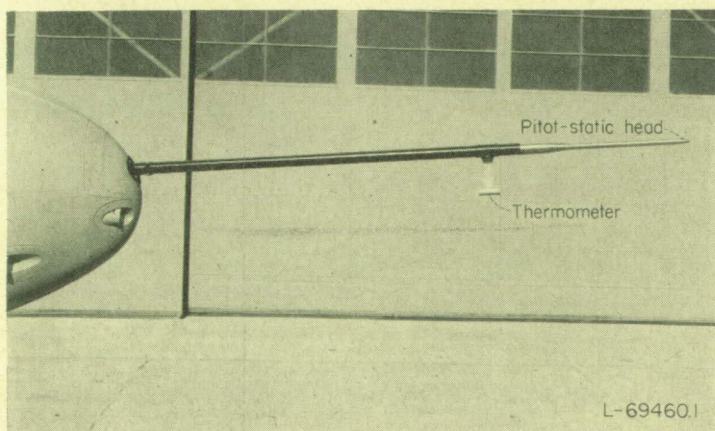


FIGURE 12.—Pitot-static head and thermometer mounted on the fuselage nose boom.

The instruments installed in the airplane and the range of each instrument are as follows:

Thermometer.....	—40° to 40° F
Static-pressure recorder.....	95 to 422 inches of water
Impact-pressure recorder.....	30 to 84 inches of water
Normal accelerometer.....	0 to 2g
Normal accelerometer.....	2 to 4g
Longitudinal accelerometer.....	0.5 to 0.2g
Sun camera.....	30°

All of these instruments recorded the measurements continuously on films which were synchronized by a $\frac{1}{10}$ -second timer. An identification code, which synchronized the radar measurements with the measurements taken in the airplane, was transmitted to the ground radar station by the aircraft radio.

The sun camera was mounted in the airplane below an opening in the skin ahead of the pilot's canopy. The camera was designed to record continuously the attitude of the airplane relative to the sun. A simple sundial, shown in figure 13, was installed to aid the pilot in maintaining the lateral axis of the airplane normal to the rays of the sun.

The time of the start or end of a test was determined by means of an ordinary watch checked against time obtained from radio station WWV operated by the National Bureau of Standards.

The radar-phototheodolite ground equipment was the same as that described in reference 1.

ACCURACY OF RECORDING EQUIPMENT

In order to utilize adequately the accuracy of the instruments, much care was taken in the calibration of the instruments and in the reading of the film.

A flight test was made prior to the airspeed-calibration tests to check the measurements of free-stream temperature by using two thermometers and recording galvanometers of the same design. The recorders indicated an occasional difference in temperature of no more than $\frac{1}{2}$ ° F. The errors in temperature resulting from lag in the thermometer varied with the rate of change of measured temperature. The average error was only about —0.1° F in the dives; therefore, no corrections were applied since the errors were considered negligible.

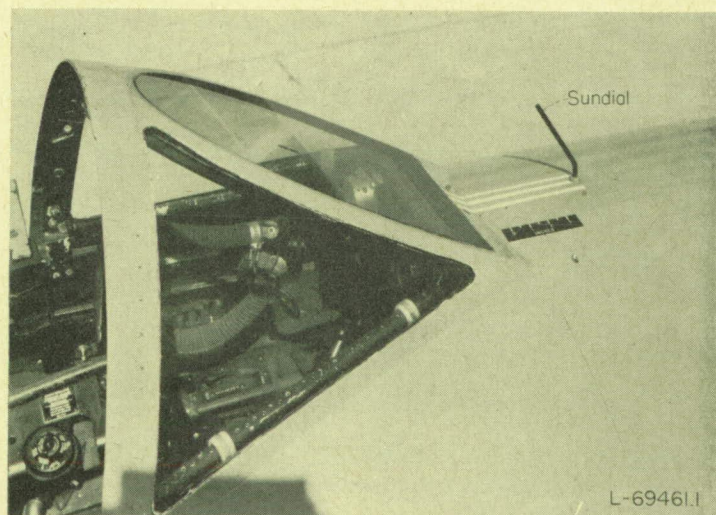


FIGURE 13.—Sundial used by the pilot as an aid in aligning the airplane with the sun.

The static-pressure recorder was specially built and had a reading accuracy of about ± 0.05 inch of water. A calibration of the static-pressure recorder with increasing and then decreasing pressure indicated a hysteresis loop of about ± 0.5 inch of water. The accuracy of the static-pressure recorder, however, is believed to be better than this value indicates since the diaphragm was put in a rested state by applying and releasing a suction of about 350 inches of water several times immediately before flight and since the static-pressure recorder was calibrated immediately after flight by using a pressure-time sequence approximating the flight tests.

The impact-pressure recorder had an accuracy greater than about ± 0.1 inch of water. The accuracy of the recorder is well within the precision required by the accelerometer method since impact pressure affects only the ratio of static-pressure error to impact pressure, the determination of Mach number M' , and the temperature T' .

The time lag in the static-pressure line connected to the static-pressure recorder was estimated to be 0.05 second for the altitude of the tests. Since this time lag corresponded to a lag in static pressure of less than 0.1 inch of water or $0.001 q_c'$ for the maximum rate of change of static pressure occurring in the maneuvers, no correction was applied. The time lag in the total-pressure line was estimated to be 0.03 second. The effect of the lag in the total-pressure and static-pressure lines on impact pressure was negligible.

A calibration of the normal accelerometer indicated effects of longitudinal acceleration and of temperature for which corrections were applied. Consistent errors in normal acceleration due to zero shift in the instrument are believed to be eliminated by use of the method described in appendix B. The normal accelerometer had a constant uncertainty of about ± 0.2 percent of the change of normal acceleration from 1 g . Uncertainty of the longitudinal accelerometer zero is believed to be about $\pm 0.002g$.

The sun camera had a reading accuracy of about 0.07° and its setting relative to the axes of the accelerometers could be measured to within about 0.2° . Although the solar time was determined to within 5 seconds, the time was taken at the midpoint of each maneuver since the resulting error in the elevation angle of the sun at the beginning and end of the maneuver was estimated to be small (less than 0.1°).

TESTS

The tests consisted of three shallow dives from an altitude of 31,000 feet to an altitude of about 26,000 feet. Two dives of about $\frac{1}{2}$ -minute duration covered a Mach number range from about 0.6 to 0.8 with 2 and $4g$ pull-ups, and the third dive of about $1\frac{3}{4}$ -minutes duration covered a range of Mach numbers from about 0.40 to 0.80 with a $3g$ pull-up. Prior to each of the short dives, a survey of atmospheric pressure for the radar-phototheodolite method was made in a climb at an airplane Mach number of about 0.45 and records were taken about every 500 feet between altitudes of 23,000 and 31,000 feet. Continuous measurements were made during the dives and pull-ups. Radar-phototheodolite equipment was not used for the third dive. The pilot attempted to hold the lateral axis of the airplane normal to the sun's rays through the use of the sundial.

EVALUATION OF MEASUREMENTS

The calibration of the airspeed installation by the radar-phototheodolite method was made as described in reference 1. Free-stream static pressure was determined from surveys of atmospheric pressure by using the static-pressure error determined in previous tests with a trailing airspeed head up to a Mach number of about 0.40. The surveys were made at a Mach number of about 0.45; therefore, extrapolation of the static-pressure error obtained from the tests with trailing airspeed head for the approximate Mach number range from 0.40 to 0.45 was necessary.

For the purpose of evaluation, the dives were divided into two parts. Data from the first part of each dive were used to determine the vertical velocity v_1 at the beginning of the last part of the dive as described in appendix B. The static-pressure error was evaluated only for the last part of the dive.

The static-pressure error determined for corresponding flight conditions from results obtained with the trailing airspeed head and the radar method was used in computing the vertical velocity v_1 from data taken during the first 12 seconds of dive 1, the first 13 seconds of dive 2, and the first 45 seconds of dive 3.

The free-stream static pressure p_1 in equation (7) was obtained from the static pressure measured at the beginning of the last part of the dives by using the static-pressure error as determined from the calibration by the radar method. Since the results of the radar calibration were obtained from Mach numbers of 0.57 to 0.78, the results of dive 3 were evaluated by the accelerometer method starting at the time at which a Mach number of 0.57 was attained (57 sec from end of maneuver).

By using vertical velocity v_1 and static pressure p_1 thus determined, the static-pressure error was evaluated for the last 22, 24, and 57 seconds of dives 1, 2, and 3, respectively.

RESULTS AND DISCUSSION

The results of the calibrations by both methods are presented as plots of $\frac{p_m - p}{q_c'}$ against indicated Mach number M' in figure 14. The static-pressure error determined by the accelerometer method for dives 1 and 2 and subsequent pull-outs was nearly constant at about 2.5 percent of impact pressure with very little scatter of data over the range of Mach numbers used in the evaluation (0.65 to 0.78). The results of dive 3, evaluated over a much larger time interval and Mach number range (0.57 to 0.78), agreed closely with the results of the other dives up to the start of the pull-out, after which the results for dive 3 showed a few points that were lower than the average for the other dives by as much as 1 percent of impact pressure. Because of the absence of similar effects in pull-outs following dives 1 and 2, this deviation is not considered to be the effect of airplane lift coefficient on the calibration. The deviation is, however, within the accuracy accepted for most calibrations. Uncertainties of the airspeed calibration, due to the estimated errors of several sources, varied from zero near the beginning of dive 3 to maximum values near the end of the dive. These maximum values are shown in table I. Since the calculations for two of the sources were necessarily probable errors and for

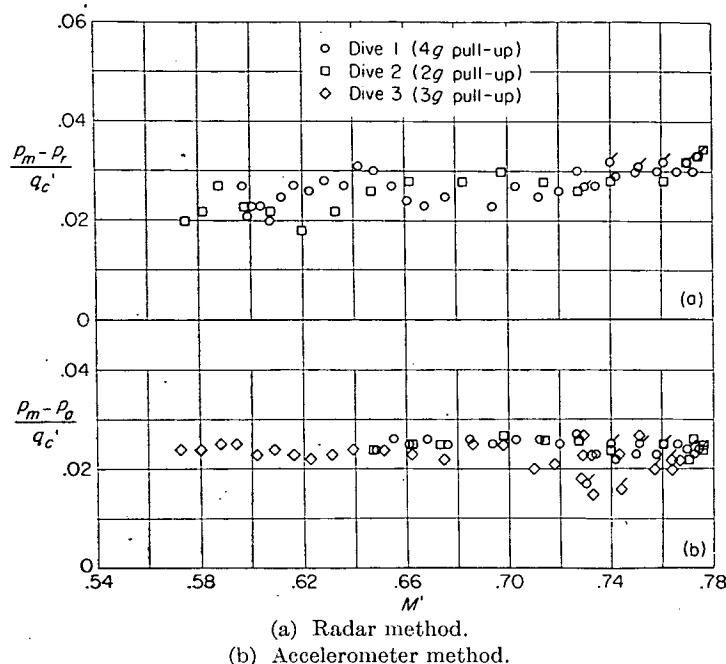


FIGURE 14.—Airspeed calibration as evaluated by the radar and accelerometer methods. Flagged symbols are in the pull-out where $a_z \geq 2g$.

the remaining sources were necessarily nominal maximum possible errors, no attempt was made to compare the data with any combination of these uncertainties.

The static-pressure error as determined by the radar method increased from about 2.5 percent of impact pressure at a Mach number of 0.57 to a little over 3.0 percent at a Mach number of 0.78. The scatter of about ± 0.5 percent at the low Mach numbers and ± 0.2 percent at the high Mach numbers is about $\frac{1}{2}$ the maximum possible scatter due to inaccuracies of measuring static pressure and height by radar (± 45 ft for slant range and ± 0.2 mil for elevation angle). On the basis of the faired data, the results of the radar-method calibration show the greatest difference (0.5 percent) from the accelerometer-method calibration at the high Mach numbers. Differences of this magnitude have been noted between two tests for a radar calibration in reference 1, although in the present tests there was no consistent difference between the two successive dives. It should

TABLE I.—ESTIMATED ERRORS IN AIRSPEED CALIBRATION NEAR THE END OF DIVE 3 DUE TO VARIOUS SOURCES OF CONSTANT ERROR IN THE EVALUATION BY THE ACCELEROMETER METHOD

Source	Source error	Error in airspeed calibration (percent impact pressure)
Initial velocity v_0 determined by least-squares method	± 0.09 ft/sec (probable error)	± 0.04 (probable)
Zero of normal accelerometer determined by least-squares method	± 0.0053 ft/sec ² (probable error)	± 0.21 (probable)
Sensitivity of the normal accelerometer	± 0.002 ($a_z - g$)	± 0.11
Zero of longitudinal accelerometer	± 0.064 ft/sec ²	± 0.08
Attitude angle	$\pm 0.2^\circ$	± 0.02
Temperature	$\pm \frac{1}{2}^\circ$ F	± 0.25
Static pressure	$\pm \frac{1}{4}$ in. of water	± 0.46

be noted that the maximum uncertainty in the accelerometer method, due to the estimated possible error in static pressure (shown in table I), is about the same magnitude.

The calibration, as determined by both the radar and accelerometer methods, is typical of nose-boom installations inasmuch as there was little effect of either Mach number or lift coefficient over the ranges covered in these tests.

CONCLUDING REMARKS

A method is described for calibrating airspeed installations on airplanes at transonic and supersonic speeds in vertical-plane maneuvers in which use is made of measurements of normal and longitudinal accelerations and attitude angle. The method involves starting or ending a calibration run near level flight at a speed for which the airspeed calibration is known and, hence, for which the free-stream static pressure may be determined. Integration of the vertical acceleration computed from the normal and longitudinal accelerations and the attitude angle determines the change in altitude which, when combined with the temperature measurements, gives the change in static pressure from the start or end of the calibration run and, hence, the variation of free-stream static pressure during the calibration run. The static-pressure error is then obtained at any instant during the calibration run by subtracting the free-stream static pressure from the indicated static pressure.

In the method described herein the required instrumentation is carried within the airplane. Should the airplane at any time enter a previously unexplored flight condition in a vertical-plane maneuver, a calibration may be readily obtained.

An analytical study of the effects of various sources of error on the accuracy of an airspeed calibration by the accelerometer method indicated that the required measurements can be made accurately enough to insure a satisfactory calibration.

In order to obtain an experimental verification of the analytical study, flight tests were made and an airspeed calibration of a jet fighter airplane was evaluated using the accelerometer method and, for comparison, the radar-phototheodolite method. The tests included shallow dives up to a Mach number of about 0.80 with pull-ups of about 4, 3, and 2g normal acceleration. The calibrations of the dives by the two methods are typical of a nose-boom installation inasmuch as there was little effect of either Mach number or lift coefficient over the ranges covered in these tests. The static-pressure error as determined by the radar-phototheodolite method increased from about 2.5 percent of impact pressure at a Mach number of 0.57 to a little over 3.0 percent at a Mach number of 0.78. The static-pressure error determined by the accelerometer method was nearly constant at about 2.5 percent of impact pressure over the same Mach number range. From the results of the tests it appears that, for vertical-plane maneuvers, the accelerometer method may be used as an alternate to the radar-phototheodolite method.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., Dec. 5, 1952.

APPENDIX A

CALCULATION OF SUITABLE VALUES OF n FOR USE IN EQUATION (7)

The value of n that yields zero or nearly zero error in the computed free-stream static pressure as a result of using p_m and M' in equation (7) may be found by first differentiating equation (6) as

$$\frac{\Delta p}{p} = -\frac{1}{p^n} \int \frac{p^n}{RT} (ne_p - e_T) dh \quad (13)$$

where e_p in the integral is the error in the static-pressure source and $\Delta p/p$ on the left side of the equation is the error in the computed free-stream static pressure due to use of p_m and M' . Also in the integral e_T is the error in T due to the use of M' . For zero error in computing free-stream static pressure

$$ne_p = e_T$$

or

$$n = \frac{e_T}{e_p}$$

For $M \leq 1.0$, the value of $\frac{e_T}{e_p}$ may be determined from equation (4) and the equation

$$p_T = p \left(1 + \frac{\gamma-1}{2} M^2 \right)^{\frac{\gamma}{\gamma-1}} \quad (14)$$

Differentiating each equation and combining results in the following expression:

$$n = \frac{e_T}{e_p} = \frac{p}{T} \frac{dT}{dp} = \frac{\gamma-1}{\gamma} \left(\frac{1 + \frac{\gamma-1}{2} M^2}{1 + \frac{\gamma-1}{2} K M^2} \right) K \quad (15)$$

For $M \geq 1.0$ the value of n may be similarly computed from equation (4) and the equation

$$p_T = \frac{\gamma+1}{2} M^2 p \left[\frac{(\gamma+1)^2 M^2}{4\gamma M^2 - 2(\gamma-1)} \right]^{\frac{1}{\gamma-1}} \quad (16)$$

or

$$n = \frac{\gamma-1}{2\gamma} \frac{2\gamma M^2 - (\gamma-1)}{2M^2 - 1} \frac{M^2 K}{1 + \frac{\gamma-1}{2} K M^2} \quad (17)$$

The values of n for $K=0.9$ and $K=1.0$ are tabulated for various Mach numbers as follows:

M	n	
	K=0.9	K=1.0
0.5	0.258	0.286
1.0	.261	.286
1.5	.347	.374
2.0	.461	.490
3.0	.644	.670
4.0	.759	.779

Since, in the general case, a range of Mach number would be covered in a calibration test, a mean value of n may be taken. The possible error in the computed free-stream static pressure resulting from the use of the mean value for a range of Mach numbers from $M=1.0$ to $M=2.0$ was estimated to be less than 3 percent of the static-pressure error of the airspeed installation.

APPENDIX B

DETERMINATION OF INITIAL VERTICAL VELOCITY AND ZERO SHIFT IN A NORMAL ACCELEROMETER

If a run in, or near, level flight is made prior to, and continuous with, the calibration maneuver, the initial vertical velocity v_0 at the start of this run may be determined from integrations of the accelerometer measurements and change in geometric height computed from pressure and temperature measurements by using the equation

$$-\int_{p_0}^p \frac{RT}{p} dp = v_0 t + \int_0^t \int_0^t a_v dt dt \quad (18)$$

This method of determining vertical velocity v_0 , however, may introduce errors due to errors in vertical acceleration. An appreciable source of error in the airspeed calibration by the accelerometer method may be the zero shift of the normal accelerometer. This error can be corrected at the same time that v_0 is determined. The vertical acceleration may be written as

$$a_v = a_v' + \Delta a_v \quad (19)$$

where a_v' is indicated vertical acceleration and Δa_v is a con-

stant error in vertical acceleration. In, or near, level flight

$$\Delta a_v \approx \Delta a_z \quad (20)$$

Equation (18) may therefore be rewritten as

$$-\int_{p_0}^p \frac{RT}{p} dp - \int_0^t \int_0^t a_v' dt dt = v_0 t + \Delta a_z \frac{t^2}{2} \quad (21)$$

A solution of this equation which contains two unknowns, v_0 and Δa_z , may be determined by satisfying the equation over two time intervals. A better approach is to use the method of least squares with a large number of time intervals.

Once v_0 (and Δa_z) is determined, the vertical velocity v_1 at the start of the calibration maneuver may be determined as

$$v_1 = v_0 + \int_0^{t_1} a_v dt \quad (22)$$

Although the preceding discussion refers specifically to a maneuver starting in, or near, level flight, the method is also applicable to a maneuver ending in, or near, level flight.

APPENDIX C

CALCULATION OF ERRORS

ERROR IN ATTITUDE ANGLE DUE TO ERROR IN TIME OF CALIBRATION

An error in the time of calibration results in an error in the elevation angle of the sun and, hence, in the attitude angle. The error in the elevation angle of the sun is, after differentiation of equation (12),

$$\Delta\lambda = -\frac{\cos \epsilon \cos \tau}{\cos \lambda} \sin(\omega - \beta) \Delta\omega \quad (23)$$

or

$$\Delta\lambda = -\frac{\cos \epsilon \cos \tau}{\cos \lambda} \sin(\omega - \beta) \frac{15}{3600} \Delta t \quad (24)$$

where

$$\Delta\omega = \frac{15}{3600} \Delta t$$

and Δt is the error in time in seconds.

Since

$$\Delta\lambda = -\Delta\theta$$

$$\Delta\theta = 0.00416 \frac{\cos \epsilon \cos \tau}{\cos \lambda} \sin(\omega - \beta) \Delta t \quad (25)$$

The error in attitude angle due to an error of 60 seconds in time of the calibration is plotted in figure 1.

ERROR IN ATTITUDE ANGLE DUE TO ERROR IN LATITUDE AND LONGITUDE OF THE AIRPLANE

The error in attitude angle due to error in the latitude of the airplane is

$$\Delta\theta = -\Delta\lambda = \left[\frac{-\sin \epsilon \cos \tau + \cos \epsilon \sin \tau \cos(\omega - \beta)}{\cos \lambda} \right] \Delta\tau \quad (26)$$

or

$$\Delta\theta = \left[\frac{-\sin \epsilon \cos \tau + \cos \epsilon \sin \tau \cos(\omega - \beta)}{\cos \lambda} \right] \frac{57.3}{4000} s_\beta \quad (27)$$

where

$$\Delta\tau = \frac{57.3}{4000} s_\beta$$

and s_β is in miles.

Similarly, the error in attitude angle due to error in the longitude of the airplane is

$$\Delta\theta = \frac{-\cos \epsilon \sin(\omega - \beta)}{\cos \lambda} \frac{57.3}{4000} s_\tau \quad (28)$$

where s_τ is in miles.

The error in attitude angle due to error in the latitude and longitude of the airplane is plotted in figure 2.

ERROR IN VERTICAL COMPONENT OF ACCELERATION DUE TO ERROR IN ATTITUDE ANGLE

The error in vertical acceleration due to an error in attitude angle is, after differentiation of equation (10),

$$\Delta a_v = -(a_z \sin \theta + a_x \cos \theta) \Delta\theta \quad (20)$$

where $\Delta\theta$ is in radians. The error in the vertical component of acceleration for a $\pm 1^\circ$ error in attitude angle is shown in figure 3.

ERROR IN VERTICAL COMPONENT OF ACCELERATION DUE TO ERROR IN LONGITUDINAL AND NORMAL COMPONENTS OF ACCELERATION

The error in the vertical component of acceleration due to error in the longitudinal component of acceleration is, after differentiation of equation (10),

$$\Delta a_v = -\Delta a_x \sin \theta \quad (30)$$

Similarly, for an error in the normal component of acceleration

$$\Delta a_v = \Delta a_z \cos \theta \quad (31)$$

The error in the vertical component of acceleration due to an error of $\pm 0.01g$ in the longitudinal and normal components of acceleration is presented in figure 4.

ERROR IN VERTICAL COMPONENT OF ACCELERATION DUE TO ANGLE OF BANK

The expression for vertical acceleration, including the angle of bank, is

$$a_v = a_z \cos \theta \cos \phi - a_x \sin \theta - g \quad (32)$$

where ϕ is the angle of bank.

If the angle of bank is neglected, the error is

$$\Delta a_v = a_z \cos \theta (\cos \phi - 1) \quad (33)$$

The error in the vertical component of acceleration is plotted in figure 5 for angles of bank of 5° , 10° , and 15° .

ERROR IN VERTICAL COMPONENT OF ACCELERATION DUE TO ANGLE OF YAW

The vertical component of acceleration, including the effect of yaw, is

$$a_v = a_z \cos \theta - a_x \sin \theta \cos \psi + a_y \sin \psi \sin \theta - g \quad (34)$$

where ψ is the angle of yaw and a_y is the lateral acceleration due to yaw.

If the angle of yaw is neglected, the error is

$$\begin{aligned}\Delta a_v &= a_x \sin \theta (1 - \cos \psi) + a_y \sin \psi \sin \theta \\ &= [a_x (1 - \cos \psi) + a_y \sin \psi] \sin \theta\end{aligned}\quad (35)$$

This error in the vertical component of acceleration is shown in figure 6 for 2° and 4° of yaw.

ERROR IN FREE-STREAM STATIC PRESSURE DUE TO ERROR IN DETERMINING VERTICAL COMPONENT OF ACCELERATION

The error in free-stream static pressure due to error in the vertical component of acceleration is found by substituting equation (8) into equation (7) and then differentiating the resulting equation, or

$$\frac{\Delta p}{p} = -\frac{1}{p^n} \int_{t_1}^t \left[\frac{p^n (1 + 0.2KM^2)}{RT_m} \int_{t_1}^t \Delta a_v dt \right] dt \quad (36)$$

In the integral Δa_v , p , M , and T_m may vary with time. In order to obtain the order of magnitude of the error, however, it is sufficient to assume these quantities as constant. Then

$$\frac{\Delta p}{p} = -\frac{\Delta a_v}{2RT} t^2 \quad (37)$$

The variation of the error in static pressure with time is shown in figure 7 for an error in vertical acceleration of $\pm 0.01g$.

ERROR IN VERTICAL VELOCITY DUE TO ERROR IN DETERMINING THE TIME RATE OF CHANGE OF STATIC PRESSURE

The error in vertical velocity due to error in determining the time rate of change of static pressure is obtained from equation (9) as

$$\Delta v = -\frac{RT}{p} \Delta \left(\frac{dp}{dt} \right) \quad (38)$$

The error in vertical velocity due to an error of 0.01 inch of water per second in the time rate of change of static pressure is shown in figure 8 for various altitudes.

ERROR IN FREE-STREAM STATIC PRESSURE DUE TO ERROR IN DETERMINING VERTICAL VELOCITY OR THE TIME RATE OF CHANGE OF STATIC PRESSURE

The error in free-stream static pressure due to error in determining vertical velocity is, after substitution of equation (8) into equation (7) and differentiation of the resulting equation,

$$\frac{\Delta p}{p} = -\frac{1}{p^n} \int_{t_1}^t \frac{p^n (1 + 0.2KM^2)}{RT_m} \Delta v_1 dt \quad (39)$$

In evaluating the order of magnitude of error in static pressure, T_m , M , and p may be assumed as constant in the integral and therefore

$$\frac{\Delta p}{p} = -\frac{\Delta v_1}{RT} \quad (40)$$

In terms of error in time rate of change of static pressure, the error in static pressure is, after substitution for Δv from

equation (38),

$$\frac{\Delta p}{p} = \Delta \left(\frac{dp}{dt} \right) \frac{t}{p} \quad (41)$$

This static-pressure error is plotted in figure 9 as a function of time for an altitude of about 40,000 feet and an error of ± 0.01 inch of water in the time rate of change of static pressure.

ERROR IN FREE-STREAM STATIC PRESSURE DUE TO ERROR IN MEASURING T_m

The error in free-stream static pressure due to error in measuring T_m is, after differentiation of equation (7),

$$\frac{\Delta p}{p} = \frac{1}{p^n} \int_{p_1}^p \frac{p^n (1 + 0.2KM^2)}{RT_m} \frac{\Delta T_m}{T_m} dh \quad (42)$$

For $\frac{\Delta T_m}{T_m} = \text{Constant}$, equation (42) reduces to

$$\frac{\Delta p}{p} = \frac{\Delta T_m}{n T_m} \left[\left(\frac{p_1}{p} \right)^n - 1 \right] \quad (43)$$

The error in free-stream static pressure due to an error of ± 1 percent in T_m is plotted in figure 10 against the ratio of initial pressure p_1 to pressure p at any time during the calibration. A value of n of $\frac{\gamma-1}{\gamma}$ or 0.286 was assumed.

ERROR IN FREE-STREAM STATIC PRESSURE DUE TO ERROR IN MEASURING TOTAL PRESSURE

The error in total pressure introduces an error in the computation of temperature and, hence, in the computation of free-stream static pressure. The static-pressure error may be found by differentiating equation (6) as

$$\frac{\Delta p}{p} = \frac{1}{p^n} \int_{h_1}^h \frac{p^n}{RT} \frac{\Delta T}{T} dh \quad (44)$$

If $\frac{\Delta T}{T}$ is assumed to be a constant and is related to $\frac{\Delta p_T}{p_T}$ through the use of equations (4), (14), and (16), the static-pressure error is found to be

$$\frac{\Delta p}{p} = -\frac{\Delta p_T}{p_T} \left[\left(\frac{p_1}{p} \right)^n - 1 \right] \quad (45)$$

The error in static pressure due to ± 1 -percent error in total pressure is plotted in figure 10 against the ratio of initial pressure p_1 to pressure p at any time during the calibration for $n = \frac{\gamma-1}{\gamma}$.

ERROR IN FREE-STREAM STATIC PRESSURE DUE TO ERROR IN ELAPSED TIME

Integration of equation (8) between the limits of h and h_1 yields

$$h - h_1 = v_1 t + \int_0^t \int_0^t a_v dt dt \quad (46)$$

If the measured elapsed time t_m is substituted into equation (46), the equation becomes

$$h_m - h_1 = v_1 t_m + \int_0^{t_m} \int_0^{t_m} a_v dt_m dt_m \quad (47)$$

where h_m is computed altitude corresponding to measured elapsed time t_m . If the error in elapsed time is defined as

$$e_t = \frac{t_m - t}{t}$$

the error in altitude determined from equations (46) and (47) becomes

$$\Delta h = e_t [2(h - h_1) - v_1 t] \quad (48)$$

The corresponding error in free-stream static pressure is

$$\frac{\Delta p}{p} = \frac{e_t}{RT} [v_1 t - 2(h - h_1)] \quad (49)$$

The error in free-stream static pressure is a maximum if v_1 is zero or if v_1 has a direction opposite to the resultant change in altitude. For $v_1 = 0$,

$$\frac{\Delta p}{p} = \frac{2e_t(h_1 - h)}{RT} \quad (50)$$

ERROR IN MACH NUMBER DUE TO ERROR IN FREE-STREAM STATIC PRESSURE

The error in Mach number due to an error in static pressure is, after differentiation of equation (14) for $M \leq 1.0$,

$$\Delta M = -\frac{1 + 0.2 M^2}{1.4 M} \frac{\Delta p}{p} \quad (51)$$

and, after differentiation of equation (16) for $M \geq 1.0$,

$$\Delta M = -\frac{M}{7} \frac{7 M^2 - 1}{2 M^2 - 1} \frac{\Delta p}{p} \quad (52)$$

The error in Mach number due to a ± 1 -percent error in free-stream static pressure is shown in figure 11.

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REPORT 1146

AERODYNAMIC FORCES AND LOADINGS ON SYMMETRICAL CIRCULAR-ARC AIRFOILS WITH PLAIN LEADING-EDGE AND PLAIN TRAILING-EDGE FLAPS¹

By JONES F. CAHILL, WILLIAM J. UNDERWOOD, ROBERT J. NUBER, and GAIL A. CHEESMAN

SUMMARY

An investigation has been made in the Langley two-dimensional low-turbulence tunnel and in the Langley two-dimensional low-turbulence pressure tunnel of 6- and 10-percent-thick symmetrical circular-arc airfoil sections at low Mach numbers and several Reynolds numbers. The airfoils were equipped with 0.15-chord plain leading-edge flaps and 0.20-chord plain trailing-edge flaps. The section lift and pitching-moment characteristics were determined for both airfoils with the flaps deflected individually and in combination. The section drag characteristics were obtained for the 6-percent-thick airfoil with the flaps partly deflected as low-drag-control flaps and for both airfoils with the flaps neutral. Surface pressures were measured on the 6-percent-thick airfoil section with the flaps deflected either individually or in appropriate combination to furnish flap load and hinge-moment data applicable to the structural design of the airfoil.

The experimental results showed maximum lift coefficients of 1.95 and 2.03 for the optimum combinations of deflection of leading-edge flaps and trailing-edge flaps as compared with 0.73 and 0.67 for the plain 6- and 10-percent-thick airfoils, respectively. Scale effect on the maximum lift coefficients was, in general, small. The aerodynamic center was ahead of the quarter-chord point and moved toward the leading edge when either the leading-edge flap or the trailing-edge flap was deflected. Deflecting the leading-edge flap was more effective in extending the low-drag range to higher section lift coefficients than deflecting the trailing-edge flap. The maximum flap normal-force and hinge-moment coefficients were, respectively, 4.74 and 2.24 for the leading-edge flap as compared with 1.48 and -0.61 for the trailing-edge flap.

A generalized method is developed that permits the determination of the chordwise pressure distribution over sharp-edge airfoils with plain leading-edge flaps and plain trailing-edge flaps of arbitrary size and deflection.

INTRODUCTION

Thin sharp-edge wings designed to minimize wave resistance have been proposed for use on high-speed aircraft. If, however, the aircraft is to land safely or to fly satisfactorily in the low-speed range, means must be provided for increasing

the naturally low maximum lift of the sharp-edge airfoils. Because leading-edge separation appears to be the limiting factor, the use of leading-edge high-lift devices is indicated as a possible means for improving the maximum-lift characteristics. An investigation has accordingly been made in the Langley two-dimensional low-turbulence tunnel and in the Langley two-dimensional low-turbulence pressure tunnel of the aerodynamic forces on 6- and 10-percent-thick symmetrical circular-arc airfoil sections and of the aerodynamic loadings on the 6-percent-thick airfoil section at a low Mach number and several Reynolds numbers. The airfoils were equipped with 0.15-chord plain leading-edge flaps and 0.20-chord plain trailing-edge flaps.

The section lift and pitching-moment characteristics were determined for both airfoils with the high-lift devices deflected individually and in combination. The section drag characteristics were obtained for the 6-percent-thick airfoil with the flaps partly deflected as low-drag-control flaps and for both airfoils with the flaps neutral. Surface pressures were measured on the 6-percent-thick airfoil section with the flaps deflected either individually or in appropriate combination to furnish flap load and hinge-moment data applicable to the structural design of the airfoil.

In an effort to provide the designer with additional section-load information, a generalized method has been developed that permits the determination of the chordwise pressure distribution over sharp-edge airfoils with plain leading-edge flaps and plain trailing-edge flaps of arbitrary size and deflection.

COEFFICIENTS AND SYMBOLS

c_l	airfoil section lift coefficient, l/qc
$c_{l_{\delta}}$	change in ideal lift coefficient caused by flap deflection
c_{l_a}	airfoil section additional lift coefficient due to angle of attack, $c_l - c_{l_{\delta}}$
$\Delta c_{l_{max}}$	increment of maximum section lift coefficient due to flap deflection
c_d	airfoil section drag coefficient, d/qc
$c_{m_{c/4}}$	airfoil section pitching-moment coefficient about quarter chord, Pitching moment about quarter chord/ qc^2

¹ Based on recently declassified NACA RM L6K22, "Two-Dimensional Wind-Tunnel Investigation at High Reynolds Numbers of Two Symmetrical Circular-Arc Airfoil Sections With High-Lift Devices," by William J. Underwood and Robert J. Nuber, 1947; NACA RM L7H04, "Aerodynamic Load Measurements Over Leading-Edge and Trailing-Edge Plain Flaps on a 6-Percent-Thick Symmetrical Circular-Arc Airfoil Section," by William J. Underwood and Robert J. Nuber, 1947; and NACA RM L50H17a, "A Method for Predicting the Low-Speed Chordwise Pressure Distribution Over Sharp-Edge Airfoil Sections With Plain Flaps at the Leading and Trailing Edges," by Robert J. Nuber and Jones F. Cahill, 1950.

c_{mac}	airfoil section pitching-moment coefficient about aerodynamic center, Pitching moment about aerodynamic center/ qc^2
c_n	flap section normal-force coefficient, n/qc_f
c_c	flap section chord-force coefficient, x'/qc_f
c_h	flap section hinge-moment coefficient, h/qc_f^2
l	airfoil lift per unit span
d	drag per unit span
m	pitching moment per unit span
n	flap normal force per unit span, positive upward
x'	flap-chord force per unit span, positive toward trailing edge
h	flap hinge moment per unit span, positive when trailing edge tends to deflect downward or leading edge upward
S	surface-pressure coefficient, in incompressible flow, $\frac{H_0 - p}{q} = \left(\frac{v}{V}\right)^2$
P	pressure-difference coefficient across airfoil, $\frac{p_u - p_L}{q}$
p	local static pressure
H_0	free-stream total pressure
q	free-stream dynamic pressure, $\rho V^2/2$
V	free-stream velocity
v	local velocity on surface of basic uncambered airfoil at zero angle of attack
v'	incremental local velocity on airfoil surface due to separation
v_b	effective local velocity on surface of basic airfoil at any given lift coefficient
Δv_a	additional local velocity on airfoil surface due to departure from ideal lift coefficient
c	airfoil chord with all flaps neutral
c_f	flap chord
t	airfoil thickness
x	distance behind leading edge, in.
y	distance above or below chord, in.
$E = \frac{c_f}{c}$	
α_0	airfoil section angle of attack, deg
$\Delta\alpha_{c_{l_{max}}}$	increment of section angle of attack at maximum lift due to flap deflection
δ	flap deflection, positive when deflected below chord line, deg
ρ	free-stream density
R	Reynolds number
M	Mach number
Subscripts:	
N	leading-edge flap
F	trailing-edge flap
i	ideal
U	upper surface
L	lower surface
$b\delta$	refers to conditions at ideal lift coefficient with flap deflected
a	refers to difference between conditions at ideal lift coefficient and any arbitrary lift coefficient

MODELS

Two symmetrical circular-arc airfoil sections with thicknesses of 6 percent and 10 percent are discussed. Ordinates of the 6- and 10-percent-thick circular-arc airfoil sections are given in tables I and II, respectively. Both of the circular-arc airfoil models had a 24-inch chord and a 35.5-inch span and were made of steel. Each model was equipped with a 0.20-chord plain trailing-edge flap and a 0.15-chord plain leading-edge flap which were pivoted on leaf hinges mounted flush with the lower surface. The flaps of the 6-percent-thick airfoil were made of brass and those of the 10-percent-thick airfoil were made of duralumin. Sketches of the models are presented in figure 1. After the force tests were complete, pressure orifices were installed on the 6-percent-thick model at the midspan in a single chordwise row. The chordwise positions of these orifices are given in figure 2. Model end plates were used to facilitate setting the deflection of the plain leading-edge flap and plain trailing-edge flap. Figure 3 shows photographs of the model with and without model end plates.

The models were designed so that trailing-edge-flap deflections δ_F up to 60° and leading-edge-flap deflections δ_N up to 50° could be obtained. The flaps were sealed at the hinge line by having the flap skirt in rubbing contact with the flap. When the trailing-edge flap of the 6-percent-thick airfoil was deflected beyond 50° , the gap between the flap and skirt was sealed with modeling clay to prevent leakage.

For all tests, the surfaces of the models were finished with No. 400 carborundum paper to produce smooth surfaces; slight discontinuities, however, still existed at the leaf hinges on the lower surfaces and at the line of contact between the flaps and flap skirts.

TESTS

A summary of the tests made on the two airfoil sections is given in table III showing the model configurations and

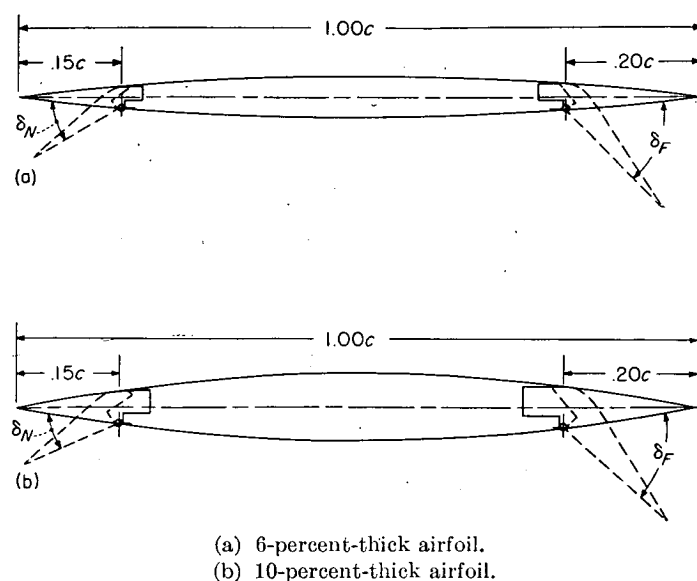
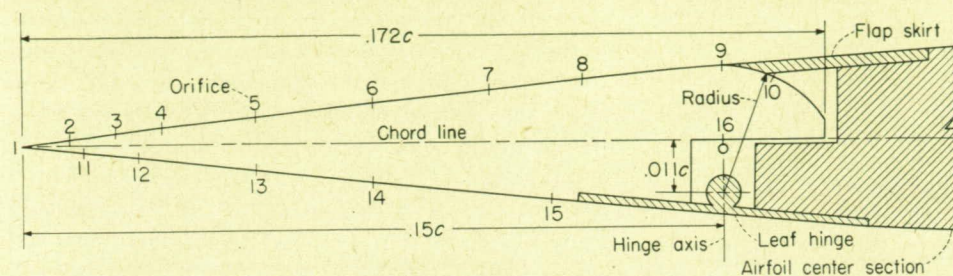
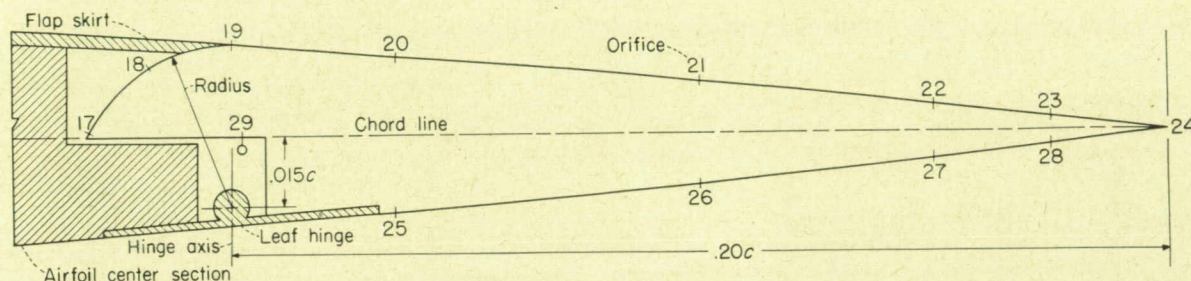


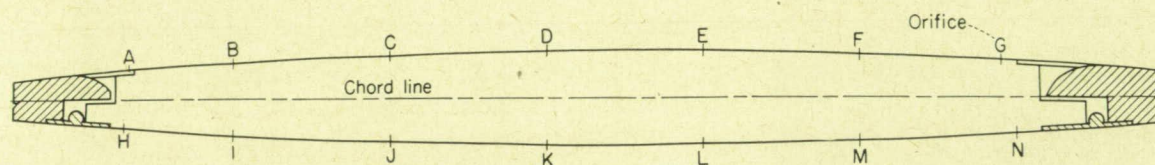
FIGURE 1.—Symmetrical circular-arc airfoils with plain leading-edge flaps and plain trailing-edge flaps.



Plain leading-edge flap



Plain trailing-edge flap



Airfoil center section

FIGURE 2.—Location of pressure orifices on 6-percent-thick airfoil with a 0.15-chord plain leading-edge flap and a 0.20-chord plain trailing-edge flap.

Plain Leading-Edge Flap

Orifice	x/c	y/c
1	0	0
2	1	.11
3	2	.23
4	3	.35
5	5	.57
6	7.5	.84
7	10	1.08
8	12	1.28
9	15	1.47
10	16.1	1.17
11	1.3	-.15
12	2.6	-.30
13	5	-.57
14	7.5	-.83
15	11.4	-1.21
16	15	-.11

Plain Trailing-Edge Flap

Orifice	x/c	y/c
17	77.03	0.25
18	78.3	1.54
19	80	1.87
20	85	1.53
21	90	1.08
22	95	.57
23	97.5	.29
24	100	0
25	85	-1.53
26	90	-1.08
27	95	-.57
28	97.5	-.29
29	80.3	-.15

Airfoil Center Section

Orifice	x/c
A	18.3
B	25
C	35
D	45
E	55
F	65
G	74
H	18.1
I	25
J	35
K	45
L	55
M	65
N	75

the figures in which the data are presented. The airfoil lift, drag, and pitching moment were measured and corrected to free-air conditions by the methods described in reference 1. The flap section normal-force, chord-force, and hinge-moment coefficients were obtained from mechanical integration of the pressure distributions. Lift measurements of the models with the flaps neutral, with and without model end plates, indicated that the model end plates had no significant effect on the measured characteristics.

RESULTS AND DISCUSSION

AIRFOILS WITH FLAPS NEUTRAL

The section aerodynamic characteristics of the 6- and 10-percent-thick symmetrical circular-arc airfoils with the flaps neutral are presented in figure 4.

The maximum section lift coefficients are 0.73 and 0.67 for the 6- and 10-percent-thick airfoils, respectively. This decrease in maximum section lift coefficient with increasing airfoil thickness is opposite to the trends that are shown by

the data for NACA 6-series airfoils (ref. 1) through the same thickness range, but it is believed to be explainable in the following manner: As the thickness of the NACA 6-series airfoils is increased from 6 to 10 percent, the corresponding increase in the airfoil leading-edge radius results in improved air-flow conditions around the leading edge at the high angles of attack. The increase in trailing-edge angle that results from increasing thickness tends to decrease the maximum section lift coefficient due to an increase in boundary-layer thickness on the upper surface. The favorable effect of a large leading-edge radius appears to predominate in this thickness range for the NACA 6-series airfoils and higher values of maximum lift are produced. For the circular-arc airfoils, however, the leading edges of both the 6- and 10-percent-thick airfoils are sharp and the air-flow conditions around the leading edges at high angles of attack are about the same. The effect of an increase in trailing-edge angle with increasing thickness therefore is a decrease of maximum lift.

The lift-curve slopes are 0.097 and 0.090 for the 6- and 10-percent-thick airfoils, respectively. Because the air-flow

conditions around the leading edge of both circular-arc airfoils are probably very nearly alike through the complete range of angle of attack, the thicker boundary layer of the 10-percent-thick airfoil is probably the cause of the decrease in the lift-curve slope. The slope of the lift curve for the 10-percent-thick airfoil was measured at small positive or negative values of the lift coefficient to avoid including the slight jog in the lift curve that occurs near zero lift. This jog in the lift curve has been noticed before in connection with sharp leading-edge airfoils (ref. 2) and appeared when the trailing-edge angle became large. Although a similar phenomenon may have existed on the 6-percent-thick airfoil, it was not of sufficient magnitude to be noticeable in the lift curve. The data (fig. 4) show no appreciable scale effect on the lift characteristics of either circular-arc airfoil with the flaps neutral through the range of Reynolds number investigated.

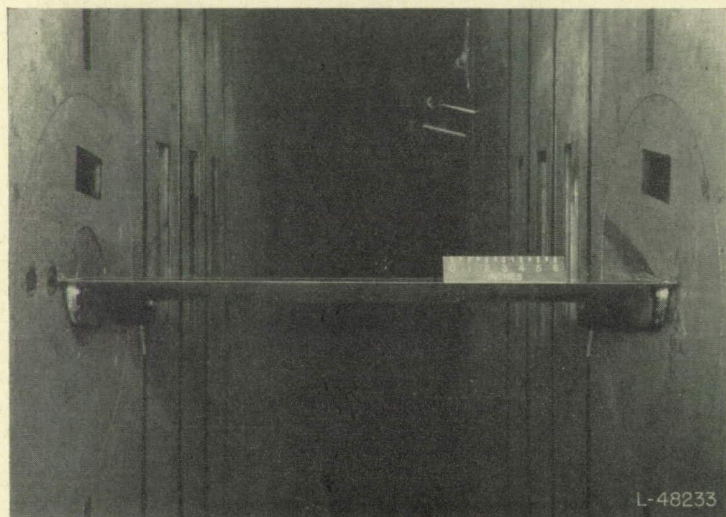
The variation of the quarter-chord pitching-moment coefficient of both the 6- and 10-percent-thick circular-arc airfoils indicates a forward position of the aerodynamic center with respect to the quarter-chord point of the airfoil. This variation of the pitching moment probably results from the relative thickening of the boundary layer near the trailing edge on the upper surface with increasing angle of attack. The aerodynamic center of the 10-percent-thick airfoil is more forward than that of the 6-percent-thick airfoil. This shift in aerodynamic-center position is in fair quantitative agreement with data presented in reference 3 which show that increases in trailing-edge angle or in the thickness of the rear portion of an airfoil cause the aerodynamic-center position to move forward. As is usually true when an airfoil stalls, the center of pressure of the circular-arc airfoils moves toward the rear and the quarter-chord moment coefficient increases negatively in the normal manner. The small negative pitching moment of both models at zero lift is attributed to asymmetrical loading resulting from very small model irregularities.

For airfoils having sharp leading edges, the drag coefficient increases fairly rapidly as the angle of attack departs from zero. In general, the drag coefficients decrease with increasing Reynolds number in approximately the manner expected for fully developed turbulent flow on both surfaces. In the case of the 6-percent-thick airfoil, however, laminar flow apparently was obtained over a fairly extensive portion of the upper surface at zero and negative angles of attack at Reynolds numbers of 3×10^6 and 6×10^6 , as indicated by the lower drag for these conditions as compared with the drag obtained at a Reynolds number of 9×10^6 .

AIRFOILS WITH FLAPS DEFLECTED INDIVIDUALLY

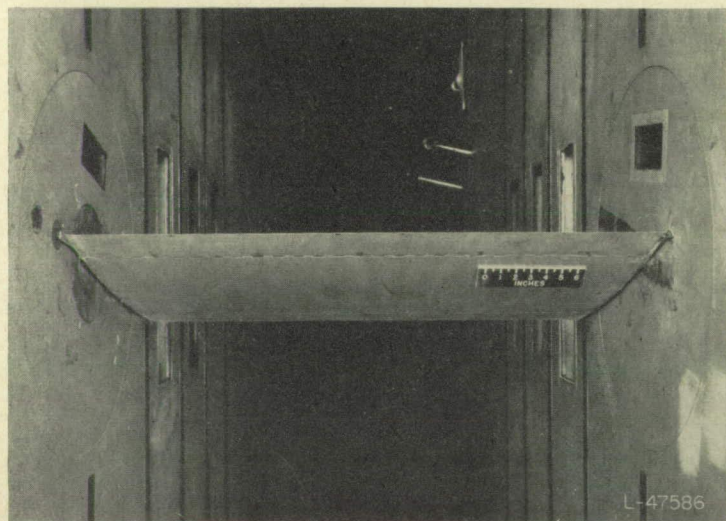
The lift and pitching-moment characteristics of the two symmetrical circular-arc airfoils with the plain trailing-edge flaps and plain leading-edge flaps deflected individually are presented in figures 5 and 6, respectively.

The maximum section lift coefficients of the 6- and 10-percent-thick airfoils increased and the angles of attack for maximum lift decreased as the 0.20-chord trailing-edge flaps were deflected. The values of the maximum lift coefficients (fig. 5) for both airfoils were substantially equivalent at corresponding flap deflections.



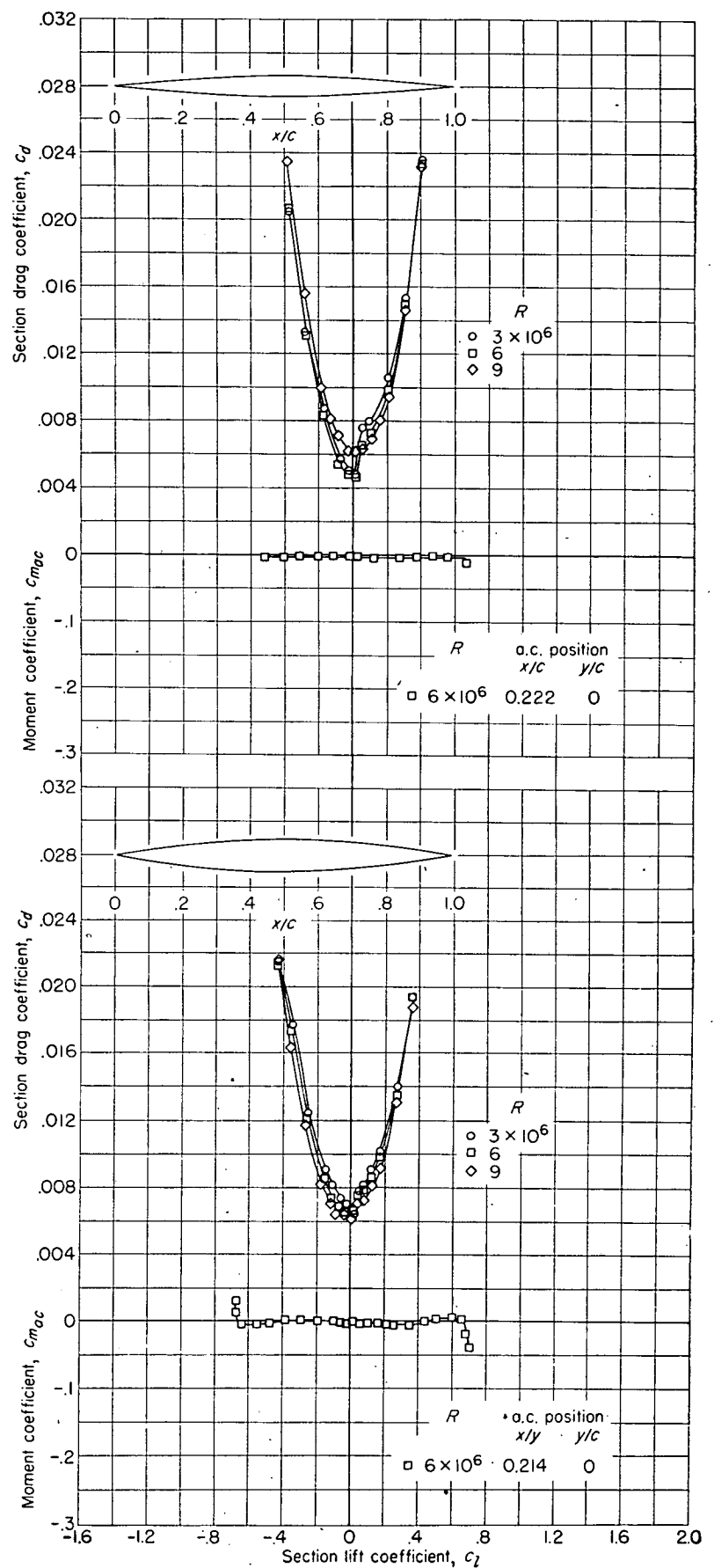
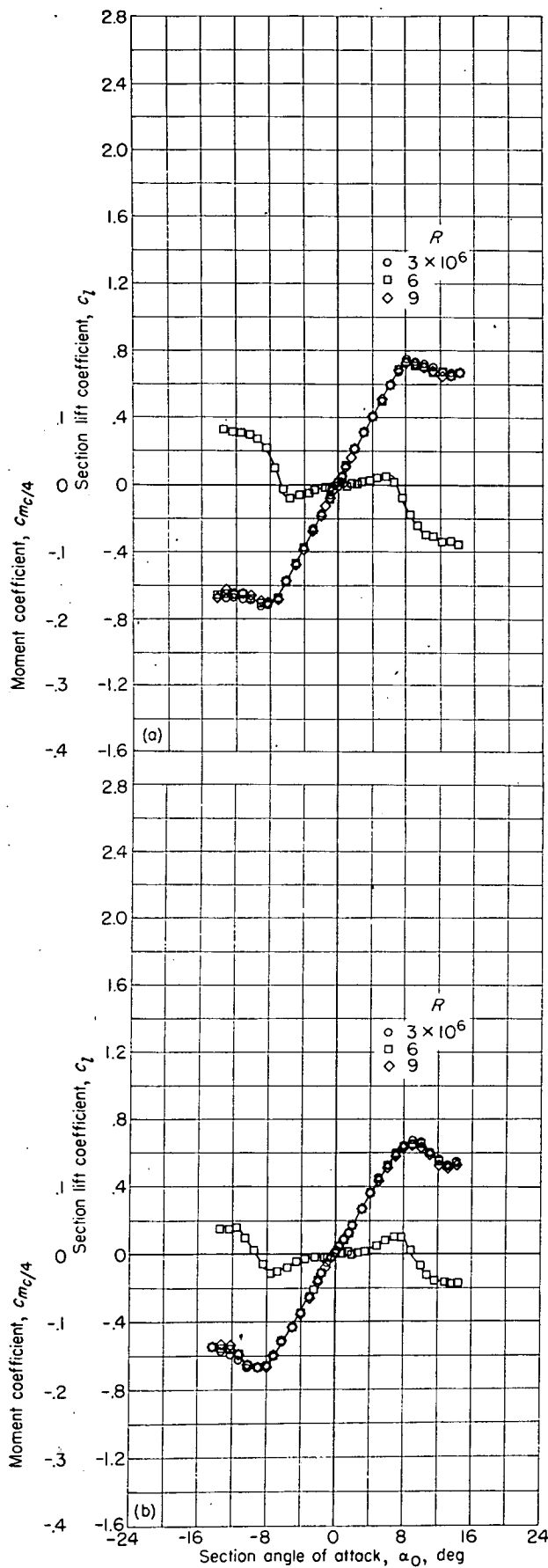
(a) With model end plates.

FIGURE 3.—Front of a symmetrical circular-arc airfoil with and without model end plates in the Langley two-dimensional low-turbulence pressure tunnel.



(b) Without model end plates.

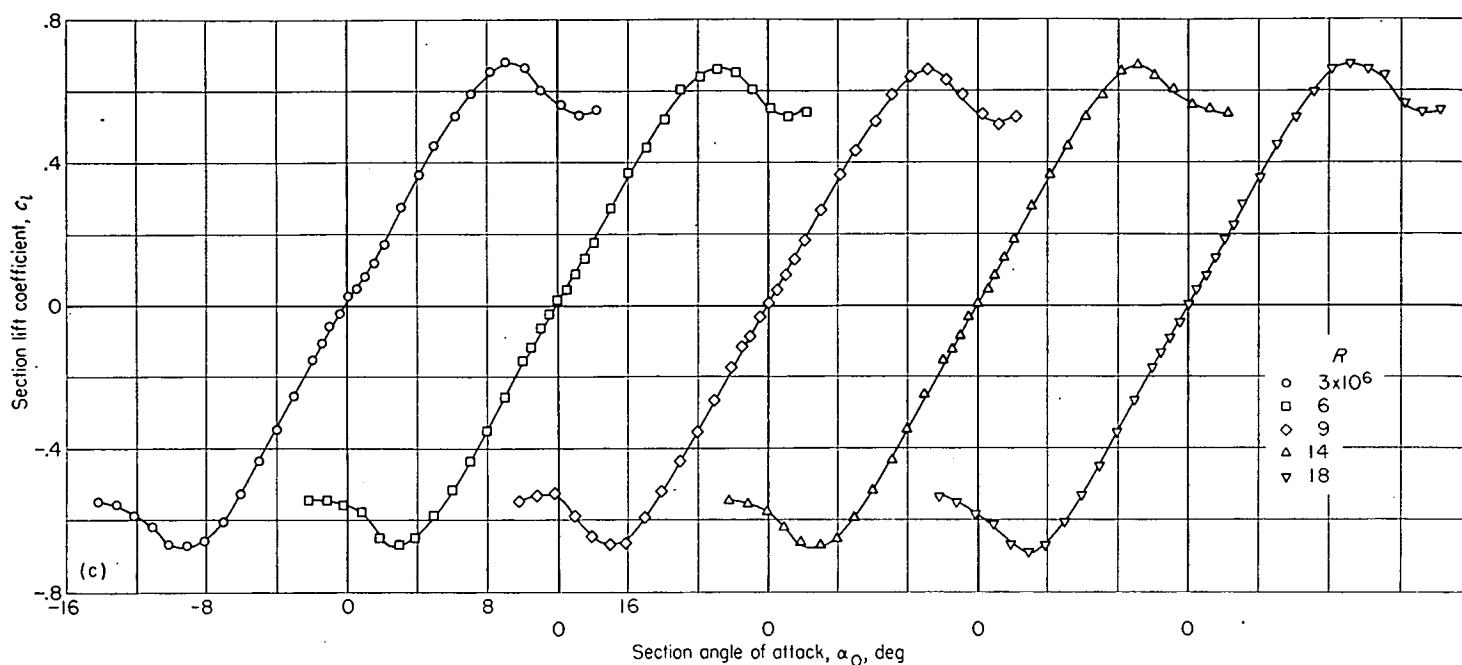
FIGURE 3.—Concluded.



(a) 6-percent-thick airfoil.

(b) 10-percent-thick airfoil.

FIGURE 4.—Aerodynamic characteristics of two symmetrical circular-arc airfoils with flaps neutral.



(c) 10-percent-thick airfoil.

FIGURE 4.—Concluded.

Deflecting the 0.15-chord leading-edge flaps increased the maximum section lift coefficients and increased the angles of attack for maximum lift (fig. 6) primarily by alleviating the negative pressure peaks that cause leading-edge separation near maximum lift. These pressure peaks are alleviated because the flow approaching the leading edge is more nearly aligned at high angles of attack when the leading-edge flap is deflected. The maximum section lift coefficients for the 6- and 10-percent-thick airfoils at the optimum deflection of the leading-edge flap, 30° , are 1.17 and 1.15, respectively. The optimum flap deflection is defined as the flap deflection for highest maximum lift. At corresponding deflections of the 0.15-chord leading-edge flap, the maximum section lift coefficients of both airfoils are essentially the same. At angles of attack well below those for maximum lift, the leading-edge flaps act as spoilers on the lower surface of the airfoils and cause some reduction in lift. These losses in lift increase as the flap deflection is increased.

The variation of the increment in maximum section lift coefficient $\Delta c_{l_{max}}$ and increment in angle of attack at maximum lift $\Delta \alpha_{c_{l_{max}}}$ for both models with deflection of the leading-edge or trailing-edge flaps individually is summarized in figure 7. From figure 7, it can be seen that the leading-edge-flap deflection for maximum lift, the optimum deflection, occurs at approximately 30° for both the 6- and 10-percent-thick airfoils. No optimum deflection was obtained for the trailing-edge flap because the highest test deflection was still the most effective. The maximum section lift coefficients of both airfoils are approximately the same at corresponding flap deflections, but the increments of maximum section lift coefficient obtained with flap deflection differ because of the lower maximum section lift coefficient of the 10-percent-thick airfoil with the flaps neutral. (See fig. 4.) Positive increments of the angle of attack for maximum lift resulted when the leading-edge flap was deflected,

but negative increments resulted when the trailing-edge flap was deflected (fig. 7).

The pitching-moment characteristics of the two models (figs. 5 and 6) show that the aerodynamic center at low α_0 (near the ideal lift coefficient) continues to move toward the leading edge as either the leading-edge or trailing-edge flaps are deflected. At higher angles of attack, the center of pressure always moves to the rear and causes the variation of pitching moment with angle of attack to become stable. The increments in angle of attack and lift coefficient at which this change in stability occurs show approximately the same variation with flap deflection as is shown in figure 7 for maximum lift.

AIRFOILS WITH FLAPS DEFLECTED IN COMBINATION

The section lift characteristics of the two symmetrical circular-arc airfoils with the plain leading-edge flaps and plain trailing-edge flaps deflected in various combinations are presented in figure 8. The flap deflections that resulted in the highest maximum section lift coefficient were $\delta_N = 30^\circ$, $\delta_F = 60^\circ$ (fig. 8 (a)) and $\delta_N = 36^\circ$, $\delta_F = 60^\circ$ (fig. 8 (c)) for the 6- and 10-percent-thick airfoils, respectively. The data for the 10-percent-thick airfoil with the trailing-edge flap deflected 60° indicate no important changes in the maximum section lift coefficient with small departures from the optimum deflection of the leading-edge flap. A comparison between the lift characteristics of the two airfoils with the leading-edge flap deflected 30° and the trailing-edge flap deflected 60° (fig. 8) with those for the airfoil with the leading-edge flap neutral and the trailing-edge flap deflected 60° (fig. 5) shows that the maximum section lift coefficients were increased 0.32 and 0.30 (to 1.95 and 2.03) and the angles of attack for maximum lift were increased 6.5° and 6° , respectively, for the 6- and 10-percent-thick airfoils by deflection of the leading-edge flap.

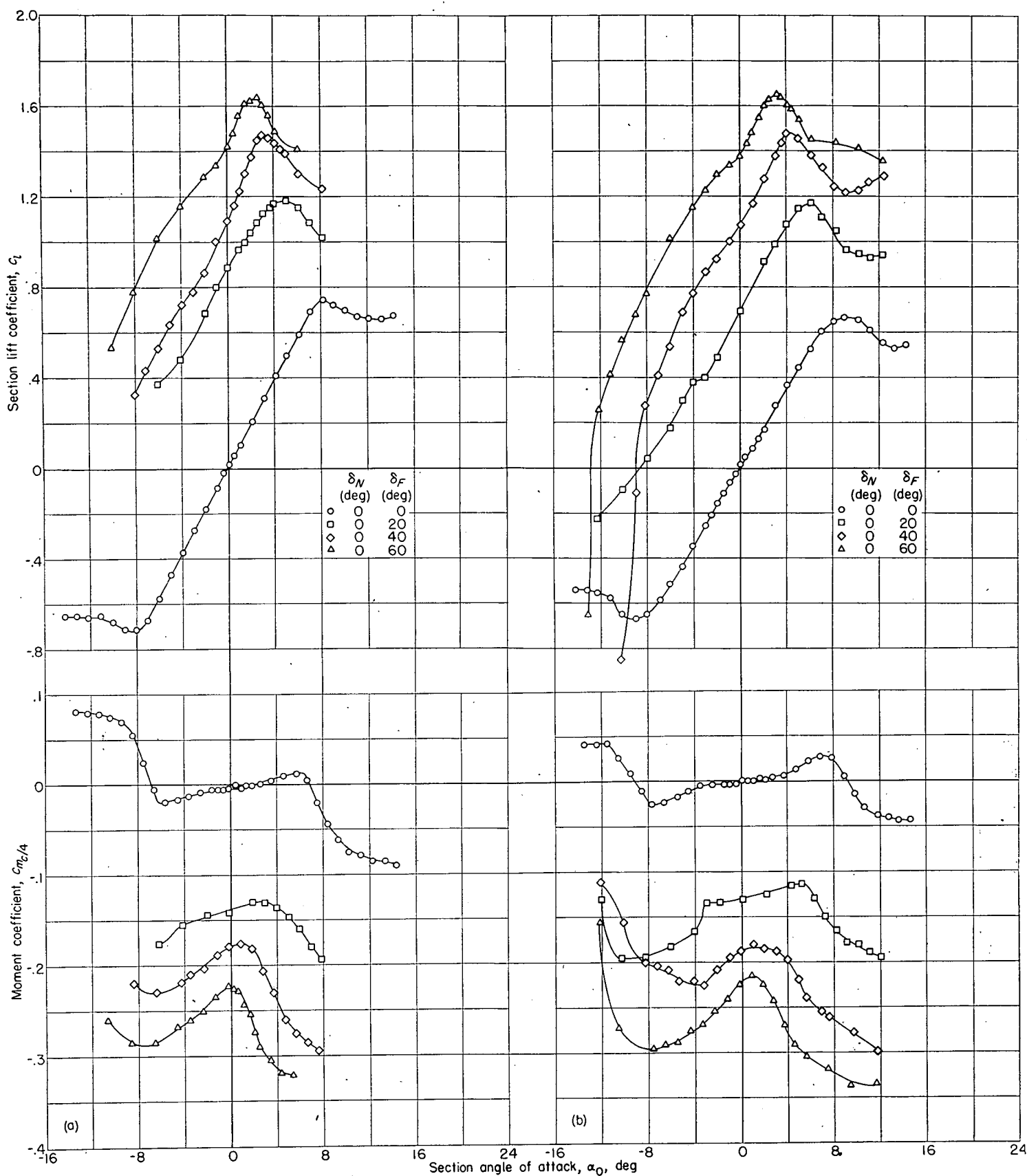


FIGURE 5.—Section lift and pitching-moment characteristics of two symmetrical circular-arc airfoils for various deflections of the 0.20-chord plain trailing-edge flap; $R=6 \times 10^5$.

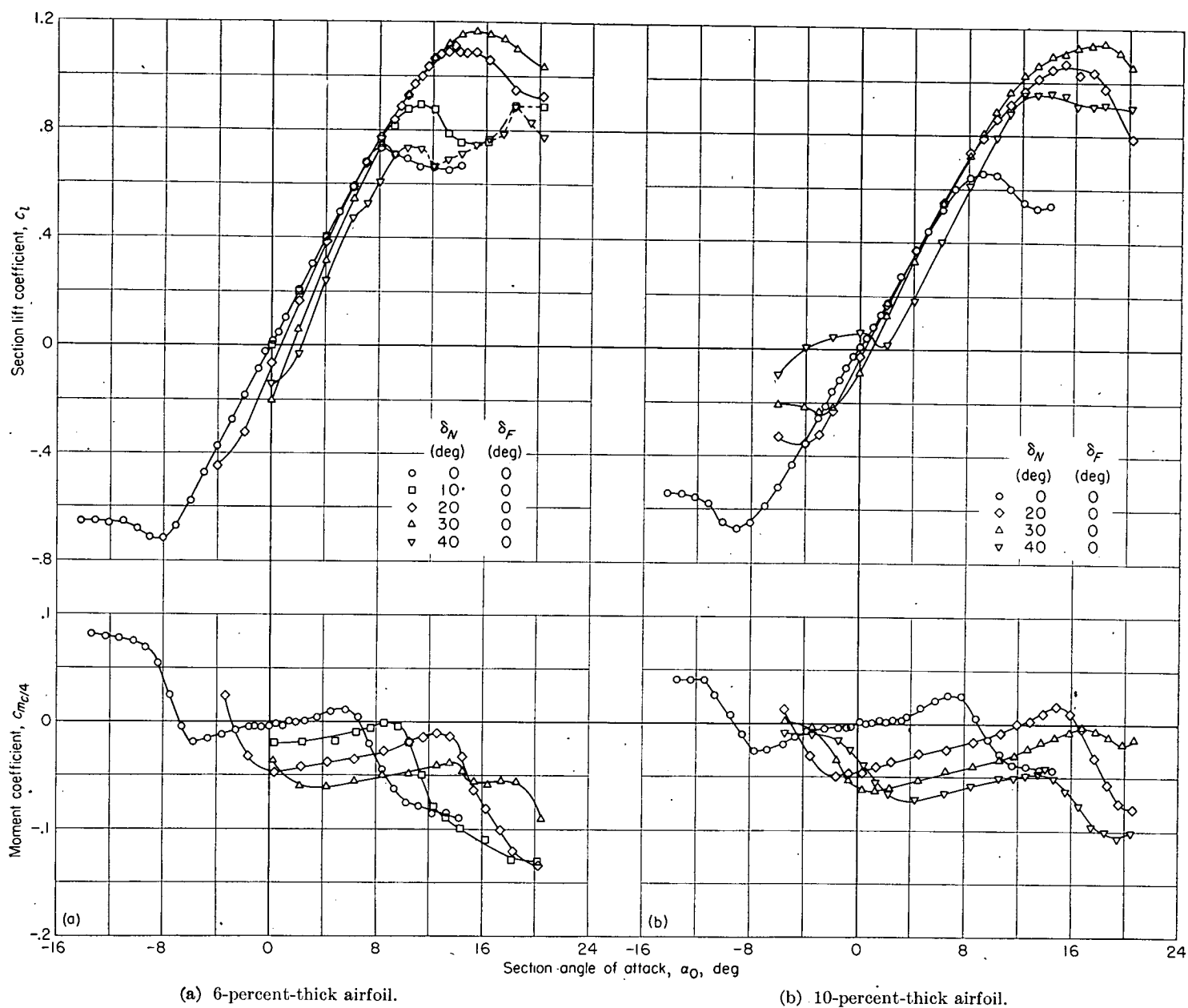


FIGURE 6.—Section lift and pitching-moment characteristics of two symmetrical circular-arc airfoils for various deflections of the 0.15-chord plain leading-edge flap; $R=6 \times 10^5$.

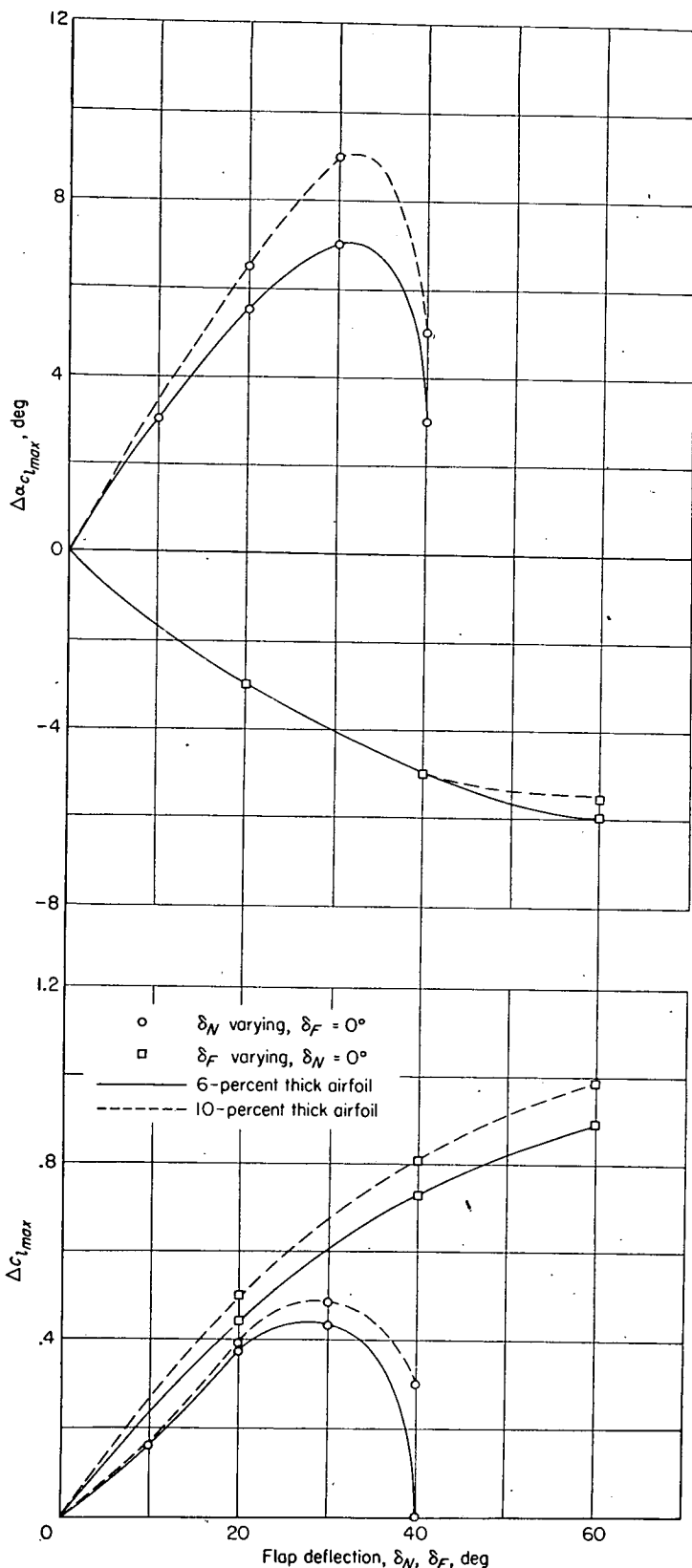


FIGURE 7.—Variation of the increment in maximum section lift coefficient and angle of stall with deflection of the plain leading-edge flap and plain trailing-edge flap; $R=6 \times 10^6$.

The section lift characteristics of the two models with the plain leading-edge flaps and plain trailing-edge flaps deflected 30° and 60° , respectively, obtained at Reynolds numbers of 3×10^6 , 6×10^6 , and 9×10^6 are presented in figure 9. At Reynolds numbers between 3×10^6 and 9×10^6 , the data (fig. 9 (a)) show no appreciable scale effect on the maximum lift coefficient of the 6-percent-thick airfoil. The section lift characteristics of the 6-percent-thick airfoil with the leading- and trailing-edge flaps deflected 27° and 60° , respectively, are presented in figure 10 for Reynolds numbers from 0.70×10^6 to 2.29×10^6 . In this range of Reynolds numbers, the maximum section lift characteristics of the 6-percent-thick airfoil are independent of scale. In the case of the 10-percent-thick airfoil (fig. 9 (b)), however, some adverse scale effect (nearly 0.1) is indicated in the maximum section lift coefficient at Reynolds numbers between 3×10^6 and 6×10^6 . Similarly, some adverse scale effect (fig. 8 (c)) is indicated in the maximum section lift coefficient at Reynolds numbers between 3×10^6 and 9×10^6 with the leading- and trailing-edge flaps deflected 36° and 60° , respectively. At Reynolds numbers above 9×10^6 , however, the maximum section lift coefficient of this combination remained approximately constant.

The section pitching-moment characteristics of the two airfoils with the leading- and trailing-edge flaps deflected 30° and 60° , respectively, (fig. 9) show that the aerodynamic center remains ahead of the quarter-chord point for angles of attack greater than zero. In addition, the combined action of the leading- and trailing-edge flaps caused the moment coefficients to increase negatively with increasing lift coefficient until the angle of attack was high enough to alleviate the spoiler action of the leading-edge flap. As the lift coefficient was increased beyond this point, the moment became less negative until approximately 2.5° beyond the angle of attack for maximum lift, whereupon the moment curve breaks.

LOW-DRAG-CONTROL FLAPS

The lift and drag characteristics of the 6-percent-thick symmetrical circular-arc airfoil with the leading- and trailing-edge flaps deflected are presented in figure 11. Deflecting the leading-edge flap to 10° decreased the section drag coefficient of the 6-percent-thick airfoil at a lift coefficient of 0.3 about 40 percent by delaying the formation of a negative pressure peak at the leading edge which causes separation. In general, the leading-edge flap was more effective in extending the low drag range to higher section lift coefficients than was the trailing-edge flap.

AIRFOIL LOADING

Pressure coefficients obtained from orifice static-pressure measurements made on the 6-percent-thick symmetrical circular-arc airfoil with the plain leading- and trailing-edge flaps deflected in various combinations and at several angles of attack are presented in table IV.

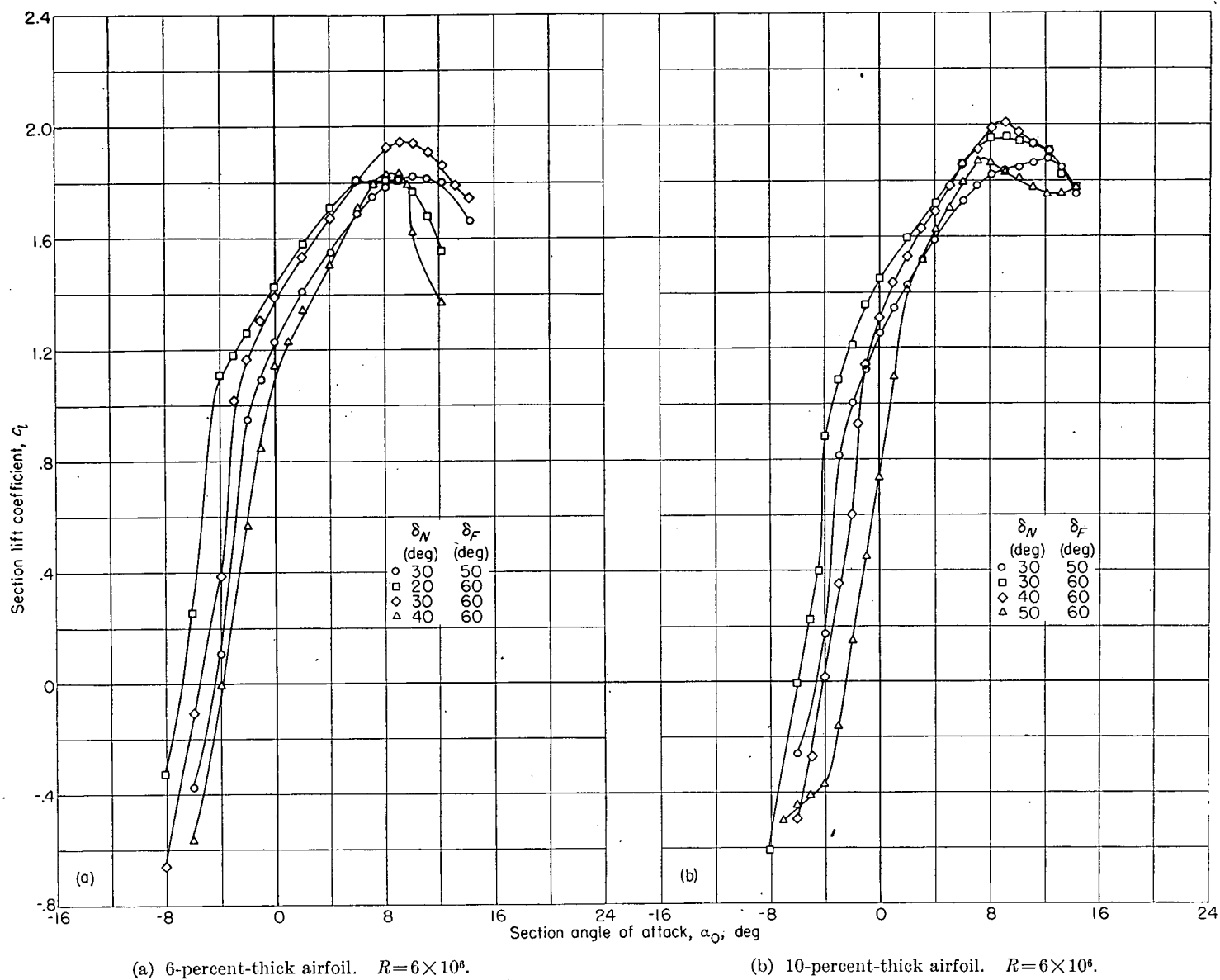
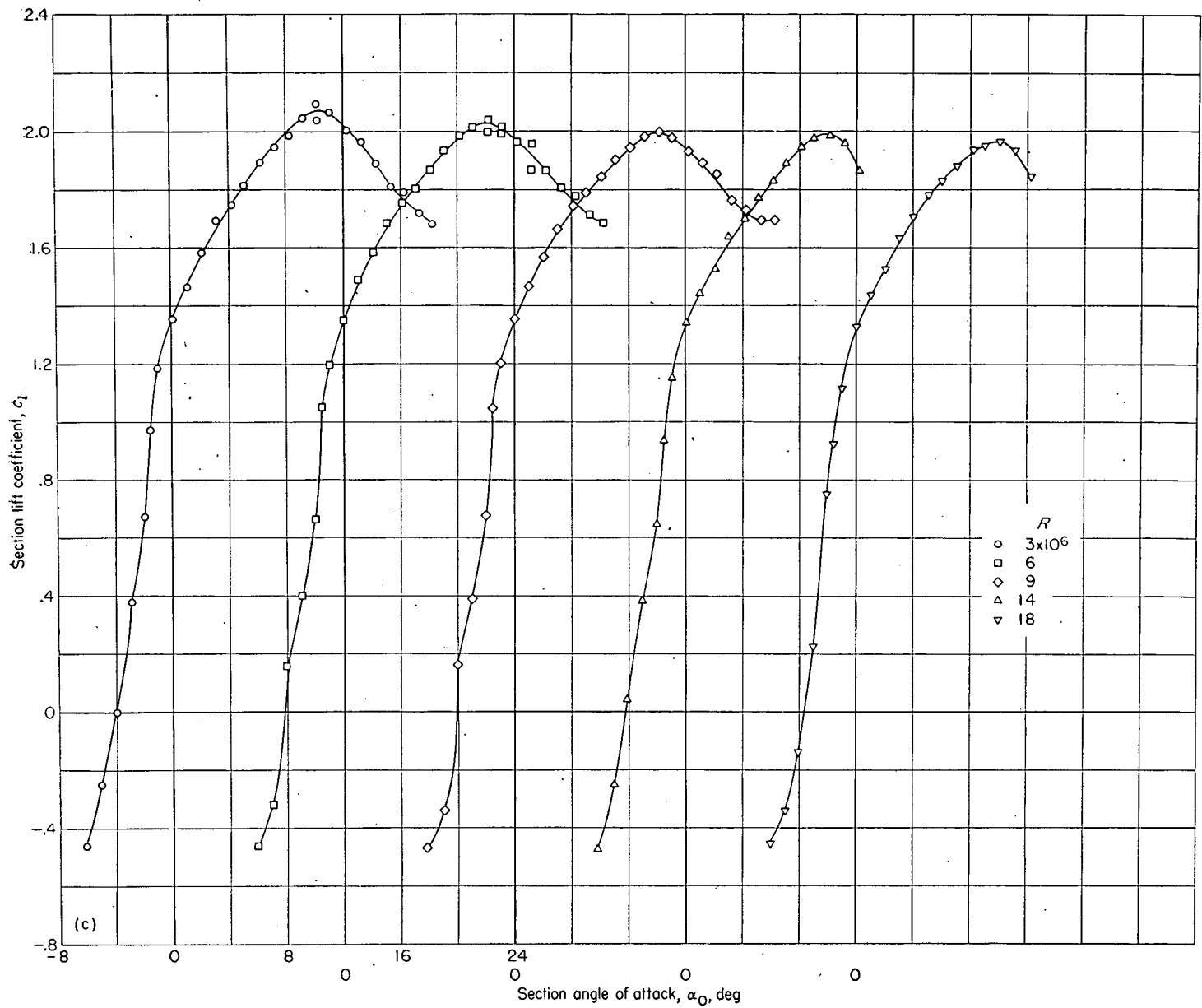


FIGURE 8.—Section lift characteristics of two symmetrical circular-arc airfoils for various deflections of the plain leading-edge flap and plain trailing-edge flap.



(c) 10-percent-thick airfoil. $\delta_N = 36^\circ$, $\delta_P = 60^\circ$.

FIGURE 8.—Concluded.

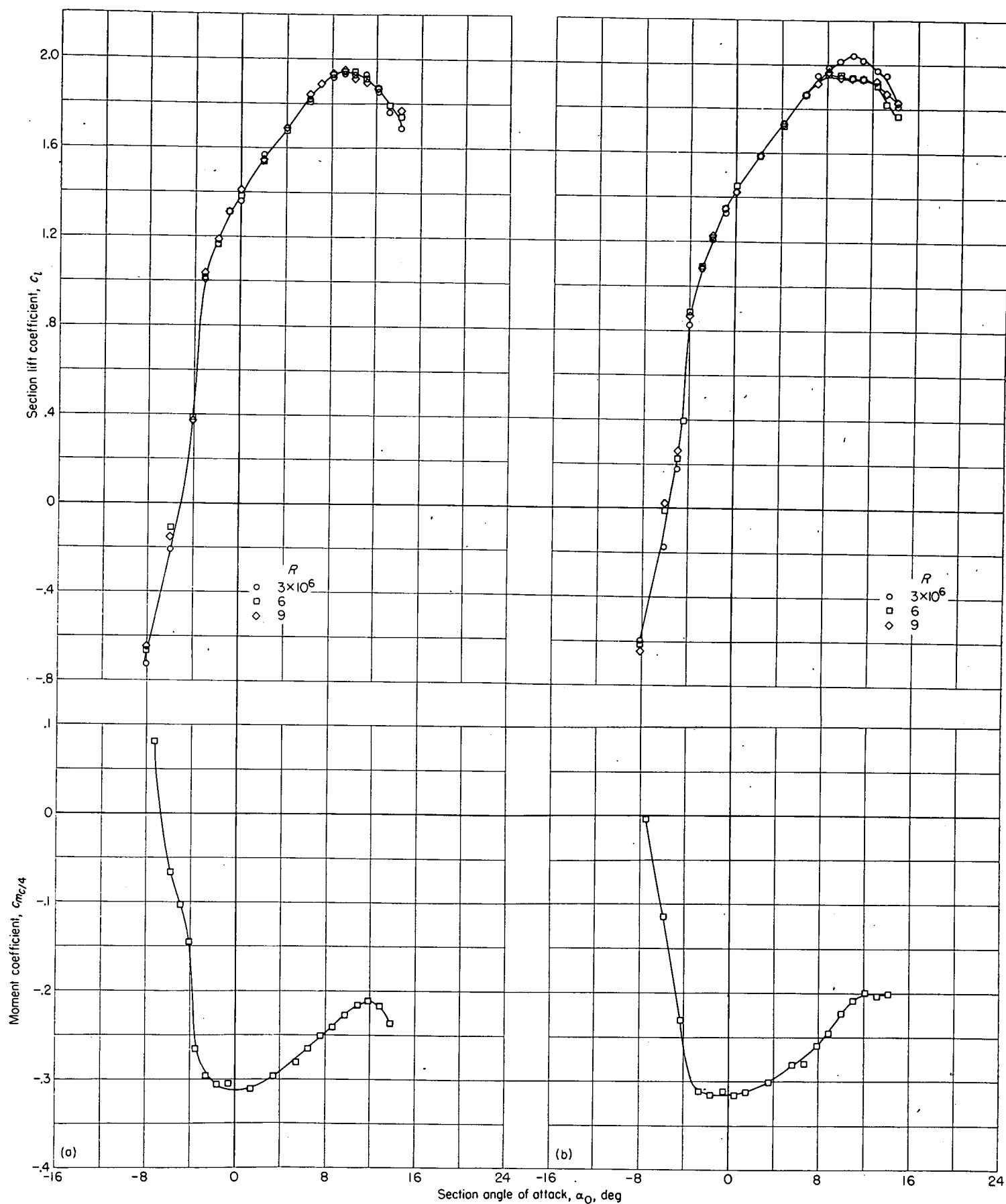


FIGURE 9.—Section lift and pitching-moment characteristics of two symmetrical circular-arc airfoils with the plain leading-edge flap deflected 30° and the plain trailing-edge flap deflected 60° .

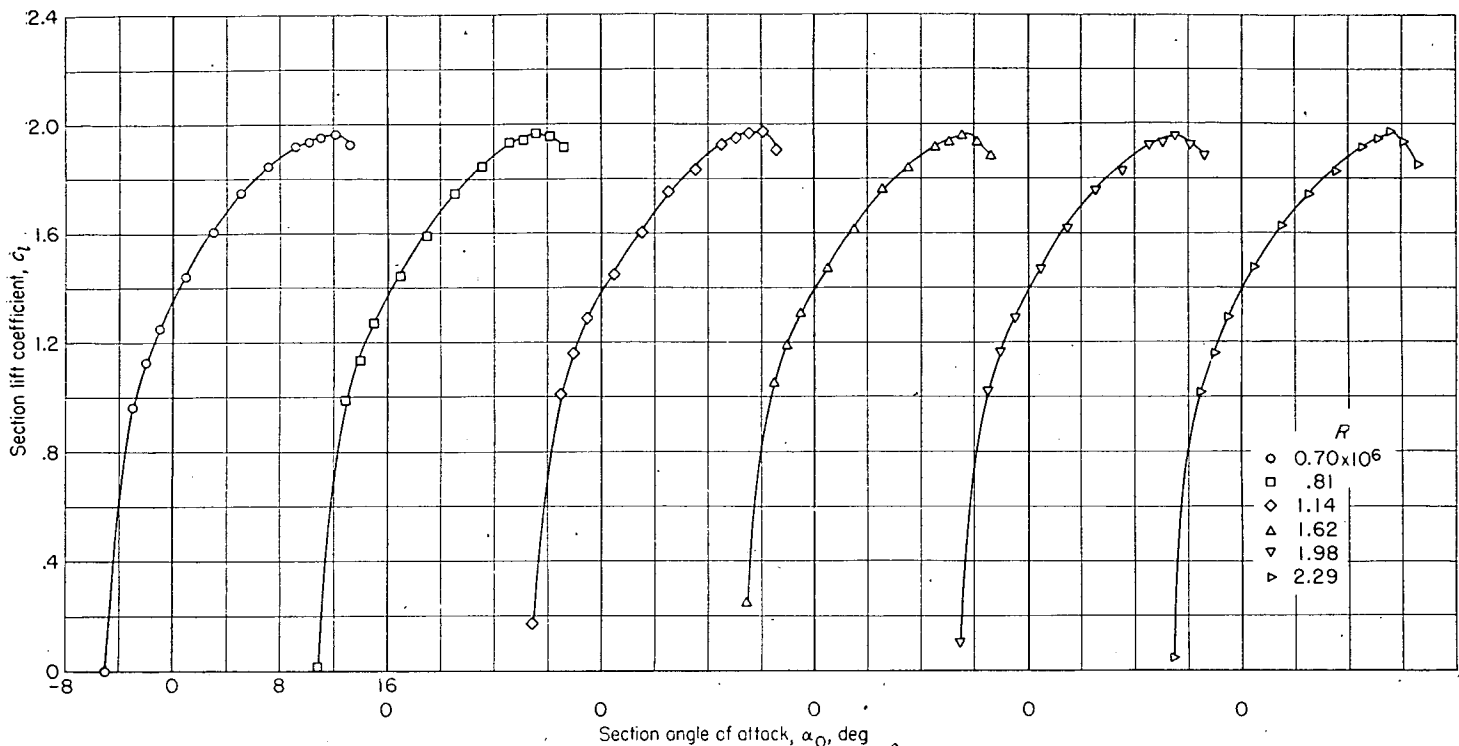


FIGURE 10.—Section lift characteristics at several values of the Reynolds number for 6-percent-thick symmetrical circular-arc airfoil with plain leading-edge flap and plain trailing-edge flap; $\delta_N=27^\circ$, $\delta_F=60^\circ$.

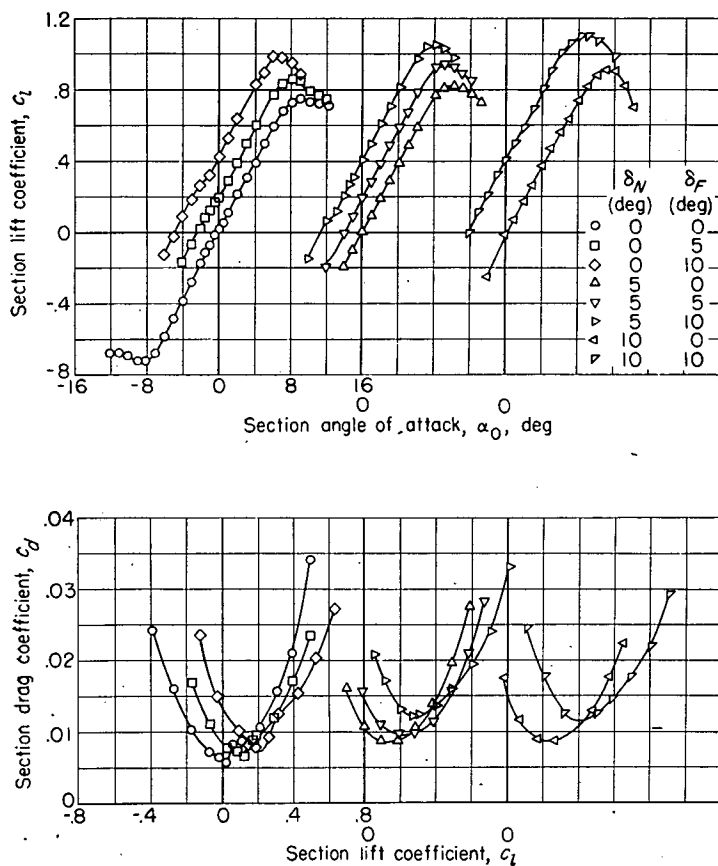
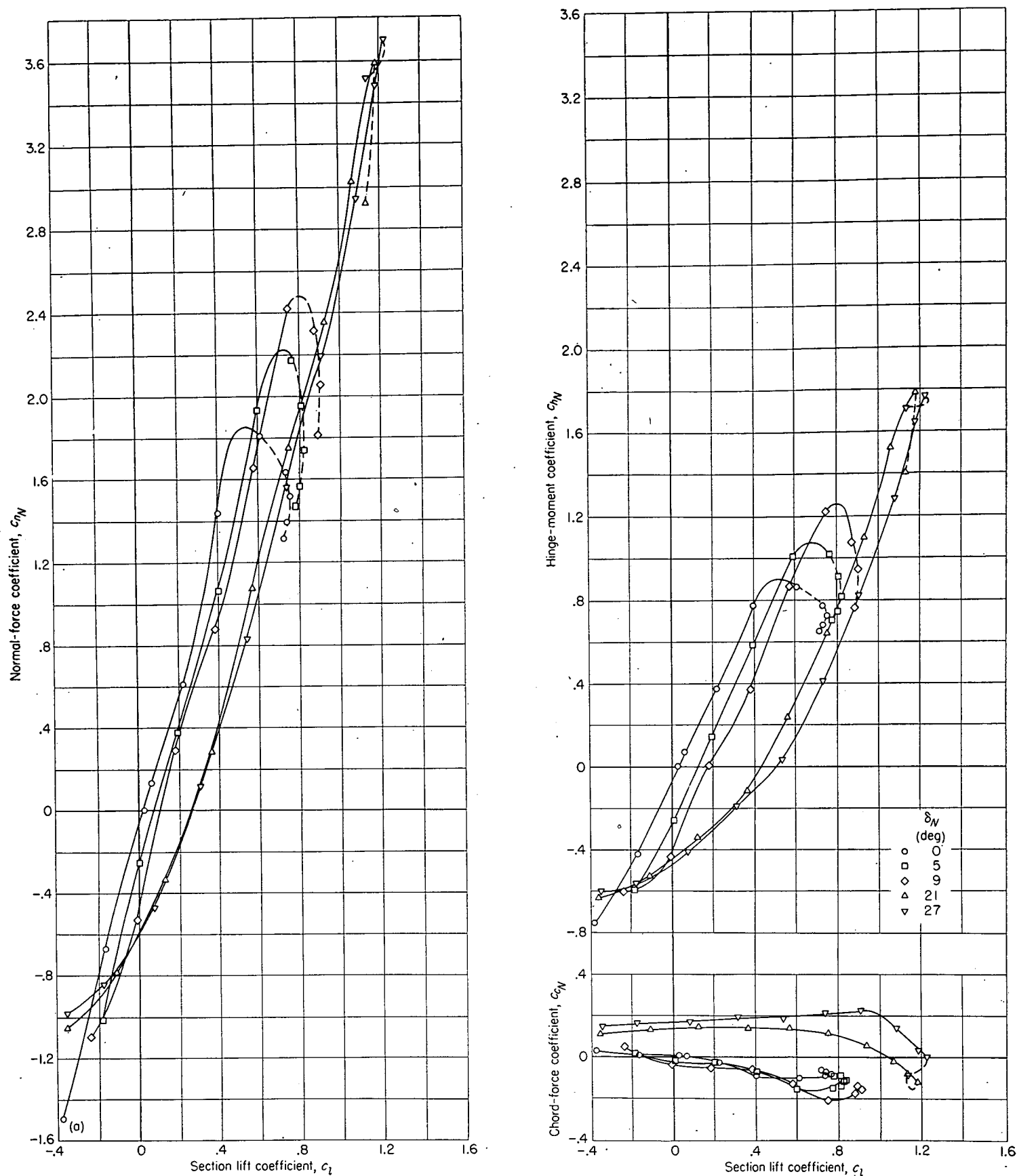


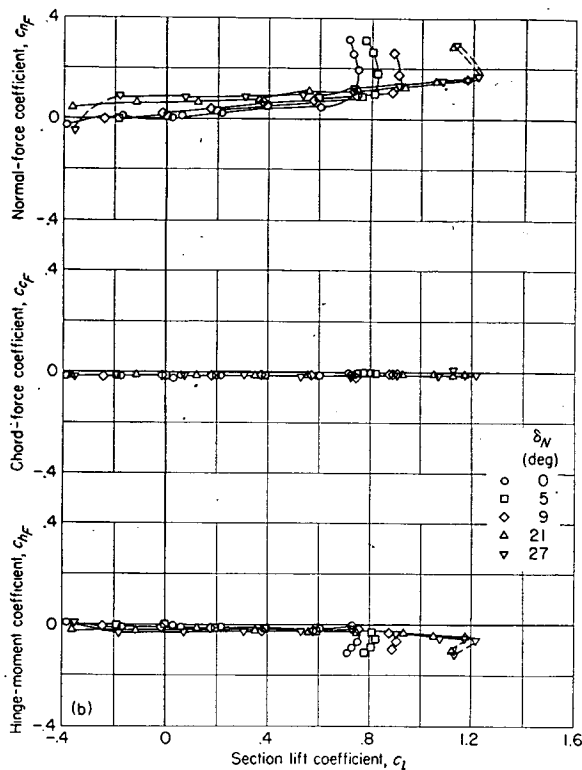
FIGURE 11.—Section lift and drag characteristics of a 6-percent-thick circular-arc airfoil for various deflections of the plain leading-edge flap and plain trailing-edge flap; $R=2.1 \times 10^6$.

The flap section normal-force, chord-force, and hinge-moment characteristics with the flaps deflected obtained from integrations of these pressure distributions are presented in figures 12 to 16. The loads on the leading-edge and trailing-edge flaps varied qualitatively in the same manner which would be indicated by the thin-airfoil theory. As is shown subsequently, however, separation at the sharp leading edge caused rather large changes in the pressure distributions, and the quantitative agreement between the experimental loads and those predicted by thin-airfoil theory is not good. For a given flap configuration, the normal force and moment on the leading-edge flap increased rapidly in a positive direction with increasing lift coefficient; whereas, in comparison, the normal force and moment on the trailing-edge flap remained almost constant. For a given lift coefficient, increasing the downward deflection of either flap produced downward increments in both the normal force and moment on the leading-edge flap in contrast to the usual characteristic of the conventional trailing-edge flap where the increments of the normal force and moment increase in the upward direction with increased trailing-edge-flap deflection. Deflection of the leading-edge flap had very little effect on normal-force and hinge-moment characteristics of the trailing-edge flap. The magnitude of the loads and moments on the plain trailing-edge flap are of a similar magnitude to those of the plain flaps on an NACA 0009 airfoil (ref. 4). As shown in figure 16 for a combined deflection of the leading-edge and trailing-edge flaps ($\delta_N=27^\circ$; $\delta_F=60^\circ$), the maximum flap normal-force and hinge-moment coefficients were, respectively, 4.74 and 2.24 for the leading-edge flap as compared with 1.48 and -0.61 for the trailing-edge flap.



(a) Plain leading-edge flap.

FIGURE 12.—Flap-section load and hinge-moment characteristics of a 6-percent-thick symmetrical circular-arc airfoil for various deflections of the 0.15-chord plain leading-edge flap; $R=2.1 \times 10^6$; $\delta_P=0^\circ$.



(b) Plain trailing-edge flap.

FIGURE 12.—Concluded.

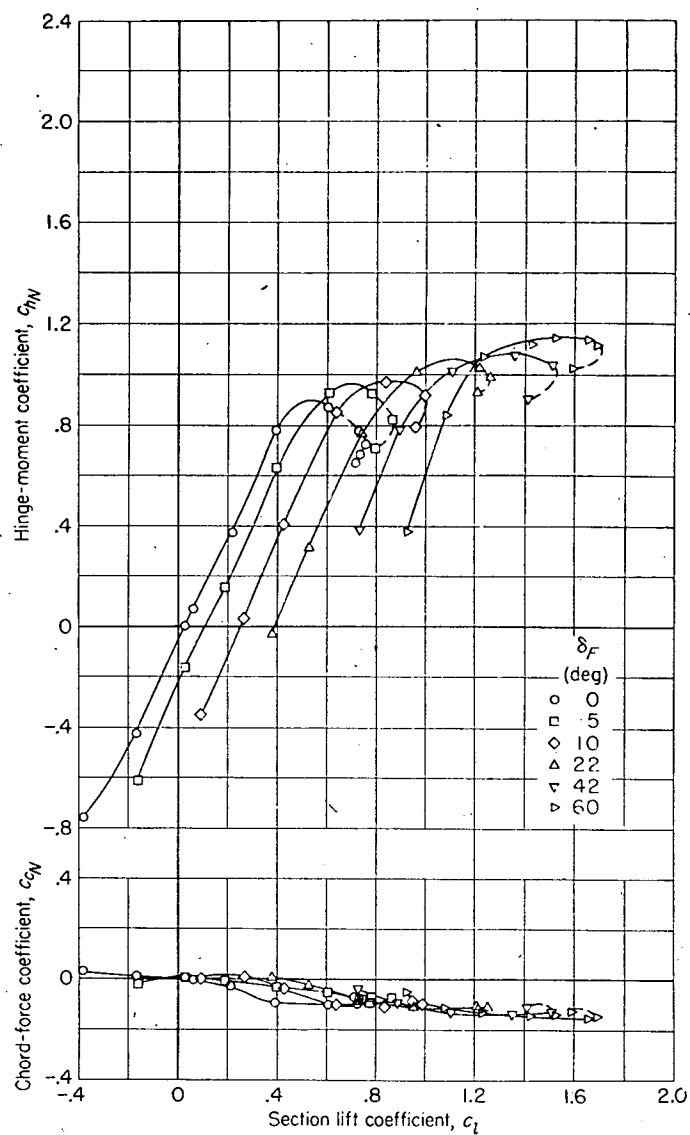
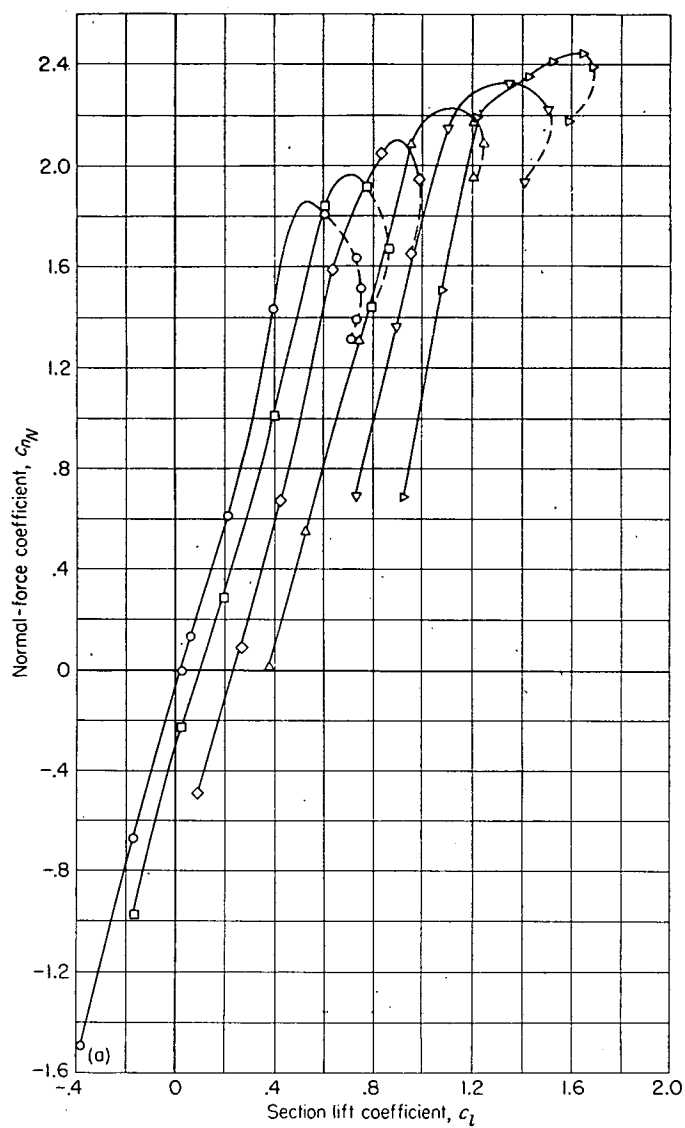
The chord-force coefficients of both flaps are negative in sign with the exception of the leading-edge-flap chord forces at deflections of 21° and 27°. The chordwise forces due to skin friction have not been included in these results. This omission is considered to be of minor importance because of the large magnitude of the normal-force coefficients. The pressure chord force, however, especially for the leading-edge flap, should not be neglected if the resultant air load is to be obtained.

The variation of the maximum flap loads and hinge moments at or below maximum lift with increasing deflection of either the leading-edge flap or trailing-edge flap is summarized in figures 17 and 18. In figure 17, it is shown that deflecting the leading-edge flap has no appreciable effect on the maximum normal-force and hinge-moment coefficients of the plain trailing-edge flap. Large increases in the corresponding coefficients of the leading-edge flap, however, are evident as the leading-edge flap is deflected. In contrast, deflecting the plain trailing-edge flap increased the maximum normal force and moment of both the leading-edge and trailing-edge flaps. The magnitudes of the maximum normal-force and moment coefficients of the plain trailing-edge flap are shown to increase more rapidly than the corresponding forces and moments of the leading-edge flap regardless of the deflection of the leading-edge flap (figs. 17 and 18).

Typical pressure-distribution diagrams are presented in figures 19 and 20 where the flap pressure coefficients are plotted against the projected chordwise position of the flap orifices on the airfoil chord. Use of the projected position accounts for the shorter effective chord in figure 20 as the flaps were deflected. The load-distribution diagram for the optimum maximum-lift configuration, presented in figure 21,

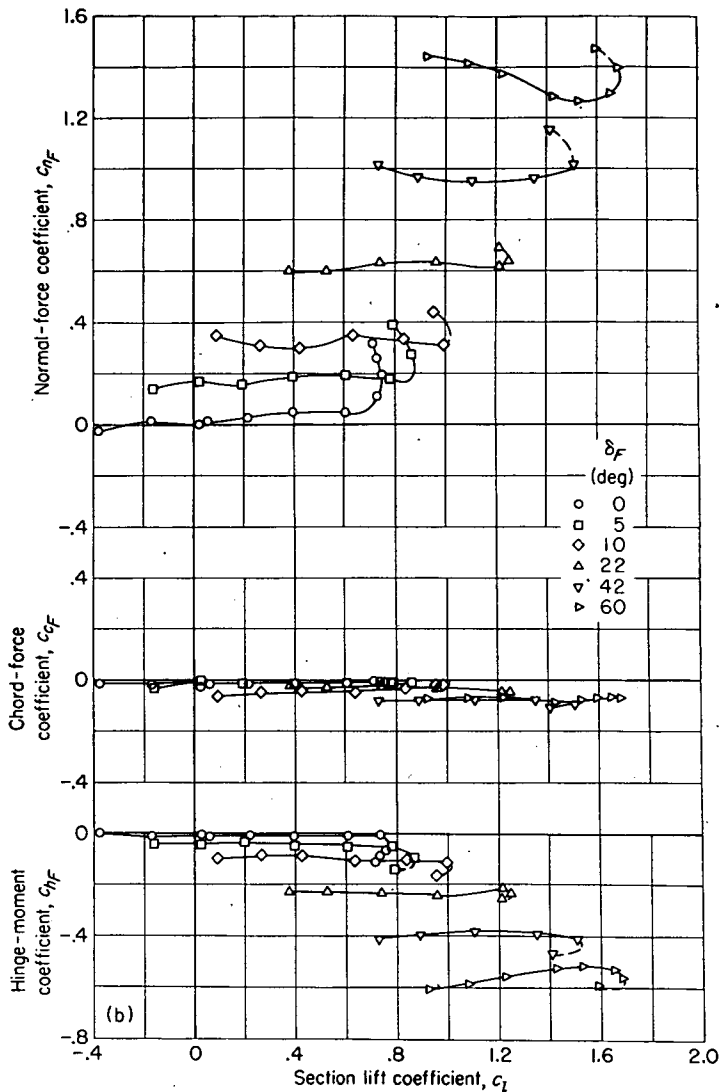
shows the comparatively larger load over the leading-edge flap than over the trailing-edge flap. This load over the leading-edge flap is the result of the additional normal load that occurs as the airfoil-flap configuration departs from the ideal angle of attack or lift coefficient. Thin-airfoil theory indicates that this additional normal load is infinite at the leading-edge but decreases rapidly with distance along the chord to zero at the trailing edge. Actually, because of the bubble of separation at the leading edge, the load has a finite value. A study of table IV shows that this local separation, as indicated by approximately constant values of the pressure coefficients on the upper surface near the leading edge, occurs for all the configurations investigated at an angle of attack well below that for maximum lift.

In order to obtain some indication of the flow pattern existing in the neighborhood of the leading edge of sharp-edge airfoils when supporting a finite lift load, observations were made of the local velocity and of the action of tufts in the airstream near the leading edge of the 6-percent-thick airfoil at several angles of attack. At 2° angle of attack, where a fairly sharp, well-defined peak occurs in the pressure distribution near the leading edge, no evidence of a separation bubble was apparent in the data obtained. The velocity distributions in the flow field above the airfoil and the pressure distribution at the airfoil upper surface for angles of attack of 4° and 6° are shown in figure 22. Pressure distributions computed from approximate potential-flow relations are also shown in this figure. At these angles of attack, where local regions of separated flow are indicated by the nearly constant values of surface pressure coefficient near the leading edge, the flow surveys show that a reversed flow existed just above the surface of the airfoil. The pressure coefficients are much lower than the computed values at the leading edge but are higher than the computed values in a region just behind the leading edge. The chordwise extent of the region of reversed flow coincides approximately with the extent of the region in which the experimental pressure coefficients are higher than the computed coefficients. Farther downstream the flow reattaches to the surface of the airfoil, no reverse flow is observed, and the pressure coefficients are slightly less than those computed. The existence of this reversed flow near the surface of the airfoil suggests the presence of a "captured" vortex imbedded in the flow, similar to that occurring on highly swept wings which experience leading-edge separation. Although the presence of this vortex causes an increase in loading over a portion of the airfoil, its effect is not large enough to cause an increase in lift-curve slope, the decreases in loading ahead of and behind the vortex apparently compensating for any increase in loading at the vortex. As the angle of attack is increased from 4° to 6°, the extent of both the flat spot in the pressure distribution and the region of reversed flow increases in the chordwise direction. Further increases in angle of attack cause the extent of this separated region to increase until it encloses the whole chord of the airfoil at maximum lift. The upper boundary of the reduced velocity in the flow over the airfoil is also shown in figure 22 and indicates very large losses in momentum occurring in the flow as a result of the local separation. These large losses in the flow are of course responsible for the very rapid variation in drag coefficient with lift coefficient shown in figure 4.



(a) Plain leading-edge flap.

FIGURE 13.—Flap-section load and hinge-moment characteristics of a 6-percent-thick symmetrical circular-arc airfoil for various deflections of the 0.20-chord plain trailing-edge flap; $R=2.1 \times 10^6$; $\delta_N=0^\circ$.



(b) Plain trailing-edge flap.

FIGURE 13.—Concluded.

METHOD FOR PREDICTING THE LOW-SPEED CHORDWISE PRESSURE DISTRIBUTION

DERIVATION OF THE METHOD

Velocity distributions as calculated by potential-flow methods generally bear little resemblance to those obtained experimentally on sharp-edge airfoils because of the existence of extensive regions of separated flow. If the velocity distributions about sharp-edge airfoils with flaps are to be analyzed, the resultant distributions can be broken down into various component parts as is done in the case of airfoils in potential flow (ref. 1). The most generally used breakdown considers the resultant velocity distribution to be made up of the following three components:

(1) Distribution of velocity about the basic symmetrical airfoil at zero angle of attack, v/V

(2) Incremental-additional-velocity distribution due to departure of the airfoil from the ideal lift coefficient, $\Delta v_a/V$ (The ideal lift coefficient is defined as the lift coefficient at which the stagnation point occurs at the leading edge.)

(3) Mean-line velocity distribution

(a) Caused by airfoil camber, $\Delta v/V$

(b) Caused by flap deflection, $(\Delta v/V)_{\delta\delta}$

In the present report, the only type of mean-line velocity distribution considered is that resulting from flap deflection, since the data used in the analysis are for a symmetrical airfoil section. It is believed, however, that the method may also be applicable to cambered sections.

In terms of the three component velocities, the complete velocity distribution about an airfoil at any lift coefficient is given approximately by

$$\sqrt{S_v} = \frac{v}{V} + \frac{\Delta v_a}{V} + \left(\frac{\Delta v}{V}\right)_{\delta\delta} \quad (1)$$

$$\sqrt{S_L} = \frac{v}{V} - \frac{\Delta v_a}{V} - \left(\frac{\Delta v}{V}\right)_{\delta\delta} \quad (2)$$

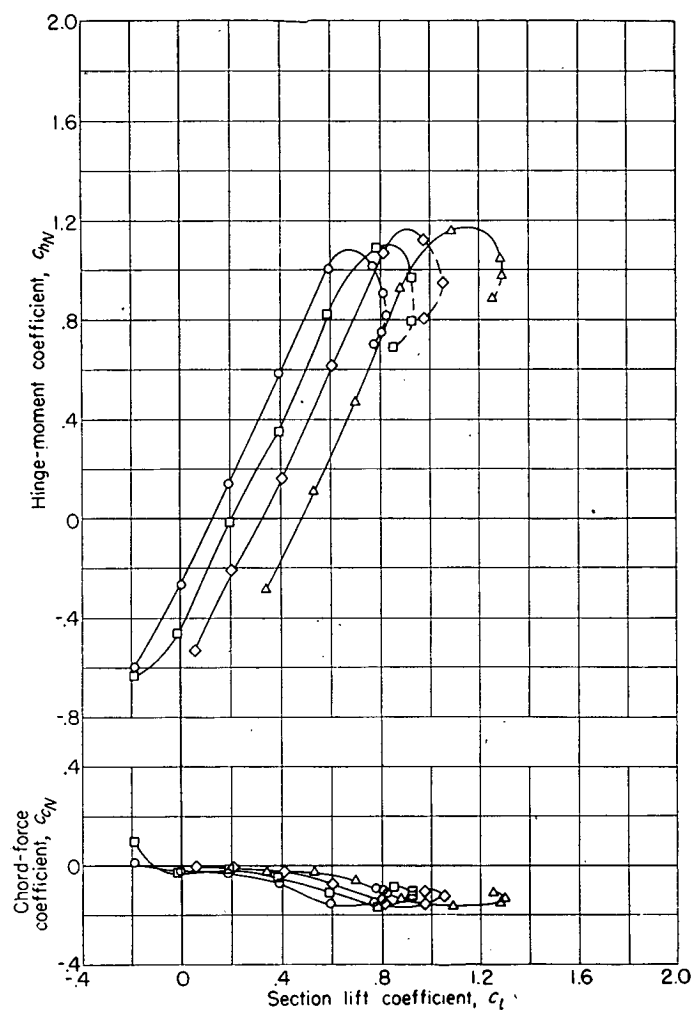
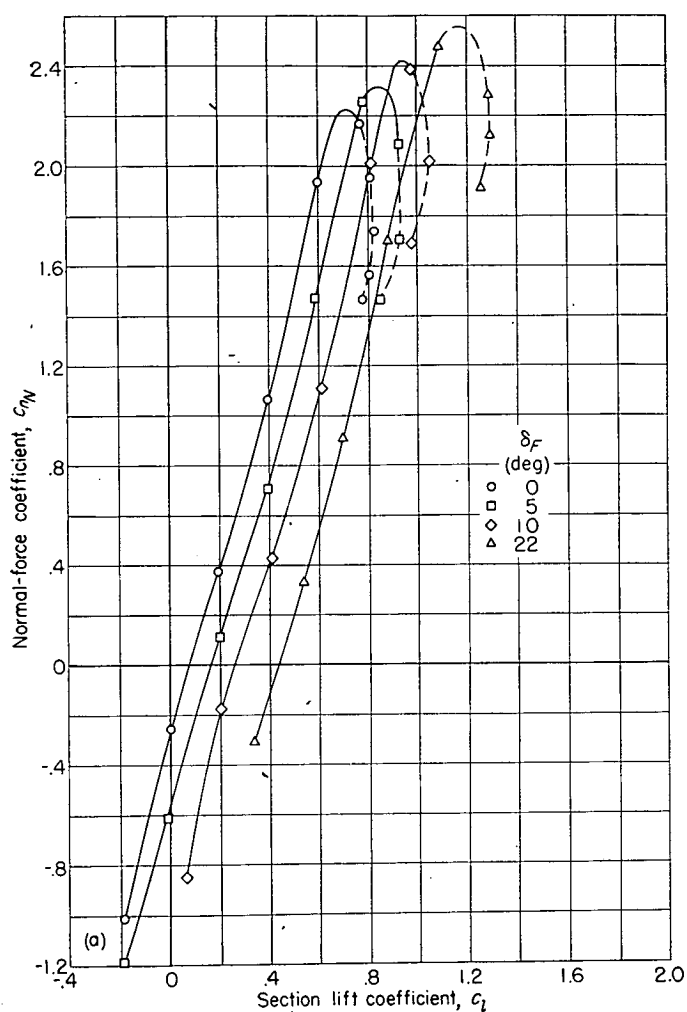
For the basic thickness form at zero lift, the velocity distribution v/V can, in any case, be calculated by the methods of references 5 and 6. In the absence of flow separation, the component $\Delta v_a/V$ is usually taken to be a linear function of the additional lift coefficient c_{la} , that is, the difference between any arbitrary lift coefficient and the ideal lift coefficient, and can be calculated by thick-airfoil theory. If extensive regions of separation do not exist, the components resulting from airfoil camber $\Delta v/V$ or flap deflection $(\Delta v/V)_{\delta\delta}$ can also be calculated. The methods of thin-airfoil theory (refs. 7 and 8) are usually employed for this purpose.

For sharp-edge airfoils for which flow separation limits the applicability of potential-flow methods, the problem of developing a general method of determining the velocity distribution resolves itself into a determination of the manner in which the various component distributions vary with c_{la} and δ . First, the velocity distribution about the basic thickness form at zero lift must be determined; that is, by definition

$$\frac{v}{V} = \frac{\sqrt{S_v} + \sqrt{S_L}}{2} \quad (3)$$

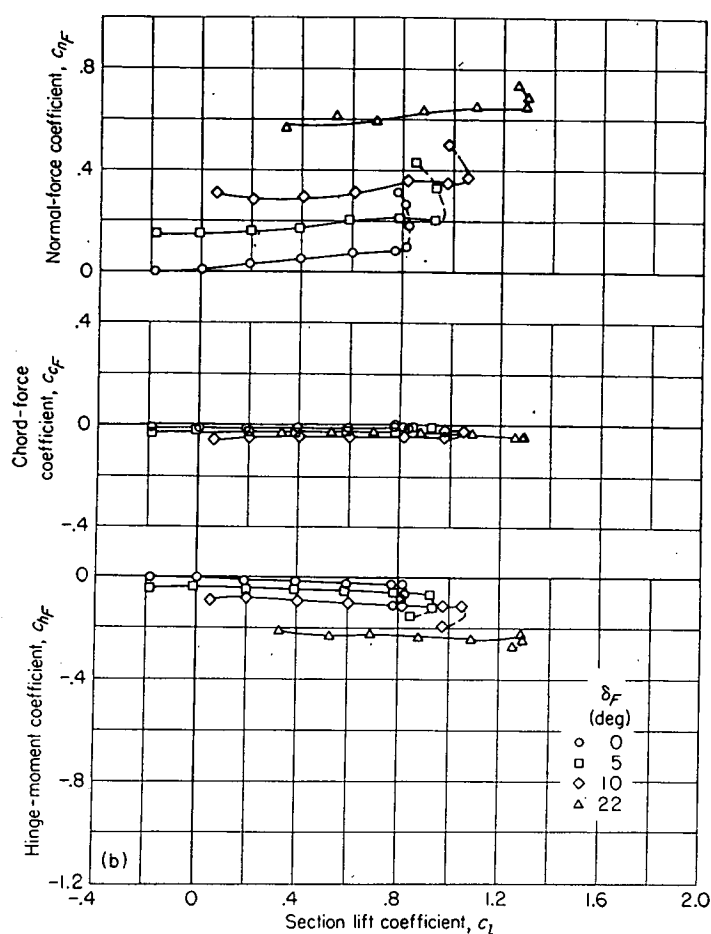
The value of v/V for the symmetrical airfoil at zero lift can, of course, be calculated by potential-flow methods; however, the extent of the separated flow on the upper and lower surfaces and, therefore, the effective value of v/V which must be used in equations (1) and (2) varies with lift coefficient and flap deflection. Consequently, the value of v/V for the symmetrical airfoil at zero lift, determined theoretically, must be corrected by an increment v'/V which is a function of lift coefficient and flap deflection. The value of v'/V can be determined from the experimental data. Next, the manner in which the additional velocity distribution $\Delta v_a/V$ varies with the additional lift coefficient must be found. The use of the experimental pressure-distribution data and the following relation obtained from equations (1) and (2) provides the solution:

$$\frac{\Delta v_a}{V} = \frac{\sqrt{S_v} - \sqrt{S_L}}{2} - \left(\frac{\Delta v}{V}\right)_{\delta\delta} \quad (4)$$



(a) Plain leading-edge flap.

FIGURE 14.—Flap-section load and hinge-moment characteristics of a 6-percent-thick symmetrical circular-arc airfoil for various deflections of the 0.20-chord plain trailing-edge flap; $R=2.1 \times 10^6$; $\delta_N=5^\circ$.



(b) Plain trailing-edge flap.

FIGURE 14.—Concluded.

Finally, the extent to which the theoretical velocity distribution due to flap deflection is realized experimentally must be determined. In order to determine the variation of $\Delta v_a/V$ with lift coefficient and to compare the experimental and theoretical velocity distributions, the ideal lift coefficient must be known. For any combination of leading-edge and trailing-edge deflections, the ideal lift coefficient can be calculated by the methods of reference 7; however, because of flow separation, a correlation must be made between the theoretical and experimental ideal lift coefficients.

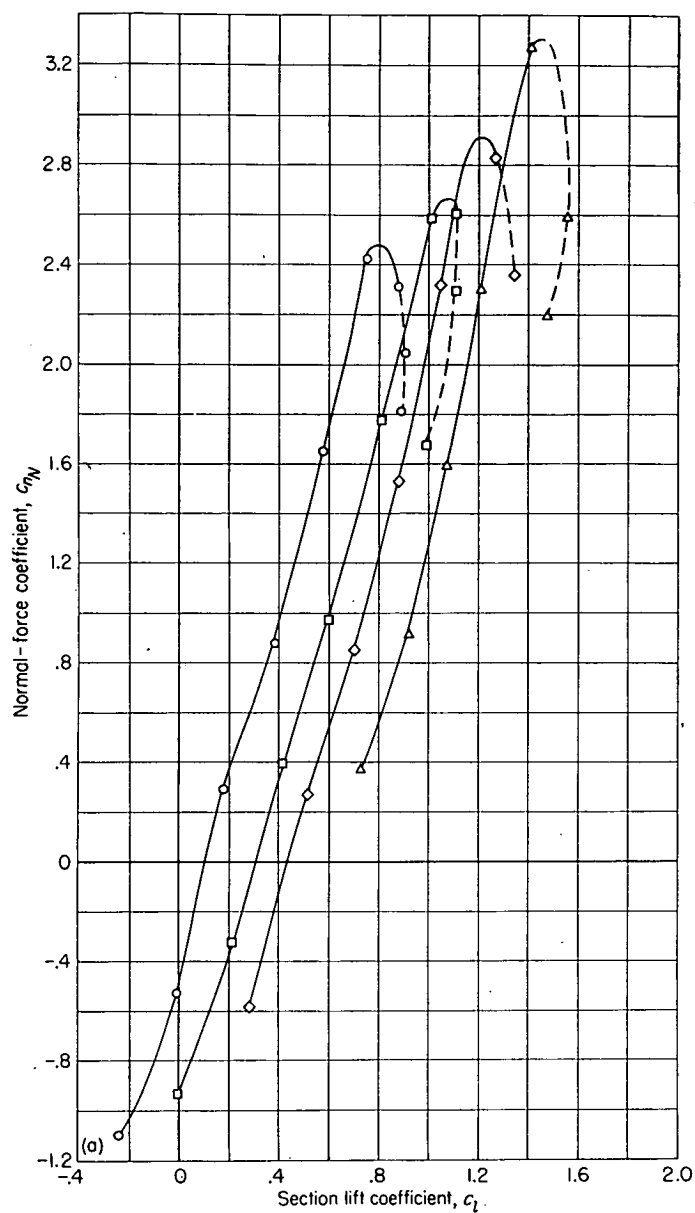
Ideal lift coefficient.—The change in ideal lift coefficient is equal to the sum of two component changes, one resulting from leading-edge-flap deflection and the other from trailing-edge-flap deflection. Each of these components may be calculated separately and added linearly. For each leading-edge- and trailing-edge-flap deflection investigated, the ideal lift coefficient $c_{l_{\delta\delta}}$ has been determined from the experimental data. The results are compared in figure 23 with those calculated from thin-airfoil theory. As shown in figure 23, the theoretical coefficients $c_{l_{\delta\delta}}$ for the leading-edge flap are identical with those obtained experimentally. In calculating the ideal lift coefficients $c_{l_{\delta\delta_N}}$ resulting from deflection of a leading-edge flap, the theoretical value may therefore be used.

For trailing-edge-flap deflections above 10° , the experimentally determined values of the ideal lift coefficient $c_{l_{\delta\delta_P}}$ are considerably lower than indicated by the theory. In order to determine the change in ideal lift coefficient associated with deflection of trailing-edge flaps of different chords, the method used by Allen in reference 9 to obtain $c_{l_{\delta\delta_P}}$ was applied to a large amount of experimental data from various sources. In this method, the ideal normal-force coefficient is related to the pitching-moment increment resulting from flap deflection and the center of pressure of the flap load for given values of flap-chord ratio and flap deflection. Values of $c_{l_{\delta\delta_P}}$ (the normal-force coefficient was taken to be essentially the same as the lift coefficient) obtained by this method are plotted in figure 24 against trailing-edge-flap deflection for flap-chord ratios ranging between 10 and 50 percent. The values of the quarter-chord pitching-moment increment required for the determination of these curves were obtained from numerous experimental data. These ideal lift coefficients (fig. 24) represent average values obtained from a series of tests of plain flaps on a large number of conventional airfoil sections. Similar computations were also made for the 0.20-chord flap on the circular-arc airfoil used in the present analysis and, as expected, the results agreed with the corresponding data in figure 24. For any profile with plain flaps, therefore, the results of figure 24 can be used for the determination of the ideal lift coefficient.

Mean-line velocity distribution.—The distribution of velocity over the surface of the airfoil resulting from flap deflection $(\Delta v/V)_{\delta\delta}$ was computed from the experimental data for various flap deflections by means of the following equation:

$$\left(\frac{\Delta v}{V}\right)_{\delta\delta} = \frac{\left(\frac{v}{V}\right)_{t_U} - \left(\frac{v}{V}\right)_{t_L}}{2}$$

This expression was obtained by subtracting equation (2) from equation (1) because, by definition, $\Delta v_a/V$ is zero at the ideal lift coefficient. The data thus obtained for various deflections of the leading-edge flap were found to be very nearly independent of flap deflection when expressed in the form of $\left(\frac{\Delta v}{V}\right)_{\delta\delta_N}/c_{l_{\delta\delta_N}}$. A comparison of the mean value of $\left(\frac{\Delta v}{V}\right)_{\delta\delta_N}/c_{l_{\delta\delta_N}}$ plotted against percent chord as determined by theory and experiment (fig. 25 (a)) shows good agreement. It is concluded, therefore, that the mean-line velocity distribution resulting from deflection of leading-edge flaps of various chords can be calculated theoretically with a sufficiently high degree of accuracy. Because of the effects of separation near the trailing edge, however, the experimental velocity distributions resulting from deflection of the plain trailing-edge flap differed markedly from those predicted by the theory, particularly for large flap deflections. A different distribution for each trailing-edge-flap deflection was determined, therefore, and the results are presented in figure 25(b) in the form of $(\Delta v/V)_{\delta\delta_P}$ plotted against percent chord.



(a) Plain leading-edge flap.

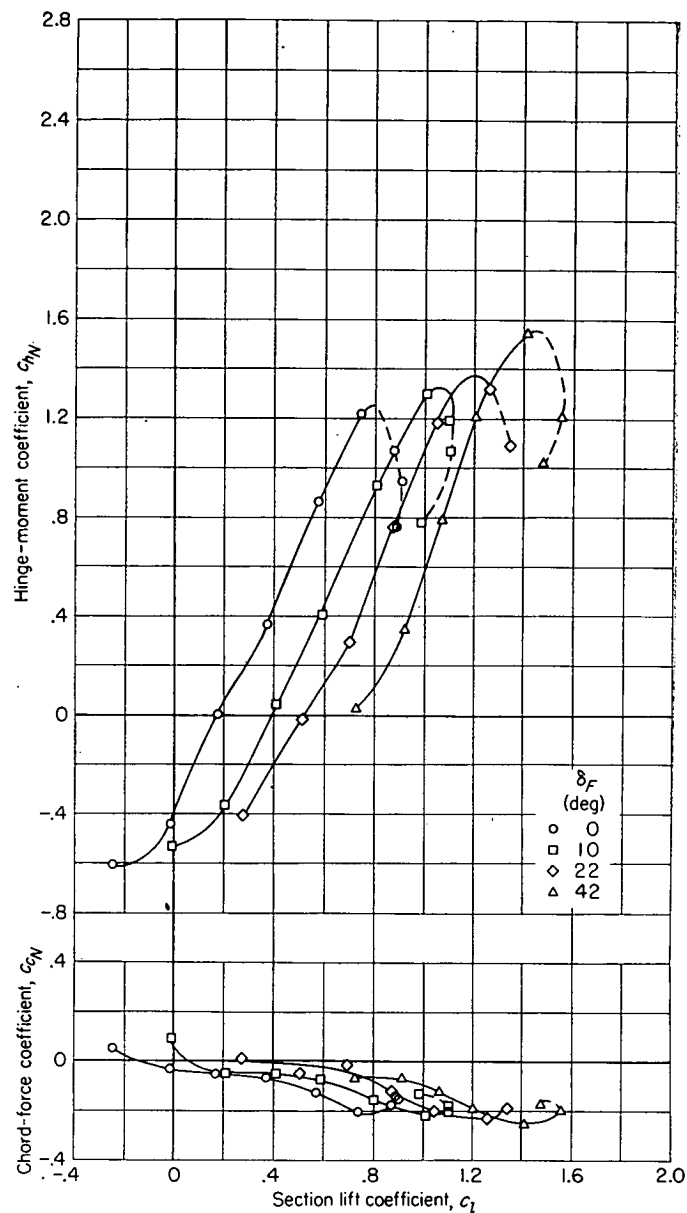
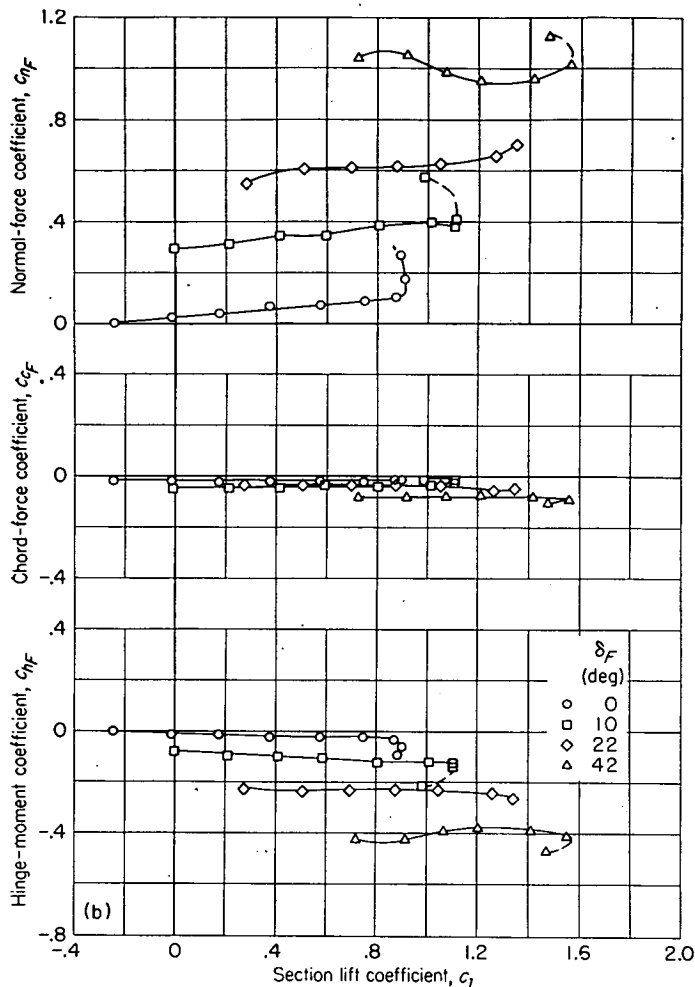


FIGURE 15.—Flap-section load and hinge-moment characteristics of a 6-percent-thick symmetrical circular-arc airfoil for various deflections of the 0.20-chord plain trailing-edge flap; $R=2.1 \times 10^6$; $\delta_N=9^\circ$.



(b) Plain trailing-edge flap.

FIGURE 15.—Concluded.

As a basis for extending the analysis to apply to sharp-edge airfoils having trailing-edge flap-chord ratios other than 0.20, the normal-force distribution $P_{bdF}/c_{l_{bdF}}$ was determined from the pressure distribution at the ideal lift coefficient for several trailing-edge flap deflections by the following relation:

$$\frac{P_{bdF}}{c_{l_{bdF}}} = \frac{\left(\frac{v}{V}\right)_{i_U}^2 - \left(\frac{v}{V}\right)_{i_L}^2}{c_{l_{bdF}}} = 4 \left(\frac{\Delta v}{V}\right)_{bd} \left(\frac{v}{V}\right)_{bd}$$

When compared with the distributions presented in reference 9 for a 0.20-chord trailing-edge flap, good agreement was obtained. It is probable, therefore, that the normal-force distribution $P_{bdF}/c_{l_{bdF}}$ and, consequently, the velocity distribution may be determined with satisfactory precision for any desired trailing-edge flap-chord ratio and deflection from table V (taken from ref. 9, table III).

Additional velocity distribution.—The values of the local incremental velocity ratio $\Delta v_a/V$ were determined from the

experimental data and equation (4). When plotted as a function of the additional lift coefficient ($c_{l_a} = c_l - c_{l_{bd}}$), these values were found to be essentially independent of both leading-edge-flap and trailing-edge-flap deflections. Average values of $\Delta v_a/V$ are plotted against c_{l_a} in figure 26 for various chordwise positions. It is thought that these values of $\Delta v_a/V$ (fig. 26) can be used for various flap-chord ratios because, after the leading edge has caused separation of the flow, any differences in airfoil contour behind that point would have only secondary effects on $\Delta v_a/V$.

Effective basic velocity distribution.—The theoretical velocity distribution v/V about any symmetrical airfoil at zero lift can be calculated by the general methods of references 5 and 6. The effective values of v/V which must be employed in equations (1) and (2), however, vary with both additional lift coefficient and trailing-edge-flap deflection because of separation phenomena. The increment v'/V which must be added to the theoretical basic velocity distribution was determined from the following relation:

$$\frac{v'}{V} = \frac{\left(\frac{v}{V}\right)_U + \left(\frac{v}{V}\right)_L}{2} - \frac{v}{V} \quad (5)$$

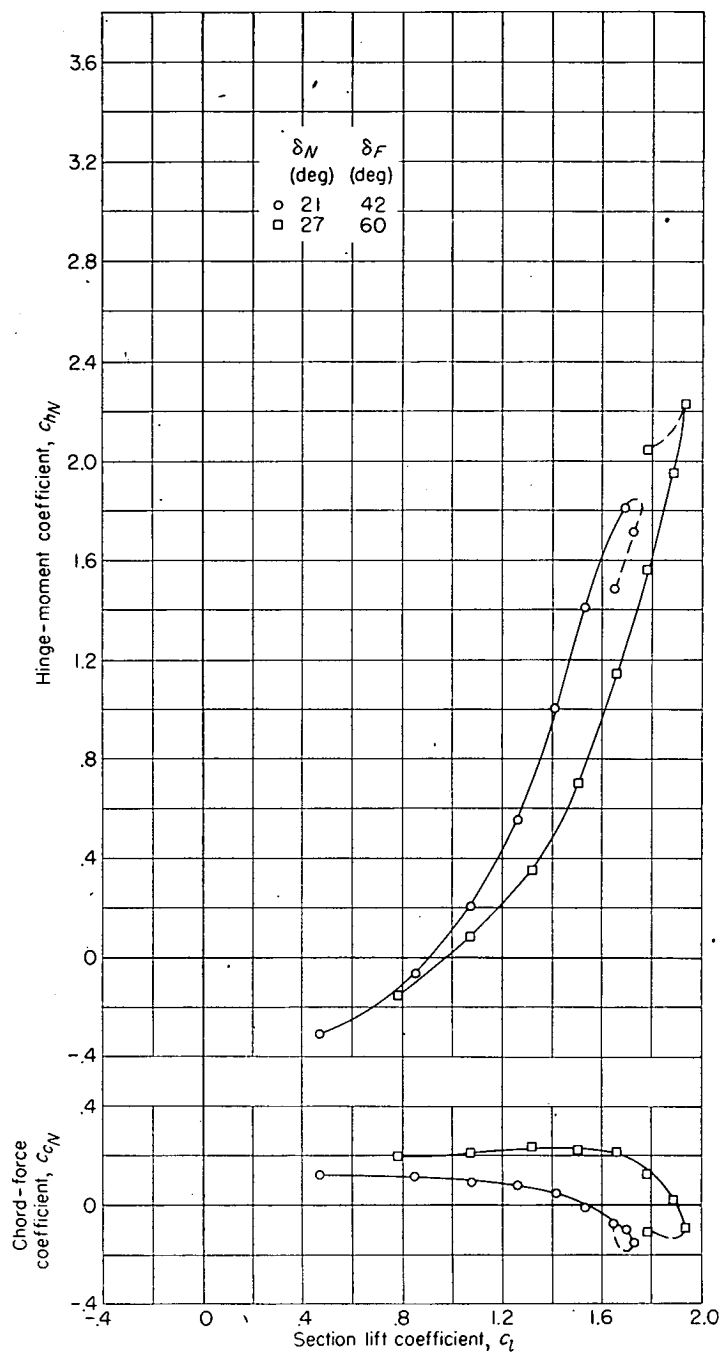
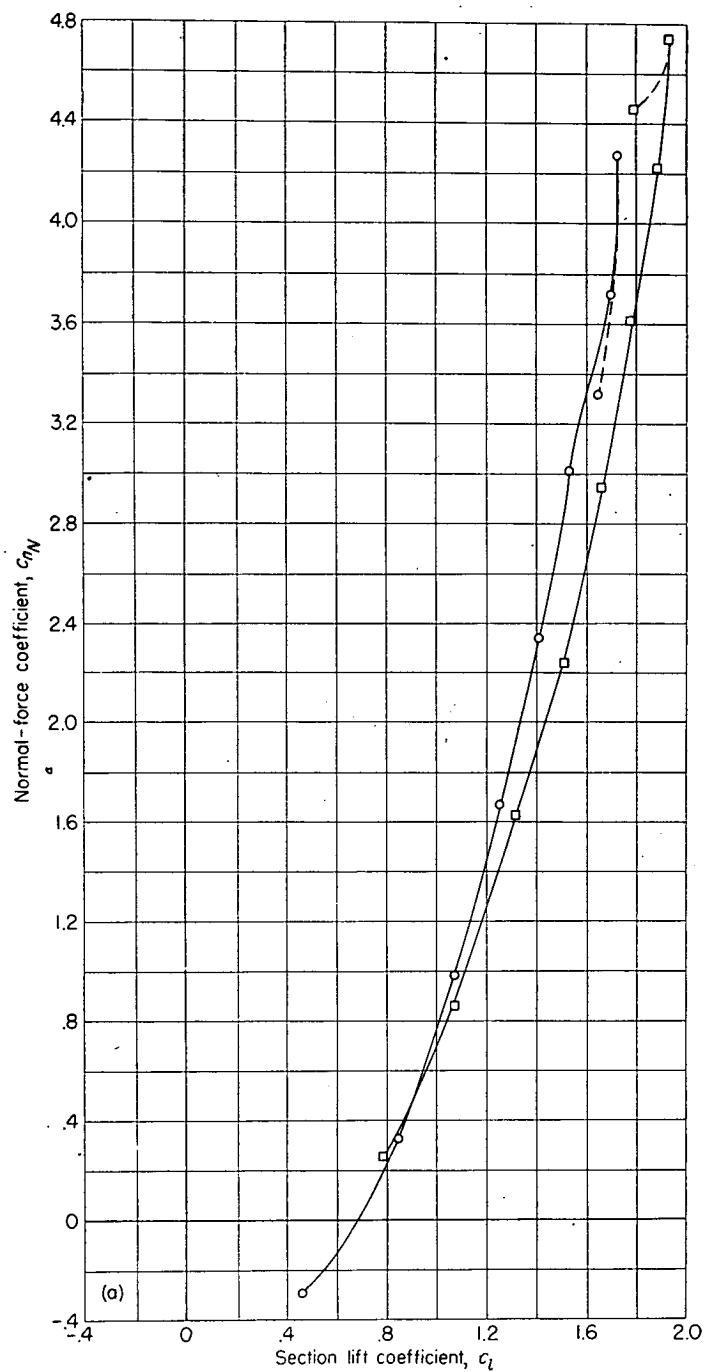
Since v'/V is a function of both trailing-edge-flap deflection and lift coefficient, it may be broken down into two components $(v'/V)_F$ and $(v'/V)_a$, respectively. The values of $(v'/V)_F$ were determined first from equation (5) by using the experimental pressure distributions at the design lift coefficient. Values of the total change in basic velocity distribution $\frac{v'}{V} = \left(\frac{v'}{V}\right)_F + \left(\frac{v'}{V}\right)_a$ were determined from equation (5) by using the experimental pressure distributions at various lift coefficients. The values of $(v'/V)_F$ were subtracted from the results thus obtained to obtain $(v'/V)_a$. It should be pointed out that deflection of the leading-edge flap had no appreciable effect on the shape of these velocity distributions when expressed as a function of c_{l_a} .

For various chordwise positions, values of $(v'/V)_F$ are presented in figure 27 as a function of trailing-edge-flap deflection. Forward of the 40-percent-chord station, values of this component of velocity were found to be negligibly small. The chordwise position of $(v'/V)_F$ is expressed in

terms of $\frac{x}{c}$ and $\frac{1-x}{E_F}$ for points ahead of and behind the

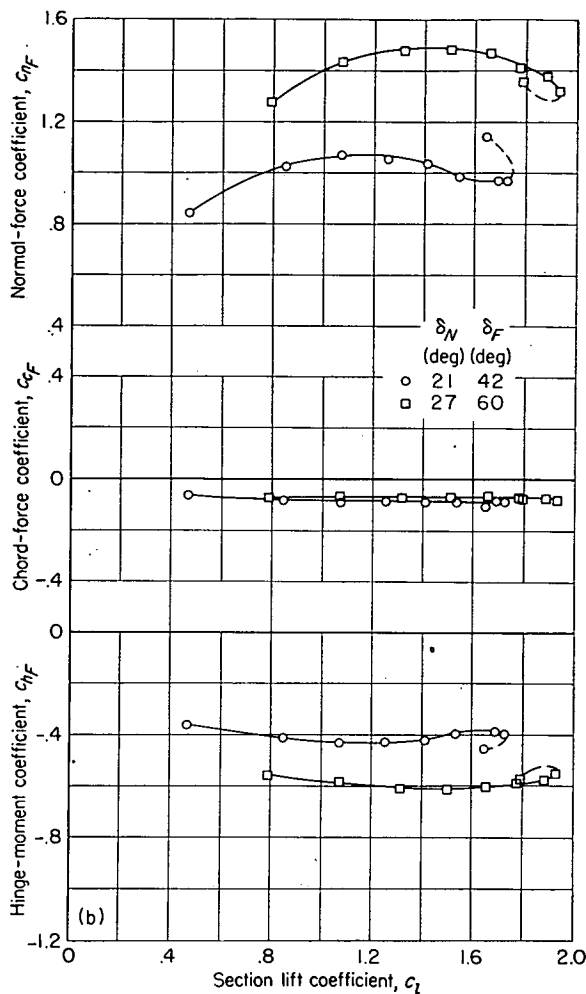
hinge, respectively. In this form, the results are correlated so that they may be applied to sharp-edge airfoils having trailing-edge flaps of varying chord. This method of correlation is thought to be justified since the distribution of $(v'/V)_F$ is a result of separation at the flap hinge and has been shown (ref. 9) to be similar for various hinge locations.

The results of the determination of $(v'/V)_a$ are shown in figure 28. As would be expected, the values are independent of flap deflection when expressed in terms of c_{l_a} .



(a) Plain leading-edge flap.

FIGURE 16.—Section flap load and hinge-moment characteristics of a 6-percent-thick symmetrical circular-arc airfoil for various deflections of the 0.15-chord plain leading-edge flap and 0.20-chord plain trailing-edge flap; $R=2.1 \times 10^6$.



(b) Plain trailing-edge flap.

FIGURE 16.—Concluded.

OUTLINE OF THE METHOD

A method has been developed for the calculation of the low-speed chordwise pressure distribution over various sharp-edge airfoils equipped with plain leading-edge and trailing-edge flaps of arbitrary size and deflection. The assumption has been made that for sharp-edge airfoils the separation phenomena controlling those components of the pressure distribution which cannot be calculated from potential-flow theory do not vary appreciably with variations in the detailed shape of the airfoil.

If the airfoil section for which calculations are to be made satisfies the conditions of the assumptions, the following data may be noted in preparation for the calculation:

- E_N leading-edge flap-chord ratio, $(c_f/c)_N$
- δ_N leading-edge flap deflection, deg
- E_F trailing-edge flap-chord ratio, $(c_f/c)_F$
- δ_F trailing-edge flap deflection, deg
- c_l section lift coefficient
- x/c chordwise locations at which the pressure is to be calculated

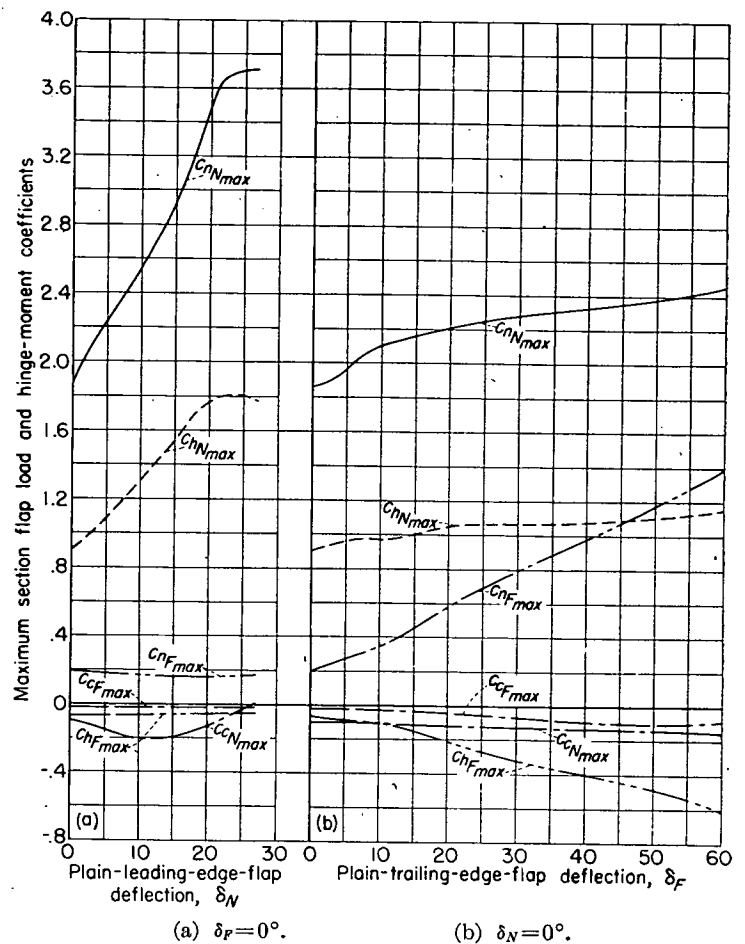


FIGURE 17.—Variation of maximum section flap load and hinge-moment coefficients of a 6-percent-thick symmetrical circular-arc airfoil for various deflections of the 0.15-chord plain leading-edge flap and 0.20-chord plain trailing-edge flap; $R=2.1 \times 10^6$.

The calculations are made in the following manner:

- (1) Find change in ideal lift coefficient caused by leading-edge flap deflection $c_{l_{\delta\delta_N}}$ from the following equation (derived from ref. 9):

$$c_{l_{\delta\delta_N}} = \frac{\pi}{45} \sqrt{E_N(1-E_N)} \delta_N$$

- (2) Find change in ideal lift coefficient caused by trailing-edge flap deflection $c_{l_{\delta\delta_F}}$ from figure 24.
- (3) Find additional lift coefficient c_{l_a}

$$c_{l_a} = c_l - (c_{l_{\delta\delta_N}} + c_{l_{\delta\delta_F}})$$

- (4) Find incremental additional velocity $\Delta v_a/V$ from figure 26.
- (5) Obtain airfoil basic velocity at x/c , v/V from references 5 and 6 or from the following equation (ref. 10) for circular-arc airfoils:

$$\frac{v}{V} = \frac{4}{n^2} \left[\frac{\cosh \eta - \cos \frac{n\pi}{2}}{\cosh \frac{2\eta}{n} + 1} \right]$$

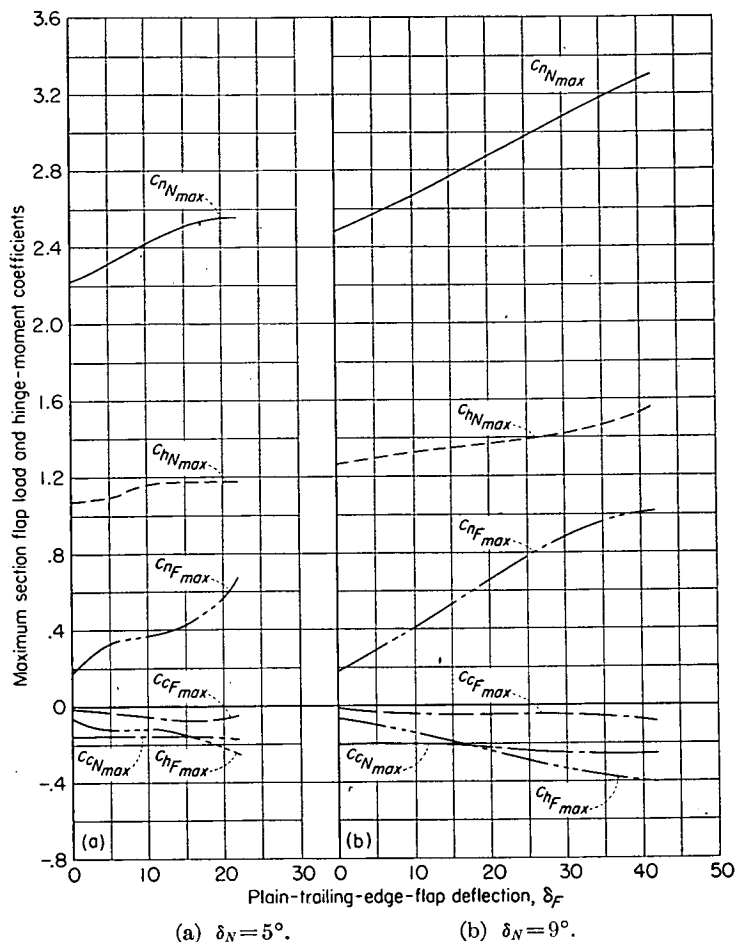


FIGURE 18.—Variation of maximum section flap load and hinge-moment coefficients with plain-trailing-edge-flap deflections of a 6-percent-thick symmetrical circular-arc airfoil for various deflections of the 0.15-chord plain leading-edge flap; $R=2.1 \times 10^6$.

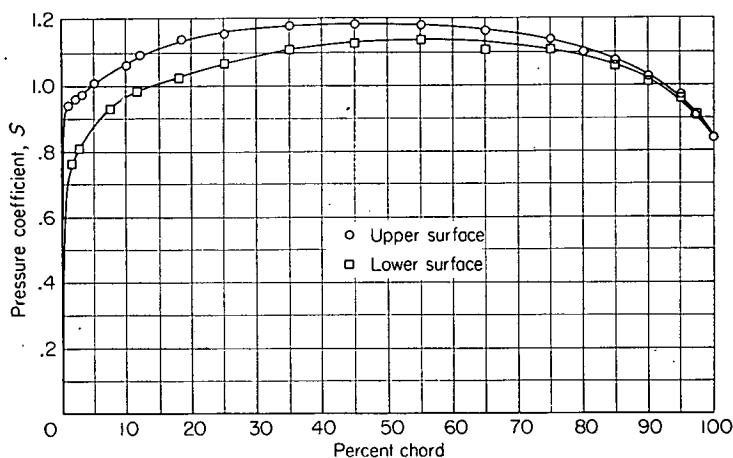


FIGURE 19.—Chordwise variation of pressure coefficient for the 6-percent-thick symmetrical circular-arc airfoil with the flaps neutral; $R=2.1 \times 10^6$; $M=0.15$; $\alpha_0=0.5^\circ$. See table IV (a).

where

$$n = 2 - \frac{4}{\pi} \tan^{-1} \frac{t}{c}$$

$$\eta = \log_e \left(\frac{x/c}{1 - x/c} \right)$$

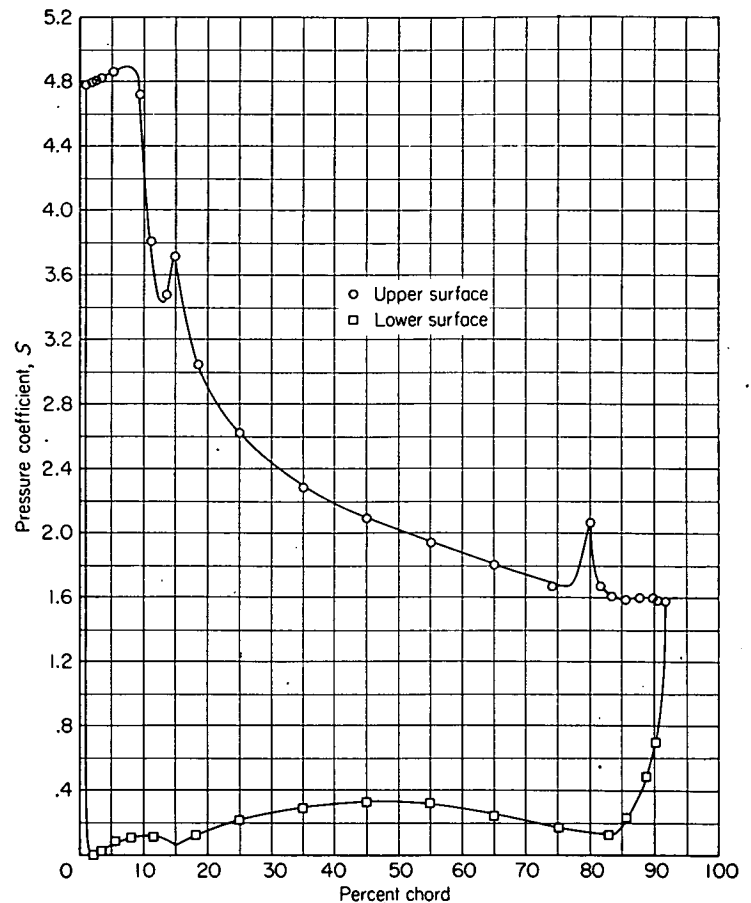


FIGURE 20.—Chordwise variation of pressure coefficient for the 6-percent-thick symmetrical circular-arc airfoil with the plain leading-edge flap deflected 27° and the plain trailing-edge flap deflected 60° ; $R=2.1 \times 10^6$; $M=0.15$; $\alpha_0=10.2^\circ$. See table IV (r).

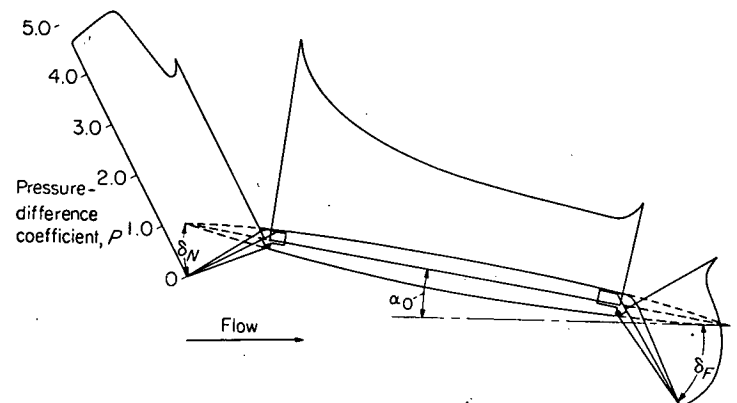


FIGURE 21.—Load distribution over a 6-percent-thick symmetrical circular-arc airfoil; $\alpha_N=27^\circ$; $\delta_F=60^\circ$. $R=2.1 \times 10^6$; $\alpha_0=10.2^\circ$.

(6) Find the pressure-difference coefficient due to the leading-edge flap $P_{b\delta_N}$ from the following equation (derived from ref. 9):

$$P_{b\delta_N} = \frac{1}{45} \delta_N \log_e \left[\frac{\sqrt{(1-E_N) \left(1 - \frac{x}{c}\right)} + \sqrt{E_N x/c}}{\sqrt{(1-E_N) \left(1 - \frac{x}{c}\right)} - \sqrt{E_N x/c}} \right]$$

(7) Find the pressure-difference coefficient due to the trailing-edge flap $P_{b\delta_F}$ from table V and step (2).

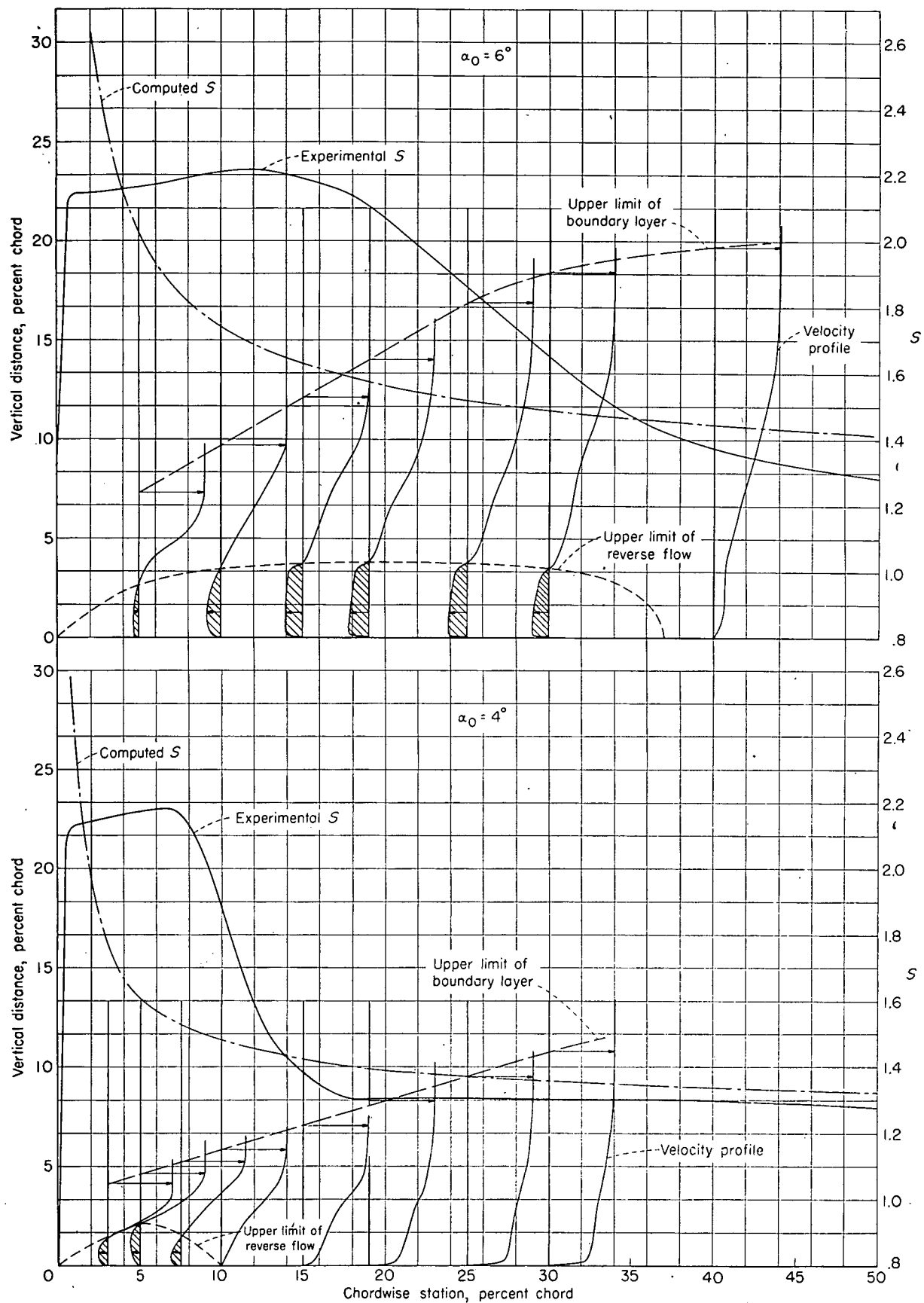


FIGURE 22.—Pressure distributions and velocity profiles over the upper surface of a 6-percent-thick symmetrical circular-arc airfoil at two angles of attack; $\delta_P = \delta_N = 0^\circ$.

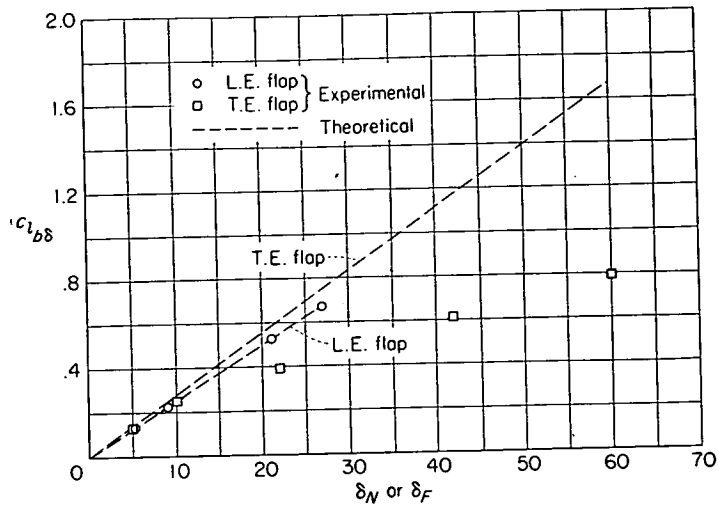


FIGURE 23.—Variation of change in ideal lift coefficient with deflection of the 0.15-chord plain leading-edge flap and 0.20-chord plain trailing-edge flap on the 6-percent-thick symmetrical circular-arc airfoil.

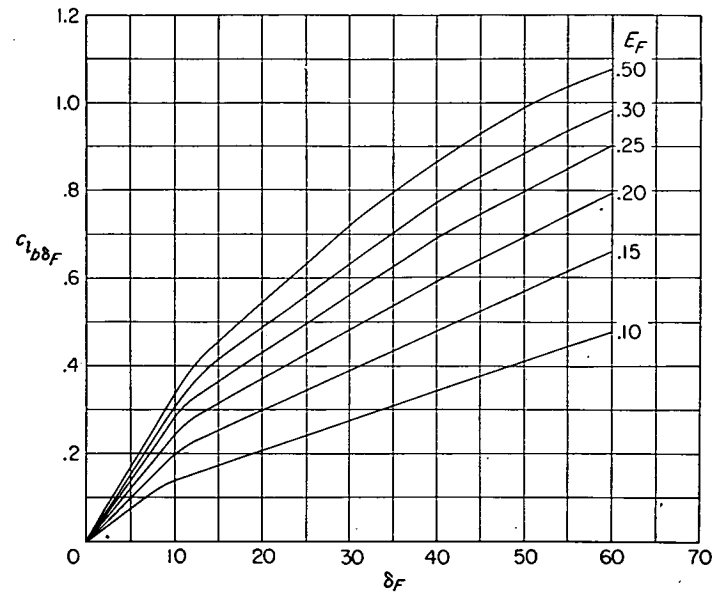


FIGURE 24.—Variation of change in ideal lift coefficient with plain-trailing-edge-flap deflection.

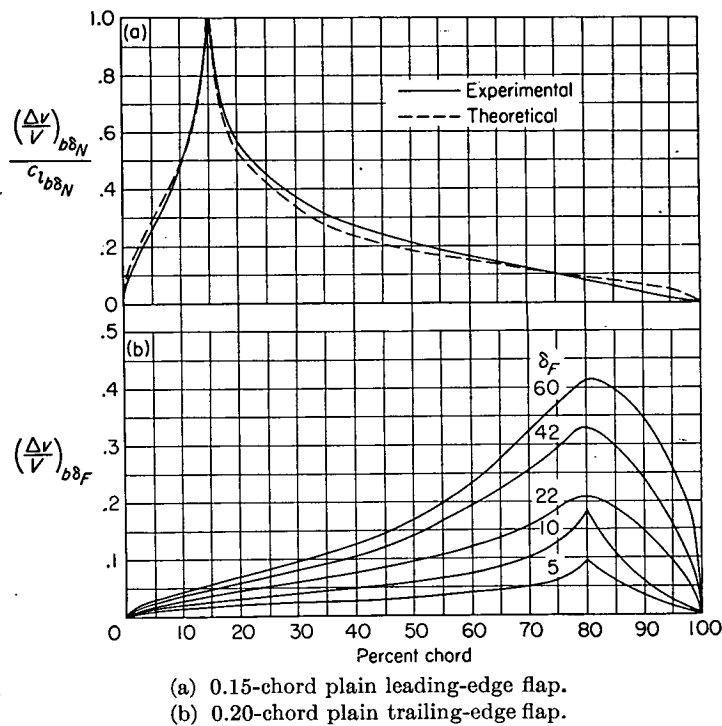


FIGURE 25.—Mean-line velocity distribution for various flap deflections.

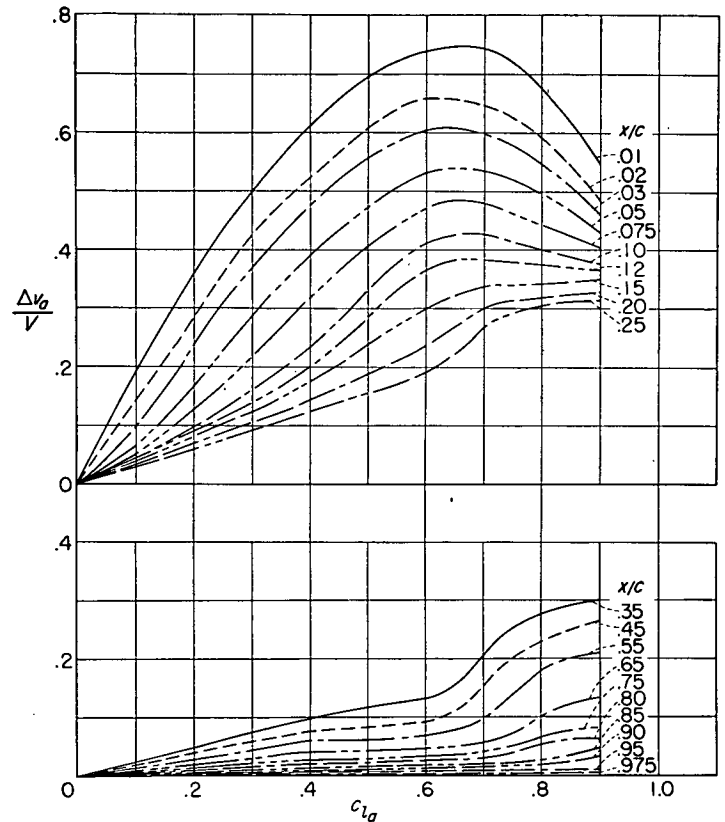


FIGURE 26.—Variation of incremental-additional-velocity ratio with additional lift coefficient.

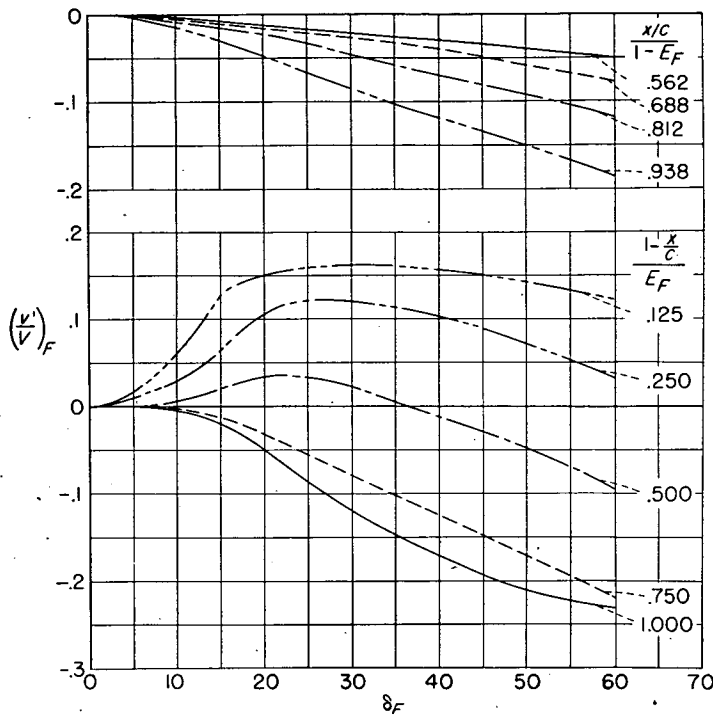


FIGURE 27.—Variation of $(v'/V)_F$ due to separation in the region of the trailing edge with plain-trailing-edge-flap deflection.

(8) Find incremental velocity due to separation at the leading edge $(v'/V)_a$ from figure 28.

(9) Find incremental velocity due to separation at the trailing edge $(v'/V)_F$ from figure 27.

(10) Add the incremental velocities obtained in steps (4), (5), (8), and (9) to obtain the effective velocity on the surface of the basic airfoil at the desired lift coefficient, v_b/V .

(11) Add the pressure-difference coefficients obtained in steps (6) and (7), P_{bs} .

(12) Substitute values from steps (10) and (11) into the following equations obtained from reference 11:

$$S_U = \left(\frac{v_b}{V} + \frac{P_{bs}}{4 \frac{v_b}{V}} \right)^2$$

$$S_L = \left(\frac{v_b}{V} - \frac{P_{bs}}{4 \frac{v_b}{V}} \right)^2$$

EXAMPLE

In order to demonstrate the method, the following example is presented. It is required to determine the pressure coefficients S at 55 percent of the chord of a 6-percent-thick symmetrical circular-arc airfoil section with a 10-percent-chord plain leading-edge flap deflected 30° and a 30-percent-

chord plain trailing-edge flap deflected 40° . The section lift coefficient is assumed to be 1.65.

The airfoil section obviously satisfies the general assumptions of the method. The following data are then assembled:

$$E_N = 0.1 \quad \delta_N = 30^\circ$$

$$E_F = 0.3 \quad \delta_F = 40^\circ$$

$$c_l = 1.65 \quad \frac{x}{c} = 0.55$$

$$(1) \quad c_{l_{bsN}} = \frac{\pi}{45} \sqrt{(0.1)(1-0.1)}(30) = 0.628$$

(2) From figure 24,

$$c_{l_{bsF}} = 0.772$$

$$(3) \quad c_{l_a} = 1.65 - (0.628 + 0.772) = 0.25$$

(4) From figure 26,

$$\frac{\Delta v_a}{V} = 0.034$$

(5) The basic velocity distribution for the 6-percent-thick symmetrical circular-arc airfoil at zero lift has been computed and plotted in figure 29. At $\frac{x}{c} = 0.55$

$$\frac{v}{V} = 1.078$$

$$(6) \quad P_{bsN} = \frac{30}{45} \log_e \left[\frac{\sqrt{(1-0.1)(1-0.55)} + \sqrt{(0.1)(0.55)}}{\sqrt{(1-0.1)(1-0.55)} - \sqrt{(0.1)(0.55)}} \right] = 0.516$$

(7) From table V, for $\frac{x/c}{1-E_F} = 0.786$,

$$\frac{P_{bsF}}{c_{l_{bsF}}} = 1.265$$

then

$$P_{bsF} = (1.265)(0.772) = 0.977$$

(8) From figure 28,

$$\left(\frac{v'}{V} \right)_a = -0.006$$

(9) From figure 27,

$$\left(\frac{v'}{V} \right)_F = -0.062$$

$$(10) \quad \frac{v_b}{V} = 0.034 + 1.078 - 0.006 - 0.062 = 1.044$$

$$(11) \quad P_{bs} = 0.516 + 0.977 = 1.493$$

$$(12) \quad S_U = \left[1.044 + \frac{1.493}{4(1.044)} \right]^2 = 1.97$$

$$S_L = \left[1.044 - \frac{1.493}{4(1.044)} \right]^2 = 0.47$$

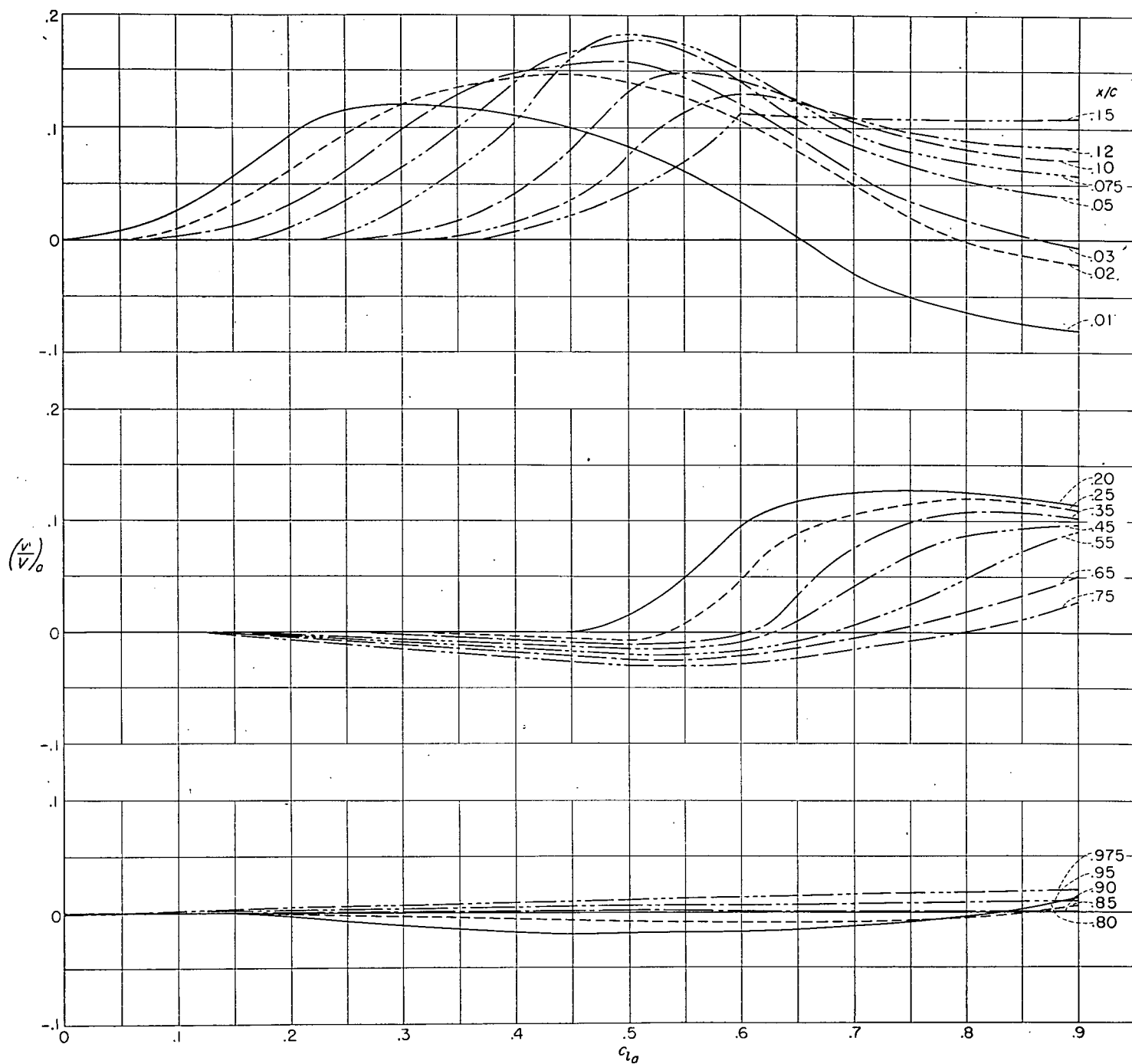


FIGURE 28.—Variation of $(v'/V)_0$ due to separation near the leading edge with additional lift coefficient.

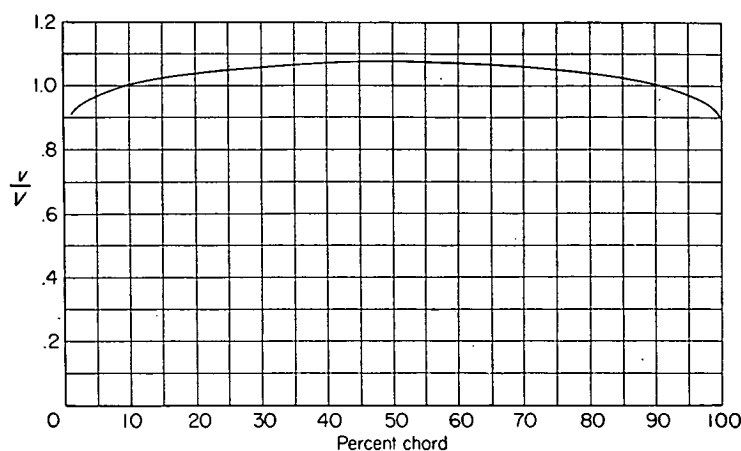


FIGURE 29.—Velocity distribution for the 6-percent-thick symmetrical circular-arc airfoil section; $\alpha_0=0^\circ$; $\delta_N=\delta_F=0^\circ$.

ACCURACY AND LIMITATIONS OF METHOD

In order to justify the method of correlation employed in the development of the present method, the calculated pressure distributions over the 6-percent-thick symmetrical circular-arc airfoil section and the integrated flap normal-force and hinge-moment coefficients for several individual and combined deflections of the plain leading-edge and plain trailing-edge flaps are compared with those obtained experimentally in figures 30 to 32. The flap pressure coefficients (fig. 30) are plotted against the projected chordwise position of the flap orifices on the airfoil chord. The dispersion of the normal-force and hinge-moment results shown in figures 31 and 32 may be considered typical of the accuracy to be expected from the present method. In every case the calculated values of the flap loadings are shown to provide a reasonable quantitative prediction of the experimental loads. For individual deflections of the leading-edge and trailing-edge flaps, the normal-force and hinge-moment characteristics as a rule are within 10 percent of the experimental values. For combined deflections of the leading-edge and trailing-edge flaps, the predicted values of the loads and hinge moments over the trailing-edge flap remain within 10 percent; whereas for the leading-edge flap, the method tends to underestimate these characteristics to a larger degree, depending upon the magnitude of the flap deflections.

The flap hinges were located on the lower surface of the airfoil and the flaps were in contact with the flap skirts so that, in effect, there was no leakage of air between the upper and lower surfaces. Changes in the vertical location of the hinge line are believed to have negligible effects on the airfoil characteristics. If leakage at the flap hinge were present, however, the effects may be such as to alter the separation phenomena particularly at low trailing-edge-flap deflections.

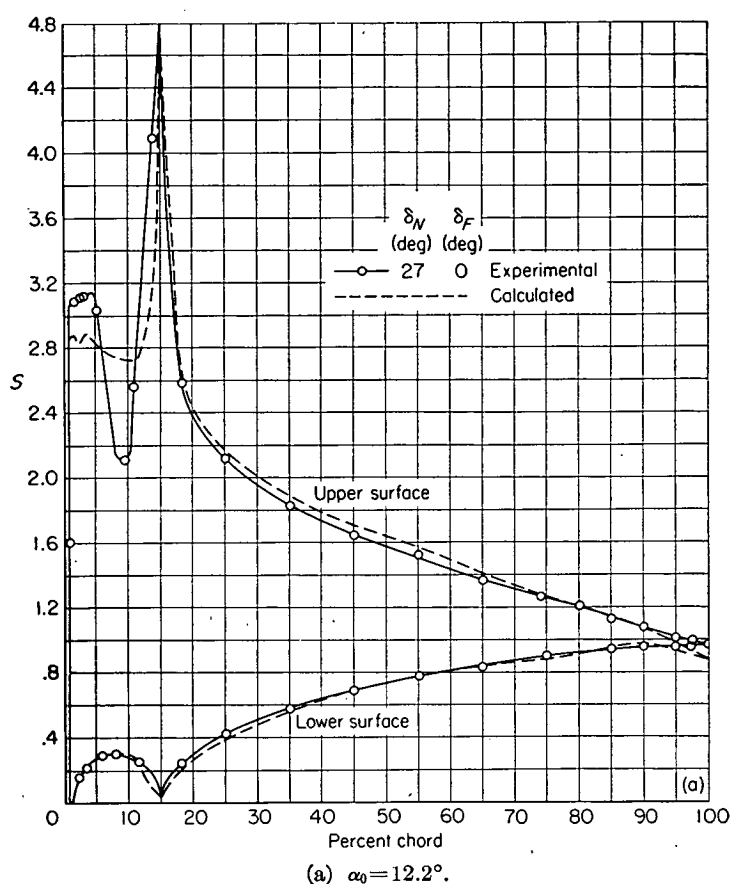


FIGURE 30.—Variation of surface pressure coefficient with percent chord for the 6-percent-thick symmetrical circular-arc airfoil section with 0.15-chord plain leading-edge flap and 0.20-chord plain trailing-edge flap.

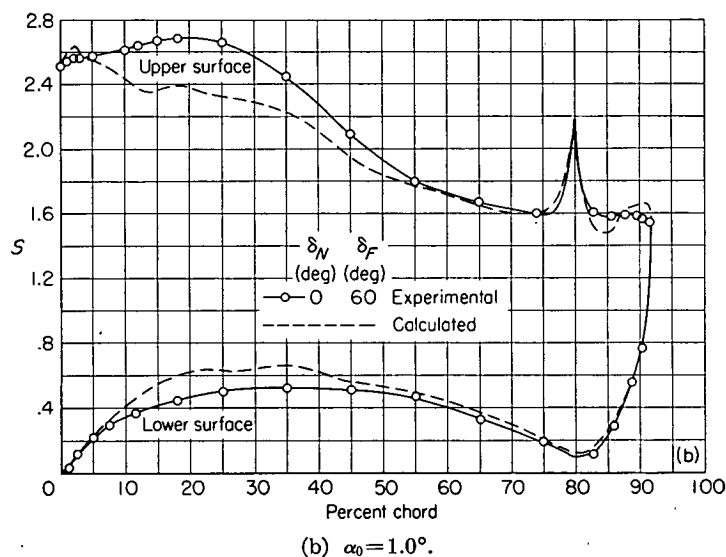


FIGURE 30.—Continued.

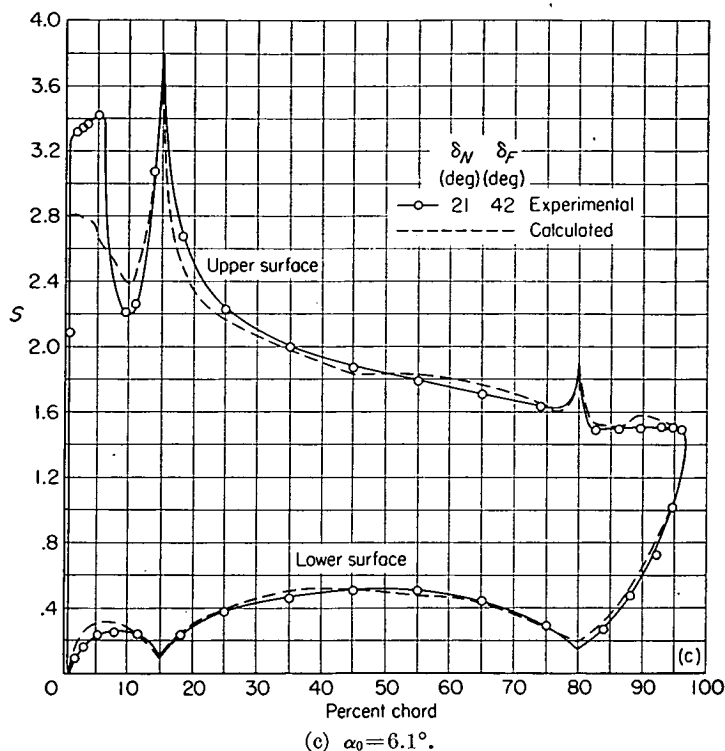
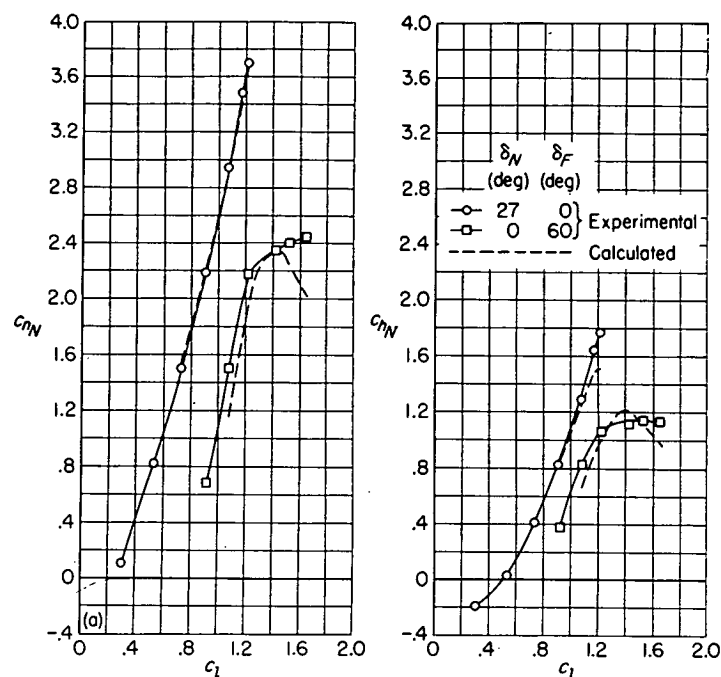
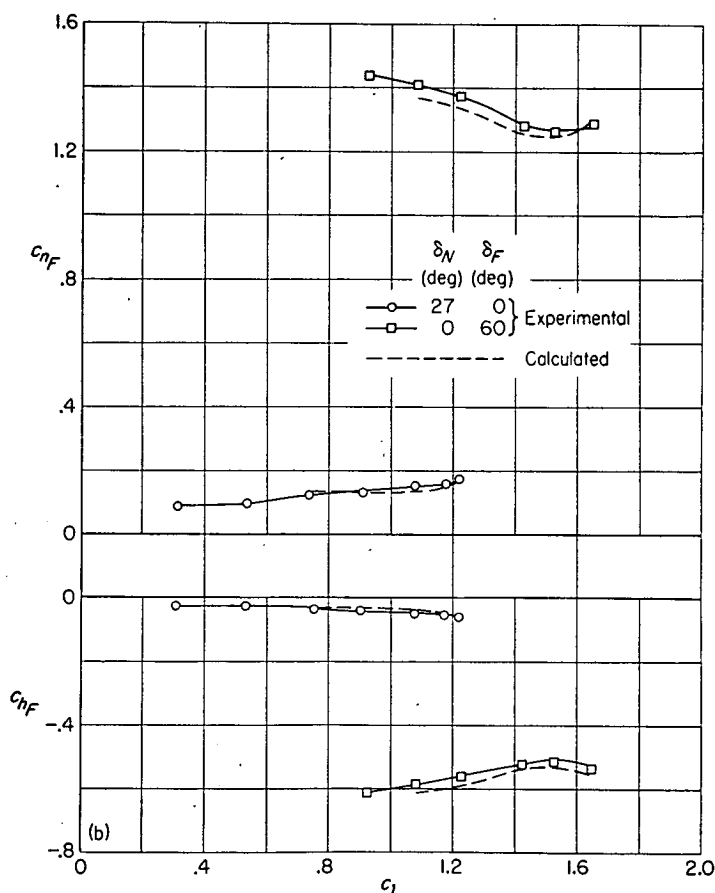


FIGURE 30.—Concluded.



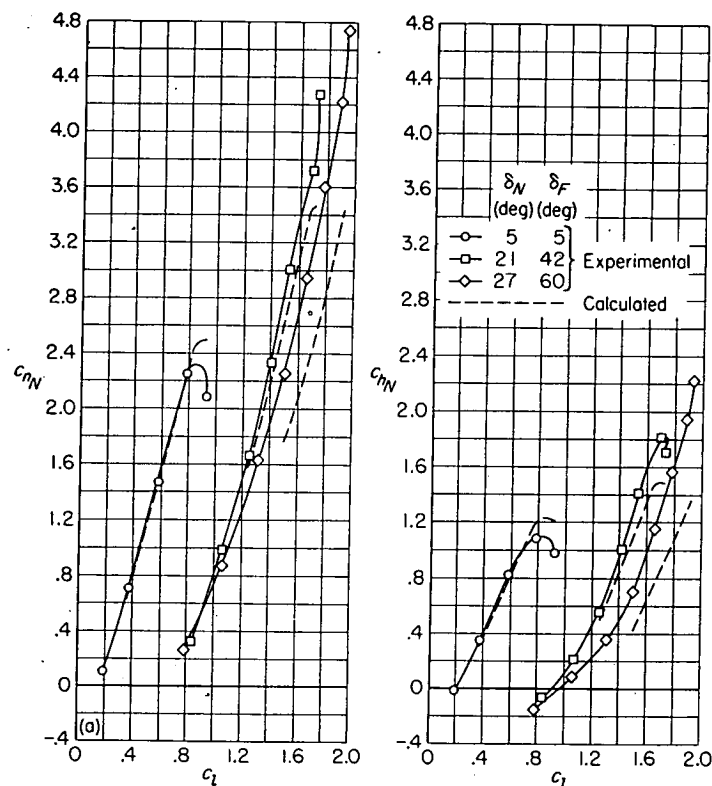
(a) 0.15-chord plain leading-edge flap.

FIGURE 31.—Flap-section normal-force and hinge-moment characteristics of a 6-percent-thick symmetrical circular-arc airfoil section for individual deflections of the plain leading-edge flap and plain trailing-edge flap.



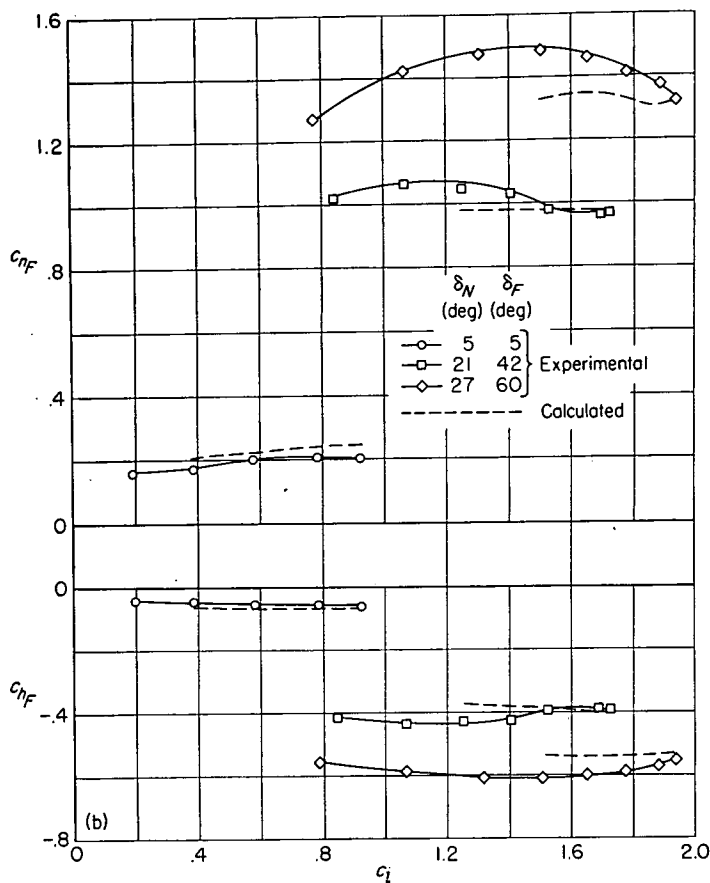
(b) 0.20-chord plain trailing-edge flap.

FIGURE 31.—Concluded.



(a) 0.15-chord plain leading-edge flap.

FIGURE 32.—Flap-section normal-force and hinge-moment characteristics of a 6-percent-thick symmetrical circular-arc airfoil section for combined deflections of the plain leading-edge flap and plain trailing-edge flap.



(b) 0.20-chord plain trailing-edge flap.

FIGURE 32.—Concluded.

Although there may be some tendency of increased Reynolds number to alter the conditions of the boundary layer, the effects of scale will probably be insignificant particularly in view of the negligible variations in section lift coefficient associated with sharp-edge airfoils. (See figs. 9 and 10, and ref. 12.)

CONCLUSIONS

A two-dimensional wind-tunnel investigation was made of two symmetrical circular-arc airfoils 6- and 10-percent thick, with plain leading- and trailing-edge flaps at Reynolds numbers from 0.70×10^6 to 18×10^6 . The results obtained indicated the following conclusions:

1. Maximum lift coefficients of 1.95 and 2.03 were obtained for the optimum combination of leading- and trailing-edge-flap deflection for the 6- and 10-percent-thick airfoils, respectively. The corresponding maximum lift coefficients for the plain airfoils were 0.73 and 0.67, respectively.

2. The optimum combinations of flap deflection for the 6- and 10-percent-thick airfoils were found to be $\delta_N = 30^\circ$, $\delta_F = 60^\circ$ and $\delta_N = 36^\circ$, $\delta_F = 60^\circ$, respectively, where δ_N represents the leading-edge-flap and δ_F the trailing-edge-flap deflections. The results for the 10-percent-thick airfoil with the trailing-edge flap deflected 60° indicate no important changes in the maximum section lift coefficient with small departures from the optimum deflection of the leading-edge flap.

3. The scale effects on the maximum lift coefficient were, in general, small, the largest change being a decrease of about 0.1 for the 10-percent-thick airfoil for Reynolds numbers from 3×10^6 to 9×10^6 .

4. The section pitching-moment characteristics indicated that the aerodynamic center was ahead of the quarter-chord point and moved toward the leading edge when either the leading-edge flap or the trailing-edge flap was deflected.

5. Deflecting the leading-edge flap was more effective in extending the low-drag range to higher section lift coefficients than deflecting the trailing-edge flap.

6. The leading-edge-flap section normal-force and hinge-moment coefficients increased rapidly in a positive direction with increasing lift coefficient; for a given lift coefficient, however, increasing the downward deflection of either flap produced downward increments in the leading-edge flap force and moment coefficients.

7. The trailing-edge flap section normal-force and hinge-moment coefficients are of a similar magnitude to those for a plain trailing-edge flap on a subsonic type of airfoil.

8. The maximum flap normal-force and hinge-moment coefficients were, respectively, 4.74 and 2.24 for the leading-edge flap as compared with 1.48 and -0.61 for the trailing-edge flap.

9. A method for predicting the pressure distribution over sharp-edge airfoils equipped with plain trailing-edge and leading-edge flaps has been developed from a generalization of the pressure-distribution measurements made for this investigation. A comparison of the measured flap loads with those obtained by the generalized method indicates that the methods by which the data were generalized give overall results which are in reasonable agreement with experiment.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., April 16, 1953.

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12. Powter, G. J., and Young, A. D.: Wind Tunnel Tests on a 7.5 Per Cent Thick Bi-Convex Wing With Leading and Trailing Edge Flaps. Rep. No. Aero. 2157, British R.A.E., Sept. 1946.

TABLE I.—ORDINATES FOR THE 6-PERCENT-THICK SYMMETRICAL CIRCULAR-ARC AIRFOIL

[Stations and ordinates given in percent of airfoil chord]

Upper surface		Lower surface	
Station	Ordinate	Station	Ordinate
0	0	0	0
5	.572	5	-.572
10	1.082	10	-1.082
15	1.533	15	-1.533
20	1.922	20	-1.922
25	2.252	25	-2.252
30	2.521	30	-2.521
35	2.731	35	-2.731
40	2.880	40	-2.880
45	2.970	45	-2.970
50	3.000	50	-3.000
55	2.970	55	-2.970
60	2.880	60	-2.880
65	2.731	65	-2.731
70	2.521	70	-2.521
75	2.252	75	-2.252
80	1.922	80	-1.922
85	1.533	85	-1.533
90	1.082	90	-1.082
95	.572	95	-.572
100	0	100	0

Radius of circular arc: 4.182c

TABLE II.—ORDINATES FOR THE 10-PERCENT-THICK SYMMETRICAL CIRCULAR-ARC AIRFOIL

[Stations and ordinates given in percent of airfoil chord]

Upper surface		Lower surface	
Station	Ordinate	Station	Ordinate
0	0	0	0
5	.958	5	-.958
10	1.812	10	-1.812
15	2.562	15	-2.562
20	3.211	20	-3.211
25	3.759	25	-3.759
30	4.207	30	-4.207
35	4.554	35	-4.554
40	4.802	40	-4.802
45	4.951	45	-4.951
50	5.000	50	-5.000
55	4.951	55	-4.951
60	4.802	60	-4.802
65	4.554	65	-4.554
70	4.207	70	-4.207
75	3.759	75	-3.759
80	3.211	80	-3.211
85	2.562	85	-2.562
90	1.812	90	-1.812
95	.958	95	-.958
100	0	100	0

Radius of circular arc: 2.525c

TABLE III.—SUMMARY OF TESTS OF CIRCULAR-ARC AIRFOIL SECTIONS

Airfoil thickness	Tunnel (a)	δ_N (deg)	δ_P (deg)	R	Measurements	Source of data
0.06c	TDT	0	0	3, 6, and 9×10^6	Lift—Drag	Figure 4 (a)
0.10c	TDT	0	0	3, 6, and 9×10^6	Lift—Drag	Figure 4 (b)
0.10c	TDT	0	0	14 and 18×10^6	Lift	Figure 4 (c)
0.06c	TDT	0	0, 20, 40, 60	6×10^6	Lift—Pitching moment	Figure 5 (a)
0.10c	TDT	0	0, 20, 40, 60	6×10^6	Lift—Pitching moment	Figure 5 (b)
0.06c	TDT	0, 10, 20, 30, 40	0	6×10^6	Lift—Pitching moment	Figure 6 (a)
0.10c	TDT	0, 20, 30, 40	0	6×10^6	Lift—Pitching moment	Figure 6 (b)
0.06c	TDT	30	50	6×10^6	Lift	Figure 8 (a)
0.06c	TDT	20, 30, 40	60	6×10^6	Lift	Figure 8 (a)
0.10c	TDT	30	50	6×10^6	Lift	Figure 8 (b)
0.10c	TDT	30, 40, 50	60	6×10^6	Lift	Figure 8 (b)
0.10c	TDT	36	60	3, 6, 9, 14, and 18×10^6	Lift	Figure 8 (c)
0.06c	TDT	30	60	3, 6, and 9×10^6	Lift—Pitching moment	Figure 9 (a)
0.10c	TDT	30	60	3, 6, and 9×10^6	Lift—Pitching moment	Figure 9 (b)
0.06c	LTT	27	60	0.7 and 2.3×10^6	Lift	Figure 10
0.06c	LTT	0	0, 5, 10	2.1×10^6	Lift—Drag	Figure 11
0.06c	LTT	5	0, 5, 10	2.1×10^6	Lift—Drag	Figure 11
0.06c	LTT	10	0, 10	2.1×10^6	Lift—Drag	Figure 11
0.06c	LTT	0, 5, 9, 21, 27	0	2.1×10^6	Pressure distribution	Table IV
0.06c	LTT	0	5, 10, 22, 42, 60	2.1×10^6	Pressure distribution	Table IV
0.06c	LTT	5	5, 10, 22	2.1×10^6	Pressure distribution	Table IV
0.06c	LTT	9	10, 22, 42	2.1×10^6	Pressure distribution	Table IV
0.06c	LTT	21	42	2.1×10^6	Pressure distribution	Table IV
0.06c	LTT	27	60	2.1×10^6	Pressure distribution	Table IV

* TDT Langley two-dimensional low-turbulence pressure tunnel.
 LTT Langley two-dimensional low-turbulence tunnel.

TABLE IV.—PRESSURE COEFFICIENTS FOR THE AIRFOIL WITH THE LEADING-EDGE FLAP AND TRAILING-EDGE FLAP DEFLECTED IN VARIOUS COMBINATIONS AND AT SEVERAL ANGLES OF ATTACK

[$R=2.1 \times 10^6$; $M=0.15$](a) $\delta_N=0^\circ$, $\delta_F=0^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—								
		0°	0.5°	2.0°	4.1°	6.1°	8.1°	9.1°	10.2°	12.2°
1	0	0.17	0	0.11	0.78	1.36	1.42	1.33	1.25	1.24
2	1	.81	.94	1.96	2.13	2.14	1.94	1.82	1.73	1.64
3	2	.87	.96	1.72	2.14	2.15	1.94	1.83	1.73	1.64
4	3	.89	.98	1.39	2.15	2.15	1.95	1.83	1.74	1.64
5	5	.94	1.01	1.17	2.18	2.16	1.96	1.84	1.74	1.65
6	7.5	.98	1.04	1.19	2.19	2.18	1.97	1.85	1.75	1.65
7	10	1.01	1.06	1.21	1.88	2.21	1.98	1.86	1.75	1.65
8	12	1.04	1.10	1.22	1.59	2.22	2.00	1.87	1.76	1.66
9	15	1.03	1.02	1.14	1.38	2.18	2.00	1.87	1.77	1.67
b 10	16.1	1.02	.98	.90	.79	.74	.69	.68	.55	.55
A	18.3	1.07	1.14	1.25	1.31	2.14	2.03	1.90	1.79	1.67
B	25	1.12	1.16	1.25	1.31	1.86	2.02	1.91	1.81	1.69
C	35	1.16	1.18	1.25	1.31	1.47	1.93	1.89	1.81	1.71
D	45	1.17	1.18	1.24	1.29	1.32	1.76	1.81	1.78	1.72
E	55	1.17	1.18	1.23	1.26	1.26	1.59	1.71	1.73	1.72
F	65	1.16	1.17	1.20	1.22	1.22	1.44	1.59	1.72	1.71
G	74	1.13	1.14	1.16	1.17	1.17	1.32	1.49	1.61	1.71
b 17	77.03	1.10	1.10	1.10	1.10	1.11	1.23	1.37	1.49	1.61
b 18	78.3	1.10	1.10	1.10	1.10	1.11	1.23	1.36	1.48	1.60
b 19	80	1.10	1.10	1.12	1.13	1.14	1.26	1.42	1.54	1.67
20	85	1.06	1.08	1.09	1.09	1.09	1.22	1.38	1.53	1.69
21	90	1.02	1.03	1.04	1.04	1.05	1.18	1.34	1.49	1.66
22	95	.97	.97	.96	.97	1.01	1.15	1.29	1.44	1.60
23	97.5	.92	.92	.92	.93	.98	1.13	1.28	1.39	1.56
24	100	.82	.84	.85	.86	.94	1.11	1.24	1.35	1.50
11	1.3	.87	.77	.50	.26	.15	.11	.11	.12	.09
12	2.6	.89	.81	.59	.37	.26	.22	.21	.21	.19
13	5	.95	.89	.72	.52	.40	.35	.35	.35	.33
14	7.5	.98	.93	.78	.61	.50	.45	.44	.44	.42
15	11.4	1.01	.98	.85	.69	.59	.54	.55	.55	.53
H	18.1	1.05	1.02	.93	.79	.71	.66	.67	.67	.66
I	25	1.09	1.07	.99	.88	.79	.76	.77	.77	.77
J	35	1.12	1.11	1.04	.94	.88	.85	.87	.88	.88
K	45	1.15	1.13	1.07	1.00	.95	.93	.95	.97	.99
L	55	1.15	1.14	1.09	1.02	.99	.99	1.02	1.04	1.07
M	65	1.12	1.11	1.09	1.02	1.01	1.02	1.06	1.09	1.13
N	75	1.10	1.11	1.09	1.04	1.04	1.06	1.11	1.15	1.21
25	85	1.05	1.06	1.05	1.02	1.04	1.09	1.15	1.20	1.29
26	90	1.01	1.02	1.02	1.00	1.03	1.09	1.16	1.23	1.33
27	95	.94	.96	.96	.95	1.00	1.09	1.17	1.26	1.38
28	97.5	.89	.93	.93	.93	.78	1.09	1.19	1.29	1.42
b 16	15	1.02	.99	.89	.77	.68	.64	.63	.63	.62
b 29	80.3	1.07	1.07	1.06	1.03	1.02	1.08	1.13	1.20	1.28

* Angle of attack for maximum lift.
 b Internal pressure.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(b) $\delta_N=5^\circ$, $\delta_F=0^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—							
		-2.0°	0°	2.0°	4.1°	6.1°	8.1°	9.1°	*10.2°
1	0	1.78	1.83	0.02	0.34	1.26	1.67	1.56	1.39
2	1	.28	.49	.99	2.14	2.47	2.37	2.15	1.97
3	2	.42	.63	1.01	2.16	2.48	2.37	2.15	1.97
4	3	.50	.69	1.03	2.08	2.49	2.38	2.16	1.98
5	5	.62	.80	1.03	1.46	2.53	2.40	2.18	1.98
6	7.5	.74	.90	1.15	1.37	2.56	2.42	2.19	2.00
7	10	.83	.98	1.19	1.37	2.29	2.50	2.21	2.02
8	12	.93	1.07	1.27	1.45	1.92	2.46	2.24	2.04
9	15	1.07	1.19	1.38	1.54	1.67	2.42	2.25	2.05
b 10	16.1	1.29	.88	.81	.69	.60	.55	.53	.53
A	18.3	1.03	1.15	1.33	1.49	1.55	2.29	2.23	2.05
B	25	1.05	1.15	1.23	1.41	1.48	2.02	2.14	2.03
C	35	1.09	1.16	1.27	1.37	1.44	1.63	1.93	1.94
D	45	1.11	1.17	1.26	1.33	1.39	1.42	1.69	1.82
E	55	1.13	1.18	1.24	1.30	1.34	1.33	1.49	1.67
F	65	1.12	1.15	1.20	1.25	1.27	1.26	1.36	1.54
G	74	1.11	1.13	1.17	1.19	1.21	1.20	1.27	1.44
b 17	77.03	1.09	1.09	1.10	1.11	1.12	1.14	1.18	1.33
b 18	78.3	1.09	1.09	1.10	1.11	1.12	1.13	1.18	1.32
19	80	1.07	1.09	1.13	1.15	1.17	1.16	1.22	1.37
20	85	1.07	1.07	1.09	1.10	1.12	1.13	1.18	1.34
21	90	1.04	1.03	1.04	1.05	1.06	1.08	1.15	1.29
22	95	.99	.97	.97	.98	.99	1.04	1.12	1.26
23	97.5	.94	.93	.92	.93	.95	1.01	1.10	1.23
24	100	.87	.85	.85	.87	.90	.98	1.07	1.20
11	1.3	1.80	1.71	.72	.38	.19	.11	.11	.11
12	2.6	1.81	1.19	.75	.48	.31	.21	.20	.21
13	5	1.83	1.04	.82	.60	.44	.35	.34	.34
14	7.5	1.81	1.04	.84	.65	.52	.42	.41	.41
15	11.4	1.61	.99	.84	.69	.57	.49	.43	.47
H	18.1	1.23	.99	.85	.64	.65	.67	.63	.69
I	25	1.12	1.07	.96	.85	.76	.69	.68	.69
J	35	1.17	1.11	1.02	.94	.86	.80	.79	.80
K	45	1.20	1.14	1.07	.99	.93	.88	.88	.89
L	55	1.19	1.15	1.09	1.03	.97	.94	.94	.96
M	65	1.16	1.13	1.09	1.05	1.00	.98	.98	1.01
N	75	1.12	1.11	1.03	1.05	1.01	1.01	1.03	1.07
25	85	1.07	1.06	1.05	1.03	1.01	1.03	1.06	1.11
26	90	1.04	1.02	1.02	1.01	1.00	1.03	1.06	1.12
27	95	.98	.96	.96	.96	.97	1.01	1.06	1.15
28	97.5	.94	.91	.93	.94	.94	1.00	1.06	1.16
b 16	15	1.29	.88	.80	.69	.58	.50	.50	.50
b 29	80.3	1.08	1.06	1.05	1.04	1.01	1.01	1.05	1.10

* Angle of attack for maximum lift.
 b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(c) $\delta_N=9^\circ$, $\delta_P=0^\circ$

Orifice	x/c	Pressure coefficients for section angle of attack—								
		-2.0°	0°	2.0°	4.1°	6.1°	8.1°	10.2°	$\approx 11.2^\circ$	12.2°
1	0	1.70	1.62	1.00	0.04	0.28	1.89	1.92	1.65	0.88
2	1	.18	.35	.69	1.55	2.42	2.81	2.45	2.19	1.98
3	2	.33	.51	.80	1.19	2.45	2.82	2.46	2.20	1.99
4	3	.40	.58	.85	1.20	2.46	2.83	2.46	2.20	1.99
5	5	.55	.71	.96	1.26	2.36	2.86	2.48	2.21	2.00
6	7.5	.69	.85	1.06	1.33	1.68	2.90	2.51	2.24	2.01
7	10	.80	.96	1.17	1.40	1.53	2.74	2.53	2.26	2.03
8	12	.94	1.10	1.31	1.53	1.64	2.35	2.53	2.27	2.04
9	15	1.25	1.43	1.66	1.90	1.94	1.96	2.49	2.27	2.06
10	16.1	1.18	.92	.63	.62	.51	.42	.38	.39	.40
A	18.3	1.07	1.20	1.34	1.53	1.67	1.75	2.37	2.21	2.03
B	25	1.06	1.16	1.29	1.43	1.55	1.62	2.17	2.14	2.01
C	35	1.09	1.17	1.27	1.37	1.47	1.53	1.83	1.97	1.94
D	45	1.11	1.18	1.25	1.33	1.40	1.45	1.57	1.78	1.85
E	55	1.12	1.18	1.24	1.29	1.34	1.38	1.42	1.62	1.75
F	65	1.12	1.16	1.20	1.24	1.28	1.30	1.31	1.49	1.65
G	74	1.11	1.13	1.16	1.18	1.21	1.23	1.23	1.39	1.56
b17	77.03	1.07	1.09	1.10	1.11	1.12	1.12	1.16	1.29	1.44
b18	78.3	1.07	1.09	1.10	1.11	1.12	1.12	1.16	1.28	1.43
19	80	1.07	1.09	1.12	1.15	1.17	1.18	1.20	1.33	1.49
20	85	1.06	1.07	1.09	1.10	1.11	1.12	1.16	1.29	1.47
21	90	1.04	1.03	1.04	1.04	1.05	1.06	1.12	1.26	1.43
22	95	.99	.98	.96	.96	.98	1.00	1.09	1.22	1.39
23	97.5	.95	.92	.91	.91	.93	.96	1.07	1.20	1.36
24	100	.89	.85	.85	.87	.88	.92	1.04	1.17	1.30
11	1.3	1.71	1.62	.93	.53	.28	.14	.10	.10	.11
12	2.6	1.72	1.64	.90	.59	.37	.24	.19	.19	.20
13	5	1.72	1.62	.89	.66	.49	.35	.30	.30	.31
14	7.5	1.74	1.47	.87	.68	.53	.42	.36	.37	.37
15	11.4	1.75	1.13	.80	.66	.54	.45	.41	.41	.41
H	18.1	1.74	.92	.81	.69	.59	.52	.48	.48	.49
I	25	1.45	1.01	.94	.83	.73	.66	.61	.62	.63
J	35	1.15	1.09	1.01	.91	.84	.77	.74	.75	.77
K	45	1.12	1.12	1.06	.98	.91	.85	.83	.85	.87
L	55	1.14	1.13	1.08	1.01	.96	.90	.92	.92	.96
M	65	1.13	1.12	1.09	1.04	.99	.96	.95	.98	1.02
N	75	1.11	1.10	1.07	1.04	1.01	.99	.99	1.04	1.09
25	85	1.06	1.06	1.04	1.01	1.00	1.00	1.03	1.08	1.15
26	90	1.02	1.02	1.01	.99	.99	.99	1.04	1.10	1.18
27	95	.97	.95	.95	.95	.95	.96	1.04	1.11	1.21
28	97.5	.94	.92	.91	.93	.93	.95	1.04	1.12	1.24
b16	15	1.73	.91	.63	.62	.50	.42	.39	.39	.39
b29	80.3	1.07	1.06	1.05	1.02	1.00	1.00	1.03	1.09	1.16

* Angle of attack for maximum lift.
 b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(d) $\delta_N=21^\circ$, $\delta_P=0^\circ$

Orifice	x/c	Pressure coefficients for section angle of attack—								
		0°	2.0°	4.1°	6.1°	8.1°	10.2°	12.2°	$\approx 14.2^\circ$	16.2°
1	0	1.54	1.31	1.17	0.47	0.16	1.18	2.46	3.44	2.57
2	1	.09	.23	.45	.83	0.20	2.82	3.41	3.82	3.02
3	2	.23	.39	.61	.94	1.65	2.86	3.43	3.84	3.04
4	3	.31	.48	.70	1.00	1.38	2.88	3.45	3.86	3.05
5	5	.47	.65	.87	1.14	1.45	2.67	3.50	3.89	3.08
6	7.5	.71	.87	1.09	1.36	1.64	1.88	3.42	3.94	3.12
7	10	.86	1.05	1.26	1.50	1.80	1.89	2.64	3.77	3.01
8	12	1.10	1.31	1.53	1.78	2.20	2.22	2.24	3.08	2.74
9	15	1.86	2.13	2.40	2.67	3.07	3.18	2.83	2.70	2.55
10	16.1	1.99	2.33	2.74	3.03	3.21	3.10	2.86	2.45	2.15
A	18.3	1.36	1.54	1.75	1.98	2.22	2.38	2.42	2.40	2.30
B	25	1.24	1.38	1.51	1.66	1.83	1.97	2.03	2.07	2.10
C	35	1.21	1.31	1.42	1.51	1.63	1.72	1.77	1.79	1.91
D	45	1.21	1.28	1.36	1.43	1.51	1.58	1.60	1.61	1.80
E	55	1.20	1.26	1.32	1.37	1.43	1.47	1.47	1.47	1.65
F	65	1.18	1.21	1.26	1.29	1.34	1.36	1.35	1.35	1.56
G	74	1.15	1.17	1.20	1.22	1.25	1.26	1.24	1.26	1.50
b17	77.03	1.09	1.10	1.11	1.12	1.14	1.15	1.14	1.17	1.39
b18	78.3	1.09	1.10	1.11	1.12	1.13	1.13	1.13	1.17	1.38
19	80	1.12	1.14	1.16	1.19	1.20	1.20	1.19	1.20	1.43
20	85	1.09	1.10	1.11	1.12	1.12	1.13	1.12	1.17	1.42
21	90	1.05	1.05	1.05	1.05	1.06	1.07	1.07	1.13	1.39
22	95	.99	.98	.98	.98	.98	1.01	1.03	1.11	1.35
23	97.5	.95	.93	.93	.93	.94	.98	1.01	1.10	1.33
24	100	.89	.87	.87	.88	.90	.94	.98	1.07	1.29
11	1.3	1.54	1.31	1.14	.73	.38	.20	.07	.03	.06
12	2.6	1.55	1.32	1.15	.70	.44	.28	.16	.10	.12
13	5	1.55	1.33	1.13	.66	.49	.35	.25	.19	.20
14	7.5	1.56	1.33	1.05	.62	.47	.37	.28	.23	.24
15	11.4	1.57	1.34	.90	.49	.42	.33	.26	.23	.24
H	18.1	1.58	1.37	.68	.47	.39	.32	.26	.24	.25
I	25	1.61	1.23	.72	.67	.58	.51	.45	.40	.42
J	35	1.50	.91	.85	.79	.72	.65	.59	.55	.57
K	45	1.21	.93	.94	.88	.82	.75	.69	.67	.69
L	55	1.02	.99	.99	.94	.88	.83	.78	.76	.80
M	65	1.02	1.02	1.01	.97	.93	.87	.84	.83	.89
N	75	1.01	1.03	1.02	.99	.96	.93	.90	.90	.98
25	85	1.00	1.01	1.00	.98	.97	.96	.94	.96	1.06
26	90	.98	.98	.99	.98	.96	.96	.96	.99	1.11
27	95	.95	.95	.95	.94	.95	.96	.96	1.02	1.16
28	97.5	.93	.93	.92	.92	.93	.95	.98	1.04	1.20
b16	15	1.58	1.35	.78	.33	.32	.34	.28	.24	.24
b29	80.3	1.01	1.02	1.01	.99	.97	.95	.93	.95	1.05

* Angle of attack for maximum lift.
 b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(e) $\delta_N=27^\circ$, $\delta_P=0^\circ$

Orifice	x/c	Pressure coefficients for section angle of attack—								
		0°	2.0°	4.1°	6.1°	8.1°	10.2°	12.2°	15.2°	18.3°
1	0	1.58	1.39	1.14	1.20	0.04	0.35	1.59	3.39	3.41
2	1	.04	.12	.29	.52	1.02	2.41	3.08	3.88	3.69
3	2	.15	.26	.46	.69	1.09	2.29	3.11	3.89	3.71
4	3	.23	.36	.55	.79	1.15	1.88	3.14	3.91	3.72
5	5	.39	.54	.74	.97	1.30	1.53	3.03	3.97	3.76
6	7.5	.60	.75	1.00	1.22	1.52	1.77	2.30	4.01	3.82
7	10	.81	.99	1.21	1.45	1.76	2.02	2.10	3.25	3.54
8	12	1.10	1.31	1.55	1.82	2.16	2.34	2.56	2.64	2.84
9	15	2.04	2.35	2.73	3.14	3.68	4.01	4.10	3.47	2.63
10	16.1	2.26	2.60	3.00	3.43	4.10	4.42	4.43	3.76	2.72
A	18.3	1.38	1.55	1.74	1.95	2.21	2.42	2.59	2.64	2.19
B	25	1.28	1.39	1.54	1.69	1.85	1.99	2.12	2.18	1.85
C	35	1.23	1.32	1.43	1.53	1.64	1.74	1.83	1.86	1.62
D	45	1.23	1.29	1.37	1.45	1.53	1.59	1.64	1.65	1.52
E	55	1.22	1.26	1.32	1.38	1.44	1.47	1.52	1.49	1.49
F	65	1.20	1.23	1.26	1.30	1.34	1.36	1.37	1.35	1.46
G	74	1.16	1.18	1.20	1.23	1.26	1.26	1.27	1.25	1.43
b17	77.03	1.10	1.10	1.11	1.13	1.14	1.15	1.15	1.18	1.38
b18	78.3	1.10	1.10	1.11	1.12	1.12	1.14	1.13	1.17	1.37
19	80	1.14	1.15	1.17	1.19	1.21	1.20	1.21	1.20	1.42
20	85	1.11	1.10	1.11	1.12	1.12	1.12	1.13	1.17	1.42
21	90	1.07	1.05	1.05	1.06	1.06	1.06	1.07	1.13	1.42
22	95	1.01	.99	.98	.98	.97	.98	1.01	1.11	1.42
23	97.5	.97	.94	.93	.92	.93	.96	.99	1.11	1.40
24	100	.94	.88	.87	.88	.89	.92	.98	1.09	1.36
11	1.3	1.58	1.39	1.12	1.15	.56	.29	.15	.04	.03
12	2.6	1.58	1.39	1.12	1.10	.56	.35	.20	.11	.09
13	5	1.59	1.40	1.13	.84	.53	.38	.29	.18	.18
14	7.5	1.59	1.40	1.13	.67	.50	.37	.29	.21	.19
15	11.4	1.60	1.41	1.15	.54	.37	.34	.26	.19	.20
H	18.1	1.61	1.42	1.17	.47	.36	.30	.24	.20	.20
I	25	1.64	1.46	.99	.63	.56	.48	.42	.37	.36
J	35	1.65	1.27	.77	.78	.70	.62	.58	.51	.52
K	45	1.53	.99	.87	.87	.80	.73	.69	.64	.66
L	55	1.30	.95	.94	.92	.87	.81	.77	.73	.77
M	65	1.19	.98	.98	.96	.92	.87	.84	.80	.87
N	75	1.02	1.00	1.00	.98	.95	.91	.90	.88	.97
25	85	.99	.99	.99	.98	.96	.94	.94	.95	1.07
26	90	.98	.98	.98	.98	.95	.95	.95	.98	1.13
27	95	.95	.95	.95	.95	.94	.94	.96	1.01	1.21
28	97.5	.94	.92	.92	.92	.92	.93	.96	1.04	1.26
b16	15	1.58	1.40	1.14	.45	.20	.21	.23	.18	.18
29	80.3	1.01	1.00	1.00	.99	.96	.94	.93	.93	1.05

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(g) $\delta_N=0^\circ$, $\delta_P=10^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—						
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°
1	0	1.39	0.91	0.58	0.79	1.23	1.56	1.66
2	1	.45	.85	1.98	2.24	2.30	2.17	1.91
3	2	.58	.90	1.80	2.26	2.30	2.17	1.92
4	3	.65	.93	1.48	2.27	2.31	2.18	1.92
5	5	.75	.98	1.20	2.29	2.33	2.19	1.92
6	7.5	.83	1.02	1.21	2.29	2.36	2.21	1.94
7	10	.89	1.06	1.24	2.02	2.39	2.23	1.95
8	12	.94	1.09	1.26	1.70	2.40	2.25	1.97
9	15	.96	1.06	1.12	1.46	2.37	2.28	1.99
b10	16.1	1.04	.99	.88	.83	.87	.81	.77
A	18.3	1.03	1.16	1.31	1.40	2.26	2.28	2.00
B	25	1.09	1.19	1.31	1.40	1.91	2.27	2.02
C	35	1.16	1.23	1.33	1.42	1.52	2.11	2.00
D	45	1.21	1.27	1.34	1.42	1.42	1.85	1.93
E	55	1.26	1.31	1.36	1.42	1.41	1.61	1.83
F	65	1.30	1.32	1.36	1.42	1.39	1.45	1.71
G	74	1.37	1.37	1.38	1.42	1.39	1.35	1.60
b17	77.03	1.40	1.34	1.31	1.34	1.28	1.21	1.40
b18	78.3	1.44	1.38	1.36	1.39	1.32	1.26	1.47
19	80	1.75	1.64	1.58	1.62	1.50	1.33	1.50
20	85	1.33	1.27	1.25	1.30	1.28	1.26	1.49
21	90	1.16	1.13	1.14	1.16	1.17	1.22	1.46
22	95	1.01	1.05	1.08	1.06	1.09	1.18	1.42
23	97.5	.94	1.02	1.06	1.03	1.06	1.16	1.39
24	100	.85	1.00	1.04	1.00	1.03	1.14	1.36
11	1.3	1.85	.82	.47	.24	.14	.09	.09
12	2.6	1.50	.84	.57	.37	.24	.19	.19
13	5	1.10	.90	.69	.50	.39	.32	.31
14	7.5	1.11	.94	.75	.59	.47	.41	.41
15	11.4	1.10	.96	.87	.66	.56	.50	.50
H	18.1	1.11	.99	.87	.75	.66	.62	.61
I	25	1.13	1.02	.93	.82	.75	.70	.71
J	35	1.12	1.04	.96	.88	.82	.79	.80
K	45	1.10	1.04	.98	.91	.87	.84	.87
L	55	1.05	1.01	.98	.91	.88	.87	.91
M	65	.97	.96	.95	.90	.87	.87	.91
N	75	.88	.86	.84	.81	.81	.81	.87
25	85	.84	.85	.84	.82	.82	.84	.92
26	90	.89	.91	.91	.88	.89	.93	1.02
27	95	.95	.94	.96	.93	.94	1.00	1.13
28	97.5	.88	.96	.98	.95	.96	1.04	1.20
b16	15	1.04	.98	.85	.74	.67	.61	.60
b29	80.3	1.27	1.23	1.20	1.20	1.15	1.10	1.27

* Angle of attack for maximum lift.
 b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(h) $\delta_N=0^\circ$, $\delta_P=22^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—						
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°
1	0	0.74	-0.34	0.61	1.08	1.64	2.00	1.98
2	1	.75	1.78	2.19	2.36	2.35	2.27	2.14
3	2	.83	1.23	2.21	2.37	2.36	2.28	2.14
4	3	.86	1.16	2.23	2.38	2.36	2.28	2.14
5	5	.93	1.18	2.22	2.40	2.37	2.29	2.15
6	7.5	.99	1.21	1.92	2.43	2.39	2.32	2.17
7	10	1.04	1.23	1.54	2.46	2.42	2.34	2.19
8	12	1.08	1.24	1.36	2.44	2.44	2.36	2.21
9	15	1.05	1.10	1.23	2.32	2.47	2.38	2.23
b10	16.1	.96	.87	.80	.85	.79	.77	.75
A	18.3	1.16	1.31	1.40	2.14	2.48	2.39	2.24
B	25	1.20	1.32	1.42	1.66	2.46	2.41	2.26
C	35	1.26	1.36	1.43	1.48	2.25	2.32	2.23
D	45	1.31	1.38	1.44	1.48	1.93	2.12	2.13
E	55	1.36	1.41	1.45	1.49	1.67	1.88	1.97
F	65	1.39	1.42	1.45	1.48	1.53	1.67	1.79
G	74	1.42	1.43	1.44	1.47	1.48	1.54	1.65
b17	77.03	1.27	1.25	1.24	1.26	1.28	1.29	1.38
18	78.3	1.42	1.42	1.44	1.47	1.51	1.51	1.58
19	80	1.41	1.41	1.42	1.45	1.54	1.49	1.55
20	85	1.39	1.39	1.40	1.41	1.37	1.41	1.51
21	90	1.43	1.42	1.43	1.42	1.33	1.39	1.50
22	95	1.45	1.43	1.42	1.40	1.29	1.37	1.47
23	97.5	1.44	1.42	1.39	1.37	1.28	1.35	1.46
24	100	1.37	1.36	1.34	1.31	1.25	1.32	1.41
11	1.3	.91	.53	.28	.14	.07	.06	.06
12	2.6	.90	.62	.40	.25	.16	.14	.15
13	5	.94	.72	.53	.39	.29	.27	.28
14	7.5	.96	.78	.61	.47	.37	.37	.36
15	11.4	.96	.82	.67	.56	.46	.45	.45
H	18.1	.99	.87	.74	.65	.55	.55	.55
I	25	1.00	.90	.80	.72	.64	.64	.64
J	35	1.00	.92	.83	.77	.70	.70	.72
K	45	.98	.91	.85	.79	.73	.74	.76
L	55	.93	.88	.83	.78	.74	.75	.77
M	65	.88	.83	.80	.77	.73	.75	.77
N	75	.85	.80	.77	.74	.70	.72	.76
25	85	.87	.82	.79	.76	.72	.74	.78
26	90	.90	.85	.82	.79	.75	.78	.84
27	95	.99	.90	.96	.94	.90	.94	1.00
28	97.5	1.09	1.09	1.07	1.04	1.00	1.04	1.11
b16	15	.95	.85	.73	.65	.57	.56	.56
b29	80.3	1.26	1.23	1.21	1.23	1.24	1.25	1.33

* Angle of attack for maximum lift.
 b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(i) $\delta_N=0^\circ$, $\delta_P=42^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—						
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°
1	0	0.34	1.18	2.07	2.38	2.30	2.30	2.05
2	1	1.94	2.19	2.38	2.45	2.32	2.32	2.07
3	2	1.53	2.21	2.40	2.45	2.33	2.33	2.08
4	3	1.26	2.23	2.40	2.46	2.34	2.34	2.08
5	5	1.20	2.23	2.42	2.47	2.34	2.34	2.09
6	7.5	1.21	2.22	2.46	2.49	2.36	2.36	2.10
7	10	1.25	1.53	2.48	2.52	2.37	2.37	2.11
8	12	1.28	1.36	2.43	2.54	2.39	2.39	2.12
9	15	1.14	1.25	2.27	2.56	2.41	2.41	2.14
b10	16.1	.83	.75	.76	.70	.67	.67	.65
A	18.3	1.34	1.40	2.02	2.57	2.43	2.43	2.14
B	25	1.37	1.43	2.02	2.52	2.46	2.46	2.16
C	35	1.42	1.48	1.51	2.22	2.44	2.44	2.19
D	45	1.46	1.50	1.54	1.85	2.31	2.31	2.18
E	55	1.50	1.52	1.56	1.64	2.11	2.11	2.12
F	65	1.54	1.54	1.56	1.55	1.89	1.89	2.00
G	74	1.60	1.56	1.56	1.54	1.74	1.74	1.92
b17	77.03	1.58	1.51	1.49	1.50	1.65	1.65	1.91
18	78.3	1.59	1.51	1.49	1.69	1.66	1.66	1.81
19	80	1.59	1.51	1.47	1.49	1.60	1.60	1.81
20	85	1.61	1.53	1.50	1.50	1.57	1.57	1.77
21	90	1.61	1.53	1.51	1.51	1.56	1.56	1.74
22	95	1.64	1.53	1.50	1.50	1.54	1.54	1.71
23	97.5	1.62	1.53	1.50	1.50	1.50	1.50	1.67
24	100	1.56	1.50	1.47	1.47	1.50	1.50	1.67
11	1.3	.46	.27	.14	.06	.03	.03	.04
12	2.6	.57	.38	.24	.15	.11	.11	.12
13	5	.68	.51	.36	.27	.22	.22	.24
14	7.5	.71	.57	.45	.35	.31	.31	.31
15	11.4	.74	.64	.53	.43	.39	.39	.40
H	18.1	.79	.69	.60	.52	.48	.48	.51
I	25	.82	.73	.66	.58	.54	.54	.57
J	35	.80	.75	.68	.63	.61	.61	.64
K	45	.79	.73	.68	.63	.62	.62	.66
L	55	.71	.68	.63	.60	.59	.59	.64
M	65	.58	.55	.52	.53	.51	.51	.54
N	75	.42	.39	.34	.33	.33	.33	.35
25	85	.34	.34	.31	.30	.31	.31	.33
26	90	.56	.54	.54	.53	.53	.53	.56
27	95	.84	.81	.79	.78	.79	.79	.87
28	97.5	1.02	.93	.91	.91	.95	.95	1.06
b16	15	.79	.69	.59	.53	.49	.49	.50
b29	80.3	1.33	1.27	1.25	1.24	1.36	1.36	1.56

* Angle of attack for maximum lift.
 b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(j) $\delta_N=0^\circ$, $\delta_P=60^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—							
		−6.1°	−4.1°	−2.0°	0°	1.0°	2.0°	3.0°	4.1°
1	0	0.29	1.36	2.23	2.45	2.52	2.50	2.44	2.30
2	1	1.87	2.26	2.46	2.52	2.55	2.52	2.47	2.30
3	2	1.36	2.29	2.48	2.52	2.56	2.53	2.47	2.30
4	3	1.20	2.30	2.49	2.53	2.57	2.54	2.48	2.30
5	5	1.19	2.31	2.51	2.55	2.58	2.55	2.49	2.31
6	7.5	1.21	2.08	2.54	2.56	2.60	2.56	2.49	2.32
7	10	1.25	1.67	2.56	2.60	2.62	2.59	2.52	2.34
8	12	1.28	1.44	2.49	2.62	2.64	2.61	2.54	2.35
9	15	1.13	1.31	2.31	2.64	2.67	2.63	2.57	2.37
b 10	16.1	.78	.70	.73	.69	.67	.63	.62	.66
A	18.3	1.34	1.47	2.00	2.63	2.68	2.64	2.57	2.38
B	25	1.39	1.51	1.62	2.47	2.67	2.67	2.61	2.41
C	35	1.45	1.56	1.58	1.99	2.46	2.61	2.61	2.42
D	45	1.51	1.61	1.62	1.69	2.09	2.40	2.51	2.37
E	55	1.58	1.66	1.66	1.61	1.80	2.10	2.31	2.26
F	65	1.64	1.69	1.68	1.61	1.66	1.85	2.06	2.10
G	74	1.66	1.68	1.65	1.59	1.60	1.70	1.86	1.96
17	77.03	2.19	2.12	1.97	2.07	2.08	2.03	2.07	2.04
18	78.3	1.92	1.87	1.81	1.75	1.72	1.70	1.80	1.89
19	80	2.00	1.93	1.82	1.65	1.61	1.64	1.75	1.87
20	85	1.86	1.83	1.72	1.58	1.58	1.61	1.72	1.81
21	90	1.86	1.83	1.72	1.59	1.59	1.62	1.71	1.79
22	95	1.85	1.81	1.71	1.58	1.59	1.61	1.69	1.76
23	97.5	1.82	1.78	1.69	1.57	1.58	1.60	1.66	1.74
24	100	1.79	1.75	1.65	1.55	1.55	1.57	1.64	1.71
11	1.3	.47	.24	.13	.07	.03	.02	.02	.02
12	2.6	.56	.35	.23	.14	.11	.09	.08	.09
13	5	.65	.47	.35	.26	.22	.19	.18	.19
14	7.5	.69	.53	.42	.34	.30	.26	.25	.26
15	11.4	.71	.57	.49	.41	.37	.35	.33	.35
H	18.1	.74	.64	.56	.48	.46	.42	.44	.44
I	25	.75	.67	.59	.53	.51	.48	.48	.49
J	35	.72	.66	.60	.55	.53	.51	.51	.53
K	45	.66	.63	.57	.53	.52	.50	.50	.52
L	55	.55	.51	.49	.49	.48	.47	.46	.47
M	65	.37	.35	.35	.34	.34	.33	.33	.34
N	75	.34	.31	.27	.21	.20	.19	.19	.20
25	85	.31	.28	.18	.11	.12	.14	.13	.15
26	90	.32	.30	.28	.28	.29	.29	.29	.30
27	95	.61	.59	.58	.56	.56	.56	.57	.60
28	97.5	.86	.84	.80	.77	.77	.78	.79	.83
b 16	15	.75	.64	.57	.50	.47	.44	.43	.45
b 20	80.3	.59	.56	.50	.49	.47	.46	.47	.51

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(k) $\delta_N=5^\circ$, $\delta_P=5^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—								
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°	* 10.2°	12.2°
1	0	1.77	1.78	1.10	0.54	0.73	1.76	1.83	1.53	1.23
2	1	.20	.36	.71	1.77	2.35	2.58	2.27	1.93	1.72
3	2	.34	.51	.80	1.23	2.37	2.59	2.29	1.94	1.72
4	3	.42	.58	.85	1.19	2.39	2.60	2.29	1.94	1.73
5	5	.55	.71	.94	1.23	2.36	2.62	2.31	1.96	1.74
6	7.5	.69	.82	1.03	1.28	1.80	2.64	2.34	1.97	1.75
7	10	.77	.91	1.10	1.32	1.53	2.64	2.37	1.99	1.76
8	12	.88	1.00	1.17	1.39	1.47	2.50	2.40	2.01	1.77
9	15	1.10	1.09	1.20	1.39	1.44	2.23	2.40	2.03	1.79
b 10	16.1	1.73	.97	.86	.78	.70	.68	.64	.62	.62
A	18.3	.99	1.10	1.26	1.46	1.57	1.87	2.36	2.03	1.80
B	25	1.02	1.12	1.25	1.39	1.50	1.59	2.26	2.03	1.82
C	35	1.05	1.16	1.26	1.36	1.45	1.50	1.98	1.99	1.82
D	45	1.13	1.18	1.27	1.35	1.42	1.45	1.69	1.90	1.82
E	55	1.17	1.21	1.27	1.34	1.39	1.41	1.49	1.80	1.80
F	65	1.19	1.22	1.27	1.31	1.35	1.36	1.37	1.67	1.76
G	74	1.22	1.23	1.27	1.30	1.32	1.32	1.30	1.58	1.76
b 17	77.03	1.21	1.18	1.20	1.20	1.21	1.20	1.18	1.42	1.59
b 18	78.3	1.20	1.18	1.20	1.19	1.20	1.18	1.18	1.40	1.58
19	80	1.39	1.38	1.39	1.28	1.40	1.36	1.28	1.49	1.66
20	85	1.17	1.18	1.20	1.20	1.21	1.22	1.48	1.69	1.69
21	90	1.10	1.09	1.09	1.09	1.11	1.12	1.17	1.44	1.67
22	95	1.02	1.00	.99	1.00	1.02	1.04	1.14	1.40	1.62
23	97.5	.96	.94	.94	.96	.97	1.00	1.12	1.38	1.53
24	100	.88	.89	.90	.92	.93	.96	1.10	1.34	1.53
11	1.3	1.78	1.80	.94	.53	.28	.14	.10	.11	.11
12	2.6	1.79	1.82	.93	.61	.39	.25	.20	.21	.21
13	5	1.81	1.70	.94	.70	.51	.38	.33	.33	.33
14	7.5	1.82	1.36	.94	.74	.57	.46	.40	.41	.41
15	11.4	1.83	1.05	.90	.75	.62	.58	.46	.47	.48
H	18.1	1.71	1.01	.90	.77	.67	.58	.46	.55	.56
I	25	1.36	1.09	1.00	.88	.78	.70	.66	.68	.69
J	35	1.15	1.13	1.05	.95	.87	.80	.77	.79	.82
K	45	1.13	1.14	1.07	.99	.92	.86	.84	.88	.91
L	55	1.12	1.12	1.07	1.00	.95	.90	.90	.94	.99
M	65	1.09	1.08	1.05	1.00	.95	.91	.92	.98	1.04
N	75	1.03	1.01	.98	.95	.92	.91	.91	.99	1.07
25	85	.99	.99	.96	.94	.92	.91	.95	1.05	1.15
26	90	.98	.93	.97	.96	.95	.95	1.00	1.12	1.23
27	95	.95	.95	.95	.95	.95	.96	1.03	1.18	1.33
28	97.5	.92	.94	.93	.94	.94	.96	1.05	1.22	1.38
b 16	15	1.68	.96	.82	.73	.62	.55	.51	.51	.51
b 29	80.3	1.07	1.04	1.04	1.02	1.01	.99	.99	1.12	1.28

* Angle of attack for maximum lift.
b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(l) $\delta_N=5^\circ$, $\delta_P=10^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—								
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	* 8.1°	10.2°	
1	0	1.64	1.72	0.70	0.37	1.10	1.59	1.84	1.59	
2	1	.29	.51	1.00	2.18	2.57	2.54	2.19	1.91	
3	2	.44	.64	1.02	2.20	2.59	2.55	2.20	1.91	
4	3	.52	.71	1.05	2.12	2.60	2.56	2.20	1.92	
5	5	.64	.83	1.10	1.59	2.63	2.58	2.22	1.94	
6	7.5	.77	1.00	1.17	1.41	2.66	2.66	2.25	1.94	
7	10	.87	1.07	1.23	1.41	2.37	2.66	2.27	1.96	
8	12	.96	1.10	1.31	1.48	1.96	2.66	2.29	1.97	
9	15	1.15	1.15	1.34	1.46	1.67	2.61	2.31	1.99	
b 10	16.1	1.17	.89	.82	.73	.67	.70	.66	.63	
A	18.3	1.08	1.21	1.40	1.56	1.62	2.43	2.31	2.00	
B	25	1.11	1.22	1.35	1.49	1.57	2.08	2.27	2.01	
C	35	1.16	1.25	1.35	1.45	1.54	1.67	2.14	1.99	
D	45	1.21	1.28	1.36	1.44	1.50	1.52	1.93	1.94	
E	55	1.26	1.31	1.36	1.43	1.48	1.47	1.72	1.86	
F	65	1.29	1.32	1.36	1.41	1.45	1.42	1.56	1.77	
G	74	1.37	1.37	1.38	1.40	1.44	1.39	1.45	1.70	
b 17	77.03	1.37	1.34	1.31	1.31	1.32	1.27	1.28	1.50	
b 18	78.3	1.42	1.38	1.34	1.34	1.37	1.32	1.33	1.55	
19	80	1.71	1.61	1.55	1.54	1.29	1.27	1.34	1.59	
20	85	1.32	1.27	1.24	1.26	1.17	1.19	1.31	1.57	
21	90	1.15	1.13	1.16	1.17	1.09	1.12	1.27	1.53	
22	95	1.02	1.06	1.11	1.11	1.06	1.10	1.26	1.50	
23	97.5	.95	1.03	1.09	1.10	1.06	1.10	1.26	1.50	
24	100	.88	1.01	1.07	1.07	1.02	1.07	1.23	1.45	
11	1.3	1.70	1.61	.71	.37	.18	.10	.09	.10	
12	2.6	1.71	1.11	.74	.47	.29	.20	.18	.19	
13	5	1.73	1.02	.79	.58	.42	.33	.30	.31	
14	7.5	1.70	1.01	.82	.63	.49	.40	.38	.38	
15	11.4	1.47	.95	.80	.66	.54	1.46	.44	.45	
H	18.1	1.09	.94	.81	.70	.60	.53	.52	.53	
I	25	1.04	1.01	.90	.80	.71	.64	.64	.66	
J	35	1.07	1.04	.95	.87	.79	.74	.74	.77	
K	45	1.10	1.05	.98	.91	.83	.80	.81	.84	
L	55	1.04	1.02	.97	.91	.85	.83	.85	.90	
M	65	.98	.98	.94	.89	.84	.83	.85	.91	
N	75	.89	.88	.84	.82	.78	.78	.81	.88	
25	85	.87	.87	.85	.83	.80	.79	.85	.95	
26	90	.90	.92	.92	.90	.87	.88	.95	1.06	
27	95	.91	.95	.97	.99	.93	.95	1.04	1.18	
28	97.5	.90	.98	1.00	.99	.95	.98	1.10	1.26	
b 16	15	1.16	.91	.78	.66	.56	.51	.49	.50	
b 29	80.3	1.25	1.22	1.19	1.18	1.19	1.15	1.16	1.34	

* Angle of attack for maximum lift.
b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(m) $\delta_N=5^\circ$, $\delta_P=22^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—								
		−4.1°	−2.0°	0°	2.0°	4.1°	6.1°	* 7.1°	8.1°	
1	0	1.09	0.71	0.34	0.73	1.09	1.43	1.90	1.74	
2	1	.43	.86	2.06	2.48	2.68	2.40	2.24	2.09	
3	2	.57	.92	.83	2.50	2.69	2.40	2.25	2.09	
4	3	.64	.96	1.47	2.51	2.70	2.41	2.25	2.10	
5	5	.76	1.03	1.26	2.53	2.72	2.42	2.26	2.11	
6	7.5	.88	1.11	1.28	2.16	2.76	2.46	2.29	2.13	
7	10	.97	1.18	1.38	1.81	2.79	2.49	2.31	2.15	
8	12	1.06	1.25	1.44	1.56	2.73	2.51	2.34	2.17	
9	15	1.11	1.26	1.42	1.49	2.55	2.53	2.36	2.19	
b10	16.1	.89	.84	.72	.66	.69	.64	.63	.62	
A	18.3	1.18	1.35	1.53	1.64	2.20	2.53	2.36	2.20	
B	25	1.20	1.33	1.46	1.59	1.80	2.49	2.35	2.21	
C	35	1.24	1.36	1.44	1.55	1.61	2.31	2.27	2.18	
D	45	1.29	1.38	1.45	1.54	1.57	2.03	2.12	2.10	
E	55	1.33	1.40	1.44	1.52	1.55	1.78	1.93	1.98	
F	65	1.36	1.41	1.44	1.49	1.51	1.60	1.75	1.85	
G	74	1.39	1.42	1.43	1.45	1.48	1.50	1.61	1.73	
b17	77.03	1.27	1.28	1.24	1.26	1.28	1.28	1.36	1.47	
18	78.3	1.39	1.42	1.40	1.43	1.47	1.50	1.56	1.66	
19	80	1.38	1.40	1.39	1.40	1.43	1.49	1.53	1.62	
20	85	1.37	1.39	1.37	1.38	1.40	1.39	1.48	1.60	
21	90	1.40	1.43	1.40	1.42	1.40	1.37	1.47	1.59	
22	95	1.42	1.43	1.40	1.42	1.39	1.34	1.44	1.56	
23	97.5	1.40	1.42	1.39	1.40	1.37	1.33	1.43	1.54	
24	100	1.36	1.37	1.34	1.36	1.31	1.29	1.39	1.49	
11	1.3	1.72	.78	.43	.22	.10	.06	.07	.07	
12	2.6	1.87	.79	.52	.34	.20	.14	.14	.15	
13	5	1.16	.82	.61	.45	.33	.26	.26	.27	
14	7.5	.99	.83	.65	.52	.40	.33	.34	.34	
15	11.4	.94	.80	.67	.56	.45	.39	.40	.40	
H	18.1	.91	.79	.69	.61	.51	.46	.46	.47	
I	25	.98	.88	.78	.71	.62	.56	.57	.59	
J	35	.99	.91	.83	.77	.69	.64	.66	.68	
K	45	.97	.91	.84	.79	.73	.69	.71	.73	
L	55	.92	.87	.82	.79	.73	.70	.73	.75	
M	65	.83	.80	.76	.74	.71	.67	.69	.72	
N	75	.68	.64	.62	.60	.55	.53	.56	.60	
25	85	.69	.63	.63	.62	.59	.58	.61	.64	
26	90	.84	.82	.80	.79	.76	.75	.79	.84	
27	95	.99	.99	.97	.96	.93	.91	.97	1.04	
28	97.5	1.09	1.09	1.07	1.07	1.02	1.02	1.09	1.16	
b16	15	.86	.78	.65	.56	.50	.45	.45	.45	
b29	80.3	1.24	1.24	1.22	1.22	1.22	1.24	1.31	1.41	

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(o) $\delta_N=9^\circ$, $\delta_P=22^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—						
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°
1	0	1.44	1.42	0.48	0.55	1.18	2.23	2.29
2	1	.33	.60	1.11	2.36	2.84	2.92	2.44
3	2	.47	.73	1.15	2.39	2.85	2.94	2.45
4	3	.55	.80	1.16	2.36	2.87	2.94	2.46
5	5	.69	.92	1.24	1.92	2.91	2.96	2.47
6	7.5	.86	1.08	1.31	1.55	2.96	3.01	2.50
7	10	.96	1.17	1.42	1.57	2.46	3.02	2.51
8	12	1.11	1.32	1.56	1.72	2.00	3.00	2.52
9	15	1.48	1.71	1.97	2.10	1.92	2.90	2.54
b10	16.1	1.19	.62	1.16	.49	.41	.34	.33
A	18.3	1.24	1.38	1.59	1.75	1.81	2.63	2.51
B	25	1.23	1.36	1.51	1.64	1.72	2.23	2.44
C	35	1.26	1.37	1.48	1.58	1.66	1.81	2.29
D	45	1.30	1.39	1.47	1.55	1.61	1.66	2.08
E	55	1.35	1.42	1.47	1.54	1.57	1.60	1.88
F	65	1.37	1.42	1.47	1.50	1.52	1.55	1.72
G	74	1.40	1.43	1.46	1.47	1.47	1.51	1.61
b17	77.03	1.25	1.26	1.26	1.24	1.23	1.28	1.34
18	78.3	1.41	1.43	1.43	1.44	1.44	1.52	1.57
19	80	1.41	1.42	1.41	1.41	1.40	1.50	1.55
20	85	1.39	1.40	1.39	1.39	1.37	1.40	1.48
21	90	1.42	1.44	1.42	1.41	1.39	1.38	1.46
22	95	1.43	1.44	1.43	1.41	1.39	1.36	1.44
23	97.5	1.41	1.42	1.42	1.40	1.38	1.34	1.42
24	100	1.35	1.36	1.37	1.36	1.33	1.29	1.39
11	1.3	1.47	.97	.58	.31	.15	.06	.06
12	2.6	1.47	.92	.63	.41	.25	.14	.14
13	5	1.49	.89	.68	.50	.36	.25	.24
14	7.5	1.48	.85	.68	.54	.41	.33	.30
15	11.4	1.38	.78	.65	.53	.43	.36	.34
H	18.1	1.05	.77	.66	.58	.49	.42	.41
I	25	.91	.77	.69	.61	.54	.53	.53
J	35	.95	.90	.82	.76	.69	.63	.62
K	45	.95	.90	.84	.79	.73	.68	.68
L	55	.92	.87	.83	.78	.73	.70	.70
M	65	.85	.80	.77	.74	.71	.67	.68
N	75	.73	.65	.62	.60	.57	.53	.58
25	85	.73	.65	.64	.62	.60	.57	.60
26	90	.85	.82	.80	.79	.77	.74	.78
27	95	1.00	.98	.98	.96	.94	.91	.96
28	97.5	1.09	1.09	1.09	1.08	1.05	1.02	1.08
b16	15	1.18	.62	.60	.50	.40	.34	.33
b29	80.3	1.24	1.23	1.23	1.20	1.18	1.23	1.29

a Angle of attack for maximum lift.

b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(p) $\delta_N=9^\circ$, $\delta_P=42^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—						
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°
1	0	1.00	0.34	0.52	1.47	2.41	2.57	2.25
2	1	.63	1.21	2.38	2.84	3.08	2.63	2.26
3	2	.77	1.17	2.41	2.86	3.09	2.63	2.27
4	3	.83	1.18	2.39	2.88	3.10	2.64	2.27
5	5	.95	1.26	1.99	2.92	3.12	2.65	2.28
6	7.5	1.08	1.33	1.63	2.94	3.17	2.66	2.28
7	10	1.21	1.45	1.59	2.41	3.18	2.68	2.30
8	12	1.36	1.60	1.75	1.96	3.10	2.71	2.32
9	15	1.77	2.02	2.14	1.95	2.86	2.74	2.34
b10	16.1	.58	.56	.45	.37	.32	.28	.29
A	18.3	1.44	1.57	1.79	1.84	2.45	2.70	2.33
B	25	1.41	1.60	1.69	1.77	2.01	2.66	2.34
C	35	1.45	1.55	1.65	1.70	1.77	2.49	2.32
D	45	1.47	1.56	1.63	1.67	1.71	2.26	2.24
E	55	1.52	1.58	1.62	1.65	1.67	2.00	2.14
F	65	1.56	1.60	1.61	1.62	1.63	1.80	2.01
G	74	1.61	1.64	1.62	1.59	1.60	1.68	1.90
b17	77.03	1.61	1.61	1.55	1.50	1.51	1.62	1.87
18	78.3	1.92	1.90	1.78	1.70	1.68	1.68	1.81
19	80	1.64	1.62	1.54	1.47	1.47	1.59	1.78
20	85	1.64	1.62	1.55	1.48	1.48	1.56	1.75
21	90	1.65	1.64	1.56	1.49	1.49	1.56	1.75
22	95	1.66	1.65	1.57	1.50	1.50	1.55	1.72
23	97.5	1.65	1.64	1.56	1.49	1.49	1.54	1.70
24	100	1.61	1.60	1.54	1.47	1.47	1.51	1.67
11	1.3	.89	.52	.28	.14	.05	.03	.04
12	2.6	.86	.57	.38	.23	.14	.11	.12
13	5	.83	.62	.47	.33	.24	.21	.22
14	7.5	.80	.63	.50	.38	.29	.26	.27
15	11.4	.72	.59	.50	.40	.33	.31	.31
H	18.1	.71	.60	.53	.45	.38	.36	.37
I	25	.79	.70	.62	.55	.49	.46	.48
J	35	.80	.73	.67	.60	.56	.53	.55
K	45	.78	.72	.68	.62	.58	.56	.59
L	55	.70	.67	.63	.59	.56	.55	.58
M	65	.67	.65	.63	.59	.56	.55	.58
N	75	.44	.37	.35	.32	.31	.29	.33
25	85	.38	.33	.32	.29	.29	.29	.32
26	90	.56	.55	.54	.52	.50	.50	.55
27	95	.84	.82	.80	.77	.76	.77	.83
28	97.5	1.03	1.01	.99	.94	.93	.94	1.04
b16	15	.57	.56	.45	.37	.32	.28	.29
b29	80.3	1.36	1.35	1.29	1.25	1.25	1.31	1.51

a Angle of attack for maximum lift.

b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Continued

(q) $\delta_N=21^\circ$, $\delta_P=42^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—									
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°	*10.2°	12.2°	
1	0	1.19	0.94	1.02	0.28	0.88	2.09	3.24	3.87	2.84	
2	1	.17	.43	.72	1.78	2.70	3.32	3.94	4.21	3.17	
3	2	.32	.58	.85	1.28	2.73	3.35	3.97	4.22	3.18	
4	3	.40	.67	.93	1.29	2.73	3.37	3.98	4.24	3.19	
5	5	.57	.84	1.09	1.42	2.20	3.42	4.03	4.27	3.22	
6	7.5	.80	1.05	1.20	1.56	1.93	2.42	4.10	4.32	3.27	
7	10	.98	1.25	1.49	1.79	1.96	2.21	3.60	4.35	3.28	
8	12	1.25	1.52	1.75	2.09	2.29	2.26	2.77	3.97	3.14	
9	15	1.95	2.23	2.35	2.97	3.18	3.08	2.78	3.26	2.95	
10	16.1	2.03	2.34	2.45	2.67	2.80	2.74	2.62	2.56	2.37	
A	18.3	1.56	1.97	2.58	2.41	2.59	2.67	2.70	2.76	2.69	
B	25	1.44	1.63	1.77	1.97	2.13	2.23	2.32	2.40	2.52	
C	35	1.43	1.59	1.71	1.82	1.93	2.00	2.06	2.10	2.33	
D	45	1.45	1.58	1.67	1.76	1.83	1.87	1.92	1.93	2.16	
E	55	1.49	1.60	1.67	1.73	1.77	1.79	1.81	1.80	2.03	
F	65	1.51	1.62	1.66	1.70	1.72	1.71	1.70	1.68	1.91	
G	74	1.56	1.66	1.68	1.70	1.68	1.64	1.61	1.59	1.83	
b17	77.03	1.55	1.65	1.64	1.57	1.60	1.54	1.50	1.51	1.78	
18	78.3	1.82	1.96	1.90	1.87	1.80	1.68	1.62	1.60	1.80	
19	80	1.58	1.65	1.62	1.61	1.57	1.49	1.46	1.46	1.73	
20	85	1.59	1.65	1.62	1.61	1.58	1.49	1.46	1.45	1.70	
21	90	1.60	1.67	1.65	1.63	1.59	1.50	1.47	1.47	1.69	
22	95	1.59	1.67	1.66	1.64	1.60	1.51	1.48	1.47	1.67	
23	97.5	1.58	1.66	1.64	1.62	1.59	1.50	1.47	1.47	1.66	
24	100	1.57	1.62	1.60	1.60	1.55	1.49	1.46	1.45	1.63	
11	1.3	1.19	.94	.73	.39	.20	.09	.03	.01	.02	
12	2.6	1.19	.95	.68	.43	.27	.17	.09	.05	.07	
13	5	1.20	.95	.63	.45	.33	.24	.17	.12	.14	
14	7.5	1.20	.95	.57	.44	.34	.26	.20	.16	.18	
15	11.4	1.21	.94	.45	.39	.30	.24	.19	.17	.18	
H	18.1	1.23	.85	.40	.34	.28	.24	.20	.18	.19	
I	25	1.26	.65	.56	.49	.43	.38	.33	.29	.31	
J	35	1.16	.65	.64	.57	.51	.46	.42	.39	.41	
K	45	.96	.68	.66	.60	.55	.51	.47	.44	.47	
L	55	.82	.65	.62	.57	.54	.51	.47	.45	.48	
M	65	.74	.56	.53	.49	.47	.44	.42	.40	.43	
N	75	.68	.45	.37	.34	.32	.29	.28	.27	.29	
25	85	.63	.43	.33	.36	.29	.27	.26	.26	.28	
26	90	.69	.60	.53	.51	.50	.48	.47	.46	.50	
27	95	.85	.83	.80	.78	.77	.73	.72	.71	.78	
28	97.5	1.00	1.02	1.00	.98	.95	.92	.89	.88	.98	
b16	15	1.23	.90	.37	.28	.31	.28	.22	.18	.20	
b29	80.3	1.34	1.37	1.37	1.35	1.33	1.27	1.24	1.24	1.45	

a Angle of attack for maximum lift.

b Internal pressures.

TABLE IV.—PRESSURE COEFFICIENTS—Concluded

(r) $\delta_N=27^\circ$, $\delta_P=60^\circ$

Orifice	z/c	Pressure coefficients for section angle of attack—									
		-4.1°	-2.0°	0°	2.0°	4.1°	6.1°	8.1°	* 10.2°	12.2°	
1	0	0.42	0.75	0.29	0.18	1.30	2.73	4.07	4.79	4.55	
2	1	.21	.46	.82	2.13	2.87	3.56	4.26	4.80	4.32	
3	2	.36	.62	.95	1.51	2.01	3.59	4.28	4.82	4.33	
4	3	.45	.72	1.02	1.35	2.93	3.61	4.31	4.83	4.35	
5	5	.64	.90	1.20	1.50	2.49	3.67	4.36	4.87	4.37	
6	7.5	.94	1.42	1.48	1.80	2.13	3.00	4.36	4.92	4.40	
7	10	1.13	1.18	1.71	2.02	2.18	2.40	3.62	4.73	4.41	
8	12	1.48	1.81	2.13	2.47	2.69	2.68	2.91	3.82	3.97	
9	15	2.69	3.21	3.68	4.20	4.44	4.33	3.88	3.49	3.33	
10	16.1	2.99	3.54	4.16	4.68	4.88	4.70	4.26	3.72	3.21	
A	18.3	1.78	2.05	2.30	2.57	2.79	2.93	3.02	3.06	2.86	
B	25	1.63	1.83	2.02	2.20	2.35	2.46	2.57	2.63	2.54	
C	35	1.59	1.74	1.88	2.00	2.11	2.19	2.26	2.30	2.25	
D	45	1.61	1.72	1.83	1.92	1.99	2.05	2.08	2.10	2.05	
E	55	1.64	1.74	1.82	1.88	1.92	1.94	1.96	1.94	1.91	
F	65	1.71	1.74	1.80	1.87	1.86	1.85	1.84	1.81	1.80	
G	74	1.84	1.69	1.76	1.90	1.83	1.70	1.69	1.67	1.71	
17	77.03	2.29	2.16	2.14	2.13	2.06	2.02	2.05	2.08	1.99	
18	78.3	1.88	1.91	1.89	1.88	1.85	1.80	1.78	1.67	1.68	
19	80	1.79	1.95	1.94	1.92	1.87	1.80	1.72	1.61	1.66	
20	85	1.81	1.86	1.85	1.83	1.78	1.72	1.65	1.60	1.65	
21	90	1.81	1.85	1.83	1.82	1.78	1.70	1.66	1.60	1.64	
22	95	1.82	1.83	1.82	1.80	1.77	1.70	1.65	1.60	1.64	
23	97.5	1.82	1.81	1.81	1.78	1.76	1.69	1.64	1.59	1.63	
24	100	1.82	1.75	1.76	1.76	1.72	1.66	1.61	1.58	1.62	
11	1.3	.96	.75	.58	.30	.15	.04	.01	0	0	
12	2.6	.96	.75	.54	.34	.21	.12	.06	.02	.03	
13	5	.97	.75	.50	.36	.25	.18	.12	.08	.09	
14	7.5	.98	.74	.45	.34	.25	.19	.14	.11	.11	
15	11.4	.98	.73	.31	.30	.23	.17	.13	.11	.11	
H	18.1	1.00	.67	.29	.24	.20	.16	.13	.12	.12	
I	25	1.02	.51	.44	.37	.33	.28	.24	.22	.22	
J	35	.91	.51	.50	.45	.40	.35	.32	.30	.30	
K	45	.75	.53	.50	.45	.42	.37	.35	.33	.34	
L	55	.63	.48	.44	.41	.38	.35	.33	.32	.33	
M	65	.58	.40	.32	.29	.29	.26	.25	.25	.25	
N	75	.52	.32	.26	.23	.22	.19	.18	.17	.18	
25	85	.47	.22	.20	.22	.20	.18	.15	.14	.14	
26	90	.49	.38	.29	.26	.25	.24	.23	.23	.24	
27	95	.65	.62	.57	.55	.53	.51	.50	.48	.51	
28	97.5	.84	.84	.81	.79	.77	.73	.71	.69	.72	
b 16	15	.98	.67	.18	.15	.19	.15	.12	.10	.10	
b 29	80.3	.68	.49	.49	.46	.46	.44	.42	.41	.42	

TABLE V.— $P_{ds}/c_{l_{ds}}$ DISTRIBUTION¹

(a) Plain flap at $\delta=5^\circ, 10^\circ$, and 15°																(b) Plain flap at $\delta=20^\circ$															
$E \rightarrow$		0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60				
$\uparrow$ $\frac{x/c}{1-E}$ (ahead of hinge) $\downarrow$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
	.05	.15	.15	.16	.16	.17	.17	.18	.19	.20	.21	.22	.23	.25	.27	.15	.15	.16	.16	.17	.17	.18	.19	.20	.21	.22	.23				
	.10	.22	.23	.23	.24	.25	.25	.26	.27	.29	.30	.32	.34	.36	.39	.22	.23	.23	.24	.25	.25	.26	.27	.29	.30	.32	.34				
	.20	.33	.34	.34	.35	.36	.38	.39	.40	.42	.44	.46	.49	.52	.56	.33	.34	.34	.35	.36	.38	.39	.40	.42	.44	.46	.49				
	.30	.43	.44	.45	.46	.47	.49	.50	.52	.54	.57	.60	.63	.67	.72	.43	.44	.45	.46	.47	.49	.50	.52	.54	.57	.60	.63				
	.40	.54	.55	.56	.57	.58	.60	.62	.64	.67	.70	.73	.77	.82	.88	.54	.55	.56	.57	.58	.60	.62	.64	.67	.70	.73	.77				
	.50	.66	.66	.67	.69	.71	.73	.75	.77	.80	.84	.88	.92	.98	1.05	.66	.66	.67	.69	.71	.73	.75	.77	.80	.84	.88	.92				
	.60	.79	.80	.82	.83	.85	.88	.90	.93	.96	1.00	1.05	1.10	1.17	1.25	.79	.80	.82	.83	.85	.88	.90	.93	.96	1.00	1.05	1.10				
	.70	.97	.98	1.00	1.01	1.03	1.05	1.08	1.11	1.15	1.19	1.24	1.30	1.39	1.48	.97	.98	1.00	1.01	1.03	1.05	1.08	1.11	1.15	1.19	1.24	1.30				
	.80	1.23	1.24	1.26	1.27	1.29	1.31	1.34	1.38	1.42	1.46	1.52	1.59	1.67	1.78	1.23	1.24	1.26	1.27	1.29	1.31	1.34	1.38	1.42	1.46	1.52	1.59				
$\uparrow$ $1-\frac{x}{c}$ (back of hinge) $\downarrow$	1.00	8.74	6.04	4.89	4.40	4.01	3.71	3.50	3.35	3.23	3.15	3.11	3.06	3.04	3.02	5.83	4.05	3.38	3.02	2.83	2.70	2.63	2.58	2.56	2.56	2.58	2.62				
	.90	6.45	4.48	3.78	3.32	2.99	2.77	2.60	2.48	2.39	2.32	2.26	2.22	2.19	2.16	5.08	3.53	2.98	2.61	2.36	2.18	2.05	1.96	1.88	1.83	1.78	1.75				
	.80	4.96	3.51	2.92	2.57	2.31	2.12	2.00	1.90	1.81	1.74	1.69	1.64	1.60	1.57	4.23	2.99	2.50	2.19	1.97	1.81	1.70	1.62	1.55	1.49	1.44	1.40				
	.70	3.85	2.72	2.23	1.97	1.77	1.64	1.53	1.44	1.37	1.32	1.27	1.22	1.19	1.16	3.71	2.63	2.16	1.90	1.71	1.58	1.48	1.39	1.32	1.27	1.22	1.18				
	.60	3.09	2.18	1.78	1.57	1.41	1.30	1.21	1.14	1.08	1.04	1.00	.96	.93	.91	3.33	2.35	1.92	1.69	1.52	1.40	1.30	1.22	1.17	1.12	1.08	1.04				
	.50	2.46	1.72	1.42	1.24	1.11	1.02	.95	.89	.85	.81	.77	.75	.72	.70	2.98	2.09	1.73	1.50	1.35	1.24	1.15	1.08	1.03	.98	.94	.91				
	.40	1.90	1.34	1.10	.96	.86	.78	.73	.69	.65	.62	.59	.57	.55	.53	2.65	1.87	1.54	1.34	1.19	1.09	1.02	.96	.91	.86	.83	.80				
	.30	1.40	.99	.81	.70	.63	.57	.53	.50	.47	.45	.43	.42	.40	.39	2.32	1.63	1.34	1.16	1.04	.95	.88	.82	.78	.75	.72	.69				
	.20	.92	.65	.53	.46	.41	.38	.35	.33	.31	.29	.28	.27	.26	.25	1.93	1.35	1.11	.96	.86	.79	.73	.68	.65	.62	.59	.56				
	.10	.48	.34	.27	.24	.21	.20	.18	.17	.16	.15	.15	.14	.13	.13	1.39	.98	.79	.69	.62	.57	.53	.50	.47	.45	.43	.41				
.05	.32	.19	.16	.14	.12	.11	.10	.10	.09	.08	.08	.08	.07	.07	1.16	.70	.58	.50	.45	.40	.38	.35	.33	.31	.30	.28					

(c) Plain flap at $\delta=30^\circ$																(d) Plain flap at $\delta=40^\circ$									
$E \rightarrow$		0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40						
$\uparrow$ $\frac{x/c}{1-E}$ (ahead of hinge) $\downarrow$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0						
	.05	.15	.15	.16	.16	.17	.17	.18	.19	.20	.21	.15	.15	.16	.16	.17	.17	.18	.19						
	.10	.22	.23	.23	.24	.25	.25	.26	.27	.29	.30	.22	.23	.23	.24	.25	.25	.26	.27						
	.20	.33	.34	.34	.35	.36	.38	.39	.40	.42	.44	.33	.34	.35	.36	.38	.39	.40	.42						
	.30	.43	.44	.45	.46	.47	.49	.50	.52	.54	.57	.43	.44	.45	.46	.47	.49	.51	.52						
	.40	.54	.55	.56	.57	.58	.60	.62	.64	.67	.70	.54	.55	.56	.57	.58	.60	.62	.64						
	.50	.66	.66	.67	.69	.71	.73	.75	.77	.80	.84	.66	.66	.67	.69	.71	.73	.75	.77						
	.60	.79	.80	.82	.83	.85	.88	.90	.93	.96	1.00	.79	.80	.82	.83	.85	.88	.90	.93						
	.70	.97	.98	1.00	1.01	1.03	1.05	1.08	1.11	1.15	1.19	.97	.98	1.00	1.01	1.03	1.05	1.08	1.11						
	.80	1.23	1.24	1.26	1.27	1.29	1.31	1.34	1.38	1.42	1.46	1.22	1.23	1.25	1.26	1.28	1.30	1.33	1.36						
$\uparrow$ $1-\frac{x}{c}$ (back of hinge) $\downarrow$	1.00	4.50	3.12	2.66	2.36	2.22	2.13	2.08	2.05	2.01	1.99	4.10	2.90	2.42	2.12	1.99	1.88	1.81	1.76						
	.90	4.61	3.20	2.70	2.37	2.15	2.01	1.90	1.83	1.77	1.73	4.24	3.01	2.52	2.23	2.02	1.88	1.78	1.72						
	.80	4.34	3.07	2.56	2.24	2.02	1.86	1.75	1.66	1.59	1.52	4.29	3.04	2.53	2.22	2.00	1.84	1.73	1.66						
	.70	4.08	2.89	2.37	2.09	1.88	1.73	1.62	1.52	1.45	1.40	4.27	3.02	2.48	2.18	1.96	1.81	1.70	1.59						
	.60	3.82	2.69	2.20	1.94	1.74	1.60	1.49	1.40	1.34	1.28	4.14	2.91	2.39	2.10	1.89	1.74	1.62	1.52						
	.50	3.55	2.49	2.05	1.79	1.61	1.47	1.37	1.29	1.22	1.17	3.92	2.75	2.27	1.98	1.77	1.62	1.51	1.42						
	.40	3.25	2.29	1.88	1.64	1.46	1.34	1.25	1.17	1.11	1.06	3.64	2.56	2.11	1.83	1.64	1.50	1.40	1.31						
	.30	2.89	2.04	1.68	1.45	1.29	1.18	1.10	1.03	.98	.93	3.28	2.31	1.90	1.64	1.47	1.34	1.24	1.17						
	.20	2.44	1.72	1.41	1.22	1.09	1.00	.93	.87	.82	.78	2.82	1.98	1.62	1.40	1.26	1.15	1.07	1.00						
	.10	1.81	1.27	1.03	.90	.81	.74	.69	.65	.61	.58	2.11	1.48	1.20	1.05	.95	.87	.81	.76						
.05	1.53	.92	.76	.66	.59	.53	.49	.46	.44	.41	1.79	1.08	.89	.77	.69	.63	.58	.54							

(e) Plain flap at $\delta=50^\circ$										(f) Plain flap at $\delta=60^\circ$									
$E \rightarrow$		0.5	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40		
$\uparrow$ $\frac{x/c}{1-E}$ (ahead of hinge) $\downarrow$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
	.05	.15	.15	.16	.16	.17	.17	.18	.19	.15	.15	.16	.16	.17	.17	.18	.19		
	.10	.22	.23	.23	.24	.25	.25	.26	.27	.22	.23	.23	.24	.25	.25	.26	.27		
	.20	.33	.34	.35	.36	.38	.39	.40	.42	.33	.34	.35	.36	.38	.39	.40	.42		
	.30	.43	.44	.45	.46	.47	.49	.51	.52	.43	.44	.45	.46	.47	.49	.51	.52		
	.40	.54	.55	.56	.57	.58	.60	.62	.64	.54	.55	.56	.57	.58	.60	.62	.64		
	.50	.66	.66	.68	.69	.71	.73	.75	.77	.66	.67	.68	.69	.71	.73	.75	.77		
	.60	.79	.80	.82	.83	.85	.88	.90	.93	.79	.80	.82	.83	.85	.88	.90	.93		
	.70	.97	.98	1.00	1.03	1.05	1.08	1.11	1.14	.97	.98	.99	1.01	1.02	1.05	1.07	1.10		
	.80	1.21	1.22	1.23	1.25	1.27	1.29	1.32	1.35	1.20	1.21	1.22	1.24	1.25	1.28	1.29	1.32		
$\uparrow$ $1-\frac{x}{c}$ (back of hinge) $\downarrow$	1.00	1.54	1.55	1.56	1.57	1.57	1.57	1.57	1.57	1.52	1.53	1.54	1.55	1.54	1.53	1.51	1.50		
	.90	3.81	2.68	2.27	2.00	1.86	1.77	1.71	1.67	3.84	2.59	2.22	1.97	1.81	1.71	1.63	1.59		
	.80	4.18	2.90	2.45	2.15	1.95	1.81	1.72	1.66	4.04	2.80	2.36	2.08	1.88	1.76	1.68	1.62		
	.70	4.25	3.00	2.50	2.20	1.98	1.82	1.72	1.64	4.16	2.94	2.45	2.15	1.94	1.78	1.68	1.59		
	.60	4.27	3.02	2.48	2.18	1.96	1.81	1.70	1.59	4.27	3.02	2.48	2.18	1.96	1.81	1.70	1.59		
	.50	4.21	2.97	2.43	2.14	1.92	1.77	1.65	1.55	4.27	3.01	2.46	2.17	1.95	1.79	1.67	1.57		
	.40	4.07	2.86	2.36	2.06	1.84	1.69	1.57	1.48	4.18	2.93	2.42	2.11	1.89	1.73	1.62	1.52		
	.30	3.86	2.71	2.24	1.95														

REPORT 1147

THE SIMILARITY LAW FOR HYPERSONIC FLOW AND REQUIREMENTS FOR DYNAMIC SIMILARITY OF RELATED BODIES IN FREE FLIGHT¹

By FRANK M. HAMAKER, STANFORD E. NEICE, and THOMAS J. WONG

SUMMARY

The similarity law for nonsteady, inviscid, hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. The conclusions drawn from the potential analysis used in the development of the law are shown to be valid for rotational flow. A direct consequence of the hypersonic similarity law is that the ratio of the local static pressure to the free-stream static pressure is the same at corresponding points in similar flow fields.

Requirements for dynamic similarity of related shapes in free flight, including the correlation of their flight paths, are obtained using the aerodynamic forces and moments as correlated by the hypersonic similarity law. In addition to the conditions of hypersonic similarity, dynamic similarity depends upon conditions derived from the inertial properties of the bodies and the immersing fluids. In order to have dynamic similarity, however, rolling motions must not occur in combination with other motions.

The law is examined for steady flow about related three-dimensional shapes. The results of a computational investigation showed that the similarity law as applied to nonlifting cones and ogives is applicable over a wide range of Mach numbers and fineness ratios. In the special case of inclined bodies of revolution, the law is extended to include some significant effects of the viscous cross force. Results of a limited experimental investigation of the pressures acting on two inclined cones are found to check the law as it applies to bodies of revolution.

INTRODUCTION

The hypersonic similarity law for steady potential flows about thin airfoil sections and slender nonlifting bodies of revolution was first developed by Tsien in reference 1. Hayes (ref. 2) investigated this law from the standpoint of analogous nonsteady flows and concluded that it would also apply to nonpotential flows containing shock waves and vorticity, provided the local Mach number was everywhere large with respect to 1. He also reasoned that similitude could be obtained in hypersonic flows about slender three-dimensional bodies of arbitrary shape; however, the form of the similarity law in terms of customary aerodynamic parameters was not determined. Oswatitsch (ref. 3) investigated the law for two-dimensional steady flow in the limiting case where the Mach number tends toward infinity and, hence, ceases to be a flow parameter. His formulation of the

law, therefore, involves only thickness ratio and angle of attack. Goldsworthy (ref. 4) investigated the effects of rotation on the hypersonic similarity law for two-dimensional steady flow. His results corroborated, in part, the previous findings of Hayes and showed the potential analysis of Tsien to be valid.

An investigation of the law as it applies in nonsteady flow was made by Lin, Reissner, and Tsien (ref. 5). In particular, the necessary conditions for similarity of hypersonic flow about oscillating two-dimensional bodies were determined. The analysis for more arbitrary motion of two- or three-dimensional bodies is apparently not available.

Ehret, Rossow, and Stevens (ref. 6) investigated the hypersonic similarity law for steady flow about nonlifting bodies of revolution by comparing pressure distributions calculated by means of the method of characteristics. They found the law to be applicable over a wide range of Mach numbers and thickness ratios. Their investigation did not, however, include the effects of vorticity arising from the curvature of the nose shock wave. Rossow (ref. 7) continued this investigation and found that the law was equally valid when the effects of vorticity were included in the calculations. These findings corroborated, in part, the observations of Hayes and indicated that the law may be used with confidence to investigate the aerodynamic characteristics for steady flow about nonlifting bodies of revolution at hypersonic speeds.

It appears desirable, therefore, to attempt to unify the different treatments of the similarity law into a single formulation. The primary purpose of this report is, then, to determine the form of the hypersonic similarity law for nonsteady flow about slender three-dimensional bodies of arbitrary shape and to present the results in terms of customary aerodynamic parameters. It is further undertaken to examine the hypersonic similarity law in some detail as it applies to steady flow.

The possibility of obtaining a hypersonic similarity law for correlating the aerodynamic forces and moments on related shapes in free flight suggests a more general dynamic problem, that of correlating their motions with the aid of this law. Hence, it is also undertaken in this report to determine the requirements on the inertial properties of related bodies and the immersing fluids in order that such bodies may have similar free-flight paths, that is, dynamic similarity.

¹ Supersedes NACA TN 2443, "The Similarity Law for Hypersonic Flow About Slender Three-Dimensional Shapes," by Frank M. Hamaker, Stanford E. Neice, and A. J. Eggers, Jr., 1951, and NACA TN 2631, "The Similarity Law for Nonsteady Hypersonic Flows and Requirements for the Dynamical Similarity of Related Bodies in Free Flight," by Frank M. Hamaker and Thomas J. Wong, 1952.

SYMBOLS

a	speed of sound
A	characteristic reference area of body, $A=bt$
b	characteristic width of body
C_z	side-force coefficient, $\frac{\text{side force}}{\frac{1}{2}\rho_0 V_0^2 A}$
$\tilde{C}_c$	side-force function
C_D	drag coefficient, $\frac{\text{drag}}{\frac{1}{2}\rho_0 V_0^2 A}$
$\tilde{C}_D$	drag function
C_l	rolling-moment coefficient, $\frac{\text{rolling moment}}{\frac{1}{2}\rho_0 V_0^2 A b}$
$\tilde{C}_l$	rolling-moment function
C_L	lift coefficient, $\frac{\text{lift}}{\frac{1}{2}\rho_0 V_0^2 A}$
$\tilde{C}_L$	lift function
C_m	pitching-moment coefficient, $\frac{\text{pitching moment}}{\frac{1}{2}\rho_0 V_0^2 A c}$
$\tilde{C}_m$	pitching-moment function
C_n	yawing-moment coefficient, $\frac{\text{yawing moment}}{\frac{1}{2}\rho_0 V_0^2 A b}$
$\tilde{C}_n$	yawing-moment function
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
c	characteristic length of body
c_{d_c}	section drag coefficient of circular cylinder with axis perpendicular to the flow
$\bar{c}_{d_c}$	mean c_{d_c} for a body of revolution
D	displaced-fluid-mass factor, $\frac{c b t \rho_0}{2}$
d	length of flight path
F	viscous force or moment function
f	dimensionless perturbation potential function
fn	general functional designation
G	body-shape function
g	dimensionless body-shape function
$\bar{h}$	vector from the origin of the coordinate system to any point on the body

$\bar{i}, \bar{j}, \bar{k}$ unit vectors along coordinate axes x, y, z , respectively

$\left. \begin{matrix} I_{x-x} \\ I_{y-y} \\ I_{z-z} \end{matrix} \right\}$ moments of inertia of body about the x, y, z axes, respectively

$\left. \begin{matrix} K_t = M_0 \frac{t}{c}, K_b = M_0 \frac{b}{c} \\ K_\alpha = M_0 \alpha, K_\beta = M_0 \beta \\ K_\delta = \delta, K_p = M_0 \left(\frac{pb}{V_0} \right) \\ K_q = M_0 \left(\frac{qc}{V_0} \right), K_r = M_0 \left(\frac{rc}{V_0} \right) \end{matrix} \right\}$ hypersonic similarity parameters

$\left. \begin{matrix} K_{x-x} = \frac{c^2 D}{I_{x-x}}, K'_{x-x} = \frac{b^2 D}{I_{x-x}}, K_{y-y} = \frac{c^2 D}{I_{y-y}} \\ K_{z-z} = \frac{c^2 D}{I_{z-z}}, K_\mu = \frac{D}{\mu} \end{matrix} \right\}$ dynamic similarity parameters

l, m, n direction cosines of the unit outer normal vector to the body surface

M Mach number

M_x, M_y, M_z	moments acting on body about x, y, z axes, respectively
$\bar{N}$	unit outer normal vector to surface of body
P	static pressure
p, q, r	rolling, pitching, and yawing velocities, respectively
R	radius of curvature of flight path
R_c	cross Reynolds number based on maximum body diameter and the component of the free-stream velocity normal to the body axis
r	radius of body of revolution at any station x
s	cross force per unit length
t	characteristic depth of body
u, v, w	components of body velocity along the x, y, z axes, respectively
V	resultant velocity
x, y, z	Cartesian coordinates fixed relative to the body
X, Y, Z	forces on body along x, y, z axes, respectively
α	angle of attack
β	angle of sideslip
γ	ratio of the specific heats, $\gamma = \frac{C_p}{C_v}$
δ	angle of roll
ϵ	orifice location on the test cones
ξ, η, ζ	dimensionless coordinates corresponding to x, y, z , respectively
θ	time coordinate
μ	mass of body
ρ	density of the fluid
τ	dimensionless time coordinate, $\frac{a_0 M_0 \theta}{c}$
φ	perturbation potential function
Φ	potential function
ψ, Ω	alternate time variables
ω	angular velocity of the body

SUBSCRIPTS

0	free-stream conditions
v	viscous cross-force effects
1, 2, 3	different functions F , C_m , or C_n , except as noted

SUPERScript

— vector quantities

Except for symbols noted above, all variables used as subscripts indicate partial differentiation with respect to the subscript variable.

THE SIMILARITY LAW FOR NONSTEADY THREE-DIMENSIONAL FLOW

DEVELOPMENT OF THE LAW

The hypersonic similarity law is derived from the equations of motion and energy and from the boundary conditions. In deriving the law, the following assumptions are made: (1) The Mach number of the uniform stream is large compared to 1; (2) the disturbance velocities are small compared

to the free-stream velocities; and (3) the flow is of the potential type. These assumptions imply that the analysis is restricted to hypersonic flow over slender bodies at small angles of attack and to irrotational flows, respectively. As was indicated in the introduction, the law has been extended to rotational flows by both Hayes and Goldsworthy. An analysis is presented in Appendix A to show that the rotational effects in a three-dimensional nonsteady flow obey the hypersonic similarity law as formulated by the potential analysis. Hence, the conclusions derived from the analysis based upon potential flow will also be valid for rotational flow. The purpose of making the assumption of potential flow is merely to simplify the analysis.

The coordinate system is fixed with respect to the body, as shown in figure 1. Also shown are the possible angular velocities of the body and the direction of the velocity vector of the free stream. The angles have the conventional positive sense of angles of attack and sideslip. Under assumption (2), these angles must be small.

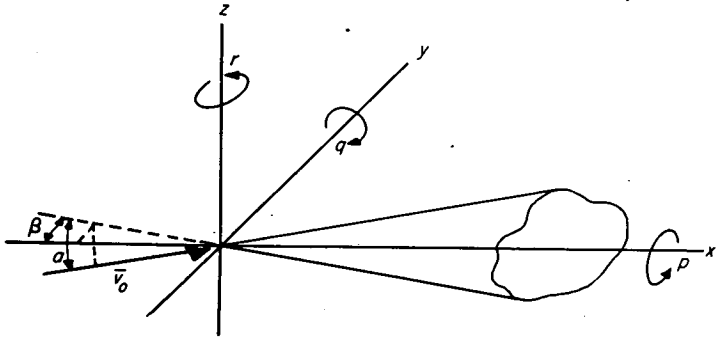


FIGURE 1.—Schematic diagram of orientation of body in flow.

The development of the law involves, first, derivation of a simplified potential equation describing the flow, second, the statement of the boundary conditions, and third, the transformation of these equations into nondimensional coordinates.

The simplified potential equation is obtained from the nonsteady equation of motion and the energy equation which are written in the following potential form:

$$\begin{aligned} &\Phi_{\theta\theta} + \Phi_{xx}(\Phi_x^2 - a^2) + \Phi_{yy}(\Phi_y^2 - a^2) + \Phi_{zz}(\Phi_z^2 - a^2) + \\ &2(\Phi_{xy}\Phi_x\Phi_y + \Phi_{yz}\Phi_y\Phi_z + \Phi_{xz}\Phi_z\Phi_x) + \\ &2(\Phi_x\Phi_{x\theta} + \Phi_y\Phi_{y\theta} + \Phi_z\Phi_{z\theta}) = 0 \end{aligned} \quad (1a)$$

$$\Phi_\theta + \frac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) + \frac{a^2}{\gamma - 1} = \frac{V_0^2}{2} + \frac{a_0^2}{\gamma - 1} \quad (1b)$$

These equations are expanded by expressing the potential-function derivatives in the following perturbation form:

$$\left. \begin{aligned} \Phi_x &= V_0 - \frac{V_0 \alpha^2}{2} - \frac{V_0 \beta^2}{2} + \varphi_x \\ \Phi_y &= -V_0 \beta + \varphi_y \\ \Phi_z &= V_0 \alpha + \varphi_z \\ \Phi_\theta &= \varphi_\theta \end{aligned} \right\} \quad (2)$$

The local speed of sound a can be eliminated by combining the expanded forms of equations (1b) and (1a). The resulting equation can be simplified by neglecting higher order terms keeping in mind that for hypersonic flows about slender shapes φ_x , φ_y , φ_z , and a_0 are small compared to V_0 and that φ_x is small compared to φ_y and φ_z . The simplified potential equation then assumes the following form:

$$\begin{aligned} &\frac{\varphi_{\theta\theta}}{a_0^2} + M_0^2 \varphi_{xx} + \varphi_{yy} \left[M_0^2 \beta^2 + (\gamma - 1) \frac{M_0}{a_0} \varphi_x - (\gamma + 1) \frac{M_0}{a_0} \beta \varphi_y + \right. \\ &(\gamma - 1) \frac{M_0}{a_0} \alpha \varphi_z + \frac{\gamma + 1}{2} \frac{\varphi_y^2}{a_0^2} + \frac{\gamma - 1}{2} \frac{\varphi_z^2}{a_0^2} + (\gamma - 1) \frac{\varphi_\theta^2}{a_0^2} - 1 \left. \right] + \\ &\varphi_{zz} \left[M_0^2 \alpha^2 + (\gamma - 1) \frac{M_0}{a_0} \varphi_x - (\gamma - 1) \frac{M_0}{a_0} \beta \varphi_y + (\gamma + 1) \frac{M_0}{a_0} \alpha \varphi_z + \right. \\ &\frac{\gamma - 1}{2} \frac{\varphi_y^2}{a_0^2} + \frac{\gamma + 1}{2} \frac{\varphi_z^2}{a_0^2} + (\gamma - 1) \frac{\varphi_\theta^2}{a_0^2} - 1 \left. \right] + 2 \left[M_0 \varphi_{xy} \left(-M_0 \beta + \frac{\varphi_y}{a_0} \right) + \right. \\ &\varphi_{yz} \left(-M_0^2 \alpha \beta - \frac{M_0 \beta \varphi_z}{a_0} + \frac{M_0 \alpha \varphi_y}{a_0} + \frac{\varphi_y \varphi_z}{a_0^2} \right) + M_0 \varphi_{xz} \left(-M_0 \alpha + \right. \\ &\left. \frac{\varphi_z}{a_0} \right) \left. \right] + 2 \left(\frac{M_0 \varphi_{x\theta}}{a_0} - \frac{M_0 \beta \varphi_{y\theta}}{a_0} + \frac{\varphi_y \varphi_{y\theta}}{a_0^2} + \frac{M_0 \alpha \varphi_{z\theta}}{a_0} + \frac{\varphi_z \varphi_{z\theta}}{a_0^2} \right) = 0 \end{aligned} \quad (3)$$

The shape of the body can be expressed by the functional relation

$$G(x, y, z) = 0 \quad (4)$$

The unit outer normal at a point on the body surface is given by the vector

$$\bar{N} = \bar{l}i + \bar{m}j + \bar{n}k \quad (5)$$

and the requirement that the body be slender is satisfied by the condition

$$l \ll 1 \quad (6)$$

There are two boundary conditions which must be satisfied. The first of these is that the perturbation velocity, imposed by the presence of the body, must vanish at large distances ahead of the body. Consequently,

$$\varphi_x = \varphi_y = \varphi_z = 0 \text{ at } x = -\infty \quad (7)$$

The other boundary condition is given by the fact that the flow is tangent to the body at the surface, that is, for no angular velocity

$$\bar{N} \cdot \bar{V} = 0 \quad (8)$$

The angular velocity of a body will cause an apparent distortion of the velocity vector at the surface of the body. By expressing the angular velocity in the form,

$$\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k} \quad (9)$$

the velocity of each point on the surface of the body is then given by the vector cross product

$$\bar{\omega} \times \bar{h} = (qz - ry)\bar{i} + (rx - pz)\bar{j} + (py - qx)\bar{k} \quad (10)$$

The boundary condition on the surface of the body then becomes

$$(\bar{V} - \bar{\omega} \times \bar{h}) \cdot \bar{N} = 0 \quad (11)$$

After equation (10) is expanded and combined with equation (11), and higher order terms neglected in accordance with equation (6), the second boundary condition assumes the following form:

$$V_0 G_z - (V_0 \beta - \varphi_y + rx - pz) G_y + (V_0 \alpha + \varphi_z + qx - py) G_x = 0 \text{ at } G=0 \quad (12)$$

In obtaining the similarity law for flow about related bodies, the equations of motion and boundary conditions are expressed in a nondimensional form. A nondimensional coordinate system is introduced by the following affine transformation:

$$\xi = \frac{x}{c}, \eta = \frac{y}{b}, \zeta = \frac{z}{t}, \tau = \frac{\theta a_0 M_0}{c} \quad (13)$$

and a nondimensional perturbation potential function is defined by the relation

$$f(\xi, \eta, \zeta, \tau) = \frac{\varphi(x, y, z, \theta)}{a_0 M_0 c \left(\frac{t}{c}\right)^2} \quad (14)$$

where c , b , and t are a characteristic length, width, and depth of the body, respectively. Under the coordinate transformation given above, equation (4) takes the form

$$g(\xi, \eta, \zeta) = 0 \quad (15)$$

By substitution of equations (13) and (14), equations (3), (7), and (12) become, respectively,

$$\begin{aligned} & K_t^2(f_{\tau\tau} + f_{\xi\xi}) + \left(\frac{K_t}{K_b}\right)^2 f_{\eta\eta} \left[K_\beta^2 + (\gamma-1) K_t^2 f_\xi - \right. \\ & (\gamma+1) \left(\frac{K_t}{K_b}\right) K_t K_\beta f_\eta + (\gamma-1) K_t K_\alpha f_\xi + \\ & \left. \frac{\gamma+1}{2} \left(\frac{K_t}{K_b}\right)^2 K_t^2 f_\eta^2 + \frac{\gamma-1}{2} K_t^2 f_\xi^2 + (\gamma-1) K_t^2 f_\tau - 1 \right] + f_{\xi\xi} \left[K_\alpha^2 + \right. \\ & (\gamma-1) K_t^2 f_\xi - (\gamma-1) \left(\frac{K_t}{K_b}\right) K_t K_\beta f_\eta + (\gamma+1) K_t K_\alpha f_\xi + \\ & \left. \frac{\gamma-1}{2} \left(\frac{K_t}{K_b}\right)^2 K_t^2 f_\eta^2 + \frac{\gamma+1}{2} K_t^2 f_\xi^2 + (\gamma-1) K_t^2 f_\tau - 1 \right] + \\ & 2 \left\{ \left(\frac{K_t}{K_b}\right) f_{\xi\eta} \left[\left(\frac{K_t}{K_b}\right) K_t^2 f_\eta - K_t K_\beta \right] + \left(\frac{K_t}{K_b}\right) f_{\eta\tau} \left[-K_\alpha K_\beta - \right. \right. \\ & K_t K_\beta f_\xi + K_t \left(\frac{K_t}{K_b}\right) f_\eta (K_\alpha + K_t f_\tau) \left. \right] + K_t f_{\xi\tau} (K_\alpha + K_t f_\tau) \left. \right\} + \\ & 2 \left\{ K_t^2 f_{\xi\tau} - \left(\frac{K_t}{K_b}\right) f_{\eta\tau} \left[K_t K_\beta + \left(\frac{K_t}{K_b}\right) K_t^2 f_\eta \right] + \right. \\ & \left. K_t f_{\xi\tau} (K_\alpha + K_t f_\tau) \right\} = 0 \end{aligned} \quad (16)$$

$$f_\tau = f_\xi = f_\eta = f_\zeta = 0 \text{ at } \xi = -\infty \quad (17)$$

$$\begin{aligned} & g_\xi - \left[\left(\frac{K_\beta}{K_b}\right) - \left(\frac{K_t}{K_b}\right) f_\eta + \left(\frac{K_\tau}{K_b}\right) \xi - \frac{K_p}{K_t} \left(\frac{K_t}{K_b}\right)^2 \zeta \right] g_\eta + \left[\left(\frac{K_\alpha}{K_t}\right) + \right. \\ & \left. f_\xi + \left(\frac{K_q}{K_t}\right) \xi - \left(\frac{K_p}{K_t}\right) \eta \right] g_\zeta = 0 \end{aligned} \quad (18)$$

where

$$K_t = M_0 \frac{t}{c} \quad (19)$$

$$K_b = M_0 \frac{b}{c} \quad (20)$$

$$K_\alpha = M_0 \alpha \quad (21)$$

$$K_\beta = M_0 \beta \quad (22)$$

$$K_p = M_0 \left(\frac{pb}{V_0} \right) \quad (23)$$

$$K_q = M_0 \left(\frac{qc}{V_0} \right) \quad (24)$$

$$K_r = M_0 \left(\frac{rc}{V_0} \right) \quad (25)$$

It is seen then that, if two related bodies are flying with given motions and attitudes so that the parameters, equations (19) through (25), are the same for both bodies, the flows are characterized by the same function $f(\xi, \eta, \zeta, \tau)$ and are therefore similar. The requirements expressed by the nondimensional form of the body-shape function, equation (15), and the similarity parameters, equations (19) through (25), therefore constitute the similarity law of hypersonic flow.

A closer examination of the parameters K_t , K_b , K_α , and K_β reveals that an essential property of similarity is that the lateral dimensions and the slopes of a body with respect to the flow direction are in inverse proportion to the flight Mach number. In fact, the remaining parameters K_p , K_q , and K_r , which relate to nonsteady motion of the body, can be interpreted by means of the same property. In rolling, for example, points on the body surface perform helical motions, and the quantity pb/V_0 in equation (23) is simply proportional to the slope of the helix with respect to the flow direction. This slope must be inversely proportional to the flight Mach number. Similar arguments may be applied to K_q and K_r .

Because of the complexities of algebra involved, the effects of angle of roll were not included in the previous equations. Had they been included, however, the result would be the same as above with the additional requirement that the angle of roll must be the same for the related bodies. Hence, the additional hypersonic similarity parameter is

$$K_\delta = \delta \quad (26)$$

CORRELATION OF AERODYNAMIC FORCES AND MOMENTS

The correlation of aerodynamic forces and moments on related bodies in unsteady hypersonic flows can be developed by consideration of the pressure distribution over the bodies. The pressure relation is obtained from the energy equation, equation (2), and is given in the following form:

$$\frac{P}{P_0} = \left[\frac{1 + \frac{\gamma-1}{2a_0^2} V_0^2}{1 + \frac{\gamma-1}{2a^2} (V^2 + 2\phi_\theta)} \right]^{\frac{\gamma}{\gamma-1}} \quad (27)$$

When this expression is simplified (in a manner paralleling the development of the preceding section) to include only higher order terms and put into nondimensional form, it

reduces to a function only of the nondimensional coordinates and the similarity parameters (for a constant γ).

$$\frac{P}{P_0} = \frac{P}{P_0}(\xi, \eta, \zeta, \tau; K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \quad (28)$$

It is clear from this relation that for similar flows, the ratio of the local to the free-stream static pressure is the same at corresponding points in the flow fields.²

The correlation of the aerodynamic forces and moments is then obtained with the aid of equation (28) by integration of the appropriate components of the pressure forces over the related shapes. This correlation can be given in the following forms:

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ M_0 C_C &= \tilde{C}_C = \tilde{C}_C(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_m(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) + \frac{1}{M_0^2} \tilde{C}_{m_2}(K_t, \dots, K_r) \\ C_n &= \tilde{C}_n = \tilde{C}_n(K_t, \dots, K_r) + \frac{1}{M_0^2} \tilde{C}_{n_2}(K_t, \dots, K_r) \\ M_0 C_i &= \tilde{C}_i = \tilde{C}_i(K_t, \dots, K_r) \end{aligned} \right\} \quad (29)$$

It appears, from the equations for the pitching-moment and yawing-moment functions of equation (29), that these two functions cannot be correlated for related but otherwise arbitrary body shapes. However, a careful examination of the order of magnitude involved in the analysis indicates that the second term on the right in these expressions becomes negligible in magnitude in all but two very special cases. In the case of the pitching-moment function, both terms on the right side become of the same order of magnitude when the l and n components of the unit normal to the body surface are very small. This condition corresponds to an isolated vertical fin as shown in figure 2 (a). If, however, the vertical fin is mounted on a body, or used in combination with a body equipped with horizontal wings, the contribution of the vertical fin to the total pitching moment will be very small indeed. The contribution of the second term in the pitching-moment function for the entire body will, of

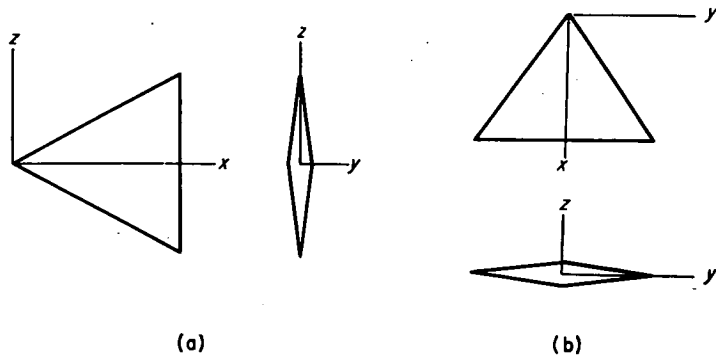


FIGURE 2.—Bodies excluded from similarity considerations as applied to pitching and yawing moments.

course, be correspondingly small. An analogous situation exists in the yawing-moment function for an isolated wing (fig. 2 (b)) in which the l and m components of the unit normal vector are both small. For most practical aerodynamic shapes, therefore, the offending terms can be neglected, and correlation of the aerodynamic coefficients can be achieved as shown in the following relations:

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ M_0 C_C &= \tilde{C}_C = \tilde{C}_C(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_m(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ C_n &= \tilde{C}_n = \tilde{C}_n(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\ M_0 C_i &= \tilde{C}_i = \tilde{C}_i(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \end{aligned} \right\} \quad (30)$$

DETERMINATION OF REQUIREMENTS FOR DYNAMIC SIMILARITY OF RELATED BODIES IN FREE FLIGHT

The requirements for dynamic similarity of related bodies in free flight are developed on the assumption that the forces and moments on such bodies are correlated by the law of hypersonic similarity. In order to determine the conditions for dynamic similarity to be coexistent with hypersonic similarity, the dynamic equations of motion should be transformed to the same dimensionless coordinate system that was used in developing the requirements for hypersonic similarity. In addition, the velocity and force quantities should be expressed in terms of hypersonic similarity parameters.

In this dynamic system, only those forces are considered which correspond to the "power-off" conditions in free flight. The coordinate axes are taken to be principal axes of the body so that the products of inertia vanish. The dynamic equations of motion of the body are given by the relations

$$\left. \begin{aligned} u_\theta - rv + qw &= \frac{X}{\mu} \\ v_\theta - pw + ru &= \frac{Y}{\mu} \\ w_\theta - qu + pv &= \frac{Z}{\mu} \end{aligned} \right\} \quad (31)$$

$$\left. \begin{aligned} p_\theta I_{x-x} - qr(I_{y-y} - I_{z-z}) &= M_x \\ q_\theta I_{y-y} - pr(I_{z-z} - I_{x-x}) &= M_y \\ r_\theta I_{z-z} - pq(I_{x-x} - I_{y-y}) &= M_z \end{aligned} \right\} \quad (32)$$

The translational and rotational velocities may be expressed in terms of hypersonic similarity parameters, the Mach number, and the speed of sound of the free stream by the relations

$$\left. \begin{aligned} u &= a_0 M_0, \quad v = -a_0 K_\beta, \quad w = a_0 K_\alpha \\ p &= a_0 \frac{K_p}{b}, \quad q = a_0 \frac{K_q}{c}, \quad r = a_0 \frac{K_r}{c} \end{aligned} \right\} \quad (33)$$

² Analogous statements can be made for the ratios of local to free-stream values of temperature, density, and Mach number.

Similarly, the aerodynamic forces and moments are given in terms of the correlation functions by the relations

$$\left. \begin{aligned} X &= \tilde{C}_D \left(\frac{a_0^2 M_0 b t \rho_0}{2} \right) \\ Y &= \tilde{C}_C \left(\frac{a_0^2 M_0 b t \rho_0}{2} \right) \\ Z &= \tilde{C}_L \left(\frac{a_0^2 M_0 b t \rho_0}{2} \right) \\ M_x &= \tilde{C}_i b \left(\frac{a_0^2 M_0 b t \rho_0}{2} \right) \\ M_y &= \tilde{C}_m c \left(\frac{a_0^2 M_0 b t \rho_0}{2} \right) \\ M_z &= \tilde{C}_n b M_0 \left(\frac{a_0^2 M_0 b t \rho_0}{2} \right) \end{aligned} \right\} \quad (34)$$

By substituting equations (33) and (34) into equations (31) and (32), and by treating only that length of flight path over which M_0 can be considered constant, the following set of equations is obtained:

$$K_q K_\alpha + K_r K_\beta = K_\mu \tilde{C}_D \quad (35)$$

$$-\frac{dK_\beta}{d\tau} + K_r - \frac{K_p K_\alpha}{K_b} = K_\mu \tilde{C}_C \quad (36)$$

$$\frac{dK_\alpha}{d\tau} - K_q - \frac{K_p K_\beta}{K_b} = K_\mu \tilde{C}_L \quad (37)$$

$$\frac{1}{K_{x-z} K_b} \frac{dK_p}{d\tau} - \left(\frac{1}{K_{y-y}} - \frac{1}{K_{z-z}} \right) \frac{K_q K_r}{M_0^2} = \frac{K_b}{M_0^2} \tilde{C}_i \quad (38)$$

$$\frac{1}{K_{y-y}} \frac{dK_q}{d\tau} - \left(\frac{1}{K_{z-z}} - \frac{1}{K_{x-x}} \right) \frac{K_p K_r}{K_b} = \tilde{C}_m \quad (39)$$

$$\frac{1}{K_{z-z}} \frac{dK_r}{d\tau} - \left(\frac{1}{K_{x-x}} - \frac{1}{K_{y-y}} \right) \frac{K_p K_q}{K_b} = K_b \tilde{C}_n \quad (40)$$

where K_μ is given by ³

$$K_\mu = \frac{D}{\mu} \quad (41)$$

and where

$$K_{x-z} = \frac{c^2 D}{I_{x-z}} \quad (42)$$

$$K_{y-y} = \frac{c^2 D}{I_{y-y}} \quad (43)$$

$$K_{z-z} = \frac{c^2 D}{I_{z-z}} \quad (44)$$

$$D = \frac{c b t \rho_0}{2} \quad (45)$$

The initial conditions to this set of equations are the initial values of the hypersonic similarity parameters.

If both hypersonic similarity and dynamic similarity are to be achieved, it is required that equations (35) through (40) be independent of the Mach number as a separate variable. The elimination of M_0^2 from equation (38) is impossible in the general case, even approximately, because

all the terms involved may be of comparable order of magnitude. Consequently, since equation (38) is the relation for rolling effects, it is indicated that flight paths which include rolling cannot be correlated by this method for obtaining dynamic similarity. For motions that do not involve roll, it is seen that dynamic similarity will exist for related shapes if the hypersonic similarity parameters and the dynamic similarity parameters given in equations (41) through (44) remain invariant. These dynamic similarity parameters relate the masses of the bodies and the immersing fluids, as well as the distribution of the mass in the body.

For rolling motions only, correlation can again be achieved but with a slightly different set of parameters. In this case, only equation (38) remains and can be rewritten as

$$\frac{1}{K'_{x-z}} \frac{dK_p}{d\tau} = \tilde{C}_i \quad (46)$$

where, now

$$K'_{x-z} = \frac{b^2 D}{I_{x-z}} \quad (47)$$

so that correlation for pure rolling motions is now given by the hypersonic similarity parameters and the parameter K'_{x-z} .

A familiar example of motions where rolling effects would be absent is the case of motions confined to the plane of symmetry of the body, the so-called longitudinal motions. To extend the application of this law to the more general case where there are lateral motions as well as longitudinal ones, but no roll, it is necessary to have a suitable symmetry of shape and to have the inertial properties satisfy the relation

$$K_{y-y} = K_{z-z} \quad (48)$$

When these conditions are fulfilled, the flight paths of related bodies can be correlated. As an illustrative example, the disturbed motions of related missile shapes can be examined. The lengths of corresponding portions of related flight paths would be proportional to the corresponding lengths of the shapes. This property can be used to relate the amount of damping in the disturbed flight paths. As shown in Appendix B, the radii of curvature at corresponding points of the flight paths would be proportional to the product of the body length and the flight Mach number. Some of these points are illustrated in the example given in figure 3.

APPLICATION OF THE LAW TO PARTICULAR SHAPES IN STEADY FLOW

In steady flow, the three similarity parameters K_p , K_q , and K_r are zero and equations (30) reduce to the following form:

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_i, K_b, K_\alpha, K_\beta, K_\delta) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_i, K_b, K_\alpha, K_\beta, K_\delta) \\ M_0 C_C &= \tilde{C}_C = \tilde{C}_C(K_i, K_b, K_\alpha, K_\beta, K_\delta) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_m(K_i, K_b, K_\alpha, K_\beta, K_\delta) \\ C_n &= \tilde{C}_n = \tilde{C}_n(K_i, K_b, K_\alpha, K_\beta, K_\delta) \\ M_i C_i &= \tilde{C}_i = \tilde{C}_i(K_i, K_b, K_\alpha, K_\beta, K_\delta) \end{aligned} \right\} \quad (49)$$

³ The parameter K_μ is equivalent to a familiar stability-analysis term known as the relative mass factor.

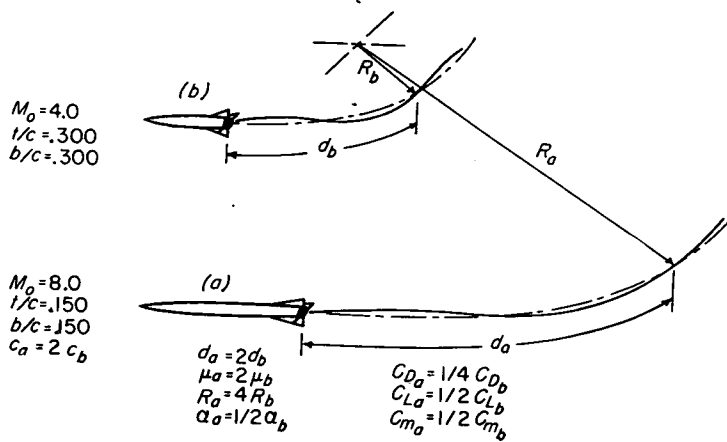


FIGURE 3.—Related wing-body combinations at hypersonic speeds.

It is important to note that the correlation of the aerodynamic coefficients given by equations (30) was obtained on the basis of two restrictions as to allowable body shapes. (See section Correlation of Aerodynamic Forces and Moments and also fig. 2). These restrictions apply equally well to equations (49).

BODIES OF REVOLUTION

For bodies of revolution, equations (49) reduce to ⁴

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_a) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_a) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_m(K_t, K_a) \end{aligned} \right\} \quad (50)$$

where K_b is eliminated as it is identical to K_t .⁵ It is apparent from these relations that the corresponding force and moment parameters have identical values for related bodies of revolution, provided the corresponding similarity parameters have identical values. It will now be shown that this conclusion can be generalized to include significant effects of the viscous cross forces on related inclined bodies.

The viscous cross force arises from the boundary-layer flow transverse to the body axis. A method of estimating this force along with the lift, drag, and pitching-moment coefficients associated with it has been suggested by Allen in reference 8 and is presented in Appendix C. The resulting expressions for these coefficients (see eqs. (C3) in Appendix C) are transformed to the nondimensional form, and the following relations are obtained:

$$\left. \begin{aligned} M_0 C_{L_v} &= \hat{c}_{d_c} F_1(K_t, K_a) \\ M_0^2 C_{D_v} &= \hat{c}_{d_c} F_2(K_t, K_a) \\ M_0 C_{m_v} &= \hat{c}_{d_c} F_3(K_t, K_a) \end{aligned} \right\} \quad (51)$$

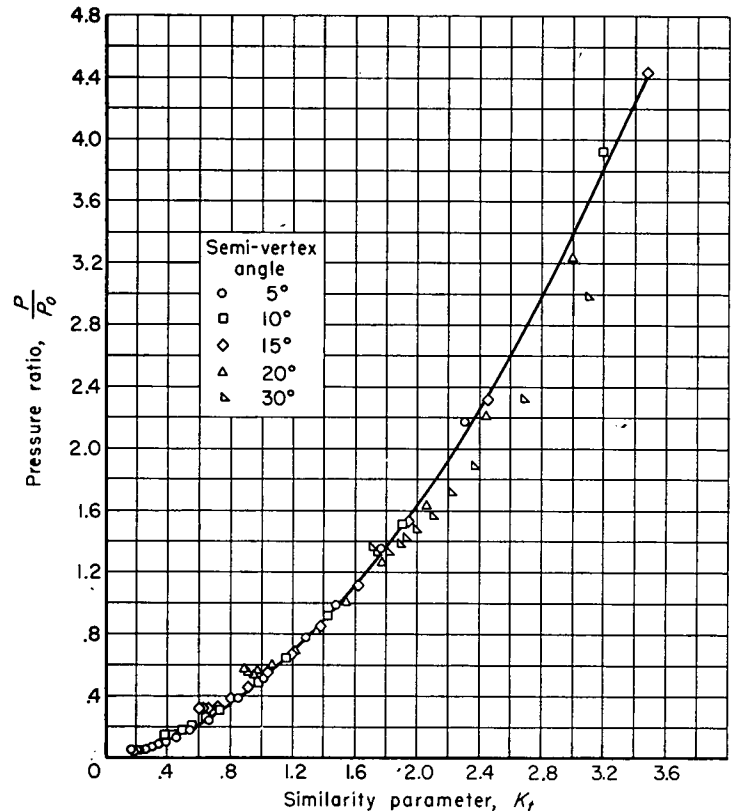
For slender bodies of revolution of the type under consideration, $\hat{c}_{d_c}$ is primarily a function of the Mach number and Reynolds number of the flow component normal to the body

axis. Consequently, these expressions can be reduced to the form

$$\left. \begin{aligned} M_0 C_{L_v} &= \tilde{C}_{L_v} = \tilde{C}_{L_v}(K_t, K_a, R_c) \\ M_0^2 C_{D_v} &= \tilde{C}_{D_v} = \tilde{C}_{D_v}(K_t, K_a, R_c) \\ M_0 C_{m_v} &= \tilde{C}_{m_v} = \tilde{C}_{m_v}(K_t, K_a, R_c) \end{aligned} \right\} \quad (52)$$

where R_c is the cross Reynolds number. For small angles of attack, the cross Mach number is identical to K_a . It is clear, when comparing these relations with those of equation (50), that the latter relations apply with equal validity when viscous cross-flow effects are considered, provided that R_c is included as a similarity parameter.⁶

Nonlifting cones and ogives.—In reference 6 an analysis was performed to determine the limits of applicability of the hypersonic similarity law for nonlifting cones and ogives.⁷ To determine this limit for cones, surface pressures were calculated using reference 9 and were plotted as a function of the similarity parameter K_t as shown in figure 4. A single curve favoring the slender cones was faired through the calculated points. It is apparent that the similarity in pressure holds for a wide range of values of K_t for slender cones. If it is assumed that a pressure deviation of 5 percent from the faired curve can be tolerated in using the similarity law, then limits of similarity can be determined as a function

FIGURE 4.—Variation of pressure ratio, P/P_0 , with similarity parameter, K_t , for nonlifting cones.

⁴ Because of the axial symmetry of bodies of revolution, only angles of attack are considered. This latter consideration obviates a discussion of force and moment characteristics at angles of sideslip or combined angles of attack and sideslip, while roll, of course, has no meaning. It is clear, then, that the similarity parameters K_b and K_s are eliminated from this analysis.

⁵ If the angle of attack is zero, K_a is also zero, and the expression for the drag parameter reduces to a form equivalent to that obtained by Tsien in reference 1.

⁶ It is assumed that the viscous flow considered here does not significantly influence the potential, inviscid flow discussed previously. Hence, the force and moment coefficients resulting from these flows may be superimposed.

⁷ It should be noted that ogives are not exactly a related set of bodies; nevertheless, they were chosen in this study since the configuration is of interest, and the deviation in thickness distribution is not significant for slender bodies.

of Mach number and fineness ratio c/t of the cone. The limits determined in this way are illustrated in figure 5 (a). The shaded area indicates the regions of Mach number and fineness ratio where the similarity law as applied to pressure on the cone will be in error 5 percent or more.

Since the surface slope of an ogive is largest at the vertex, the pressures at this point should provide a critical test for similarity of pressures. Accordingly, the limits of applicability of the law for ogives were determined in reference 6 from consideration of the pressures over a cone tangent to the ogive at the vertex. Figure 5 (b) presents the limits of applicability for ogives as obtained by this method. These results illustrate the conclusions of reference 4 that the law, as applied to nonlifting cones and ogives, is applicable over a wide range of Mach numbers and fineness ratios in spite of the simplifying assumptions made in the derivation.

A check of the applicability of the hypersonic similarity law in a rotational flow field was performed in reference 7 by comparing the pressure distributions, obtained by the method of characteristics, over ogive cylinders at several values of K_t . The pressure distributions for two ogive cylinders at a value of K_t of 2.0 are presented in figure 6 and serve to illustrate the general results obtained in reference 7. The high degree of correlation of pressures in figure 6 indicates that the hypersonic similarity law applies in a rotational flow field and verifies the analysis presented by Hayes in reference 2.

Lifting cones.—For bodies of revolution at angles of attack, a limited experimental check was made in the Ames 10-by 14-inch supersonic wind tunnel. Two cones having fineness ratios of 3.0 and 4.9 were tested at Mach numbers of 2.75 and 4.46, respectively; thus, the value of K_t was 0.91. Overlapping values of K_a up to 14° were obtained. Pressure measurements were made at the locations shown in figure 7 for angles up to 5° . The results are shown in figure 8 as a function of K_a . Agreement with the prediction of the similarity law is generally observed, in that the values of p/p_0 for corresponding points on the two bodies lie essentially along the same curve. The exception to this agreement is on the lee sides of the cones ($\epsilon=180^\circ$) where it is noted that significantly different curves are defined. This difference

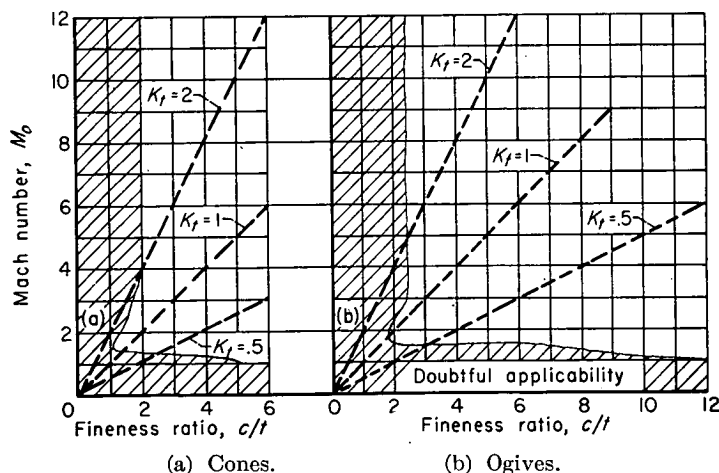


FIGURE 5.—Range of applicability of similarity law for nonlifting cones and ogives.

is believed to be the result of the dissimilar flow separation from the two cones, caused by the fact that identical values of R_c could not be obtained for the two cones at the same value of K_a . This difference in R_c should not affect the pressures appreciably where separation does not occur.

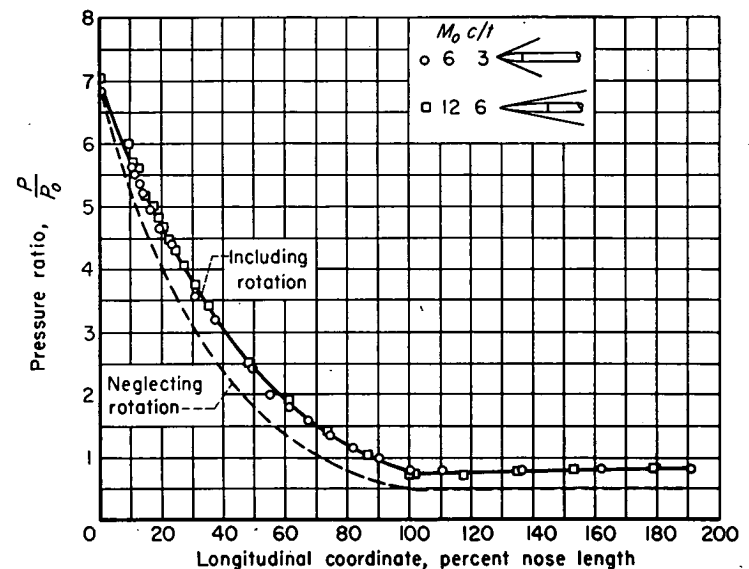


FIGURE 6.—Variation of pressure ratio, P/P_0 , along nonlifting ogive cylinders for a value of the similarity parameter, K_t , of 2.0.

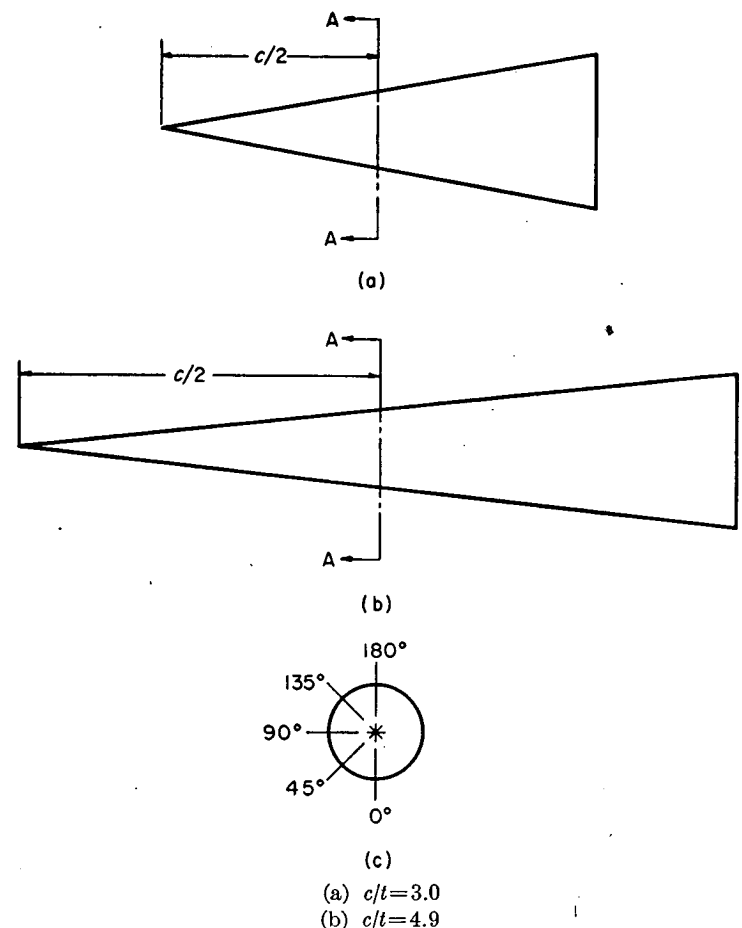


FIGURE 7.—Location of orifices on two cones tested at $K_t=0.91$.

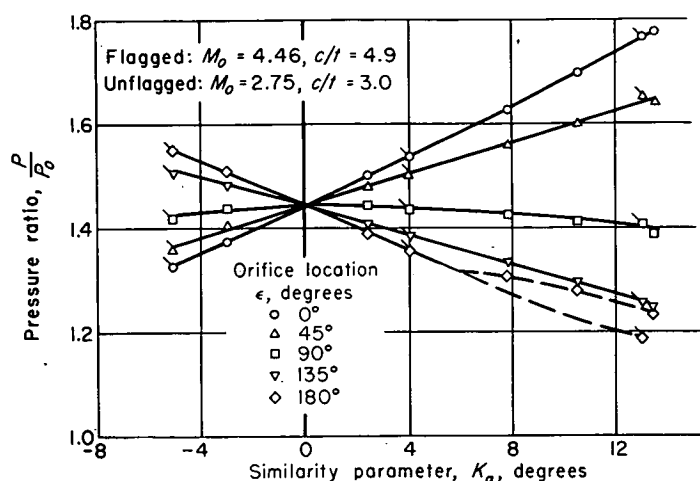


FIGURE 8.—Variation of pressure ratio, P/P_0 , with K_α , for two cones tested at $K_t = 0.91$.

WINGS AND WING-BODY COMBINATIONS

If, for spanwise symmetric wings, only angle of attack is considered, the similarity parameters K_β and K_δ vanish from equations (49) and only three of the aerodynamic coefficients remain. The corresponding force and moment functions are reduced to the following form:⁸

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_b, K_\alpha) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_b, K_\alpha) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_m(K_t, K_b, K_\alpha) \end{aligned} \right\} \quad (53)$$

These relations also apply, of course, to wing sections. In this case, b and therefore K_b are infinite and it is seen from equations (16) through (18) that the terms involving K_b vanish and the equations reduce to the two-dimensional equations for hypersonic flow. The similarity parameter K_b is thus eliminated from equation (53). This result is equivalent to that presented in reference 1.⁹

Of practical importance is the conclusion to be drawn from application of the dimensionless equation of motion (eq. (16)) and the dimensionless boundary condition (eq. (18)), to steady flow about thin wings at zero angle of yaw. It is noticed in the equations that the parameter, K_b , always appears in the form

$$\left(\frac{K_t}{K_b}\right)^2 = \frac{t^2}{b^2}$$

If b is of the same order of magnitude as c , then, consistent with the other approximations made in developing this equation, the terms involving $(K_t/K_b)^2$ are to be neglected. Performing this operation, however, yields the equation of motion for two-dimensional flow. Thus, it is indicated that, if the aspect ratio is of the order of magnitude of one or greater, hypersonic flow about wings may be treated approximately as a two-dimensional-flow problem. The latter problem is, of course, relatively simple to solve.

From a physical point of view, this conclusion stems from the fact that, in supersonic flow, the effect of a disturbance at a point is confined to the conical zone formed by the

Mach lines from that point. For very high Mach numbers, this zone of influence is a narrow region behind the disturbance. Consequently, conditions along a streamline are, for the most part, independent of the conditions along adjacent streamlines.¹⁰ For thin wings in hypersonic flow, therefore, it can readily be seen that the zone of influence of disturbances caused by wing tips will, for example, be small compared to the wing area if the aspect ratio is greater than one. The effect of the tip disturbances on the aerodynamic characteristics of the wing will, of course, be correspondingly small.

Wing-body combinations may be thought of merely as irregular-shaped bodies. As such, the aerodynamic coefficients are correlated by equations (49) with the restrictions discussed in relation to these equations. The illustrative example, given in figure 3 in connection with the free-flight motion of a wing-body configuration, can be re-examined on the basis of steady flow. It is seen that in going from a Mach number of 4 to a Mach number of 8, the wing and body lengths are doubled, the angle of attack is decreased by one-half, while the body thickness and wing spans remain the same. The changes in some of the aerodynamic coefficients are also shown in the figure.

CONCLUDING REMARKS

The similarity law for nonsteady, inviscid hypersonic flow about slender three-dimensional shapes has been derived in terms of customary aerodynamic parameters. The conclusions drawn from the potential analysis used to derive the law were found to apply also to rotational flows. As a direct consequence of this law, it was found that the ratio of the local static pressure to the free-stream static pressure is the same at corresponding points in similar flow fields. With the aid of this law, expressions were obtained for correlating the forces and moments acting on related shapes in hypersonic flows.

It was found that the motions of related bodies in free flight could be correlated using the hypersonic similarity parameters and additional parameters relating the inertial properties of the bodies and the air densities. The dynamic similarity of the free flight of related bodies can be obtained for motions which include pitching and yawing but no rolling. For pure rolling motions, similarity can again be achieved.

In the case of steady flow about inclined bodies of revolution, the correlations of forces and moments derived from the similarity law can be generalized to include the significant effects of the viscous cross force.

The results of a computational analysis, using the method of characteristics, showed that the similarity law as applied to nonlifting cones and ogives is applicable over a wider range of Mach numbers and fineness ratios than might be expected from the assumptions made in the derivation.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., June 5, 1951.

⁸ Parameters equivalent to these were obtained by Tsien and, although not published, were presented in the form of lecture notes which were brought to the attention of the authors after completion of this investigation.

⁹ The exponents of M_0 obtained here are different from those obtained in reference 1, because $b-t$ is used as a reference area, rather than $c-b$.

¹⁰ This result holds, in fact, for nonsteady as well as steady hypersonic flow about thin wings, as pointed out by Eggers in reference 10.

APPENDIX A

EXTENSION OF POTENTIAL FLOW ANALYSIS TO ROTATIONAL FLOW

The hypersonic similarity law can be extended to rotational flows by the method of Hayes (ref. 2). This extension is in fact demonstrated by Hayes' results. However, to understand fully the reasoning involved, it is instructive to elaborate on his analysis. Hayes showed that the hypersonic potential equation for steady flow about slender shapes was identical to the nonsteady potential equation in one less spatial coordinate under the transformation

$$x = a_0 M_0 \theta \quad (A1)$$

In the case of two-dimensional flow, the transformation, equation (A1), allows, for example, the upper surface of the body profile to be replaced by the upper surface of a moving piston as shown in figure 9. The piston motion must be such that a given piston displacement y_1 at time θ_1 will be the same as the ordinate on the body profile at the coordinate x_1 given by the relation $x_1 = a_0 M_0 \theta_1$.

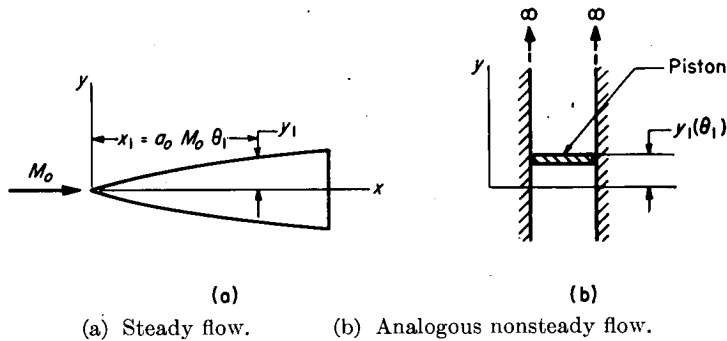


FIGURE 9.—Two-dimensional steady flow and analogous one-dimensional nonsteady flow.

In investigating the physical significance of this transformation, Hayes pointed out that its existence resulted from the basic assumptions of slender bodies and large Mach numbers. Since, as a result of these assumptions, the x component of the fluid velocity does not change appreciably and is always much greater than the local speed of sound, there is essentially no chance for disturbances to propagate in the x direction. This is the essential feature that permits the replacement of x by the time variable θ and, hence, the existence of an analogous nonsteady flow.

Hayes further showed that in hypersonic flow about slender shapes the local Mach number remains large compared to one, even in the presence of strong shock waves caused by small surface inclinations. Consequently, the consideration of the hypersonic flow about a slender body as a nonstationary problem in one less spatial dimension remains valid when shock waves and the resultant entropy gradients are present.

One further feature of Hayes' analysis, which is not explicitly stated in reference 2, is that similarity follows directly from the existence of the analogous nonsteady flow.

This feature is illustrated for two-dimensional flows as follows: The motion of the nonsteady boundary (in this case, the piston face) can be expressed in the following dimensionless form:

$$\frac{y}{t} = f n \left(\frac{a_0 \theta}{t} \right) \quad (A2)$$

Upon transforming to the two-dimensional steady flow system, by the substitution of equation (A1) into the functional relationship on the right side of equation (A2), we obtain

$$\frac{y}{t} = f n \left(\frac{a_0}{t} \frac{x}{a_0 M_0} \right) = f n \left(\frac{x}{c} \frac{1}{M_0 \frac{t}{c}} \right) = f n \left(\frac{\dot{x}}{c} \frac{1}{K_t} \right) \quad (A3)$$

or

$$\frac{y}{t} = f n \left(\frac{x}{c} \right), \quad K_t = \text{constant} \quad (A4)$$

Equation (A4) expresses the conditions for which the nonsteady flow system can replace a steady flow system; namely, that the body profile must be expressible in a specific non-dimensional form and that the parameter, K_t , must be constant for all profiles given by this form. These are, of course, the conditions of hypersonic similitude in two-dimensional steady flow. The extension of these considerations to three-dimensional steady flow is straightforward.

To extend these concepts and results to three-dimensional, nonsteady flow, the nonsteady part of the flow may be considered, in the analogous nonsteady flow, as a nonsteady increment on the already nonsteady boundary. This can be demonstrated with reference to the potential analysis as follows: If the transformation, $x = a_0 M_0 \psi$, is used on the equation for steady-state hypersonic flow in perturbation form¹¹

$$\begin{aligned} M_0^2 \varphi_{xx} - \left[1 - (\gamma - 1) \frac{M_0}{a_0} \varphi_x - \frac{\gamma + 1}{2} \frac{\varphi_y^2}{a_0^2} - \frac{\gamma - 1}{2} \frac{\varphi_z^2}{a_0^2} \right] \varphi_{yy} - \\ \left[1 - (\gamma - 1) \frac{M_0}{a_0} \varphi_x - \frac{\gamma - 1}{2} \frac{\varphi_y^2}{a_0^2} - \frac{\gamma + 1}{2} \frac{\varphi_z^2}{a_0^2} \right] \varphi_{zz} + 2 \left(\frac{M_0}{a_0} \varphi_y \varphi_{xy} + \right. \\ \left. \frac{\varphi_y \varphi_z \varphi_{yz}}{a_0^2} + \frac{M_0}{a_0} \varphi_z \varphi_{xz} \right) = 0 \end{aligned} \quad (A5)$$

there is obtained the equation

$$\begin{aligned} \frac{\varphi_{\psi\psi}}{a_0^2} - \left[1 - (\gamma - 1) \frac{\varphi_{\psi}}{a_0^2} - \frac{\gamma + 1}{2} \frac{\varphi_y^2}{a_0^2} - \frac{\gamma - 1}{2} \frac{\varphi_z^2}{a_0^2} \right] \varphi_{yy} - \\ \left[1 - (\gamma - 1) \frac{\varphi_{\psi}}{a_0^2} - \frac{\gamma - 1}{2} \frac{\varphi_y^2}{a_0^2} - \frac{\gamma + 1}{2} \frac{\varphi_z^2}{a_0^2} \right] \varphi_{zz} + \\ 2 \left(\frac{\varphi_y \varphi_{\psi y}}{a_0^2} + \frac{\varphi_y \varphi_z \varphi_{yz}}{a_0^2} + \frac{\varphi_z \varphi_{\psi z}}{a_0^2} \right) = 0 \end{aligned} \quad (A6)$$

By applying the same transformation to the nonsteady flow equation

¹¹ In all the equations of this section, the wind axes are made to coincide with the body axes in order not to obscure the argument.

$$\begin{aligned} & \varphi_{\theta\theta} + M_0^2 \varphi_{zz} - \left[1 - (\gamma - 1) \frac{M_0}{a_0} \varphi_x - \frac{\gamma + 1}{2} \frac{\varphi_y^2}{a_0^2} - \frac{\gamma - 1}{2} \frac{\varphi_z^2}{a_0^2} - \right. \\ & (\gamma - 1) \frac{\varphi_\theta}{a_0^2} \left. \right] \varphi_{yy} - \left[1 - (\gamma - 1) \frac{M_0}{a_0} \varphi_x - \frac{\gamma - 1}{2} \frac{\varphi_y^2}{a_0^2} - \frac{\gamma + 1}{2} \frac{\varphi_z^2}{a_0^2} - \right. \\ & (\gamma - 1) \frac{\varphi_\theta}{a_0^2} \left. \right] \varphi_{zz} + 2 \left(\frac{M_0}{a_0} \varphi_y \varphi_{xy} + \frac{M_0}{a_0} \varphi_z \varphi_{xz} + \frac{\varphi_y \varphi_z}{a_0^2} \varphi_{yz} \right) + \\ & 2 \left(\frac{M_0}{a_0} \varphi_{x\theta} + \frac{\varphi_y}{a_0^2} \varphi_{y\theta} + \frac{\varphi_z}{a_0^2} \varphi_{z\theta} \right) = 0 \end{aligned} \quad (A7)$$

with an additional variable change of

$$\Omega = \theta + \psi \quad (A8)$$

the same equation (A6) is obtained with ψ replaced by Ω . Hence, Hayes' conclusions concerning steady-state, three-dimensional flow should apply equally well to nonsteady, three-dimensional flows.

APPENDIX B

CORRELATION OF THE FLIGHT-PATH CURVATURE

Consider related bodies moving through properly related fluids in paths of finite radii of curvature. Equating the centrifugal force to the side force, the following relation is obtained:

$$\mu \frac{V_0^2}{R} = C_c \frac{1}{2} \rho_0 V_0^2 A \quad (B1)$$

After rearranging in terms of similarity parameters, equation (B1) becomes

$$\frac{M_0 c}{R} = \tilde{C}_c K_\mu \frac{A}{bt} = \text{constant} \quad (B2)$$

The parameter $M_0 c/R$ correlates the radii of curvature at corresponding points of similar flight paths.

This conclusion is also true for curved flight in the vertical plane.

APPENDIX C

FORCES AND MOMENTS DUE TO VISCOUS CROSSFLOWS ON BODIES OF REVOLUTION

In reference 11, Prandtl demonstrated that laminar viscous flows over infinitely long inclined cylinders may be treated by considering, independently, the components of the flow normal and parallel to the axis of the cylinder. Jones, in reference 12, applied this concept to the study of boundary-layer flows over yawed cylinders. The work of Prandtl and Jones suggests, as indicated by Allen in reference 8, that the cross force on slender inclined bodies of revolution may be estimated in the following manner: Each cross section of the body is treated as an element of an infinite cylinder of the same radius. The cross force per unit length on such a cylinder is given by the following equation:

$$s_r = r c_{d_c} \rho_0 V_0^2 \sin^2 \alpha \quad (C1)$$

The incremental lift, drag, and moment produced by this cross force are then given by the relations

$$\left. \begin{aligned} \text{lift} &= r c_{d_c} \rho_0 V_0^2 \sin^2 \alpha \cos \alpha \\ \text{drag} &= r c_{d_c} \rho_0 V_0^2 \sin^3 \alpha \\ \text{moment} &= r x c_{d_c} \rho_0 V_0^2 \sin^2 \alpha \end{aligned} \right\} \quad (C2)$$

Retaining leading terms in α and integrating over the body, where $r = r(x)$, the aerodynamic coefficients are given by the equations

$$\left. \begin{aligned} C_{L_v} &= \frac{2 \hat{c}_{d_c} \alpha^2}{A} \int_0^c r dx \\ C_{D_v} &= \frac{2 \hat{c}_{d_c} \alpha^3}{A} \int_0^c r dx \\ C_{m_v} &= \frac{2 \hat{c}_{d_c} \alpha^2}{A c} \int_0^c r x dx \end{aligned} \right\} \quad (C3)$$

where the reference area is proportional to the maximum cross-sectional area of the body, and the reference length is the body length. The coefficient $\hat{c}_{d_c}$ is the mean c_{d_c} for the body of revolution, and has therefore been taken outside the integral.

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REPORT 1148

A SPECIAL INVESTIGATION TO DEVELOP A GENERAL METHOD FOR THREE-DIMENSIONAL PHOTOELASTIC STRESS ANALYSIS¹

By M. M. FROCHT and R. GUERNSEY, JR

SUMMARY

The method of strain measurement after annealing is reviewed and found to be unsatisfactory for the materials available in this country. A new, general method is described for the photoelastic determination of the principal stresses at any point of a general body subjected to arbitrary loads. The method has been applied to a sphere subjected to diametral compressive loads. The results show possibilities of high accuracy.

INTRODUCTION

It is known that purely photoelastic procedures cannot solve the general three-dimensional stress problem. The photoelastic method furnishes five independent equations, whereas the complete specification of the state of stress at a point requires six relations to determine six unknown stress components.

In order to obtain a sixth relation it has been suggested that the frozen slices removed from the model be annealed and strain measurements be made after annealing. This suggestion has recently received a rather extensive treatment from Prigorovsky and Preiss in Russia (reference 1). A careful analysis of this suggested method shows that its successful application requires model materials having relatively low values of Poisson's ratio at the elevated temperatures used in the freezing process. Such materials are not available in this country. Fosterite and Bakelite, which are the best available materials, have Poisson's ratios approximately equal to 1/2. It is further shown that the method of strain measurement after annealing breaks down when this ratio approaches 1/2.

In this report a new method is described which does not depend on Poisson's ratio and therefore can be used with models made of Fosterite and Bakelite. This method employs frozen stress patterns from normal and oblique incidence. The separation of the principal stresses is obtained by the numerical integration of one of the differential equations of equilibrium in Cartesian coordinates rather than by strain measurement after annealing which involves Poisson's ratio. It will be shown that this permits the determination of all six stress components at each point of a body.

The report consists of three parts. The first part comprises a survey and analysis of the method in three-dimensional photoelasticity which rests on the freezing and slicing processes and strain measurement after annealing. The

second part presents the theory of the new method. The third part contains the application of the new method to the determination of stresses in a diametrically compressed sphere.

The investigation was conducted in the Photoelastic Laboratory of the Mechanics Department at the Illinois Institute of Technology under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. The Research Corporation provided the funds for the fellowship held by Mr. Roscoe Guernsey, Jr. Mr. David Landsberg, Assistant Research Engineer in Experimental Stress Analysis, assisted in all experimental phases of the work. It is a pleasure to acknowledge his cooperation. Acknowledgment is also due Mrs. Dora L. Frocht for her assistance in the translation of the paper by Prigorovsky and Preiss (reference 1).

SYMBOLS

$\sigma_x, \sigma_y, \sigma_z$	normal stresses, psi
$\tau_{xy}, \tau_{yz}, \tau_{zx}$	shearing stresses, psi
$\sigma_\theta, \tau_\theta$	stresses on an inclined plane, psi
p, q	principal stresses, psi
p', q'	secondary principal stresses in XY-plane, psi
p'', q''	secondary principal stresses in XZ-plane, psi
n_x, n_y, n_z	normal stresses in fringes
n_{xy}, n_{yz}, n_{zx}	shearing stresses in fringes
n', n'', n'''	fringe orders at normal incidence for slices parallel to XY-plane, XZ-plane, and YZ-plane, respectively
$n_{\theta y}, n_{\theta z}$	fringe orders at oblique incidence for rotations about Y-axis and Z-axis, respectively
θ	angle of rotation of a slice; also angle defining an inclined plane
ϕ', ϕ'', ϕ'''	isoclinic parameters at normal incidence for slices parallel to XY-plane, XZ-plane, and YZ-plane, respectively
$\phi_{\theta y}$	isoclinic parameter at oblique incidence for rotation about Y-axis
F	shear fringe value of slice, psi per fringe
F', F''	shear fringe value for slices parallel to XY-plane and XZ-plane, respectively
$F_{\theta y}$	shear fringe value for actual light path in slices rotated about Y-axis
f	shear fringe value of material, psi per fringe per inch

¹ Supersedes NACA TN 2822, "A Special Investigation to Develop a General Method for Three-Dimensional Photoelastic Stress Analysis" by M. M. Frocht and R. Guernsey, Jr., 1952.

$\epsilon_x, \epsilon_y, \epsilon_z$	linear strains
E	Young's modulus, psi
ν	Poisson's ratio
P	load, pounds
A	area of equatorial plane of sphere, square inches
A_c	area of surfaces of contact, square inches
R	radius of sphere, inches
R_c	radius of contact areas, inches
$\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_z$	normal stresses in terms of P/A
$\bar{\tau}_{xy}, \bar{\tau}_{yz}, \bar{\tau}_{zx}$	shearing stresses in terms of P/A
σ_y'	contact pressure in terms of P/A_c
τ_{yz}'	contact shearing stress in terms of P/A_c

SURVEY AND ANALYSIS OF EXISTING METHODS

FROZEN STRESSES AND OBLIQUE INCIDENCE

Frozen pattern.—It is now well established that elastic stress systems can be fixed or frozen into models made of certain diphase plastics and that such models with frozen stresses can be sliced into thin sections without disturbing the fixed pattern (references 2 to 5). Observations of such slices in a polariscope yield the relative retardations as well as the isoclinic parameters at each point of the pattern.

Oblique incidence.—The use of oblique incidence of a collimated beam of light, as suggested by Drucker and Mindlin, adds materially to the information obtainable photoelastically (references 6 and 7). The retardation and isoclinics at normal incidence are a function of the secondary principal stresses in the plane of the slice, while those at oblique incidence depend on the secondary principal stresses in a plane perpendicular to the wave normal at each point of the slice.

The basic relation for oblique incidence with rotation about the Z -axis is given by the following expression

$$(2Fn_{\theta_z})^2 = \frac{1}{\cos^2 \theta_z} \{ [(\sigma_x - \sigma_z) + (\sigma_y - \sigma_z) \sin^2 \theta_z + \tau_{xy} \sin 2\theta_z]^2 + 4(\tau_{xz} \cos \theta_z + \tau_{yz} \sin \theta_z)^2 \} \quad (1)$$

The system of notation used in this report is shown in figure 1. Normal stresses are positive when tensile and negative when compressive. The four components of shear in the XY -plane are referred to either as the τ_{xy} or the τ_{yx} shear system, and the sign of this system is positive when the shearing components are as shown in figure 1 (reference 8, par. 1.3). Similarly the shear system in the YZ -plane is positive if the components are as shown in figure 1. No signs are attached to individual shearing stress components, their directions being determined by inspection (reference 8, par. 8.2).

By combining the data from five stress patterns of different obliqueness it is possible to determine the three differences between the normal stress components and the three systems of shearing stresses at each point in the slice (for convenience the plane of the slice is taken as one of the coordinate planes).

It can be shown that from the five quantities obtained with the aid of oblique incidence it is possible in turn to

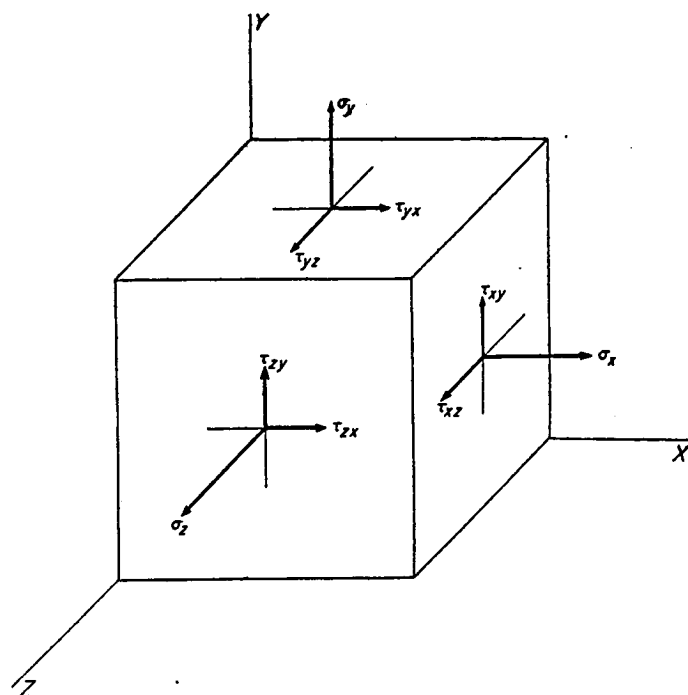


FIGURE 1.—Positive normal stress components and positive systems of shearing stresses.

obtain the three principal shears at all points of the section. This is equivalent to determining Mohr's circle for a three-dimensional state of stress except for its origin which remains indeterminate.

Limitations of purely photoelastic data.—Except for special cases, the optical data by themselves are insufficient for the determination of the individual principal stresses. This limitation results from the fact that isotropic stress systems produce no photoelastic effects. Consequently, two states of stress differing by an arbitrary isotropic system produce equal photoelastic effects.

The method employing scattered light, or the Tyndall effect, which was developed in this country by Weller (references 9 and 10) and independently by Menges (reference 11) suffers from the same limitation.

The method of convergent light employed by Hiltcher (reference 12) and by Kuske (reference 13) makes it possible to determine also the directions of the principal stresses but not their magnitudes.

SEPARATION OF PRINCIPAL STRESSES

Free surfaces.—The limitation mentioned above does not hold at free boundary surfaces. A free surface is subjected to only two principal stresses, similar to those in plane stress systems. Tangential slices yield directly the difference between the principal surface stresses. If in addition a slice is taken normal to the surface and parallel to one principal stress, it is possible to determine the individual principal stresses on the surface (fig. 2). This method has been employed by Leven and Frocht (reference 14) to determine the principal stresses on the surface of Diesel engine valves. Leven (reference 15) has also applied this method to the

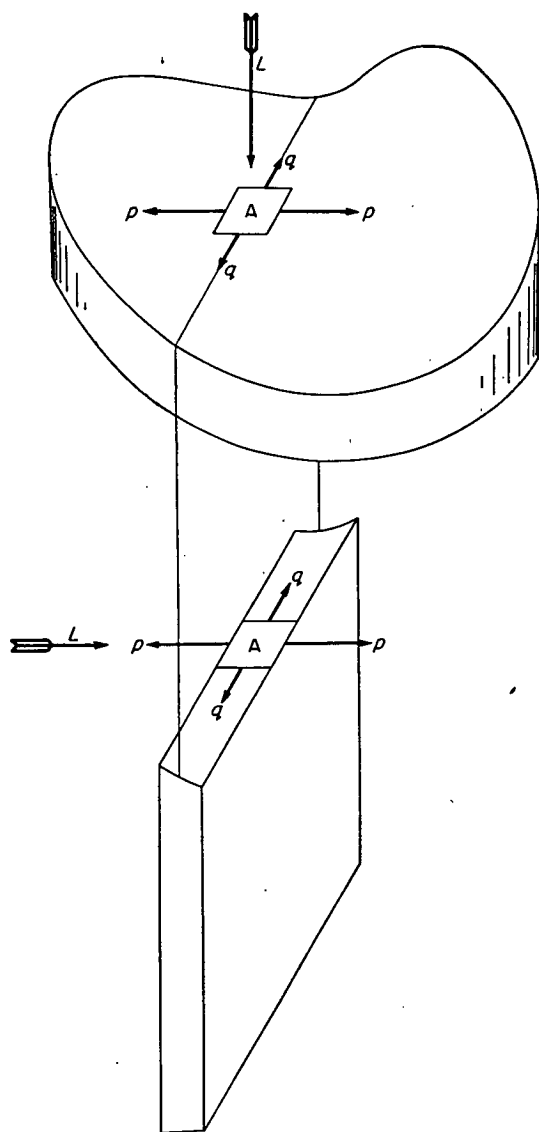


FIGURE 2.—Slices and directions of light for determination of surface stresses. L , direction of light.

problem of surface stresses in torsion, and Hetényi (reference 16) has applied it to threaded connections. In these applications the faces of the slice were oriented to be normal to the direction of a collimated polarized beam.

In special cases the combination of oblique and normal incidence leads to a complete determination of the principal stresses. Using this combination, the stress distribution in Saint Venant torsion was determined (reference 17).

Planes of symmetry.—For the special case where a plane of symmetry exists Jessop (reference 18) has developed an extension of the Lamé-Maxwell equations (Filon's graphical integration) to three-dimensional cases. By means of these extended equations it is possible to determine the stresses along the axis of symmetry. However, the method lacks generality.

Strain measurement after annealing.—It has been suggested by Kuske (reference 5) that mechanical strain measurements after annealing in conjunction with the

freezing method might be used to provide the additional relation necessary for the determination of the principal stresses at a general point. If it be assumed that the differences between the three normal stress components at a point have been found photoelastically from equation (1), there result:

$$\left. \begin{aligned} \sigma_z - \sigma_y &= C_{zy} \\ \sigma_y - \sigma_x &= C_{yx} \\ \sigma_z - \sigma_x &= C_{zx} \end{aligned} \right\} \quad (2)$$

where the C 's represent constants. If now the slice from the model containing the frozen stresses is annealed, the state of stress is relieved and the thickness of the slice at each point will return to its original unstressed value. From this change in thickness, if it can be measured with accuracy, the strain at a point in a direction perpendicular to the slice can be computed. Taking this direction as the Z -axis, for instance, the strain would be ϵ_z . Then from Hooke's law,

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad (3a)$$

which may be written as

$$\epsilon_z = \frac{1}{E} [(1 - 2\nu)\sigma_z + \nu(\sigma_z - \sigma_y) + \nu(\sigma_z - \sigma_x)] \quad (3b)$$

from which

$$\sigma_z = \frac{E\epsilon_z - \nu(\sigma_z - \sigma_y) - \nu(\sigma_z - \sigma_x)}{1 - 2\nu} \quad (4a)$$

In view of equations (2) this gives the stress component σ_z after which σ_x and σ_y are readily found. The entire state of stress has thus been determined.

LIMITATIONS OF MECHANICAL STRAIN MEASUREMENT

Poisson's ratio equal 1/2.—The method outlined above would seem to solve the problem and offer a powerful method of attack. Closer examination discloses certain serious difficulties. For the photoelastic materials used in this country, such as Fosterite and Bakelite, Poisson's ratio is very nearly 1/2 at the elevated temperatures used in the freezing process, and for this value of ν the method breaks down.

Thus, inspection of the general equations of Hooke's law

$$\left. \begin{aligned} \epsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_z + \sigma_x)] \\ \epsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \right\} \quad (5)$$

shows that when $\sigma_x = \sigma_y = \sigma_z$, that is, when the stresses form an isotropic system, and, in addition, the value of Poisson's ratio ν is 1/2, then

$$\epsilon_x = \epsilon_y = \epsilon_z = 0$$

In other words, isotropic systems of stresses produce no strains. Hence, two stress systems differing by an arbitrary isotropic system produce the same strains. Thus, when Poisson's ratio is $1/2$ a given strain field does not determine a unique stress field, although the converse is not true. Strain measurement when $\nu=1/2$ thus adds nothing to the general solution of the three-dimensional problem.

Poisson's ratio nearly $1/2$.—If Poisson's ratio is slightly less than $1/2$, the method of strain measurement after annealing should theoretically lead to a solution. However, other difficulties arise. Equation (4a) may be written as

$$\sigma_z = \frac{E\epsilon_z - \nu(C_{zy} + C_{zx})}{1 - 2\nu} \quad (4b)$$

If ν is the true value of Poisson's ratio and $\nu + \Delta\nu$ is its experimentally determined value, then the error $\Delta\sigma_z$ in the computed stress $\sigma'_z = \sigma_z + \Delta\sigma_z$ for a measured ϵ_z is

$$\sigma'_z - \sigma_z = \Delta\sigma_z = \frac{\Delta\nu}{1 - 2\nu - 2\Delta\nu} (\sigma_x + \sigma_y) \quad (6a)$$

With ν very close to $1/2$, a very small error $\Delta\nu$ in Poisson's ratio may lead to large errors $\Delta\sigma_z$ in the computed stress. For instance, if $\nu=0.48$ is assumed, which is the approximate value for Fosterite, and $\Delta\nu$ is taken as only 0.01 , then

$$\Delta\sigma_z = \frac{0.01}{0.02} (\sigma_x + \sigma_y) = 0.50(\sigma_x + \sigma_y) \quad (6b)$$

Experimental measurements of ν .—Experiments with the determination of Poisson's ratio for Fosterite and similar plastics indicate that it will be rather difficult to determine the value of ν closer than ± 5 percent. The error $\Delta\sigma_z$ in equation (6b) would be particularly large when σ_x and σ_y happen to be of the same sign and each is large in comparison with σ_z . All things considered, no great accuracy can be expected from this method so long as Poisson's ratio is nearly $1/2$.

It must be pointed out, however, that strain measurements may serve a useful purpose. Assuming that, in some way or other, the normal stress components have been found, the strains can be calculated and compared with those found experimentally. Here the error in the computed strains due to an error in Poisson's ratio is given by

$$\Delta\epsilon_z = -\frac{1}{E} (\sigma_x + \sigma_y) \Delta\nu \quad (7)$$

which is not large.

METHOD SUGGESTED BY PRIGOROVSKY AND PREISS

The method outlined above for the separation of the principal stresses which employs oblique and normal incidence of collimated polarized light and strain measurement after annealing is not the only possible procedure. Prigorovsky and Preiss suggest two alternative methods in reference 1. Their procedures combine (1) stress patterns from normal and oblique incidence with (2) axis patterns from

convergent polarized light and (3) strain measurements after annealing. The significant point lies in the fact that their method utilizes strain measurement after annealing and therefore breaks down when Poisson's ratio equals $1/2$.

THEORY OF SHEAR DIFFERENCE METHOD

General theory.—A method for determining stresses in three-dimensional problems is now proposed which is completely general. With this method the six stress components at any point may be found. It is essentially an extension to three dimensions of the method, long and effectively used for plane problems, which is known as the shear difference method (reference 8, ch. 8).

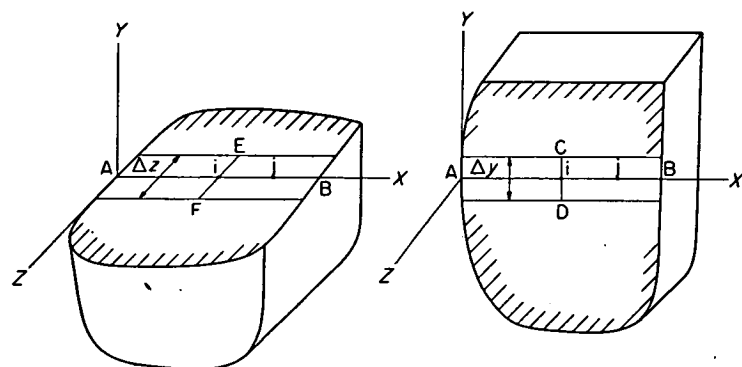


FIGURE 3.—Auxiliary lines in XY- and XZ-planes for shear difference method.

Consider an arbitrarily loaded unsymmetrical model with the set of coordinate axes as shown in figure 3. Let a straight line AB be drawn through point i from boundary to boundary and let this line be taken as the X-axis. At any point along this line the first differential equation of equilibrium, with body forces neglected, is

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \quad (8)$$

and upon integration the stress at any point j is given by

$$(\sigma_x)_j = (\sigma_x)_a - \int_a^j \frac{\partial \tau_{yx}}{\partial y} dx - \int_a^j \frac{\partial \tau_{zx}}{\partial z} dx \quad (9)$$

where $(\sigma_x)_a$ denotes the stress at point A and $(\sigma_x)_j$, the stress at any other point j on the line AB. The partial derivative $\partial \tau_{yx} / \partial y$ is the rate of change of τ_{yx} with respect to y and $\partial \tau_{zx} / \partial z$ is the rate of change of τ_{zx} with respect to z. Thus if values of τ_{yx} were computed along a line through i parallel to the Y-axis and the curve $\tau_{yx} = f(y)$ were plotted, then $\left(\frac{\partial \tau_{yx}}{\partial y} \right)_i$ would be the slope of the curve $f(y)$ at point i.

Similarly $\left(\frac{\partial \tau_{zx}}{\partial z} \right)_i$ is the slope of the curve $\tau_{zx} = g(z)$ at point i.

As in the plane problem, these slopes may be approximated by computing the shearing stresses at points near i on opposite sides of the line AB and forming finite difference quotients. Thus, choosing neighboring points C and D in the XY-plane and similarly points E and F in the XZ-plane,

$$\left(\frac{\partial \tau_{yz}}{\partial y}\right)_i = \frac{(\tau_{yz})_C - (\tau_{yz})_D}{\Delta y} = \left(\frac{\Delta \tau_{yz}}{\Delta y}\right)_i \quad (10a)$$

$$\left(\frac{\partial \tau_{zx}}{\partial z}\right)_i = \frac{(\tau_{zx})_F - (\tau_{zx})_E}{\Delta z} = \left(\frac{\Delta \tau_{zx}}{\Delta z}\right)_i \quad (10b)$$

Thus if the shearing stresses can be determined along four auxiliary lines, parallel to and on opposite sides of AB, two lines in the XY -plane and two in the XZ -plane, one has all the data necessary to obtain the quotients on the right side of equations (10) and hence good approximations to the partial derivatives. In evaluating equations (10), care must be taken to attach the proper signs to the shear systems τ_{yz} and τ_{zx} , as in figure 1.

Substituting the above approximations for the partial derivatives in equation (9) and replacing the integrals by summations, the following equation is obtained:

$$(\sigma_x)_j = (\sigma_x)_a - \sum_a^j \frac{\Delta \tau_{yz}}{\Delta y} \Delta x - \sum_a^j \frac{\Delta \tau_{zx}}{\Delta z} \Delta x \quad (11a)$$

The summations are evaluated graphically in the same manner as in plane problems. For convenience, Δy and Δz may be taken numerically equal to Δx . Then equation (11a) becomes

$$(\sigma_x)_j = (\sigma_x)_a \pm \sum_a^j \Delta \tau_{yz} \pm \sum_a^j \Delta \tau_{zx} \quad (11b)$$

in which $\Delta \tau_{yz}$ and $\Delta \tau_{zx}$ have the mean value in each interval Δx .

Shearing stresses in first slice.—In order to carry out this integration, it is necessary to determine the magnitude and direction of shearing stresses τ_{yz} along the two auxiliary lines in the XY -plane and of τ_{zx} along the two auxiliary lines in the XZ -plane. The shearing stresses τ_{yz} are obtained from a slice in the XY -plane containing AB in its middle surface. The stress pattern of this slice from normal incidence will give the difference between the secondary principal stresses in the plane of the slice at all points, and the corresponding isoclinics furnish their orientation. The magnitude of the shearing stress τ_{yz} at any point will then be given by

$$\tau_{yz} = \frac{1}{2} (p' - q') \sin 2\phi' \quad (12a)$$

where p' and q' are the secondary principal stresses in the XY -plane and ϕ' is the isoclinic parameter. The directions are determined by inspection as in paragraph 8.2 of reference 8. Using equation (12a), the shearing stresses τ_{yz} along the auxiliary lines and along AB itself may be found.

Shearing stresses in second slice.—A second slice lying in the XZ -plane and containing line AB would furnish similar information for τ_{zx} . Here a practical difficulty arises since the first slice removes an essential part of the second slice. One of several procedures may be used to eliminate this difficulty.

(1) In the general case two identical models, identically loaded, may be used, one for the XY slice and one for the XZ slice. The shearing stresses τ_{yz} for the XY slice are

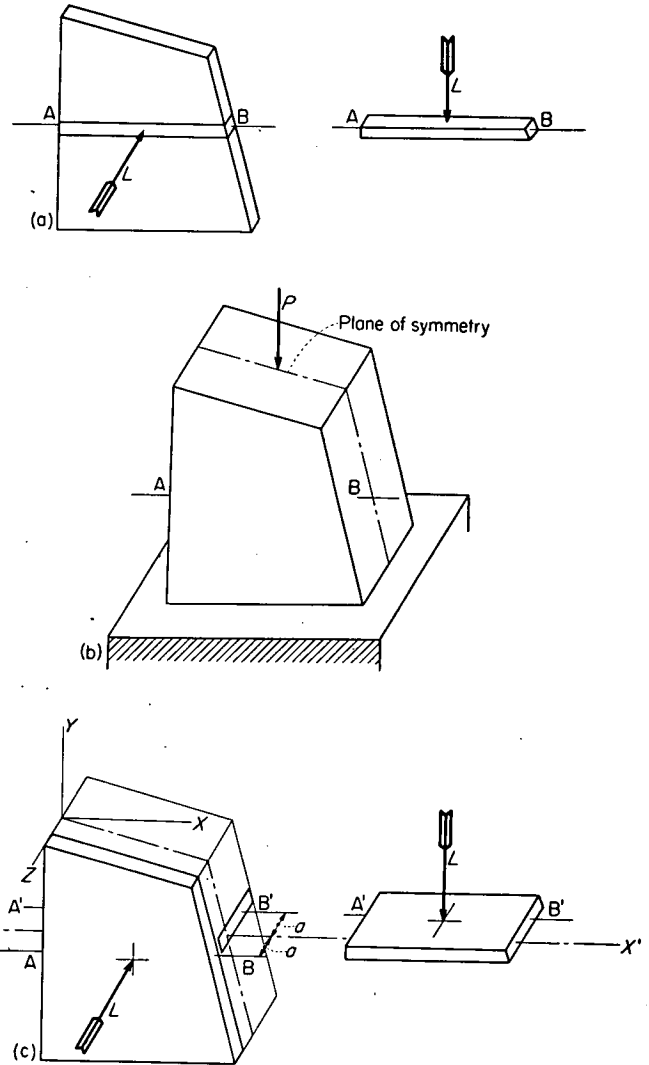
calculated from equation (12a). Similarly, the shearing stresses τ_{zx} for the XZ slice are given by

$$\tau_{zx} = \frac{1}{2} (p'' - q'') \sin 2\phi'' \quad (12b)$$

where p'' and q'' are the secondary principal stresses and ϕ'' is the isoclinic parameter in the XZ slice.

(2) In large models it may be possible to use a sub slice from the main slice for determining τ_{zx} . After the data are obtained from the main slice, a small section containing the line AB is cut from it, as shown in figure 4 (a). The necessary data in the XZ -plane are then obtained from normal incidence on the sub slice, as shown. This procedure is feasible if the model is large so that the main slice can be made of sufficient thickness.

(3) In the particular case where a plane of stress symmetry exists, such as the XY -plane in figure 4 (b), advantage can be taken of this symmetry. Referring to figure 4 (b), let it



(a) Sub slice.
(b) Loaded model.
(c) Slices from opposite sides of plane of symmetry.

FIGURE 4.—Schematic diagram of necessary slices.

be assumed that the stresses on line AB are required. The first slice is made parallel to the XY -plane and contains the line AB as shown in figure 4 (c). The orthogonal slice is cut from the opposite side of the body so that it contains the symmetrically placed line $A'B'$, along which the stresses are the same as along AB itself. The necessary shearing stresses are calculated as outlined in procedure (1), above.

Normal stresses.—The starting value $(\sigma_x)_a$ will be obtained from boundary conditions and boundary fringe orders. The integration may then be carried out and values of σ_x obtained along AB. Further, from Mohr's circle or other considerations:

$$(\sigma_x - \sigma_y)_j = (p' - q')_j \cos 2\phi'_j = 2F'_j n'_j \cos 2\phi'_j \quad (13a)$$

$$(\sigma_x - \sigma_z)_j = (p'' - q'')_j \cos 2\phi''_j = 2F''_j n''_j \cos 2\phi''_j \quad (13b)$$

where the F 's and n 's denote, respectively, fringe value of the model in shear and fringe order at point j . From equations (13a) and (13b)

$$(\sigma_y)_j = (\sigma_x)_j - (p' - q')_j \cos 2\phi'_j \quad (13c)$$

$$(\sigma_z)_j = (\sigma_x)_j - (p'' - q'')_j \cos 2\phi''_j \quad (13d)$$

All the necessary data for the evaluation of $(\sigma_y)_j$ and $(\sigma_z)_j$ are obtained from the slices in the XY - and XZ -planes, respectively.

Use of oblique incidence.—At this stage five of the six stress components, namely σ_x , σ_y , σ_z , τ_{yz} , and τ_{zx} , have been found at all points of AB. There remains one unknown stress component τ_{yx} . The shear system τ_{yz} has no influence on the stress patterns from normal incidence for either one of the slices but will have an effect on the patterns from oblique incidence.

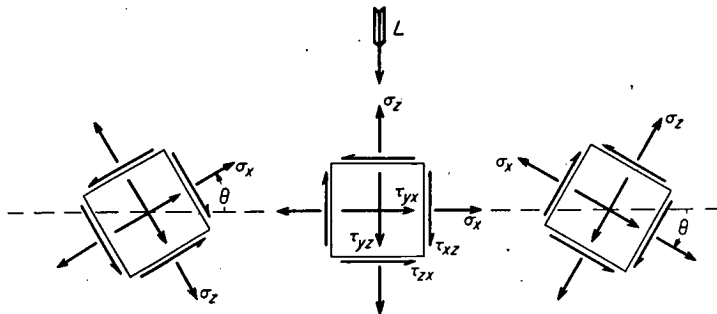


FIGURE 5.—Normal incidence and oblique incidence for different directions of rotation.

In order to find the shear system τ_{yz} an oblique stress pattern is obtained from either one of the two slices. For concreteness assume that the slice parallel to the XY -plane is used and that it is rotated in a clockwise direction about the Y -axis through an arbitrary angle θ_y . Figure 5 shows a view of a small element as seen from the positive end of the Y -axis. From equation (1) the fringe order n_{θ_y} at any point due to the oblique incidence is given by

$$2F_{\theta_y} n_{\theta_y} = \sqrt{[\sigma_y - (\sigma_x \cos^2 \theta_y + \sigma_z \sin^2 \theta_y - \tau_{zx} \sin 2\theta_y)]^2 + 4(\tau_{yz} \cos \theta_y - \tau_{yx} \sin \theta_y)^2} \quad (14a)$$

in which σ_x , σ_y , and σ_z are the known normal stresses and τ_{yz} and τ_{zx} are the known shear systems. Also the

isoclinic parameter ϕ_{θ_y} for the oblique incidence is given by the expression

$$\sin 2\phi_{\theta_y} = \frac{\tau_{yz} \cos \theta_y - \tau_{yx} \sin \theta_y}{F_{\theta_y} n_{\theta_y}} \quad (15a)$$

If the rotation about the Y -axis be made in the counterclockwise direction then

$$2F_{\theta_y} n_{\theta_y} = \sqrt{[\sigma_y - (\sigma_x \cos^2 \theta_y + \sigma_z \sin^2 \theta_y + \tau_{zx} \sin 2\theta_y)]^2 + 4(\tau_{yz} \sin \theta_y + \tau_{yx} \cos \theta_y)^2} \quad (14b)$$

and

$$\sin 2\phi_{\theta_y} = \frac{\tau_{yz} \cos \theta_y + \tau_{yx} \sin \theta_y}{F_{\theta_y} n_{\theta_y}} \quad (15b)$$

Similar equations may be written for rotation of the slice parallel to the XZ -plane about the Z -axis.

It is to be noted that in general the retardation observed at any point depends on the direction of rotation of the slice. For one direction of rotation the fringe order at a point will be different from its value for the other direction. In the particular case when the slice contains a principal plane, then $\tau_{zx} = \tau_{zy} = 0$ and equations (14a) and (14b) become identical. In such cases the direction of rotation is immaterial. In dealing with general slices it is important to note carefully the direction of rotation relative to the wave normal and to attach the proper signs to all the stresses.

Each of equations (14a), (14b), (15a), and (15b) may be solved for the unknown shear system τ_{yz} . It is necessary only to determine the fringe order and the isoclinic parameter ϕ_{θ_y} along the line AB. If the rotation is counterclockwise equation (14b) or (15b) is appropriate. Of these, equation (15b) is much the simpler. Using equation (15b) and $\theta = 45^\circ$ there is obtained

$$\tau_{yz} = \sqrt{2} F_{\theta_y} n_{\theta_y} \sin 2\phi_{\theta_y} - \tau_{yx} \quad (15c)$$

With this, τ_{yz} is easily computed. All six components of stress are thus determined for the point i , and therefore the principal stresses themselves are determined at the given point.

Extension to the plastic state.—It should be noted that the method described in this report is not limited to a linear stress-optic law. With minor modifications, which are stated below, the method is equally valid for a nonlinear stress-optic law. Thus, the method is applicable not only to the elastic state but also to the plastic state of the model. This follows from the fact that the only equations, in addition to the stress-optic law, are the equations of equilibrium which are independent of stress-strain relations.

In order to adapt the equations to a nonlinear stress-optic law it is necessary to observe that whereas in the linear range fringes can be used as the unit of stress, since the stress is proportional to the fringe order, in the nonlinear range fringes cannot serve as the unit of stress, since proportionality between stress and birefringence no longer exists. To obviate this difficulty all fringe orders in the equations should be converted into standard units, say pounds per square inch, as was done in all preceding equations.

Now, let the nonlinear stress-optic law be given by

$$\tau_{max} = (p - q)/2 = \tau(n)$$

where $\tau(n)$ is a known function of n . If one replaces the products $F \times n$ by $\tau(n)$ in equations (14) and (15) these equations are directly applicable to a nonlinear stress-optic law. It should, however, be noted that the results will apply to the model only and are not directly transferable to the prototype. It is also observed that the photoelastic models are assumed to be free from strain-hardening.

Effect of Poisson's ratio.—In conclusion, it should be noted that in transferring the results from three-dimensional photoelastic models to metal prototypes the effect of Poisson's ratio will have to be considered. It is fortunate, as shown by the theoretical solutions obtained to date, that the effect of Poisson's ratio on the most significant stresses is small (references 19 and 20).

APPLICATION OF SHEAR DIFFERENCE METHOD TO A DIAMETRICALLY COMPRESSED SPHERE

DESCRIPTION OF APPARATUS

The apparatus used in this investigation consisted of the following items:

- (1) An electric furnace with temperature controls and built-in loading frame with special jigs
- (2) An 8-inch photoelastic polariscope with a special immersion tank
- (3) An oblique incidence jig
- (4) A Babinet-Soleil compensator

A photograph of the electric furnace is shown in figure 6. This is a relatively large oven 46 inches high, 42 inches wide, and 19 inches deep. It is fitted with automatic temperature controls by means of which any desired thermal cycle could be imposed on the model. The furnace was equipped with a built-in loading frame suitable for the application of all basic types of loading.

A special loading jig built for the investigation is shown in figure 7. It consisted of a smooth circular shaft about 1/2 inch in diameter passing through a pair of smooth, lubricated guide holes carefully aligned so that the axis of the shaft was perpendicular to the base. The load was applied to the top

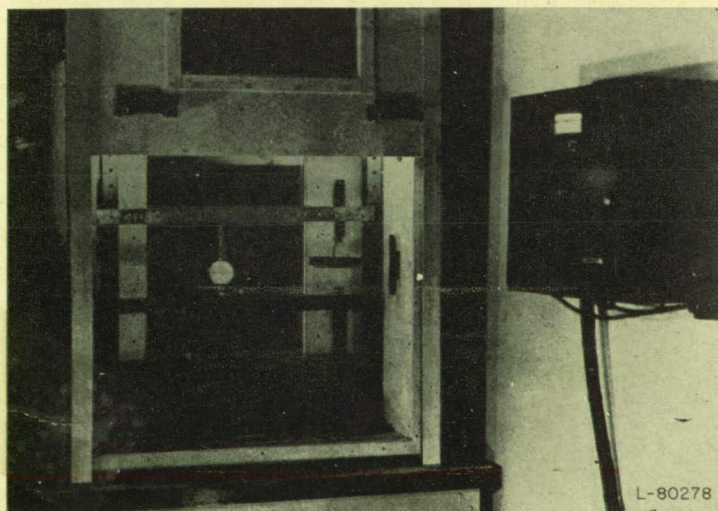


FIGURE 6.—Electric furnace and control panel.

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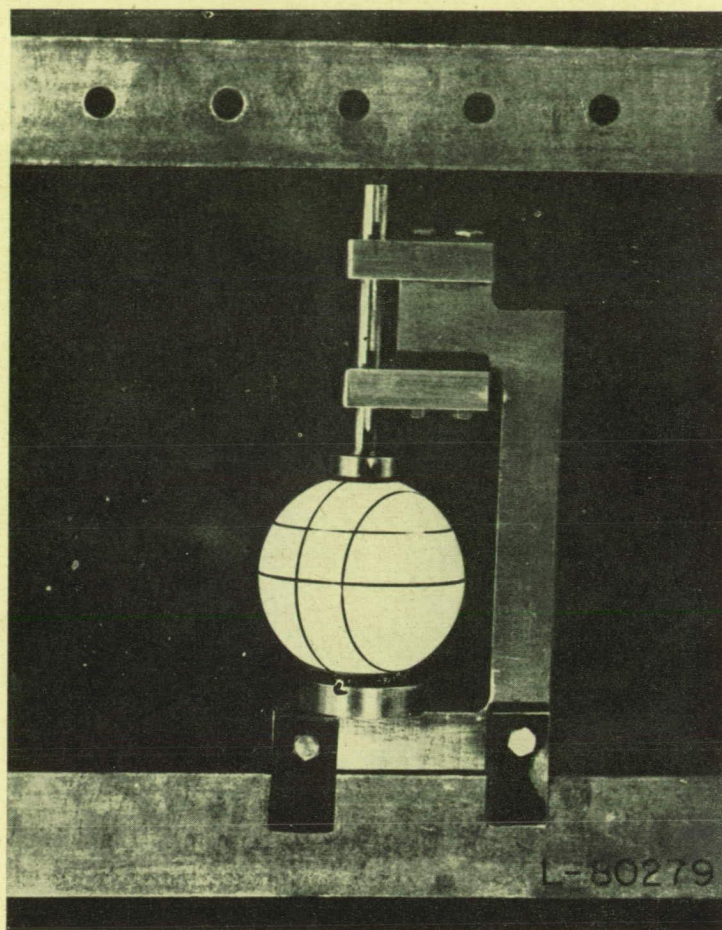


FIGURE 7.—Loading jig and model of sphere.

of the shaft through a hard steel ball. This jig was found to give almost perfect vertical loading and the friction was negligible.

A special jig was also built for oblique incidence. The frame of the jig can be rotated about a vertical axis through any desired angle which can easily be measured to one-tenth of a degree. The slice was mounted in the frame of the jig and the whole unit was placed in an immersion tank with a suitable mixture of Halowax and mineral oil.

The remaining equipment was standard apparatus in photoelastic laboratories, the descriptions of which are available in the literature.

TEST PROCEDURE

Model.—The sphere was machined from a cylinder of Fosterite which was previously annealed to reduce initial stresses. The machining was carried out in a lathe. The rough cylinder of Fosterite was first turned to a true cylinder. In order to form the sphere a tool bit was set in a special jig riding on the carriage of the lathe. This bit could be rotated about a vertical axis lying in the plane of the lathe centers. The cutting was performed by swinging the tool bit by hand around its vertical axis while the cylinder was rotating, and the radius of the sphere was slowly reduced by bringing the tool bit gradually closer to its axis. In this way it was possible to shape the complete sphere except for a relatively small nipple near the chuck. The final diameter of the sphere was 3.313 ± 0.002 inches.

Loading.—The sphere was placed in the loading jig and carefully aligned for diametral compressive loading. The model was then heated to 162° F in the electric furnace, the rate of heating being about 10° F per hour. A load of 172 pounds was then applied to the model. After a soaking period of about 2 hours the temperature was lowered at the rate of 4° F per hour to room temperature. The final diameter of the equatorial plane was found to be 3.334 inches and the load axis was measured as 3.102 inches. Although relatively large deformations were developed in the loaded regions, the main body of the sphere was not notably distorted from its original shape. The stress pattern of the whole sphere in figure 8 shows that the loads and the stresses were rotationally symmetrical.

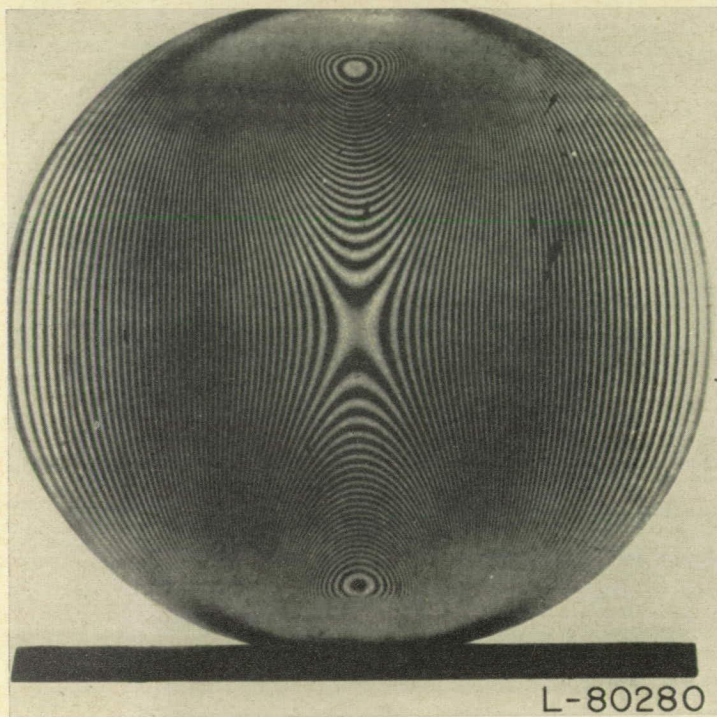


FIGURE 8.—Stress pattern of whole sphere.

Slicing.—In preparation for slicing the center lines of all the slices were carefully scribed on the sphere using the flat spots in the loaded regions as datum planes. The slices were then sawed out roughly on a bandsaw to a thickness of about 3/8 inch. They were subsequently ground by hand to about 1/8-inch thickness in most cases. Great care was taken to keep the slices symmetrical with respect to their center lines.

Figure 9 shows the slicing plan. The first slice removed was parallel to the equator and midway between the equator and the load point. Then from the opposite side of the sphere a meridian slice was removed. Next a slice containing the equatorial plane was cut. Finally a slice parallel to the meridian slice and halfway out on the radius was removed.

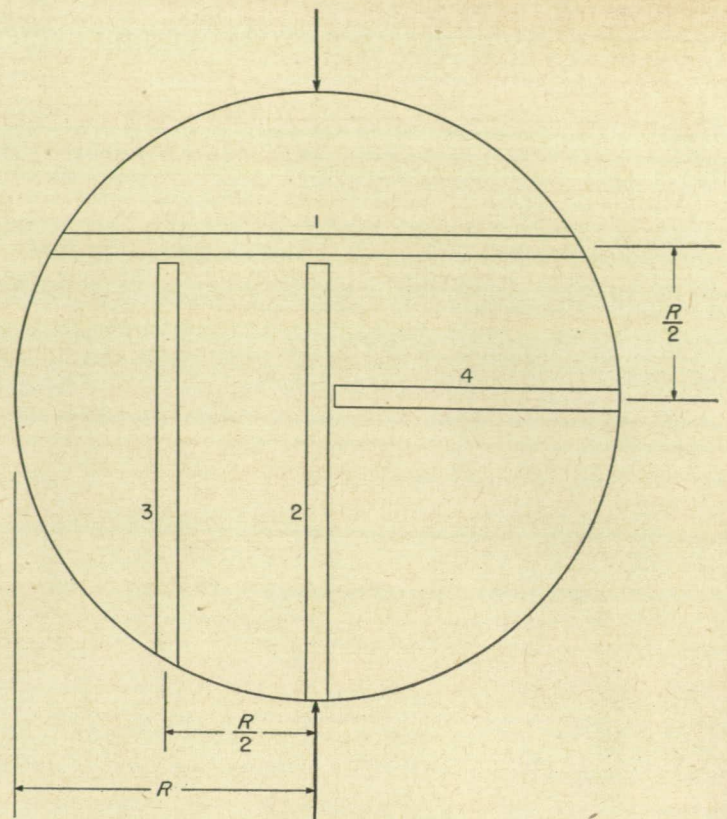


FIGURE 9.—Slicing plan for sphere. 1, slice parallel to equator; 2, meridian slice; 3, slice parallel to meridian; and 4, equatorial slice.

Stress patterns and isoclinics.—The slices were mounted in the oblique incidence jig and stress patterns at normal and oblique incidence were recorded photographically. Typical stress patterns are shown in figures 10 to 15. Most of the normal incidence patterns show very few fringes. In order to obtain accurate data in these cases a Babinet-Soleil compensator was used to obtain the fringe-order distribution along the lines of interest by point-by-point exploration.

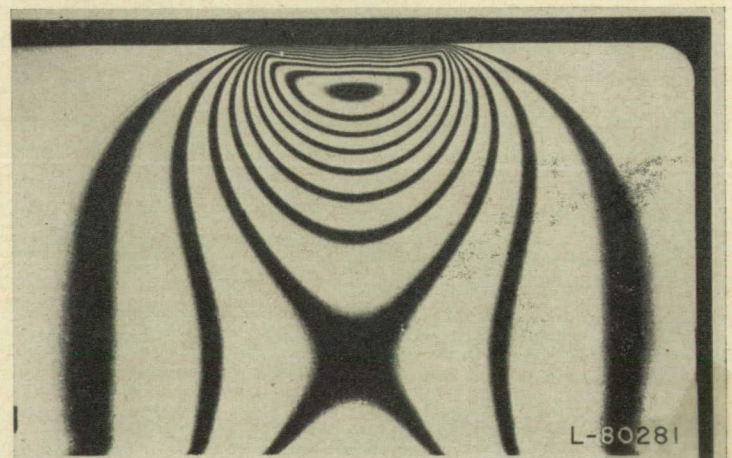


FIGURE 10.—Stress pattern of meridian slice at normal incidence.

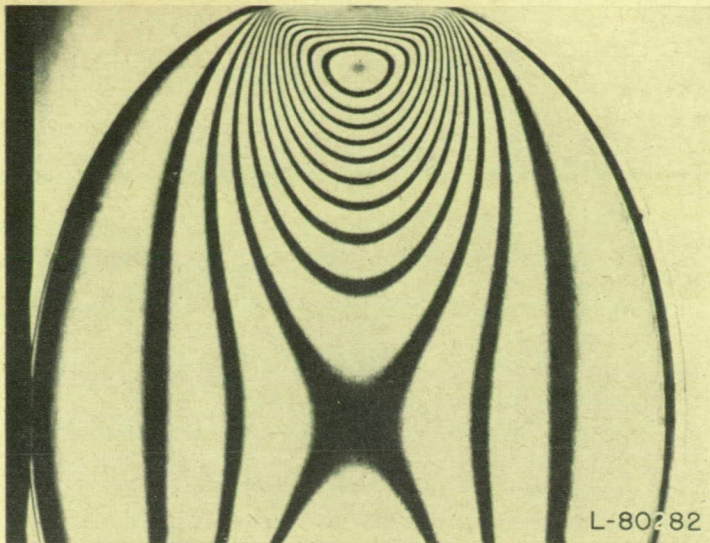


FIGURE 11.—Stress pattern of meridian slice for a rotation of 45° about Y-axis.

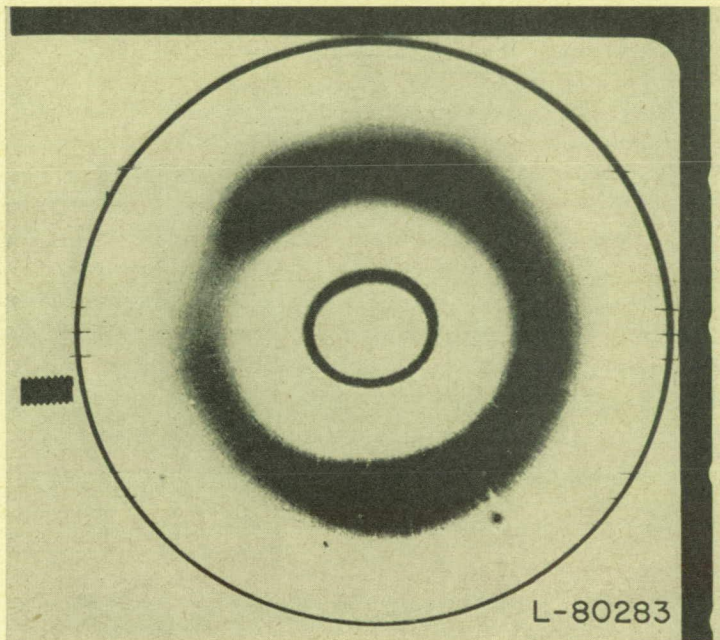


FIGURE 12.—Stress pattern of slice parallel to equator at normal incidence.

A black cross was made on the screen to mark the point on which attention should be centered. The model slice was then adjusted until the line of interest on the image ran true on the intersection of the cross as the straining frame was moved laterally. Then beginning at the outer edge and moving the straining frame by a known amount after each observation the fringe order was obtained at a series of points along the line. From these data the curve of fringe-order distribution was plotted.

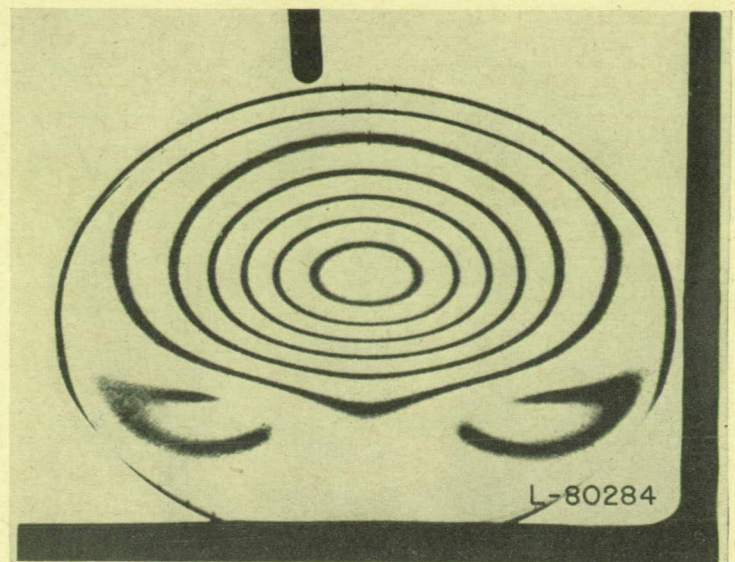


FIGURE 13.—Stress pattern of slice parallel to equator for a rotation of 45° about X-axis.

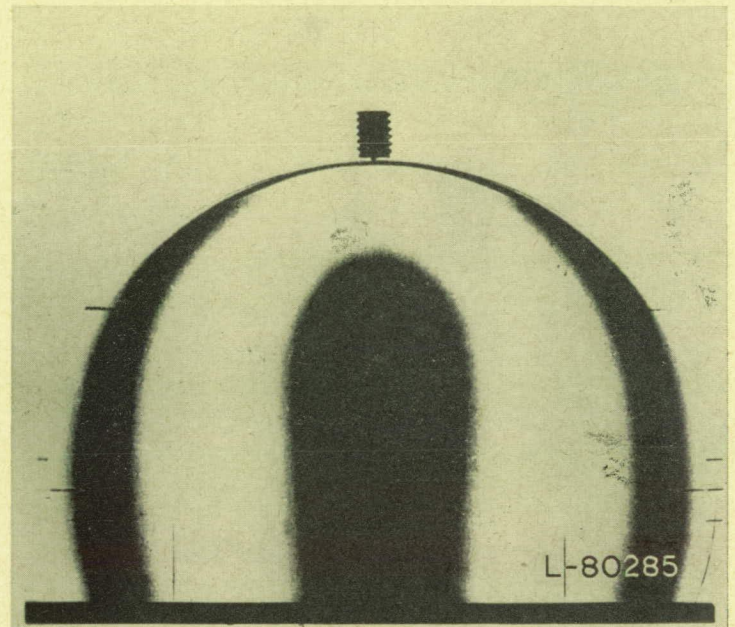


FIGURE 14.—Stress pattern of slice parallel to meridian at normal incidence.

It may be noted that fractional fringe orders can also be obtained by the Tardy method of compensation, the accuracy being comparable with that of the Babinet-Soleil compensator.

Isoclinic lines were recorded by one of two methods. For the over-all picture the isoclinic lines were photographed in most cases (figs. 16 and 17). From the photographs averaged sketches were prepared and used in making the calculations. On several lines direct sketching of the isoclinics

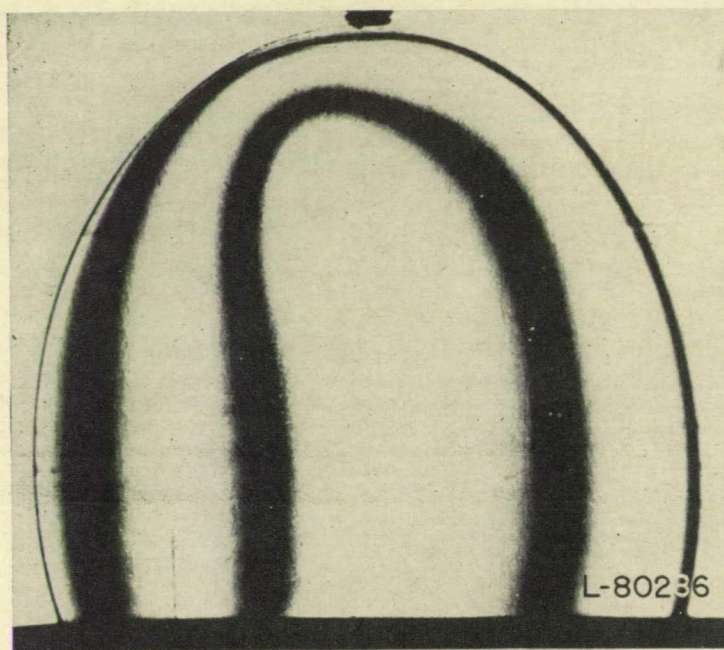


FIGURE 15.—Stress pattern of slice parallel to meridian for a rotation of 45° about Y-axis.

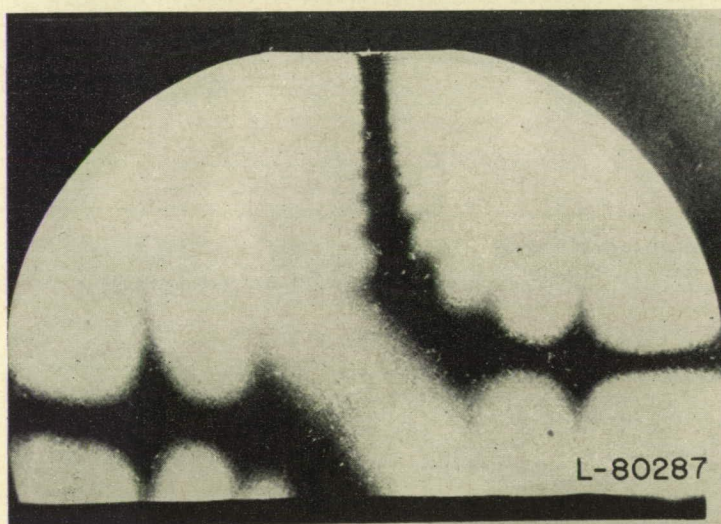


FIGURE 16.—Typical isoclinic for meridian slice.

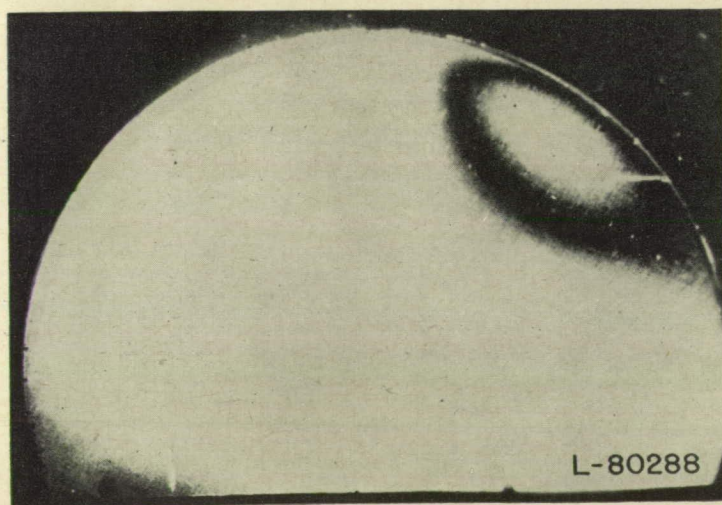


FIGURE 17.—Typical isoclinic for slice parallel to meridian.

was used with attention being confined to the particular line of interest. Here the intersections of successive isoclinic lines with the line of interest were obtained at very short intervals, from which the distribution of the isoclinic parameters along the line could be plotted. This method was found to be accurate and considerably less time consuming than the photographic method. White light was used in all isoclinic work.

In plane stress systems the isoclinic parameter at a point on a free boundary is determined by the tangent to the boundary at the point. The isoclinic parameter thus changes from point to point along the boundary in general. This is not necessarily true for isoclinics of secondary principal stresses. In the slice parallel to the meridian the secondary principal stresses at the boundary consist solely of one normal stress σ_z which is horizontal. The boundary is therefore a zero isoclinic and no other isoclinic may intersect the boundary at any point. The higher order isoclinics therefore all lie within the boundary forming closed loops in this case, as shown in figure 17.

Fringe value.—The fringe value of the material was obtained from a small cylinder about $1/2$ inch in diameter and $1\frac{1}{2}$ inches long. This was loaded in compression in the special loading jig used for the sphere and subjected to the same thermal cycle as the sphere. A portion of the cylinder was machined away to leave a V shape as shown in figure 18. The resulting stress pattern was then photographed (fig. 19). The V shape was used to make clearly visible the fringe of zero order occurring at the sharp edge of the wedge. In the cylinder itself the first few fringes crowd together near the boundary of the cylinder and it is practically impossible to identify the zero fringe. From the stress pattern in figure 19 it was a simple matter to plot fringe order against position, which for the wedge described is a straight line (fig. 20).

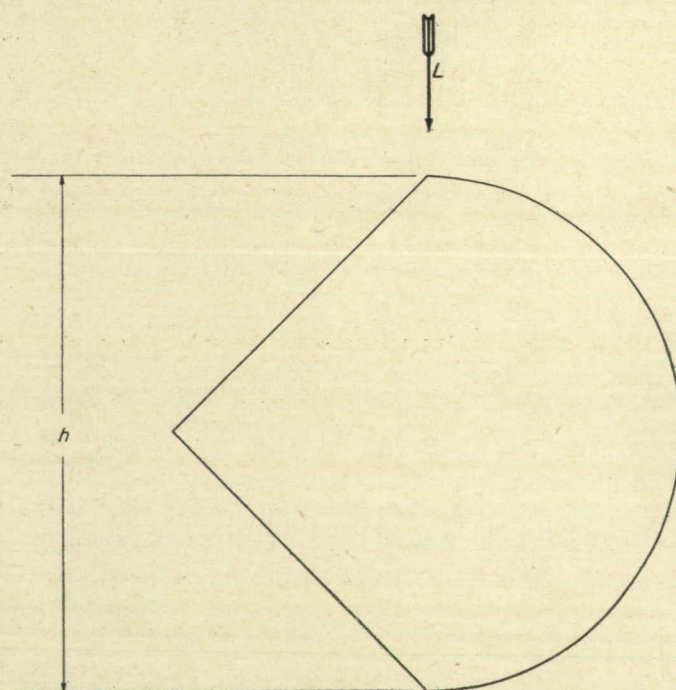


FIGURE 18.—Cross section of calibration member after machining of wedge.

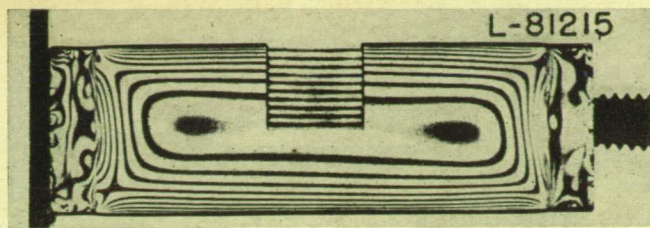


FIGURE 19.—Stress pattern of calibration member.

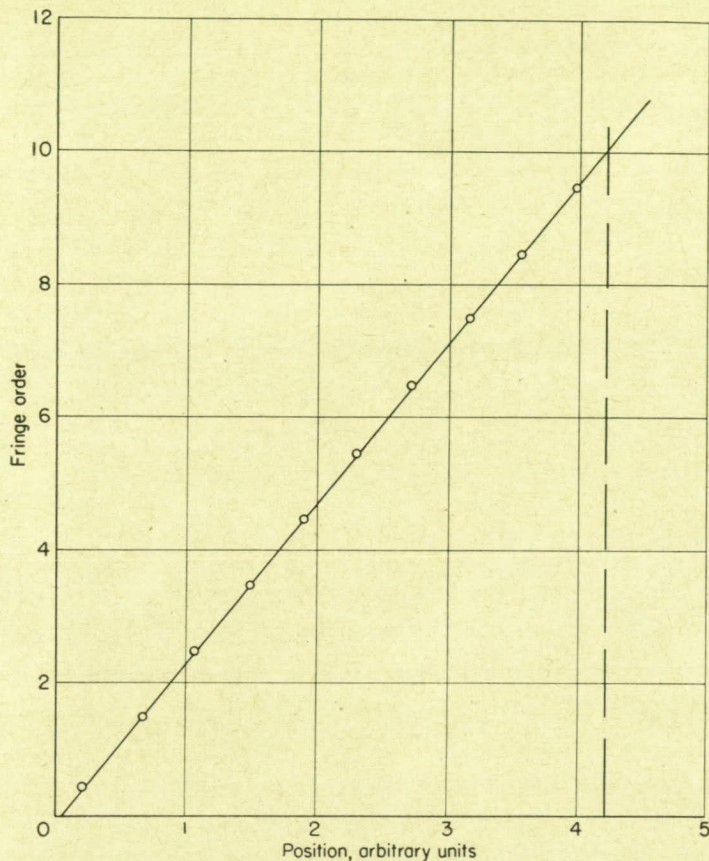


FIGURE 20.—Fringe order for calibration member as a function of distance from edge of wedge.

In this way the fringe order at the point where the plane faces of the V intersect the curved boundary of the cylinder could be accurately determined. At the same point the thickness h could also be measured accurately. With the measured fringe order and thickness the fringe value $2f$ of the material was found to be 3.0 psi per fringe, per inch compression.

RESULTS

Interior stresses.—The stress distribution was obtained along six lines passing through the interior of the sphere. The lines are indicated in figure 21 by the letters A-A, B-B, C-C, D-D, E-E, and F-F. Thus the six lines are defined as follows:

- A-A intersection of meridian slice with equatorial slice
- B-B intersection of meridian slice with slice parallel to equator
- C-C intersection of slice parallel to equator with slice parallel to meridian
- D-D load line
- E-E intersection of equatorial slice with slice parallel to meridian

F-F intersection of slice parallel to the meridian slice with a meridian plane which is perpendicular to it

In this problem advantage was taken of the symmetry of the sphere to eliminate the need for two models as discussed in the theoretical part of the report. In particular because of the rotational symmetry of the stresses one meridian slice could be taken to represent all meridian slices.

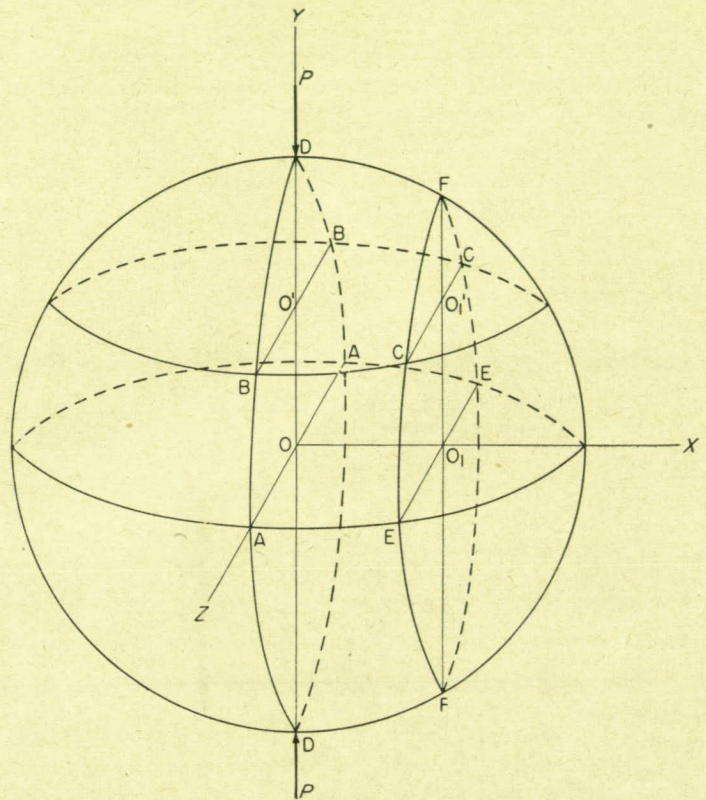


FIGURE 21.—Lines in sphere along which stress distributions were determined.

Typical calculation.—In order to make clear the application of the method the complete calculations for line C-C will now be given. The evaluation of the stresses along this line requires all the generality which would be encountered in a body devoid of symmetry. The basic data for the determination of the stresses on this line are obtained from the stress patterns and isoclinics at normal incidence of the two slices defining the line C-C and from the stress pattern and isoclinics of one of the slices at oblique incidence with rotation about an axis perpendicular to C-C and lying in the plane of the plate.

Because of the symmetry of the stresses along C-C it is necessary only to deal with half the length of the line. This half length was divided into 10 equal subdivisions. The two necessary auxiliary lines were drawn parallel to it in each of the two orthogonal planes and spaced the length of one subdivision apart.

The first step is to obtain the distribution of the shearing stresses along line C-C and along the four auxiliary lines. This requires the determination of the distribution of the secondary principal stress differences and of the isoclinics along these lines. Figure 22 shows the curves of n''' and ϕ''' for the slice parallel to the meridian, and figure 23

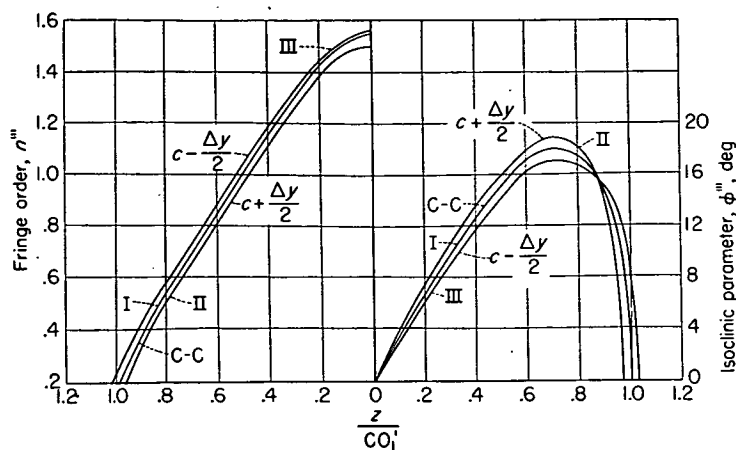


FIGURE 22.—Distribution of fringe order n''' and isoclinic parameter ϕ''' for line C-C and two auxiliary lines in slice parallel to meridian. In curves II and III the letter c denotes the Y -coordinate of line C-C

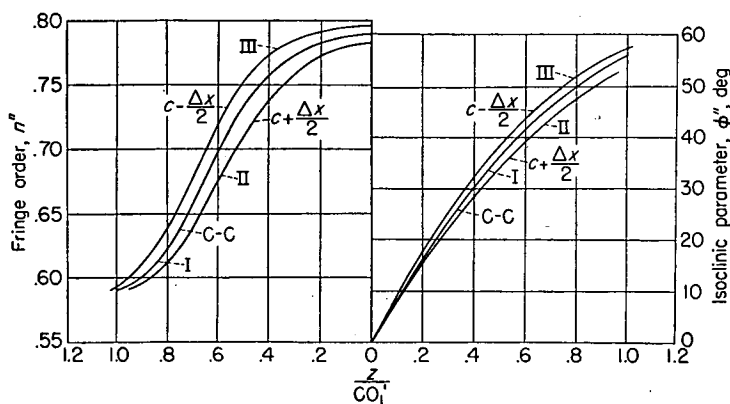


FIGURE 23.—Distribution of fringe order n'' and isoclinic parameter ϕ'' for line C-C and two auxiliary lines in slice parallel to equator. In curves II and III the letter c denotes the X -coordinate of line C-C

shows n'' and ϕ'' for the slice parallel to the equator. With these data the required shearing stresses are computed at each division point of C-C. Thus, following equations (12) and expressing the stresses in terms of fringes,

$$n_{yz} = \frac{1}{2} n'' \sin 2\phi'' \quad (16a)$$

$$n_{xz} = \frac{1}{2} n'' \sin 2\phi'' \quad (16b)$$

It will be noted that for positive values of z the shear system n_{yz} is positive and n_{xz} is negative.

As noted in the theoretical part of the report the integration requires the use of the difference between the shearing stresses at the center of each subdivision. These differences are obtained from the curves representing the distribution of the shearing stresses just found. Figure 24 shows the curves of the shear differences for the two slices.

The next step is to obtain the value of the normal stress n_z at each division point by an integration of one of the dif-

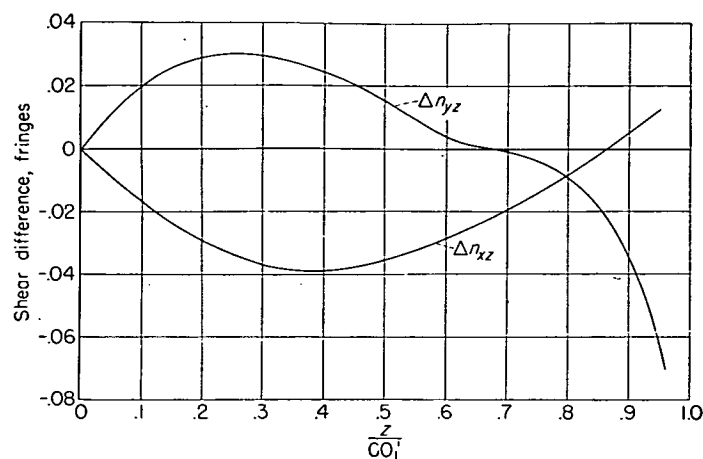


FIGURE 24.—Distribution of shear differences Δn_{yz} and Δn_{xz} for line C-C.

ferential equations of equilibrium. The integration is carried out by approximation using summations to replace the integrals. The appropriate equation for line C-C is similar to equation (11a), that is,

$$(n_z)_j = (n_z)_c - \sum_c^j \Delta n_{yz} \frac{\Delta z}{\Delta y} - \sum_c^j \Delta n_{xz} \frac{\Delta z}{\Delta x} \quad (17a)$$

Choosing $\Delta x = \Delta y = \Delta z$ the ratios of these quantities are unity. The signs of the ratios depend on the choice of axes and the direction of integration. The integration begins at the boundary and proceeds inward. Consequently Δz is negative. The shear differences have been formed in such a way that Δy is positive and Δx negative. The final form of the equation is therefore

$$(n_z)_j = (n_z)_c + \sum_c^j \Delta n_{yz} - \sum_c^j \Delta n_{xz} \quad (17b)$$

The signs of the shear differences are as shown by the curves.

In order to start the integration the value of $(n_z)_c$ is required. This must be determined from the boundary conditions and boundary fringe orders. From the fact that the boundary is unloaded it is evident that the principal stress normal to the boundary is zero. Inspection of the meridian slice shows that, excepting the loaded regions, the boundary stresses in the meridian section are also zero. The fringe order at the boundary of the slice parallel to the equator is 0.58 fringe tension, and the direction of this stress makes an angle of 55.6° with line C-C. The boundary value of n_z is thus found from the equations of stress transformation as follows:

$$(n_z)_c = 0.58 \cos^2 55.6^\circ = 0.185 \quad (18)$$

The expression for $(n_z)_j$ therefore takes the form

$$(n_z)_j = 0.185 + \sum_c^j \Delta n_{yz} - \sum_c^j \Delta n_{xz} \quad (17c)$$

The integration is easily carried out in tabular form as shown in table I.

Once n_z has been found the values of n_x and n_y are found from expressions similar to equation (13c), that is,

$$n_x = n_z - n'' \cos 2\phi'' \quad (19a)$$

$$n_y = n_z - n''' \cos 2\phi''' \quad (19b)$$

This computation is shown in table II.

The last step is to determine the remaining shearing stress system n_{xy} . This was obtained in this case from oblique incidence on the slice parallel to the meridian with rotation through 45° about the Y-axis. Figure 25 shows the fringe order and isoclinic distributions along C-C for this case. With these data and the known values of n_{yz} previously determined the required shearing stress component is found from an expression similar to equation (15c). Thus

$$n_{xy} = \frac{1}{2} n_{\theta y} \sin 2\phi_{\theta y} - n_{yz} \quad (20)$$

The results of this computation are shown in table III. This completes the solution for all six stress components along line C-C. In order to determine the stress components in pounds per square inch it is necessary only to multiply the stresses in fringes by the proper fringe value of the slice.

Using methods similar to those just explained the stress components for all six lines have been determined. With the exception of line D-D integration began at the boundary and proceeded inward. For line D-D the starting point was

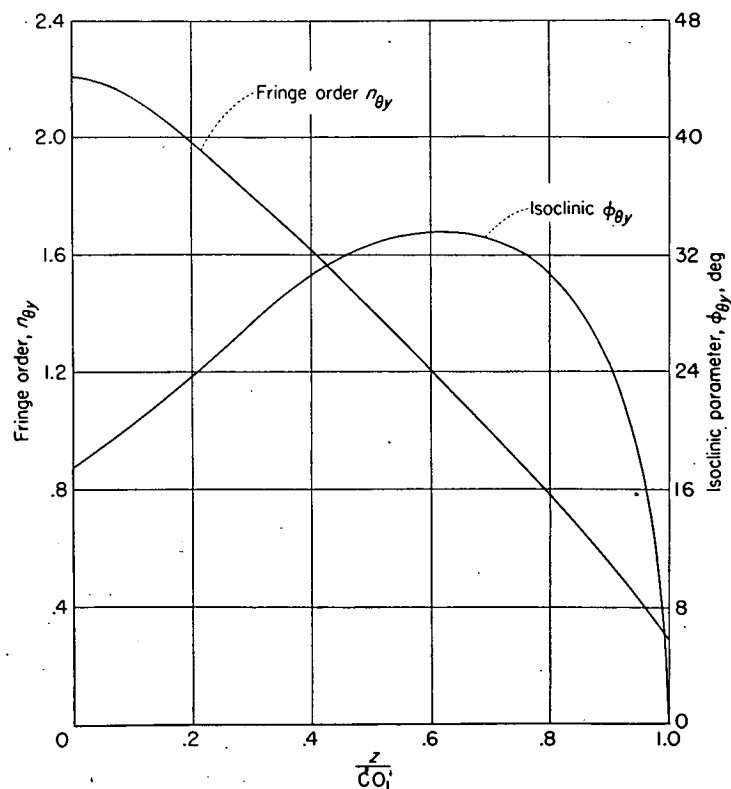


FIGURE 25.—Distribution of $n_{\theta y}$ and $\phi_{\theta y}$ along line C-C for a 45° rotation of slice parallel to the meridian about the Y-axis.

taken as the center of the sphere and integration proceeded upward. The starting value of n_y for this line was taken to be that obtained from line A-A. The results of these computations are shown in figures 26 to 32. At the center of the sphere the stress components were found to be $\sigma_y = -2.59P/A$ and $\sigma_x = \sigma_z = 0.45P/A$. These values may be compared with the stresses at the center of a disk under diametral compression which are $\sigma_y = -1.91P/A$ and $\sigma_x = 0.64P/A$.

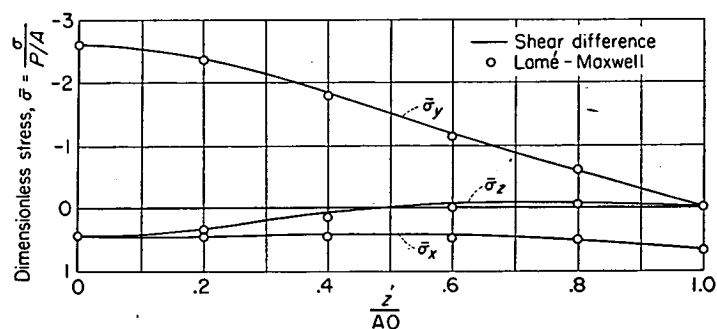


FIGURE 26.—Distribution of normal stresses along line A-A.

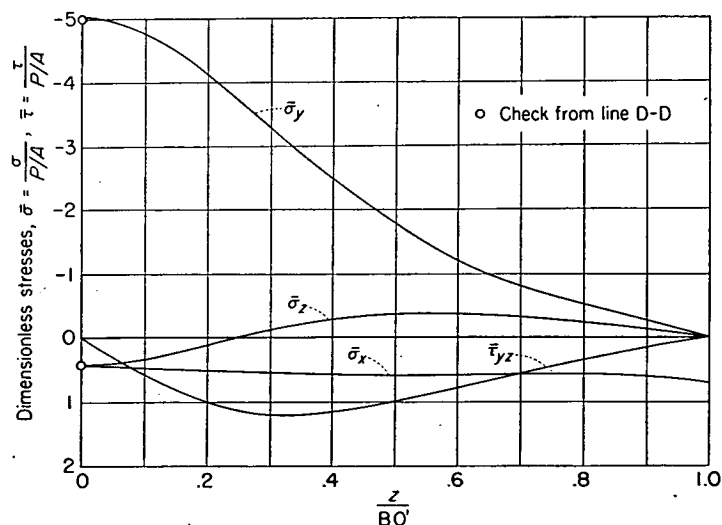


FIGURE 27.—Distribution of stresses along line B-B.

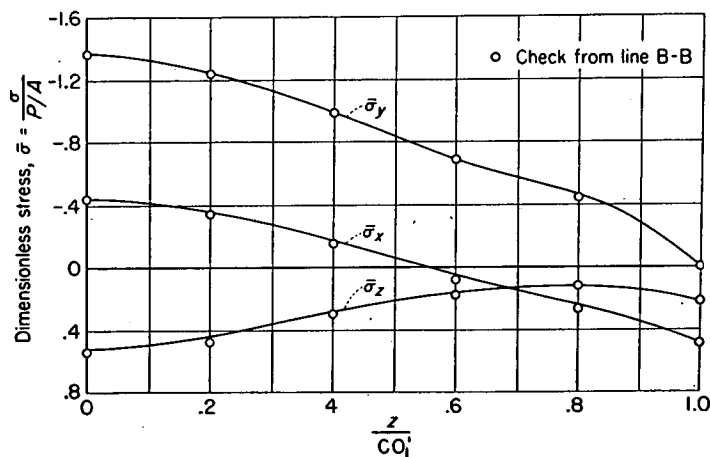


FIGURE 28.—Distribution of normal stresses along line C-C.

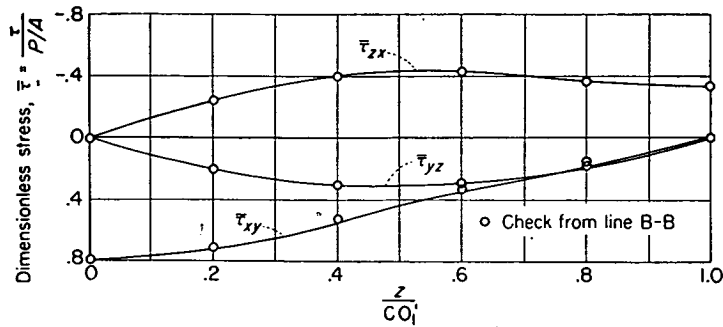


FIGURE 29.—Distribution of shearing stresses along line C-C.

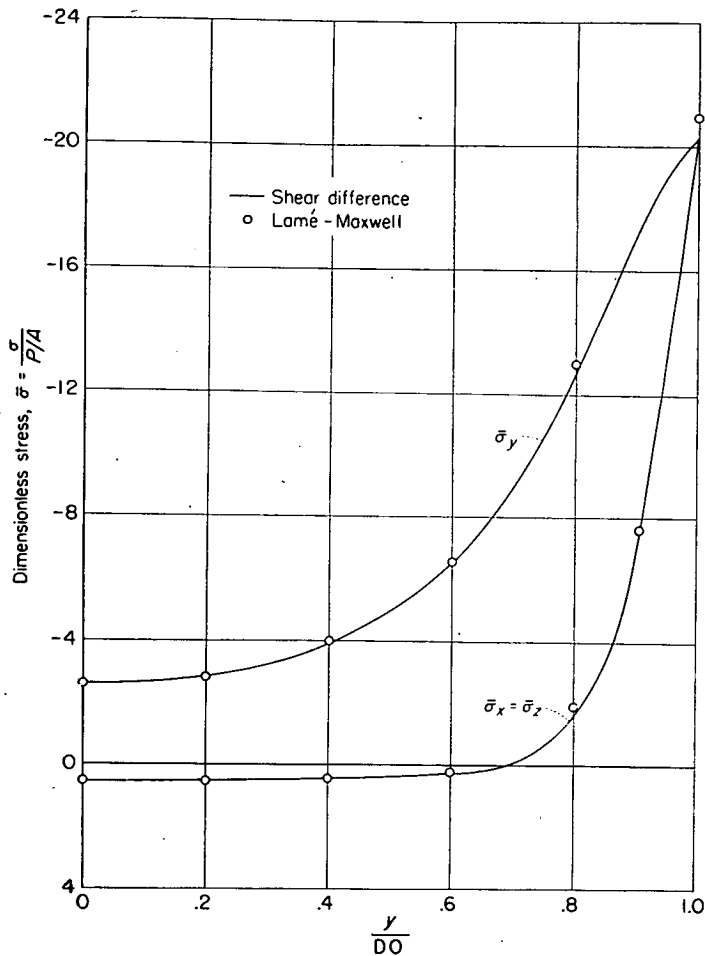


FIGURE 30.—Distribution of stresses along line D-D.

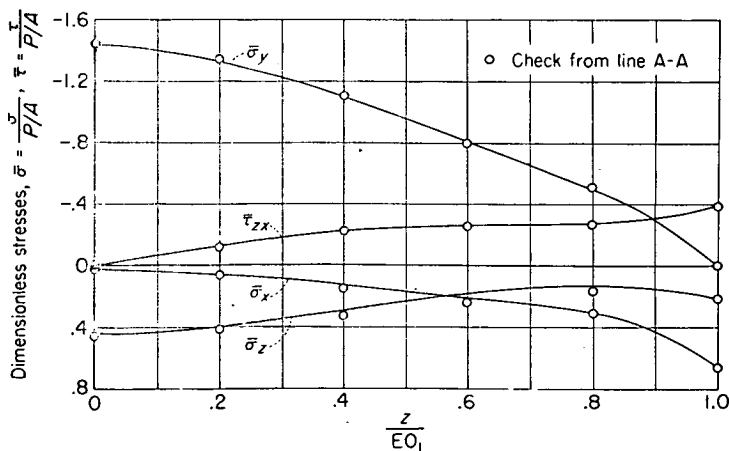


FIGURE 31.—Distribution of stresses along line E-E

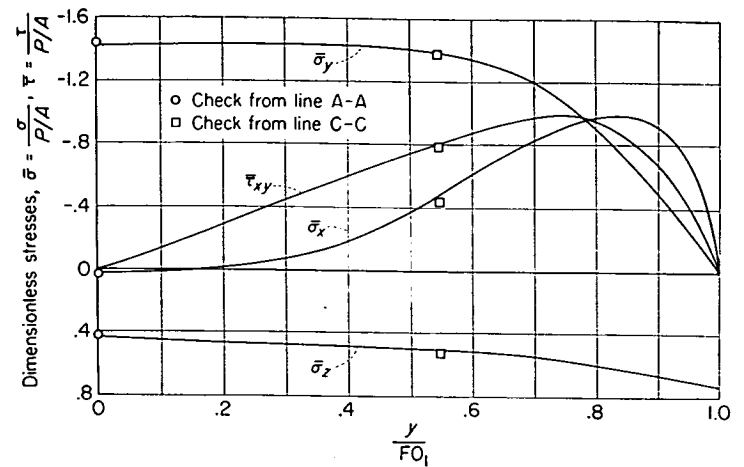


FIGURE 32.—Distribution of stresses along line F-F.

Contact stresses at load points.—As noted previously the loads on the sphere produced considerable local deformation which resulted in flattened areas at the poles. Upon observation of the meridian slice it was found that fringes and isoclinics were unusually clear right to the loaded boundary. It was therefore possible to continue the integration all the way to the loaded boundary along lines normal to the areas of contact and thereby to obtain approximations to the contact stresses. The stress components at the ends of line D-D represent the contact stresses at the poles, that is, at the centers of the loaded areas. In order to determine, at least roughly, the actual distribution of the contact stresses over the loaded areas two additional slices normal to the loaded zones were cut from the remaining material of the sphere. The intersections of these slices with the meridian slice then define two lines parallel to line D-D which extend from line B-B to the loaded boundary. Starting values were taken from the stress distribution on line B-B previously obtained, and integration proceeded to the loaded boundary in the usual fashion. In this way the contact stresses at two points at different distances from the center were obtained. With these three points the distribution of the normal stresses on the contact surface could be pretty well determined. The shearing stresses acting on the surface of contact were found from the values of n' and the isoclinics ϕ' in a meridian section in the region of contact. The results are shown in figure 33, the directions of the shearing stresses being from the poles outward.

Checks on accuracy.—Two types of checks are available in this problem, static checks and checks between stresses on different lines. Static checks were made from the stresses on lines A-A and B-B and from those acting on a diameter in the surfaces of contact. Since these stresses are rotationally symmetrical the resultant force acting on the equatorial plane and on the plane containing line B-B parallel to the equator as well as on the plane of contact can be determined by integration. From the stresses on line A-A the resultant load on the equatorial plane was computed as 176 pounds, which is 2.3 percent higher than the applied load of 172 pounds. The stresses on line B-B gave a resultant of 168 pounds which is 2.3 percent low. Lastly the resultant of the normal stresses on the surface of contact was found to be 170 pounds, or 1.2 percent low.

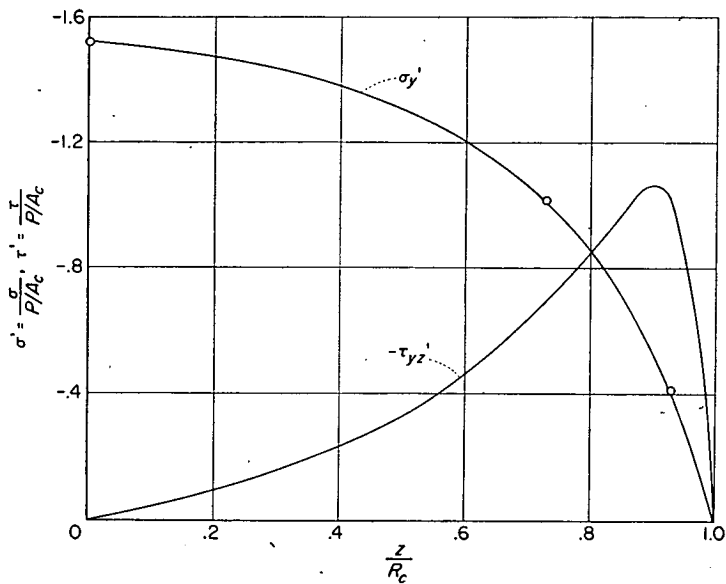


FIGURE 33.—Distribution of stresses on surfaces of contact.

Points O' and O_1' (fig. 21) are common points on different lines. The stresses at these points can be determined from each line and the results compared. The stresses at O' were found by integrating along line B-B and also by integration along the path AOO'. From line B-B the stresses were found to be $n_y = -4.08$ and $n_x = n_z = 0.40$. From the path AOO' they were found to be $n_y = -4.06$ and $n_x = n_z = 0.42$. At O_1' the stresses are found from lines C-C and F-F. From line C-C the stresses were computed as $n_y = -1.12$, $n_x = -0.36$, $n_z = 0.43$, and $n_{xy} = 0.62$. From line F-F they were $n_y = -1.14$, $n_x = -0.39$, $n_z = 0.40$, and $n_{xy} = 0.66$. The agreement in these values is seen to be quite good.

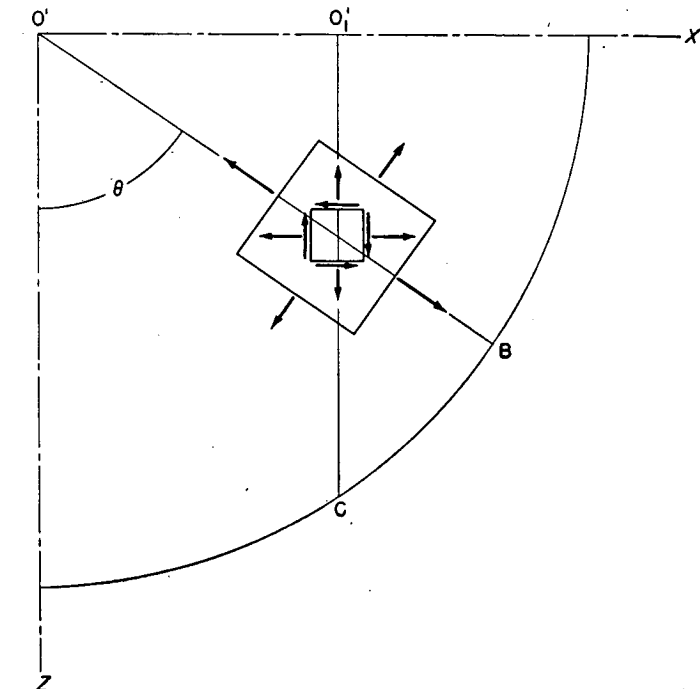


FIGURE 34.—Sketch relating stress components at a point on line B-B to those on line C-C.

It is also possible to compute the stresses on lines C-C and E-E from the stresses on lines A-A and B-B. Figure 34 shows the rectangular stress components on lines C-C and B-B. The necessary equations for transformation are similar to the familiar equations for inclined planes in plane stress systems, that is,

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad (21a)$$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \quad (21b)$$

The stresses computed by transformation were compared with the stresses independently determined on lines C-C and E-E by integration. The comparative values are shown in figures 28, 29, and 31. The general agreement is seen to be excellent.

Check by the Lamé-Maxwell equations.—Lines A-A and D-D are lines of symmetry for the sphere. For these special lines the stresses can be computed by the method outlined by Jessop (reference 18) using the extension of the Lamé-Maxwell equations to three dimensions. This computation has been carried out using fringe orders from the meridian and equatorial slices and the 5° isoclinic from the meridian slice. The comparative values are given in figure 26 for line A-A and in figure 30 for line D-D. Inspection of the figures shows that in both cases the agreement is close.

ANALYSIS AND DISCUSSION

The primary objective of the project under discussion was to develop a general method for solving three-dimensional problems photoelastically. In the theoretical part of this report such a method is described. The experimental work shows that the proposed method is practical.

It is too early to draw broad conclusions regarding the general accuracy of the new method. However, the excellence of the static checks and the consistency of the results, as shown by the close cross checks between the results from the various lines, seem to indicate possibilities of high accuracy. Unfortunately there is as yet no theoretical solution available for this particular problem to furnish conclusive checks and a measure of the errors.² Nevertheless there is a reasonable degree of certainty that the major stresses are free from significant error.

It must be pointed out that the stresses as found here represent the solution for a material for which Poisson's ratio is 0.48, whereas most structural materials have Poisson's ratios of about 0.3. This is an inherent limitation of three-dimensional photoelasticity. However, as noted previously, the theoretical solutions available to date indicate that Poisson's ratio has only a small influence on the major stresses although the effect on the minor stresses may be pronounced (references 19 and 20).

Although no theoretical solution is available for the sphere Hertz's solution can be used to check the contact stresses

² A theoretical solution of this problem has recently been published. (See reference 21.)

determined photoelastically. According to Hertz's solution the contact pressure is representable by the ordinates to a hemisphere erected on the contact surface. Further the Hertz theory predicts that the maximum contact pressure should be $1.5P/A_c$. Reference to figure 33 shows that the experimentally determined value of the maximum pressure is $1.53P/A_c$, which is 2 percent high. When it is considered that the path of integration used in determining the maximum pressure led along an equatorial radius to the center and thence up the load axis to the surface this is indeed a remarkable check. The general distribution of the pressures is also seen to be substantially correct. The proposed method would therefore seem to hold considerable promise for the determination of contact stresses.

According to the Hertz theory the two principal stresses in the plane of the contact surface at the pole, σ_x and σ_y , should each equal $\frac{1+2\nu}{2}\sigma_y$. For Poisson's ratio of 1/2 this reduces to $\sigma_x=\sigma_y$ which indicates that an isotropic point exists at the pole, and therefore the shearing stress is zero at this point. This is borne out by the photoelastic results.

Along the circular boundary of the surface of contact the stress consists of a pure shear of the amount $\left(\frac{1-2\nu}{3}\right)\frac{P}{A_c}$ according to the Hertz theory. If Poisson's ratio is 1/2 this expression vanishes. Hence the normal and shearing stresses at the boundary should be zero. The photoelastic results are seen to give this value.

The problem treated in this report has complete rotational symmetry which simplifies the experimental technique by eliminating the use of two models. No problem has as yet been solved which requires two models. The use of two models will no doubt introduce complications, but no insurmountable difficulties are anticipated. However, further work must be done to demonstrate the effectiveness of the proposed method with two models.

There remains also the possibility of using sub slices from the main slice as discussed in the theoretical part. The model used in this investigation was not large enough to make this procedure feasible although some attempts were made. This possibility also needs to be further explored.

It will perhaps also be desirable to repeat the solution of the sphere with smaller loads in order to reduce the relatively large local deformations.

SUMMARY OF RESULTS

The results from this investigation to develop a general method for three-dimensional photoelastic stress analysis may be summarized as follows:

1. The method of strain measurement after annealing cannot be used with the materials now available in this country.

2. A general photoelastic method for obtaining six stress components at any point of an unsymmetrical body arbitrarily loaded has been developed. This method does not depend on Poisson's ratio, although the results reflect the physical constants of the model.

3. The new method is applicable in the plastic range of the model.

4. The method shows possibilities for the determination of contact stresses.

5. The stresses existing in a sphere subjected to diametral compression have been determined with considerable accuracy.

6. At the center of the sphere the stress components were found to be $\sigma_y = -2.59P/A$ and $\sigma_x = \sigma_z = 0.45P/A$, where P is the load on the sphere and A is the area of the equatorial plane of the sphere. These values may be compared with the stresses at the center of a disk under diametral compression which are $\sigma_y = -1.91P/A$ and $\sigma_x = 0.64P/A$.

7. Further work is needed to determine the full potentialities of the method when two models are used. Further work is also needed to determine the practicability of sub slices.

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CHICAGO 16, ILL., September 15, 1951.

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TABLE I

NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATION (17c) FOR LINE C-C

$\frac{z}{CO_1'}$	Mean Δn_{yz}	Mean Δn_{xz}	n_x
0			0.431
	0.012	-0.009	
0.1			0.410
	0.025	-0.024	
0.2			0.361
	0.030	-0.034	
0.3			0.297
	0.028	-0.038	
0.4			0.231
	0.021	-0.038	
0.5			0.172
	0.009	-0.033	
0.6			0.130
	0.001	-0.024	
0.7			0.105
	-0.004	-0.014	
0.8			0.095
	-0.019	-0.001	
0.9			0.113
	-0.060	0.012	
1.0			0.185

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TABLE II

CALCULATION OF NORMAL STRESS COMPONENTS FOR LINE C-C

$\frac{z}{CO_1'}$	n_x	n''	$\cos 2\phi''$	n'''	$\cos 2\phi'''$	$n_x - n_x^{(1)}$	$n_x - n_y^{(2)}$	$n_x^{(3)}$	$n_y^{(4)}$
0	0.43	0.79	1.000	1.55	1.000	0.79	1.55	-0.36	-1.12
0.1	0.41	0.79	0.957	1.52	0.993	0.76	1.51	-0.35	-1.10
0.2	0.36	0.78	0.841	1.43	0.972	0.66	1.39	-0.30	-1.03
0.3	0.30	0.77	0.674	1.30	0.942	0.52	1.22	-0.22	-0.92
0.4	0.23	0.76	0.488	1.16	0.906	0.37	1.05	-0.14	-0.82
0.5	0.17	0.74	0.302	1.00	0.863	0.22	0.86	-0.05	-0.69
0.6	0.13	0.70	0.122	0.84	0.827	0.08	0.69	0.05	-0.56
0.7	0.10	0.66	-0.035	0.70	0.809	-0.02	0.57	0.12	-0.47
0.8	0.10	0.62	-0.167	0.56	0.819	-0.10	0.46	0.20	-0.36
0.9	0.11	0.60	-0.282	0.40	0.866	-0.17	0.35	0.28	-0.24
1.0	0.18	0.59	-0.375	0.18	1.000	-0.22	0.18	0.40	0

$$^1 n_x - n_x = n'' \cos 2\phi''$$

$$^2 n_x - n_y = n''' \cos 2\phi'''$$

$$^3 n_x = n_x - (n_x - n_x)$$

$$^4 n_y = n_x - (n_x - n_y)$$

TABLE III

CALCULATION OF SHEARING STRESS BY OBLIQUE INCIDENCE FOR LINE C-C

$\frac{z}{CO_1'}$	n_{xy}	$2\phi_{xy}$	$\frac{1}{2} n_{xy} \sin 2\phi_{xy}$	n_{yz}	$n_{zy}^{(1)}$
0	2.22	34.8	0.63	0	0.63
0.1	2.14	41.0	0.70	0.09	0.61
0.2	1.99	47.4	0.73	0.17	0.57
0.3	1.82	54.8	0.74	0.22	0.52
0.4	1.63	61.4	0.71	0.25	0.46
0.5	1.42	65.6	0.65	0.25	0.40
0.6	1.20	67.0	0.55	0.24	0.31
0.7	0.99	66.0	0.45	0.21	0.24
0.8	0.79	62.0	0.34	0.16	0.18
0.9	0.55	49.0	0.20	0.10	0.10
1.0	0.28	0	0	0	0

$$^1 n_{zy} = \frac{1}{2} n_{xy} \sin 2\phi_{xy} - n_{yz}$$

REPORT 1149

ON TRANSONIC FLOW PAST A WAVE-SHAPED WALL¹

By CARL KAPLAN

SUMMARY

The present report is an extension of a previous investigation (described in NACA Rep. 1069) concerned with the solution of the nonlinear differential equation for transonic flow past a wavy wall. In the present work several new notions are introduced which permit the solution of the recursion formulas arising from the method of integration in series. In addition, a novel numerical test of convergence, applied to the power series (in the transonic similarity parameter) representing the local Mach number distribution at the boundary, indicates that smooth symmetrical potential flow past the wavy wall is no longer possible once the critical value of the stream Mach number has been exceeded.

INTRODUCTION

In NACA Report 1069 (ref. 1) the writer considered the problem of two-dimensional transonic flow past an infinitely long sinusoidal solid boundary. The problem was treated in the physical plane and the purpose was to investigate whether or not the flow past the wavy wall remains a smooth symmetrical type of potential flow when the undisturbed-stream Mach number exceeds its critical value. By a smooth type of potential flow is meant one for which the velocity potential, say, and its first derivatives are single-valued and continuous; that is, there are no discontinuities of the nature of shock waves.

Initially, the Prandtl-Busemann small-perturbation method was applied and the velocity potential developed inclusive of the third power in the "thickness" coefficient $\epsilon = \frac{a}{\lambda/2\pi}$

where

a amplitude of wavy wall

λ wave length of wavy wall

The velocity potential was then referred to the critical speed of sound c_{cr} , the coefficient ϵ was replaced by $\frac{k(1-M_\infty^2)^{3/2}}{\gamma+1}$

where $k = \frac{(\gamma+1)\epsilon}{(1-M_\infty^2)^{3/2}}$ is the transonic similarity parameter, and terms involving powers of $1-M_\infty^2$ higher than the first were neglected (main assumption for transonic flow). This simplified or transonic form of the Prandtl-Busemann solution was shown to be identical with the one obtained by solving the simplified nonlinear differential equation for transonic flow, with the boundary condition taken not at the wave-shaped wall but at the flat plate corresponding to vanishing amplitude. The calculation was carried through the sixth power in the tran-

sonic similarity parameter k and corresponds to the insuperable task of obtaining the Prandtl-Busemann solution to the sixth power in the thickness coefficient ϵ . Thus each iteration step of the Prandtl-Busemann method contributes to the transonic form of the solution, which may therefore be considered a result of thin-profile theory with disturbances not necessarily small compared with $1-M_\infty^2$. The main conclusion reached in reference 1 was that the transonic similarity parameter k must be less than $\frac{4}{3}$ —a value still somewhat greater than the critical value 0.83770 calculated there.

The purpose of the present work is to express the solution of the problem of transonic flow past the wavy wall in a form more suitable for general considerations and to prove that the assumed smooth symmetrical type of potential flow cannot exist at stream Mach numbers beyond the critical value.

SYMBOLS

x, y	nondimensional rectangular Cartesian coordinates
a	amplitude of wavy wall
λ	wave length of wavy wall
$\lambda/2\pi$	reference element of length
ϵ	"thickness" coefficient of wavy wall, $\frac{a}{\lambda/2\pi}$
γ	ratio of specific heats at constant pressure and constant volume
U	stream velocity
M	local Mach number
M_∞	undisturbed-stream Mach number
k	transonic similarity parameter, $\frac{(\gamma+1)\epsilon}{(1-M_\infty^2)^{3/2}}$
c_{cr}	critical speed of sound
ϕ	velocity potential of flow
$f(x, y)$	transonic disturbance potential
$f_n(y)$	functions of y only, related to $f(x, y)$
A_q^p	numerical coefficients
$A_q^{n,p}$	generating functions of k , $\sum_{r=0}^{\infty} A_q^{n,p+r} k^{2r}$
$A_{m,p}$	functions of dummy variable r , $\sum_{n=1,m}^{\infty} A_{n-2m}^{n,p} r^n$, lower label starts from 1 when m is negative and from m when m is positive

Primes denote differentiation with respect to independent variable.

¹ Supersedes NACA TN 2748, "On Transonic Flow Past a Wave-Shaped Wall" by Carl Kaplan, 1952.

ANALYSIS

GENERAL FORMULAS

If the undisturbed stream is in the direction of the positive x -axis, then the velocity potential ϕ referred to the critical velocity c_{cr} can be written as (see ref. 1)

$$\phi = x + \frac{1}{\gamma+1} (1-M_\infty^2) f(x, y)$$

where the second term on the right-hand side is a disturbance velocity potential and implies that terms involving powers of $1-M_\infty^2$ higher than the first have been neglected. The differential equation for transonic flow satisfied by the function $f(x, y)$ is obtained from the general differential equation for compressible flow and takes the following simplified nonlinear form: .

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial x^2} \quad (1)$$

The boundary conditions to be fulfilled by $f(x, y)$ are as follows:

$$\left. \begin{aligned} \frac{\partial f}{\partial x} &= -1 \\ \frac{\partial f}{\partial y} &= 0 \\ \frac{\partial f}{\partial y} &= -k \sin x \end{aligned} \right\} \quad \left. \begin{aligned} & \\ & \\ & \end{aligned} \right\} \quad \begin{aligned} & \\ & \\ & \text{(at } y = \infty) \\ & \text{(at } y = 0, -\infty < x < \infty) \end{aligned} \quad (2)$$

Here x and y are nondimensional rectangular Cartesian coordinates simply related to the physical plane coordinates X and Y by means of the transformation

$$\begin{aligned} x &= X \\ y &= (1-M_\infty^2)^{1/2} Y \end{aligned}$$

and the equation of the wavy wall is $Y = \epsilon \cos X$ or $y = (1-M_\infty^2)^{1/2} \epsilon \cos x$. Clearly, $f(x, y)$ involves only the variables x and y and the transonic similarity parameter k .

The most general form for the function $f(x, y)$ to ensure symmetrical potential flow past the wavy wall is the following (see fig. 1):

$$f(x, y) = -x + \sum_{n=1}^{\infty} f_n \sin nx \quad (3)$$

where the f_n are functions of y only. When this form for $f(x, y)$ is substituted into equation (1) and the coefficients of the separate harmonic terms $\sin nx$ are equated to zero,

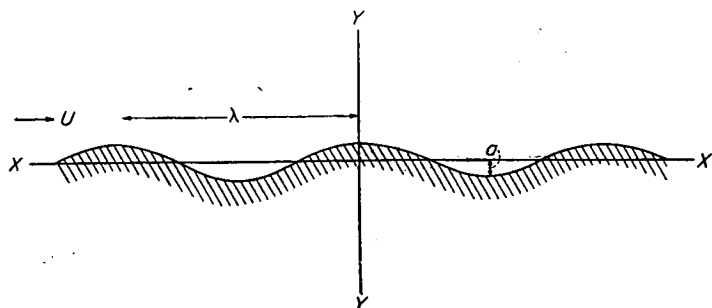


FIGURE 1.—Wave-shaped wall.

the following system of second-order nonlinear ordinary differential equations for f_n results:

$$\begin{aligned} f_n'' - n^2 f_n &= -\frac{1}{4} n \sum_{m=1}^{n-1} m(n-m) f_m f_{n-m} - \\ &\quad \frac{1}{2} n \sum_{m=1}^{\infty} m(n+m) f_m f_{n+m} \quad (n=1, 2, \dots, \infty) \end{aligned} \quad (4)$$

Before proceeding to the solution of these equations, several formulas of general interest and subsequent use are given. They have been derived in reference 1. The local Mach number distribution is given by

$$1 - M^2 = -(1 - M_\infty^2) \frac{\partial f}{\partial x} \quad (5)$$

The equation from which the critical value of the transonic similarity parameter is calculated follows from equation (5) by taking $x=0$, $y=0$, and $M=1$; that is,

$$\left(\frac{\partial f}{\partial x} \right)_{x=0, y=0} = 0$$

or with the aid of equation (3),

$$\sum_{n=1}^{\infty} n f_n(0) = 1 \quad (6)$$

The pressure coefficient

$$C_{p, M_\infty} = \frac{p - p_\infty}{\frac{1}{2} \rho_\infty U^2}$$

is given by

$$C_{p, M_\infty} = -\frac{2}{\gamma+1} (1 - M_\infty^2) \left(1 + \frac{\partial f}{\partial x} \right) \quad (7)$$

INTEGRATION IN SERIES FOR THE FUNCTIONS f_n

In reference 1, equations (4) were solved by an iteration procedure and approximate expressions for f_1 to f_6 were obtained. An examination of these expressions showed that the general form of f_n is

$$f_n = \sum_{p=0}^{\infty} e^{-(2p+n)y} \sum_{q=0}^{n-1} y^q \sum_{r=p}^{\infty} A_{q,r}^n k^{n+2r} \quad (n=1, 2, \dots, \infty) \quad (8)$$

where, if $p=0$, the upper limit of q is $n-1$ and, if $p \neq 0$, the upper limit of q is $2p+n-2$. The four-labeled coefficients $A_{q,r}^n$ are real numbers calculated from recursion formulas obtained from the system of differential equations (4) and the boundary condition at the surface of the wavy wall. The boundary condition at $y=\infty$ is automatically satisfied by the form of f_n ; whereas the boundary condition at the wall takes the form

$$(f_n')_{y=0} = \begin{cases} -k & (n=1) \\ -0 & (n \neq 1) \end{cases} \quad (9)$$

Inserting the expression for f_n given by equation (8) into equations (9) yields immediately the following results:

$$\left. \begin{aligned} A_{0,0}^1 &= 1 \\ \sum_{p=0}^r (2p+n) A_{0,p}^n &= \sum_{p=0,1}^r A_{1,p}^n \end{aligned} \right\} \quad (n=1, 2, \dots, \infty) \quad (10)$$

where, if $n=1$, the lower limit of p on the right-hand side is unity and, if $n \neq 1$, the lower limit of p is zero. Also, if $n=1$, the upper limit r of p goes from 1 to ∞ and, if $n \neq 1$, r goes from 0 to ∞ .

By elementary manipulation of series, the second of equations (10) can be replaced by the following more useful forms:

$$\left. \begin{aligned} nA_{0,0}^n &= A_{1,0}^n & (n=2, 3, \dots, \infty) \\ nA_{0,p}^n &= \sum_{q=0}^r A_{1,p}^n - \sum_{q=1}^r (2p+n)A_{0,p}^n & \left(\begin{matrix} r=1, 2, \dots, \infty \\ n=1, 2, \dots, \infty \end{matrix} \right) \end{aligned} \right\} \quad (11)$$

where in the first term on the right-hand side the lower label p starts from 1 when $n=1$ and from 0 when $n \neq 1$.

In reference 1, recursion formulas were derived for the coefficients $A_{q,p}^n$. In the present report a much more significant approach is introduced. Note that equation (8) can be rewritten in the following form:

$$f_n = \sum_{p=0}^{\infty} (ke^{-\nu})^{2p+n} \sum_{q=0}^{2p+n-2} A_{q,p}^n y^q \quad (12)$$

where

$$A_{q,p}^n = \sum_{r=0}^{\infty} A_{q,p+r}^n k^{2r}$$

In a manner similar to that described in reference 1 for the coefficients $A_{q,p}^n$, recursion formulas can be obtained for the power series $A_{q,p}^n$. Indeed, the two types of recursion formulas are intimately connected and, for a given value of p , the one can be obtained from the other by mere inspection. A single recursion formula can be written for the general quantity $A_{q,p}^n$ but the resulting expression is cumbersome and serves no practical purpose. It is much more desirable to obtain separate recursion formulas for each value of p . As examples of procedure, the recursion formulas for $p=0$, 1, and 2 are considered in the sections which follow.

RECURSION FORMULA FOR $A_{q,0}^n$

With $p=0$, the recursion formula is (compare with eq. (57) of ref. 1)

$$\begin{aligned} 2n(q+1)A_{q+1,0}^n &= (q+1)(q+2)\delta_{q,n-2}^n A_{q+2,0}^n + \\ &\frac{1}{4}n \sum_{q_1=0}^q \sum_{m=q-q_1}^{n-2-q_1} (m+1)(n-m-1)A_{q_1,0}^{n-m-1} A_{q-q_1,0}^{m+1} \\ &(n=2, 3, \dots, \infty; q=0, 1, \dots, n-2) \end{aligned} \quad (13)$$

where

$$\delta_{q,n-2}^n = \begin{cases} 0 & (q=n-2) \\ 1 & (q \neq n-2) \end{cases}$$

This recursion formula can be solved, the solution starting with $q=n-2$ and descending towards $q=0$. Thus, for $q=n-2$, equation (13) becomes

$$8(n-1)A_{n-1,0}^n = \sum_{m=0}^{n-2} (m+1)(n-m-1)A_{m,0}^{m+1} A_{n-m-2,0}^{n-m-1} \quad (n=2, 3, \dots, \infty) \quad (14)$$

Now, multiply both sides of this equation by r^n where r is a dummy variable and sum from $n=2$ to $n=\infty$. Then

$$8 \sum_{n=1}^{\infty} (n-1)A_{n-1,0}^n r^n = \sum_{n=2}^{\infty} r^n \sum_{m=0}^{n-2} (m+1)(n-m-1)A_{m,0}^{m+1} A_{n-m-2,0}^{n-m-1} \quad (15)$$

Let

$$A_{1,0} = \sum_{n=1}^{\infty} A_{n-1,0}^n r^n$$

then

$$A_{1,0}' = \sum_{n=1}^{\infty} nA_{n-1,0}^n r^{n-1}$$

and

$$(A_{1,0}')^2 = \sum_{n=2}^{\infty} r^{n-2} \sum_{m=1}^{n-1} m(n-m)A_{m-1,0}^m A_{n-m-1,0}^{n-m}$$

By observation, it can be seen that the right-hand side of equation (15) is equal to $(rA_{1,0}')^2$. Equation (15) can thus be replaced by the following first-order nonlinear differential equation:

$$(rA_{1,0}')^2 - 8rA_{1,0}' + 8A_{1,0} = 0 \quad (16)$$

The solution of this equation is

$$A_{1,0}' = c_0 e^{\frac{1}{4}rA_{1,0}'}$$

where c_0 is the arbitrary constant of integration. From the definition of $A_{1,0}$ it follows that, with $r=0$,

$$c_0 = A_{1,0}^1$$

and hence

$$\frac{A_{1,0}'}{A_{1,0}^1} = e^{\frac{1}{4}rA_{1,0}^1 \frac{A_{1,0}'}{A_{1,0}^1}} \quad (17)$$

Note that by definition $A_{1,0}^1$ involves the coefficients $A_{0,0}^n$ which are ultimately calculated by means of the auxiliary relations, equations (11), engendered by the boundary conditions at the surface of the wavy wall. In solutions of the recursion formulas, therefore, the coefficients $A_{0,0}^n$ appear as undetermined quantities.

The expansion of $\frac{A_{1,0}'}{A_{1,0}^1}$ in powers of $rA_{1,0}^1$ is a nontrivial problem which fortunately can be solved with the aid of Lagrange's investigations on the reversion of power series. In reference 2, Lagrange's problem is illustrated by the following example:

To expand e^{ax} in powers of $y = xe^{bx}$. The result is expressed as

$$\begin{aligned} e^{ax} &= 1 + ay + \frac{a(a-2b)}{2!}y^2 + \frac{a(a-3b)^2}{3!}y^3 + \dots + \\ &\frac{a(a-nb)^{n-1}}{n!}y^n + \dots \end{aligned} \quad (18)$$

In particular, with $a=1$, $b=-1$, and $\xi=e^x$, the solution of the equation $\log \xi = y\xi$ or $\xi = e^{y\xi}$ (generally referred to as Eisenstein's problem) is given by the series

$$\xi = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!} y^{n-1}$$

Then, if ξ is replaced by $\frac{A_{1,0}'}{A_{1,0}^1}$ and y by $\frac{1}{4}rA_{1,0}^1$, the solution of equation (17) is

$$\frac{A_{1,0}'}{A_{1,0}^1} = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)! 4^{n-1}} (A_{1,0}^1)^{n-1} r^{n-1}$$

or

$$\sum_{n=1}^{\infty} n A_{n-1}^n r^{n-1} = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)! 4^{n-1}} (A_0^1)^n r^{n-1}$$

Hence, by equating coefficients of equal powers of r on both sides of this equation,

$$A_{n-1}^n = \frac{n^{n-2}}{n! 4^{n-1}} (A_0^1)^n \quad (n=2, 3, \dots, \infty) \quad (19)$$

which is the solution of the recursion formula, equation (14). The coefficient A_{n-1}^n can be considered as the generating function for the set of coefficients A_{n-1}^n . Thus, equation (19) can be written as

$$\sum_{r=0}^{\infty} A_{n-1}^n k^{2r} = \frac{n^{n-2}}{n! 4^{n-1}} \left(\sum_{r=0}^{\infty} A_0^1 k^{2r} \right)^n \quad (20)$$

Then, by equating coefficients of equal powers of k on both sides of this equation and putting $A_0^1 = 1$ (see boundary condition, eqs. (10)), the following equations are obtained:

$$\left. \begin{aligned} A_{n-1}^n &= \frac{n^{n-2}}{n! 4^{n-1}} \\ A_{n-1}^n &= \frac{n^{n-2}}{(n-1)! 4^{n-1}} A_0^1 \\ A_{n-1}^n &= \frac{n^{n-2}}{(n-1)! 4^{n-1}} A_0^1 + \frac{n^{n-2}}{2!(n-2)! 4^{n-1}} (A_0^1)^2 \\ &\dots \end{aligned} \right\} \quad (21)$$

Note that from the first of these formulas $A_1^2 = \frac{1}{8}$ and that from the boundary condition, equations (11), $A_0^2 = \frac{1}{16}$.

Consider now $q=n-3$. Then equation (13) becomes

$$2n(n-2) A_{n-2}^n = (n-1)(n-2) A_{n-1}^n + \frac{1}{2} n \sum_{m=2}^{n-1} m(n-m) A_{m-2}^m A_{n-m-1}^n \quad (22)$$

Multiply both sides of this equation by r^n and sum from $n=3$ to $n=\infty$. Then

$$2 \sum_{n=2}^{\infty} n(n-2) A_{n-2}^n r^n = \sum_{n=1}^{\infty} (n-1)(n-2) A_{n-1}^n r^n + \frac{1}{2} \sum_{n=3}^{\infty} n r^n \sum_{m=2}^{n-1} m(n-m) A_{m-2}^m A_{n-m-1}^n \quad (23)$$

Let

$$A_1 = \sum_{n=1}^{\infty} A_{n-1}^n r^n$$

and

$$A_2 = \sum_{n=2}^{\infty} A_{n-2}^n r^n$$

then

$$A_1' A_2' = \sum_{n=1}^{\infty} r^n \sum_{m=1}^n m(n-m+2) A_{m-1}^m A_{n-m+2}^n$$

It can be shown easily that the last term on the right-hand side of equation (23) is equal to $\frac{1}{2} r(r^2 A_1' A_2')'$. Hence, equation (23) can be rewritten as the following differential equation:

$$2r^2 A_2'' - 2r A_2' = r^2 A_1'' - 2r A_1' + 2A_1 + \frac{1}{2} r(r^2 A_1' A_2')' \quad (24)$$

Now, from equation (16), it follows that

$$\left. \begin{aligned} (r A_1')^2 &= 8(r A_1' - A_1) \\ (r A_1')' &= \frac{4 A_1'}{4 - r A_1'} \\ r A_1' (r A_1')' &= 4 r A_1'' \end{aligned} \right\} \quad (25)$$

Then equation (24) can be written as

$$(r A_2')' - \frac{8 + r(r A_1')'}{r(4 - r A_1')} (r A_2') = \frac{1}{r(4 - r A_1')} [2r(r A_1')' - 6r A_1' + 4A_1] \quad (26)$$

Let $r A_2' = v \frac{(r A_1')^2}{4 - r A_1'}$, where v is the new dependent variable. Then equation (26) becomes

$$v' = \frac{1}{r(r A_1')^2} [2r(r A_1')' - 6r A_1' + 4A_1]$$

or with the aid of equations (25),

$$v' = \frac{1}{8} (r A_1')'$$

Hence

$$v = \frac{1}{8} r A_1' + \frac{1}{8} c_1$$

or

$$8r A_2' = \frac{(r A_1')^3}{4 - r A_1'} + c_1 \frac{(r A_1')^2}{4 - r A_1'}$$

The arbitrary constant c_1 is determined by differentiating this last equation twice and evaluating for $r=0$. Thus,

$$c_1 = 64 \frac{A_0^2}{(A_0^1)^2}$$

Finally,

$$A_2' = \frac{1}{16} r^2 [(A_1')^2]' + 8 \frac{A_0^2}{(A_0^1)^2} r A_1'' \quad (27)$$

Now, from equation (18) with $a=2$, $b=-1$, and $\xi=e^x$, it follows that

$$\xi^2 = 2 \sum_{n=2}^{\infty} \frac{n^{n-3}}{(n-2)!} y^{n-2}$$

Or with

$$\xi = \frac{A_{1,0}'}{A_{1,0}^0} \quad y = \frac{1}{4} r A_{1,0}^0$$

$$(A_{1,0}')^2 = 2 \sum_{n=2}^{\infty} \frac{n^{n-3}}{(n-2)!4^{n-2}} (A_{1,0}^0)^n r^{n-2}$$

and

$$[(A_{1,0}')^2]' = 2 \sum_{n=3}^{\infty} \frac{n^{n-3}}{(n-3)!4^{n-2}} (A_{1,0}^0)^n r^{n-3}$$

Thus, equation (27) becomes

$$\sum_{n=2}^{\infty} n A_{n-2}^0 r^{n-1} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{n^{n-3}}{(n-3)!4^{n-2}} (A_{1,0}^0)^n r^{n-1} + 8 \frac{A_{1,0}^0}{(A_{1,0}^0)^2} \sum_{n=2}^{\infty} n(n-1) A_{n-1}^0 r^{n-1}$$

Hence, replacing A_{n-1}^0 by its value from equation (19) and equating the coefficients of equal powers of r on both sides of this equation yields

$$A_{n-2}^0 = 8 \frac{n^{n-3}}{(n-2)!4^{n-1}} (A_{1,0}^0)^{n-2} A_{0,0}^0 + \frac{1}{2} \frac{n^{n-4}}{(n-3)!4^{n-1}} (A_{1,0}^0)^n \quad (n=3, 4, \dots, \infty) \quad (28)$$

the solution of the recursion formula, equation (22). Again, equation (28) can be written as

$$\sum_{r=0}^{\infty} A_{n-2}^0 k^{2r} = 8 \frac{n^{n-3}}{(n-2)!4^{n-1}} \sum_{r=0}^{\infty} A_{0,0}^0 k^{2r} \left(\sum_{r=0}^{\infty} A_{1,0}^0 k^{2r} \right)^{n-2} + \frac{1}{2} \frac{n^{n-4}}{(n-3)!4^{n-1}} \left(\sum_{r=0}^{\infty} A_{1,0}^0 k^{2r} \right)^n \quad (29)$$

Then, equating the coefficients of equal powers of k on both sides of this equation and putting $A_{0,0}^0 = 1$ and $A_{0,0}^0 = \frac{1}{16}$ yields:

$$\left. \begin{aligned} A_{n-2}^0 &= \frac{n^{n-3}}{2(n-2)!4^{n-1}} + \frac{n^{n-4}}{2(n-3)!4^{n-1}} \\ A_{n-2}^0 &= \frac{8n^{n-3}}{(n-2)!4^{n-1}} A_{0,0}^0 + \frac{n^{n-3}}{(n-3)!4^{n-1}} A_{1,0}^0 \\ A_{n-2}^0 &= \frac{8n^{n-3}}{(n-2)!4^{n-1}} A_{0,0}^0 + \frac{n^{n-3}}{(n-3)!4^{n-1}} \left[A_{0,0}^0 + 8A_{1,0}^0 A_{0,0}^0 + \frac{1}{2} (A_{1,0}^0)^2 \right] + \frac{n^{n-3}}{2(n-4)!4^{n-1}} (A_{1,0}^0)^2 \end{aligned} \right\} \quad (30)$$

Note that from the first of these formulas $A_{1,0}^0 = \frac{1}{24}$ and that from the boundary conditions, equations (11), $A_{0,0}^0 = \frac{1}{72}$.

The procedure followed in order to obtain equation (28) for A_{n-2}^0 is a general one and with very little difficulty other members of the family A_n^0 can be obtained. For example,

$$A_{n-3}^0 = 48 \frac{n^{n-4}}{(n-3)!4^{n-1}} (A_{1,0}^0)^{n-3} A_{0,0}^0 + \frac{1}{8} \frac{n^{n-4}}{(n-4)!4^{n-1}} (A_{1,0}^0)^{n-4} [16A_{0,0}^0 + (A_{1,0}^0)^2] \quad (31)$$

From this equation it follows that

$$\left. \begin{aligned} A_{n-3}^0 &= \frac{2}{3} \frac{n^{n-4}}{(n-3)!4^{n-1}} + \frac{1}{2} \frac{n^{n-4}}{(n-4)!4^{n-1}} \\ A_{n-3}^0 &= 48 \frac{n^{n-4}}{(n-3)!4^{n-1}} A_{0,0}^0 + 8 \frac{n^{n-4}}{(n-4)!4^{n-1}} A_{0,0}^0 + \frac{n^{n-4}}{4^{n-1}} A_{1,0}^0 \left[\frac{5}{3} \frac{1}{(n-4)!} + \frac{1}{2} \frac{1}{(n-5)!} \right] \\ &\dots \end{aligned} \right\} \quad (32)$$

From the first of these formulas and the boundary conditions, equations (11), $A_{1,0}^0 = \frac{7}{384}$ and $A_{0,0}^0 = \frac{7}{1536}$.

At this point, it is noted that the coefficients of the form A_n^0 are calculated from the first formula of each set, equations (21), (30), (32), and so forth. A number of this type of coefficient have been evaluated (see ref. 1) by means of the recursion formulas for the coefficients A_n^0 themselves. They are listed as follows:

$$\begin{aligned} A_{1,0}^0 &= \frac{1}{8} & A_{1,0}^0 &= \frac{7 \times 13}{72 \times 256} \\ A_{1,0}^0 &= \frac{1}{24} & A_{1,0}^0 &= \frac{13}{18 \times 256} \\ A_{1,0}^0 &= \frac{7}{384} & A_{1,0}^0 &= \frac{13 \times 19}{576 \times 256} \\ A_{1,0}^0 &= \frac{7}{768} & & \dots \end{aligned}$$

A careful examination of these numerical values leads to the general rule,

$$A_{1,0}^0 = \frac{\{3n-5\}}{n!4^{n-1}} \quad (n=2, 3, \dots, \infty)$$

and from the boundary conditions, equations (11),

$$A_{0,0}^0 = \frac{\{3n-5\}}{nn!4^{n-1}} \quad (n=2, 3, \dots, \infty)$$

where by definition

$$\{3n-5\} = 1 \times 4 \times 7 \times 10 \times 13 \times \dots \times (3n-5)$$

In the expression for the local Mach number distribution

evaluated at the crest of the wavy wall ($x=0, y=0$), there occurs the following power series:

$$F = \sum_{n=1}^{\infty} n A_0^{n-1} k^n$$

or

$$F = k + \sum_{n=2}^{\infty} \frac{\{3n-5\}}{n! 4^{n-1}} k^n$$

This power series can be expressed in the closed form

$$F = 2 - 2 \left(1 - \frac{3}{4} k \right)^{2/3}$$

The graph of F against k is a semicubical parabola with the cusp point at $k = \frac{4}{3}$ and $F = 2$. With the necessary condition that one and only one value of k correspond to a given value of F , the transonic similarity parameter k cannot be greater than $4/3$. Moreover, the lower limit of k is zero when the amplitude of the wavy wall is zero but M_{∞} is different from unity.

RECURSION FORMULA FOR A_n^{q-1}

With $p=1$, the recursion formula for A_n^{q-1} is (compare with eq. (69) of ref. 1)

$$4(n+1)A_n^{q-1} - 2(n+2)(q+1)\delta_q^n A_{n+1}^{q-1} + (q+1)(q+2)\delta_q^{n-1} A_{n+2}^{q-1} = -\frac{1}{2}n\delta_q^n \sum_{q_1=0}^q \sum_{m=q-q_1}^{n-1-q_1} m(n-m)A_n^{m-1} A_{q_1}^{n-m-1} - \frac{1}{2}n(n+1)A_0^{1,0} A_n^{q+1,0} \quad (n=1, 2, \dots, \infty; q=0, 1, \dots, n) \quad (33)$$

where

$$\delta_q^{n-1,n} = \begin{cases} 0 & (q=n-1 \text{ or } n) \\ 1 & (q \neq n-1 \text{ or } n) \end{cases}$$

$$\delta_q^n = \begin{cases} 0 & (q=n) \\ 1 & (q \neq n) \end{cases}$$

The solution of this recursion formula proceeds as in the case $p=0$, the solution starting with $q=n$ and descending towards $q=0$. Thus, for $q=n$, equation (33) becomes

$$A_n^{n,1} = -\frac{1}{8}n A_0^{1,0} A_n^{n+1,0} \quad (n=1, 2, \dots, \infty)$$

or

$$A_{n-1}^{n-1,1} = -\frac{1}{8}(n-1) A_0^{1,0} A_n^{n,0} \quad (n=2, 3, \dots, \infty)$$

Hence, inserting the expression for $A_{n-1}^{n,0}$ given by equation (19) gives

$$A_{n-1}^{n-1,1} = -\frac{1}{8} \frac{n^{n-3}}{(n-2)! 4^{n-1}} (A_0^{1,0})^{n+1} \quad (n=2, 3, \dots, \infty) \quad (34)$$

or

$$\sum_{r=0}^{\infty} A_{n-1}^{n-1,1} k^{2r} = -\frac{1}{8} \frac{n^{n-3}}{(n-2)! 4^{n-1}} \left(\sum_{r=0}^{\infty} A_0^{1,0} k^{2r} \right)^{n+1}$$

From this relation, by equating coefficients of equal powers of k on both sides of the equation, the following equations are obtained:

$$\left. \begin{aligned} A_{n-1}^{n-1,1} &= -\frac{1}{8} \frac{n^{n-3}}{(n-2)! 4^{n-1}} \\ A_{n-1}^{n-1,2} &= -\frac{3}{8} \frac{n^{n-3}}{(n-2)! 4^{n-1}} A_0^{1,0} - \frac{1}{8} \frac{n^{n-3}}{(n-3)! 4^{n-1}} A_0^{1,0} \\ A_{n-1}^{n-1,3} &= -\frac{3}{8} \frac{n^{n-3}}{(n-2)! 4^{n-1}} [A_0^{1,0} + (A_0^{1,0})^2] - \\ &\quad \frac{1}{8} \frac{n^{n-3}}{(n-3)! 4^{n-1}} [A_0^{1,0} + 3(A_0^{1,0})^2] - \\ &\quad \frac{1}{16} \frac{n^{n-3}}{(n-4)! 4^{n-1}} (A_0^{1,0})^2 \\ &\dots \dots \dots (n=2, 3, \dots, \infty) \end{aligned} \right\} \quad (35)$$

Consider now $q=n-1$; equation (33) becomes

$$4(n+1)A_n^{n-1,1} = 2n(n+2)A_n^{n,1} - \frac{1}{2}n \sum_{m=0}^{n-1} (m+1)(n-m-1)A_m^{m+1,0} A_{n-m-1}^{n-1,1} - \frac{1}{2}n(n+1)A_0^{1,0} A_n^{n+1,0} \quad (n=1, 2, \dots, \infty)$$

Multiply both sides of this equation by r^n and sum from $n=1$ to $n=\infty$. Then

$$4 \sum_{n=1}^{\infty} (n+1)A_n^{n-1,1} r^n = 2 \sum_{n=1}^{\infty} n(n+2)A_n^{n,1} r^n - \frac{1}{2} \sum_{n=1}^{\infty} n r^n \sum_{m=0}^{n-1} (m+1)(n-m-1)A_m^{m+1,0} A_{n-m-1}^{n-1,1} - \frac{1}{2} A_0^{1,0} \sum_{n=1}^{\infty} n(n+1)A_n^{n+1,0} r^n \quad (36)$$

Let

$$A_0 = \sum_{n=1}^{\infty} A_n^{n,1} r^n \quad A_1 = \sum_{n=1}^{\infty} A_n^{n-1,1} r^n$$

$$A_1 = \sum_{n=1}^{\infty} A_n^{n,0} r^n \quad A_2 = \sum_{n=2}^{\infty} A_n^{n-2,0} r^n$$

Then it can easily be shown that the second term on the right-hand side of equation (36) is $-\frac{1}{2} r(r^2 A_0 A_1)'$ and that equation (36) can be replaced by

$$4(r A_1)' = 2r^2 A_0'' + 6r A_0 A_1' - \frac{1}{2} r(r^2 A_0 A_1)' - \frac{1}{2} r A_0^{1,0} A_2'' \quad (37)$$

Now,

$$A_0 = -\frac{1}{8} A_0^{1,0} \sum_{n=1}^{\infty} (n-1)A_n^{n,0} r^{n-1}$$

or, with the aid of equations (25),

$$r A_0 = -\frac{1}{64} A_0^{1,0} (r A_1)'$$

Hence, from the use of equations (25) and equation (27) for $A_{2,0}'$ it follows, after routine calculations, that the integral of equation (37) is

$$r A_{1,1} = \frac{1}{128} A_{1,0}^{1,0} \left\{ -\frac{1}{2} (r A_{1,0}')^2 - \frac{1}{4} (r A_{1,0}')^3 - \frac{1}{64} (r A_{1,0}')^4 + (c_1 + 4) \left[\frac{1}{4} (r A_{1,0}')^2 - 2r^2 A_{1,0}'' \right] \right\}$$

where

$$c_1 = 64 \frac{A_{1,0}^{2,0}}{(A_{1,0}^{1,0})^2}$$

Now from equation (18),

$$(r A_{1,0}')^m = m \sum_{n=m}^{\infty} \frac{n^{n-m-1}}{(n-m)! 4^{n-m}} (A_{1,0}^{1,0})^n r^n \quad (38)$$

Then

$$\sum_{n=2}^{\infty} A_{n-2,1}^{n-1,1} r^n = \frac{1}{128} (A_{1,0}^{1,0})^{n+1} \left\{ -\sum_{n=2}^{\infty} \frac{n^{n-3}}{(n-2)! 4^{n-2}} r^n - \frac{3}{4} \sum_{n=3}^{\infty} \frac{n^{n-4}}{(n-3)! 4^{n-3}} r^n - \frac{1}{16} \sum_{n=4}^{\infty} \frac{n^{n-5}}{(n-4)! 4^{n-4}} r^n + (c_1 + 4) \left[\frac{1}{2} \sum_{n=2}^{\infty} \frac{n^{n-3}}{(n-2)! 4^{n-2}} r^n - 2 \sum_{n=2}^{\infty} \frac{n^{n-2}}{(n-2)! 4^{n-1}} r^n \right] \right\} \quad (39)$$

Equating coefficients of equal powers of r on both sides of this equation gives

$$A_{n-2,1}^{n-1,1} = -\frac{1}{64} \left\{ 4[2 + (c_1 + 4)] \frac{1}{(n-2)!} + 14[2 + (c_1 + 4)] \frac{1}{(n-3)!} + 2[5 + 4(c_1 + 4)] \frac{1}{(n-4)!} + (c_1 + 4) \frac{1}{(n-5)!} \right\} \frac{n^{n-5}}{4^{n-1}} (A_{1,0}^{1,0})^{n+1} \quad (n=2, 3, \dots, \infty) \quad (40)$$

From this equation the expressions for all the coefficients of the type $A_{n-2,1}^{n-1,1}$ with $n=2, 3, \dots, \infty$ and $r=0, 1, \dots, \infty$ can be obtained. Thus, for example,

$$\left. \begin{aligned} A_{n-2,1}^{n-1,1} &= -\frac{1}{32} \frac{n^{n-5}}{4^{n-1}} \left[\frac{20}{(n-2)!} + \frac{70}{(n-3)!} + \frac{37}{(n-4)!} + \frac{4}{(n-5)!} \right] \\ A_{n-2,2}^{n-1,2} &= -\frac{1}{32} \frac{n^{n-5}}{4^{n-1}} \left\{ [(5n+1) A_{0,1}^{1,0} + 32 A_{0,1}^{2,0}] \frac{4}{(n-2)!} + [(5n+1) A_{0,1}^{1,0} + 32 A_{0,1}^{2,0}] \frac{14}{(n-3)!} + [(37n+5) A_{0,1}^{1,0} + 256 A_{0,1}^{2,0}] \frac{1}{(n-4)!} + (n A_{0,1}^{1,0} + 8 A_{0,1}^{2,0}) \frac{1}{(n-5)!} \right\} \\ &\dots \end{aligned} \right\} \quad (41)$$

For $q=n-2$, equation (33) becomes

$$4(n+1) A_{n-2}^{n-1} = 2(n-1)(n+2) A_{n-1}^{n-1} - n(n-1) A_n^{n-1} - \frac{1}{2} n \sum_{m=0}^{n-2} [(m+2)(n-m-2) A_m^{m+2,0} A_{n-m-2}^{n-m-2,1} + (m+1)(n-m-1) A_m^{m+1,0} A_{n-m-1}^{n-m-1,1}] - \frac{1}{2} n(n+1) A_{0,1}^{1,0} A_{n-2}^{n+1,0} \quad (42)$$

The solution of this recursion formula follows along the same lines as that for equation (36) and leads to the following result:

$$A_{n-3,1}^{n-1,1} = -\frac{1}{512} \left\{ \left[\frac{162}{(n-3)!} + \frac{414}{(n-4)!} + \frac{205}{(n-5)!} + \frac{47}{2} \frac{1}{(n-6)!} \right] \frac{n^{n-7}}{4^{n-2}} + (c_1 + 4) \left[\frac{21}{(n-3)!} + \frac{24}{(n-4)!} + \frac{9}{2} \frac{1}{(n-5)!} \right] \frac{n^{n-5}}{4^{n-2}} + 12 \times 128 \frac{A_{0,1}^{3,0}}{(A_{1,0}^{1,0})^3} \left[\frac{1}{(n-3)!} + \frac{1}{2} \frac{1}{(n-4)!} \right] \frac{n^{n-4}}{4^{n-2}} + \frac{1}{8} (c_1 + 4)^2 \left[\frac{3}{(n-4)!} + \frac{1}{(n-5)!} \right] \frac{n^{n-4}}{4^{n-2}} \right\} (A_{1,0}^{1,0})^{n+1} \quad (n=3, 4, \dots, \infty) \quad (43)$$

From this equation there follow, in the usual manner, expressions for coefficients of the type $A_{n-3,1}^{n-1,1}$. Thus, for example, with $r=0$,

$$A_{n-3,1}^{n-1,1} = -\left[\left(\frac{81}{64} + \frac{21}{16} n^2 + \frac{1}{6} n^3 \right) \frac{1}{(n-3)!} + \left(\frac{207}{64} + \frac{3}{2} n^2 + \frac{13}{48} n^3 \right) \frac{1}{(n-4)!} + \left(\frac{205}{128} + \frac{9}{32} n^2 + \frac{1}{16} n^3 \right) \frac{1}{(n-5)!} + \frac{47}{256} \frac{1}{(n-6)!} \right] \frac{n^{n-7}}{4^{n-1}} \quad (44)$$

RECURSION FORMULA FOR A_n^q

With $p=2$, the recursion formula for A_n^q is

$$\begin{aligned}
 & 8(n+2)A_n^{q+2} - 2(n+4)(q+1)\delta_n^{q+2} A_{n+1}^{q+2} + \\
 & (q+1)(q+2)\delta_n^{q+1, n+2} A_{n+2}^{q+2} \\
 & = -\frac{1}{2} n \delta_n^{q+2} \delta_{n-1, n+1}^{q+1} \delta_{n-2, n+2}^{q+2} \sum_{q_1=0}^q \sum_{m=q-q_1}^{n+1-q_1} (m-2)(n-m+1) \\
 & \quad 2) A_{n-m+2}^{q+2} A_{n-q_1}^{m-2} - \frac{1}{4} n \delta_n^{q+1, n+2} \delta_{n-1}^q \sum_{q_1=0}^q \sum_{m=q-q_1}^{n-1} m(n-m) \\
 & \quad A_{n-m+1}^{q+1} A_{n-q_1}^{m-1} - \frac{1}{2} n(n+1)(\delta_n^{q+2} A_{n+1}^{q+2} A_{n+1}^{q+1} + \\
 & \quad \delta_n^{q+1, n+2} A_{n+1}^{q+1} A_{n+1}^{q+2} + \delta_n^{q+2} A_{n+1}^{q+2} A_{n+1}^{q+1}) - \\
 & \quad n(n+2)(\delta_n^{q+2} A_{n+1}^{q+2} A_{n+1}^{q+2} + \delta_n^{q+2} A_{n+1}^{q+2} A_{n+1}^{q+2}) \\
 & \quad (n=1, 2, \dots, \infty; q=0, 1, \dots, n+2) \quad (45)
 \end{aligned}$$

where δ is defined in the usual manner. The solution of this recursion formula proceeds as in the previous cases, starting with $q=n+2$ and descending towards $q=0$. Thus, for $q=n+2$, equation (45) becomes:

$$A_{n+2}^{n+2} = -\frac{1}{8} n A_{n+1}^{n+2} A_{n+1}^{n+2} \quad (n=1, 2, \dots, \infty)$$

or

$$A_{n+2}^{n+2} = -\frac{1}{8} (n-2) A_{n+1}^{n+2} A_{n+1}^{n+2} \quad (n=3, 4, \dots, \infty)$$

Then, from equation (19),

$$A_{n+2}^{n+2} = -\frac{1}{64} (n-2) \frac{n^{n-2}}{n! 4^{n-1}} (A_{n+1}^{n+2})^{n+2} \quad (n=3, 4, \dots, \infty) \quad (46)$$

or

$$\sum_{r=0}^{\infty} A_{n+2}^{n+2} k^{2r} = -\frac{(n-2)n^{n-2}}{n! 4^{n-1}} \left(\sum_{r=0}^{\infty} A_{n+1}^{n+2} k^{2r} \right)^{n+2}$$

By equating coefficients of equal powers of k on both sides of this equation, the following equations are obtained:

$$\left. \begin{aligned}
 A_{n+2}^{n+2} &= -\frac{(n-2)n^{n-2}}{n! 4^{n-1}} \\
 A_{n+2}^{n+2} &= -\frac{(n^2-4)n^{n-2}}{n! 4^{n-1}} A_{n+1}^{n+2} \\
 A_{n+2}^{n+2} &= -\frac{(n^2-4)n^{n-2}}{n! 4^{n-1}} \left[A_{n+1}^{n+2} + \frac{1}{2} (n+1)(A_{n+1}^{n+2})^2 \right] \\
 &\dots
 \end{aligned} \right\} \quad (47)$$

Consider now $q=n+1$; equation (45) becomes

$$\begin{aligned}
 & 8(n+2)A_{n+1}^{n+1} = 2(n+2)(n+4)A_{n+2}^{n+1} - \\
 & \frac{1}{2} n \sum_{m=0}^{n-2} (m+1)(n-m-1) A_{n-m+1}^{m+1} A_{n-m+1}^{n-m-1} - \\
 & \frac{1}{2} n(n+1)(A_{n+1}^{n+1} A_{n+1}^{n+1} + A_{n+1}^{n+1} A_{n+1}^{n+1}) - \\
 & n(n+2)(A_{n+1}^{n+1} A_{n+1}^{n+1} + A_{n+1}^{n+1} A_{n+1}^{n+1}) \\
 & (n=1, 2, \dots, \infty) \quad (48)
 \end{aligned}$$

With the introduction of

$$\begin{aligned}
 A_{-2} &= \sum_{n=1}^{\infty} A_{n+2}^{n+2} r^n & A_{20} &= \sum_{n=2}^{\infty} A_{n-2}^{n-2} r^n \\
 A_{-1} &= \sum_{n=1}^{\infty} A_{n+1}^{n+1} r^n & A_{01} &= \sum_{n=1}^{\infty} A_n^{n+1} r^n \\
 A_{10} &= \sum_{n=1}^{\infty} A_{n-1}^{n-1} r^n & \dots
 \end{aligned}$$

equation (48) is replaced by the following ordinary linear differential equation:

$$\begin{aligned}
 8(r^2 A_{-1})' &= 2r^2 (r A_{-2})' + 12r (r A_{-2})' + 16r A_{-2} - \\
 & \frac{1}{2} r^2 (r^2 A_{10}' A_{-2}) - \frac{1}{2} A_{10}^2 r^2 A_{01}'' - \frac{1}{2} A_{11}^2 r^2 A_{10}'' - \\
 & A_{20}^2 r A_{10}'' + A_{20}^2 A_{10}' - A_{10}^2 A_{20}'' + \\
 & A_{10}^2 A_{20}' - A_{10}^2 A_{20}^2 \quad (49)
 \end{aligned}$$

By repeated use of equations (25), the solution of this differential equation is found to be as follows:

$$\begin{aligned}
 8r^2 A_{-1} &= (A_{10}^2)' \left\{ \frac{3}{32} (r A_{10}') - \frac{1}{32} (r A_{10}')^2 - \frac{1}{768} (r A_{10}')^3 - \right. \\
 & \frac{1}{2048} (r A_{10}')^4 + \frac{1}{16} r^2 A_{10}'' + \frac{1}{8} A_{20} + \\
 & \left. \frac{1}{64} (c_1 + 4) \left[\frac{1}{8} (r A_{10}')^2 - r^2 A_{10}'' \right] \right\} - \frac{3}{32} r (A_{10}^2)' + \\
 & A_{20}^2 \left[A_{10} - \frac{1}{8} (r A_{10}')^2 \right] - r A_{10}^2 A_{20}^2 \quad (50)
 \end{aligned}$$

Finally, with the aid of equation (38) and the definition of A_{-1} ,

$$\begin{aligned}
 A_{n+1}^{n+1} &= \left\{ \frac{1}{256} \left[\frac{1}{(n-1)!} - \frac{1}{(n-2)!} + \frac{2}{(n-3)!} \right] \frac{n^{n-3}}{4^{n-1}} - \right. \\
 & \frac{1}{64} \frac{n^{n-5}}{(n-4)! 4^{n-1}} + \frac{1}{512} (c_1 + 4) \left[\frac{1}{(n-1)!} - \right. \\
 & \left. \left. \frac{1}{(n-2)!} \right] \frac{n^{n-2}}{4^{n-1}} \right\} (A_{10}^2)^{n+2} \\
 & \quad \left(n=3, 4, \dots, \infty; c_1 = 64 \frac{A_{20}^2}{(A_{10}^2)^2} \right) \quad (51)
 \end{aligned}$$

From this equation, there follows in the usual manner formulas for the coefficients of the type A_{n+1}^{n+1} . Thus, for $r=0$,

$$\begin{aligned}
 A_{n+1}^{n+1} &= \left[\frac{1}{(n-1)!} - \frac{1}{(n-2)!} + \frac{2}{(n-3)!} \right] \frac{n^{n-3}}{4^{n+3}} - \frac{n^{n-5}}{(n-4)! 4^{n+2}} + \\
 & \left[\frac{1}{(n-1)!} - \frac{1}{(n-2)!} \right] \frac{n^{n-2}}{4^{n+2}} \quad (n=3, 4, \dots, \infty)
 \end{aligned}$$

DISCUSSION OF CONVERGENCE OF SMOOTH TYPE OF POTENTIAL FLOW PAST WAVY WALL

In the preceding sections, a number of examples of recursion formulas and their solutions have been given in considerable detail. The purpose of this exposition is threefold:

First, to show the inherent elegance of the method of integration in series although the equations concerned are nonlinear in character; second, to present a type of analysis which may be useful in other problems involving nonlinear differential equations; and third, to indicate that an analytical proof of convergence may ultimately be obtained by careful examination of the recursion formulas for the quantities A_n^p and their solutions. One example is the obtaining of the general expression for the coefficients A_1^0 and the subsequent conclusion that $k \leq \frac{4}{3}$.

In actual practice it has been found more convenient to evaluate the coefficients A_n^p from their separate recursion formulas rather than to derive the general formulas. The appendix contains the exact numerical values of the coefficients necessary for the development of the functions f_n to the eighth power in the transonic similarity parameter k . These coefficients are utilized to demonstrate numerically the test for convergence of the smooth symmetrical type of potential flow (eq. (3)) assumed in this report. Thus, when the form of f_n given by equation (8) is inserted into equation (5) for the local Mach number distribution and evaluated at the surface of the sinusoidal wall ($y=0$), the following result is obtained:

$$\frac{M^2-1}{1-M_\infty^2} = -1 + \sum_{n=1}^{\infty} k^n \sum_{m=0}^{\left[\frac{n}{2}\right]} (n-2m) \cos(n-2m)x \sum_{p=0}^m A_n^{-2m} p \quad (52)$$

where $\left[\frac{n}{2}\right]$ denotes the integral part of $n/2$. At the crest of the wavy wall ($x=0$), the point of maximum fluid velocity, with the numerical values for the coefficients inserted, equation (52) becomes:

$$\begin{aligned} \frac{M^2-1}{1-M_\infty^2} + 1 = & k + \frac{1}{8} k^2 + \frac{25}{384} k^3 + \frac{337}{9216} k^4 + \frac{4043}{576 \times 256} k^5 + \\ & \frac{359381}{270 \times 256^2} k^6 + \frac{7326757}{6480 \times 256^2} k^7 + \\ & \frac{81688733}{86400 \times 256^2} k^8 + \dots \end{aligned} \quad (53)$$

The critical value of k (that is, when $M=1$) calculated from this equation is

$$k_{cr} = 0.83244$$

(Note that in ref. 1 the value $k_{cr} = 0.83770$ corresponds to the first six terms of eq. (53).) Consider now the infinite series

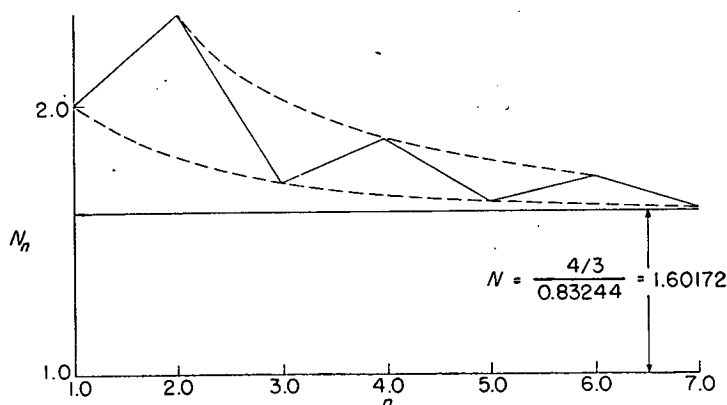


FIGURE 2.—Numerical test of convergence.

$$\sum_{n=1}^{\infty} A_n^0 k^n$$

where

$$A_n^0 = \frac{\{3n-5\}}{nn! 4^{n-1}} \quad (n=2, 3, \dots, \infty)$$

The Cauchy ratio test $R_{1n} = \frac{A_{n+1}^0}{A_n^0}$ yields in the limit $n \rightarrow \infty$ the result that the radius of convergence R_1 is equal to or less than $4/3$. If the corresponding ratio R_{2n} are formed for the right-hand side of equation (53) and the quotient $N_n = \frac{R_{1n}}{R_{2n}}$ is calculated, the resulting sequence of numbers is as follows:

n	R_{1n}	R_{2n}	N_n
1	16.0	8.0	2.0
2	4.5	1.92	2.34375
3	3.04762	1.78041	1.71175
4	2.5	1.33366	1.87472
5	2.21546	1.35000	1.64104
6	2.04167	1.17721	1.73432
7	1.92481	1.19588	1.60950

The noteworthy feature of this table is that, although R_{1n} (and presumably R_{2n}) is converging quite slowly toward $R_1 = \frac{4}{3}$ (and R_2), the quotient N_n exhibits a strong tendency to approach an asymptotic value N for a relatively low value of n . Figure 2 shows this tendency in a graphic manner. The apparent asymptote represented by the straight line is the ratio of $4/3$, the limit of R_{1n} as $n \rightarrow \infty$, and of 0.83244 , the critical value of k . Certainly, the rapid approach of the lower dotted curve toward the apparent asymptote and the decreasing oscillations represented by the upper dotted curve indicate that the critical value of k is the radius of convergence R_2 of the power series on the right-hand side of equation (53). Thus, the critical value of the stream Mach number marks the limit of convergence of the smooth symmetrical type of potential flow assumed. The ability to approximate closely the limiting value N is a matter of luck; namely, the choice of the known comparison series. Once, however, the proper comparison series has been selected and the approach to an asymptote indicated, there can be no question of the meaningfulness of the approximate value of N obtained. It may be that one would like to extend figure 2 to $n=9$. This extension would entail the forbidding calculation of an additional 185 coefficients A_n^p . The result presumably would be to decrease slightly the approximate critical value of k and thereby raise slightly the straight-line asymptote of figure 2. This extension of figure 2 would show still more convincingly the approach to an asymptote and the dying-out of the oscillations. Perhaps more important still, figure 2 definitely shows that conclusions based on less than six or eight terms are mere speculations in this field.

CONCLUDING REMARKS

If the numerical test of convergence presented is acceptable, the conclusion to be drawn is that smooth symmetrical potential flow past the wavy wall exists only for the purely subsonic range. Moreover no such flow can possibly represent the transonic or mixed type for which a local region of supersonic flow near the solid boundary is imbedded in the

otherwise subsonic stream. It follows as a corollary that the transonic or mixed type of flow past the wavy wall is necessarily an asymmetric one. This asymmetry in the flow pattern entails a resistance usually defined as wave drag. As shown by experimental observations, the shock wave associated with wave drag closes the downstream portion of the local supersonic zone.

As a final remark—although the analysis and conclusions of the present work refer directly to the wavy wall, the

suggested result that the critical stream Mach number marks the limit of smooth potential flow very likely applies to other boundaries. This conclusion is based on the idea that the gradual transition from a purely sinusoidal wall to a boundary composed of a single bump, say, introduces no essential changes in the analysis.

LANGLEY AERONAUTICAL LABORATORY,

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,

LANGLEY FIELD, VA., May 6, 1952.

APPENDIX

NUMERICAL VALUES OF THE COEFFICIENTS $A_{\alpha}^{\beta\gamma}$

Coefficients for f_1 :

$$A_{00}^{10} = 1$$

$$A_{01}^{11} = -\frac{5}{256}$$

$$A_{02}^{12} = \frac{65}{576 \times 256}$$

$$A_{03}^{13} = \frac{3385}{1152 \times 256^2}$$

$$A_{01}^{10} = \frac{11}{256}$$

$$A_{02}^{11} = -\frac{2765}{9 \times 256^2}$$

$$A_{03}^{12} = \frac{26435}{81 \times 256^3}$$

$$A_{13}^{13} = \frac{3907}{432 \times 256^2}$$

$$A_{02}^{10} = \frac{1861}{3 \times 256^2}$$

$$A_{03}^{11} = -\frac{184345}{1728 \times 256^2}$$

$$A_{12}^{12} = -\frac{7}{72 \times 256}$$

$$A_{23}^{13} = \frac{23}{12 \times 256^2}$$

$$A_{03}^{10} = \frac{4896755}{81 \times 256^3}$$

$$A_{11}^{11} = -\frac{1}{64}$$

$$A_{13}^{12} = -\frac{53995}{1728 \times 256^2}$$

$$A_{33}^{13} = -\frac{583}{54 \times 256^2}$$

$$A_{12}^{11} = -\frac{33}{64 \times 256}$$

$$A_{22}^{12} = -\frac{9}{32 \times 256}$$

$$A_{43}^{13} = -\frac{119}{12 \times 256^2}$$

$$A_{13}^{11} = -\frac{139}{4 \times 256^2}$$

$$A_{23}^{12} = -\frac{2765}{96 \times 256^2}$$

$$A_{53}^{13} = -\frac{1}{96 \times 256}$$

$$A_{32}^{12} = -\frac{1}{8 \times 256}$$

$$A_{33}^{12} = -\frac{55}{8 \times 256^2}$$

Coefficients for f_2 :

$$A_{00}^{20} = \frac{1}{16}$$

$$A_{01}^{21} = -\frac{125}{36 \times 256}$$

$$A_{02}^{22} = \frac{12245}{144 \times 256^2}$$

$$A_{03}^{23} = -\frac{17999}{90 \times 256^3}$$

$$A_{01}^{20} = \frac{419}{72 \times 256}$$

$$A_{02}^{21} = -\frac{45895}{108 \times 256^2}$$

$$A_{03}^{22} = \frac{35023583}{1620 \times 256^3}$$

$$A_{13}^{23} = \frac{72577}{5760 \times 256^2}$$

$$A_{02}^{20} = \frac{234215}{432 \times 256^2}$$

$$A_{03}^{21} = -\frac{4210409}{81 \times 256^3}$$

$$A_{12}^{22} = \frac{6245}{72 \times 256^2}$$

$$A_{23}^{23} = \frac{45409}{2160 \times 256^2}$$

$$A_{03}^{20} = \frac{42791533}{648 \times 256^3}$$

$$A_{11}^{21} = -\frac{5}{256}$$

$$A_{13}^{22} = \frac{104063}{90 \times 256^3}$$

$$A_{33}^{23} = \frac{3515}{864 \times 256^2}$$

$$A_{10}^{20} = \frac{1}{8}$$

$$A_{12}^{21} = -\frac{815}{576 \times 256}$$

$$A_{22}^{22} = -\frac{79}{3 \times 256^2}$$

$$A_{43}^{23} = -\frac{10229}{864 \times 256^2}$$

$$A_{11}^{20} = \frac{11}{1024}$$

$$A_{13}^{21} = -\frac{456185}{3456 \times 256^2}$$

$$A_{23}^{22} = -\frac{167}{4 \times 256^2}$$

$$A_{53}^{23} = -\frac{73}{8 \times 256^2}$$

$$A_{12}^{20} = \frac{4085}{24 \times 256^2}$$

$$A_{22}^{21} = -\frac{1}{128}$$

$$A_{32}^{22} = -\frac{47}{192 \times 256}$$

$$A_{63}^{23} = -\frac{25}{12 \times 256^2}$$

$$A_{13}^{20} = \frac{21287}{324 \times 256^2}$$

$$A_{23}^{21} = -\frac{11}{32 \times 256}$$

$$A_{33}^{22} = -\frac{24199}{864 \times 256^2}$$

$$A_{23}^{21} = -\frac{4811}{192 \times 256^2}$$

$$A_{42}^{22} = -\frac{1}{12 \times 256}$$

$$A_{43}^{22} = -\frac{11}{2 \times 256^2}$$

Coefficients for f_3 :

$$A_{00}^{30} = \frac{1}{72}$$

$$A_{01}^{31} = -\frac{1765}{3 \times 256^2}$$

$$A_{02}^{32} = \frac{475823}{15 \times 256^3}$$

$$A_{01}^{30} = \frac{23603}{27 \times 256^2}$$

$$A_{02}^{31} = -\frac{1351393}{2880 \times 256^2}$$

$$A_{12}^{32} = \frac{74353}{320 \times 256^2}$$

$$A_{02}^{30} = \frac{9843883}{81 \times 256^3}$$

$$A_{11}^{31} = -\frac{155}{32 \times 256}$$

$$A_{22}^{32} = \frac{3083}{24 \times 256^2}$$

$$A_{10}^{30} = \frac{1}{24}$$

$$A_{12}^{31} = -\frac{30365}{48 \times 256^2}$$

$$A_{32}^{32} = -\frac{41}{2 \times 256^2}$$

$$A_{11}^{30} = \frac{259}{72 \times 256}$$

$$A_{21}^{31} = -\frac{117}{32 \times 256}$$

$$A_{42}^{32} = -\frac{1785}{40 \times 256^2}$$

$$A_{12}^{30} = \frac{149317}{432 \times 256^2}$$

$$A_{22}^{31} = -\frac{29545}{96 \times 256^2}$$

$$A_{52}^{32} = -\frac{25}{2 \times 256^2}$$

$$A_{20}^{30} = \frac{1}{32}$$

$$A_{31}^{31} = -\frac{1}{256}$$

$$A_{21}^{30} = \frac{33}{32 \times 256}$$

$$A_{32}^{31} = -\frac{55}{256^2}$$

$$A_{22}^{30} = \frac{139}{2 \times 256^2}$$

Coefficients for f_4 :

$$A_{00}^{40} = \frac{7}{6 \times 256}$$

$$A_{01}^{41} = -\frac{35251}{90 \times 256^2}$$

$$A_{02}^{42} = \frac{741115}{5184 \times 256^2}$$

$$A_{01}^{40} = \frac{390547}{720 \times 256^2}$$

$$A_{02}^{41} = -\frac{80872738}{675 \times 256^3}$$

$$A_{12}^{42} = \frac{38381086}{405 \times 256^3}$$

$$A_{02}^{40} = \frac{1704729613}{16200 \times 256^3}$$

$$A_{11}^{41} = -\frac{2465}{576 \times 256}$$

$$A_{22}^{42} = \frac{423833}{1152 \times 256^2}$$

$$A_{10}^{40} = \frac{7}{384}$$

$$A_{12}^{41} = -\frac{30596538}{135 \times 256^3}$$

$$A_{32}^{42} = \frac{20845}{144 \times 256^2}$$

$$A_{11}^{40} = \frac{32947}{36 \times 256^2}$$

$$A_{21}^{41} = -\frac{455}{96 \times 256}$$

$$A_{42}^{42} = -\frac{173}{12 \times 256^2}$$

$$A_{12}^{40} = \frac{20527213}{162 \times 256^3}$$

$$A_{22}^{41} = -\frac{127825}{192 \times 256^2}$$

$$A_{52}^{42} = -\frac{30}{256^2}$$

$$A_{20}^{40} = \frac{3}{128}$$

$$A_{31}^{41} = -\frac{119}{48 \times 256}$$

$$A_{62}^{42} = -\frac{36}{5 \times 256^2}$$

$$A_{21}^{40} = \frac{617}{288 \times 256}$$

$$A_{32}^{41} = -\frac{51343}{216 \times 256^2}$$

$$A_{22}^{40} = \frac{369587}{1728 \times 256^2}$$

$$A_{41}^{41} = -\frac{25}{48 \times 256}$$

$$A_{30}^{40} = \frac{1}{96}$$

$$A_{42}^{41} = -\frac{275}{8 \times 256^2}$$

$$A_{31}^{40} = \frac{11}{24 \times 256}$$

$$A_{32}^{40} = \frac{4811}{144 \times 256^2}$$

Coefficients for f_3 :

$$A_{00}^{50} = \frac{7}{3840}$$

$$A_{01}^{50} = \frac{90354659}{259200 \times 256^2}$$

$$A_{10}^{50} = \frac{7}{768}$$

$$A_{11}^{50} = \frac{886243}{1080 \times 256^2}$$

$$A_{20}^{50} = \frac{25}{1536}$$

$$A_{21}^{50} = \frac{24375}{32 \times 256^2}$$

$$A_{30}^{50} = \frac{5}{384}$$

$$A_{31}^{50} = \frac{8950}{27 \times 256^2}$$

$$A_{40}^{50} = \frac{25}{24 \times 256}$$

$$A_{41}^{50} = \frac{1375}{24 \times 256^2}$$

Coefficients for f_6 :

$$A_{00}^{60} = \frac{91}{432 \times 256}$$

$$A_{01}^{60} = \frac{285522407}{4860 \times 256^3}$$

$$A_{10}^{60} = \frac{91}{72 \times 256}$$

$$A_{11}^{60} = \frac{144862403}{810 \times 256^3}$$

$$A_{20}^{60} = \frac{23}{8 \times 256}$$

$$A_{21}^{60} = \frac{2553091}{2880 \times 256^2}$$

$$A_{30}^{60} = \frac{13}{1024}$$

$$A_{31}^{60} = \frac{56387}{96 \times 256^2}$$

$$A_{40}^{60} = \frac{15}{8 \times 256}$$

$$A_{41}^{60} = \frac{815}{4 \times 256^2}$$

$$A_{50}^{60} = \frac{9}{20 \times 256}$$

$$A_{51}^{60} = \frac{297}{10 \times 256^2}$$

$$A_{01}^{51} = -\frac{5470394}{81 \times 256^3}$$

$$A_{11}^{51} = -\frac{88735}{96 \times 256^2}$$

$$A_{21}^{51} = -\frac{25 \times 15587}{288 \times 256^2}$$

$$A_{31}^{51} = -\frac{25 \times 997}{24 \times 256^2}$$

$$A_{41}^{51} = -\frac{105}{64 \times 256}$$

$$A_{51}^{51} = -\frac{9}{32 \times 256}$$

$$A_{01}^{61} = -\frac{2483629}{54 \times 256^3}$$

$$A_{11}^{61} = -\frac{4371541}{5760 \times 256^2}$$

$$A_{21}^{61} = -\frac{265901}{192 \times 256^2}$$

$$A_{31}^{61} = -\frac{403535}{288 \times 256^2}$$

$$A_{41}^{61} = -\frac{79841}{96 \times 256^2}$$

$$A_{51}^{61} = -\frac{2205}{8 \times 256^2}$$

$$A_{61}^{61} = -\frac{2401}{60 \times 256^2}$$

Coefficients for f_7 :

$$A_{00}^{70} = \frac{13}{126 \times 256}$$

$$A_{10}^{70} = \frac{13}{18 \times 256}$$

$$A_{20}^{70} = \frac{287}{144 \times 256}$$

$$A_{30}^{70} = \frac{833}{288 \times 256}$$

$$A_{40}^{70} = \frac{343}{144 \times 256}$$

$$A_{50}^{70} = \frac{343}{320 \times 256}$$

$$A_{60}^{70} = \frac{2401}{45 \times 256^2}$$

Coefficients for f_8 :

$$A_{00}^{80} = \frac{13 \times 19}{18 \times 256^2}$$

$$A_{10}^{80} = \frac{13 \times 19}{576 \times 256}$$

$$A_{20}^{80} = \frac{89}{64 \times 256}$$

$$A_{30}^{80} = \frac{59}{24 \times 256}$$

$$A_{40}^{80} = \frac{47}{18 \times 256}$$

$$A_{50}^{80} = \frac{19}{2880}$$

$$A_{60}^{80} = \frac{7}{2880}$$

$$A_{70}^{80} = \frac{1}{2520}$$

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REPORT 1150

CONSIDERATIONS ON THE EFFECT OF WIND-TUNNEL WALLS ON OSCILLATING AIR FORCES FOR TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOW¹

By HARRY L. RUNYAN and CHARLES E. WATKINS

SUMMARY

This report treats the effect of wind-tunnel walls on the oscillating two-dimensional air forces in a compressible medium. The walls are simulated by the usual method of placing images at appropriate distances above and below the wing. An important result shown is that, for certain conditions of wing frequency, tunnel height, and Mach number, the tunnel and wing may form a resonant system so that the forces on the wing are greatly changed from the condition of no tunnel walls. It is pointed out that similar conditions exist for three-dimensional flow in circular and rectangular tunnels and apparently, within certain Mach number ranges, in tunnels of nonuniform cross section or even in open tunnels or jets.

INTRODUCTION

The understanding of flutter and other nonsteady phenomena requires a knowledge of the associated unsteady flow. In the underlying theories of unsteady flow, such assumptions as small displacements, linearizations, and an inviscid fluid are made in order to obtain workable and usable results. When it is necessary to investigate the effect of these assumptions on analytical results by measurements of the forces and moments on an oscillating wing in a wind tunnel or to treat cases that do not conform to theory, the question of the effect of the tunnel walls naturally arises. In the case of steady flow the problem of the effect of tunnel walls is more or less classic and has been treated by many investigators. In general, these investigators have been able to obtain relatively simple factors which can be used to modify measurements of the air forces on a wing in a tunnel to correspond to free-air conditions. The extension of the results to compressible flow presents no difficulties since the results for incompressible flow can be corrected according to Prandtl-Glauert correction factors.

In the case of unsteady flow, Reissner, reference 1, and W. P. Jones, reference 2, have published papers showing the effect of wind-tunnel walls for the incompressible case. In both papers, the influence of the tunnel walls is found to be comparatively small for most cases, although indications are given that, for some ranges of a reduced-frequency parameter, the effect may be quite large. In the unsteady case, unlike the steady case, the transition from results for incompressible flow to those for compressible flow cannot be accomplished by simple transformations. This difficulty is a result

of the fact that, in an incompressible fluid, the velocity of propagation of a disturbance is infinite and no time lag occurs between the initiation of a disturbance and its effect at another position in the field, but, in a compressible fluid, a definite time is required for a signal to reach a distant field point so that both a phase lag and a change in magnitude result. Under certain conditions this phase lag can result in a resonant condition which would involve large corrections.

The purpose of this report is to consider the effect of wind-tunnel walls on the forces on an oscillating airfoil of infinite span with considerations of the compressibility of the fluid. The usual method of images is employed in order to satisfy the condition of no normal velocity at the tunnel walls. First, the effect of tunnel walls on the induced vertical velocity, hereinafter referred to as downwash, of an oscillating doublet is determined and this result is used to formulate the integral equation for the downwash of an oscillating airfoil in a tunnel. This report is not intended to give numerical values or any detailed calculations of final tunnel-wall correction factors but mainly to show the existing need for such calculations and to present equations for calculating corrections for the two-dimensional case.

SYMBOLS

A	constant
b	semichord
c	velocity of sound
H	tunnel height
$H_0^{(2)}, H_1^{(2)}$	Hankel functions
M	Mach number
Δp	local pressure difference
t	time
$u = \frac{\omega \xi}{V \beta^2}$	
V	velocity
w, w_0, w_1	downwash or vertical induced velocity
ξ, x, y, z	Cartesian coordinates
$\beta = \sqrt{1 - M^2}$	
γ	Euler's constant
ω	angular frequency
λ	wave length
ψ	acceleration potential
φ	velocity potential
ρ	fluid density

¹ Supersedes NACA TN 2532, "Considerations on the Effect of Wind-Tunnel Walls on Oscillating Air Forces for Two-Dimensional Subsonic Compressible Flow" by Harry L. Runyan and Charles E. Watkins, 1951.

ANALYSIS

EFFECT OF TUNNEL WALLS ON THE DOWNWASH OF A SINGLE DOUBLET

The differential equation that governs flow due to small nonsteady perturbations imposed on a steady, uniform flow field is the wave equation. Referred to rectangular coordinates, fixed relative to the undisturbed stream at infinity, this equation is

$$(1-M^2) \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{2M}{c} \frac{\partial^2 \psi}{\partial x \partial t} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (1)$$

In this equation the independent variable ψ may be regarded as either a perturbation velocity potential or as an acceleration potential. In treating the boundary conditions of the second section of this analysis it is convenient to regard ψ as an acceleration potential. Thus, in order to be consistent, ψ is hereinafter regarded as an acceleration potential. Accordingly ψ is directly proportional to a perturbation pressure field and is therefore related to a perturbation velocity potential φ as follows:

$$\psi = \frac{\partial \varphi}{\partial t} + V \frac{\partial \varphi}{\partial x} \quad (2)$$

In order to calculate the downwash $w = \frac{\partial \varphi}{\partial y}$ associated with ψ , it is necessary to solve equation (2) for φ in terms of ψ .

When ψ and φ are sinusoidal functions of time, such that

$$\left. \begin{aligned} \psi(x, y, t) &= e^{i\omega t} \bar{\psi}(x, y) \\ \varphi(x, y, t) &= e^{i\omega t} \bar{\varphi}(x, y) \end{aligned} \right\} \quad (3)$$

equation (2) becomes independent of time and thus reduces to an equation with one independent variable, namely

$$\bar{\psi} = i\omega \bar{\varphi} + V \frac{\partial \bar{\varphi}}{\partial x} \quad (4)$$

This equation can be integrated with respect to x to give

$$\bar{\varphi} = \frac{e^{-\frac{i\omega}{V}x}}{V} \int_{-\infty}^x \bar{\psi}(\xi, y) e^{\frac{i\omega}{V}\xi} d\xi \quad (5)$$

where the lower limit of integration is chosen for later convenience so that φ vanishes far ahead of the point of disturbance. The downwash may be readily calculated with the use of this equation. In the absence of tunnel walls the retarded potential ψ_0 (that is, the potential corresponding to outgoing waves) of a harmonically pulsating pressure doublet located, for simplicity, at $(0, 0)$ that satisfies equation (1) is

$$\begin{aligned} \psi_0 &= -A\beta e^{i\omega \left(t + \frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \frac{e^{-\frac{i\omega}{c\beta^2} \sqrt{x^2 + \beta^2 y^2 + \beta^2 z^2}}}{\sqrt{x^2 + \beta^2 y^2 + \beta^2 z^2}} dz \\ &= A\pi i e^{i\omega \left(t + \frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2 y^2} \right) \\ &= -\frac{i\omega A\pi}{c} e^{i\omega \left(t + \frac{Mx}{c\beta^2}\right)} \frac{y}{\sqrt{x^2 + \beta^2 y^2}} H_1^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2 y^2} \right) \quad (6) \end{aligned}$$

where $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel functions as defined in reference 3, A is an arbitrary constant denoting doublet strength, ω is circular frequency, and $\beta = \sqrt{1-M^2}$. The Hankel function $H_1^{(2)}$ in equation (6) becomes infinite (as $\frac{1}{\sqrt{x^2 + \beta^2 y^2}}$) as its argument approaches zero. Otherwise $H_1^{(2)}$ is continuous and approaches zero as its argument approaches infinity. Thus the only discontinuity in ψ_0 is at the location of the doublet, that is, at $(x=0, y=0)$.

In the presence of plane tunnel walls located parallel to the x -axis at $H/2$ units above and $H/2$ units below the doublet position, the potential ψ of a pressure doublet may be represented by the potential of an infinite system of appropriately chosen reflecting doublets, namely (see fig. 1)

$$\psi = A\pi i e^{i\omega \left(t + \frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y} \sum_{n=-\infty}^{\infty} (-1)^n H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2 (y - nH)^2} \right] \quad (7)$$

In this equation the term corresponding to $n=0$ is the potential ψ_0 , equation (6), discussed in the preceding paragraph. It may be noted that only this term of the infinite summation in equation (7) gives rise to a discontinuity in ψ at any point within the tunnel $\left(-\frac{H}{2} \leq y \leq \frac{H}{2}, -\infty < x < \infty\right)$. The infinity of terms corresponding to $n \neq 0$ is necessary to cause the downwash w to vanish at all points of the tunnel walls, $y = \pm \frac{H}{2}$.

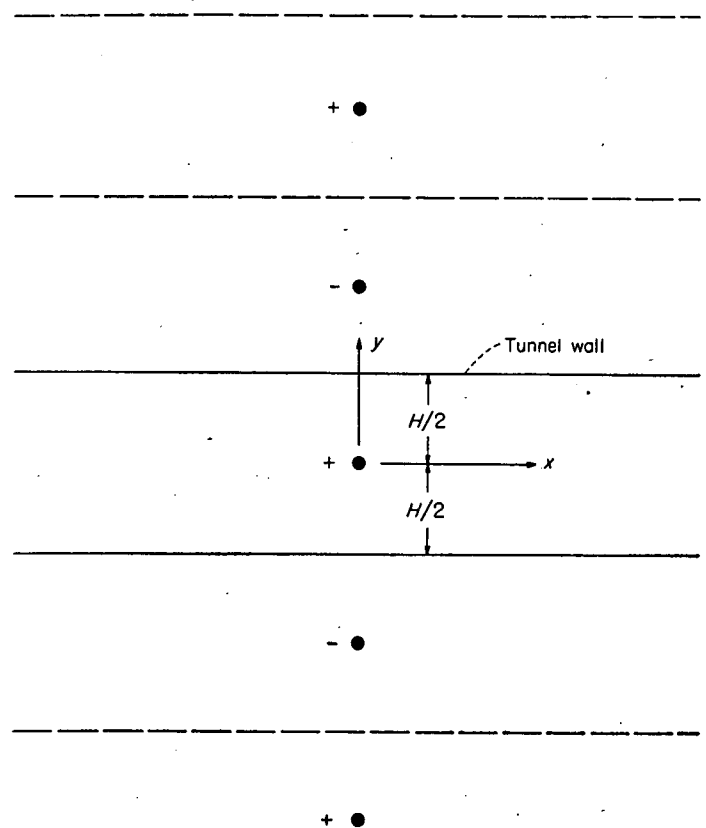


FIGURE 1.—Sketch showing reflecting system of doublets simulating two-dimensional tunnel walls.

The downwash along the midsection of the tunnel $y=0$ is given by

$$w = \frac{A\pi i}{V} e^{i\omega(t - \frac{z}{V})} \lim_{y \rightarrow 0} \int_{-\infty}^x \sum_{n=-\infty}^{\infty} (-1)^n e^{\frac{i\omega \xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2(y-nH)^2} \right] d\xi = w_0 + w_1 \quad (8)$$

where

$$w_0 = \frac{A\pi i}{V} e^{i\omega(t - \frac{z}{V})} \lim_{y \rightarrow 0} \int_{-\infty}^x e^{\frac{i\omega \xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) d\xi \quad (9)$$

represents the downwash associated with the pressure doublet in the absence of tunnel walls and

$$w_1 = \frac{2A\beta\pi i}{V} e^{i\omega(t - \frac{z}{V})} \lim_{y \rightarrow 0} \int_{-\infty}^x \sum_{n=1}^{\infty} (-1)^n e^{\frac{i\omega \xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2(y-nH)^2} \right] d\xi \quad (10)$$

represents the additional downwash due to the presence of tunnel walls. Thus the relative value of w_0 as compared with $w_0 + w_1$ is the main item of interest here.

The integrals appearing in equations (9) and (10) can be reduced to simpler form for evaluation but since the steps required to reduce one of the integrals are the same as required to reduce the other, only the integral appearing in equation (9) will be treated in detail. The reduced form of the other integral can then be obtained by simple comparison. The Hankel function in equation (9) satisfies the following identity:

$$\frac{\partial^2}{\partial y^2} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) = -\beta^2 \frac{\partial^2}{\partial \xi^2} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) - \frac{\omega^2}{\beta^2 c^2} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) \quad (11)$$

Substituting this relation into equation (9) gives

$$w_0 = -\frac{A\pi i}{V} e^{i\omega(t - \frac{z}{V})} \lim_{y \rightarrow 0} \left[\beta^2 \int_{-\infty}^x e^{\frac{i\omega \xi}{V\beta^2}} \frac{\partial^2}{\partial \xi^2} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) d\xi + \frac{\omega^2}{\beta^2 c^2} \int_{-\infty}^x e^{\frac{i\omega \xi}{V\beta^2}} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) d\xi \right] \quad (12)$$

In equation (12) the first integral can be integrated twice by parts to give for w_0

$$w_0 = -\frac{A\pi i}{V} \lim_{y \rightarrow 0} e^{i\omega(t - \frac{z}{V})} \left[-\frac{\omega}{c} e^{\frac{i\omega x}{V\beta^2}} \frac{x}{\sqrt{x^2 + \beta^2 y^2}} H_1^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2 y^2} \right) - \frac{i\omega}{V} e^{\frac{i\omega x}{V\beta^2}} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2 y^2} \right) - \frac{\omega^2}{V^2} \int_{-\infty}^x e^{\frac{i\omega \xi}{V\beta^2}} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) d\xi \right] \quad (13)$$

By writing the integral in equation (13) as the sum of two integrals, namely

$$\int_{-\infty}^x = \int_{-\infty}^0 + \int_0^x \quad (14)$$

and making a change of variable

$$\frac{\omega \xi}{V\beta^2} = u \quad (15)$$

the expression for w_0 may be further reduced to

$$w_0 = \frac{A\pi\omega}{V^2} \lim_{y \rightarrow 0} e^{i\omega(t - \frac{z}{V})} \left\{ iMe^{\frac{i\omega x}{V\beta^2}} \frac{x}{\sqrt{x^2 + \beta^2 y^2}} H_1^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2 y^2} \right) - e^{\frac{i\omega x}{V\beta^2}} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2 y^2} \right) + \frac{2i\beta}{\pi} e^{-\frac{\omega|y|}{V}} \log \left(\frac{1 + \sqrt{1 - M^2}}{M} \right) + i\beta^2 \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)} \left[M \sqrt{u^2 + \left(\frac{\omega y}{V\beta} \right)^2} \right] du \right\} \quad (16)$$

In the limit $y=0$ the expression in braces in equation (16) reduces to the kernel of Possio's integral equation relating pressure and downwash for the oscillating airfoil in compressible flow. (This result checks the results for this expression given, for example, in ref. 4.)

The value of the integrals in equation (10) may be similarly reduced to give

$$\begin{aligned}
 w_1 = & \frac{2A\pi\omega}{V^2} e^{i\omega(t-\frac{x}{V})} \lim_{y \rightarrow 0} \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{iMx}{\sqrt{x^2 + \beta^2(y-nH)^2}} e^{\frac{i\omega x}{V\beta^2}} H_1^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2(y-nH)^2} \right] - e^{\frac{i\omega x}{V\beta^2}} H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{x^2 + \beta^2(y-nH)^2} \right] + \right. \\
 & \left. i\beta^2 \int_0^{\infty} e^{-iu} H_0^{(2)} \left[M \sqrt{u^2 + \frac{\omega^2}{V^2\beta^2} (y-nH)^2} \right] du + i\beta^2 \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)} \left[M \sqrt{u^2 + \frac{\omega^2}{V^2\beta^2} (y-nH)^2} \right] du \right\} \\
 = & \frac{2A\pi\omega}{V^2} e^{i\omega(t-\frac{x}{V})} \sum_{n=1}^{\infty} (-1)^n \left\{ -e^{\frac{i\omega x}{V\beta^2}} \left(1 + \frac{iV\beta^2}{\omega} \frac{\partial}{\partial x} \right) H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{x^2 + (\beta nH)^2} \right] + \right. \\
 & \left. i\beta^2 \int_0^{\infty} e^{-iu} H_0^{(2)} \left[M \sqrt{u^2 + \left(\frac{\omega nH}{V\beta} \right)^2} \right] du + i\beta^2 \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)} \left[M \sqrt{u^2 + \left(\frac{\omega nH}{V\beta} \right)^2} \right] du \right\} \quad (17)
 \end{aligned}$$

In general, this infinite-series representation of w_1 , equation (17), converges to a finite value. However, for certain critical values of the frequency parameter $\omega H/V$, it is found that the value of w_1 becomes infinite. This fact can be readily made evident by use of relations given in reference 5 where it is shown that an infinite series of Hankel functions of the type appearing in equation (17) can be replaced by an equivalent series of exponential functions as follows:

$$\begin{aligned}
 \sum_{n=1}^{\infty} (-1)^n H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{x^2 + (\beta nH)^2} \right] &= \sum_{n=1}^{\infty} (-1)^n H_0^{(2)} \left[\frac{M\omega H}{V\beta} \sqrt{n^2 + \left(\frac{x}{\beta H} \right)^2} \right] \\
 &= -\frac{1}{2} H_0^{(2)} \left(\frac{Mx}{\beta^2 H} \frac{\omega H}{V} \right) + \frac{i\beta}{M} e^{-\frac{Mx}{\beta^2 H} \sqrt{\left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2}} \\
 &\quad + \frac{i\beta}{M} \sum_{m=1}^{\infty} \left[\frac{e^{-\frac{Mx}{\beta^2 H} \sqrt{(2m+1)^2 \left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2}}}{\sqrt{(2m+1)^2 \left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2}} + \frac{e^{-\frac{Mx}{\beta^2 H} \sqrt{(2m-1)^2 \left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2}}}{\sqrt{(2m-1)^2 \left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2}} \right] \quad (18)
 \end{aligned}$$

If relation (18) is substituted into equation (17) the value of w_1 becomes

$$\begin{aligned}
 w_1 = & \frac{2A\pi\omega}{V^2} e^{i\omega(t-\frac{x}{V})} = \sum_{n=1}^{\infty} \left(-e^{\frac{i\omega x}{V\beta^2}} (-1)^n \left(1 + \frac{iV\beta^2}{\omega} \frac{\partial}{\partial x} \right) H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{x^2 + (\beta nH)^2} \right] + \frac{i\beta\omega}{V} \left\{ -\frac{V}{\pi\omega} \log \left(\frac{1 + \sqrt{1-M^2}}{M} \right) + \right. \right. \\
 & \left. \frac{2i\beta^2 H}{M^2} \left[\frac{1}{\sqrt{\left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2}} \frac{1}{\sqrt{\left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2} + \frac{i\omega H}{MV}} + \frac{1}{\sqrt{(2n+1)^2 \left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2}} \frac{1}{\sqrt{(2n+1)^2 \left(\frac{\pi\beta}{M} \right)^2 - \left(\frac{\omega H}{V} \right)^2} + \frac{i\omega H}{MV}} \right] \right\} + \\
 & \left. i\beta^2 (-1)^n \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)} \left[M \sqrt{u^2 + \left(\frac{\omega nH}{V\beta} \right)^2} \right] du \right) \quad (19)
 \end{aligned}$$

It may be seen that this expression becomes infinite for all values of x when the frequency parameter $\omega H/V$ has any of the values given by

$$\frac{\omega H}{V} = (2m-1) \frac{\pi\beta}{M} \quad (m=1, 2, 3, \dots) \quad (20)$$

These critical values of the frequency parameter correspond to a condition of pure resonance in the tunnel which in the present case implies that a harmonic disturbance of any finite amplitude may lead to a downwash of infinite amplitude.

Of course these infinite values of w_1 would never be realized under practicable conditions because factors such as finite tunnel length, absorption through walls, fluid viscosity, and so forth that would give rise to damping would make pure resonance unobtainable; however, with damping present, resonant frequencies yielding values of $\omega H/V$ in the neighborhood of those given in equation (20) would exist and it is not likely that quantitative agreement or even possibly qualitative agreement between calculated and measured downwash (or forces) can be realized when the value of $\omega H/V$ is in the neighborhood of these critical values.

It is interesting to note that the effect of boundary conditions such as section geometry, tunnel-wall flexibility, and so forth is to change the value of the critical frequency but not to do away with the possibility of resonance. Also, by treatments similar to those employed herein, it can be shown that under idealized conditions resonance can occur in three-dimensional flow in both round and rectangular tunnels or apparently, within certain Mach number ranges, in tunnels of nonuniform cross section (expanding or contracting section) or even in open tunnels or jets.

The fundamental or smallest critical values of $\omega H/V$, corresponding to $m=1$ in equation (20), are shown plotted as functions of Mach number M in figure 2. This figure indicates that there is no finite critical value of $\omega H/V$ for the conditions $M=0$, $V \neq 0$, and $c = \infty$, which correspond to a flow of incompressible fluid in the tunnel. This result agrees with those found in references 1 and 2.

The frequency parameter

$$\frac{\omega H}{c} = (2m-1)\pi\beta \quad (m=1, 2, 3, \dots) \quad (21)$$

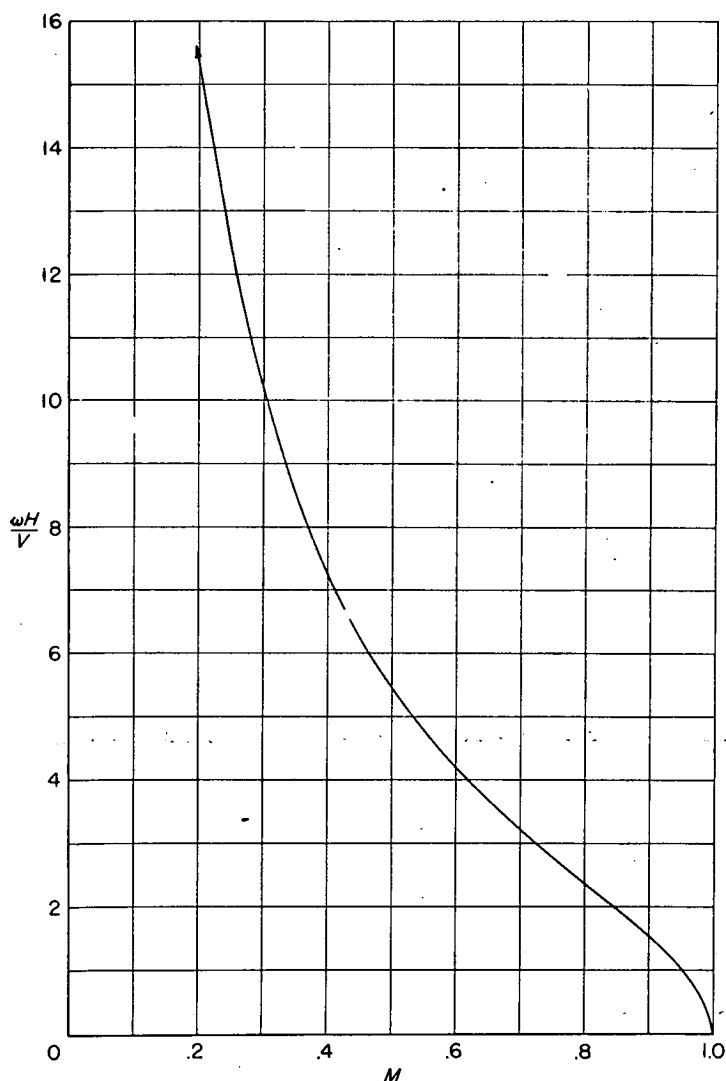


FIGURE 2.—Fundamental critical values of frequency parameter $\omega H/V$ plotted as a function of Mach number M .

which may be derived from equation (20) is shown plotted, for $m=1$, as a function of Mach number in figure 3. Equations (21) and figure 3 show that finite values of the critical frequency exist for the conditions $M=0$, $V=0$, and $c \neq \infty$. These conditions correspond to a compressible fluid at zero velocity in the tunnel. For these conditions equations (21) and the corresponding wave lengths

$$\lambda = \frac{2\pi c}{\omega} = \frac{2H}{2m-1} \quad (m=1, 2, 3, \dots) \quad (22)$$

agree, respectively, with results found in the literature for the characteristic frequencies and wave lengths associated with transverse acoustic vibrations in rectangular chambers when the location of the source of disturbance is excluded as a nodal point. See, for example, reference 6.

It may be of interest to note that equation (21) can be derived from the principle of standing waves as follows: The condition for resonance for the type of disturbance considered implies that the standing transverse waves have a maximum velocity at the midsection of the tunnel and zero velocity at

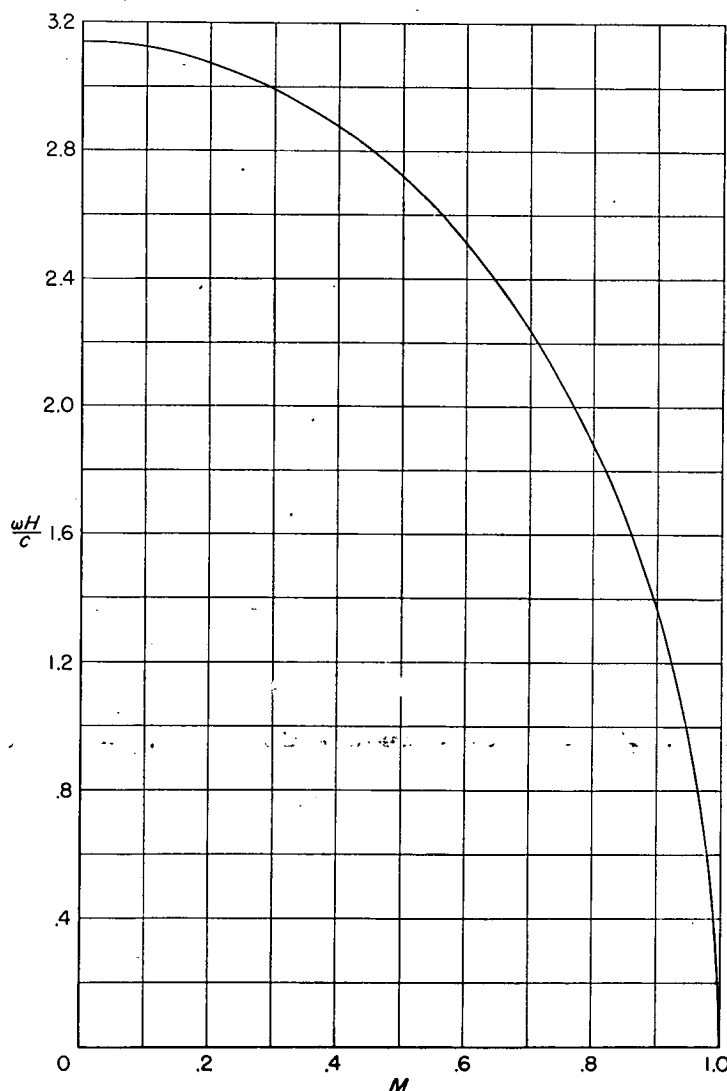


FIGURE 3.—Fundamental critical values of frequency parameter $\omega H/c$ plotted as a function of Mach number M .

the boundaries. A half-sine wave of wave length $\lambda=2H$ or any odd divisor of this length, namely, $\lambda=\frac{2H}{2m-1}$ satisfies this condition. If c is the velocity of sound in the medium and V the velocity of the medium, the velocity of propagation of a disturbance in a fixed plane perpendicular to the air flow is $\sqrt{c^2-V^2}$. Since the frequency is given by the speed of propagation divided by the wave length there is obtained

$$f = \frac{(2m-1)\sqrt{c^2-V^2}}{2H} = \frac{\omega}{2\pi}$$

or

$$\frac{\omega H}{c} = \pi\beta(2m-1)$$

INTEGRAL EQUATION FOR AN AIRFOIL OF INFINITE ASPECT RATIO OSCILLATING IN A WIND TUNNEL

In order to present equations from which tunnel-wall corrections for two-dimensional flow can be calculated, use is made of the foregoing analysis to derive the integral equation, relating downwash distributions and lift distributions, for the effect of tunnel walls on the lift distribution associated with a given downwash distribution.

The resultant pressure or local lift Δp associated with the acceleration potential of a single doublet located at $(x_0, 0)$ with strength depending on streamwise position x_0 may be expressed simply as (compare with eq. (6)):

$$\begin{aligned} \Delta p &= -2\rho \lim_{y \rightarrow 0} \psi[x_0, (x-x_0), y, t] \\ &= -2\rho\pi i \lim_{y \rightarrow 0} A(x_0) e^{i\omega\left(t + \frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y} H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{(x-x_0)^2 + \beta^2 y^2} \right] \end{aligned} \quad (23)$$

where $A(x_0)$ denotes local doublet strength or lift density. The downwash due to a distribution of such doublets between $x_0 = -b$ and $x_0 = b$ is

$$w(x, t) = \frac{-2\rho\pi i}{V} e^{i\omega t} \lim_{y \rightarrow 0} \int_{-b}^b A(x_0) e^{-\frac{i\omega}{V}(x-x_0)} dx_0 \int_{-\infty}^{x-x_0} e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) d\xi \quad (24)$$

For a given value of the lift density $A(x_0)$, this equation determines the downwash. For a given or prescribed expression of $w(x, t)$, the distribution of lift density must be determined. Thus, in this case, equation (24) is a form of Possio's integral equation relating downwash and pressure for an airfoil oscillating in compressible flow. In passing it may be well to point out that Possio's equation has not yet been solved in closed form but has been evaluated by different methods of approximation by several authors. Reference 4 gives a résumé of these methods of approximation.

For an airfoil inside a two-dimensional tunnel the relation between downwash and local lift becomes (compare with eq. (8))

$$\begin{aligned} w(x, t) &= \frac{-2\rho\pi i}{V} e^{i\omega t} \lim_{y \rightarrow 0} \int_{-b}^b A(x_0) e^{-\frac{i\omega}{V}(x-x_0)} dx_0 \left\{ \int_{-\infty}^{x-x_0} e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)} \left(\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 y^2} \right) d\xi + \right. \\ &\quad \left. 2 \int_{-\infty}^{x-x} \sum_{n=1}^{\infty} (-1)^n e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)} \left[\frac{\omega}{c\beta^2} \sqrt{\xi^2 + \beta^2 (y-nH)^2} \right] d\xi \right\} \end{aligned} \quad (25)$$

For a given value of lift density $A(x_0)$, this equation determines the effect of tunnel walls on the corresponding downwash. For a given downwash distribution, the more pertinent effect of tunnel walls on the distribution of lift density is obtained by comparing the solution of equation (24) with the solution of equation (25). In either case the summation in the second integral in braces in equation (25) is the same summation that was found in the preceding section to have critical values of the frequency parameter $\omega H/V$ that cause the summation to become infinite. Consequently, evaluations of equation (25) for values of the frequency parameter in the neighborhood of these critical values would lead to the same resonant effects found in the treatment of a single doublet. Otherwise, for values of the frequency parameter not too near critical values, it is

proposed that a fairly close approximation to solutions of equations (24) and (25) for effects of tunnel walls on lift density (or lift) will generally yield results from which tunnel-wall correction factors for two-dimensional flow can be obtained. Expressions from which correction factors for three-dimensional flow can be obtained may be similarly derived when the downwash of a three-dimensional pressure doublet is employed instead of the downwash of a two-dimensional pressure doublet.

It appears desirable to solve equations (24) and (25) by collocation or some other approximate method to obtain tunnel-wall corrections for some particular cases of prescribed downwash and to determine experimentally the range, if any, of frequency parameter in which quantitative results can be obtained for these cases.

CONCLUDING REMARKS

The important result shown is that, in a tunnel of infinite length containing a flowing fluid, a resonant condition involving a transverse oscillation of the fluid across the tunnel is possible and measured air forces at or near this condition of resonance might be greatly modified from those measured in free air. This resonant condition is a (simple) function of Mach number, tunnel height, and wing frequency and brings to attention a new type of tunnel-wall interference.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *September 24, 1951.*

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REPORT 1151

THE EFFECTS ON DYNAMIC LATERAL STABILITY AND CONTROL OF LARGE ARTIFICIAL VARIATIONS IN THE ROTARY STABILITY DERIVATIVES¹

By ROBERT O. SCHADE AND JAMES L. HASSELL, Jr.

SUMMARY

An investigation has been conducted in the Langley free-flight tunnel to determine the effects of large artificial variations of several rotary lateral-stability derivatives on the dynamic lateral stability and control characteristics of a 45° sweptback-wing airplane model. The derivatives investigated were the damping-in-yaw derivative C_{n_r} (the yawing moment due to yawing), the damping-in-roll derivative C_{l_p} (the rolling moment due to rolling), and the two cross derivatives C_{l_r} (the rolling moment due to yawing) and C_{n_p} (the yawing moment due to rolling). Flight tests of a free-flying model were made in which the derivatives were varied over a wide range by means of an artificial-stabilization device incorporating a gyroscope sensitive to rolling or yawing velocity. Calculations of the period and damping of the lateral motions and of the response to roll and yaw disturbances were made for correlation with the experimental results. In order to simplify the analysis, most of the calculations were based on the assumption of idealized artificial-stabilization systems, but a few check calculations were made in which the small constant time lag of the stabilization device used in the tests was taken into account. Extensive calculations were not made by this method, however, because of the extremely laborious process involved and because a systematic determination of the effect of time lag on stability throughout the variation of the four derivatives was considered beyond the scope of the present investigation.

The calculated results were in qualitative agreement with the experimental results in predicting the general trends in flight characteristics produced by large changes in the stability derivatives, but in some cases the theory with the assumption of zero lag was not in good quantitative agreement with the experimental results. In these cases the check calculations with time lag taken into account indicated that the discrepancies could be attributed to the effect of the small constant time lag in the stabilization device used. The results showed that the only derivative which provided a large increase in damping of the lateral oscillation without adversely affecting other flight characteristics was C_{n_r} . (Because of the limitations imposed by the relatively small size of the test section of the Langley free-flight tunnel, however, the flight characteristics of the model were not appreciably influenced by the stiffness in turning maneuvers that has been found objectionable in some airplanes equipped with yaw dampers.)

Increasing C_{l_p} to moderately large negative values produced substantial increases in the damping of the lateral oscillation but caused an objectionable stiffness in roll. Further negative increases in C_{l_p} did not cause additional increases in damping of the lateral oscillation and made the stiffness in roll more objectionable. Increases in C_{l_r} or C_{n_p} in the positive direction produced an increase in damping of the lateral oscillation but caused an undesirable spiral tendency.

INTRODUCTION

Many present-day high-speed airplanes have exhibited unsatisfactory damping of the lateral oscillation, partly because of the configurations required for high-speed flight and partly because of the more severe operating conditions encountered (high altitude and high wing loading). Since in many cases satisfactory oscillatory stability cannot be obtained by making reasonable geometric changes to the airplane, much interest has been shown in the use of artificial-stabilization devices as a means of obtaining satisfactory damping of the lateral oscillation.

Yaw dampers have been installed in some airplanes in an effort to improve the lateral oscillatory stability. This artificial-stabilization device provides rudder deflection in response to a signal from a gyroscope sensitive to yawing velocity so that the yawing moment of the rudder tends to damp the lateral motion of the airplane. In an idealized system such a device produces the damping-in-yaw derivative C_{n_r} (the yawing moment due to yawing). Similar devices can be considered, in an idealized case, to vary the damping-in-roll derivative C_{l_p} (the rolling moment due to rolling) and the two cross derivatives C_{n_p} (the yawing moment due to rolling) and C_{l_r} (the rolling moment due to yawing). In a practical case, of course, the actual characteristics of the artificial-stabilization device should be taken into account rather than considering that the device produces a simple change in one of these derivatives. References 1 and 2 present some results of theoretical investigations of large derivative variations as produced by idealized artificial-stabilization systems and references 3 and 4 present methods for taking into account the effect of constant time lag in the stabilization systems.

¹ Supersedes NACA TN 2781, "The Effects on Dynamic Lateral Stability and Control of Large Artificial Variations in the Rotary Stability Derivatives" by Robert O. Schade and James L. Hassell, Jr., 1952.

Varying the value of either of the damping derivatives C_{n_r} and C_{l_p} changes the total damping of the airplane. Varying the value of either of the cross derivatives C_{n_p} and C_{l_r} primarily causes a redistribution of the natural damping of the system for cases in which the airplane has low values of the product of inertia. For high values of the product of inertia, variations in C_{n_p} or C_{l_r} can cause sizable changes in the total damping of the airplane.

In order to study the relative effects of large independent variations of these four rotary stability derivatives on the dynamic stability and control characteristics of airplanes, an investigation has been carried out in the Langley free-flight tunnel on a free-flying dynamic airplane model equipped with an artificial-stabilization device incorporating a rate-sensitive gyroscope. This investigation is a part of a general research program to determine the effects of several of the lateral-stability derivatives, both independently and in combination, on dynamic lateral stability and control.

Force tests were made to determine all the lateral-stability derivatives of the model in the basic condition for use in making calculations and establishing flight-test conditions. Calculations were made to determine the period and damping of the lateral motions and the lateral response to rolling and yawing disturbances for correlation with flight-test results. In order to simplify the analysis, most of the calculations were based on the assumption of idealized artificial-stabilization systems although the stabilization device used in the tests did have a small constant time lag. Additional calculations including the effect of constant time lag were made for some conditions in which the idealized theory was not in good quantitative agreement with the experimental results. All tests and calculations were made at a lift coefficient of 1.0.

Although the results do not apply directly to airplanes or flight conditions other than those investigated, the trends of the results presented are believed to give a qualitative indication of the general effects of large independent variations of the four stability derivatives under consideration.

SYMBOLS AND COEFFICIENTS

All force and moment measurements were obtained with respect to the stability axes. A sketch showing the axes and the positive directions of the forces, moments, and angles is given in figure 1.

C_L	lift coefficient, Lift/ qS
C_n	yawing-moment coefficient, Yawing moment/ qSb
C_l	rolling-moment coefficient, Rolling moment/ qSb
C_Y	lateral-force coefficient, Lateral force/ qS
L	rolling moment, about X-axis, ft-lb
N	yawing moment, about Z-axis, ft-lb
Y	lateral force, lb
q	dynamic pressure, $\frac{1}{2} \rho V^2$, lb/sq ft
S	wing area, sq ft
l	distance from airplane center of gravity to vertical-tail center of pressure, ft

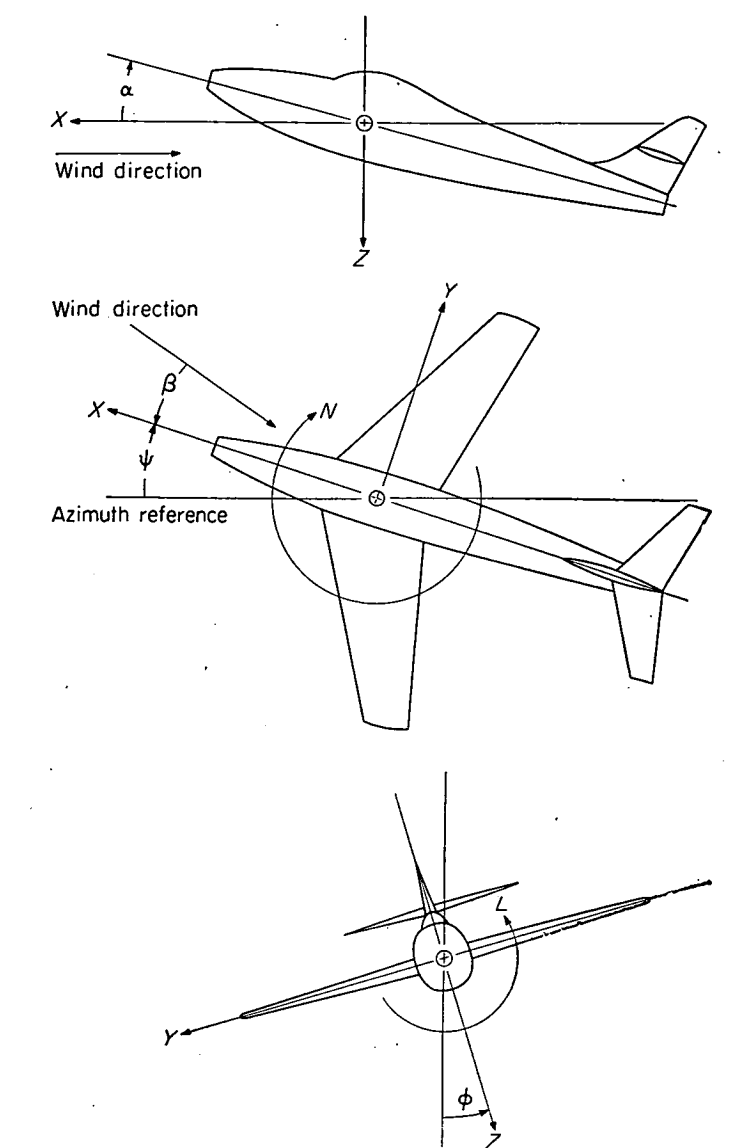


FIGURE 1.—The stability system of axes. Arrows indicate positive directions of moments, forces, and angles. This system of axes is defined as an orthogonal system having the origin at the center of gravity and in which the Z-axis is in the plane of symmetry and perpendicular to the relative wind, the X-axis is in the plane of symmetry and perpendicular to the Z-axis, and the Y-axis is perpendicular to the plane of symmetry. At a constant angle of attack, these axes are fixed in the airplane.

b	wing span, ft
t	time, sec
y	sidewise displacement from center line of test section, ft
ρ	mass density of air, slugs/cu ft
V	airspeed, ft/sec
β	angle of sideslip, radians except where other- wise noted
ψ	angle of yaw, deg
ϕ	angle of bank, deg
α	angle of attack, deg
δ	control deflection, deg
δ_a	total aileron deflection, deg

δ_i	vertical-tail deflection, deg
μ	relative density factor, $m/\rho Sb$
m	mass of airplane, slugs
η	angle of attack of principal longitudinal axis of airplane, deg
ω	frequency, radians/sec
ω_0	natural frequency of model, radians/sec
δ_a/p	amplitude ratio, $\frac{\text{Aileron deflection}}{\text{Rolling velocity}}$, deg/radian/sec
γ	inclination of flight path to horizontal axis, positive in a climb, deg
I_{x_0}	moment of inertia about principal longitudinal axis, slug-ft ²
I_{z_0}	moment of inertia about principal normal axis, slug-ft ²
k_{x_0}	radius of gyration in roll about principal longitudinal axis, ft
k_{z_0}	radius of gyration in yaw about principal vertical axis, ft
K_x	nondimensional radius of gyration in roll about longitudinal stability axis,

$$\sqrt{\left(\frac{k_{x_0}}{b}\right)^2 \cos^2 \eta + \left(\frac{k_{z_0}}{b}\right)^2 \sin^2 \eta}$$

K_z	nondimensional radius of gyration in yaw about vertical stability axis,
-------	---

$$\sqrt{\left(\frac{k_{z_0}}{b}\right)^2 \cos^2 \eta + \left(\frac{k_{x_0}}{b}\right)^2 \sin^2 \eta}$$

K_{xz}	nondimensional product-of-inertia parameter,
----------	--

$$\left[\left(\frac{k_{z_0}}{b}\right)^2 - \left(\frac{k_{x_0}}{b}\right)^2 \right] \sin \eta \cos \eta$$

i_w	wing incidence, deg
$pb/2V$	rolling-angular-velocity factor, radians
$rb/2V$	yawing-angular-velocity factor, radians
p	rolling angular velocity, radians/sec
r	yawing angular velocity, radians/sec

$$C_{Y\beta} = \frac{\partial C_Y}{\partial \beta}$$

$$C_{n\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{l\beta} = \frac{\partial C_l}{\partial \beta}$$

$$C_{l_p} = \frac{\partial C_l}{\partial \frac{pb}{2V}}$$

$$C_{Y_p} = \frac{\partial C_Y}{\partial \frac{pb}{2V}}$$

$$C_{n_p} = \frac{\partial C_n}{\partial \frac{pb}{2V}}$$

$$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$$

$$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}}$$

$$C_{l_r} = \frac{\partial C_l}{\partial \frac{rb}{2}}$$

$$(C_l)_{\delta_a} = \frac{\partial C_l}{\partial \delta_a}$$

$$(C_n)_{\delta_i} = \frac{\partial C_n}{\partial \delta_i}$$

C_{l_c}	rolling-moment coefficient due to deflection of both ailerons
-----------	---

C_{n_c}	yawing-moment coefficient due to rudder deflection
-----------	--

P	period of oscillation, sec
-----	----------------------------

$T_{1/2}$	time for amplitude of lateral oscillation or aperiodic mode of motion to decrease to one-half amplitude, sec
-----------	--

A, B	coefficients of first two terms of lateral-stability quartic equation (see ref. 1)
--------	--

$$\frac{B}{A} = -\frac{1}{4\mu} \frac{\left[2K_x^2 K_z^2 C_{Y\beta} + K_x^2 C_{n_r} + K_z^2 C_{l_p} - 2K_{xz}^2 C_{Y\beta} - K_{xz} C_{l_r} - K_{xz} C_{n_p} \right]}{K_x^2 K_z^2 - K_{xz}^2}$$

APPARATUS

TUNNEL AND MODEL

The flight-test part of the investigation was carried out in the Langley free-flight tunnel which is equipped for testing free-flying dynamic models. A complete description of the tunnel and its operation is given in reference 5. The static longitudinal and lateral stability characteristics were determined in the Langley stability tunnel and the aileron- and rudder-effectiveness tests were made in the Langley free-flight tunnel. The dynamic lateral-stability derivatives were determined in the Langley stability tunnel by the yawing- and rolling-flow techniques described in references 6 and 7.

A three-view drawing of the model used in the investigation is presented in figure 2 and a photograph of the model is presented as figure 3. The dimensional and mass characteristics of the model are presented in table I. A wing having 45° sweepback of the leading edge, a taper ratio of 0.5, and an aspect ratio of 3.00 was incorporated in the design because this plan form was typical of a number of proposed fighter airplanes. The center of gravity of the model was located at 23.3 percent of the mean aerodynamic

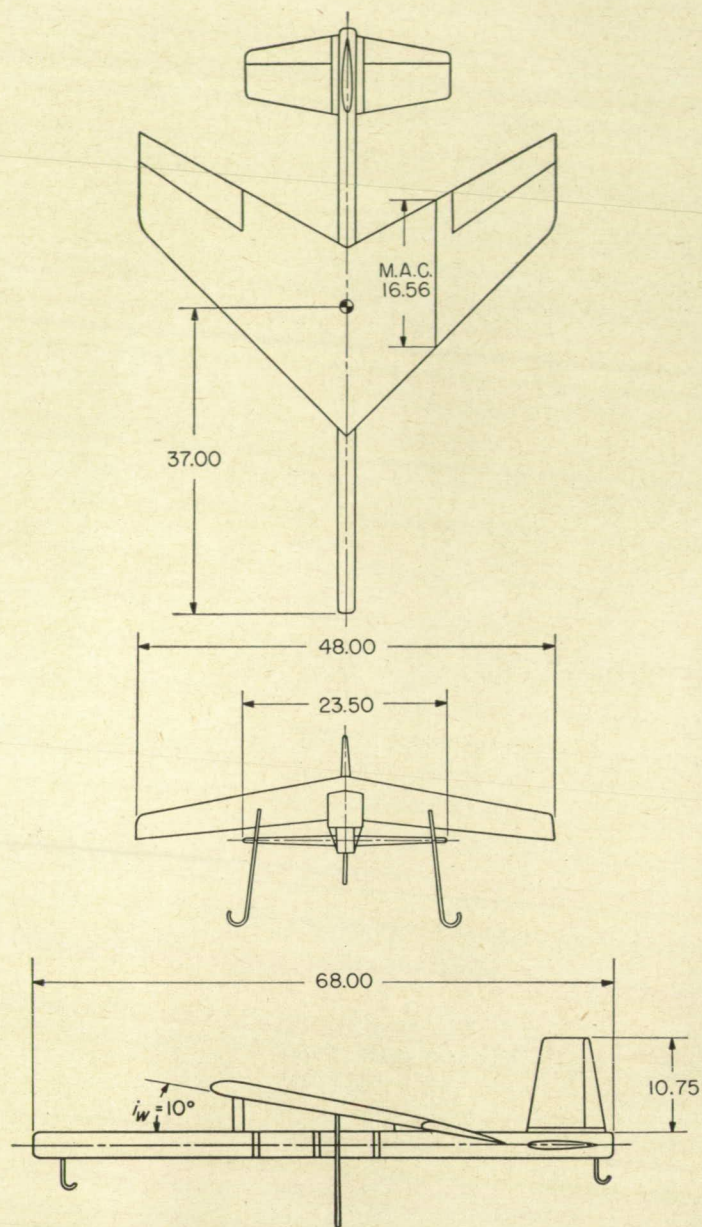


FIGURE 2.—Three-view drawing of test model. All dimensions are in inches.

chord for all tests. The model was equipped with oversize (half-span, 30-percent-chord) ailerons and an all-movable vertical tail in order to obtain the high rolling and yawing moments required for large variations of the rotary derivatives. The ailerons were also used for manual control but the all-movable tail had a flap-type rudder for manual control. Conventional horizontal stabilizing surfaces were employed. A boom-type metal fuselage was used in order to simplify the construction of the model.

For manual control the rudder and ailerons were electrically interconnected to move together in order to eliminate the adverse yawing moment of the ailerons. Aileron and rudder deflections of $\pm 21^\circ$ and $\pm 14^\circ$, respectively, were used for all flight conditions except for the highest value of C_{l_p} . In this condition the aileron deflection was $\pm 29^\circ$ and the rudder deflection was $\pm 19^\circ$.

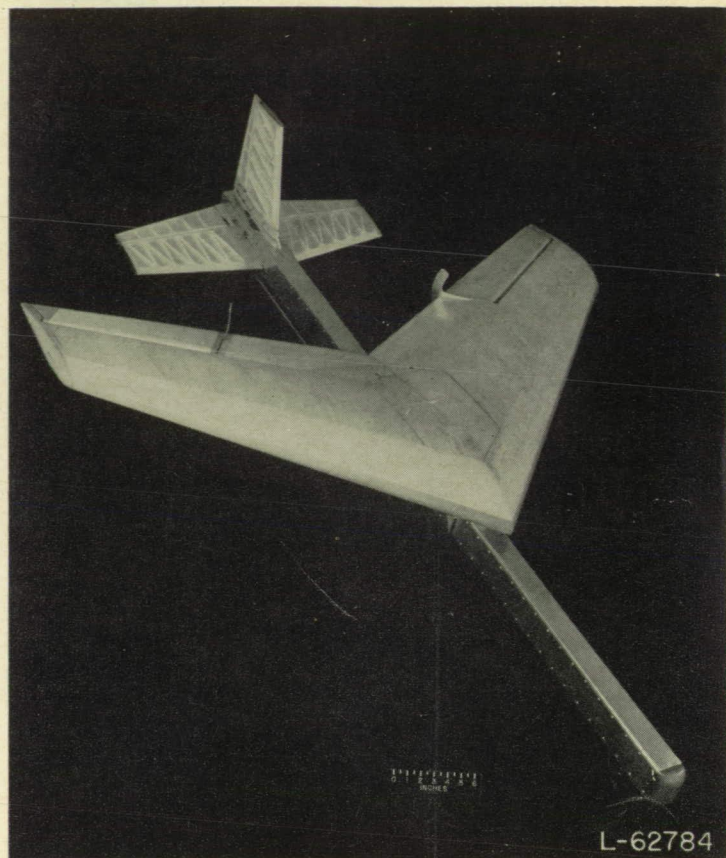


FIGURE 3.—Model used in free-flight tunnel tests.

The manually controlled rudder was operated by a flicker-type (full on or full off) electrical actuator. Although all other servoactuators were of the proportional pneumatic type, essentially flicker-type control was obtained with them because control was applied by abrupt movements of the control sticks and because very high gearing was used between the stick and control surface.

In order to have the model represent an airplane that had poor oscillatory stability and hence require an artificial-stabilization device, the wing incidence was adjusted so that the basic model had a neutrally stable lateral oscillation at the test lift coefficient of 1.0. This neutrally stable oscillation was obtained by increasing the wing incidence to 10° so that the principal axes of inertia became more closely aligned with the wind axes. (See ref. 8.)

ARTIFICIAL-STABILIZATION DEVICE

The artificial-stabilization device used in this investigation consisted of a rate gyro and a servoactuator. The rate gyro was mounted on a quadrant so that it could be aligned with either the roll or yaw stability axis; therefore it would be sensitive only to a rolling or a yawing velocity as desired. The servoactuator operated both the ailerons and the all-movable tail to produce the derivatives C_{l_p} or C_{l_r} . In order to produce pure rolling moments without adverse yaw, the all-movable tail had to be deflected simultaneously with the ailerons. For the two yawing-moment derivatives C_{n_p} and C_{n_r} the servoactuator operated only the all-movable tail.

TABLE I
DIMENSIONAL AND MASS CHARACTERISTICS OF THE MODEL

Weight, lb.....	20. 5
Wing loading, lb/sq ft.....	3. 85
Relative density factor, $m/\rho Sb$	12. 58
Moments of inertia:	
I_{x_0} , slug-ft ²	0. 220
I_{z_0} , slug-ft ²	1. 473
Wing:	
Airfoil section..... Rhode St. Genese 35	
Area, sq ft.....	5. 33
Span, ft.....	4. 00
Sweepback, leading edge, deg.....	45
Incidence, deg.....	10
Dihedral, deg.....	0
Taper ratio.....	0. 5
Aspect ratio.....	3. 00
Mean aerodynamic chord, ft.....	1. 38
Location of leading edge of mean aerodynamic chord behind leading edge of root chord, ft.....	0. 99
Root chord, ft.....	1. 78
Tip chord, ft.....	0. 89
Aileron:	
Area (total), percent wing area.....	12. 5
Span (total), percent wing span.....	50
Chord, percent wing chord.....	30
Vertical tail:	
Area:	
Square feet.....	0. 53
Percent wing area.....	10
Span, ft.....	0. 90
Aspect ratio.....	1. 50
Sweepback, 50 percent chord, deg.....	0
Root chord, ft.....	0. 75
Tip chord, ft.....	0. 44
Tail length (from 0.23 mean aerodynamic chord of wing to 0.25 mean aerodynamic chord of tail), l/b	0. 514
Airfoil section..... NACA 0009	
Horizontal tail:	
Area:	
Square feet (including area through fuselage) ..	1. 19
Percent wing area.....	22. 3
Span, ft.....	1. 96
Aspect ratio.....	3. 23
Sweepback, 50 percent chord, deg.....	0
Root chord, ft.....	0. 75
Tip chord, ft.....	0. 44
Tail length (from 0.23 mean aerodynamic chord of wing to 0.25 mean aerodynamic chord of tail), l/b	0. 514
Airfoil section..... NACA 0009	
Fuselage:	
Length, ft.....	5. 67
Cross section, in.....	2 by 3

No accompanying aileron deflection was required since at the flight-test lift coefficient of 1.0 the tail produced no rolling moment.

Deflection of the all-movable vertical tail to produce the rolling- and yawing-moment derivatives also produced changes in the lateral-force derivatives C_{Y_p} (the lateral force due to rolling) and C_{Y_r} (the lateral force due to yawing). In the calculations, however, these changes in C_{Y_p} and C_{Y_r}

were neglected because preliminary calculations indicated that even the largest changes in these derivatives did not appreciably affect the calculated results.

The value of a derivative was artificially increased or decreased by varying the gyro rotor speed or the control linkage to produce more or less control deflection for a given rolling or yawing velocity. The sign of a derivative was changed by rotating the gyro 180° about the rotor axis to give opposite response for a given velocity.

A schematic drawing of the control system used for the C_{i_p} derivative is shown in figure 4. Both ailerons were used for control but for clarity in the drawing only one aileron is shown. This drawing shows the artificial-stabilization device, the manual servoactuator, and the control linkage. This linkage allowed both the artificial-stabilization device and manual actuator to operate the same aileron surfaces. The tubes shown in figure 4 supply air to the gyro rotor to produce a given rotor speed and to the servoactuators to provide the force required to move the control surfaces. Air is also supplied to the gyro pick-off valve which varies the signal pressure to the servoactuator.

In order to explain the operation of the artificial-stabilization device, the assumptions are made that the device is set up to produce negative C_{i_p} and that the model has received a rolling disturbance causing the model to roll to the right. The operation then is as follows: In response to the rolling velocity the rate-gyro rotor produces a torque about the precessional axis of the gyro and the resulting rotation about this axis causes the pick-off valve to move. The movement of the valve varies the signal pressure to the servoactuator which deflects the control surfaces. This control deflection produces a rolling moment which tends to prevent the model from rolling to the right.

An example of the results obtained from the calibration of the artificial-stabilization device is shown in figure 5. The results presented, which are for one value of the derivative C_{i_p} , show the variation of the amplitude ratio and the phase angle with frequency. These results indicate that the amplitude ratio did not vary appreciably throughout the frequency range, but the variation of phase angle with frequency was such that the system had an essentially constant time lag of about 0.05 second.

DETERMINATION OF BASIC STABILITY AND CONTROL PARAMETERS OF THE MODEL

The stability derivatives of the model in the basic condition for a lift coefficient of 1.0 were determined from force tests made at a dynamic pressure of 25 pounds per square foot, which corresponds to a test Reynolds number of approximately 1,245,000 based on the mean aerodynamic chord of 1.38 feet. The results of these tests are given in table II.

Aileron and rudder effectiveness at a lift coefficient of 1.0 was determined from force tests made at a dynamic pressure of 3.0 pounds per square foot which corresponds to a test Reynolds number of approximately 350,000 based on the mean aerodynamic chord of 1.38 feet. The results of these tests showed that for the range of deflections used in

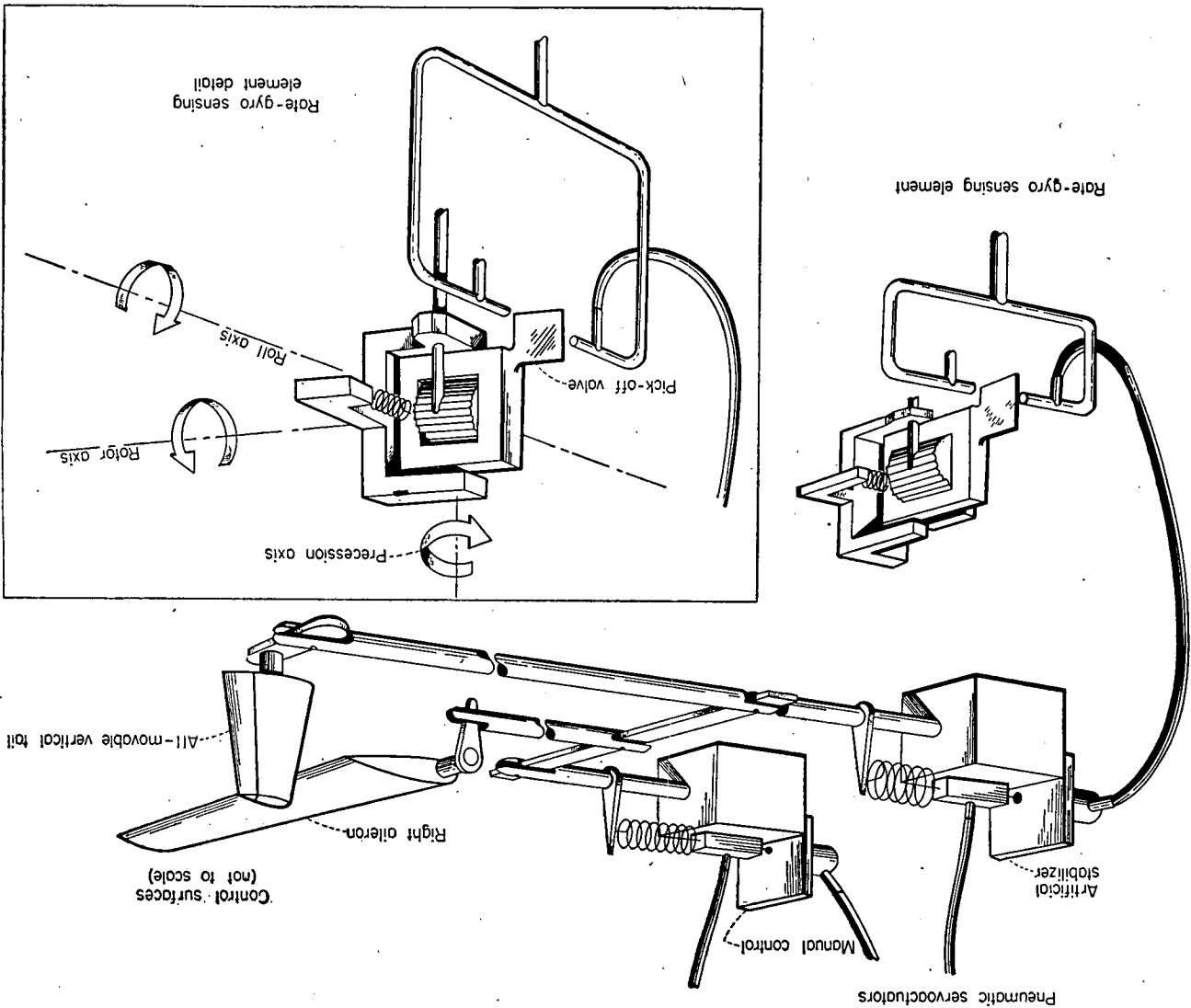


Figure 4.—Sketch of artificial-stabilization system. Arrangement for producing C_p is shown.

the flight tests the variation of control moment with con-
 (C)₀ of 0.0018 per degree and the all-movable tail produced
 a value of (C)₀ of 0.0018 per degree. These data were used
 in determining the values of the stability derivatives simu-
 lated by the artificial-stabilization device.

FLIGHT TESTS

TEST PROCEDURE AND RATINGS OF FLIGHT CHARACTERISTICS

The various flight characteristics rated in the free-flight-
 tunnel tests were the damping of the lateral oscillation,
 apparent spiral stability, apparent damping in roll, maneu-
 verability, controllability, and general flight behavior. The
 ratings are listed and defined in table II. These ratings
 merely indicate the relative effect of changes in the various
 derivatives on the flight characteristics and should not be
 considered as absolute ratings that can be used to relate
 these results with results for other models or full-scale air-
 planes. Motion-picture records were also obtained to sup-
 plement the flight ratings. One of the main uses of these
 records was to provide time histories for measuring quanti-
 tative values of damping.
 Control-fixed oscillations were initiated by rocking the
 model in roll approximately in phase with the natural fre-
 quency of the oscillation. This procedure is different from

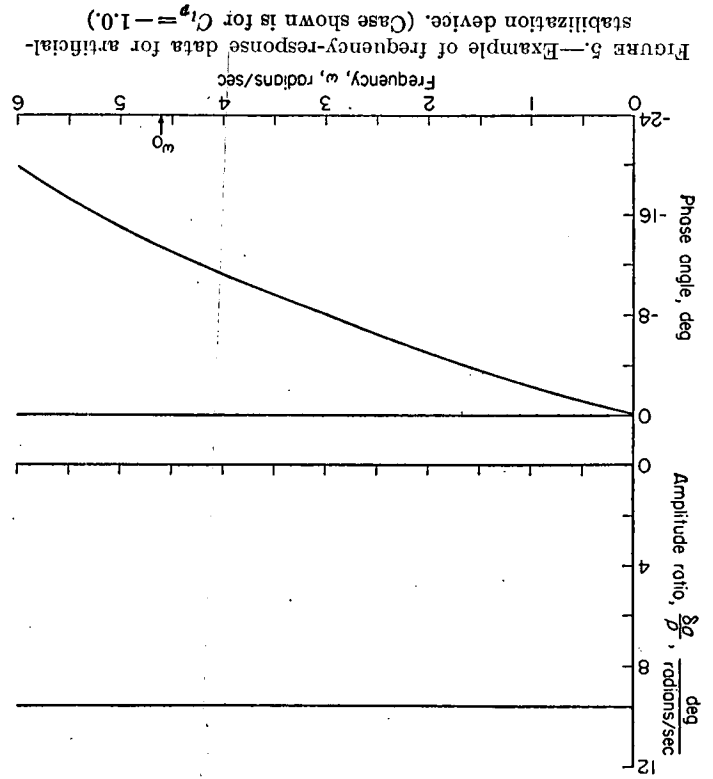


Figure 5.—Example of frequency-response data for artificial-stabilization device. (Case shown is for $C_p = -1.0$.)

TABLE II

FLIGHT RATINGS AND CALCULATED PERIOD AND TIME TO DAMP TO ONE-HALF AMPLITUDE FOR FLIGHT-TEST CONDITIONS

Constant Aerodynamic and Mass Terms Used in Calculating the Damping and Period of the Model

μ	12.58	C_L	1.0
$(kx_0/b)^2$	0.0216	$C_{Y\beta}$	-0.78
$(kz_0/b)^2$	0.1443	C_{Yp}	0.31
η	4.68	C_{Yr}	0.43
Kx^2	0.0225	$\tan\gamma$	-0.2126
Kz^2	0.1439	$C_{n\beta}$	0.2064
Kxz	0.0100	$C_{l\beta}$	-0.2180
V , ft/sec.....	56.3		

Explanation of Flight Ratings

Rating		Damping of oscillatory and nonoscillatory motions	Lateral control		General flight behavior
			Maneuverability	Controllability	
A	Satisfactory	Stable; heavily damped	Good	Good	Good.
B		Stable; moderately damped	Fair	Fair	Fair.
C	Unsatisfactory	Stable; lightly damped	Poor	Poor	Poor.
D		Neutrally stable	Very poor	Uncontrollable	Unflyable.
E		Unstable			

Derivative varied	Case	Value of derivative varied	Calculated values for—				Flight ratings for—					
			Oscillatory mode		Aperiodic mode		Damping of lateral oscillation	Damping of the non-oscillatory motion		Maneuverability	Controllability	General flight behavior
			Period, sec	$T_{1/2}$, sec	Spiral $T_{1/2}$, sec	Rolling $T_{1/2}$, sec		Apparent spiral tendency	Apparent damping in roll			
C_{n_r}	1	-7.2	2.25	-8.68	0.05	0.12	E+	A+	B	A-	B+	C-
	2	-5.2	2.22	8.18	.09	.10	C	A+	B	A-	A-	B-
	3	-3.2	2.09	1.51			B	A+	B	A-	A+	A
	4	-2.2	1.84	.88			B+	A	B	A-	A	A-
	5	-1.2	5.72	.19			B	B+	B	A-	A-	B
	6	-7	1.42	2.33	1.15	.14	C	B	B	A	B+	B-
	7	-21	1.37	-24.80	5.02	.14	D	C	B	A	B-	C
	8	.0	1.40	-4.76	44.20	.14	D-	D+	B	A	B-	C
	9	1.8	1.43	-.51	-.48	.13	E-	E-	B	A+	C-	D
C_{l_p}	10	-7.3	1.63	1.56	5.09	.01	B	C	A+	D+	A+	C-
	11	-1.3	1.64	1.52	5.44	.04	C	C	A+	C	A+	C
	12	-8	1.54	1.69	5.31	.07	B	C	A-	B	A	B
	13	-6	1.46	2.35	5.20	.09	B	C	A-	B+	A	B-
	14	-5	1.42	3.47	5.14	.11	B-	C	B+	A-	A-	B-
	7	-32	1.37	-24.80	5.02	.14	D	C	B	A	B-	C
	15	.1	1.38	-.59	4.21	.33	E	C-	C	A+	C-	C-
C_{l_r}	16	-2.9	1.35	-.40			E	A+	B	A+	C+	C-
	17	-9	1.38	-1.16	.65	.16	E+	B+	B	A+	B-	C
	7	.13	1.37	-24.80	5.02	.14	D	C	B	A	B	C
	18	.3	1.37	13.28	35.40	.14	D+	D	B	(9)	B	C+
	19	1.1	1.36	6.63	-12.31	.13	C	E	B	(9)	C	C
	20	3.1	1.31	.57	-.49	.10	B+	E-	B	(9)	D+	D+
C_{n_p}	21	-7	1.10	-.73	5.41	.11	E-	C	B	A+	C	C-
	7	-.07	1.37	-24.80	5.02	.14	D	C-	B	A	B+	C
	22	.0	1.41	7.05	4.97	.16	C	C-	B	A	B-	C+
	23	.3	1.81	.70	4.17	.21	B+	D	B-	(9)	A-	B-
	24	.4	2.00	.44	3.52	.32	A	E	C+	(9)	B	C+
	25	.9	1.56	.19	-.43	8.36	A+	E-	C	(9)	C-	D+

* Basic condition.

* No definite estimate of maneuverability could be made; see "Results and Discussion" section.

the normal full-scale flight-testing procedure in which the airplane is released from a sideslipped attitude or disturbed by an abrupt rudder deflection. Because of the limited size of the test section in the free-flight tunnel, the model usually struck the tunnel wall after a sideslip disturbance before enough cycles of an oscillation could be obtained for determining the damping.

Apparent spiral stability is a measure of the ability of the model to fly, controls fixed, without an aperiodic divergence into the tunnel wall. One indication of spiral instability in the flight tests was the necessity for almost continuous corrective control to prevent an aperiodic divergence into the tunnel wall. Apparent damping in roll is the measure of the stiffness in roll of the model in response to aileron control.

In this investigation maneuverability is considered a measure of the ability to maneuver the model with aileron control

easily and quickly. Controllability is a measure of the ease with which the model can be kept flying satisfactorily in a wings-level attitude.

The general flight behavior is an indication of the overall flight characteristics as affected by all the various stability and control characteristics. A proper balance of oscillatory and aperiodic stability, controllability, and maneuverability is necessary to give satisfactory flying characteristics. The general-flight-behavior ratings are therefore considered the best basis for judging the relative merit of the various flight-test conditions.

RANGES OF VARIABLES

All flight tests were made at a lift coefficient of 1.0 and a wing loading of 3.85 pounds per square foot which corresponds to a value for the relative density parameter μ of 12.58 at sea level. The ranges of values of the four artificially varied

The experimental and calculated results are presented in figures 6 to 17. The flight ratings are presented in table II and examples of time histories showing the changes in the flight characteristics of the model throughout the variation of each derivative are presented in figures 6, 9, 12, and 15.

The calculated dynamic lateral stability characteristics of the model for the range of each derivative covered in the investigation are presented in figures 7, 10, 13, and 16 in the form of period and damping of the lateral oscillation and experimental values of period and damping of the short-period lateral oscillation determined from the flight-test records are also shown in these figures for comparison with the theoretical results. The damping of both the oscillatory motion and the aperiodic motion is expressed in terms of the damping factor $1/T_{1/2}$.

The calculated response of the model to rolling and yawing disturbances for various values of each derivative is presented in figures 8, 11, 14, and 17. The primary reason for making these calculations was to obtain a theoretical indication of the effect of changes in the various derivatives on the initial response and resulting motions for use in explaining the flight-test results.

The effects on dynamic stability, control, and general flight behavior of artificially varying the derivatives are discussed independently for each derivative. Results are presented for a wide range of values (both positive and negative) for each derivative; however, since damping of the lateral oscillation is the primary function of any artificial-stabilization system, only variations of the derivatives in the direction which produces improvement in oscillatory stability are discussed in detail.

The experimental results, based on flight ratings for oscillatory stability and general flight behavior, are summarized in figure 18. In this summary a comparison is made of the improvements in oscillatory stability and of the accompanying changes in general flight behavior obtained by varying the different derivatives.

The effects of each derivative on the total damping of the system are presented in figure 19. These results are presented in order to provide a better understanding of the effects of the different derivatives on oscillatory stability and general flight behavior.

A comparison of the calculated effects of the four derivatives is shown in figure 20 in order to show the relative effectiveness of each derivative in providing satisfactory oscillatory stability. For this comparison the period and damping factor have been scaled up so that the results can also be compared directly with the Air Force and Navy damping requirements (refs. 10 and 11). In scaling up these values the model was assumed to be a $\frac{1}{9}$ -scale model of an airplane; therefore, the period of the model was multiplied by 3 and the damping factor was divided by 3.

derivatives for which flight tests were made are given in the following table:

Derivative	Value for model in basic condition	Range tested
C_{n_r}	-0.21	-7.2 to 1.8
C_{l_p}	-0.32	-7.3 to 1.1
C_{l_r}	.13	-2.9 to 3.1
C_{n_p}	-.07	-.7 to .9

The values of the derivatives for the model in the basic condition were determined from force tests to an accuracy of two decimal places. For the artificial variation of the derivatives, however, the values could be determined to an accuracy of only one decimal place.

CALCULATIONS

Most of the calculations were made, time lag being neglected, by the method of reference 1 to determine the effects of large variations of the four derivatives on period and damping for the flight-test conditions listed in table II. The mass and aerodynamic parameters used in the calculations are also listed in table II.

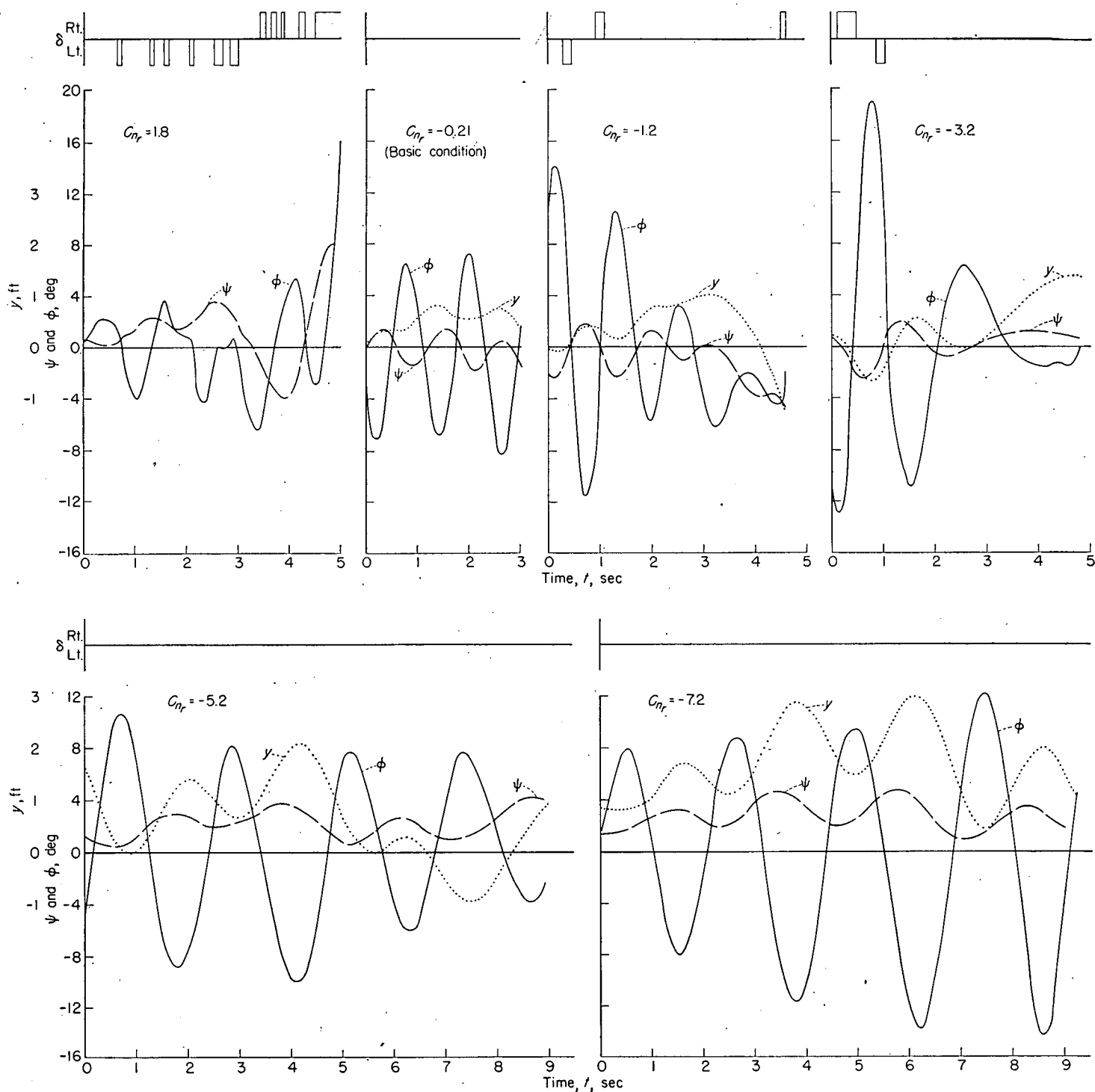
For certain conditions in which the experimental and calculated results were not in good quantitative agreement, additional calculations were made in which the effect of time lag in the artificial-stabilization device was considered. These calculations were made for a constant time lag of 0.05 second by the method of reference 3. Extensive calculations were not made by this method, however, because of the extremely laborious process involved and because a systematic determination of the effect of time lag on stability throughout the variation of the four derivatives was considered beyond the scope of the present investigation.

The damping of both the oscillatory and aperiodic motions is expressed in terms of the damping factor $1/T_{1/2}$, the reciprocal of the time to damp to one-half amplitude. Positive values of this damping factor indicate stability and negative values indicate instability (or time to double amplitude). Calculations of motions were also made by the method of reference 9 on a Reeves Electronic Analog Computer for some representative flight-test conditions (table II) to determine the response to a rolling- or a yawing-moment disturbance of 0.01. In these motion calculations the disturbance was applied in one direction for approximately one-half the calculated period of the oscillation and then applied in the opposite direction for an equal length of time.

RESULTS AND DISCUSSION

PRESENTATION OF RESULTS

The experimental results are presented primarily in the form of ratings for the dynamic stability, control, and general flight behavior based on the pilot's comments and, in some cases, these ratings are supplemented by time histories of the motions of the model taken from motion-picture records.

FIGURE 6.—Flight records of the lateral motions for various values of C_{nr} .

Improvement in damping of the short-period lateral oscillation. Further negative increases in the value of C_{nr} caused a reduction in damping of the oscillation; in fact, oscillatory instability was obtained with a value of C_{nr} of -7.2. It appeared to the pilot that the best damping of the oscillation was obtained with values of C_{nr} between -1 and -3.

When C_{nr} was varied in the positive direction from the basic condition, the lateral oscillation became unstable. This instability increased until, at a value of C_{nr} of 1.8, the model became so unstable that sustained flight was impossible. Neither the period nor the time for the oscillation to double amplitude could be estimated from the flight-test results in this range of C_{nr} , because the model could not be allowed to fly uncontrolled for more than a second or two at a time.

The comparison of the calculated and experimental values of period and damping of the lateral oscillation shown in figure 7 indicates good agreement for the various values of C_{nr} covered in the tests. These results indicate that maximum damping of the oscillation was obtained with a value of C_{nr} of about -2.0. For this value of C_{nr} , the lateral oscillation damped to one-half amplitude in about 0.9 second. These results also show that the period of the oscillation increased from about 1.4 seconds to about 2.2 seconds as C_{nr} was varied from -0.21 to -7.2.

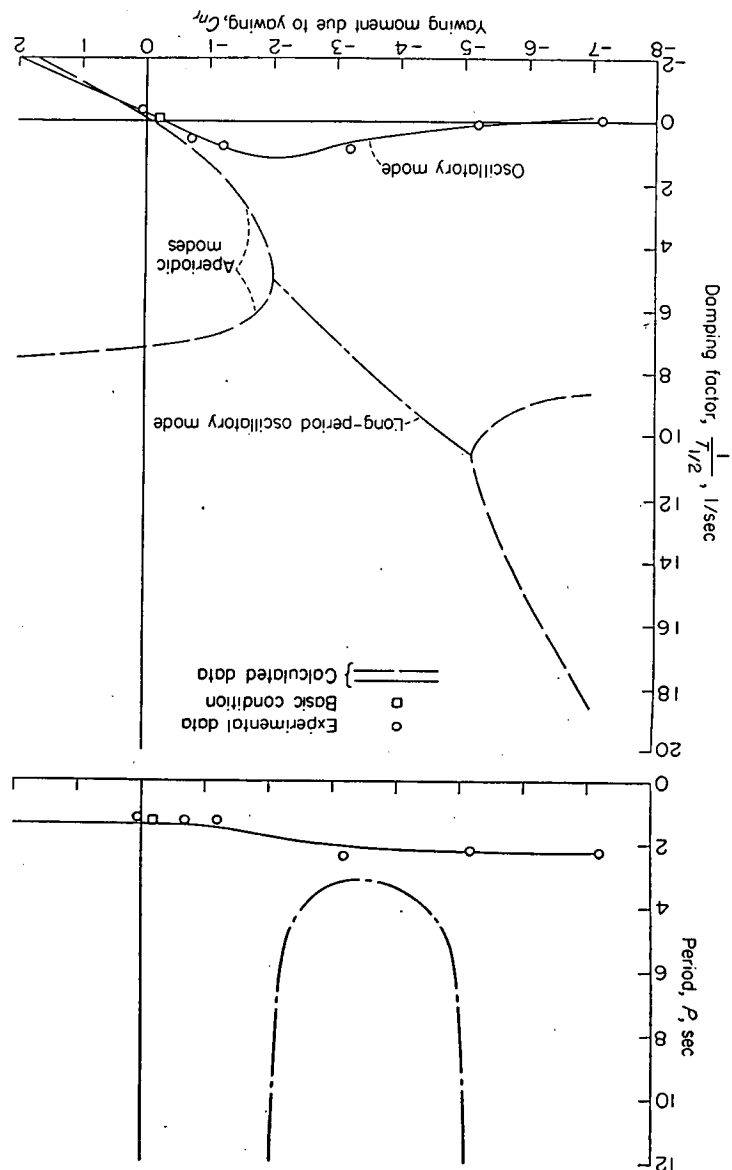
For the higher negative values of C_{nr} (-3.2 to -7.2), the flight tests indicated that the lateral motion of the model progressively changed from the normal Dutch roll oscillation to a pendulum type of oscillation that consisted mainly of roll and sideways displacement. The time histories of figure 6 show that at a value of C_{nr} of -7.2 the ratio of yaw to roll is approximately one-half the value obtained in the basic condition. This decrease in the ratio of yaw to roll is attributed to the fact that increasing the damping in yaw causes partial restraint of the yawing motions. This change in the nature of the lateral oscillation is also shown in the calculated motions in figure 8.

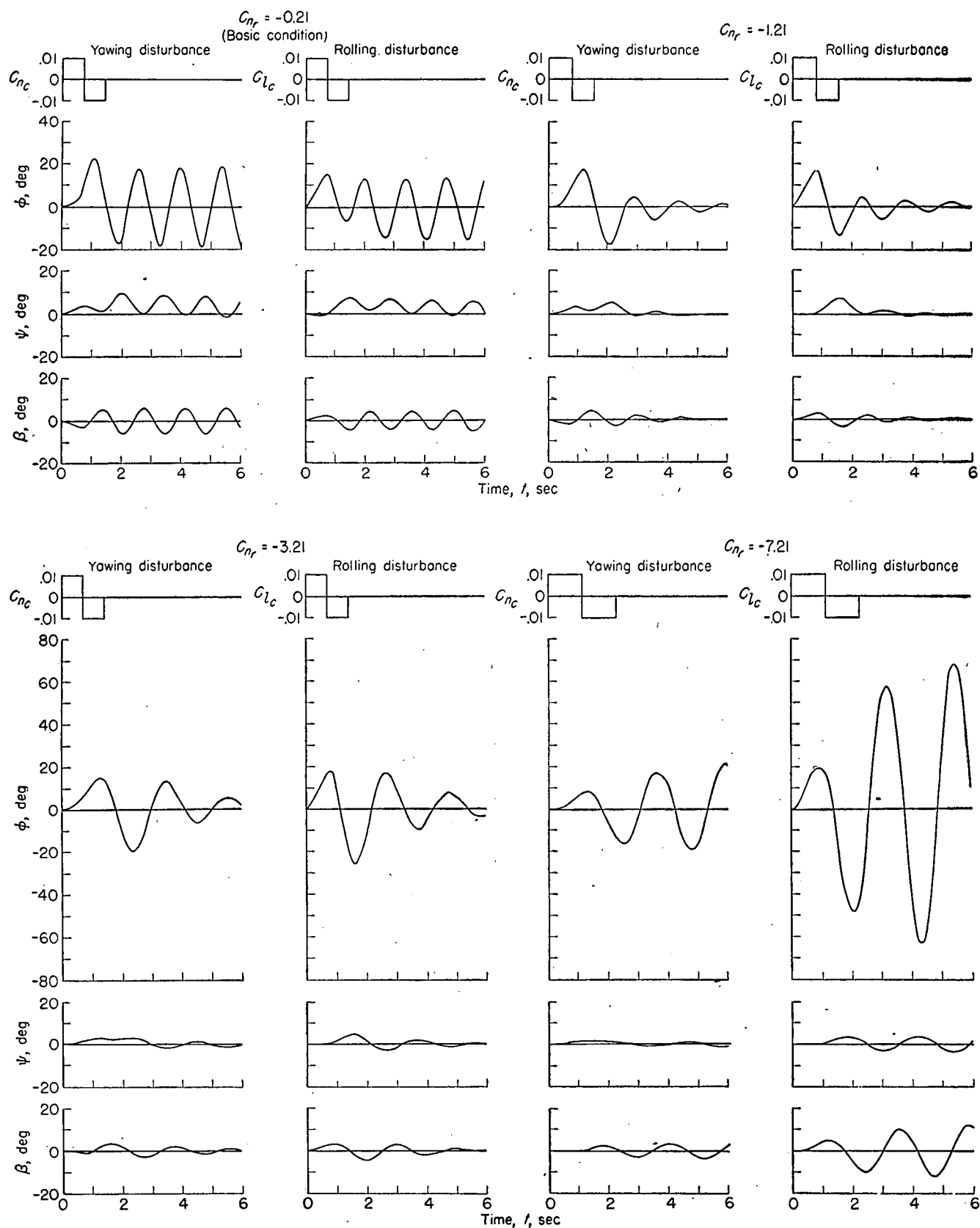
During the flight tests a change was also noted in the nonoscillatory dynamic lateral stability of the model as C_{nr} was varied. Although the damping of the aperiodic modes of motion could not be measured from the flight-test records, the pilot was aware of increasingly better apparent spiral stability as C_{nr} was increased negatively since the model would fly for long periods of time with controls fixed despite the natural gustiness of the air flow. This increase in apparent spiral stability is shown by the time histories (fig. 6) which indicate that it was possible to obtain longer uncontrolled flight records as C_{nr} was increased negatively despite the highly damped or unstable oscillations at the higher negative values of C_{nr} . Since the pilot considered this flight characteristic desirable, the best spiral-stability ratings were obtained with the higher negative values of C_{nr} .

As C_{nr} was increased in the negative direction, the damping of the lateral oscillation increased up to an optimum value and then decreased while the apparent spiral stability continued to improve. In this range the lateral control was good, and no apparent loss of maneuverability occurred with increasing C_{nr} . The best general flight behavior was obtained with a value of C_{nr} slightly greater than that which produced the greatest damping of the oscillation. A detailed discussion of the changes in dynamic stability, control, and general flight behavior is given in the following sections.

Dynamic stability.—In the basic condition ($C_{nr} = -0.21$), the model had neutral oscillatory stability. The flight records of figure 6 indicate that moderate increases in the value of C_{nr} in the negative direction caused a marked im-

Figure 7.—Calculated effect of C_{nr} on stability and comparison with experimental data.



FIGURE 8.—Calculated response of model to yawing or rolling disturbance for various values of C_{nr} .

characteristics obtained were a little better because of the better spiral stability and because the model appeared to be somewhat easier to control. As C_{nr} was further increased negatively, the progressive decrease in oscillatory stability and the appearance of the objectionable pendulum type of oscillation resulted in poorer general flight behavior. With values of C_{nr} greater than -5.2 (cases 1 and 2), the overall flight characteristics of the model were unsatisfactory because of the lightly damped or unstable oscillation. When C_{nr} was increased in the positive direction from the basic condition (cases 8 and 9), the general flight behavior became very poor because of both the oscillatory and spiral instability.

EFFECT OF ROLLING MOMENT DUE TO ROLLING C_l

Small negative increases in the value of C_l caused the damping of the lateral oscillation to improve rapidly, but further negative increases in C_l resulted in no further improvement in the oscillatory stability. Most of the damping added to the system by these further increases in C_l was absorbed by the aperiodic rolling mode so that the model appeared to be very stiff in roll. Although this flight characteristic caused the model to have very poor maneuverability in roll, the model was very easy to control in a wings-level attitude. The general flight behavior was considered satisfactory only for small negative values of C_l .

Dynamic stability.—The results for the damping of the lateral oscillation indicate that as C_l was increased in the negative direction from the basic value of -0.32 the damping rapidly improved for values of C_l up to about -0.6 (case 13, table II). As C_l was further increased, the oscillation could not be initiated because the rolling mode was so heavily damped that the model was essentially restrained from rolling (cases 10 and 11). The time histories in figure 9 show this change in the nature of the motion. In the high negative range of C_l (-1.0 to -7.0), some flights were made in which the initiation of oscillations by rudder deflection was attempted, but these attempts to obtain oscillations were not successful because the model sideslipped into the tunnel wall before enough cycles of the oscillation were obtained to permit measurement of the damping. With a value of C_l of -0.8 , the oscillation damped to one-half amplitude in about 1.4 seconds.

The flight records show that increasing C_l in the positive direction caused the lateral oscillation of the model to become unstable. This instability increased very rapidly and, with a rather small positive value of C_l (0.1), sustained flight was impossible. A comparison of the experimental and calculated values of period and damping of the model is presented in figure 10. These results show that the experimental values of period and damping are in fairly good agreement with the calculated values for the limited range of negative C_l , where the period and damping could be measured. The calculations show that

The calculated stability in the negative C_{nr} range (fig. 7) indicates that the aperiodic modes merge to form a second oscillation for values of C_{nr} between -2.0 and -5.2 . This oscillation was so heavily damped that it was never observed in the model flights. The constantly increasing apparent spiral stability observed in the flight tests as C_{nr} was varied in the negative direction appears to correspond to the increasing stability of first the spiral mode and then the long-period oscillation.

Control.—The lateral control characteristics are presented in table II in the form of ratings based on the pilot's opinion of the controllability and maneuverability of the model for various values of C_{nr} . It may be seen from this table that as C_{nr} was varied in the negative direction the controllability improved. In the basic condition (case 7), despite the undamped oscillation, the model could be flown with only occasional corrective control deflections to keep the model in the center of the test section. As C_{nr} was increased in the negative direction, the model required progressively less control and with the higher values of C_{nr} would fly uncontrolled for relatively long periods of time. (See fig. 6.) The best lateral control of the model was obtained with a value of C_{nr} of -3.2 , when the lateral motion of the model following a disturbance would completely die out before any corrective control was required. When C_{nr} was varied in the positive direction, the lateral control characteristics became worse. The model was barely controllable with the most positive value of C_{nr} tested (case 9) since the unstable oscillatory motion and the unstable spiral mode necessitated constant corrective control deflections to prevent the model from crashing.

In the opinion of the pilot the model had adequate maneuverability throughout the range of C_{nr} tested in that the model could be maneuvered to any desired position in the tunnel quickly and easily. In fact, had the model not been easily maneuverable, flight with positive values of C_{nr} might have been impossible because of both oscillatory and spiral instability. In the negative range of C_{nr} , it was not possible to note the decreased maneuverability or increased stiffness in making turns which has been experienced with some airplanes equipped with yaw dampers (ref. 12) since steady turning maneuvers cannot be made in the Langley free-flight tunnel because of restrictions imposed by the size of the test section.

General flight behavior.—The general flight behavior of the model in the basic condition (case 7) was not satisfactory because of the undamped lateral oscillation. As C_{nr} was increased negatively, the general flight behavior of the model improved as a result of both the increased damping of the oscillation and the improved spiral stability. The best general flight behavior was obtained with a value of C_{nr} of -3.2 (case 3). Although this value of C_{nr} produced less damping of the oscillation than the maximum obtained with C_{nr} equal to -2.2 , the pilot felt that the overall flight

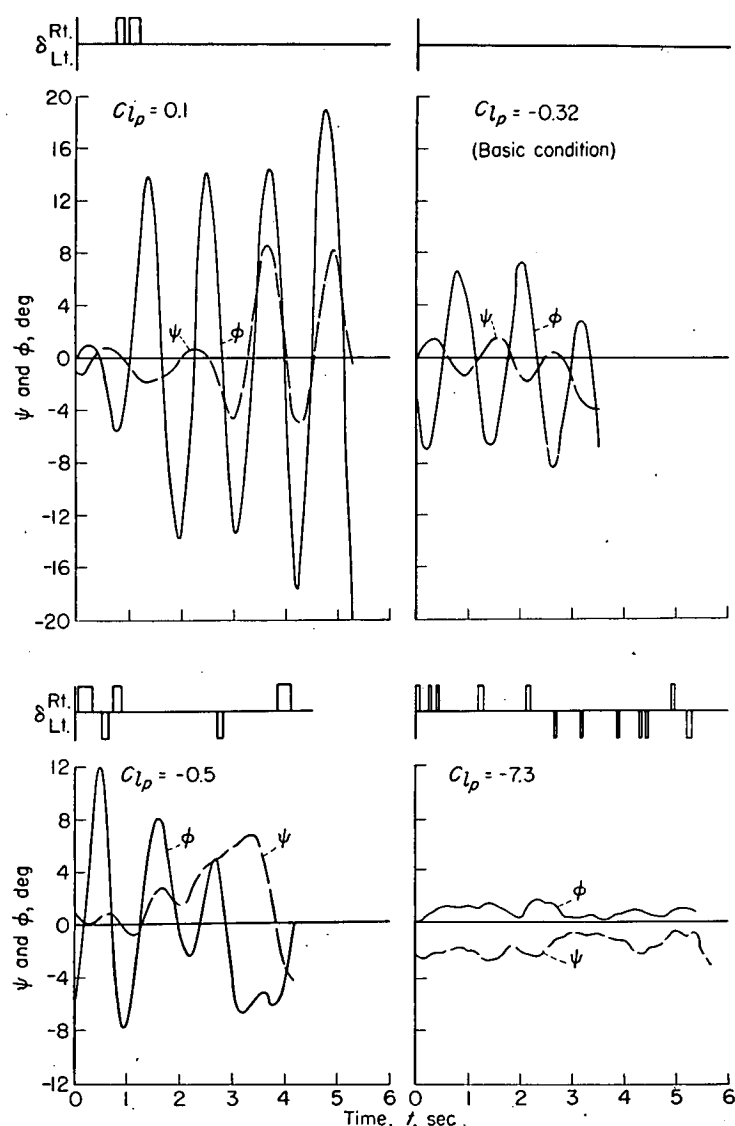


FIGURE 9.—Flight records of the lateral motions for various values of C_{l_p} .

for negative values of C_{l_p} greater than -0.9 damping of the oscillation did not increase further. Although the calculations correctly predicted the existence of an unstable oscillation in the positive C_{l_p} region, the oscillatory instability ($\frac{1}{T_{1/2}} = -0.50$) determined from the flight-test results for $C_{l_p} = 0.10$ was not so severe as that predicted by the calculations in which time lag was assumed to be negligible ($\frac{1}{T_{1/2}} = -1.70$). Additional calculations showed that for this same value of C_{l_p} a value of $1/T_{1/2}$ of zero (neutral stability) would be obtained with a time lag of about 0.10 second. By interpolation the calculated results can be assumed to indicate that the actual time lag of 0.05 second known to exist in the stabilization device would result in a value of $1/T_{1/2}$ of about -0.85 , which is in better agreement with the experimentally determined value of -0.50 . The discrepancy

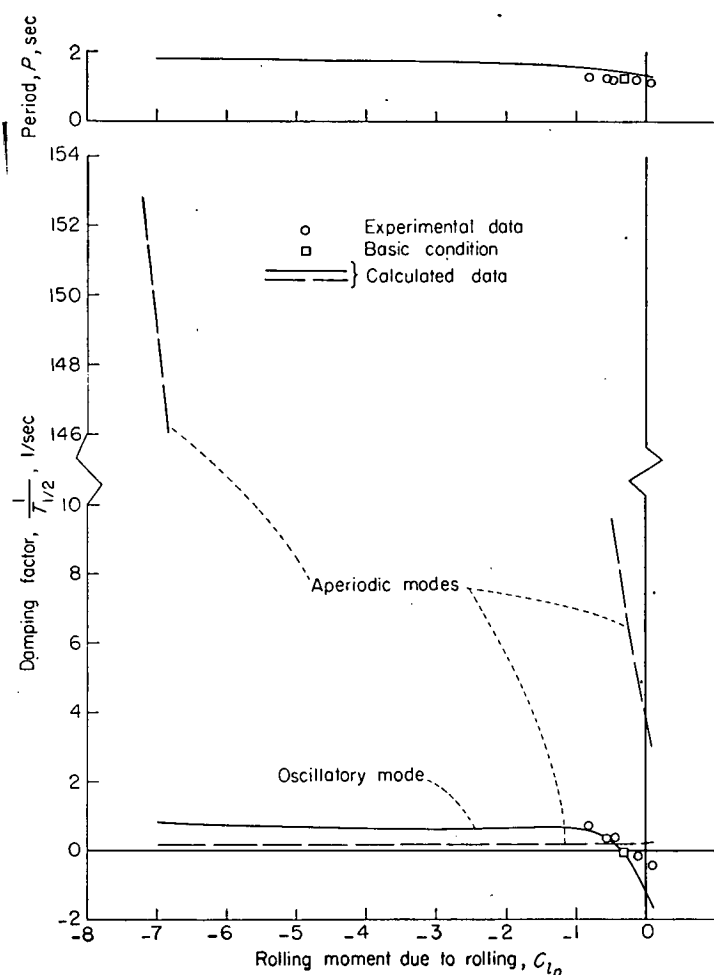


FIGURE 10.—Calculated effect of C_{l_p} on stability and comparison with experimental data.

between the measured and calculated values of damping shown in figure 10 may therefore be attributed at least partly to the effect of time lag in the stabilization system.

The most significant change in the dynamic stability of the model as C_{l_p} was varied in the negative direction was the very rapid increase in stability of the rolling mode. In the flight tests this increase in rolling stability was evidenced by an increase in the stiffness in roll as C_{l_p} was increased negatively. With very large negative values of C_{l_p} the model was essentially restrained from rolling. When C_{l_p} was increased in the positive direction from the basic condition, the model became overly sensitive to aileron control; this sensitivity indicates that the stability of the rolling mode decreased. No noticeable change in damping of the spiral mode of motion occurred throughout the C_{l_p} range covered in the tests.

The tendency toward restraint in roll experienced in the flight tests is indicated in the calculated results (fig. 10) which show that one of the aperiodic modes (the rolling mode) became increasingly stable as C_{l_p} was increased negatively. The calculated response for various values of C_{l_p} presented in figure 11 shows the reduction in amplitude of the rolling motion as C_{l_p} was increased negatively from the basic condition.

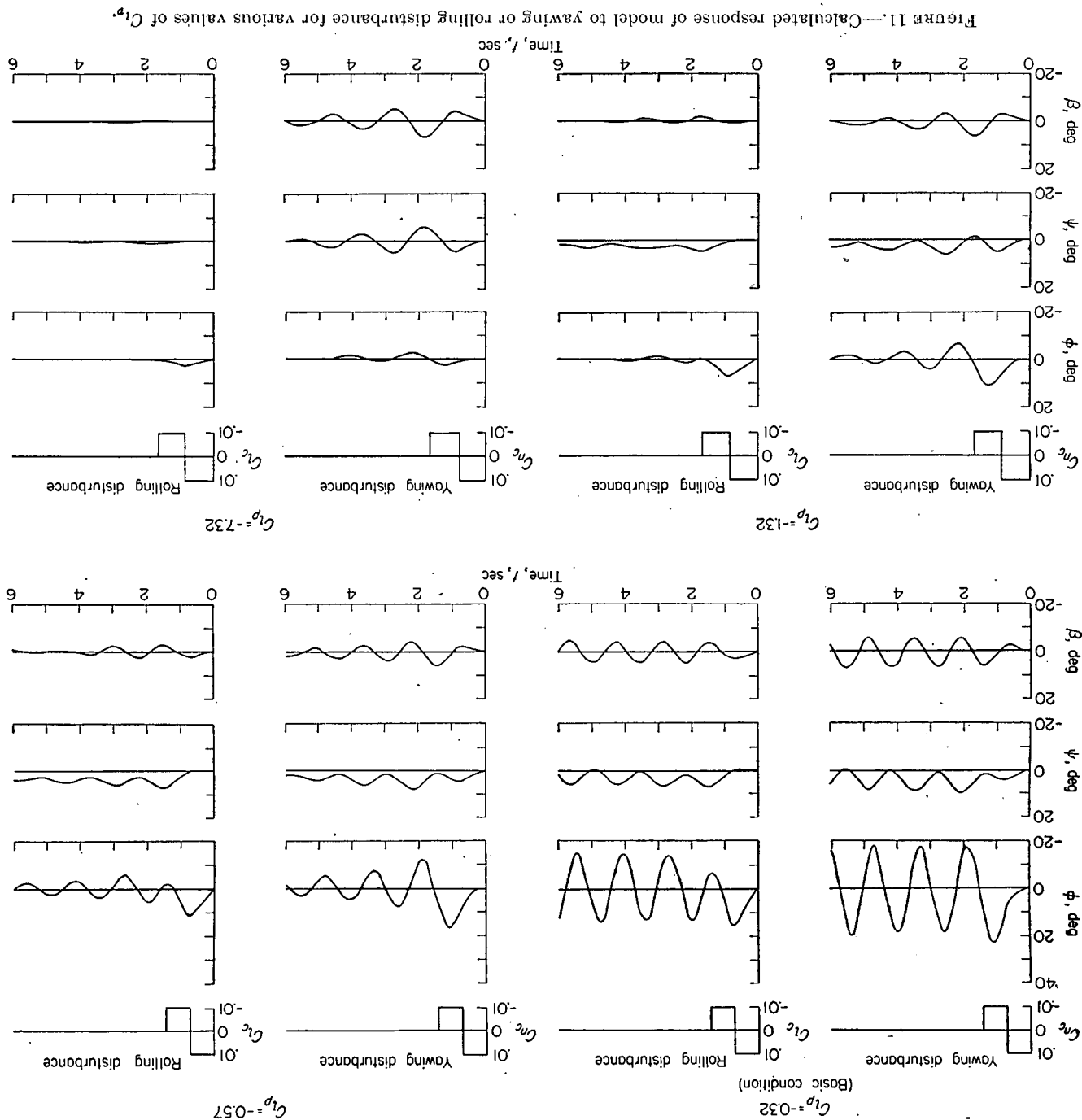


FIGURE 11.—Calculated response of model to yawing or rolling disturbance for various values of C_{lp} .

flow in a steady wings-level attitude. The best lateral control characteristics were obtained with a value of C_{lp} of about -0.6, where the oscillation required little control and the stiffness in roll was not excessive. The overall lateral control characteristics of the model with very large negative values of C_{lp} were considered unsatisfactory because of the reduced maneuverability.

The adverse effect of high negative values of C_{lp} on maneuverability might be eliminated without sacrificing the desirable steadiness in wings-level flight by utilizing a control system similar to that suggested in reference 12 for an airplane equipped with a yaw damper. In performing maneuvers with an airplane equipped with one form of such a

The flight records (fig. 9) show that the negative damping in roll (positive C_{lp}) caused the model to have an unstable oscillation rather than an aperiodic divergence or roll-off. Apparently the reason for this result is the fact that the rolling mode was still stable for the highest positive value of C_{lp} covered in the tests. (See fig. 10.)

Control.—The lateral-control ratings presented in table II indicate that increasing C_{lp} in the negative direction caused the model to have good controllability but poor maneuverability (cases 10 to 14). The tendency toward restraint in roll imposed by high negative values of C_{lp} , although undesirable for maneuverability, caused the model to be very steady and to require very little corrective control when

control system, deflection of the control stick would not directly deflect the ailerons but would modify the signal from the rate-sensing device to the servoactuator such that the aileron would be deflected in the manner required to perform the desired maneuver. The stiffness in roll apparent to the pilot could thereby be greatly reduced. In some preliminary tests with another model, results with this type of control system have been very satisfactory.

When C_{l_p} was varied in the positive direction from the basic condition (from case 7 to case 15), constant corrective control was required because of the unstable oscillation, but the model was highly maneuverable in roll. This increase in maneuverability was attributed to the reduced damping of the rolling mode. (See fig. 10.)

General flight behavior.—The two important factors affecting the overall flight characteristics of the model when C_{l_p} was varied were the damping of the oscillation and the overdamping of the rolling mode. The best general flight behavior was obtained with a value of C_{l_p} of -0.8 (case 12). For this condition, the oscillation damped to one-half amplitude in about 1.4 seconds and the tendency toward restraint in roll was not considered too objectionable, although the model did have less rolling maneuverability than is normally desired. Steady wings-level flights with this value of C_{l_p} were very smooth and the model required very little corrective control.

For values of C_{l_p} between -0.5 and 0.1 (cases 14, 7, and 15), the general flight behavior was poor because of unsatisfactory damping of the lateral oscillation. With values of C_{l_p} between -0.8 and -7.3 (cases 10 to 12), the general flight behavior was considered unsatisfactory because the rolling mode was so heavily damped that the rolling maneuverability of the model was impaired.

EFFECT OF ROLLING MOMENT DUE TO YAWING C_{l_r}

Increasing C_{l_r} in the positive direction improved the damping of the lateral oscillation but caused the model to become very spirally unstable. No flight condition in which C_{l_r} was varied was considered appreciably better than the basic flight condition.

Dynamic stability.—The flight tests show that the damping of the lateral oscillation improved very slightly when C_{l_r} was increased from the basic value of 0.13 to a value of 0.3 (fig. 12 and table II), but the model became more difficult to fly despite this increase in damping. With values of C_{l_r} greater than 0.3 , attempts to measure the damping of the oscillation were not successful because almost continuous corrective control was required to keep the model from diverging into the tunnel walls. In this range of C_{l_r} , however, it was apparent to the pilot that the damping of the oscillation was increasing with increasing C_{l_r} . (See cases 19 and 20, table II.)

When C_{l_r} was varied in the negative direction from the basic condition, oscillatory instability was obtained, but even with the highest negative value of C_{l_r} covered in the tests (-2.9), this instability was not great enough to make

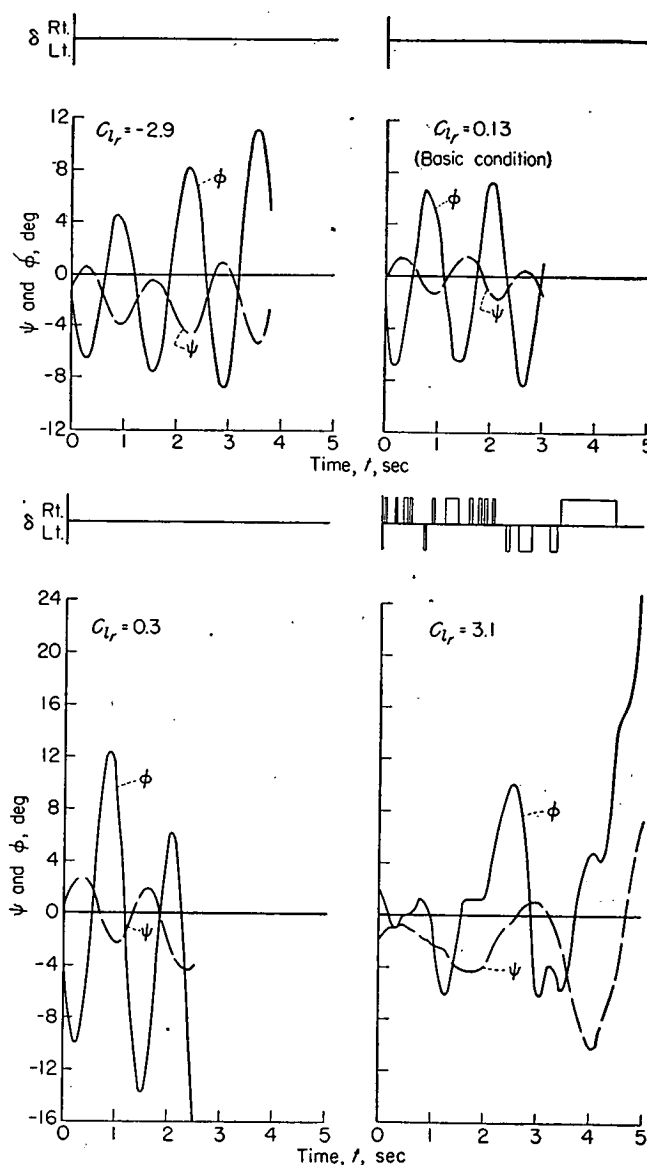


FIGURE 12.—Flight records of the lateral motions for various values of C_{l_r} .

the model unflyable. For this value of C_{l_r} , the oscillation doubled amplitude in about 3.0 seconds.

A comparison of the calculated and experimental values of period and damping as affected by changes in C_{l_r} is presented in figure 13. In the positive C_{l_r} range above 0.3 , no quantitative data on the damping of the oscillation could be obtained, as previously mentioned. The data of figure 13 show that, for all values of C_{l_r} , except those close to the basic value of 0.13 , the calculated damping of the oscillation is in rather poor agreement with the experimental results. In the negative C_{l_r} range the instability of the oscillation was not nearly as severe as that predicted by the calculations in which time lag was neglected. For the value of C_{l_r} of -2.9 , the measured value of $1/T_{1/2}$ was about -0.35 , whereas that calculated with the assumption of zero time lag was approximately -2.50 . In an effort to explain this large difference between the experimental and calculated results, additional calculations were made in

indicate that almost no variation in period occurred throughout the range of C_l . The calculated response of the model for a value of C_l of 3.13 (fig. 14) illustrates the aperiodic divergence which made it impossible to obtain a quantitative measurement of damping in the flight tests for large positive values of C_l .

The most noticeable change in stability observed in the flight tests was the severe spiral divergence encountered with high positive values of C_l . Spiral instability occurred with a value of C_l of about 0.3 and became more severe as C_l was increased. With a value of C_l of 3.1 (case 20), this spiral instability was so great that most of the flights ended in crashes. This increase in spiral instability observed in the flight tests is predicted by the damping calculations of figure 13 and is illustrated by the calculated response to rolling and yawing disturbances in figure 14.

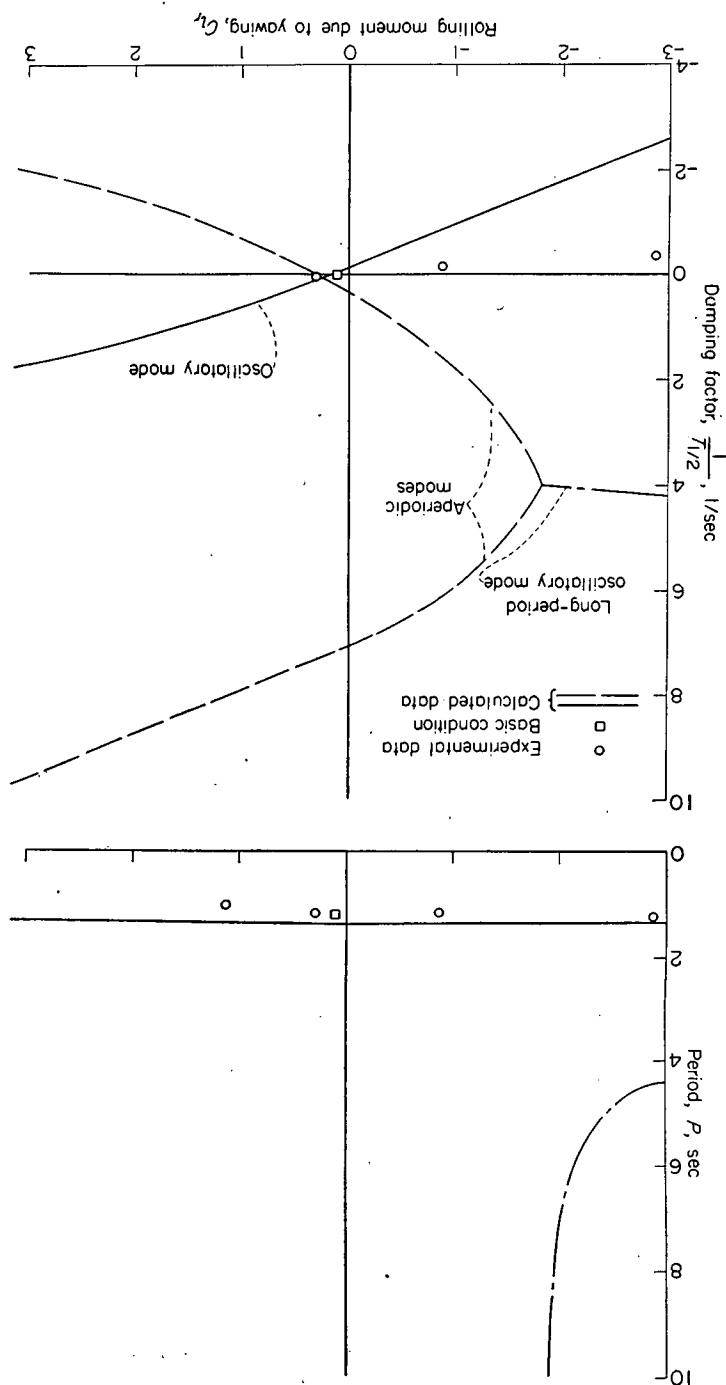
In the flight tests the spiral stability appeared to be improved as C_l was increased in the negative direction since the model would fly for long periods of time with controls fixed. This increase in spiral stability was also predicted by the calculations. (See fig. 13.) The long-period heavily damped oscillation, which the calculations show is formed from the merger of the spiral and rolling modes in the negative C_l range, was not apparent in the flight tests.

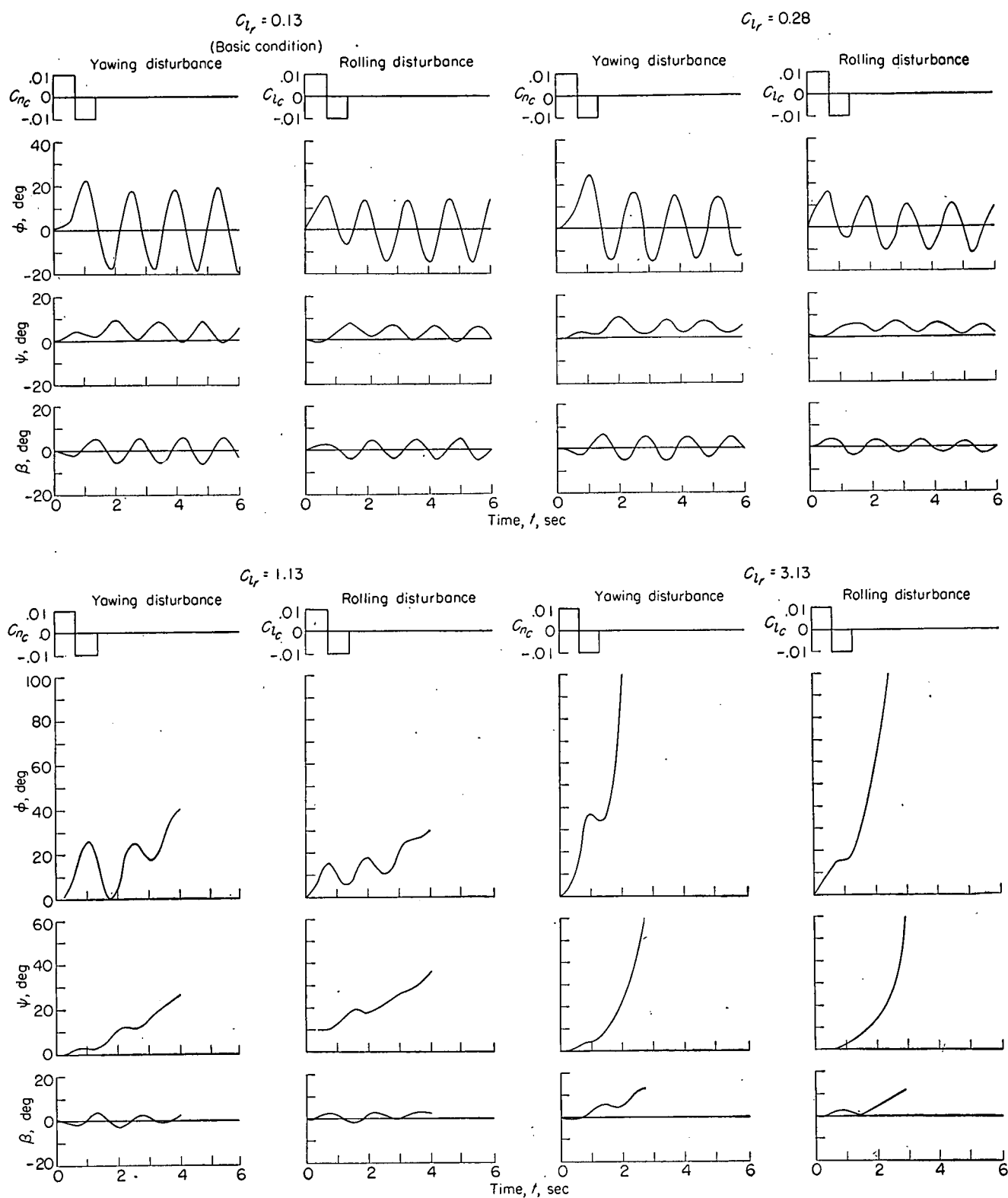
From these results the variation of the derivative C_l appears to offer very little hope for improving the overall lateral stability characteristics of an airplane. This derivative, however, may in some cases be used to redistribute the damping between the oscillatory and aperiodic modes if surplus damping of the aperiodic modes is initially present. Preliminary calculations have indicated that the damping of the oscillation obtained with C_n alone could be improved appreciably by utilizing C_l to redistribute part of the excess damping of the spiral mode to the oscillatory mode.

Control.—As C_l was varied in the positive direction, the model became more maneuverable but the controllability became worse. The increased maneuverability caused the model to be highly responsive to the slightest control deflection at the higher values of C_l so that the model became very difficult to control. Many of the flights with a value of C_l of 3.1 ended in crashes because the model was inadvertently overcontrolled; yet, reducing the control deflection did not seem advisable because at times large control deflections were required to recover from the rapid roll-off into a spiral. (See fig. 12.) In this case the model appeared to be highly maneuverable when the pilot rolled the model from an initial wings-level flight attitude; however, in the attempt to recover from a large angle of bank following such a roll-off, application of full opposite control did not produce immediate recovery. The maneuverability in this condition was therefore considered not entirely satisfactory. Because of the inability to establish a definite overall estimate of the maneuverability with positive values of C_l , no maneuverability ratings were assigned for these conditions in table II.

which the constant time lag of 0.05 second was taken into account. The result of these calculations $\left(\frac{1}{T_{1/2}} = -1.32\right)$ was in closer agreement with the experimental value of $1/T_{1/2}$. The discrepancy between the measured and calculated values of damping shown in figure 13 may therefore be attributed at least partly to the effect of time lag in the stabilization device. The calculated period, which was relatively unaffected by time lag, is in fairly good agreement with the period determined from flight records for all values of C_l where oscillations could be obtained. These results

Figure 13.—Calculated effect of C_l on stability and comparison with experimental data.



FIGURE 14.—Calculated response of model to yawing or rolling disturbance for various values of C_{lr} .

When C_l was varied in the negative direction (cases 16 and 17), the controllability became worse because corrective control was required to prevent the unstable oscillation from building up to large amplitudes. Even at the highest negative value of C_l tested, however, the oscillatory instability was easily controlled. The maneuverability of the model was satisfactory in the negative range of C_l and was not appreciably different from that of the basic condition.

General flight behavior.—The only improvement in the general flight behavior that resulted from varying C_l was obtained with a very small positive increase (from 0.13 to 0.3), and this improvement was very slight. In this condition (case 18) the slight increase in oscillatory stability was considered more important to the general flight behavior than the decrease in controllability. With further positive increases in C_l (cases 19 and 20), the general flight behavior became worse despite the increase in oscillatory stability. Poor controllability and severe spiral instability, which more than offset the increased damping of the lateral oscillation, were the causes of this poor general flight behavior. As C_l was increased in the negative direction, the unstable oscillation caused the general flight behavior of the model to become worse.

These results indicate that very little improvement in overall flight behavior of an airplane can be obtained with a change in C_l , except, perhaps, in the case of an airplane with a substantial amount of aperiodic stability in the basic condition.

EFFECT OF YAWING MOMENT DUE TO ROLLING C_{np}

Increasing the value of C_{np} in the positive direction caused a very rapid improvement in damping of the lateral oscillation, but this improvement in damping was obtained at the expense of the normally well-damped rolling mode. The decrease in the stability of the rolling mode caused the controllability and hence the general flight behavior of the model to become progressively worse.

Dynamic stability.—The results of flight tests indicated that a small positive increase in the value of C_{np} caused a large improvement in damping of the lateral oscillation. The time histories of figure 15 show that as the value of C_{np} was increased from -0.07 (basic condition) to 0.3 the neutrally stable oscillation became well-damped. For this value of C_{np} the oscillation damped to one-half amplitude in about 0.8 second. With further increases in the value of C_{np} to 0.9, quantitative values for damping of the oscillation could not be measured from the flight records because the poor lateral flight behavior of the model required almost constant corrective control. In this range of C_{np} (cases 23 to 25), however, it was apparent to the pilot that the damping of the oscillation was increasing with increasing C_{np} .

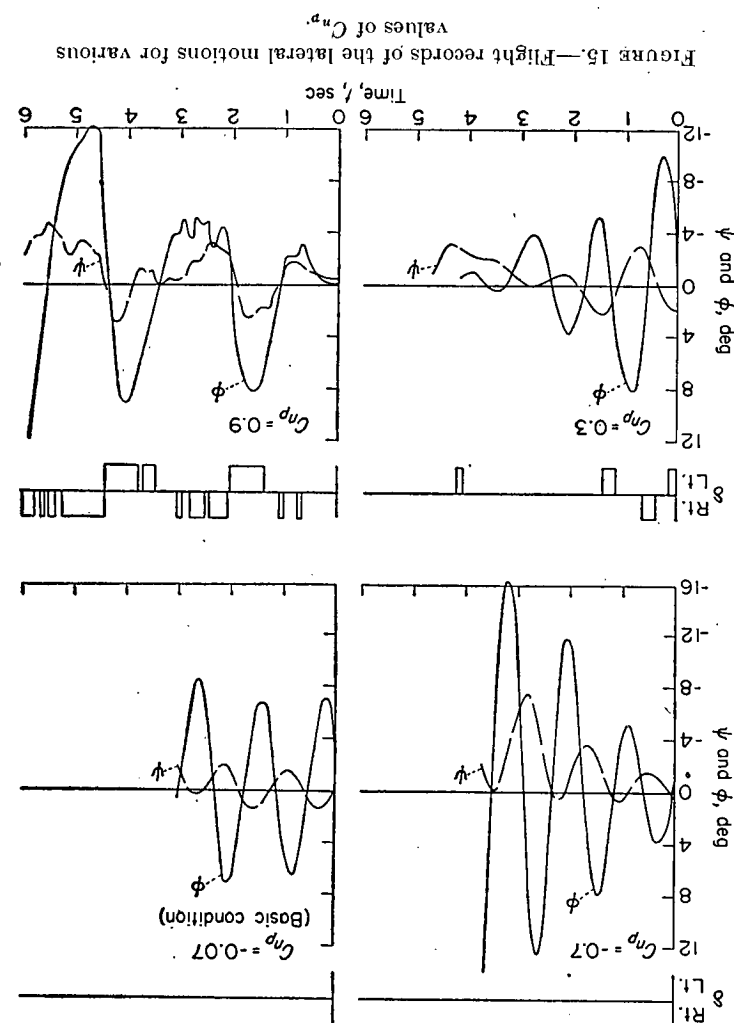


FIGURE 15.—Flight records of the lateral motions for various values of C_{np} .

(See table II.) Sustained flight was impossible with values of C_{np} greater than 0.9. The results of flight tests indicated that as C_{np} was varied in the negative direction from the basic condition the lateral oscillation became unstable. Because of this increase in oscillatory instability, the negative range of C_{np} that could be experimentally investigated was very limited. With negative values of C_{np} larger than -0.7 , the model was unflyable.

In the comparison of the experimental and calculated oscillatory stability of the model for various values of C_{np} (fig. 16), the theory is seen to be in fairly good agreement with the experimental results for damping of the oscillation. The increase in period for small positive values of C_{np} predicted by the calculations, however, was not observed in the flight tests. The calculations predict a continued increase in damping of the short-period oscillation for positive values of C_{np} larger than the maximum value tested (0.9) for which flights could be made. The calculated response of the model (fig. 17) for the value of C_{np} of 0.88 shows the aperiodic divergence

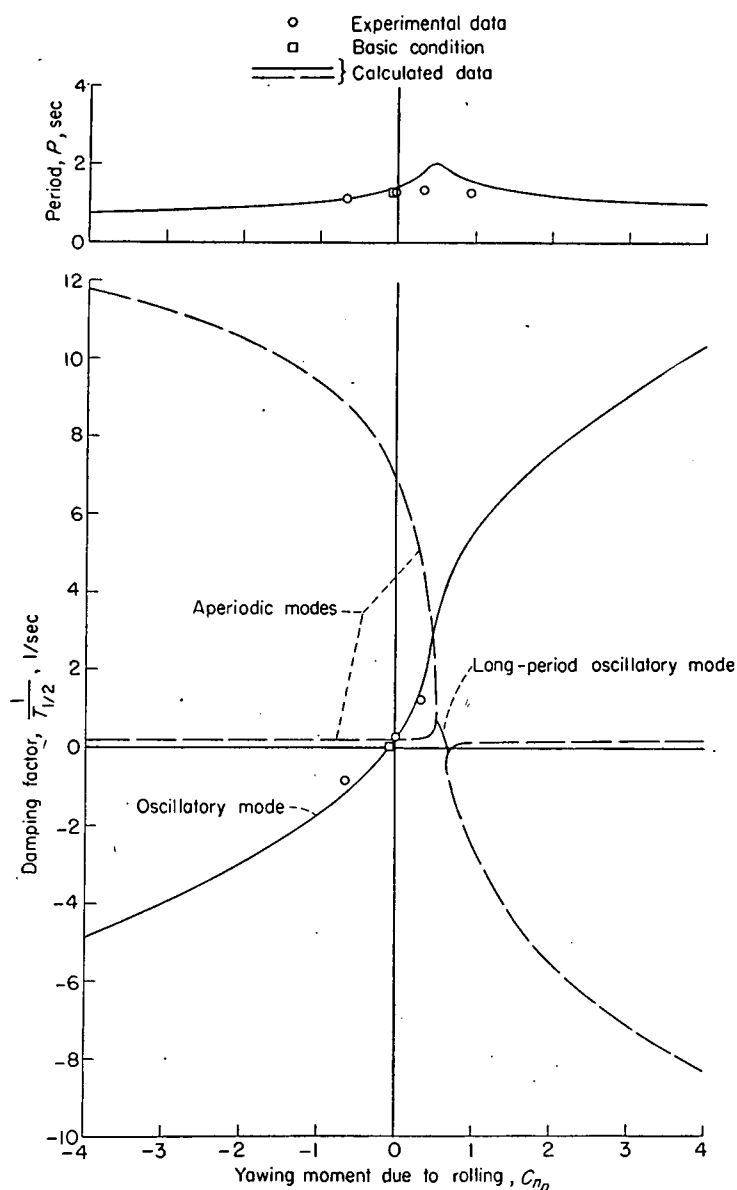


FIGURE 16.—Calculated effect of C_{n_p} on stability and comparison with experimental data.

which made a quantitative measurement of damping impossible to obtain in the flight tests for this case. In the negative range of C_{n_p} the calculations verify the highly unstable oscillation observed in the flight tests.

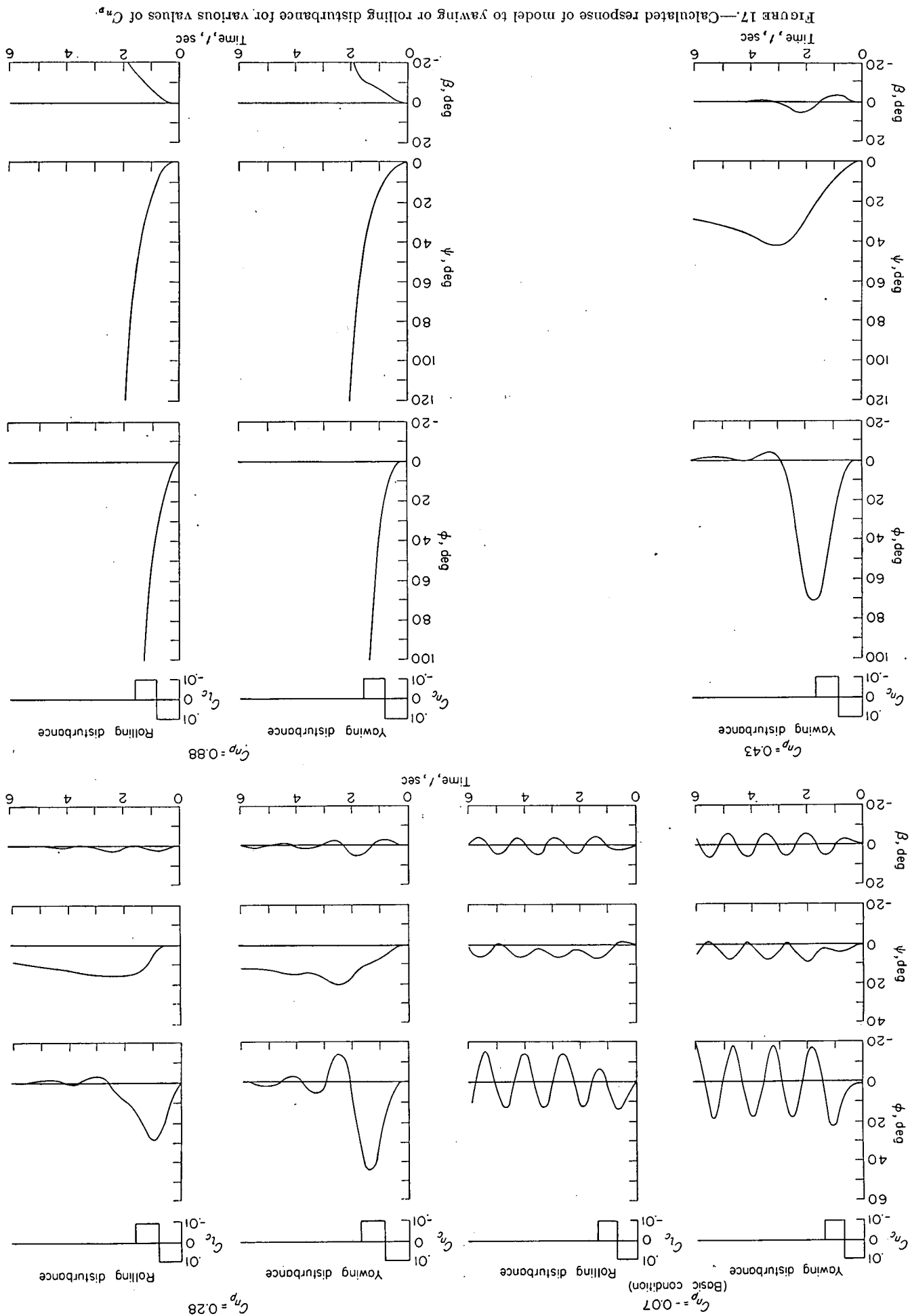
The improvement in oscillatory stability with positive values of C_{n_p} was accompanied by a decrease in the stability of the aperiodic phases of the motion. Flight tests were limited in this range by a type of instability which bore a close resemblance to the spiral instability observed in the flight tests with positive C_{l_r} . The model became very touchy to fly as C_{n_p} was increased up to 0.4 and became extremely difficult to control for values of C_{n_p} greater than 0.4. Because of this instability sustained flight was impos-

sible for values of C_{n_p} greater than 0.9. As the value of C_{n_p} was varied through this range (-0.07 to 0.9), the pilot complained of an increasingly strong tendency of the model to go into a tight turn in response to normal aileron control. To the pilot this tendency appeared to be a severe case of spiral instability. The results of calculations, however, show that the stability of the spiral mode remained unchanged up to a value of C_{n_p} of 0.5 , whereas the stability of the rolling mode decreased rapidly. (See fig. 16.) A decrease in stability of the rolling mode therefore might sometimes be mistaken for spiral instability.

The calculated results in figure 16 show that, although the rolling mode remained stable up to the point of its merger with the spiral mode at a value of C_{n_p} of about 0.55 , it was considerably less damped than in any other flight condition experienced in these tests. At the value of C_{n_p} of 0.55 , the two aperiodic modes merged to form a long-period oscillation which became unstable at a value of C_{n_p} of about 0.65 . This oscillation was not observed in the flight tests because of its extremely long period of over 40 seconds. Immediately after it became unstable, the long-period oscillation broke up to form two new aperiodic modes, one of which became increasingly unstable as C_{n_p} was increased further.

The results of these tests and calculations indicate that the derivative C_{n_p} might possibly be useful for redistributing the natural damping of an airplane in cases where the airplane has more than adequate damping of the rolling mode. The results of reference 2 indicate that the use of C_{n_p} in combination with C_{l_r} will provide an increase in oscillatory stability without a loss in rolling stability since, as previously discussed, the use of the derivative C_{l_r} alone causes a large increase in the stability of the rolling mode. Use of C_{n_p} alone, however, obviously is limited to values less than those which would cause the undesirable aperiodic motions experienced in these tests.

Control.—Despite the increased damping of the lateral oscillation as C_{n_p} was increased from 0.3 to 0.4 (cases 23 and 24), the controllability of the model became worse as a result of the increase in apparent spiral instability. With small positive increases in the value of C_{n_p} , the maneuverability of the model improved, and with the highest positive value of C_{n_p} covered in the tests (0.9 , case 25), the model appeared to be highly maneuverable when the pilot rolled the model from an initial wings-level attitude. As in the case of high positive C_{l_r} , in an attempt to recover from a large angle of bank following such a roll-off, application of full opposite control did not produce immediate recovery. The maneuverability in this case was therefore considered not entirely satisfactory. Because of the inability to establish an overall estimate of the maneuverability with positive values of C_{n_p} , no maneuverability ratings were made for these conditions in table II.



When C_{n_p} was varied in the negative direction the controllability became poor because constant corrective control was required to prevent the unstable oscillation from building up to a large amplitude. The model was uncontrollable with values of C_{n_p} more negative than -0.7 . There was no appreciable change in the maneuverability of the model with negative increases in C_{n_p} .

General flight behavior.—The increased damping of the oscillation obtained with the small positive values of C_{n_p} provided an improvement in the general flight behavior despite the decrease in apparent spiral stability. The pilot felt that, with the small positive values of C_{n_p} , the slight tendency toward spiral instability (which, in reality, was decreased damping of the rolling mode) was not highly objectionable because only small amounts of corrective control were required. With further positive increases in the value of C_{n_p} , however, the unstable aperiodic tendency became so severe that the general flight behavior was unsatisfactory even though the oscillatory stability continued to improve. When C_{n_p} was varied in the negative direction from the basic condition, the general flight behavior became worse because of the unstable oscillation.

COMPARISON OF EFFECTS OF THE ROTARY DERIVATIVES

Dynamic stability and general flight behavior.—The summary of results presented in figure 18 provides an indication of the relative merit of changes in the various derivatives. This summary, which is based on the flight ratings for oscillatory stability and general flight behavior (table II), compares the improvement in oscillatory stability and the accompanying changes in general flight behavior obtained by varying the different derivatives.

Use of C_{n_r} appears to produce the most satisfactory results since it provided the greatest amount of damping of the oscillation before introducing adverse flight characteristics. Although the results of figure 18 show that C_{l_p} produced approximately the same maximum damping of the oscillation as C_{n_r} , the poor maneuverability caused by the stiffness in roll which resulted from negative increases in C_{l_p} prevented good flight behavior from being obtained. In fact, for values of the derivatives of about -2 or -3 where the damping was essentially the same for the two derivatives, the flight behavior for C_{l_p} was considered poor whereas that for C_{n_r} was good.

Although the two cross derivatives C_{l_r} and C_{n_p} actually produced a greater improvement in the damping of the oscillation than the two damping derivatives C_{n_r} and C_{l_p} , they provided less improvement in general flight behavior. In fact, because of the severe apparent spiral instability produced by increases in these derivatives, satisfactory general flight behavior could not be obtained for any condition in which C_{l_r} was varied, and only barely satisfactory general flight behavior could be obtained with C_{n_p} .

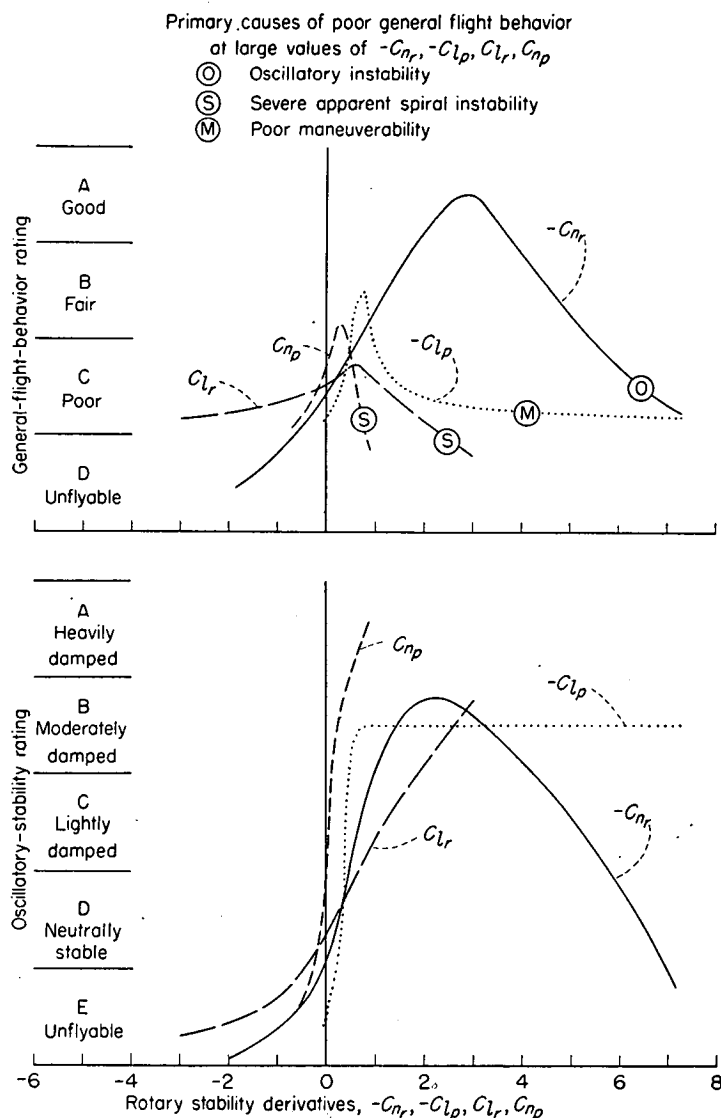


FIGURE 18.—Effect of variation in rotary derivatives on general flight behavior and oscillatory stability. (Data from table II.)

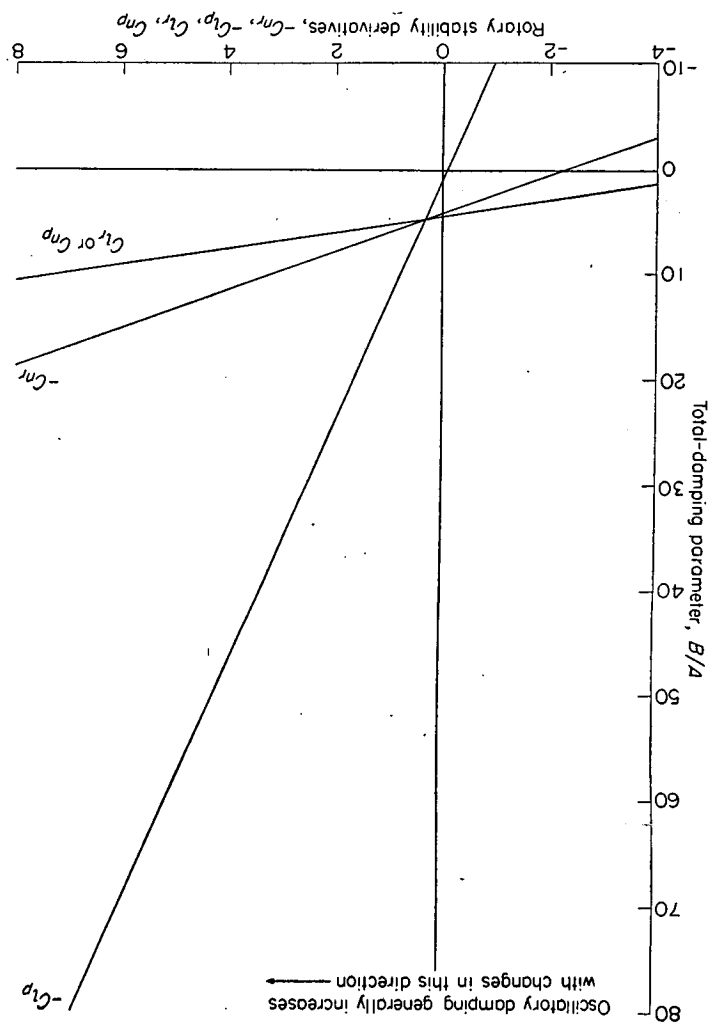
Amount and distribution of the damping of the system.—For a better understanding of the effects of the derivatives on the oscillatory stability and general flight behavior, both the changes in total damping of the system and the redistribution of this damping between the various lateral modes must be considered. The results presented in figure 19 show the way each derivative affects the total damping of the system. In this comparison the damping is expressed in terms of the ratio B/A where A and B are coefficients of the first two terms of the lateral-stability quartic equation. This ratio is proportional to the total damping. (See refs. 1 and 2.)

These results show that changes in any of the four derivatives can cause increases in the total damping. The greatest increase in damping per unit change in a derivative was obtained with negative increases in the value of C_{l_p} . Increasing the value of C_{n_r} negatively was about one-seventh

improving the damping of the lateral oscillation without possible combinations of derivatives are being considered for the various lateral modes. These effects are important when primarily caused a redistribution of the damping between did affect the total damping of the system, these changes Although changes in the values of the cross derivatives K_{xz}^2 and 14 times K_{xz} .

inertia parameters; that is, K_{z^2} is approximately 7 times of total damping is merely a reflection of the ratio of the K_{xz} . Because of this relationship the ratio of the changes to the differences in the inertia parameters K_{z^2} , K_{xz} , and of the derivatives on the total damping are directly related quartic equation indicates that the differences in the effects as increasing C_{lp} . Examination of the coefficient B of the the positive direction was about one-fourteenth as effective as effective as increasing C_{lp} , and increasing C_{lr} or C_{np} in

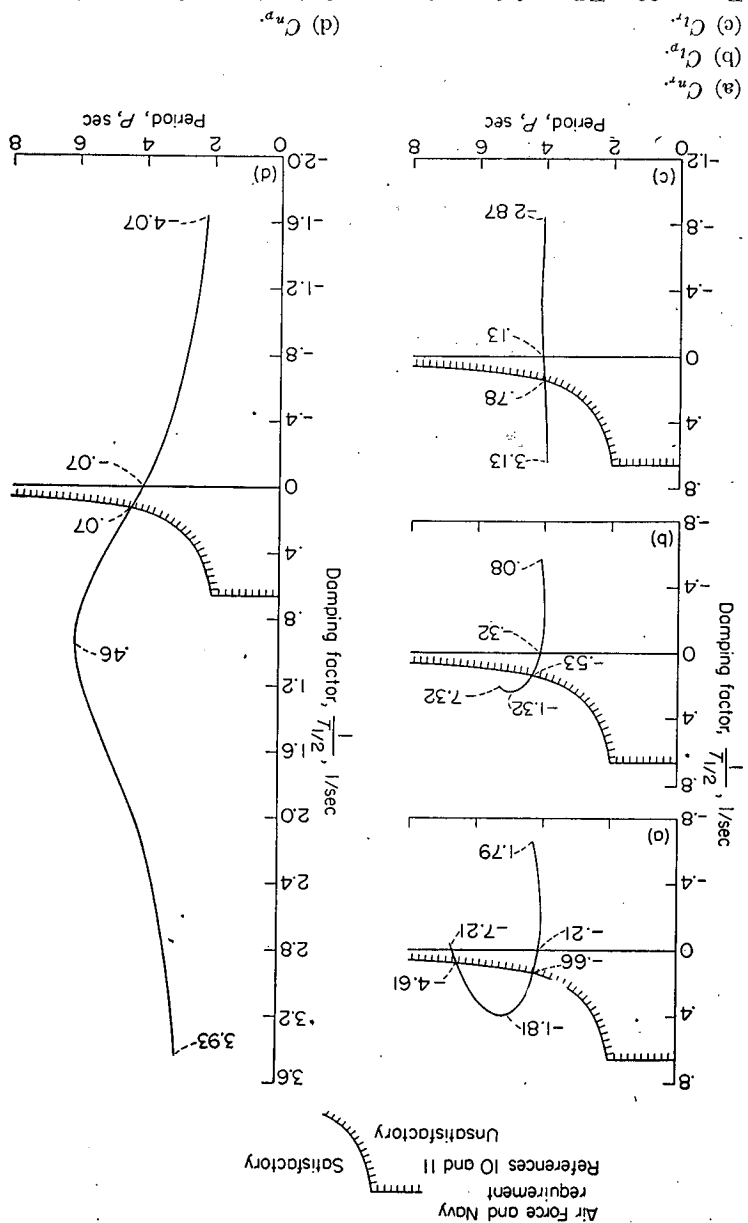
Figure 19.—Calculated effect of rotary stability derivatives on total damping.



Comparison with Air Force and Navy damping requirements.—In order to evaluate the effectiveness of the individual rotary derivatives in improving the damping of

adversely affecting the stability of the aperiodic modes. Such a balance of stability may be accomplished by artificially increasing one of the damping derivatives and then varying the proper cross derivative to provide the desired distribution of the damping between the various lateral modes.

Figure 20.—Effect of change in rotary derivatives on damping of lateral oscillation and comparison with Air Force and Navy requirement.



the oscillation for a full-scale airplane, the calculated damping has been compared with the Air Force and Navy damping requirements. (See refs. 10 and 11.) In this comparison (presented in fig. 20) the period and the damping factor have been scaled up so that the results can be compared directly with the damping requirements. In scaling up these values the model was assumed to be a $\frac{1}{9}$ -scale model of an airplane; therefore, the period of the model was multiplied by 3 and the damping factor was divided by 3.

These results indicate that, in order to satisfy the requirements, C_{n_r} would have to be changed from -0.21 to -0.66 , C_{l_p} from -0.32 to -0.53 , C_{l_r} from 0.13 to 0.78 , or C_{n_p} from -0.07 to 0.07 . A brief analysis has indicated that (if lag and nonlinearities are neglected) any of these changes can be obtained with an artificial-stabilization system utilizing conventional-size control surfaces. It should be emphasized, however, that no general conclusions should be drawn from these results since they are for one airplane and one flight condition.

A comparison of figures 18 and 20 indicates that increasing any of the derivatives except C_{l_r} increased the damping enough to meet the Air Force and Navy requirements before the general flight behavior became unsatisfactory for some other reason. Another important point that can be seen in the comparison of figures 18 and 20 is that the apparent superiority of the derivative C_{n_p} in providing damping of the lateral oscillation was not realized because of the severe apparent spiral instability that resulted with large positive values of this derivative.

SUGGESTIONS FOR FUTURE RESEARCH

The present report covers a part of an investigation to determine the best means for improving the dynamic lateral stability of airplanes by means of artificial-stabilization systems. This phase was concerned primarily with the independent variation of the four rotary stability derivatives. Another phase of the investigation should be concerned with the use of combinations of these derivatives because it appears possible to increase the total damping of the system with one of the damping derivatives and then to redistribute this damping to the various lateral modes by means of a cross derivative in order to obtain good oscillatory stability without impairing the other flight characteristics.

The present investigation was concerned with pure changes in the four rotary derivatives. Since practical artificial-stabilization systems will have a certain amount of lag and nonlinearities, they cannot produce pure changes in the derivatives. Preliminary calculations indicated that appreciable changes in stability may be caused by time lag in the artificial-stabilization system. A study should therefore be

undertaken to determine the ways in which the results of the present investigation would be altered by the introduction of these additional factors.

The results presented in the present report are for only one particular configuration and for one flight condition. Similar results for this and other configurations for a wide range of flight conditions should be obtained since the effects of artificial stabilization may vary widely with changes in the basic conditions.

CONCLUSIONS

The results of the investigation to determine the effects on dynamic lateral stability and control of large artificial variations in the rotary stability derivatives may be summarized as follows. Although these results do not apply directly to airplanes or flight conditions other than those investigated, the trends of the results presented are believed to provide a qualitative indication of the general effects of large variations of the stability derivatives.

1. The calculated results were in qualitative agreement with the experimental results in predicting the general trends in flight characteristics produced by large changes in the stability derivatives, but in some cases the calculations in which time lag was neglected were not in good quantitative agreement with the experimental results. In these cases, check calculations made by taking into account time lag indicated that these discrepancies could be attributed to the effect of the small constant time lag in the stabilization device used.

2. The only derivative which provided a large increase in damping of the lateral oscillation without adversely affecting other flight characteristics was the yawing moment due to yawing C_{n_r} . Because of the limitations imposed by the relatively small size of the test section of the Langley free-flight tunnel, however, the flight characteristics of the model were not appreciably influenced by the stiffness in turning maneuvers which has been found objectionable in some airplanes equipped with yaw dampers. Oscillatory instability was produced by extreme increases in C_{n_r} in the normally stabilizing direction (negative direction).

3. Increasing the rolling moment due to rolling C_{l_p} to moderately large negative values produced substantial increases in the damping of the lateral oscillation but caused an objectionable stiffness in roll. Further negative increases in C_{l_p} did not cause additional increases in the damping of the lateral oscillation and made the stiffness in roll more objectionable.

4. Increasing the rolling moment due to yawing C_{l_r} in the positive direction produced an increase in the damping of the lateral oscillation but caused an undesirable spiral tendency.

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5. Increasing the yawing moment due to rolling C_{np} in the positive direction produced a greater increase in the damping of the lateral oscillation than that produced by any other derivative but it caused an undesirable spiral tendency before adding a substantial amount of damping.

Some preliminary calculations have indicated that the use of combinations of derivatives such as C_{np} and C_{lp} or C_{nr} and C_{lp} should be more satisfactory than the use of single derivatives for increasing the damping of the lateral oscillation without impairing other flight characteristics.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., June 20, 1952.

REPORT 1152

THEORY AND PROCEDURE FOR DETERMINING LOADS AND MOTIONS IN CHINE-IMMERSED HYDRODYNAMIC IMPACTS OF PRISMATIC BODIES ¹

BY EMANUEL SCHNITZER

SUMMARY

A theoretical method is derived for the determination of the motions and loads during chine-immersed water landings of prismatic bodies. This method makes use of a variation of two-dimensional deflected water mass over the complete range of immersion, modified by a correction for three-dimensional flow. Equations are simplified through omission of the term proportional to the acceleration of the deflected mass for use in calculation of loads on hulls having moderate and heavy beam loading. The effects of water rise at the keel are included in these equations. In order to make a direct comparison of theory with experiment, a modification of the equations was made to include the effect of finite test-carriage mass. A simple method of computation which can be applied without reading the body of this report is presented as an appendix, along with the required theoretical plots for determination of loads and motions in chine-immersed landings.

Comparisons of theory with experiment are presented as plots of impact lift coefficient and maximum draft-beam ratio against flight-path angle and as time histories of loads and motions. These comparisons cover angles of dead rise of 0° and 30°, trims up to 45°, flight-path angles up to 90°, and beam-loading coefficients from 1 to 36.5. Fair agreement is seen to exist over these ranges. The comparisons show in general that the concept involving the two-dimensional deflected mass and a three-dimensional-flow correction can be used to predict accurately the loads and motions in landings of prismatic bodies involving immersion of the chines.

INTRODUCTION

This report is concerned with the derivation of a method for calculating impact loads and motions during water landings of narrow, heavily loaded, prismatic bodies. The problem of non-chine-immersed impacts of wide, lightly loaded, prismatic hulls has been treated in reference 1. Comparisons of the theoretical results of this reference with experimental data have been made in references 1 to 4, where reasonable agreement has been demonstrated. Although these reports are devoted largely to the non-chine-immersed case, references 2 and 3 extend the theory to cover impacts during which a small amount of chine immersion is experienced.

For impacts of heavily loaded bodies involving deep immersion of the chines, reference 5 suggests that the force be

determined in two main parts: (1) a part for the hull sections having nonimmersed chines, obtained by means of the theory of reference 1, and (2) a part for the hull sections having immersed chines, based on the two-dimensional separated flow about a wedge immersed in an infinite fluid (ref. 6, arts. 73 to 78). Preliminary analysis based on the equations proposed in reference 5 indicated that these equations were inadequate for the case of 0° dead-rise angle and showed some disagreement with experiment for practical immersions of narrow, heavily loaded, V-bottom bodies.

In view of the inadequacies of the procedure suggested in reference 5, a new analysis is made in the present report on the following basis: For a given transverse cross section of an immersing prismatic form, a unique relation is assumed to exist between the deflected water mass (two-dimensional) and the penetration of that section. A total deflected mass is determined by applying this variation to all cross sections along the hull, integrating over the wetted area, and applying an approximate correction to this integrated mass to account for three-dimensional flow. In the absence of a satisfactory relation for the continuous variation of two-dimensional deflected mass with draft through the region of chine immersion, relationships derived from the suggestions in reference 5 are employed. The purpose of this report is to present this new analysis and a procedure for applying it in impact calculations. Experimental verification of the theory is given.

The report is organized as follows: The differential equation of motion is derived in terms of a variation of two-dimensional deflected water mass and a correction for three-dimensional flow which is selected to satisfy the condition of steady planing. A suggested variation of this two-dimensional deflected mass with draft is given. The general solution of the equation of motion is then presented and modified by omitting the term involving acceleration of the deflected mass. A computational method is indicated for determining loads and motions in landings of prismatic bodies involving appreciable chine immersion. A further modification based on the assumption of constant forward velocity is made for comparison with experimental tank impact data. Finally, comparisons with experimental data are presented for angles of dead rise of 0° and 30°, trims from 6° to 45°, beam-loading coefficients from 1 to 36.5, and

¹Supersedes NACA TN 2813, "Theory and Procedure for Determining Loads and Motions in Chine-Immersed Hydrodynamic Impacts of Prismatic Bodies" by Emanuel Schnitzer, 1952.

height-path angles up to 90°. A system for determining water rise on a rectangular flat plate is given in appendix A. In appendix B a computational procedure is outlined which may be used without reading the body of the report.

SYMBOLS

(Any consistent system of units may be used)

B Bobyleff's flow coefficient, a function of dead rise
beam of hull at chines
 C ratio of water rise at keel to draft, r/z
 C_L impact lift coefficient, $\frac{(m_w)_{max}}{\rho} b^2 V_0^2$
 C_D beam-loading coefficient, $W/\rho g b^2$

F_N hydrodynamic force on hull normal to keel
 g acceleration due to gravity

J parameter involving three-dimensional deflected mass

k generalized draft parameter for free-body landing
 l wetted length along keel
 m_w two-dimensional deflected water mass in transverse plane

m_w impact load, factor measured normal to undisturbed water surface, $-\frac{g}{z}$

Q velocity-reduction parameter for free-body landing
 Q_L velocity-reduction parameter for landings involving horizontal constraint

r water rise at keel normal to undisturbed water surface
 r vertical velocity of water rise at keel

s distance from foremost immersed station along keel to flow plane
 s velocity of flow plane relative to float in direction parallel to keel

s acceleration of flow plane relative to float in direction parallel to keel

t time after water contact
 V resultant velocity of hull
 W weight of airplane

x horizontal velocity of float
 z immersion of keel at step normal to undisturbed water surface, positive down

z vertical velocity of float
 z vertical acceleration of float
 β angle of dead rise, radians

$f(\beta)$ dead-rise function
 γ height-path angle relative to undisturbed water surface

ζ immersion of keel below undisturbed water surface, normal to itself into a flow plane, positive downward

ζ velocity of float normal to keel, $x \sin \tau + z \cos \tau$
 ζ acceleration of float normal to keel
 ζ immersion of the keel normal to itself into a flow plane, corrected for water rise at keel

ζ' partial derivative of ζ' with respect to time
 κ approach parameter for free-body landing, $\frac{\sin \tau}{\sin \tau_0} \cos (\tau + \tau_0)$

κ_L approach parameter for landings involving horizontal constraint, $\tan \tau / \tan \tau_0$
 λ ratio of length of keel below undisturbed water surface to mean beam

λ' ratio of length of keel below elevated water surface to mean beam
 ρ mass density of water

σ hull cross-sectional-shape factor
 τ trim (between keel and undisturbed water surface)
 $\phi(\lambda')$ Pabst's aspect-ratio correction based on λ'

$\phi\left(\frac{\lambda}{l}\right)$ Pabst's aspect-ratio correction based on l/λ
Subscripts:

c atchine immersion
 max maximum
 0 at water contact
 s at step

THEORY FOR PRISMATIC CHINE-IMMERSED IMPACT

EQUATION OF MOTION

This derivation for the motion of a long, narrow body landing on a smooth water surface (fig. 1) is based on the concept that the flow occurs primarily in transverse planes which are fixed in space and oriented normal to the keel (refs. 1 to 5). Thus, a two-dimensional treatment with a three-dimensional-flow or aspect-ratio correction factor is made, as is usual in the calculation of the dynamics of such bodies. The effects of buoyancy, viscosity, and changes in trim are believed to be very small for practical impacts and are neglected. As in references 1 to 5, therefore, the reaction in a given flow plane of length ds (fig. 1) is defined in terms of the momentum of the fluid as

$$(1) \quad dF_N = \frac{\partial}{\partial t} (m_w \dot{\zeta}) ds$$

where $m_w ds$ is an equivalent deflected mass of fluid and $\dot{\zeta}$ is the normal velocity of the body. The total hydrodynamic force on the body is then obtained by integrating equation (1) over the wetted surface and applying the correction for three-dimensional flow $\phi(\lambda')$ (hereinafter designated three-dimensional correction):

$$(2) \quad F_N = \phi(\lambda') \frac{\partial}{\partial t} (m_w \dot{\zeta}) ds$$

Assuming that external forces, such as the wing lift force in the case of seaplanes, are equal and opposite to the weight and applying Newton's second law to equation (2) as in references 1 to 5 leads to the following differential equation for the motion normal to the keel during impact:

$$(3) \quad -\frac{g}{W} \dot{\zeta} = \phi(\lambda') \left(\int_0^{\zeta} \frac{\partial}{\partial t} m_w ds + \int_0^{\zeta} m_w ds \right)$$

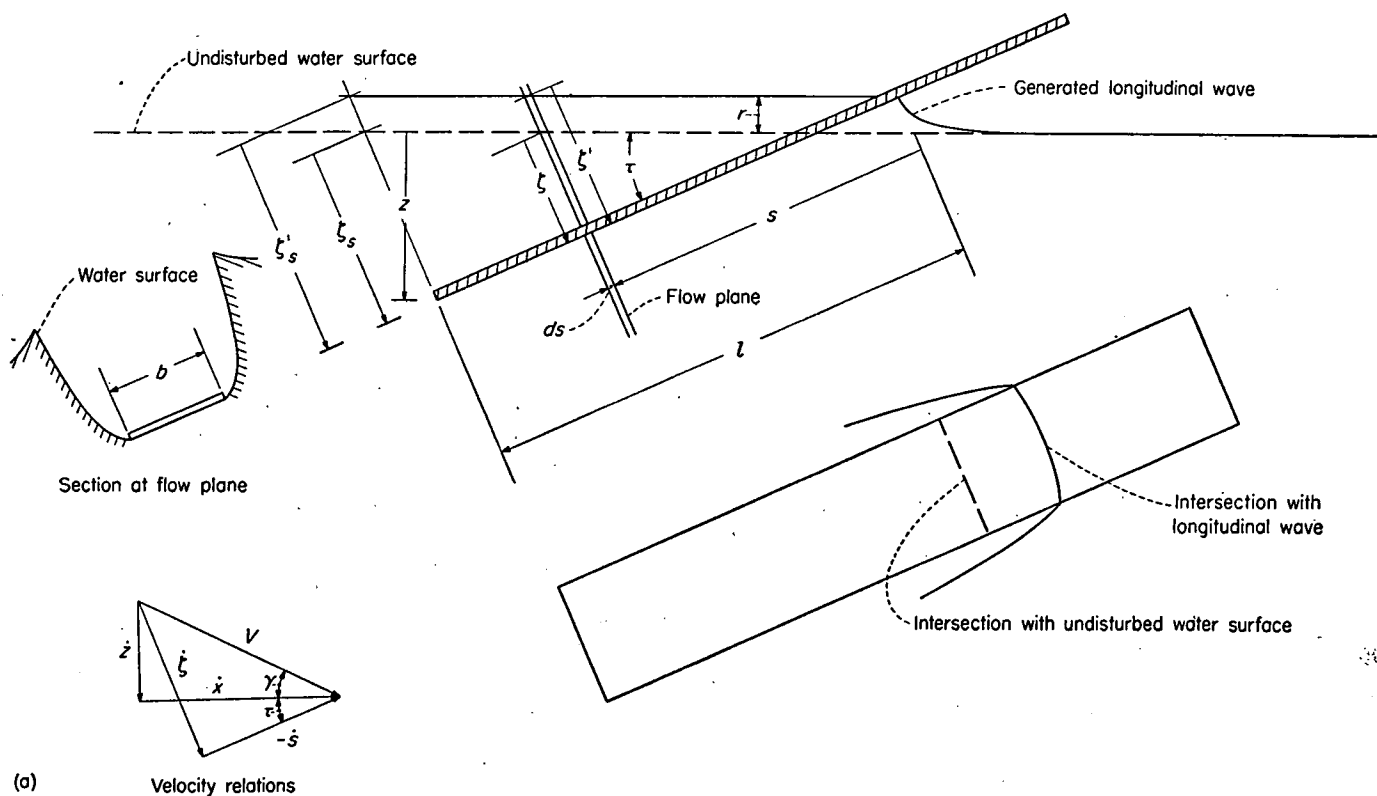


FIGURE 1.—Geometric relations during impact.

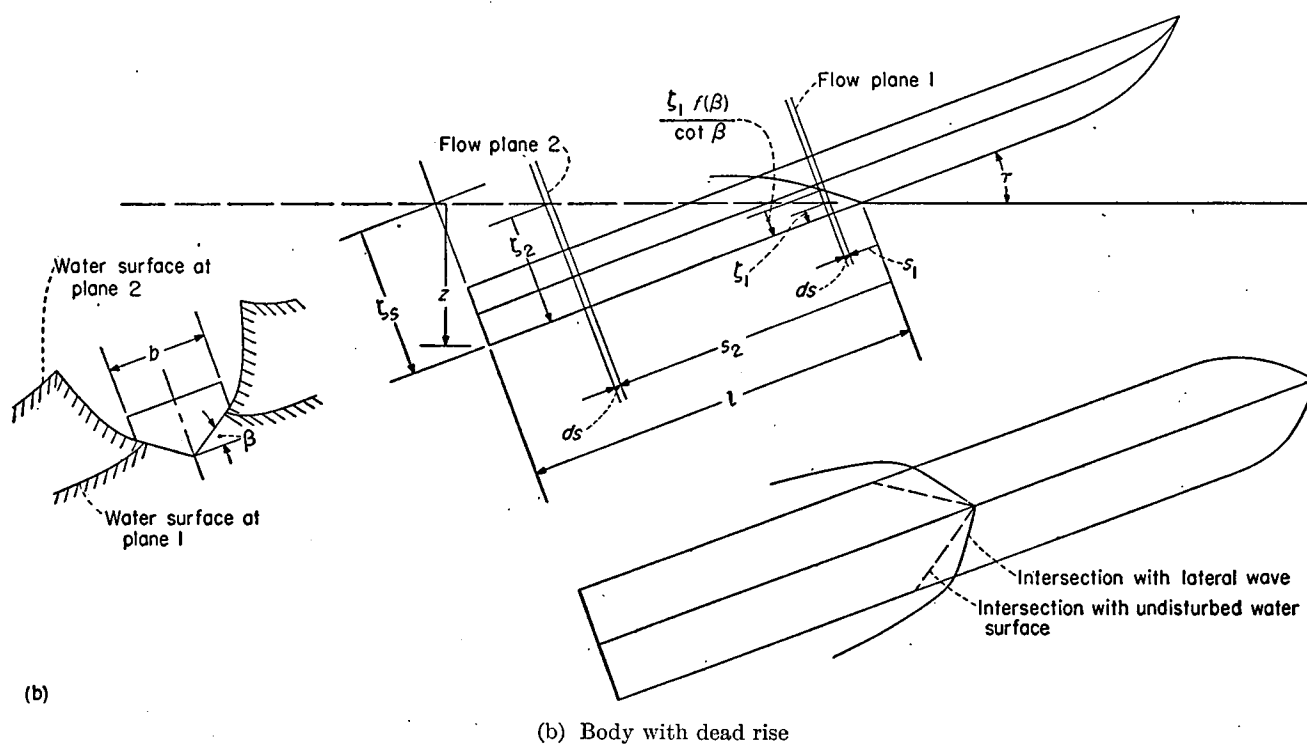


FIGURE 1.—Concluded.

This basic equation of motion can be solved specifically, provided that m_w and $\varphi(\lambda')$ are known. Although this equation may not hold for accelerated motion at very large (infinite) immersions, it is believed to apply over the practical range of impacts.

SELECTION OF THREE-DIMENSIONAL DEFLECTED MASS

The three-dimensional-flow correction $\varphi(\lambda')$ and the two-dimensional deflected mass m_w will be defined and evaluated before the procedure for solving equation (3) is presented.

Determination of three-dimensional correction.—The three-dimensional-flow correction to be used in this report was determined empirically by Pabst in reference 7 and is given by the equation

$$\phi(\lambda') = \left[\frac{1 + \frac{(\lambda')^2}{1}}{1} \right]^{1/2} \left(1 - \frac{\lambda' + \frac{1}{\lambda'}}{0.425} \right) \quad (4)$$

which is plotted in figure 2. The value λ' is taken as the ratio of the length of keel below the elevated water surface to the mean wetted width (see fig. 1). The hydrodynamic aspect ratio λ' for a rectangular flat plate is therefore expressed by the equation

$$\lambda' = \frac{b}{l} \quad (5)$$

where l is the length of keel below the elevated water surface and b is the beam of the body at the chines. For a V-bottom prismatic body the water rise at the keel is neglected so that $\lambda' = \lambda$ and $\lambda'_s = \lambda_s$ (see fig. 1). The "lateral" water rise as shown in figure 1(b) at flow plane 1 and discussed in reference 2 is, however, taken into account in determining the mean wetted width. On this basis, the ratio of the wetted keel length to the mean wetted width is found to be

$$\lambda = \frac{\tan \tau f(\beta)}{1}$$

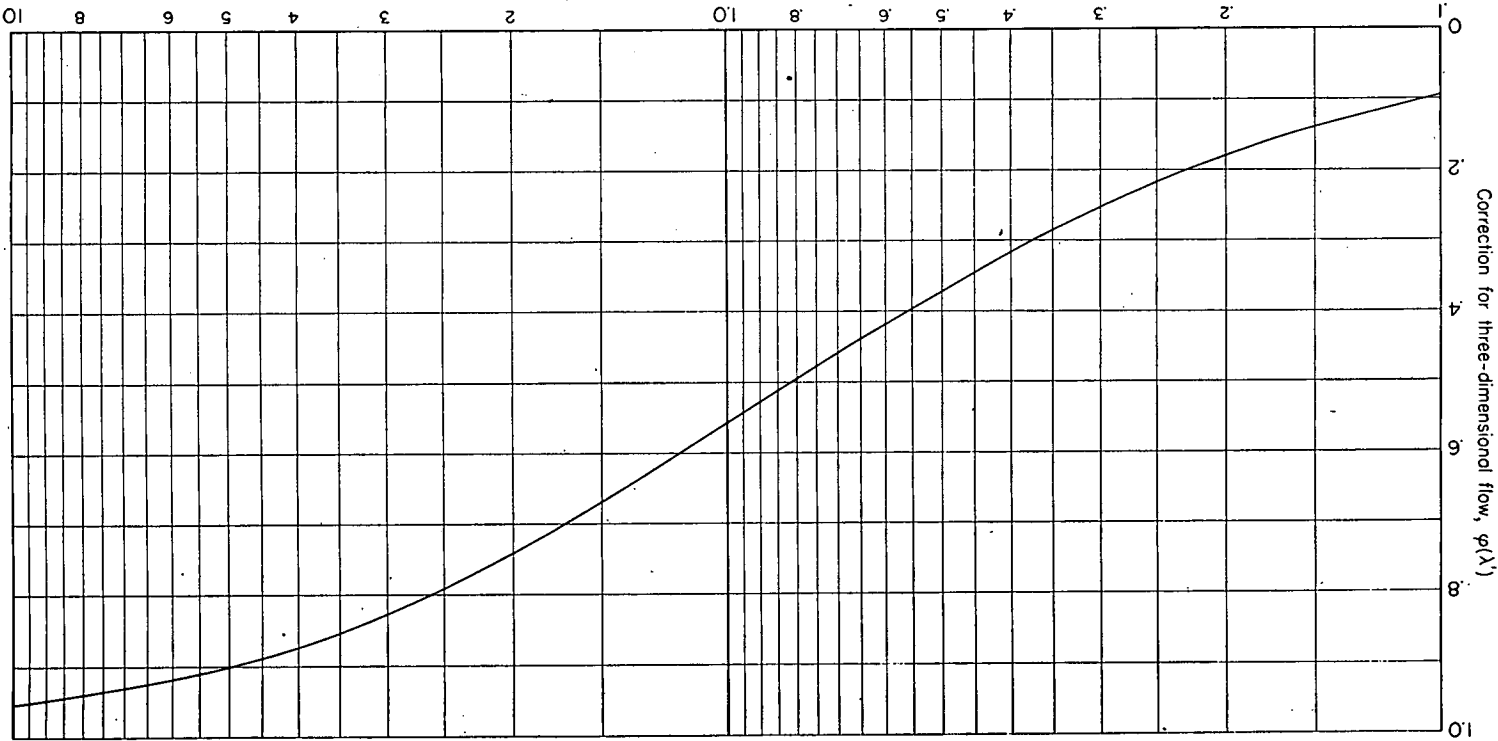
or

$$\lambda = \frac{(\lambda'_s/b)^2}{\left[\frac{b}{\lambda'_s} - \frac{4f(\beta)}{1} \right] \tan \tau}$$

$$\left(\frac{b}{\lambda'_s} \right) > \frac{2}{\tan \beta} \quad (7)$$

$$\left(\frac{b}{\lambda'_s} \right) \leq \frac{2}{\tan \beta} \quad (6)$$

FIGURE 2.—Pabst's correction factor for three-dimensional flow (ref. 7). $\phi(\lambda') = \left[\frac{1 + \frac{(\lambda')^2}{1}}{1} \right]^{1/2} \left(1 - \frac{\lambda' + \frac{1}{\lambda'}}{0.425} \right)$



where

$$f(\beta) = \frac{\pi}{2\beta} - 1 \quad (8)$$

and β is the angle of dead rise.

A certain measure of the suitability of equations (4) to (8) can be obtained by considering the case of steady planing. For this case ($\xi = 0$), equation (2) reduces to

$$F_N = \phi(\lambda') \int_0^l \frac{dm_w}{Q} ds \quad (9)$$

In order to consider this equation further, the assumption is made that the two-dimensional-deflected-mass ratio defined as $m_w/d\rho^2$ is a function of only the cross-sectional shape of the body and the ratio of normal draft to beam; that is,

$$\frac{m_w}{d\rho^2} = f\left(\sigma, \frac{b}{l}\right) \quad (10)$$

where ρ is the mass density of the fluid and σ is the cross-sectional-shape factor. On the basis of this assumption and the substitution $ds = d(\xi \cot \tau) = d(\lambda' \cot \tau)$ (see fig. 1), equation (9) may be written

$$F_N = \phi(\lambda') \int_0^{\lambda'} \frac{dm_w}{Q} d(\lambda' \cot \tau)$$

$$= \phi(\lambda') \int_0^{\lambda'} \frac{dm_w}{Q} d\lambda' \cot \tau$$

$$= \phi(\lambda') \int_0^{\lambda'} \frac{dm_w}{Q} d\lambda' \cot \tau$$

$$F_N = \phi(\lambda') \int_0^{\lambda'} \frac{dm_w}{Q} d\lambda' \cot \tau$$

(11)

where m_{w_s} is the two-dimensional deflected water mass which in seaplanes is associated with the flow plane at the step. Solution of equation (11) for m_{w_s} and division through by ρb^2 to render the equation nondimensional gives

$$\frac{m_{w_s}}{\rho b^2} = \frac{F_N \tan \tau}{\varphi(\lambda') \dot{\zeta}^2 \rho b^2} \quad (12)$$

In order to investigate this equation, use has been made of high-speed, experimental planing data (refs. 8 and 9) obtained at the Langley tank no. 1 with prismatic forms having angles of dead rise of 0° to 40° . When the right-hand side of equation (4) is substituted for $\varphi(\lambda')$ and the experimental data of references 8 and 9 are substituted into equation (12), the factor $m_{w_s}/\rho b^2$ is found to be essentially a function of only the angle of dead rise β and the ratio of normal draft at the step to beam $\dot{\zeta}'_s/b$, or

$$\frac{m_{w_s}}{\rho b^2} = f\left(\beta, \frac{\dot{\zeta}'_s}{b}\right) \quad (13)$$

This result substantiates the assumptions of equation (10) and also demonstrates the validity of the three-dimensional correction expressed by equations (4) to (8), for the case of the V-bottom prism.

Two-dimensional deflected mass.—In the absence of a theory covering the continuous variation of two-dimensional deflected mass with draft over the complete range of immersion, the two separate variations suggested in reference 5 are combined in this report to give a single mass variation with draft. This variation is selected as follows:

(1) Sections prior to chine immersion:

For non-chine-immersed sections the deflected mass is generally taken as the virtual mass, which is defined herein as the apparent additional mass observed during accelerated motion. In references 10 and 11 theories are available for determining the variation of virtual mass with draft for non-chine-immersed impacts. These theories have been checked experimentally for the prismatic V-bottom form and that of reference 10 was checked also for the scalloped-bottom form. For cross sections of arbitrary shape, the variation of the two-dimensional deflected mass may be obtained from reference 3, 10, or 12. However, experience has shown that, for small transverse concave curvature, the mass variation may be approximated by an average V-shape. The same approximation is believed to be equally valid for small convex curvatures. The two-dimensional-mass variation for a V-shape is

$$m_w = \frac{\rho \pi \dot{\zeta}^2}{2} [f(\beta)]^2 \quad (14)$$

which is taken from references 1 to 3 and is based on Wagner's work. Substitution of $\dot{\zeta}'$ for $\dot{\zeta}$ in this equation to take water rise at the keel into account gives

$$m_w = \frac{\rho \pi (\dot{\zeta}')^2}{2} [f(\beta)]^2 \quad (15)$$

which is to be used as the deflected mass prior to chine immersion of each section. Chine immersion is herein assumed to occur at the intersection of the chines in a given flow plane with a plane parallel to the undisturbed water surface and passing through the intersection point of the keel with the

actual water surface. The draft-beam ratio at chine immersion is

$$\frac{\dot{\zeta}'_c}{b} = \frac{\tan \beta}{2} \quad (16)$$

The two-dimensional deflected mass at this instant therefore becomes

$$m_w = \frac{\rho \pi b^2}{8} [f(\beta)]^2 \tan^2 \beta \quad (17)$$

(2) Sections subsequent to chine immersion:

For infinite immersion of the chines, a variation of two-dimensional deflected mass with draft may be derived from the theory of Bobyleff (ref. 6, arts. 73 to 78). However, no theory is believed to be available on the variation of two-dimensional deflected mass for moderate chine immersion. In the absence of a single accurate deflected-mass variation over the entire range of immersion, a composite deflected mass is suggested which is composed of the deflected mass present at the instant of chine immersion plus a deflected mass which is derived from Bobyleff's theory. For infinite immersions, Bobyleff has shown that the force per unit length on a two-dimensional V-shape of finite width b traveling with constant normal velocity $\dot{\zeta}'$ (point foremost) is

$$F_N = B \frac{\rho}{2} (\dot{\zeta}')^2 b \quad (18)$$

where B is a function of the angle of dead rise and is given in figure 3. This force is assumed to be equal to the rate of change of momentum of an equivalent deflected mass; that is,

$$\begin{aligned} F_N &= \frac{d}{dt} (m_w \dot{\zeta}') \\ &= (\dot{\zeta}')^2 \frac{dm_w}{d\dot{\zeta}'} \end{aligned} \quad (19)$$

where constant velocity is assumed. Thus, from equations (18) and (19),

$$\frac{dm_w}{d\dot{\zeta}'} = B \frac{\rho}{2} b \quad (20)$$

which in this report is taken as the variation of deflected mass with draft subsequent to chine immersion. Integrating to find the change in deflected mass subsequent to chine immersion Δm_w gives

$$\begin{aligned} \Delta m_w &= B \frac{\rho}{2} b \int_{\dot{\zeta}'_c}^{\dot{\zeta}'} d\dot{\zeta}' \\ &= B \frac{\rho}{2} b (\dot{\zeta}' - \dot{\zeta}'_c) \end{aligned} \quad (21)$$

The total two-dimensional deflected mass subsequent to chine immersion is found by adding to this expression the deflected mass at the instant of chine immersion. If this latter mass is assumed to be given by equation (17), the total mass becomes

$$m_w = \frac{\rho \pi b^2}{8} [f(\beta) \tan \beta]^2 + B \frac{\rho}{2} b^2 \left(\frac{\dot{\zeta}'}{b} - \frac{\tan \beta}{2} \right) \quad (22)$$

For prismatic hulls of arbitrary cross-sectional shape where the variation from a V-section is not great, approximate values of m_w are believed to be obtainable through substitution of equivalent V-sections for the arbitrary sections. The

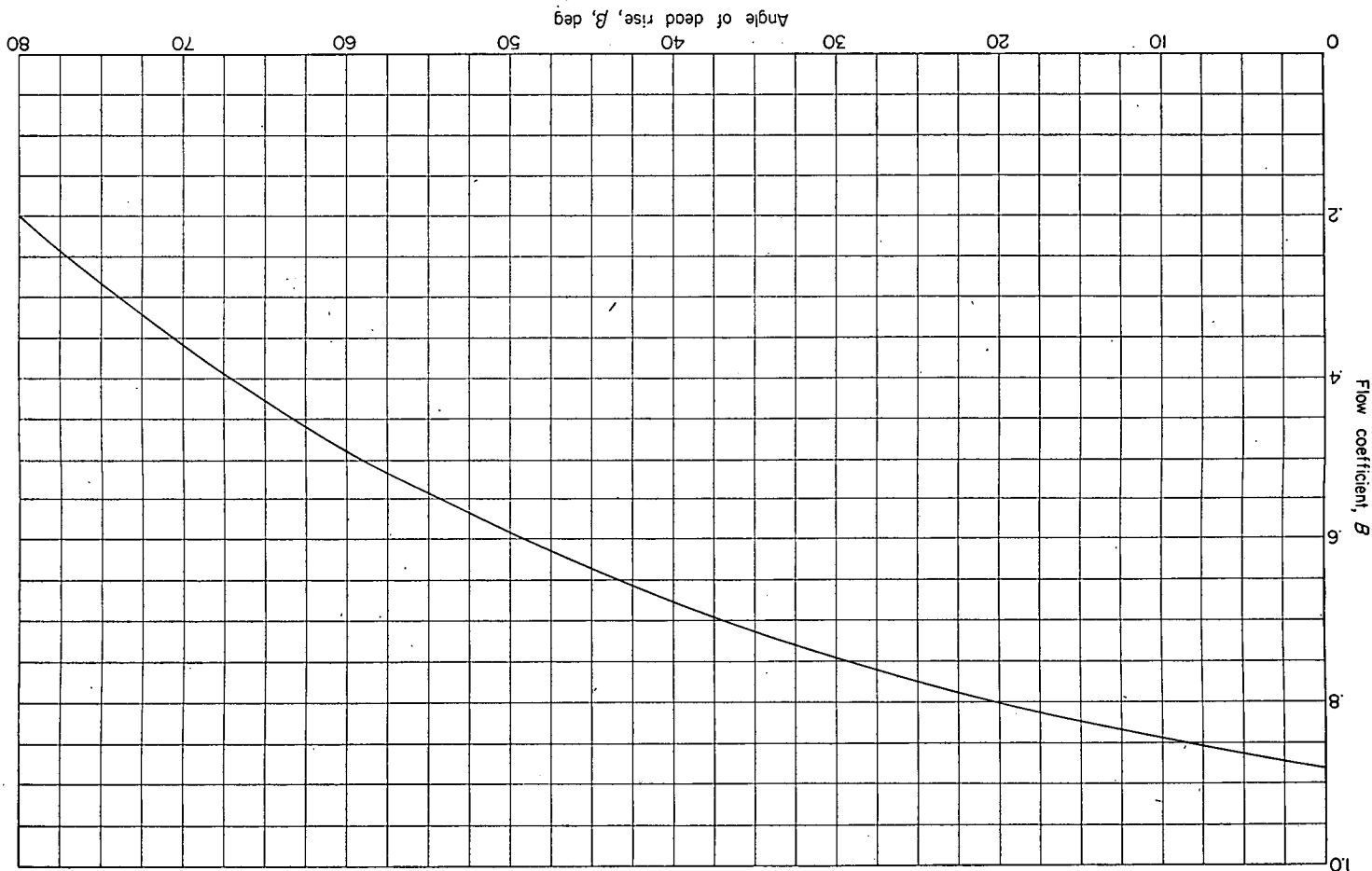


FIGURE 3.—Variation of flow coefficient with angle of dead rise (ref. 6, arts. 73 to 78).

The discussion following equation (10) substantiates the assumption expressed by that equation that the two-dimensional deflected mass for a given hull is a function only of the draft normal to the keel; that is,

$$m_w = f(\zeta') \quad (24)$$

Therefore,

$$\frac{\partial m_w}{\partial t} = \frac{\partial m_w}{\partial \zeta'} \frac{\partial \zeta'}{\partial t} = \frac{dm_w}{d\zeta'} \frac{d\zeta'}{dt} \quad (25)$$

From figure 1, $s = \frac{\zeta'}{\tan \tau}$. Differentiating this relation gives

$$ds = \frac{d\zeta'}{\tan \tau} \quad (26)$$

Substitution of equations (25) and (26) into equation (3) gives

$$-\frac{W}{\delta} \dot{\zeta} = \phi(\lambda') \left(\int_0^{\zeta'} m_w ds + \int_0^{\zeta'} \tan \tau \frac{dm_w}{d\zeta'} d\zeta' + \int_0^{\zeta'} m_w ds \right) \quad (27)$$

For impacts with deeply immersed chines, however, the water rise at the keel τ (see fig. 1) is generally small compared with the draft and is approximately constant for the

angle of dead rise would be determined by obtaining an average angle of dead rise for the arbitrary section.

For the case of 0° dead-rise angle, equation (22) reduces to

$$m_w = \rho b^2 \left(\frac{\pi}{32} + 0.44 \frac{\zeta'}{b} \right) \quad (23)$$

Equations (15), (22), and (23) are to be used in the solution of equation (3).

METHOD FOR SOLVING EQUATION OF MOTION INCLUDING COMPUTATIONAL CHARTS

GENERAL TREATMENT

The incorporation of the aspect-ratio correction $\phi(\lambda')$ and the deflected mass m_w into equation (3) permits solution of the equation by numerical methods. A large saving of time may be effected, however, by solving the equation partially by analytical means to reduce it to a form convenient for graphical integration. This reduction is given here, with equation (3) restated first for convenience:

$$-\frac{W}{\delta} \dot{\zeta} = \phi(\lambda') \left(\int_0^{\zeta'} \frac{\partial}{\partial t} m_w ds + \int_0^{\zeta'} m_w ds \right)$$

greater drafts. Therefore, its derivative with respect to time $\dot{r}$ approaches zero at these drafts, making $\dot{\zeta}'$, which is equal to $\dot{\zeta} + \dot{r} \sec \tau$, approach $\dot{\zeta}$. In this derivation, then, $\dot{\zeta}'$ is assumed equal to $\dot{\zeta}$, whereby

$$-\frac{W}{g} \ddot{\zeta} = \varphi(\lambda') \left(\frac{\dot{\zeta}^2}{\tan \tau} \int_0^{m_{ws}} dm_w + \ddot{\zeta} \int_0^l m_w ds \right) \quad (28)$$

and, after integration of the first term on the right-hand side,

$$\ddot{\zeta} \left[1 + \frac{g}{W} \varphi(\lambda') \int_0^l m_w ds \right] = -\frac{\dot{\zeta}^2 \varphi(\lambda') g m_{ws}}{W \tan \tau} \quad (29)$$

Rewriting equation (29) in terms of the coordinate system referred to the water surface is done by means of the following substitutions (see fig. 1):

$$\ddot{\zeta} = \frac{\ddot{z}}{\cos \tau} \quad (30a)$$

$$\dot{\zeta} = \frac{\dot{z}}{\cos \tau} - \dot{s} \tan \tau \quad (30b)$$

$$\zeta_s = \frac{z}{\cos \tau} \quad (30c)$$

where $\dot{s}$ is taken equal to 0 because of the assumption of frictionless flow and no external force, and $\dot{s}$ is therefore a constant. When these substitutions are made, equation (29) becomes

$$\ddot{z} \left[1 + \frac{g \varphi(\lambda')}{W} \int_0^l m_w ds \right] = -(\dot{z} - \dot{s} \sin \tau)^2 \frac{g \varphi(\lambda') m_{ws}}{W \sin \tau} \quad (31)$$

Rearranging the terms and integrating both sides with respect to z gives

$$\int_0^z \frac{\ddot{z} dz}{(\dot{z} - \dot{s} \sin \tau)^2} = - \int_0^z \frac{\frac{m_{ws}}{\sin \tau} dz}{\frac{W}{g \varphi(\lambda')} + \int_0^l m_w ds} \quad (32)$$

which can be written

$$\int_{\dot{z}_0}^{\dot{z}} \frac{\dot{z} d\dot{z}}{(\dot{z} - \dot{s} \sin \tau)^2} = - \int_0^z \frac{\frac{m_{ws}}{\sin \tau} dz}{\frac{W}{g \varphi(\lambda')} + \int_0^l m_w ds} \quad (33)$$

After the integration of the left-hand side has been performed, the following equation expressing the velocity as a function of the draft is obtained:

$$\log_e \frac{\frac{\dot{z}}{\dot{z}_0} + \kappa}{1 + \kappa} + \frac{\kappa}{\frac{\dot{z}}{\dot{z}_0} + \kappa} - \frac{\kappa}{1 + \kappa} = - \int_0^z \frac{\frac{m_{ws}}{\sin \tau} dz}{\frac{W}{g \varphi(\lambda')} + \int_0^l m_w ds} \quad (34)$$

where

$$\begin{aligned} \kappa &= -\frac{\dot{s} \sin \tau}{\dot{z}_0} \\ &= \frac{\sin \tau}{\sin \gamma_0} \cos(\tau + \gamma_0) \end{aligned}$$

If for convenience the left-hand side of equation (34) is denoted by $Q \left(\frac{\dot{z}}{\dot{z}_0}, \kappa \right)$, and if dz as obtained by differentiating equation (30c) is substituted into the right-hand side, the following equation results:

$$Q = - \int_0^{\zeta_s} \frac{\frac{m_{ws} d\zeta_s}{\tan \tau}}{\frac{W}{g \varphi(\lambda')} + \int_0^l m_w ds} \quad (35)$$

Since, from equation (24), $m_{ws} = f(\zeta'_s)$, a multiplication by $d\zeta'_s/d\zeta'_s$ is performed inside the integral to give

$$Q = - \int_0^{\zeta'_s/b} \frac{\frac{m_{ws}}{\tan \tau} \frac{d\zeta_s}{d\zeta'_s} d\zeta'_s}{\frac{W}{g \varphi(\lambda')} + \int_0^l m_w ds} \quad (36)$$

Equation (36) can be expressed in terms of nondimensional quantities through multiplication of the right-hand side by $\rho b^3/\rho b^3$; after substitution of equation (26), the following relation results:

$$Q = - \int_0^{\zeta'_s/b} \frac{\frac{m_{ws}}{\rho b^2} \frac{d\zeta_s}{d\zeta'_s} d\zeta'_s}{\frac{C_\Delta \tan \tau}{\varphi(\lambda')} + \int_0^{\zeta'_s/b} \frac{m_{ws}}{\rho b^2} d\zeta'_s} \quad (37)$$

where

$$C_\Delta = \frac{W}{\rho g b^3}$$

Since Q , which denotes the left-hand side of equation (34), contains the velocity ratio $\dot{z}/\dot{z}_0$, equation (37) represents, finally, the relation between this velocity ratio and the ratio of normal draft to beam ζ'_s/b .

The relation between the nondimensional acceleration $\ddot{z}b/\dot{z}_0^2$ and the nondimensional vertical-velocity and draft-beam ratios is determined through similar nondimensionalizing of equation (31), which results in the equation

$$\frac{\ddot{z}b}{\dot{z}_0^2} = - \frac{\left(\frac{\dot{z}}{\dot{z}_0} + \kappa \right)^2 \frac{1}{\cos \tau} \frac{m_{ws}}{\rho b^2}}{\frac{C_\Delta \tan \tau}{\varphi(\lambda')} + \int_0^{\zeta'_s/b} \frac{m_{ws}}{\rho b^2} d\zeta'_s} \quad (38)$$

Equations (37) and (38) can be used to calculate the variation of acceleration and velocity with draft during

$z/b/z_0^2$ can then be obtained for each value of z/z_0 through substitution of the derived quantities into equation (38). The ratio of vertical draft to beam can be found from the ratio of normal draft to beam by means of the equation

$$\frac{z}{z_0} = \frac{z'_s}{z'_s} \frac{b}{z'_s} \cos \tau \quad (39)$$

where z'_s/z'_s is assumed to be equal to λ/λ' , which may be obtained from figure 6 (a) for angles of dead rise smaller than 10° and is taken as unity for angles of dead rise of 10° and greater. For the case of 0° dead rise, equation (39) can also be written

$$\frac{z}{z_0} = \lambda' \frac{\lambda}{\lambda'} \sin \tau \quad (40)$$

The variations of acceleration and velocity with draft as obtained by use of equations (37) to (40) allow design maximums to be established. For calculating structural response, time histories are desirable. From the relations $dz = z' dl$, $dz = z' dl$, and $z_0 = V_0 \sin \gamma_0$ a time coefficient may be derived which is defined by either of the following equations:

$$\left\{ \begin{aligned} tV_0 &= \frac{b}{I} \int_{z/z_0}^1 \frac{\sin \gamma_0}{I} dz \\ tV_0 &= \frac{b}{I} \int_{z/z_0}^0 \frac{\sin \gamma_0}{I} dz \end{aligned} \right. \quad (41)$$

where V_0 is the initial resultant velocity at contact. Graphical integration of these equations allows the draft-time relation to be established. It should be noted that the integrand of the first equation becomes infinite for $z/z_0 = 0$ (maximum draft), whereas the integrand of the second equation becomes infinite for $z/z_0 = 1$ (water exit is obtained from the first equation. Appropriate limits of integration must of course be substituted for those given in these equations.

fixed-time impacts involving appreciablechine immersion. In order to effect specific solutions of these equations, the variables making up the equations are presented in the form of computational charts, some of which are described in the next paragraph. An indicated method of computation follows.

The variation of κ with γ_0 is given in figure 4 for various trim angles. The left-hand side of equation (34), designated as Q in equation (37), is plotted against κ for various values of z/z_0 in figure 5. The ratio $d\zeta_s/d\zeta'_s$ in equation (37) is the keel water-rise factor which in reference 13 was shown to be substantially independent of flight-path angle and therefore capable of being evaluated from planning data. For the rectangular flat plate, a large quantity of experimental planning data is available from which this factor can be computed. An analysis based on these planning data and giving the wetted length and the keel water-rise factor $\frac{d\zeta_s}{d\zeta'_s} = \frac{\partial \zeta_s}{\partial \zeta'_s}$ for landings of flat plates is presented in appendix A and the results are plotted in figure 6. For the case of finite dead rise this factor has not been fully evaluated, but since it is believed to be close to unity (no water rise), it is assigned that value in this report for angles of dead rise greater than 10° . For angles of dead rise smaller than 10° , use of the keel water-rise factor $d\zeta_s/d\zeta'_s$ for the flat plate is suggested.

In order to obtain specific solutions of equations (37) and (38) in the forms shown for flat or V-bottom prismatic bodies, the following procedure is suggested: The variation of m_w with ζ'_s may be obtained from equations (15) and (22) and figure 3, from equation (23) and figure 3, or from experimental planning data with the aid of equation (12). This mass variation may be substituted into the integral in the denominator of equations (37) and (38) and integrated analytically or graphically. The aspect ratio λ' may be determined from equations (5) to (8) as a function of the ratio of normal draft to beam (λ' is taken equal to λ for a hull with dead rise). For a flat plate, l in equation (5) is related to the normal draft (see fig. 1) by the equation $l = \frac{\tan \tau}{\zeta'_s}$. The variation of $\phi(\lambda')$ with λ' may then be obtained from figure 2. The variation of $d\zeta_s/d\zeta'_s$ may be obtained from figure 6 (b) for angles of dead rise smaller than 10° and is taken as unity for angles of dead rise of 10° and greater. After these quantities have been substituted into equation (37), it can be integrated graphically to yield the variation of Q with ζ'_s/b . The variation of z/z_0 with Q may be obtained from figure 5 after selecting a value of κ from figure 4; thus, the variation of z/z_0 with ζ'_s/b may be established. A value of

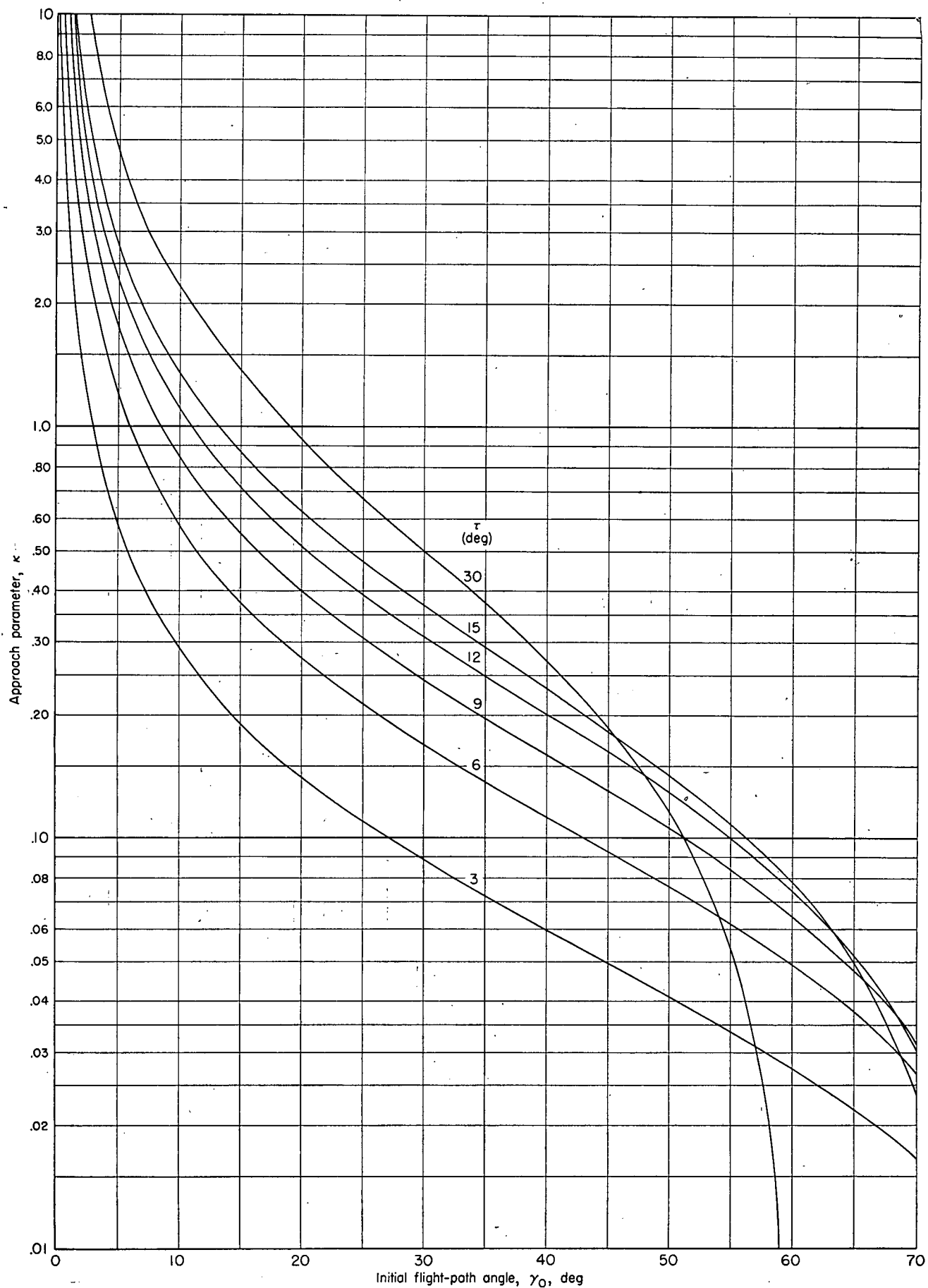


FIGURE 4.—Variation of approach parameter with trim and flight-path angle. $\kappa = \frac{\sin \tau}{\sin \gamma_0} \cos (\tau + \gamma_0)$.

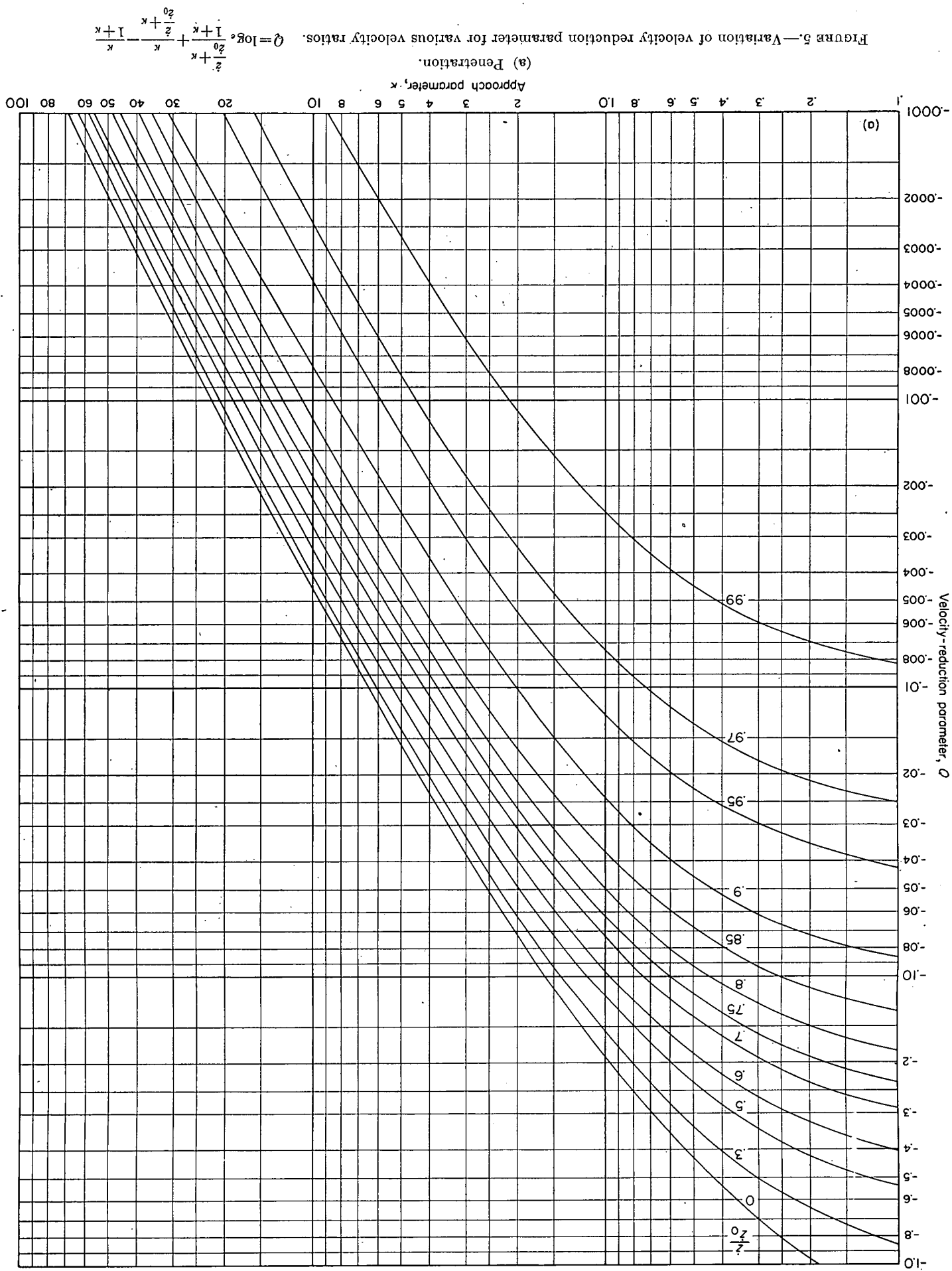
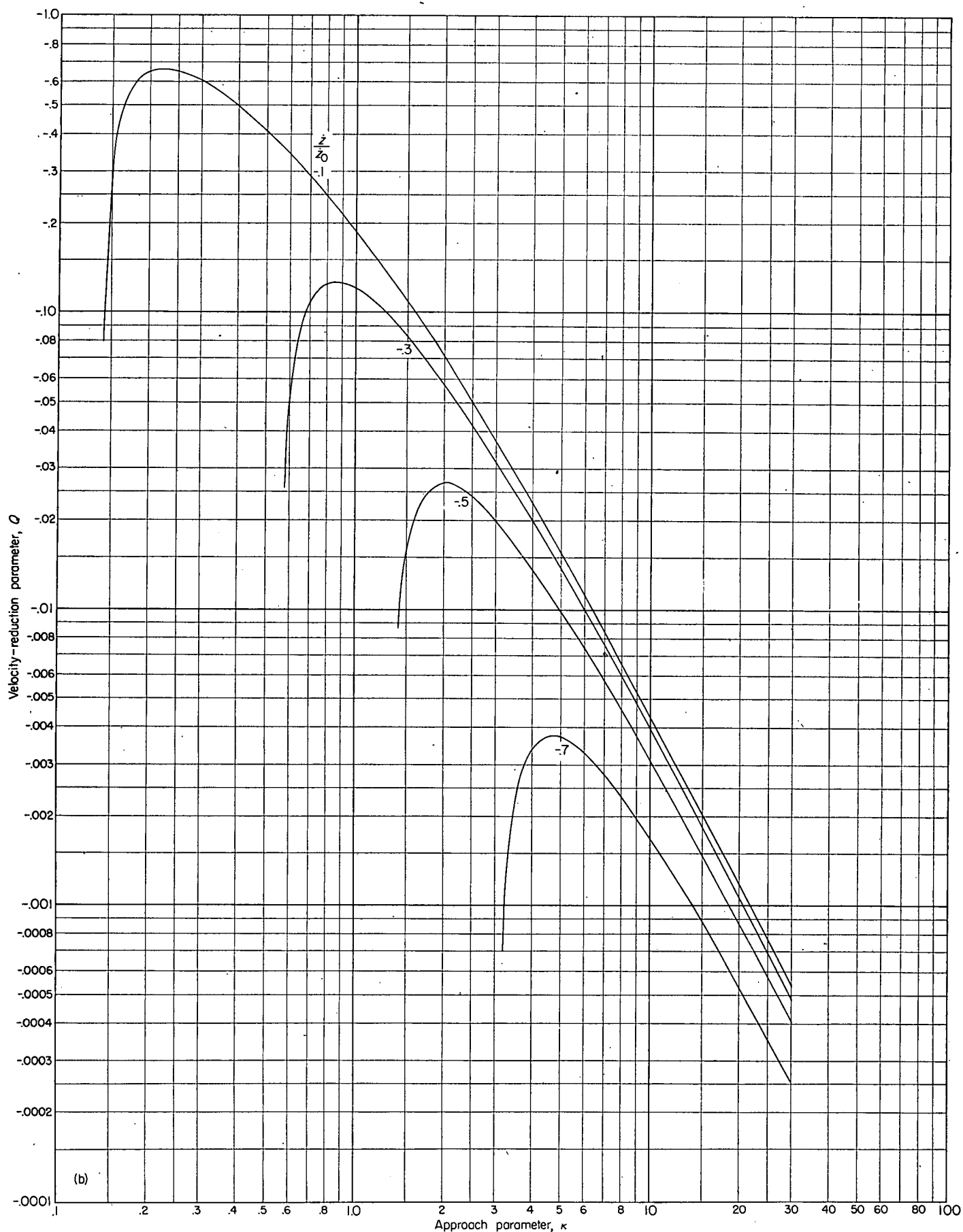
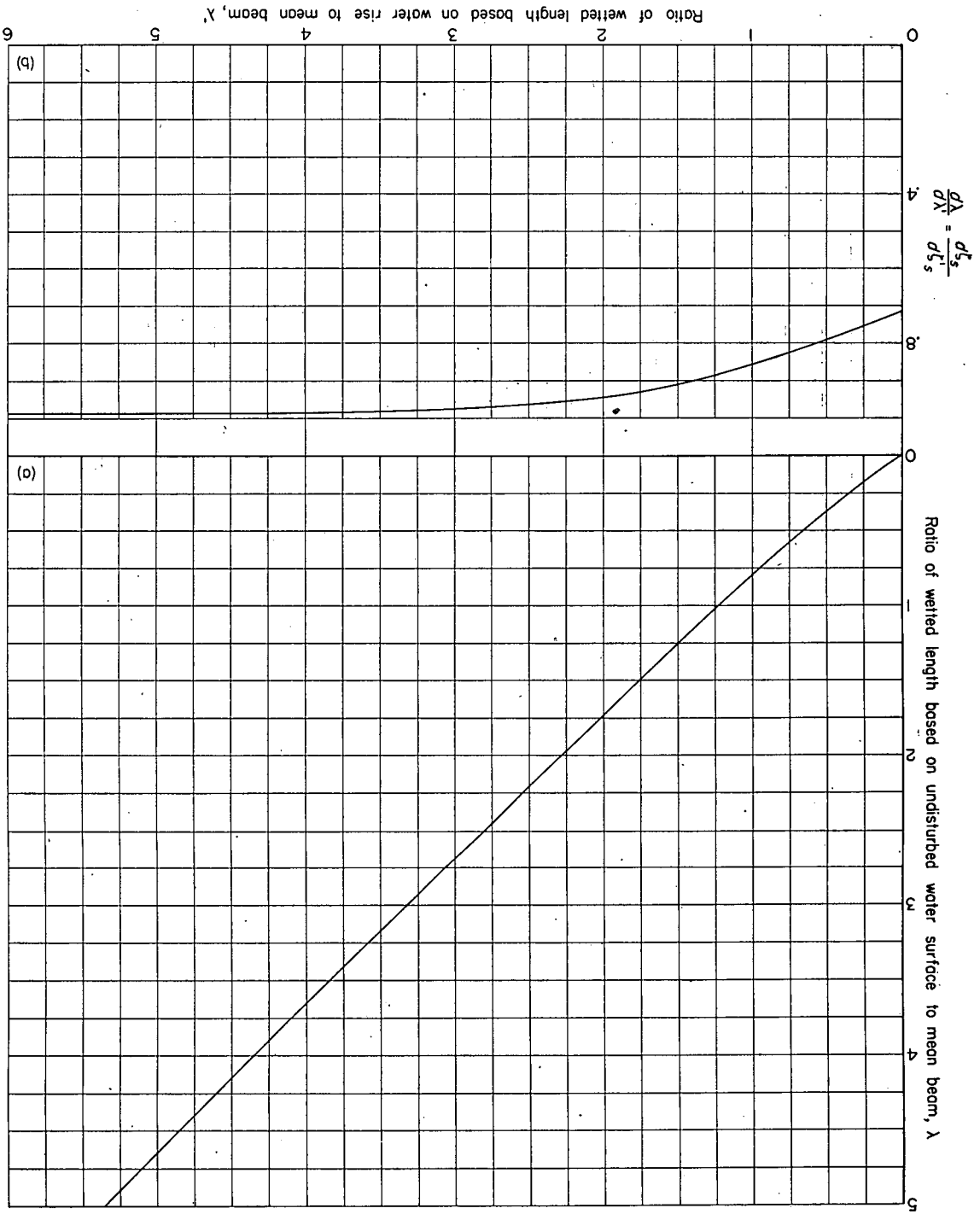


FIGURE 5.—Variation of velocity reduction parameter for various velocity ratios.



(b) Rebound.

FIGURE 5.—Concluded



(a) Wetted-length variations.
(b) Variation of derivative of wetted length ratio with ratio of wetted length based on water rise to mean beam.

FIGURE 6.—Water-rise variations for a flat plate.

[SIMPLIFICATION THROUGH OMISSION OF ACCELERATION TERM]

In order to reduce the labor required to make solutions for specific landing impacts, a simplification was effected which does not seriously reduce the accuracy of these solutions for practical landing configurations. From equation (28) it is evident that the hydrodynamic force is composed of two terms, one proportional to the square of the velocity normal to the keel and the other proportional to the acceleration normal to the keel. For impacts involving beam-loading coefficients greater than 1 and appreciable chine immersion, the ratio of the acceleration term to the velocity-squared term is usually small. This acceleration term is therefore omitted from the equation of motion, with the result that equation (37), which relates the velocity and draft, is reduced to

$$Q = -\frac{1}{C_\Delta} \int_0^{\zeta'_s/b} \frac{\varphi(\lambda')}{\tan \tau} \frac{m_{w_s}}{\rho b^2} \frac{d\zeta_s}{d\zeta'_s} d\frac{\zeta'_s}{b}$$

which may be written

$$Q = -\frac{k}{C_\Delta} \quad (42)$$

where

$$k = \int_0^{\zeta'_s/b} \frac{\varphi(\lambda')}{\tan \tau} \frac{m_{w_s}}{\rho b^2} \frac{d\zeta_s}{d\zeta'_s} d\frac{\zeta'_s}{b}$$

Omission of the acceleration term from equation (38), which relates the acceleration, velocity, and draft, reduces it to

$$\frac{\ddot{z}b}{\dot{z}_0^2} = -\left(\frac{\dot{z}}{\dot{z}_0} + \kappa\right)^2 \frac{\varphi(\lambda')}{C_\Delta \sin \tau} \frac{m_{w_s}}{\rho b^2} \quad (43)$$

Thus, a simplification of the numerical calculation is made possible through the introduction of additional computational charts. One such chart (fig. 7) shows the variation of k with ζ'_s/b for various trims and angles of dead rise. For the flat plate (0° dead rise), k was evaluated by graphical integration after substitution into equation (42) of equations (4), (5), and (23), where $l = \frac{\zeta'_s}{\tan \tau}$, and of the ratio $d\zeta_s/d\zeta'_s$, from figure 6 (b). For the case of finite dead rise, k was similarly evaluated after substitution into equation (42) of equations (4), (6) to (8), (15), and (22) for finite dead rise, where $d\zeta_s/d\zeta'_s$ was taken equal to unity. A second chart was constructed from the variation of the part of equation (43) designated as J , where

$$J = \frac{\varphi(\lambda')}{\sin \tau} \frac{m_{w_s}}{\rho b^2} \quad (44)$$

and is plotted against ζ'_s/b for various trims and angles of dead rise in figure 8.

In order to obtain specific solutions of equations (42) and (43), a computational procedure has been set forth in appendix B. This procedure is somewhat like that outlined for treating equations (37) and (38) and the labor for each solution has been considerably reduced.

EXPERIMENTAL VERIFICATION OF THEORY

MODIFICATION OF THEORY TO PERMIT COMPARISON WITH EXPERIMENT

The theory developed in this report covers free-body landings in which the velocity parallel to the keel is assumed to be constant during impact. The only available experi-

mental data that were usable for verification of this theory however, were obtained during an investigation of constrained models at the Langley impact basin. In these tests the model was mounted on a catapulted carriage in such a way that the model was free to move vertically but was constrained to move with the carriage in a horizontal direction. Since the carriage was several times as heavy as the model, the forward velocity of the carriage-model combination remained approximately constant. In order to compare the theory of this report with the available data, it was necessary to modify the equations so that the velocity component in the horizontal direction, instead of the component in the direction parallel to the keel, was considered constant during impact. The equation of motion was then solved by a procedure similar to that used in deriving the proposed free-body theory with the following results.

The equation relating the velocity of the body to its draft, which is comparable to equation (37), is

$$Q_L = -\cos^2 \tau \int_0^{\zeta'_s/b} \frac{\frac{m_{w_s}}{\rho b^2} \frac{d\zeta_s}{d\zeta'_s} d\frac{\zeta'_s}{b}}{\frac{C_\Delta \tan \tau}{\varphi(\lambda') \cos^2 \tau} + \int_0^{\zeta'_s/b} \frac{m_{w_s}}{\rho b^2} d\frac{\zeta'_s}{b}} \quad (45)$$

where

$$Q_L \equiv \log_e \frac{\frac{\dot{z}}{\dot{z}_0} + \kappa_L}{1 + \kappa_L} + \frac{\kappa_L}{\frac{\dot{z}}{\dot{z}_0} + \kappa_L} - \frac{\kappa_L}{1 + \kappa_L}$$

and

$$\kappa_L = \frac{\tan \tau}{\tan \gamma_0}$$

The equation relating the acceleration of the body to its velocity and draft, which is comparable to equation (38), is

$$\frac{\ddot{z}b}{\dot{z}_0^2} = -\frac{\left(\frac{\dot{z}}{\dot{z}_0} + \kappa_L\right)^2 \cos \tau \frac{m_{w_s}}{\rho b^2}}{\frac{C_\Delta \tan \tau}{\varphi(\lambda') \cos^2 \tau} + \int_0^{\zeta'_s/b} \frac{m_{w_s}}{\rho b^2} d\frac{\zeta'_s}{b}} \quad (46)$$

Specific solutions for impacts may be obtained with these equations as was done with equations (37) and (38). The value of Q_L can be obtained from figure 5 in place of Q when κ_L is substituted for κ .

The omission of the force term arising from acceleration of the deflected mass is handled as in the derivation of the free-body theory and leads to the following equations which are similar to equations (42) and (43): Equation (45), which relates velocity and draft, is reduced to

$$Q_L = -\frac{\cos^4 \tau}{C_\Delta} \int_0^{\zeta'_s/b} \frac{\varphi(\lambda')}{\tan \tau} \frac{m_{w_s}}{\rho b^2} \frac{d\zeta_s}{d\zeta'_s} d\frac{\zeta'_s}{b} \quad (47)$$

and equation (46), which relates the acceleration, velocity, and draft, is reduced to

$$\frac{\ddot{z}b}{\dot{z}_0^2} = -\left(\frac{\dot{z}}{\dot{z}_0} + \kappa_L\right)^2 \frac{\varphi(\lambda') \cos^4 \tau}{C_\Delta \sin \tau} \frac{m_{w_s}}{\rho b^2} \quad (48)$$

Specific impact solutions of these equations may be obtained as was done in appendix B with equations (42) and (43) for the free-body case.

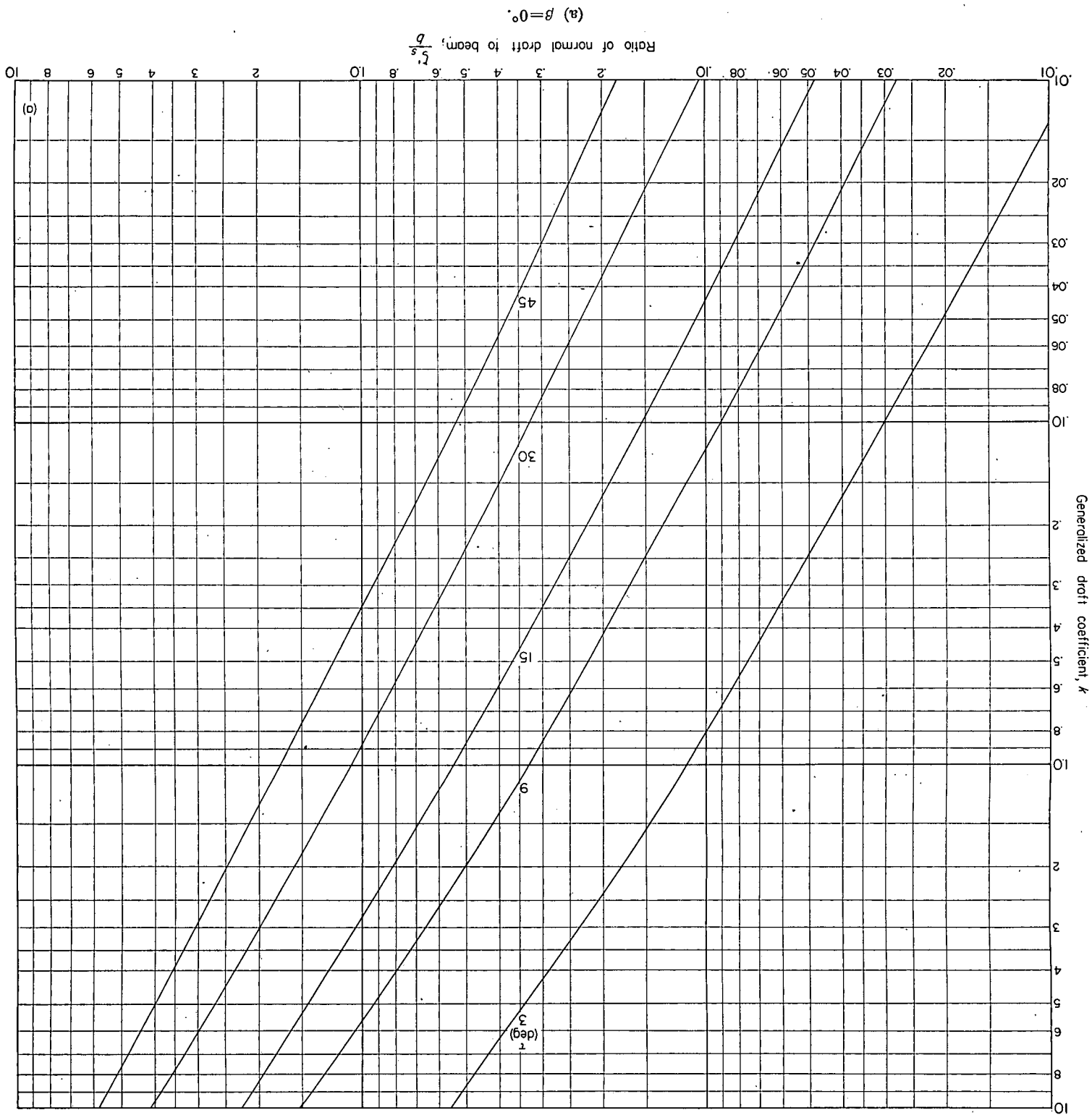


FIGURE 7.—Variation of generalized draft coefficients for various angles of dead rise.

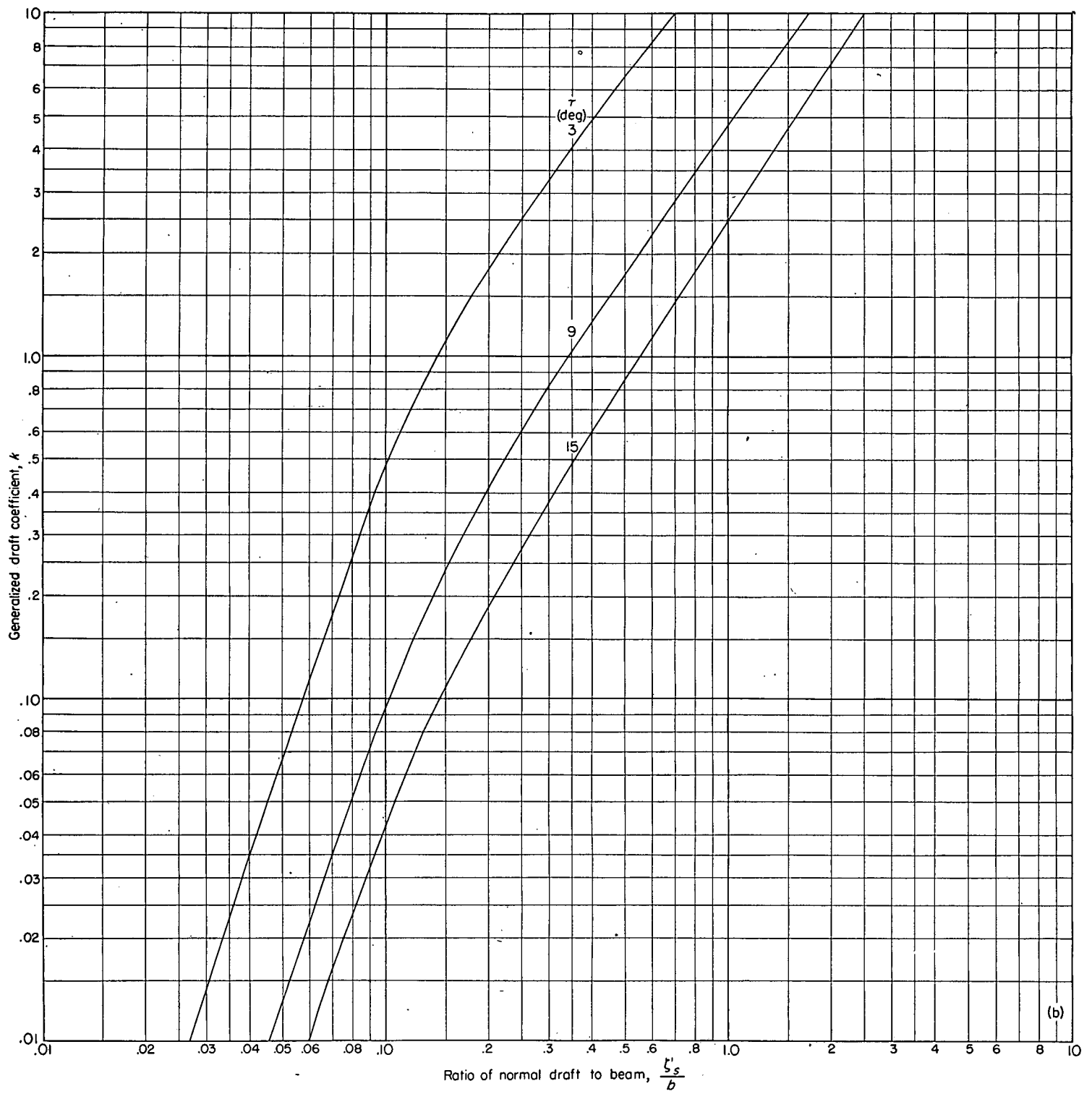
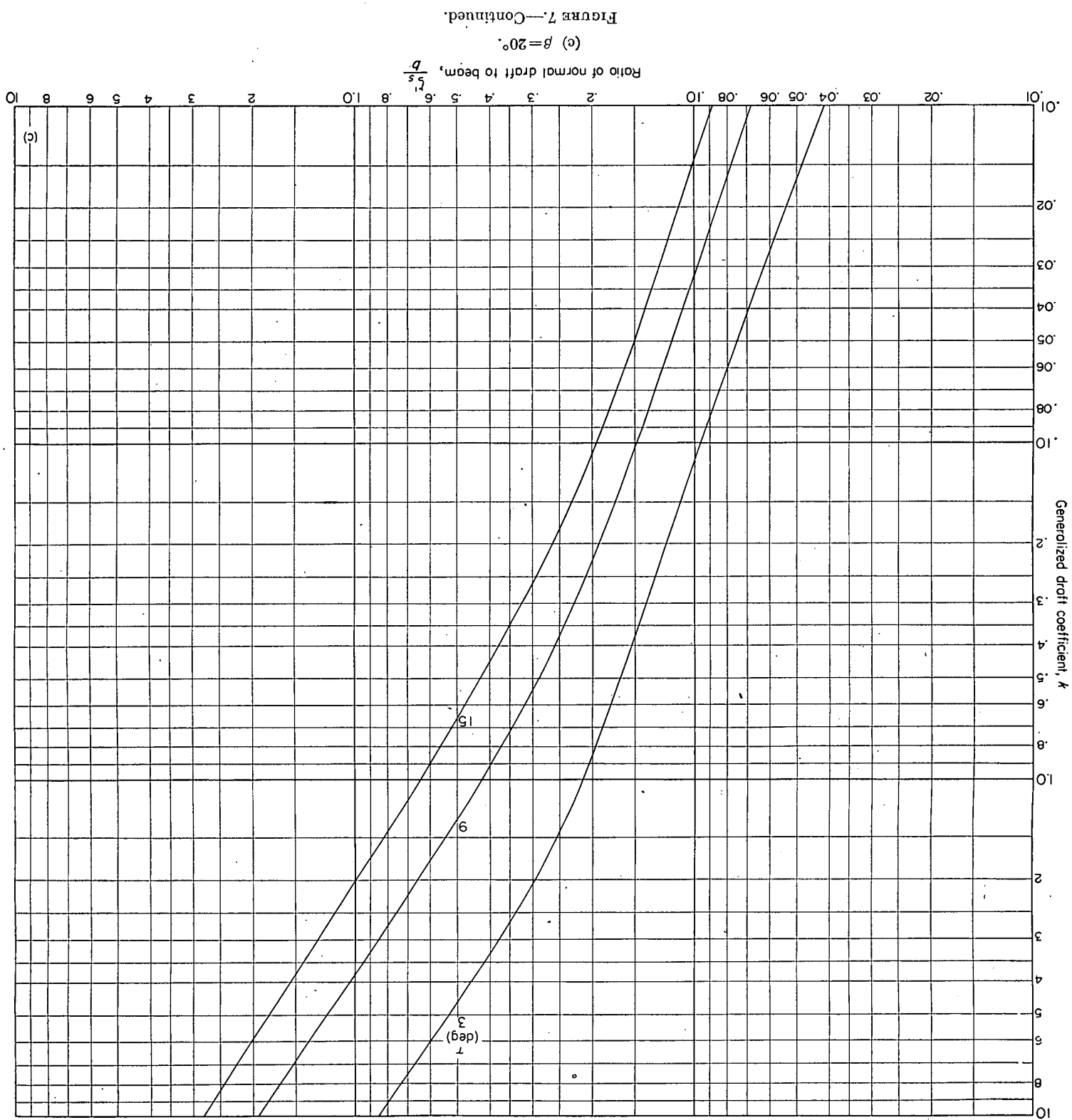
(b) $\beta = 10^\circ$.

FIGURE 7.—Continued.



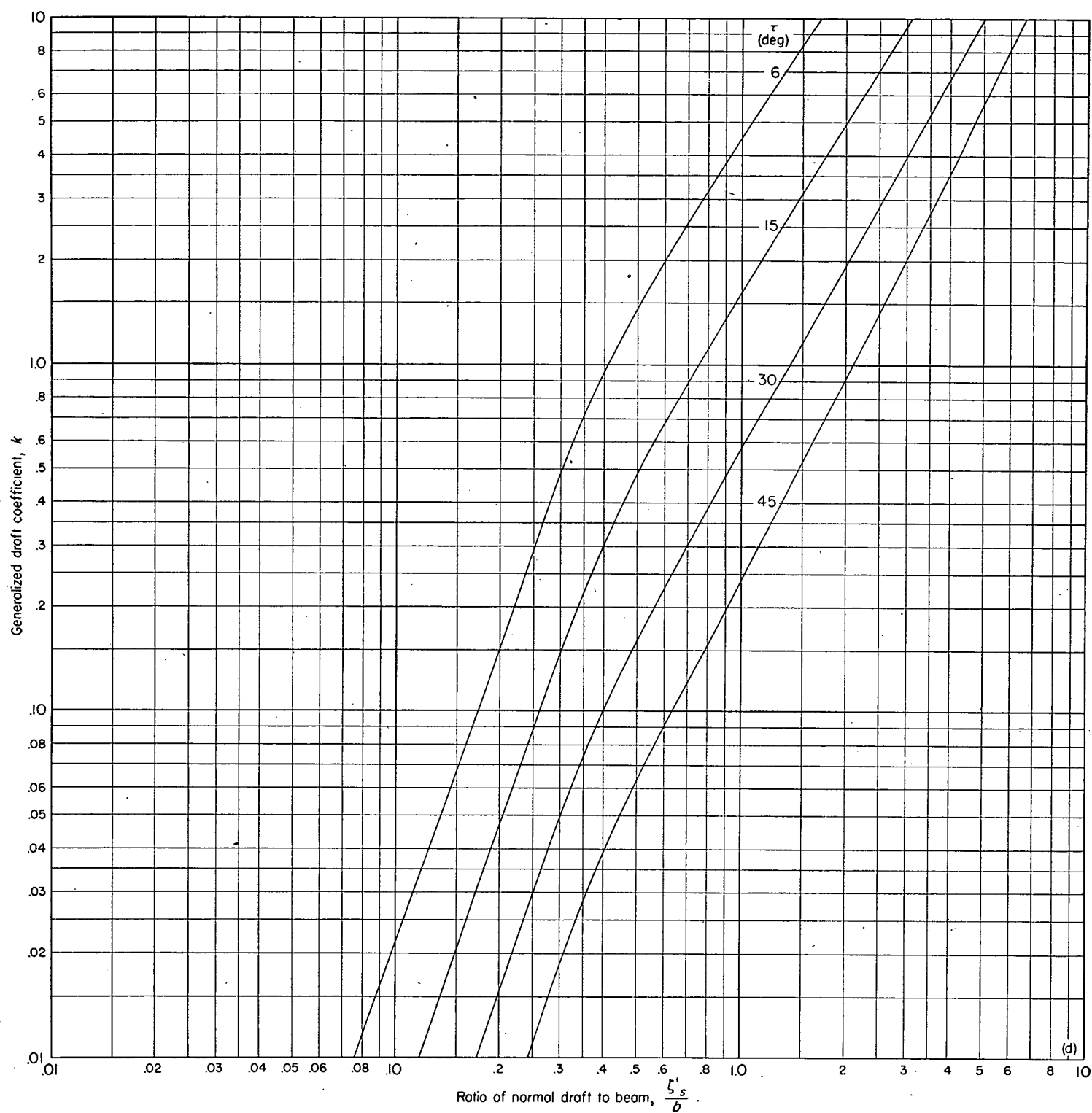


FIGURE 7.—Concluded.

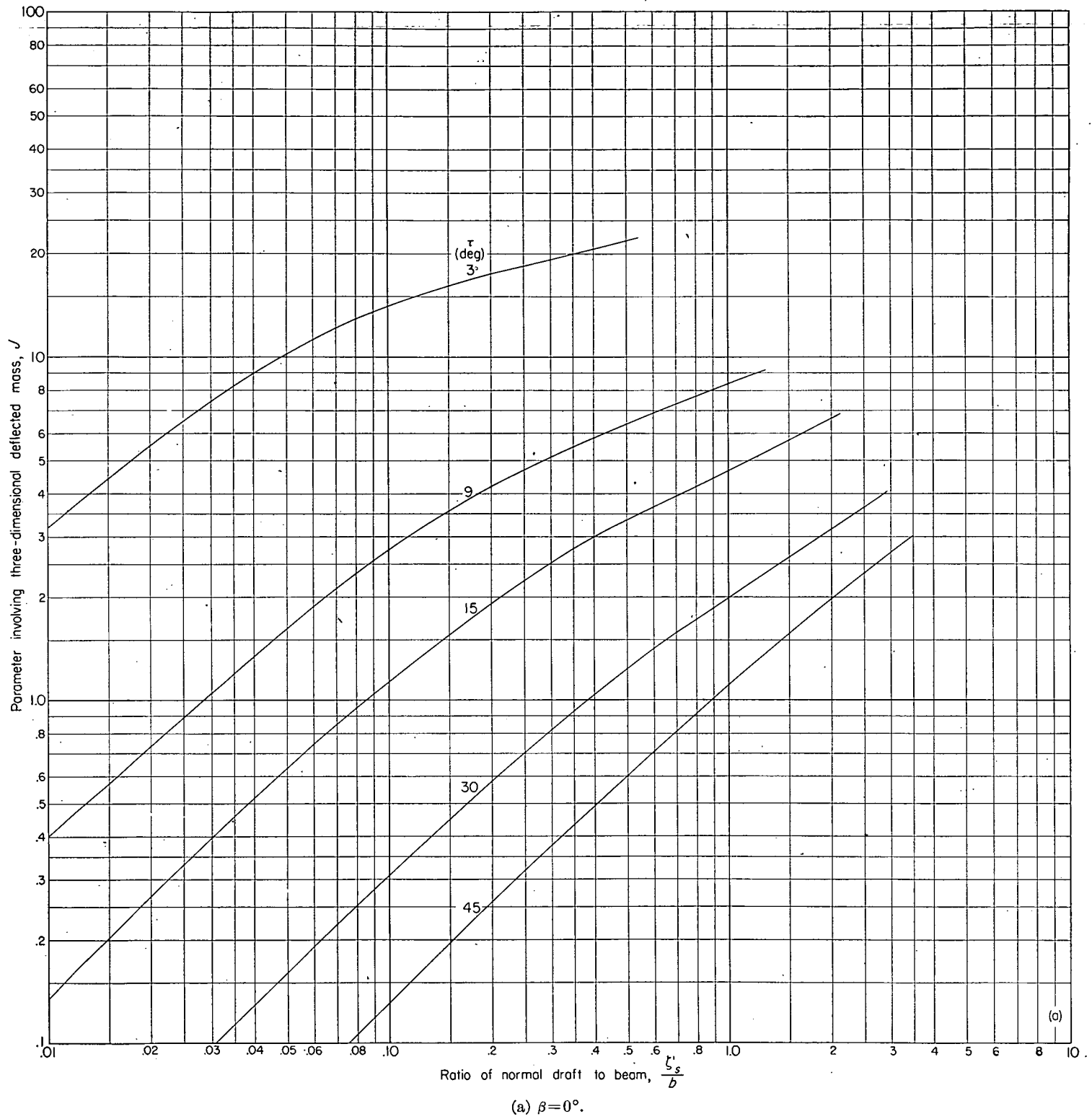


FIGURE 8.—Variation of parameter involving three-dimensional deflected mass for various angles of dead rise.

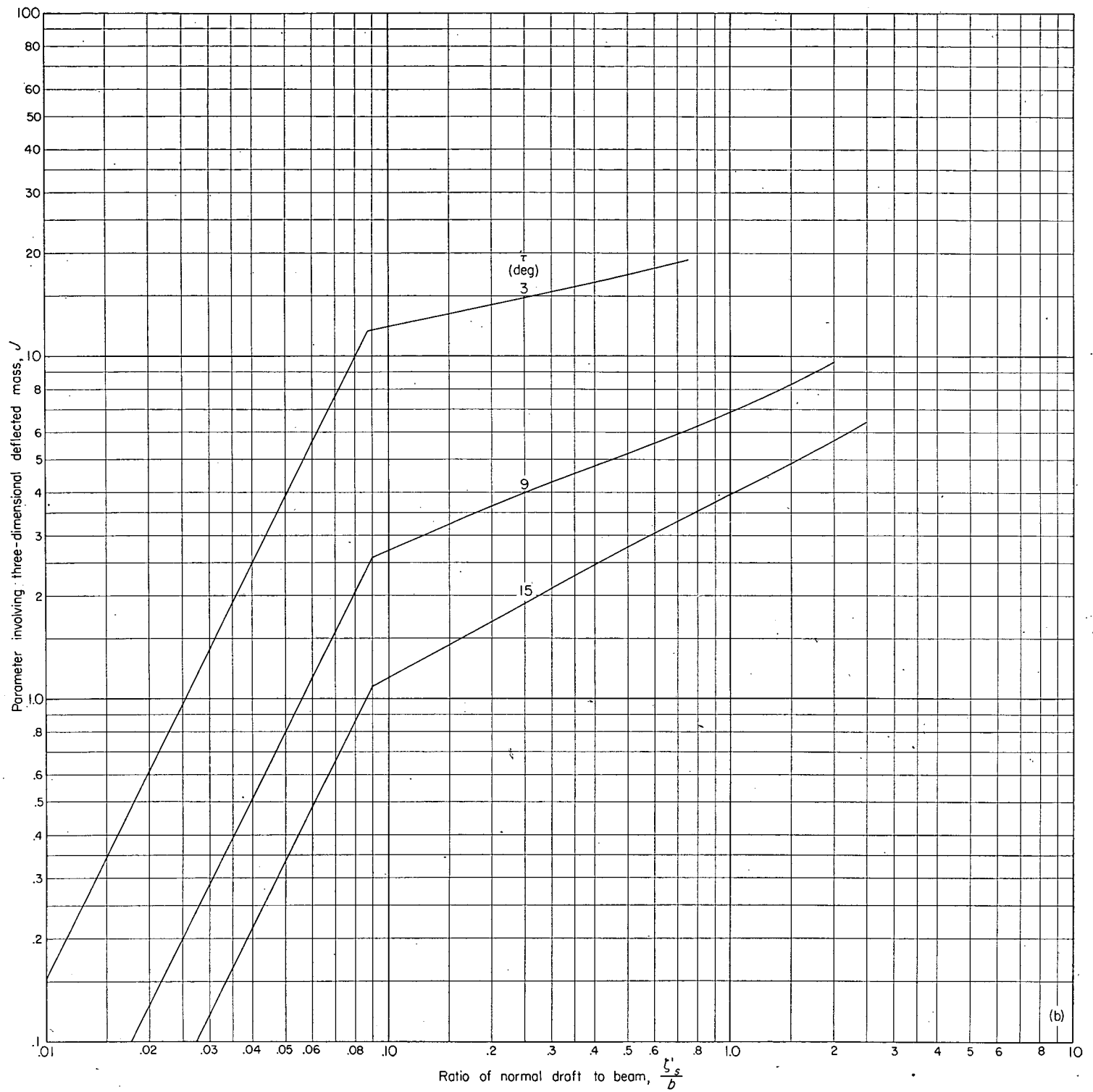
(b) $\beta = 10^\circ$.

FIGURE 8.—Continued.

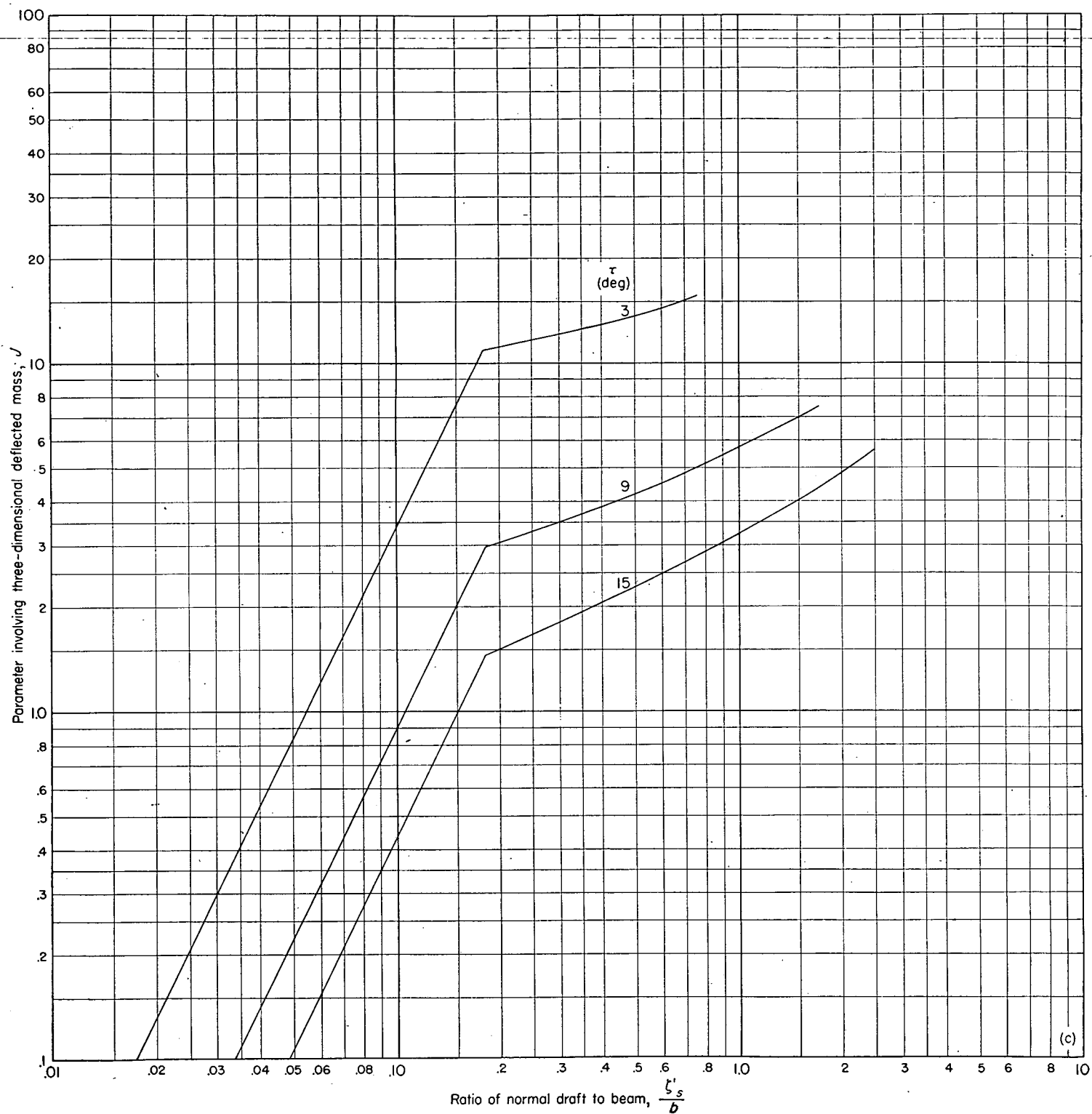
(c) $\beta = 20^\circ$.

FIGURE 8.—Continued.

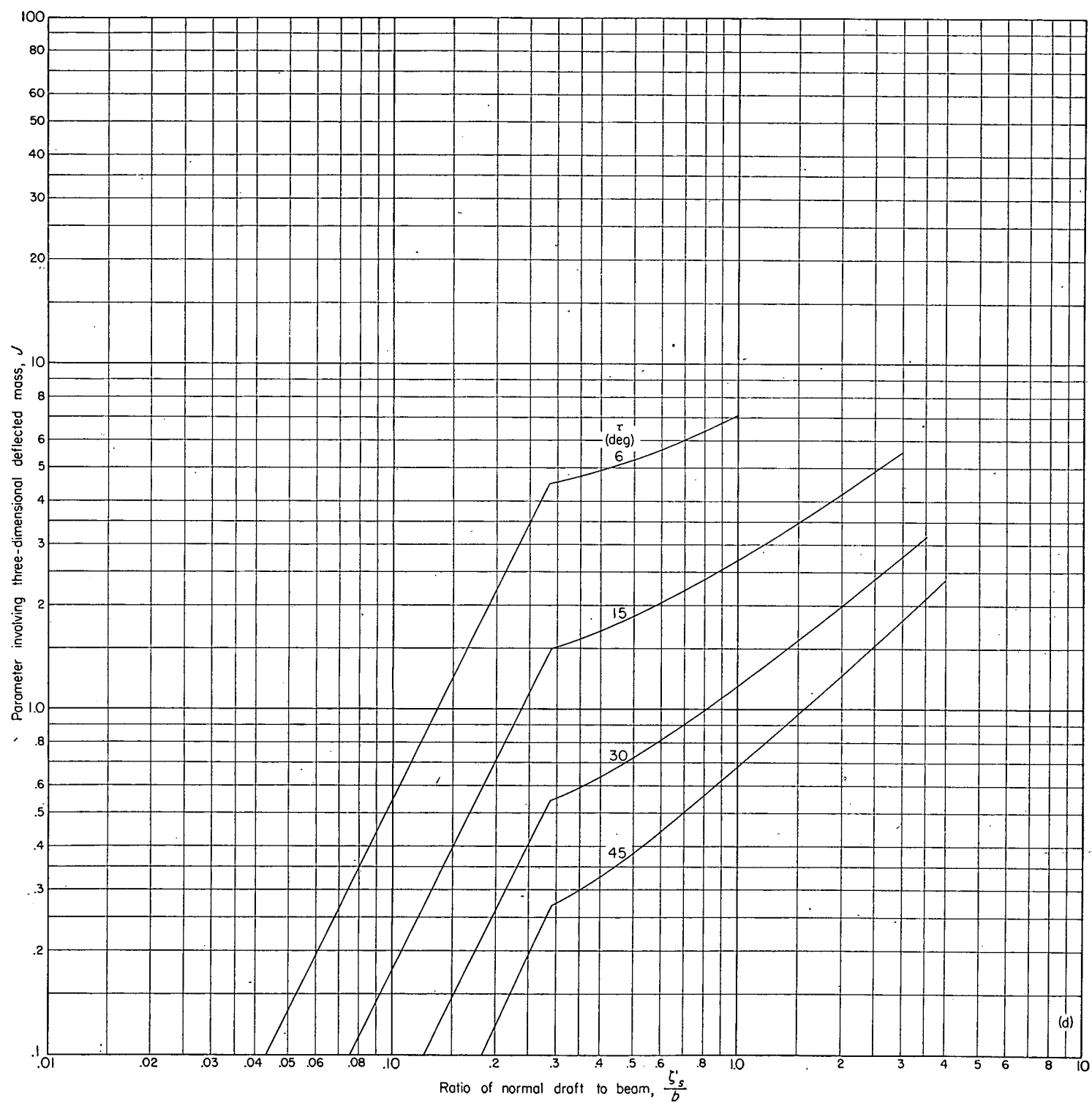
(d) $\beta = 30^\circ$.

FIGURE 8.—Concluded.

COMPARISON WITH TEST DATA

Theoretical curves, usable for actual landing-impact calculations, are presented in figures 2 to 8 and were discussed in detail in the previous sections. The succeeding figures present comparisons of experimental data, obtained at the Langley impact basin, with the proposed theory modified for constant forward velocity.

Several theoretical variations of impact load factor, vertical velocity, and vertical displacement with time are compared with experiment in figures 9 and 10. The experimental data in these plots were obtained at the Langley impact basin, and, although a portion of these data is unpublished, the rest may be found in references 13 and 14. In figures 9 (a) and 9 (b) are presented theoretical and experimental load-factor time histories of landings of a flat plate for a wide difference of trim and flight-path angle and for a beam-loading coefficient of 18.8. Fair agreement exists in each case.

An indication of the agreement between theory and experiment at the upper limit of the flight-path angle is given in figures 9 (c) and 9 (d). These figures present theoretical and experimental load-factor time histories for water landings of a flat plate having a beam-loading coefficient of 18.8 at trims of 6° and 15° for the end-point case of a vertical drop (flight-path angle of 90°). Fair agreement is also obtained in these figures.

In order to demonstrate the effect of neglecting the force term arising from the acceleration of the deflected mass, figure 9 (e) is presented for the landing of a flat plate with a medium beam-loading coefficient of 4.36, at the lowest trim for which data were available. The difference between the

two theoretical lines plotted illustrates the effect of neglecting this acceleration term. The higher line represents the solution with the deflected-mass acceleration neglected, and at maximum acceleration the difference between the two theoretical curves is seen to be about 10 percent. The deflected mass used in the acceleration term may possibly be overestimated here. If the cylindrical virtual water mass based on the beam as a diameter (ref. 2) were used instead, this difference of 10 percent would be somewhat reduced. The experimental data fall somewhere between the two curves, but no conclusion is drawn from this fact since the possible errors in the experimental data are estimated to be of the order of magnitude of $\pm 0.2g$.

The effect of water rise at the keel is demonstrated in figure 9 (f). This effect is greatest for small immersions and is therefore important for lightly loaded plates for which maximum drafts in impact are small. The upper curve includes the effect of water rise at the keel and the lower one does not. For plates with medium loading, consideration of this water rise increases the theoretical maximum load by about 9 percent and increases the initial rate of load application, so that the time to maximum load is reduced. As in figure 9 (e), the experimental data fall between the two theoretical curves up to maximum acceleration and the same reservation concerning accuracy holds. It might be mentioned that the acceleration data presented in this report have a time lag which is estimated to be approximately 0.005 second. If this also is taken into account, the theory including water rise in figure 9 (f) would give better agreement with the experimental data up to maximum load than the theory omitting water rise.

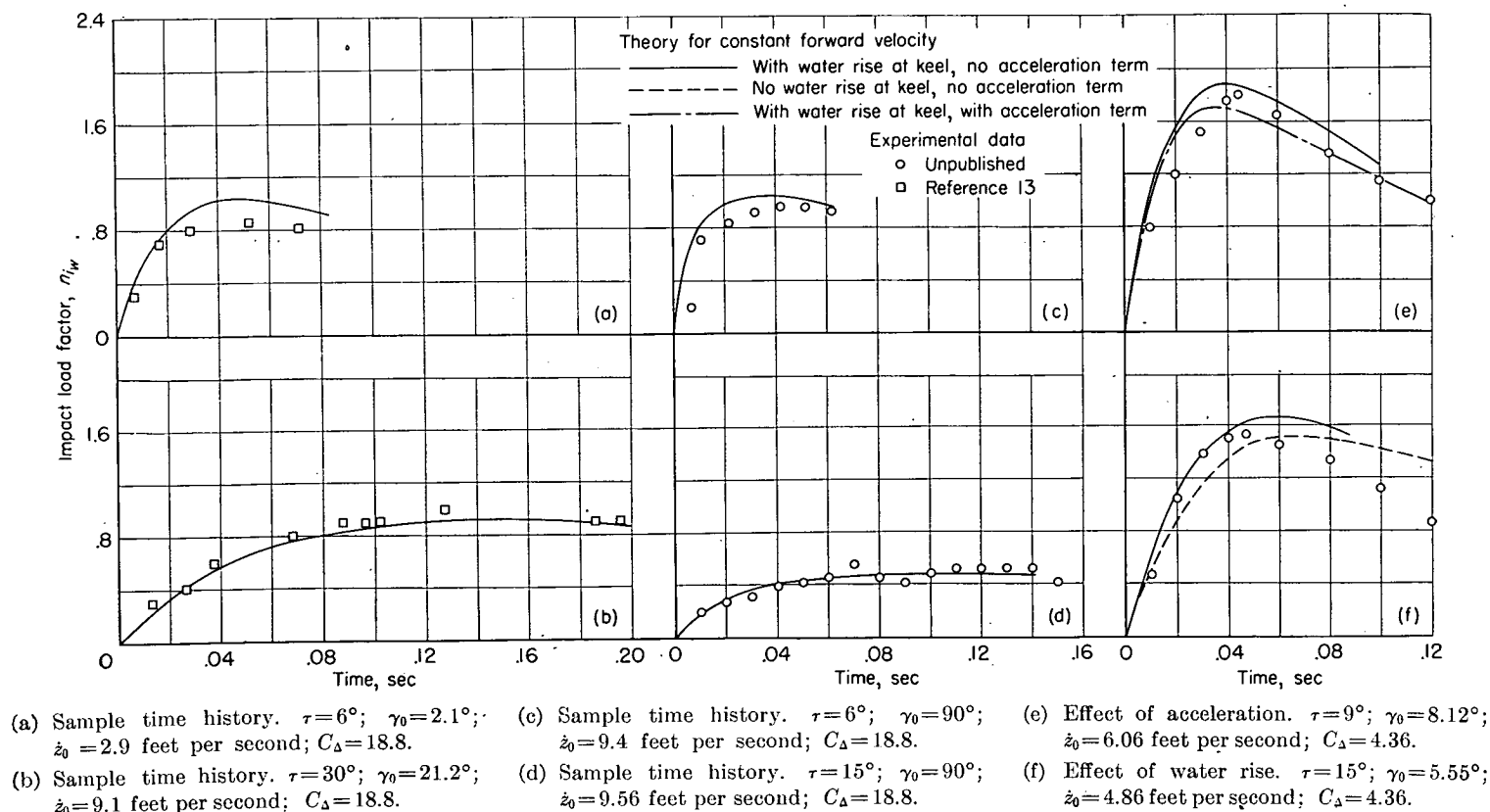


FIGURE 9.—Comparisons of theoretical and experimental impact-load time histories for flat plate.

Figure 10 (a) presents time histories of impact load factor, vertical velocity, and draft for an impacting flat plate having a beam-loading coefficient of 18.8. The effects of neglecting the acceleration of the deflected mass and water rise at the keel are again shown here, in addition to the effect of a large carriage mass. Comparison of the theoretical curves for free motion and constant forward velocity indicates that the large test-carriage mass, which causes the horizontal velocity to approach a constant value, increases the maximum load factor for this case by about 3 percent. The increase becomes considerably larger for the higher trims. The effects of carriage mass, acceleration of the deflected mass, and water rise at the keel on vertical velocity and draft are seen to be small.

In figure 10 (b), time histories of impact load factor, vertical velocity, and draft are presented for an impact of a hull with an angle of dead rise of 30° and a beam-loading coefficient of 18.8. The small effect of acceleration of the deflected mass may be noted by comparing the two curves

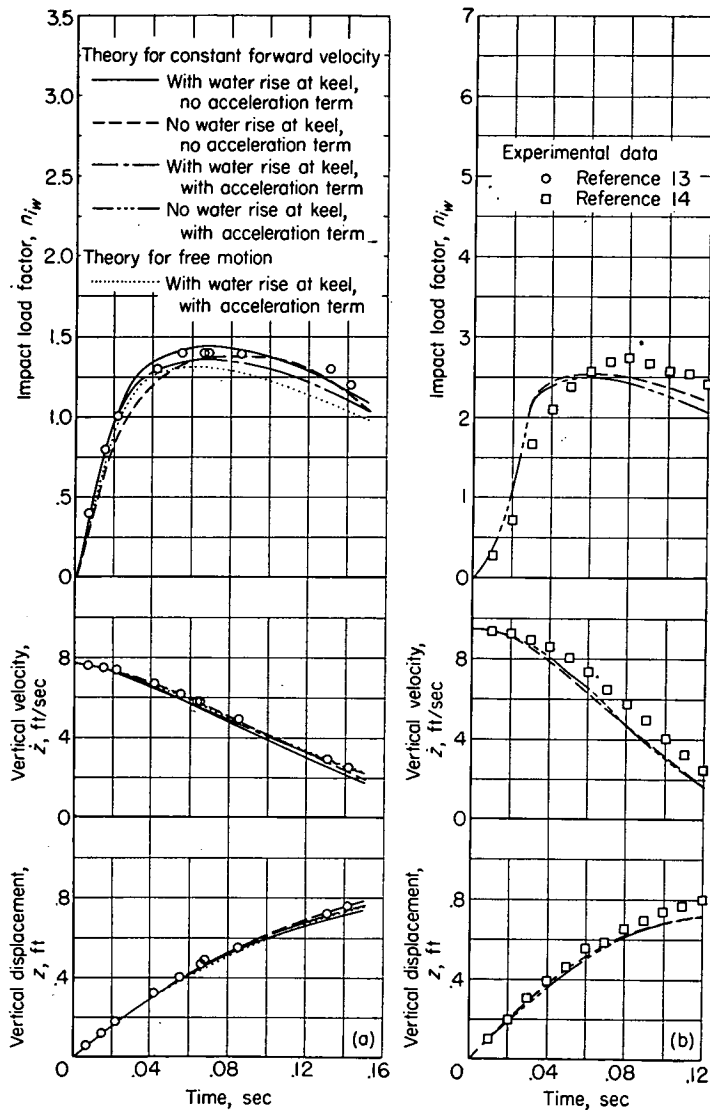


FIGURE 10.—Comparisons of theoretical and experimental impact time histories for 0° and 30° angles of dead rise.

and the experimental data on each plot of this figure. The agreement between the theoretical and experimental hull load factors would be improved if the aforementioned 0.005-second time lag were taken into account. A similar time lag exists in the vertical-velocity data and, although the exact value of this lag is unknown, it has been roughly estimated at 0.005 to 0.01 second. If this lag were taken into account, the agreement between the theoretical and the experimental velocity would be considerably improved. If more were known about the water rise at the keel for the float with dead rise and if a better variation of two-dimensional deflected mass were available, the agreement would probably be further improved. Thus, from figures 9 and 10 the agreement between theoretical and experimental time histories is seen to be fair.

The variation of impact lift coefficient with flight-path angle for wide ranges of trim and beam-loading coefficient is shown in figure 11 for 0° dead-rise angle and for flight-path angles up to 21.5° . The experimental data in this figure

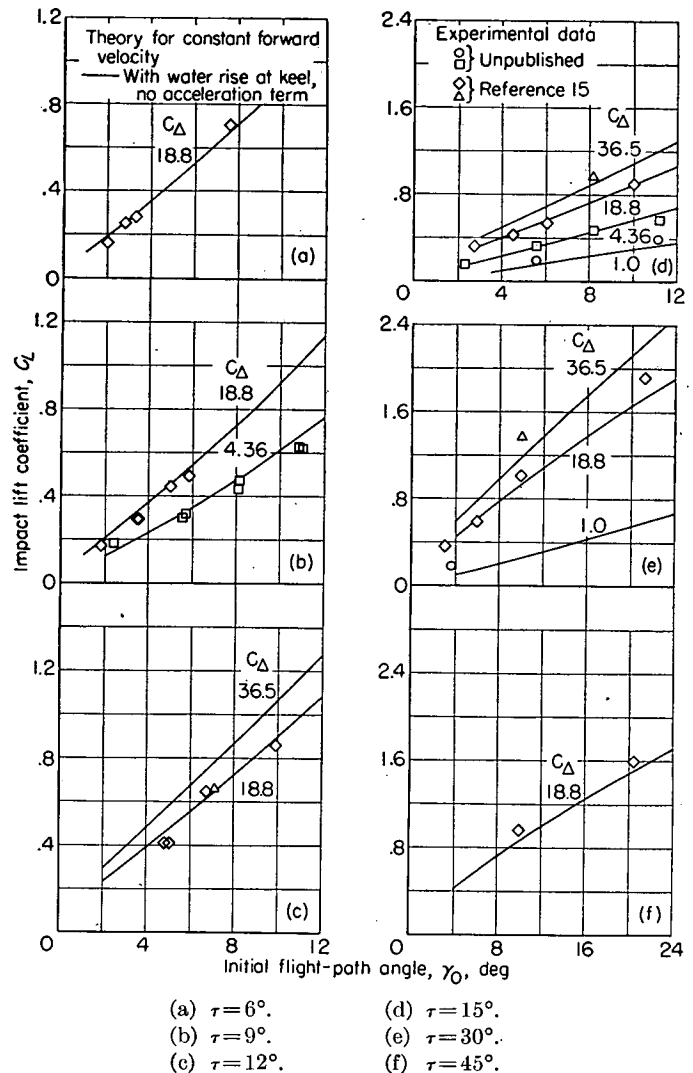


FIGURE 11.—Comparisons of theoretical and experimental impact lift coefficients C_L for a flat plate at various beam loadings.

$$C_L = \frac{(n_{vw})_{max} W}{\frac{\rho}{2} b^2 V_0^2}$$

were obtained at the Langley impact basin and the published portion may be found in reference 15. Fair agreement is seen to exist between experiment and the theory that incorporates water rise at the keel, omits the acceleration term, and assumes constant horizontal velocity, at least for beam-loading coefficients greater than 1. For beam-loading coefficients less than 1, the effects of acceleration and of flight-path angle on the deflected mass might introduce noticeable errors.

Since fair agreement was demonstrated between theoretical and experimental flat-plate landing accelerations for flight-path angles up to 21.5° in figure 11 and for the end point of 90° flight-path angle in figures 9 (c) and 9 (d), the proposed theory is believed to be applicable for all flight-path angles.

The variation of impact lift coefficient with flight-path angle for an angle of dead rise of 30° , a beam-loading coefficient of 18.8, and a wide range of trim is presented in figure 12. The solid curves represent the suggested theory for constant forward velocity including water rise at the keel and neglecting the acceleration term, with $d\xi_s/d\xi'$, taken as 1 for the body with dead rise. The dashed lines are calculated from the theory of reference 2, which predicts the occurrence of maximum load at the instant of chine immersion for impacts involving deep immersion of the chines. Each of these curves experiences a radical change of slope and shape at a certain critical flight-path angle for each trim and beam loading. For flight-path angles below this critical value, maximum load occurs prior to chine immersion. For a short range of flight-path angles immediately above this critical value, maximum load is believed to occur at or near chine immersion, and for high flight-path angles, maximum load occurs subsequent to chine immersion. Since the variation of deflected mass with draft is different before and after chine immersion, a break in the curve is expected to occur at the point of chine immersion. In figure 12 a comparison of the two theories with experimental data from reference 14 shows that for impacts involving a small degree of chine immersion the theory of reference 2 gives better results, at least for low trims, whereas for impacts involving deeply immersed chines the agreement with the theory suggested in this report is better. This disagreement of the suggested theory with experiment for small degrees of chine immersion when dead rise is present could be improved through use of a more accurate deflected-mass variation in the region of chine immersion. The general agreement between the experimental data for bodies with dead rise and the proposed theory, however, is seen to be fair, even for small amounts of chine immersion where the theory is conservative.

The variation of maximum draft with flight-path angle is presented in figure 13 for angles of dead rise of 0° and 30° and for several trims and beam loadings. The general

agreement with data from the Langley impact basin, the published portion of which may be found in references 14 and 15, is seen to be fair. This agreement indicates that the theory of the present report could be used in conjunction with the method of reference 16 to compute pitching-moment time histories during landing impacts.

CONCLUSIONS

A method has been derived for the analytical determination of the motions and hydrodynamic loads in chine-immersed waterlandings of prismatic bodies. Comparison of this theoretical work with other available theory and with experimental data obtained at the Langley impact basin has led to the following conclusions:

1. In general, the concept of a two-dimensional deflected mass with a correction for three-dimensional flow can be

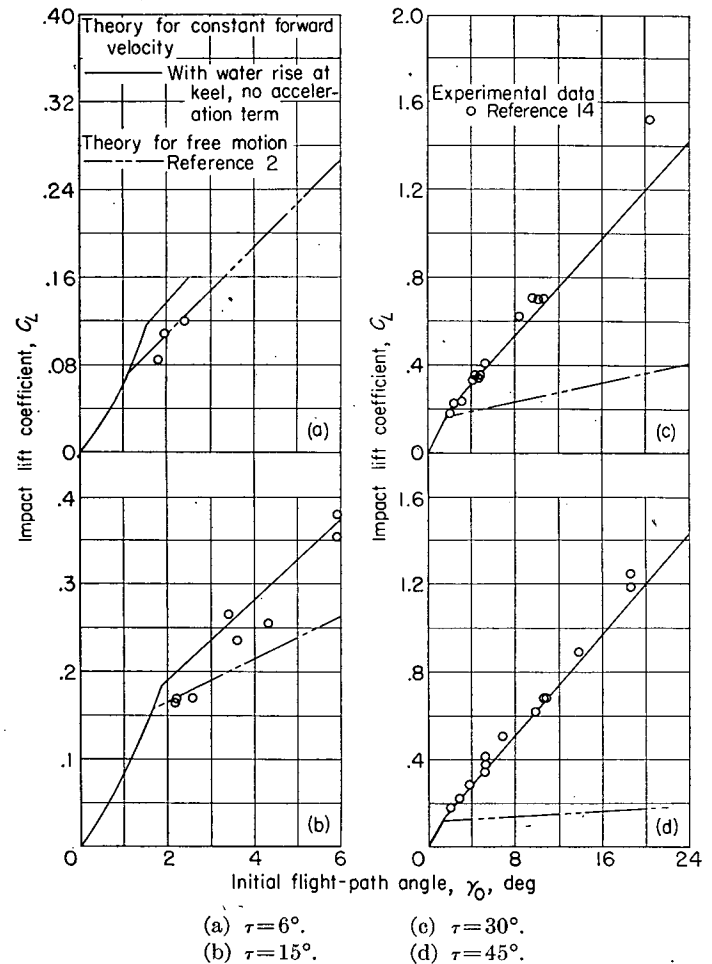


FIGURE 12.—Comparisons of theoretical and experimental impact lift coefficients for a hull with an angle of dead rise of 30° and a beam-loading coefficient of 18.8. $C_L = \frac{(n_{iw})_{max} W}{\frac{\rho}{2} b^2 V^2}$.

used to predict with reasonable accuracy the loads and motions during landings of prismatic bodies involving immersion of the chines.

2. Use of Wagner's virtual-mass variation for non-chine-immersed sections combined with a deflected-mass variation obtained from Bobyleff's solution after chine immersion of these sections, with level water as a boundary, gives fair agreement with experiment for deep impacts.

3. For shallow impacts at the lower trims, involving slight chine immersion of bodies with dead rise, the virtual-mass variation suggested in NACA TN 1516 gives better agreement with experiment than the proposed variation.

4. The effects of water rise at the keel in the case of a flat plate and the effects of the ratio of test-carriage mass to model mass in the general case are significant enough to be included in the proposed equations of motion.

5. Omission of the force arising from acceleration of the deflected mass is not serious for beam-loading coefficients larger than unity and results in a large reduction in the work required for each solution.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., June 25, 1952.

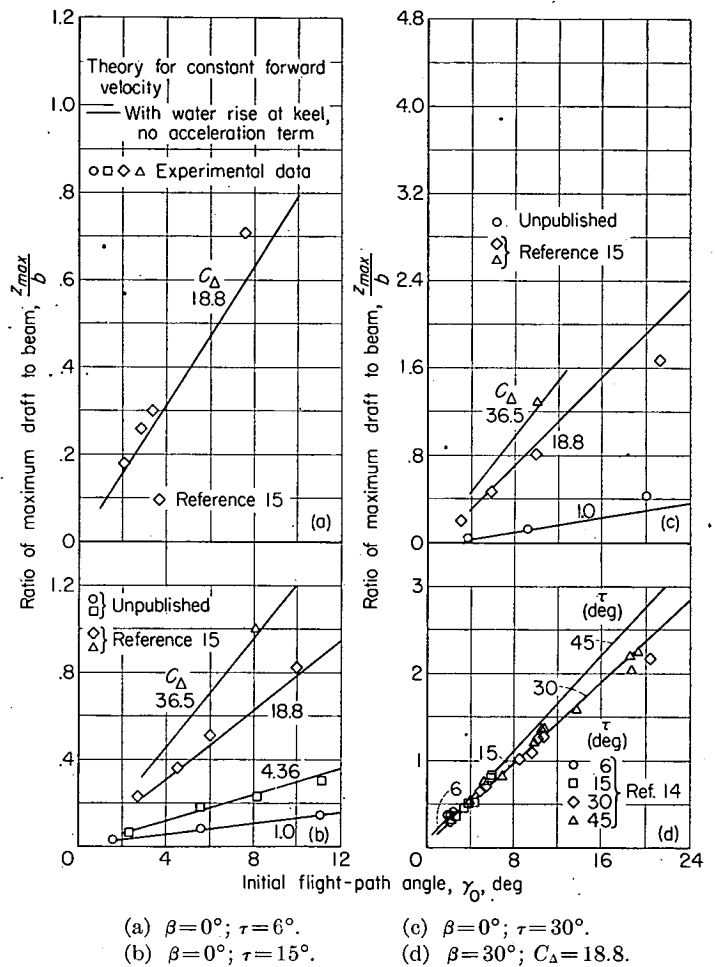


FIGURE 13.—Comparisons of theoretical and experimental maximum drafts for 0° and 30° angles of dead rise.

APPENDIX A

WATER RISE ON A RECTANGULAR FLAT PLATE

As has been mentioned in this report, reference 13 shows the water rise at the keel of an impacting body to be relatively insensitive to flight-path angle and thus capable of being evaluated from planing data. Since such data are most conveniently analyzed in terms of the wetted length of the body (l in fig. 1) and since this analysis is based on the draft z with respect to the undisturbed water surface, the relation between these two quantities must be obtained. This relation is expressed by the equation (see fig. 1)

$$z = l \sin \tau - r \quad (\text{A1})$$

where r is the water rise at the keel.

Several papers about planing discuss this phenomenon of water rise at the keel in connection with steady motion and reference 13 discusses it in connection with motion of the peak-pressure line during impact. A theoretical solution by Wagner (ref. 10) for the two-dimensional planing flat plate predicts infinite water rise for the ideal case, in which gravity and viscosity are neglected. Planing data from references 17 to 19 show the increase in wetted length due to water rise $\lambda' - \lambda$ to be a more or less constant fraction of the hull beam, with only a small variation due to changes in trim. At low length-beam ratios this result is more or less in agreement with Wagner's theory for planing, but it would not be in agreement for impact, as indicated in reference 13. This reference shows an expected gradual transition from no water rise at the instant of water contact to some constant value for the hull tested at ratios of wetted length to beam greater than 1.5. The rise variation of reference 13 is not used in this report, however, since it is based on the peak-pressure location and not on the measured wetted length. The following system is therefore used for the flat plate. The case of a hull with dead rise is not covered because of a lack of sufficient information on water rise at the keel in planing.

The assumption is made, as in reference 13, that the water rise at the keel for a two-dimensional flow about an impacting plate of infinite width is independent of the flight-path angle and is proportional to the draft; thus

$$r = Cz \quad (\text{A2})$$

where r is the rise and C is the constant of proportionality.

In nondimensional form this becomes

$$\frac{r}{b} = C\lambda \sin \tau \quad (\text{A3})$$

where $\lambda = \frac{z}{b \sin \tau}$ is the length-beam ratio below the undisturbed water surface for a flat plate. For three-dimensional flow about plates with a very small ratio of wetted length to beam, equation (A3) would also be applicable. However, as the length-beam ratio increases, this piled-up water should be relieved through flow around the sides of the model. Thus, some form of correction for three-dimensional flow $f(\lambda)$ is required to reduce the rise for high length-beam ratios, as follows:

$$\frac{r}{b} = C\lambda f(\lambda) \sin \tau \quad (\text{A4})$$

A correction for three-dimensional flow which fits the end points, reducing equation (A4) to equation (A3) at small length-beam ratios and to a constant value at large length-beam ratios, is Pabst's correction (eq. (4) in the body of the report) applied to the inverse hydrodynamic aspect ratio. Substituting this correction into equation (A4) gives

$$\frac{r}{b} = C\lambda \varphi \left(\frac{1}{\lambda} \right) \sin \tau \quad (\text{A5})$$

The effective ratio of wetted draft to beam (see fig. 1) is therefore

$$\frac{z+r}{b} = \frac{z}{b} + C\lambda \varphi \left(\frac{1}{\lambda} \right) \sin \tau \quad (\text{A6})$$

Since $\lambda = \frac{z}{b \sin \tau}$ and $\lambda' = \frac{z+r}{b \sin \tau}$, the effective ratio of wetted length to beam is

$$\lambda' = \lambda \left[1 + C\varphi \left(\frac{1}{\lambda} \right) \right] \quad (\text{A7})$$

An average value of 0.4 for C was obtained from the experimental planing data in references 17 to 19. Figure 6(a) is a plot of equation (A7) based on this value of C , and figure 6(b) is a plot of the slope of this curve, or $\frac{d\lambda}{d\lambda'} = \frac{d\xi_s}{d\xi'_s}$ against λ' .

APPENDIX B

SUGGESTED COMPUTATIONAL PROCEDURE

This section gives suggested computational methods to facilitate calculation of loads and motions during free-body impacts of a prismatic form experiencing appreciable chine immersion. These methods are based on a solution of the equation of motion which takes into account water rise at the keel, neglects the acceleration of the virtual mass, and neglects the effects of flight-path angle on water rise at the keel and on the normal-force coefficient. As a result of these omissions, the effects of which are small at the higher beam loadings, this solution is considered applicable only to those cases for which the beam-loading coefficient exceeds unity. This applicability is also restricted to drafts for which the leading edge of the prismatic surface has not penetrated beneath the water surface.

The overall computational procedure is subdivided into four less-general procedures to increase the utility of the solution. The first of these treats smooth-water landings of prismatic bodies approximating V-sections and makes use of computational charts covering specific angles of dead rise of 0°, 10°, 20°, and 30° and certain fixed trims. The second procedure applies to the same bodies for all practical angles of dead rise and fixed trims but requires more work for each solution. The third procedure covers prismatic bodies of arbitrary shape but requires experimental data from planing or drop tests. The fourth procedure accomplishes a conversion from smooth-water to rough-water landings. Explanation of symbols is given in the list of symbols following the introduction and in figure 1.

PROCEDURE 1—SMOOTH-WATER LANDINGS OF A PRISMATIC BODY HAVING A CROSS SECTION APPROXIMATING A V-SHAPE WITH A DEAD-RISE ANGLE OF 0°, 10°, 20°, OR 30° AT ONE OF SEVERAL FIXED TRIMS

1. Obtain a value of κ from figure 4 through use of appropriate values of initial flight-path angle γ_0 and trim τ .
2. Select several values of the vertical-velocity ratio $\dot{z}/\dot{z}_0$ between 1 and -1 and, with the value of κ , obtain a value of Q from figure 5 for each value of $\dot{z}/\dot{z}_0$.
3. Compute a value of k for each value of Q from the equation $k = -C_\Delta Q$, where $C_\Delta = \frac{W}{\rho g b^3}$ and is defined as the beam-loading coefficient.
4. Obtain values of the ratio of normal draft to beam ζ'_s/b for each value of k from the curve for the appropriate values of τ and average angle of dead rise β in figure 7.
5. Obtain a value of J for each value of ζ'_s/b for the appropriate values of τ and β in figure 8.
6. Calculate a value of the nondimensional acceleration $\ddot{z}b/\dot{z}_0^2$ for each combination of $\dot{z}/\dot{z}_0$ and ζ'_s/b through substitution of the appropriate quantities into the equation

$$\frac{\ddot{z}b}{\dot{z}_0^2} = -\left(\frac{\dot{z}}{\dot{z}_0} + \kappa\right)^2 \frac{J}{C_\Delta} \quad (B1)$$

7. Plot the load factor $n_{iw} = -\frac{\ddot{z}}{g}$ and the vertical velocity $\dot{z}$ against the ratio of normal draft to beam ζ'_s/b .

8. Repeat steps 1 to 7 for several other values of γ_0 and τ , covering the range of interest.

9. Compute values of $C_L = \frac{(n_{iw})_{\max} W}{\frac{\rho}{2} b^2 V_0^2}$ from the maximum

values of n_{iw} obtained from the curves obtained in step 8 and plot C_L against γ_0 for various values of τ . The resulting curves may be used as design curves.

10. Compute values of draft-beam ratio z/b from the equation

$$\frac{z}{b} = \frac{\zeta'_s}{b} \frac{\lambda}{\lambda'} \cos \tau \quad (B2)$$

where λ/λ' is obtained from figure 6(a) for $\beta=0^\circ$ and is taken as unity for $\beta \geq 10^\circ$. For the case where $\beta=0^\circ$, $\lambda' = \frac{\zeta'_s}{b \tan \tau}$.

11. For calculation of structural response, the time variation from water contact through exit may be found by incremental graphical integration of the equation

$$\frac{t V_0}{b} = \frac{1}{\sin \gamma_0} \left[\int_0^{z/b} \frac{d \frac{z}{b}}{\frac{\dot{z}}{\dot{z}_0}} + \int_{z/b}^{z_2/z_0} \frac{d \frac{z}{b}}{\frac{\dot{z}}{\dot{z}_0}} + \int_{z_2/z_0}^{z/b} \frac{d \frac{z}{b}}{\frac{\dot{z}}{\dot{z}_0}} \right]$$

where V_0 is the initial resultant velocity at water contact. The first integral gives the time to some point prior to maximum draft, the second includes maximum draft, and the third covers the remaining time to exit. Although the first and third integrals give the time as a function of draft and the second gives it as a function of vertical velocity, the previously derived relationships between draft, vertical velocity, and load factor allow the time histories of all of these quantities to be computed.

PROCEDURE 2—SMOOTH-WATER LANDINGS OF A PRISMATIC BODY HAVING A CROSS SECTION APPROXIMATING A V-SHAPE WITH ANY DEAD-RISE ANGLE, AT ANY TRIM

1. Select a series of several values of the ratio of normal draft to beam ζ'_s/b and compute values of $m_{ws}/\rho b^2$ from the equations

$$\frac{m_{ws}}{\rho b^2} = \frac{\pi^3}{32} + 0.44 \frac{\zeta'_s}{b} \quad (\beta=0^\circ)$$

$$\frac{m_{ws}}{\rho b^2} = \frac{\pi}{2} \left[\frac{\zeta'_s}{b} f(\beta) \right]^2 \quad \left(\beta > 0^\circ; \frac{\zeta'_s}{b} \leq \frac{\tan \beta}{2} \right)$$

$$\frac{m_{ws}}{\rho b^2} = \frac{\pi}{8} [f(\beta) \tan \beta]^2 + \frac{B}{2} \left(\frac{\zeta'_s}{b} - \frac{\tan \beta}{2} \right) \quad \left(\beta > 0^\circ; \frac{\zeta'_s}{b} > \frac{\tan \beta}{2} \right)$$

where

$$f(\beta) = \frac{\pi}{2\beta} - 1$$

and a value of B is obtained from figure 3 by means of the average angle of dead rise β .

2. Obtain a value of λ' for each value of ζ'_s/b selected in step 1 by means of the equations

$$\lambda' = \frac{\zeta'_s}{b \tan \tau} \quad (\beta = 0^\circ) \quad (B3)$$

$$\lambda' = \frac{1}{\tan \tau f(\beta)} \quad \left(\beta > 0^\circ; \frac{\zeta'_s}{b} = \frac{\zeta_s}{b} \leq \frac{\tan \beta}{2} \right) \quad (B4)$$

$$\lambda' = \frac{\left(\frac{\zeta'_s}{b} \right)^2}{\tan \tau \left[\frac{\zeta'_s}{b} - \frac{1}{4f(\beta)} \right]} \quad \left(\beta > 0^\circ; \frac{\zeta'_s}{b} = \frac{\zeta_s}{b} > \frac{\tan \beta}{2} \right) \quad (B5)$$

and substitute these values of λ' into figure 2 to obtain values of $\varphi(\lambda')$.

3. Obtain values of $d\zeta_s/d\zeta'_s$ from figure 6(b) for each value of λ' for cases where $0^\circ \leq \beta < 10^\circ$. For cases where $\beta \geq 10^\circ$, $d\zeta_s/d\zeta'_s$ is taken equal to unity.

4. Combine the results of steps 1 to 3 to obtain values of $\frac{\varphi(\lambda')}{\tan \tau} \frac{m_{w_s}}{\rho b^2} \frac{d\zeta_s}{d\zeta'_s}$, plot these against ζ'_s/b , and graphically evaluate the integral

$$QC_\Delta = - \int_0^{\zeta'_s/b} \frac{\varphi(\lambda')}{\tan \tau} \frac{m_{w_s}}{\rho b^2} \frac{d\zeta_s}{d\zeta'_s} d \frac{\zeta'_s}{b} \quad (B6)$$

at each value of ζ'_s/b selected in step 1.

5. Obtain a value of Q from equation (B6) for each value of ζ'_s/b where the beam-loading coefficient $C_\Delta = \frac{W}{\rho g b^3}$. For each value of Q , obtain a value of the vertical-velocity ratio $\dot{z}/\dot{z}_0$ from figure 5, using the value of κ obtained from figure 4 by means of the appropriate initial flight-path angle γ_0 and the value of fixed trim τ .

6. Calculate a value of the nondimensional acceleration $\ddot{z}b/\dot{z}_0^2$ for each combination of $\dot{z}/\dot{z}_0$ and ζ'_s/b through substitution of the appropriate quantities into the equation

$$\frac{\ddot{z}b}{\dot{z}_0^2} = - \left(\frac{\dot{z}}{\dot{z}_0} + \kappa \right)^2 \frac{\varphi(\lambda')}{C_\Delta \sin \tau} \frac{m_{w_s}}{\rho b^2}$$

PROCEDURE 3—SMOOTH-WATER LANDINGS OF A PRISMATIC BODY OF ARBITRARY CROSS SECTION WHERE SECTION CHARACTERISTICS OBTAINED FROM EXPERIMENTAL DATA ARE REQUIRED

This procedure is suggested as a rough approximation only, as it has not been verified by experimental data.

1. Select a series of several values of the ratio of normal draft to beam ζ'_s/b and compute a value of λ' for each value of ζ'_s/b , either from formulas (B3) to (B5) of procedure 2, where β is the average angle of dead rise, or from the equation $\lambda' = \frac{l^2}{S}$ where l is the wetted length and S is the wetted area projected normal to the keel. Obtain a value of $\varphi(\lambda')$ for each value of λ' from figure 2.

2. Substitute data from high-speed planing runs, vertical drops, or oblique impacts of a heavily loaded prismatic body with a cross-sectional shape similar to that of the body of interest into the equation

$$\frac{m_{w_s}}{\rho b^2} = \frac{F_N \tan \tau}{\varphi(\lambda') \dot{\zeta}^2 \rho b^2}$$

to obtain a value of $m_{w_s}/\rho b^2$ for each value of ζ'_s/b .

- 3.)
 - 4.)
 - 5.)
 - 6.)
 - 7.)
 - 8.)
 - 9.)
 - 10.)
 - 11.)
- Same as in procedure 2.
- Same as in procedure 1.

PROCEDURE 4—CONVERSION TO ROUGH-WATER STEP LANDINGS

Rough-water landings of prismatic bodies with any angle of dead rise into waves which are long compared with these bodies may be handled as in reference 20 by the following procedure:

1. Determine the wave slope at the point of contact from reference 6, articles 229 and 251, or from reference 20. The most severe landings are believed to occur on the flank of an advancing wave, in the region of the steepest slope.

2. Rotate the space coordinate system so that the z -axis is normal to the wave slope and compute an effective trim with respect to these coordinates.

3. Compute the wave-particle velocity at the point of contact from reference 6, articles 229 and 251, or from reference 20, subtract this velocity vectorially from the hull velocity, and compute an effective flight-path angle from the resultant velocity with respect to the new coordinates.

4. Using these effective values of trim and initial flight-path angle, continue as in smooth-water cases outlined in procedures 1, 2, and 3.

- 7.)
 - 8.)
 - 9.)
 - 10.)
 - 11.)
- Same as in procedure 1.

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REPORT 1153

ON THE APPLICATION OF TRANSONIC SIMILARITY RULES TO WINGS OF FINITE SPAN¹

By JOHN R. SPREITER

SUMMARY

The transonic aerodynamic characteristics of wings of finite span are discussed from the point of view of a unified small disturbance theory for subsonic, transonic, and supersonic flows about thin wings. Critical examination is made of the merits of the various statements of the equations for transonic flow that have been proposed in the recent literature. It is found that one of the less widely used of these possesses considerable advantages, not only from the point of view of a priori theoretical considerations but also of actual comparison of theoretical and experimental results. The similarity rules and known solutions of transonic-flow theory are reviewed, and the asymptotic behavior of the lift, drag, and pitching-moment characteristics of wings of large and small aspect ratio is discussed. It is shown that certain methods of data presentation are superior for the effective display of these characteristics.

INTRODUCTION

The small perturbation potential theory of transonic flow proposed apparently independently by Oswatitsch and Wieghardt, Busemann and Guderley, von Kármán (refs. 1 through 6), and others is now supplying a fund of information regarding transonic flow about aerodynamic shapes. Solutions have been given for two-dimensional flow around airfoils at both subsonic and supersonic speeds in papers by Guderley and Yoshihara, Vincenti and Wagoner, Cole, Trilling, Oswatitsch, Gullstrand (refs. 7 through 16), and others. In the application of these results to specific examples, two items of theoretical interest have been noted (see, in particular, refs. 8, 17, and 18): (a) The theoretical results appear to be applicable at Mach numbers far removed from 1 even though, in most cases, the results have been obtained from equations valid only in the immediate neighborhood of sonic speed. (b) In the application of theoretical results to specific examples at Mach numbers other than 1, it has been noted that certain ambiguities exist in the theoretical determination of aerodynamic quantities. It is one of the purposes of this report to investigate these two points. This is accomplished by examining transonic flow from the point of view of equations that are valid throughout the Mach number range rather than only in the neighborhood of sonic speed. Such an approach emphasizes the relation between the roles of linear theory and of nonlinear theory in the transonic range.

The similarity rules provided by the theory (refs. 5, 6, and 19 through 22) have also proved to be useful in the cor-

relation and interpretation of experimental data. It is with the latter aspect of the transonic-flow problem that the present paper is primarily concerned. In this paper, the similarity rules and their application to the specific problem of concise presentation of lift, drag, and pitching-moment characteristics of wings are given in detail. The known solutions of two-dimensional transonic flow are reviewed and the asymptotic behavior of the aerodynamic characteristics of wings of large and small aspect ratios is examined. It is shown that certain methods of data presentation are advantageous for displaying these characteristics.

SYMBOLS

A	aspect ratio
$\tilde{A}$	$[U_\infty k(t/c)]^{1/3} A$
a	speed of sound
a_∞	speed of sound in the free stream
a^*	critical speed of sound
b	wing semispan
C_D	drag coefficient
$\tilde{C}_D$	$\frac{(U_\infty k)^{1/3}}{(t/c)^{5/3}} C_D$
C_{D_0}	drag coefficient of symmetrical nonlifting wings
$\tilde{C}_{D_0}$	$\frac{(U_\infty k)^{1/3}}{(t/c)^{5/3}} C_{D_0}$
ΔC_D	contribution to drag coefficient due to lift
$(\Delta C_D/\alpha^2)$	$[U_\infty k(t/c)]^{1/3} (\Delta C_D/\alpha^2)$
C_L	lift coefficient
(C_L/α)	$[U_\infty k(t/c)]^{1/3} (C_L/\alpha)$
C_m	pitching-moment coefficient
(C_m/α)	$[U_\infty k(t/c)]^{1/3} (C_m/\alpha)$
C_p	pressure coefficient
$\tilde{C}_p$	$\frac{(U_\infty k)^{1/3}}{(t/c)^{2/3}} C_p$
$C_{p_{cr}}$	critical pressure coefficient
C, P	center-of-pressure function
c	wing chord
c_{d_0}	section drag coefficient of symmetrical nonlifting airfoils
$\tilde{c}_{d_0}$	$\frac{(U_\infty k)^{1/3}}{(t/c)^{5/3}} c_{d_0}$
Δc_d	contribution to section drag coefficient due to lift
$(\Delta c_d/\alpha^2)$	$[U_\infty k(t/c)]^{1/3} (\Delta c_d/\alpha^2)$

¹Supersedes NACA TN 2726 entitled "On the Application of Transonic Similarity Rules," by John R. Spreiter, 1952.

c_l	section lift coefficient
(c_l/α)	$[U_o k(t/c)]^{1/3} (c_l/\alpha)$
$\mathcal{D}$	drag function
$\mathcal{D}_o$	drag function for symmetrical nonlifting wings
$\mathcal{D}_\Delta$	drag due to lift function
d_o	section drag function for symmetrical nonlifting airfoils
d_Δ	section drag due to lift function
k	coefficient of nonlinear term of differential equation for φ . (See eqs. (7), (18), (19), (20), (23), and (35).)
$\mathcal{L}$	lift function
l	section lift function
$\mathcal{M}$	pitching-moment function
m	section pitching-moment function
M	local Mach number
M_o	free-stream Mach number
$\mathcal{P}$	pressure function
s	stretching factors defined in equation (B8)
t	maximum thickness of wing
U_o	free-stream velocity
$\bar{U}, \bar{V}$	velocity components parallel and perpendicular, respectively, to the flow direction ahead of a shock
x, y, z	Cartesian coordinates where x extends in the direction of the free-stream velocity
$x_{c, p}$	distance from wing leading edge to center of pressure
(Z/c)	ordinates of wing profiles in fractions of chord
α	angle of attack
$\tilde{\alpha}$	$\frac{\alpha}{(t/c)}$
γ	ratio of specific heats, for air $\gamma=1.4$
λ	arbitrary constant
ξ	$\frac{(M^2-1)}{[U_o k(t/c)]^{2/3}}$
ξ_o	$\frac{(M_o^2-1)}{[U_o k(t/c)]^{2/3}}$
τ	ordinate-amplitude parameter
Φ	velocity potential
φ	perturbation velocity potential

SUBSCRIPTS

l	values given by linear theory
W	conditions at the wing surface
1	conditions immediately upstream from shock
2	conditions immediately downstream from shock

FUNDAMENTAL CONCEPTS

BASIC EQUATIONS

The quasi-linear partial differential equation satisfied by the velocity potential Φ of steady isentropic flow of a perfect inviscid gas can be expressed in the form (ref. 23, p. 25)

$$(a^2 - \Phi_x^2) \Phi_{xx} + (a^2 - \Phi_y^2) \Phi_{yy} + (a^2 - \Phi_z^2) \Phi_{zz} - 2\Phi_x \Phi_y \Phi_{xy} - 2\Phi_y \Phi_z \Phi_{yz} - 2\Phi_z \Phi_x \Phi_{zx} = 0 \quad (1)$$

where the subscript notation is used to indicate differentiation and a is the local speed of sound given by the relation

$$-a^2 = a_o^2 - \frac{\gamma-1}{2} (\Phi_x^2 + \Phi_y^2 + \Phi_z^2 - U_o^2) \quad (2)$$

In this latter equation U_o and a_o are, respectively, the velocity and speed of sound in the free stream and γ is the ratio of specific heats (for air $\gamma=1.4$).

Introducing the perturbation velocity potential φ , where

$$\varphi = -U_o x + \Phi \quad (3)$$

it is possible to express equation (1) in terms of the derivatives of φ as follows:

$$(1 - M_o^2) \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \left\{ \begin{aligned} & \frac{\varphi_{xz}}{a_o^2} \left[(\gamma+1) U_o \varphi_x + \frac{\gamma+1}{2} \varphi_x^2 + \frac{\gamma-1}{2} (\varphi_y^2 + \varphi_z^2) \right] + \\ & \frac{\varphi_{xy}}{a_o^2} \left[(\gamma-1) U_o \varphi_x + \frac{\gamma-1}{2} (\varphi_x^2 + \varphi_z^2) + \frac{\gamma+1}{2} \varphi_y^2 \right] + \\ & \frac{\varphi_{zz}}{a_o^2} \left[(\gamma-1) U_o \varphi_x + \frac{\gamma-1}{2} (\varphi_x^2 + \varphi_y^2) + \frac{\gamma+1}{2} \varphi_z^2 \right] + \\ & 2 \frac{\varphi_{xy}}{a_o^2} \varphi_y (U_o + \varphi_x) + \\ & 2 \frac{\varphi_{xz}}{a_o^2} \varphi_z (U_o + \varphi_x) + \\ & 2 \frac{\varphi_{yz}}{a_o^2} \varphi_y \varphi_z \end{aligned} \right\} \quad (4)$$

If it is assumed that all perturbation velocities and perturbation velocity gradients (represented by first and second derivatives, respectively, of φ) are small and that only the first-order terms in small quantities need be retained, equation (4) simplifies to the well-known Prandtl-Glauert equation of linear theory

$$(1 - M_o^2) \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \quad (5)$$

where the free-stream velocity is directed along the positive x axis as shown in figure 1 and where M_o is the Mach number of the free stream. It is well known that equation (5) leads

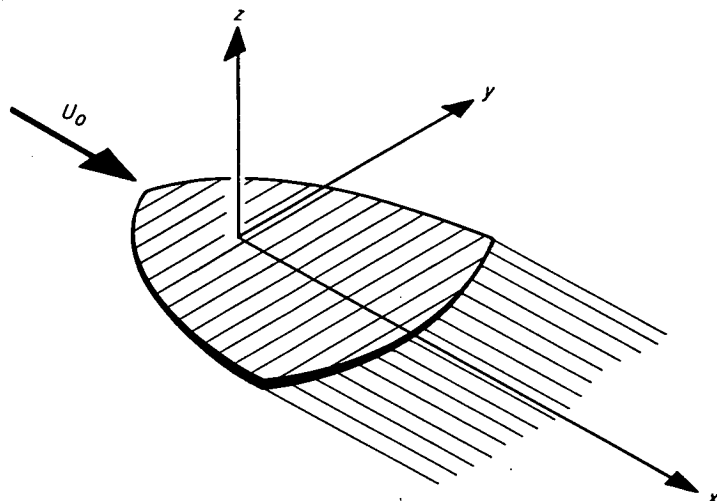


FIGURE 1.—View of wing and coordinate system.

to useful results in the study of subsonic and supersonic flows about thin wings and slender bodies but that it is incapable, in general, of treating transonic flows. The failure of linear theory in the transonic range is evidenced by the calculated value of φ_x growing to such magnitude that it can no longer be regarded as a small quantity when compared with U_o .

Second-order theory for thin wings would involve solution of the equation

$$(1 - M_o^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = M_o^2 \left[\frac{\gamma+1}{U_o} \varphi_x \varphi_{xx} + \frac{\gamma-1}{U_o} \varphi_x (\varphi_{yy} + \varphi_{zz}) + \frac{2}{U_o} (\varphi_y \varphi_{xy} + \varphi_z \varphi_{xz}) \right] \quad (6)$$

Actually, we are interested in retaining higher-order terms only to the extent that is necessary to allow the study of transonic flow. Examination of the known characteristics of transonic flow fields indicates that the first term on the right can often become of importance and should be retained. The remainder of the terms on the right can never become large for transonic flows about thin wings at small angles of attack and can be safely disregarded. The simplified equation is

$$(1 - M_o^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = M_o^2 \frac{\gamma+1}{U_o} \varphi_x \varphi_{xx} = k \varphi_x \varphi_{xx} \quad (7)$$

It follows from the basic assumptions of small-disturbance transonic theory that equation (7) is valid everywhere in the flow field, both fore and aft of any shock waves, but cannot provide any information on the discontinuities in φ that represent the shock waves. This information must be obtained through consideration of the classical equations for the velocities on either side of an oblique shock. Thus if $\bar{U}_1$ represents the velocity immediately in front of the shock wave and $\bar{U}_2$ and $\bar{V}_2$ represent the velocity components parallel and perpendicular, respectively, to $\bar{U}_1$ that occur immediately behind the shock, the equation for the shock polar (ref. 23, p. 108) provides that

$$\bar{V}_2^2 = (\bar{U}_1 - \bar{U}_2)^2 \frac{\bar{U}_1 \bar{U}_2 - a^{*2}}{\frac{2}{\gamma+1} \bar{U}_1^2 - \bar{U}_1 \bar{U}_2 + a^{*2}} \quad (8)$$

where a^* represents the velocity of sound at a point where the local Mach number is unity and is commonly designated the critical speed of sound. It is related to the free-stream velocity and speed of sound by the following equation derived from equation (2):

$$\frac{\gamma+1}{2} a^{*2} = a_o^2 + \frac{\gamma-1}{2} U_o^2 \quad (9)$$

Except for the important case of the bow wave in supersonic flow, $\bar{U}_1$ is not, in general, aligned with the direction of the x axis, but is inclined a small angle. With the resolution into components parallel to the axes of the coordinate system and the retention of only the leading terms, in a manner similar to that used in the derivation of equation (7), equation (8) provides the following relations between the velocity components (potential gradients) immediately fore and aft of the shock:

$$(\varphi_{y_1} - \varphi_{y_2})^2 + (\varphi_{z_1} - \varphi_{z_2})^2 = (\varphi_{x_1} - \varphi_{x_2})^2 \frac{[U_o^2 - a^{*2} + U_o(\varphi_{x_1} + \varphi_{x_2})]}{(a^{*2} - U_o^2 + \frac{2}{\gamma+1} U_o^2)} \quad (10)$$

This result can be put into a more desirable form by making use of the following relation derived directly from equation (9):

$$U_o^2 - a^{*2} = \frac{2}{\gamma+1} \frac{M_o^2 - 1}{M_o^2} U_o^2 \quad (11)$$

Substitution of this expression into equation (10) and rearrangement of the terms yields the desired result

$$(1 - M_o^2)(\varphi_{x_1} - \varphi_{x_2})^2 + (\varphi_{y_1} - \varphi_{y_2})^2 + (\varphi_{z_1} - \varphi_{z_2})^2 = M_o^2 \frac{\gamma+1}{U_o} \left(\frac{\varphi_{x_1} + \varphi_{x_2}}{2} \right) (\varphi_{x_1} - \varphi_{x_2})^2 = k \left(\frac{\varphi_{x_1} + \varphi_{x_2}}{2} \right) (\varphi_{x_1} - \varphi_{x_2})^2 \quad (12)$$

This equation corresponds to the shock-polar curve for weak shock waves inclined at any angle between that of normal shock waves and that of the Mach lines.

In addition to satisfying equations (7) and (12), the perturbation potential must provide flows compatible with the following physical requirements: (a) The flow must be uniform far ahead of the wing, and (b) the flow must be tangential to the wing surface. Therefore, the following boundary conditions are to be specified for the perturbation potential:

at $x = -\infty$

$$(\varphi_x)_o = (\varphi_y)_o = (\varphi_z)_o = 0 \quad (13)$$

at the wing surface W

$$\left(\frac{\varphi_z}{U_o + \varphi_x} \right)_W = \frac{\partial Z}{\partial x} \quad (14)$$

where $\partial Z/\partial x$ refers to the local slope in the x direction of the wing surface. A systematic application of the perturbation analysis indicates that the boundary conditions specified on the wing surface should be simplified by approximating the fraction on the left by φ_z/U_o . Furthermore, it is consistent with the assumption of small disturbances to satisfy this boundary condition on the two sides of the xy plane rather than on the wing surface. Equation (14) is therefore replaced by

$$\left(\frac{\varphi_z}{U_o} \right)_{z=0} = \frac{\partial Z}{\partial x} = \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \quad (15)$$

where the shape of the wing profile is given by

$$Z/c = \tau f(x/c, y/b) \quad (16)$$

where $f(x/c, y/b)$ represents the ordinate-distribution function and τ is an ordinate-amplitude parameter. Note that, in general, a variation of τ represents a simultaneous change of the thickness ratio, camber, and angle of attack. In the special case of a nonlifting wing having symmetrical sections, τ is proportional to the thickness ratio; for lifting flat-plate wings of vanishing thickness, τ is proportional to the angle

of attack. In order to obtain unique and physically important solutions, it is necessary to assume the Kutta condition (that the flow leaves all subsonic trailing edges smoothly).

Upon solving the above boundary-value problem for the potential, one may determine the pressure coefficient by means of the formula

$$C_p = -\frac{2}{U_o} \varphi_x \quad (17)$$

It should be noted that the results obtained by using the foregoing approximate equations might be expected to tend toward those of linear theory as the free-stream Mach number M_o departs far from unity. This follows from the fact that the product $\varphi_x \varphi_{xx}$ becomes small relative to the linear terms under this condition. Solutions of the equations for transonic flow found to date have all possessed this property.

COMPARISON WITH OTHER STATEMENTS OF THE TRANSONIC-FLOW EQUATIONS

As a result of minor variations in the perturbation analysis, recent papers have used at least four different relations for k , the coefficient of the nonlinear term in the simplified equation for the perturbation velocity potential. As indicated in the preceding paragraphs, straightforward development of the theory leads to the relation

$$k = M_o^2 \frac{\gamma + 1}{U_o} \quad (18)$$

This is sometimes simplified (e. g., refs. 22 and 24) to

$$k = \frac{\gamma + 1}{U_o} \quad (19)$$

by arguing that M_o can be set equal to unity in this term without much loss in accuracy, since the right-hand side of equation (7) is merely an approximation to allow the treatment of transonic flows and rapidly diminishes in magnitude as M_o departs from unity. In some treatments (e. g., refs. 16 and 21), equation (1) is divided by a^2 , and the quotient $1/a^2$ in each term is expanded in a binominal series. When this is done, the coefficient k of the term involving $\varphi_x \varphi_{xx}$ is

$$k = M_o^2 \left[\frac{2 + (\gamma - 1) M_o^2}{U_o} \right] \quad (20)$$

Still another expression for k is used by Oswatitsch in references 13 and 14. Two derivations are given, one based on mass-flow considerations (ref. 14) and the other (ref. 13) based on simplifying equation (1), under the assumption of nearly parallel flow, to

$$(1 - M^2) \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \quad (21)$$

and expanding the variable coefficient $1 - M^2$ in the series

$$1 - M^2 = 1 - M_o^2 - \frac{1 - M_o^2}{a^* - U_o} \frac{\partial \varphi}{\partial x} + \dots \quad (22)$$

where M is the local Mach number and a^* is the critical speed of sound as defined in equation (9). Comparison of equations (21) and (22) with equation (7) shows the coefficient k in this approximation is

$$k = \frac{1 - M_o^2}{a^* - U_o} \quad (23)$$

It should be noted that the four alternative relations for k are identical for $M_o = 1$ and all but that given by equation (19) are zero for $M_o = 0$.

A similar situation arises in the derivation of the simplified equation for the shock polar. Here again the precise form of the expression for the coefficient k of equation (12) depends on the details of the perturbation analysis. The most important point from a practical point of view is that the same expression for k is used in both the equation for the potential and that for the shock polar, namely equations (7) and (12). While this point has not always been explicitly stated, it is actually a necessary condition for the existence of the transonic similarity rules.

Although each of the above alternative forms of the perturbation equations has been used at least once in the recent theoretical investigations of transonic flow, the most widely used set of equations are those derived under the more restrictive assumption that all velocities are small perturbations around the critical speed of sound a^* rather than around the free-stream velocity U_o . In the latter scheme, the perturbation potential is defined by (see, e. g., ref. 6 or 20)

$$\varphi' = -a^* x + \Phi \quad (24)$$

and the resulting differential equation for φ' is

$$\varphi'_{yy} + \varphi'_{zz} = \frac{\gamma + 1}{a^*} \varphi'_x \varphi'_{xx} \quad (25)$$

The approximate relation for the shock polar is

$$(\varphi'_{v_1} - \varphi'_{v_2})^2 + (\varphi'_{z_1} - \varphi'_{z_2})^2 = \frac{\gamma + 1}{a^*} \left(\frac{\varphi'_{x_1} + \varphi'_{x_2}}{2} \right) (\varphi'_{x_1} - \varphi'_{x_2})^2 \quad (26)$$

The corresponding boundary conditions are specified as follows:

at $x = -\infty$

$$(\varphi'_x)_o = U_o - a^* \approx -\frac{a^*}{\gamma + 1} (1 - M_o^2), \quad (\varphi'_y)_o = (\varphi'_z)_o = 0$$

at the wing surface

$$(\varphi'_z)_{z=0} \approx a^* \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \quad (27)$$

where the shape of the wing profile is still given by equation (16). The equation for the pressure coefficient is approximated similarly, thus,

$$C_p' \approx -\frac{2}{a^*} [\varphi'_x - (\varphi'_x)_o] \quad (28)$$

This statement of the equations for transonic flow is clearly identical to those given previously, herein, when the free-stream Mach number is unity. Although the derivation of the a^* equations requires that the free-stream Mach number be very close to unity, these equations have been used with good success by a number of authors to calculate the aerodynamic forces on airfoils at Mach numbers considerably

removed from unity. In so doing, it has been suggested that it might be preferable to use more accurate relations for the pressure coefficient or the boundary conditions; for instance, it has been suggested that a^* be replaced with U_o in the equation for C_p . This matter has been discussed at length in references 8, 17, and 18. Since no restriction requiring the Mach number to be near unity is made in the U_o analysis, it is informative to examine the relation between the results of the a^* and the U_o analyses. This is done in Appendix A. It is found that the a^* analysis, if performed in a completely consistent manner using equations (24) through (28), yields values for C_p that are identical to those given by the more general U_o analysis using $k=(\gamma+1)/U_o$. This somewhat paradoxical result is achieved through the action of a number of compensating effects and only applies to the pressures and the forces and moments derivable therefrom. It should be noted, in particular, that the values of the local velocities and Mach numbers provided by the a^* analysis for flows having free-stream Mach numbers other than unity are in error. Throughout the remainder of this report, the discussion will be based on the U_o analysis.

A significant case where the alternative relations for k lead to different results is the prediction of the critical pressure coefficient $C_{p_{cr}}$, defined as the value of the pressure coefficient C_p at a point where the local Mach number is unity. It is important that a reasonably good approximation be maintained for the variation of $C_{p_{cr}}$ with M_o because shock waves make their first appearance, and the airfoil first experiences a pressure drag when C_p becomes more negative than $C_{p_{cr}}$ somewhere on the airfoil surface. In the present approximation, $C_{p_{cr}}$ corresponds to that value of C_p , and, hence, or φ_x , at which equation (7) changes locally from elliptic to hyperbolic type. This condition is recognized by the vanishing of the coefficient of φ_{xx} , thus

$$1 - M_o^2 - k(\varphi_x)_{cr} = 0$$

or, in view of equation (17)

$$C_{p_{cr}} = -\frac{2}{U_o} (\varphi_x)_{cr} = -\frac{2}{k U_o} (1 - M_o^2) \quad (29)$$

The exact relation for isentropic flow is (e. g., ref. 25, p. 28)

$$C_{p_{cr}} = \frac{2}{\gamma M_o^2} \left[\left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} M_o^2 \right)^{\frac{\gamma}{\gamma-1}} - 1 \right] \quad (30)$$

The variation of $C_{p_{cr}}$ with M_o has been computed by use of the exact relation and each of the four approximate relations. The results are presented graphically in figure 2. It may be seen that a reasonably good approximation for $C_{p_{cr}}$ is obtained over a wide Mach number range when k is taken as given in either equation (18) or (23), and that a somewhat greater error is incurred when equation (20) is used. On the other hand, equation (19) leads to a very poor approximation for $C_{p_{cr}}$.

Similar comparisons can be made for local Mach numbers M other than unity by noting that the coefficient $1 - M_o^2 - k\varphi_x$

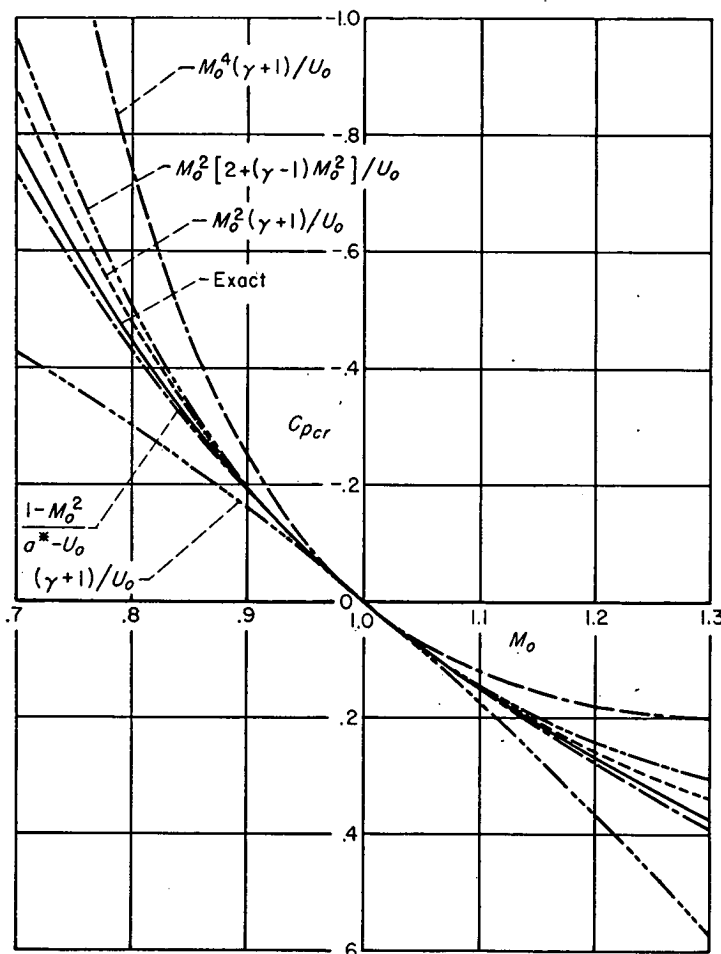


FIGURE 2.—Variation of critical pressure coefficient with Mach number.

of φ_{xx} in equation (7) corresponds, in the present approximation, to $1 - M^2$, thus

$$1 - M^2 = 1 - M_o^2 - k\varphi_x = 1 - M_o^2 + \frac{k U_o}{2} C_p \quad (31)$$

The corresponding exact relation for isentropic flow is

$$C_p = \frac{2}{\gamma M_o^2} \left[-1 + \left(\frac{1 + \frac{\gamma-1}{2} M_o^2}{1 + \frac{\gamma-1}{2} M^2} \right)^{\frac{\gamma}{\gamma-1}} \right] \quad (32)$$

The results so obtained are generally similar to those indicated in figure 2, although the relative accuracy of the better approximations changes somewhat with the situation. All the approximations are exact, of course, when $C_p = 0$; on the other hand, none of the approximations are exact, except for isolated cases, when C_p is different from zero, even though all the approximations agree among themselves when the free-stream Mach number is unity. In order to provide some information regarding the errors that are likely to be incurred when C_p is not very small, figure 3 has been prepared illustrating the variation of local Mach number with pressure coefficient for a free-stream Mach number of unity.

A second case where the exact and approximate relations may be compared is furnished by considering the velocity

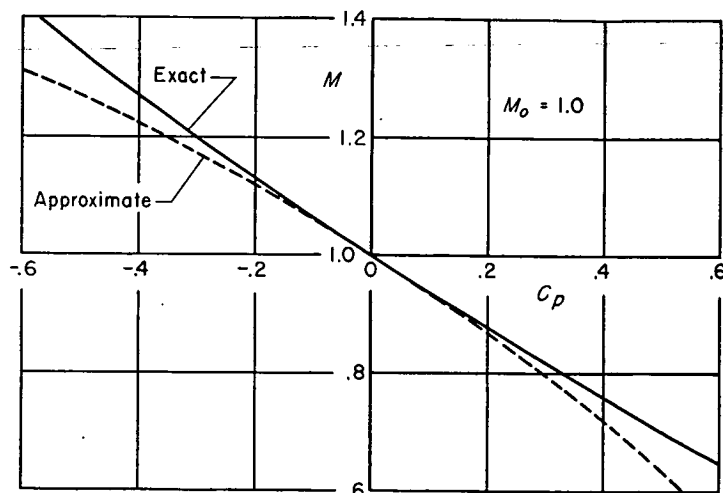


FIGURE 3.—Variation of local Mach number with pressure coefficient for $M_o = 1.0$.

jump through a shock wave. If the flow ahead of the shock wave is uniform and parallel to the x axis, the results may be conveniently represented by the shock-polar diagram in which $\bar{V}_2/a^*$ is plotted as a function of $\bar{U}_2/a^*$. The exact relation is furnished by equation (8). The corresponding approximate relations are determined from equation (12) by setting φ_{x_1} , φ_{y_1} , φ_{z_1} , and φ_{z_2} equal to zero, whereby $U_o = U_1$, $M_o = M_1$, and

$$\varphi_{z_2}^2 = \left[-(1 - M_o^2) + \frac{k}{2} \varphi_{x_2}^2 \right] \varphi_{x_2}^2 \quad (33)$$

Once the variation of φ_{z_2} with φ_{x_2} is determined for a given M_o , the corresponding variation of $\bar{V}_2$ with $\bar{U}_2$ may be readily determined since, for this case,

$$\bar{U}_2 = U_o + \varphi_{x_2}, \quad \bar{V}_2 = \varphi_{z_2} \quad (34)$$

The variation of $\frac{\bar{V}_2}{a^*}$ with $\frac{\bar{U}_2}{a^*}$ for $M_o = 1.2$ has been computed using both the exact and approximate relations, and the results are presented graphically in conventional shock-polar form in figure 4. This figure contains, in addition to curves for the four expressions for k discussed previously, a curve computed using

$$k = M_o^4 \frac{\gamma + 1}{U_o} \quad (35)$$

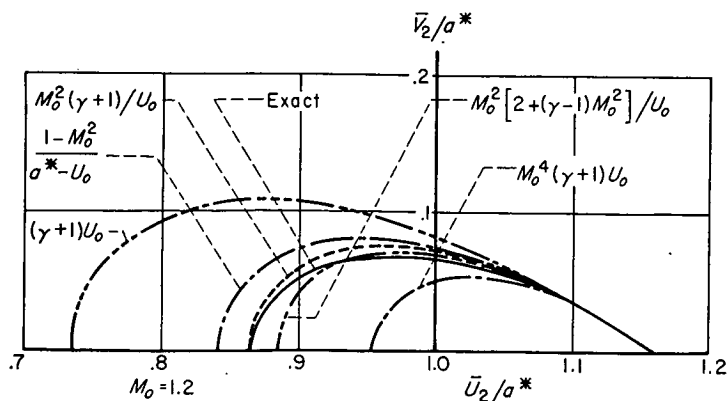


FIGURE 4.—Exact and approximate shock polars.

This expression for k arises in a series expansion of the right-hand member of equation (10). (For sake of completeness, the corresponding variation of $C_{p_{cr}}$ with M_o is also included in figure 2.) Just as with the comparison of the critical pressure coefficient, the expressions for k given by equations (19) and (35) lead to poor approximations, whereas any other of the expressions lead to reasonably good approximations. A notable point is that the expression for k given by equation (18), that is $k = M_o^2(\gamma + 1)/U_o$, leads to the exact relation for velocity jump through a normal shock wave.

In order to facilitate comparison with previous results and to achieve an economy of notation, the present analysis is carried as far as possible without specifying a particular relation for k . That is, the equations of the analysis and reduced parameters with which the results are expressed are written containing k which may be equated to any of the five stated relations, or in fact, to any coefficient that does not depend on x, y, z or φ . The actual values of the pressure coefficient and Mach number for an airfoil of specific thickness ratio, however, depend on which relation is selected for k . Wherever such values are given, they will be those obtained by use of the expression for k given in equation (18), that is,

$$k = M_o^2 \frac{\gamma + 1}{U_o} \quad (18)$$

The principal reason for this choice is that it appears to provide a set of equations, or a mathematical model, which approximates certain essential features of transonic flow with superior accuracy.

RELATION BETWEEN TRANSONIC THEORY AND LINEAR THEORY

It is important to recognize that wing theory based on equation (7) is valid for all Mach numbers below the hypersonic range. At subsonic and supersonic speeds, equation (7) is of the same order of accuracy as the Prandtl-Glauert equation of liner theory (eq. (5)) although more difficult to solve. At $M_o = 1$, equation (7) is identical with equation (25), now widely used in the study of transonic-flow problems. On the other hand, there is no a priori method for determining whether or not a solution of the equations of linear theory will be valid in the transonic range. One can only decide by solving the problem under the assumptions of linear theory and then inspecting the magnitudes of the terms, particularly of φ_x , to see whether or not they can be regarded as small quantities. If the terms are sufficiently small, the linear-theory solution is presumed valid even though the Mach number may be near unity. Linearized-theory solutions have been obtained for a great number of practical wing problems and their behavior in the transonic range is now well known. To review briefly: For unswept wings of infinite span, linear theory indicates that the magnitude of φ_x on the surface of a given airfoil is proportional to $1/\sqrt{1 - M_o^2}$; consequently, φ_x approaches infinity as M_o approaches unity and the theory is clearly inapplicable. For wings of finite span, however, the perturbation velocities may be large or small at sonic velocity, depending on the particular problem as discussed in detail in reference 26. Specifically, for three-dimensional lifting surfaces of zero

thickness, the velocities remain finite everywhere except at the leading edges, their magnitudes generally increasing with increasing aspect ratio and angle of attack. For wings of nonzero thickness, however, φ_x generally becomes large logarithmically as $1-M_o^2$ approaches zero; consequently, linear theory is inapplicable within some Mach number range surrounding unity.

Summarizing, linear theory is applicable to lifting surfaces of small or moderate aspect ratio at all transonic speeds, but fails for wings of finite thickness within a range of Mach numbers surrounding unity. The range of inapplicability diminishes to zero as the aspect ratio, thickness ratio, and angle of attack of the wing tend to zero.

In treating transonic flows for which linear theory is applicable, it is often advantageous to consider the special case of sonic flow ($M_o=1$) separately. Equation (5) for the perturbation potential then reduces to a particularly simple form

$$\varphi_{yy} + \varphi_{zz} = 0$$

Solutions of this equation, in conjunction with the boundary conditions given by equations (13) and (15), are identical to those of linear theory found by solving equation (5) and subsequently setting $M_o=1$, but can be obtained with much less effort. Since, in addition, the results of this simple theory, now generally known as slender-wing theory, are also applicable to low-aspect-ratio lifting surfaces throughout the entire Mach number range, a considerable number of solutions of slender-wing theory have been presented in the last few years (e. g., refs. 27, 28, and 29). These results are, of course, applicable to flows at $M_o=1$ to exactly the same extent as the results of linear theory.

SIMILARITY RULES

In reference 6, von Kármán derived similarity rules for the pressure distribution, lift, drag, and pitching moment of airfoils in transonic flow using equations (24) through (28). The same equations were used in reference 20 to determine the transonic similarity rules for wings of finite span. The corresponding similarity rules of linearized subsonic and supersonic wing theory were also derived and compared with the transonic similarity rules in the latter reference. It was shown that the similarity rules of linear theory contain an arbitrary parameter and can be expressed in many forms, one of which coincides with the similarity rules of transonic flow.

A derivation of the transonic similarity rules, based on the U_o equations with unspecified k , is provided in Appendix B. This derivation possesses the advantage of being based on a single statement of the problem of wing theory that is uniformly valid at subsonic, transonic, and supersonic speeds. It follows from the preceding discussion that the results so found are identical to those of references 6 and 20 if k is equated to $(\gamma+1)/U_o$. The similarity rules for C_p , C_L , C_m , and C_D are given in Appendix B as follows:

$$C_p = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{P} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A; \frac{x}{c}, \frac{y}{b} \right] \quad (36)$$

$$C_L = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{L} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A \right] \quad (37)$$

$$C_m = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{M} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A \right] \quad (38)$$

$$C_D = \frac{\tau^{5/3}}{(U_o k)^{1/3}} \mathcal{D} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A \right] \quad (39)$$

where the geometry of related wings is given by equation (16):

$$(Z/c) = \tau f(x/c, y/b)$$

Equations (36) through (39) are functional equations. For example, equation (36) is to be interpreted as stating that the pressure coefficient C_p is equal to $\tau^{2/3}/(U_o k)^{1/3}$ times some function $\mathcal{P}$ of a number of specified parameters. The foregoing equations have been written for flows where $M_o \leq 1$. If $M_o \geq 1$, the radical $\sqrt{1-M_o^2}$ should be replaced with $\sqrt{M_o^2-1}$. The functions $\mathcal{P}$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{D}$ are different, however, for subsonic and supersonic flow. Consequently, subsonic flows may be related to other subsonic flows by the similarity rules, but not to supersonic flows, and conversely.

ALTERNATIVE FORMS OF THE SIMILARITY RULES

It is important to recognize that the similarity parameters may be combined or regrouped in any manner whatsoever, provided the same number of independent parameters is always retained. For instance, in much of what follows, it will be found desirable to use the square of $\sqrt{1-M_o^2}/(U_o k \tau)^{1/3}$ and to replace $\sqrt{1-M_o^2} A$ with a new parameter $(U_o k \tau)^{1/3} A$ obtained by dividing $\sqrt{1-M_o^2} A$ by $\sqrt{1-M_o^2}/(U_o k \tau)^{1/3}$. In terms of these parameters, the similarity rules are

$$C_p = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{P} \left[\frac{M_o^2-1}{(U_o k \tau)^{2/3}}, (U_o k \tau)^{1/3} A; \frac{x}{c}, \frac{y}{b} \right] \quad (40)$$

$$C_L = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{L} \left[\frac{M_o^2-1}{(U_o k \tau)^{2/3}}, (U_o k \tau)^{1/3} A \right] \quad (41)$$

$$C_m = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{M} \left[\frac{M_o^2-1}{(U_o k \tau)^{2/3}}, (U_o k \tau)^{1/3} A \right] \quad (42)$$

$$C_D = \frac{\tau^{5/3}}{(U_o k)^{1/3}} \mathcal{D} \left[\frac{M_o^2-1}{(U_o k \tau)^{2/3}}, (U_o k \tau)^{1/3} A \right] \quad (43)$$

wherein the geometry of related wings is again given by equation (16).

The similarity rules thus formulated are totally equivalent to those given by equations (36) through (39) but possess three outstanding advantages:

(a) The indeterminacy at $M_o=1$ resulting from two parameters simultaneously vanishing is removed.

(b) The squaring of the first parameter avoids the necessity of changing parameters as sonic speed is passed.

(c) The use of the parameter $(U_\infty k \tau)^{1/3} A$ rather than $\sqrt{1-M_\infty^2} A$ aids in distinguishing the regimes in which linear theory is applicable in the transonic range from those in which nonlinear theory must be used. Thus as $(U_\infty k \tau)^{1/3} A$ approaches zero, linear theory is always applicable, provided, in some cases, that M_∞ is not precisely equal to unity. On the other hand, as $(U_\infty k \tau)^{1/3} A$ becomes large, linear theory is not applicable in the transonic range and nonlinear theory must be used.

The forms of the similarity rules given in equations (36) through (43) have been the source of some confusion, due to the multiple role that τ plays in determining the thickness ratio, camber, and angle of attack. As can be seen from equation (16), all three of these geometrical quantities are linearly proportional to τ . A more explicit statement of the symmetrical profiles of nonzero thickness may be obtained by rewriting the expression for Z/c in terms of the thickness ratio t/c and angle of attack α rather than τ , thus,

$$Z/c = t/c \left[g \left(\frac{x}{c}, \frac{y}{b} \right) - \frac{\alpha}{t/c} \frac{x}{c} \right] \quad (44)$$

Comparison of equations (16) and (44) indicates that t/c plays a similar role to τ but that it is necessary to introduce a second parameter $\alpha/(t/c)$. In this way, a new set of equations expressing the similarity rules is obtained which correspond to equations (40) through (43), although expressed in terms of α and t/c rather than τ . They are

$$\tilde{C}_p = \frac{(U_\infty k)^{1/3}}{(t/c)^{2/3}} C_p = \mathcal{P}(\xi_o, \tilde{A}, \tilde{\alpha}; x/c, y/b) \quad (45)$$

$$\tilde{C}_L = \frac{(U_\infty k)^{1/3}}{(t/c)^{2/3}} C_L = \mathcal{L}(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (46)$$

$$\tilde{C}_m = \frac{(U_\infty k)^{1/3}}{(t/c)^{2/3}} C_m = \mathcal{M}(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (47)$$

$$\tilde{C}_D = \frac{(U_\infty k)^{1/3}}{(t/c)^{5/3}} C_D = \mathcal{D}(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (48)$$

where ξ_o , $\tilde{A}$, and $\tilde{\alpha}$ are similarity parameters defined as follows:

$$\xi_o = \frac{M_\infty^2 - 1}{[(U_\infty k)(t/c)]^{2/3}}, \quad \tilde{A} = [(U_\infty k)(t/c)]^{1/3} A, \quad \tilde{\alpha} = \frac{\alpha}{t/c} \quad (49)$$

SLOPE OF PRESSURE CURVE AT $M_\infty = 1$

Liepmann and Bryson (refs. 17 and 18) have made the following observations which enable the determination by simple and intuitive considerations of the slope of the C_p versus M_∞ curve at $M_\infty = 1$. It is a well-known fact that, at slightly supersonic Mach numbers, the detached bow wave is far away from the airfoil and nearly normal. It is also well known that the Mach number downstream of weak normal shock is as much below unity as the Mach number upstream is above unity. Consequently, the Mach number distribution on the airfoil should be independent of Mach number in the neighborhood of $M_\infty = 1$, that is,

$$\left(\frac{dM}{dM_\infty} \right)_{M_\infty=1} = 0 \quad (50)$$

Now, if the isentropic flow relation for C_p in terms of M and M_∞

$$C_p = \frac{2}{\gamma M_\infty^2} \left[-1 + \left(\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M^2} \right)^{\frac{\gamma}{\gamma-1}} \right] \quad (51)$$

is differentiated with respect to M_∞ , M_∞ is equated to unity, and equation (50) is introduced, the following relation, first given in reference 8 by Vincenti and Wagoner, is found:

$$\left(\frac{dC_p}{dM_\infty} \right)_{M_\infty=1} = \frac{4}{\gamma+1} - \frac{2}{\gamma+1} (C_p)_{M_\infty=1} \quad (52)$$

Since the above-mentioned derivation is based to a certain extent on physical reasoning and makes use of the exact rather than approximate relation between pressure and Mach number, it is of interest from the present point of view to review the equivalent result contained in the model of transonic flow provided by equations (7), (12), (13), (15), and (17). In common with the other characteristics of transonic theory, the result can be expressed conveniently in the form of a similarity rule. Thus, recall the approximate relation for the local Mach number given by equation (31).

$$1 - M^2 = 1 - M_\infty^2 + \frac{k U_\infty}{2} C_p \quad (31)$$

Now, according to the similarity rules, transonic theory does not provide information about C_p or M_∞ alone, but only about parameters such as $\tilde{C}_p$ and ξ_o . Thus, the similarity rule for local Mach number is the following:

$$\xi = \frac{M^2 - 1}{[(U_\infty k)(t/c)]^{2/3}} = \frac{M_\infty^2 - 1}{[(U_\infty k)(t/c)]^{2/3}} - \frac{(U_\infty k)^{1/3}}{2(t/c)^{2/3}} C_p = \xi_o - \frac{1}{2} \tilde{C}_p \quad (53)$$

The solutions of the equations for transonic flow obtained for slightly supersonic flow by Vincenti and Wagoner in reference 8 and for slightly subsonic flow by Cole in reference 11 indicate that the approximate relation which corresponds to the Mach number freeze is the following:

$$\left(\frac{d\xi}{d\xi_o} \right)_{\xi_o=0} = 0 \quad (54)$$

It may be recognized that this relation is equivalent to the exact relation given in equation (50) if k is independent of M_∞ , as in equation (19). This is not the case, however, when the presently preferred relation for k , namely, equation (18), is used. Given equation (54), differentiation of equation (53) shows that the slope of the pressure curve in the reduced parameters is

$$\left(\frac{d\tilde{C}_p}{d\xi_o} \right)_{\xi_o=0} = 2 \quad (55)$$

It is a simple matter to derive the corresponding slope for the C_p versus M_∞ curve at $M_\infty = 1$, provided a specific expres-

sion is selected for k . If k is equated to $M_o^2(\gamma+1)/U_o$, the final result is

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1} = \frac{4}{\gamma+1} - \frac{2}{3} (C_p)_{M_o=1} \quad (56)$$

It is interesting to note that if k is independent of M_o , the slope of the pressure curve is

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1} = \frac{4}{\gamma+1} \quad (57)$$

Thus, although equation (19) leads to the exact relation for the rate of change of the local Mach number, the slope of the pressure curve is considerably less accurate than that provided by equation (18).

The range of problems to which the foregoing results apply is not known at present. Inasmuch as the entire phenomenon appears to be connected with the presence of a detached bow wave which stands normal to the flow, intuitive considerations suggest that the results are probably applicable at least to symmetrical airfoils at zero or infinitesimal angles of attack but perhaps not to airfoils at larger angles of attack or to wings of finite span.

APPLICATIONS

FUNDAMENTAL HYPOTHESES AND PRINCIPLES

The remainder of this report is principally concerned with the deduction of the qualitative, and to some extent quantitative, characteristics of thin wings in transonic flow by means of simple logical considerations based primarily on the similarity rules together with the following hypotheses:

(a) Nonlinear theory based on equation (7) is applicable to all problems.

(b) Linear theory based on equation (5) is valid for all wings at Mach numbers either appreciably below or above unity.

(c) Linear theory is valid at all Mach numbers, except possibly very near unity, for wings of small aspect ratio.

(d) The differential pressures between the upper and lower surfaces of a wing having symmetrical airfoil sections are proportional to the angle of attack for at least a small range of angles about zero.

(e) The slope of $\widetilde{C}_p$ versus ξ_o at $M_o=1$, defined by equation (55), is applicable at least to symmetrical airfoil sections at zero or infinitesimal angles of attack.

The consequences of the foregoing statements will be consistently pursued in the following sections in the discussion of the aerodynamic characteristics of airfoils and complete wings. Throughout, the analysis will be restricted to wings having symmetrical profiles. Whenever specific results are to be used to illustrate the statements, they will nearly always be for symmetrical-wedge or double-wedge profiles and for wings of triangular plan form. This choice is dictated by the availability of theoretical results.

The basic principle in the following analysis is to express the similarity rules in such forms that the lift, pitching-moment, and drag coefficients can be studied for limiting values of the parameters with no chance for ambiguity due to indeterminate forms. In this respect, the statement of the similarity

rules provided by equations (45) through (49) will be found particularly useful.

PRESSURE DRAG OF SYMMETRICAL NONLIFTING WINGS

The similarity rule for the pressure drag coefficient of symmetrical nonlifting wings having profiles given by

$$(Z/c = (t/c) g(x/c, y/b)) \quad (58)$$

is obtained from equations (44) and (48) by setting α , the angle of attack, to zero

$$\widetilde{C}_{D_o} = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}} C_{D_o} = \mathcal{D}_o(\xi_o, \tilde{A}) \quad (59)$$

where

$$\xi_o = \frac{M_o^2 - 1}{[(U_o k)(t/c)]^{2/3}}, \quad \tilde{A} = [U_o k (t/c)]^{1/3} A \quad (49)$$

Therefore, drag results for symmetrical nonlifting wings should be presented by plotting the variation of the generalized drag coefficient $\widetilde{C}_{D_o}$ with ξ_o and $\tilde{A}$.

At Mach numbers sufficiently removed from unity for linear theory to apply, C_{D_o} must be independent of k since k does not appear in either the differential equation or boundary conditions of linear theory. This implies that the parameters ξ_o and $\tilde{A}$ must be arranged in such a manner that k is canceled completely from the above relation for C_{D_o} . The only alternative inside the function $\mathcal{D}_o$ is to form the product $\sqrt{\xi_o} \tilde{A}$. Numerous possibilities exist on the outside of $\mathcal{D}_o$ depending on whether k is canceled by using ξ_o , $\tilde{A}$, or some combination thereof. Although any of these are acceptable, the first will be preferred. In this way the following results are obtained:

$$(C_{D_o})_i = \frac{(t/c)^{5/3}}{(U_o k)^{1/3}} \left\{ \frac{|M_o^2 - 1|}{[(U_o k)(t/c)]^{2/3}} \right\}^{-1/2} \mathcal{D}_{oi} \left[\left\{ \frac{|M_o^2 - 1|}{[(U_o k)(t/c)]^{2/3}} \right\}^{1/2} \times \right. \\ \left. [(U_o k)(t/c)]^{1/3} A \right] = \frac{(t/c)^2}{\sqrt{|M_o^2 - 1|}} \mathcal{D}_{oi}(\sqrt{|M_o^2 - 1|} A) \quad (60)$$

where it should be recalled that $\mathcal{D}_{oi}$ is a different function for subsonic and supersonic flow. Equation (60) is equivalent to the extended Prandtl-Glauert rule. For subsonic flow, D'Almbert's paradox requires that the drag be zero; therefore, for all wings,

$$(C_{D_o})_i = 0, \quad (\widetilde{C}_{D_o})_{i, \xi_o < 0} = 0 \quad (61)$$

For supersonic flow, wave drag exists which depends on $\sqrt{|M_o^2 - 1|} A$ as well as on the plan form and airfoil section. The general functional relation for the drag coefficient of a family of affinely related wings at zero angle of attack, as given by linear theory, is

$$(C_{D_o})_{i, M_o > 1} = \frac{(t/c)^2}{\sqrt{|M_o^2 - 1|}} \mathcal{D}_{oi}(\sqrt{|M_o^2 - 1|} A), \quad (\widetilde{C}_{D_o})_{i, \xi_o > 0} = \xi_o^{-1/2} \mathcal{D}_{oi}(\xi_o^{1/2} \tilde{A}) \quad (62)$$

Wings of infinite aspect ratio.—For wings of infinite span (or airfoils), equation (62), representing the functional

relation of linear theory for the drag coefficient, reduces to the following:

$$(c_{d_o})_{\substack{t \\ M_o > 1}} = \frac{(t/c)^2}{\sqrt{M_o^2 - 1}} \times \text{const.}, \quad (\widetilde{c_{d_o}})_{\substack{t \\ \xi_o > 0}} = \frac{\text{const.}}{\sqrt{\xi_o}} \quad (63)$$

where the value of the constant depends on the shape of the airfoil. Numerous experimental data show variations consistent with equation (63) at Mach numbers greater than about that of shock attachment. At Mach numbers closer to unity, however, the theoretical values provided by this equation are unreliable. It is evident from inspection of the results of linear theory itself that such a failure occurs, since the perturbation velocities assumed to be small in the derivation of the equations are found to become infinitely large as the Mach number approaches unity. It is apparent, therefore, that it is necessary to resort to nonlinear theory for the calculation of the drag of airfoils in the transonic speed range.

A similarity rule for the generalized section drag coefficient of symmetrical nonlifting airfoils which is valid throughout the Mach number range may be obtained from equation (59) by setting $\tilde{A}$ equal to infinity.

$$\widetilde{c_{d_o}} = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}} c_{d_o} = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}} (C_{D_o})_{\tilde{A}=\infty} = \mathcal{D}_o(\xi_o, \infty) = d_o(\xi_o) \quad (64)$$

At a Mach number of unity, the similarity parameter ξ_o vanishes and the function $d_o(\xi_o)$ is a constant

$$(\widetilde{c_{d_o}})_{\xi_o=0} = d_o(0) = \text{const.}, \quad (c_{d_o})_{M_o=1} = \frac{(t/c)^{5/3}}{(U_o k)^{1/3}} \times \text{const.} \quad (65)$$

indicating that the section drag coefficients of affinely related airfoils are proportional to the five-thirds power of their thickness ratios. If hypothesis (e) is accepted, the variation of $\widetilde{c_{d_o}}$ with ξ_o at $M_o=1$ is found to be zero for completed airfoils

$$\left(\frac{d\widetilde{c_{d_o}}}{d\xi_o} \right)_{\xi_o=0} = \oint \left(\frac{dC_p}{d\xi_o} \right)_{\xi_o=0} \left[\frac{d(Z/t)}{d(x/c)} \right] d\frac{x}{c} = 2 \oint \frac{d(Z/t)}{d(x/c)} d\frac{x}{c} = 0 \quad (66)$$

If k is taken to be $M_o^2(\gamma+1)/U_o$, $(dC_p/dM_o)_{M_o=1}$ is given by equation (56) and the slope of the drag curve is

$$\begin{aligned} \left(\frac{dc_{d_o}}{dM_o} \right)_{M_o=1} &= \oint \left(\frac{dC_p}{dM_o} \right)_{M_o=1} \frac{dZ}{dx} d\left(\frac{x}{c} \right) \\ &= \oint \left[\frac{4}{\gamma+1} - \frac{2}{3} (C_p)_{M_o=1} \right] \frac{dZ}{dx} d\left(\frac{x}{c} \right) = -\frac{2}{3} (c_{d_o})_{M_o=1} \end{aligned} \quad (67)$$

The corresponding exact relation can be determined similarly by use of equation (52) for the slope of the pressure curve. It is

$$\begin{aligned} \left(\frac{dc_{d_o}}{dM_o} \right)_{M_o=1} &= \oint \left[\frac{4}{\gamma+1} - \frac{2}{\gamma+1} (C_p)_{M_o=1} \right] \frac{dZ}{dx} d\left(\frac{x}{c} \right) \\ &= -\frac{2}{\gamma+1} (c_{d_o})_{M_o=1} \end{aligned} \quad (68)$$

Since calculations have been made of the drag in transonic flow of simple symmetrical sections at zero angle of attack, it is not necessary to speculate further regarding the variation of $\widetilde{c_{d_o}}$ with ξ_o . At present, complete theoretical information exists for the drag of symmetrical double-wedge airfoils throughout the transonic range. Solutions for this section have been obtained by transforming the nonlinear differential equation of the small disturbance transonic theory into hodograph variables and taking advantage of simplification of the normally difficult problems relating to the boundary conditions by restricting attention to polygonal profiles. The results for flows having subsonic, sonic, and supersonic free-stream velocities have been given, respectively, by Trilling (ref. 12), Guderley and Yoshihara (ref. 7), and Vincenti and Wagoner (ref. 8). The linear-theory solution for pure supersonic flows has been given by Ackeret (ref. 30). All of these results are combined on a single graph in figure 5. It may be seen that the preceding

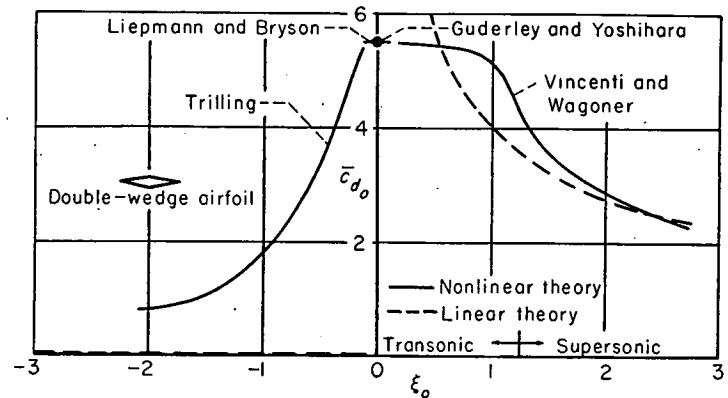


FIGURE 5.—Theoretical drag of double-wedge airfoil.

remarks concerning the relation of the results of linear theory and nonlinear theory, and the slope of the drag curve at a Mach number of unity are illustrated by this comparison.

Osawatitsch has given approximate solutions for $M_o < 1$ for the pressure distribution on symmetrical biconvex airfoils (refs. 13 and 14) and for the pressure distribution and drag on NACA four-digit symmetrical airfoils (ref. 14). This work has recently been extended to several NACA 6-series airfoils by Gullstrand (refs. 15 and 16). Their drag results are generally similar to those indicated for the double-wedge section in figure 5.

In problems such as we are considering herein, the final test is provided by comparison with experimental results. Unfortunately, experimental results for the transonic speed range are scarce, as well as difficult to obtain, and no data are available for direct comparison with the theoretical results summarized in figure 5. At present, however, complete information, both theoretical and experimental, does exist for a single-wedge section followed by a straight section extending far downstream. In accord with some of the original papers on this subject, the single-wedge section is considered as the front half of a symmetrical double-wedge airfoil having a chord c . Solutions for this section obtained using transonic flow theory have been given for flows having subsonic, sonic, and supersonic free-stream velocities, re-

spectively, by Cole (ref. 11), Guderley and Yoshihara (ref. 7), and Vincenti and Wagoner (ref. 8). These results are shown in figure 6 together with the corresponding experimental

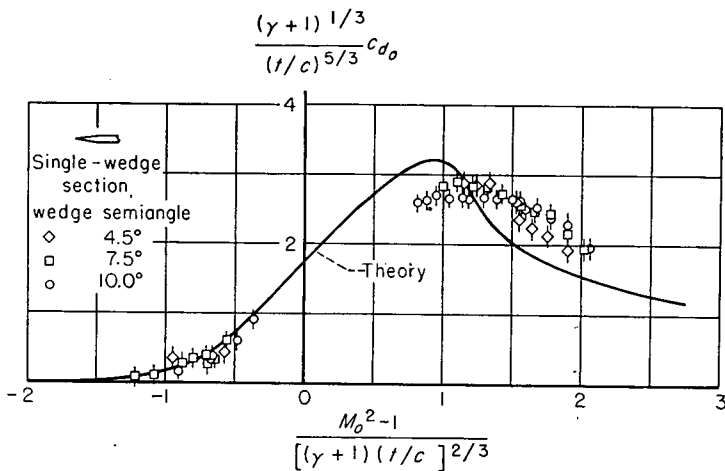


FIGURE 6.—Drag correlation of single-wedge section, $k=(\gamma+1)/U_0$.

results of Liepmann and Bryson (refs. 17 and 18). Both the theoretical curve and the experimental points are plotted in the same manner as presented in the original papers. As remarked in a preceding section, the theoretical results are based on a set of equations equivalent to the present equations with k equated to $(\gamma+1)/U_0$. The small vertical lines on the experimental data points represent the uncertainty of the values. The slope of the drag curve at $M_0=1$ is no longer zero as indicated for the complete airfoil by equation (66) but takes on a positive value given originally by Liepmann and Bryson (ref. 17) and readily derivable from equation (66).

$$\left(\frac{dC_{d0}}{d\xi_0}\right)_{\xi_0=0} = 2 \quad (69)$$

This figure indicates that the theoretical and experimental results are only in general qualitative agreement. Since it is shown in a preceding section that theoretical considerations suggest that a superior theory results if k is equated to $M_0^2(\gamma+1)/U_0$ rather than $(\gamma+1)/U_0$, it is of interest to recalculate and replot the present results accordingly so as to ascertain to what degree these thoughts are borne out by actual experiment. The results are shown in figure 7. It can be seen that the theoretical and experimental results are in nearly perfect agreement. Comparison of figures 6 and 7 provides striking evidence supporting the contention that k should be equated to $M_0^2(\gamma+1)/U_0$ rather than $(\gamma+1)/U_0$, as has been done so often in the past.

Wings of finite aspect ratio.—The similarity rules for wings of finite aspect ratio are given by equations (59) and (60). Although no essential simplification of the rules occurs for wings of small aspect ratio, the range of applicability of linear theory increases as the aspect ratio decreases. This point can be illustrated by considering the results provided by linear theory in a specific case. A good example to select for this purpose is that of the drag in supersonic flow of a triangular wing with symmetrical double-wedge airfoils. (See ref. 31.) This particular choice was made for the follow-

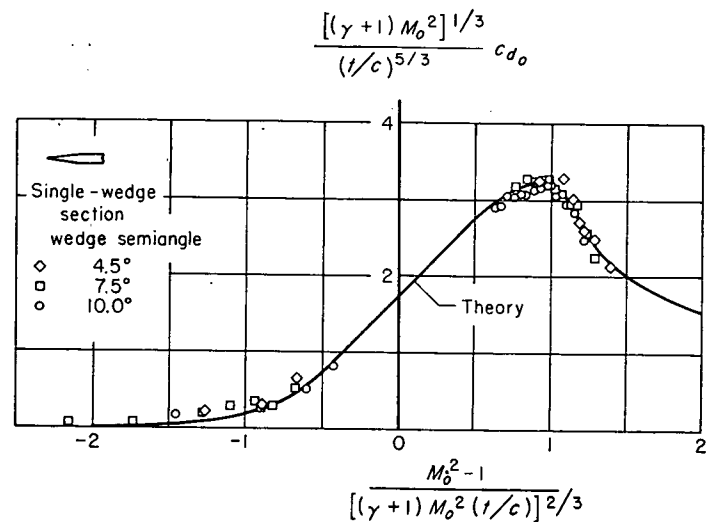


FIGURE 7.—Drag correlation of single-wedge section, $k=M_0^2(\gamma+1)/U_0$.

ing reasons: (a) Solutions are known for all supersonic Mach numbers; (b) the double-wedge airfoil discussed in the preceding sections corresponds to the limiting case of the wing of very great aspect ratio. The drag results provided for this wing by linear theory are presented in figure 8.

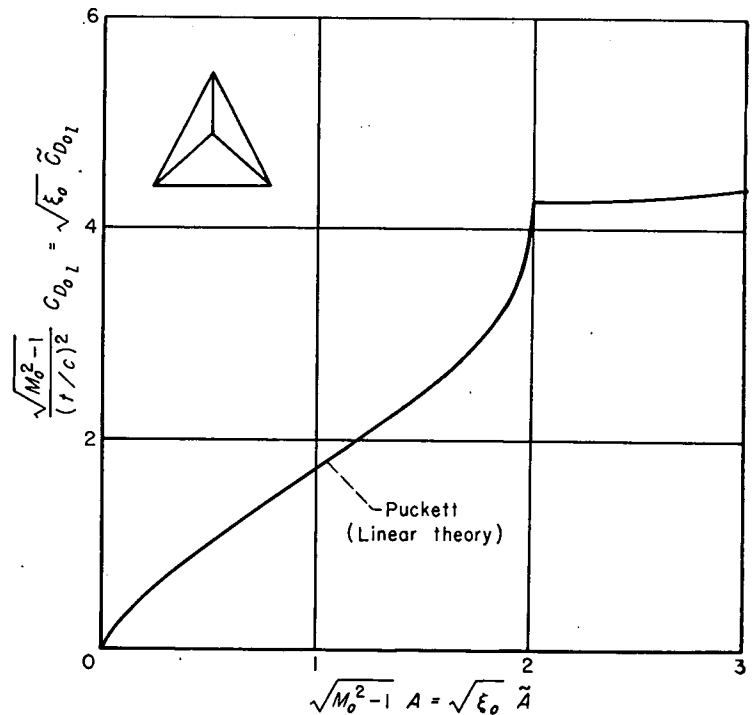
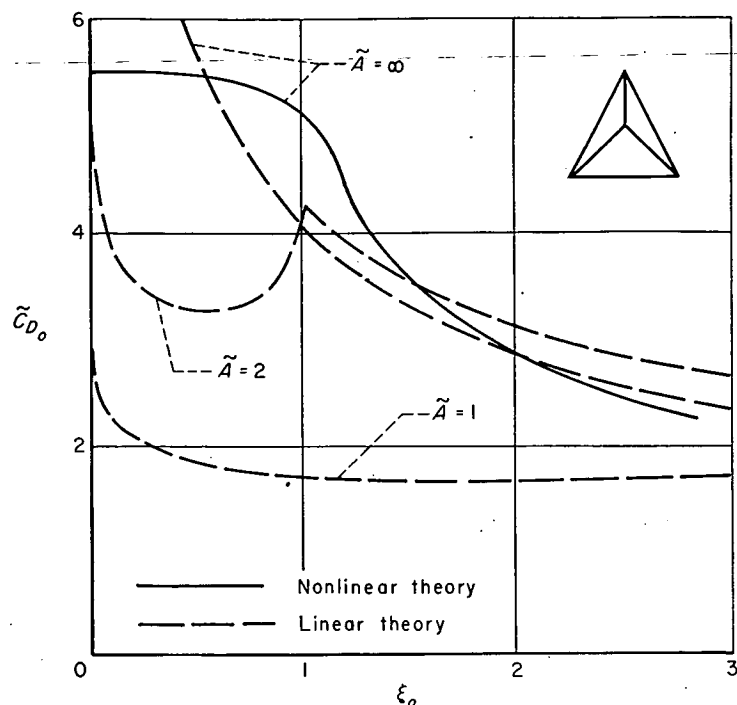


FIGURE 8.—Supersonic drag of triangular wing, linear theory.

The results of figure 8 are presented in a different manner in figure 9 wherein $\tilde{C}_{d0}$ is plotted as a function of ξ_0 for various values of $\tilde{A}$ as suggested by equation (59). For purposes of comparison, the curves for the drag of airfoils ($\tilde{A}=\infty$) computed by both linear and nonlinear theory are also included in the graph. As noted in the preceding section on airfoils, comparison of the results of linear theory with those of nonlinear theory for wings of $\tilde{A}=\infty$ shows that good agreement exists for larger ξ_0 , but that at smaller

FIGURE 9.—Variation with ξ_0 of reduced drag coefficient of triangular wing.

ξ_0 , the values predicted by linear theory become too large. For wings of finite $\tilde{A}$, only the results of linear theory are available. They also exhibit the trend of indicating infinite drag as ξ_0 approaches zero; however, the range of ξ_0 in which the value of $\tilde{C}_{D_0}$ is excessive is much less than is the case for $\tilde{A} = \infty$. In general, $\tilde{C}_{D_0}$ of wings of small aspect ratio diminishes with decreasing aspect ratio.

The drag results of figure 8 are presented in still another form in figure 10 in which is plotted the variation of $\tilde{C}_{D_0}$ with $\tilde{A}$ for various values of ξ_0 . The principal merit of this method of plotting is that it aids in distinguishing the region where nonlinear theory must be used from that where linear theory may be useful. Thus, the two-dimensional nonlinear theory results appear on the right of the graph corresponding to large $\tilde{A}$; whereas the three-dimensional linear theory results appear on the left for small $\tilde{A}$. The filling in of the remainder of the graph requires either the solution of the equations of three-dimensional nonlinear wing theory or the use of experimental data. It should again be noted that the present drag considerations apply only to the pressure drag. Before plotting experimental results in the manner indicated, it is necessary to first subtract the friction drag.

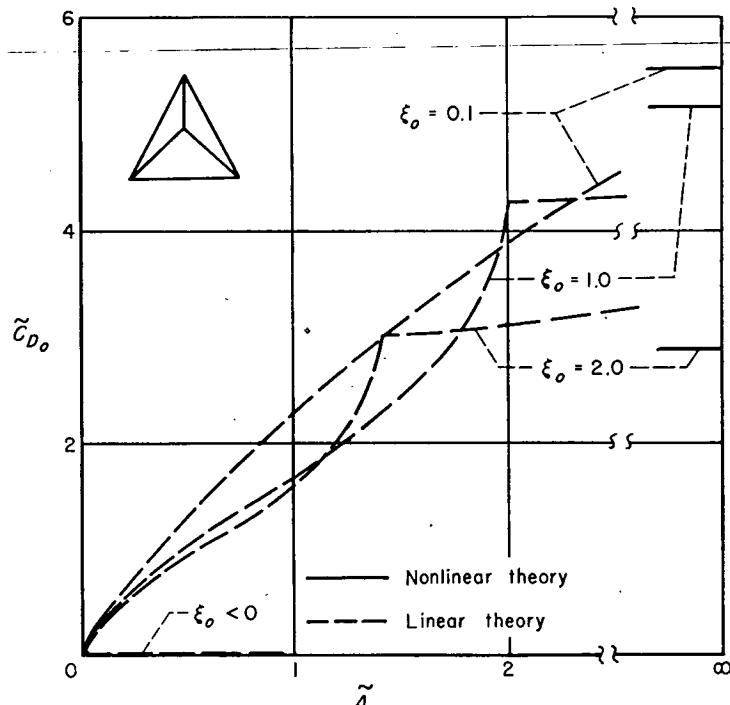
LIFT

The similarity rule for the lift coefficient C_L of a family of wings having ordinates Z given by equation (44)

$$Z/c = t/c \left[g(x/c, y/b) - \frac{\alpha}{t/c} \frac{x}{c} \right] \quad (44)$$

is given in equation (46) as

$$\tilde{C}_L = \frac{(U_0 k)^{1/3}}{(t/c)^{2/3}} C_L = \mathcal{L}(\xi_0, \tilde{A}, \tilde{\alpha}) \quad (46)$$

FIGURE 10.—Variation with $\tilde{A}$ of reduced drag coefficient of triangular wings.

where

$$\xi_0 = \frac{M_0^2 - 1}{[U_0 k(t/c)]^{2/3}}, \quad \tilde{A} = [U_0 k(t/c)]^{1/3} A, \quad \tilde{\alpha} = \frac{\alpha}{t/c} \quad (49)$$

If hypothesis (d) is accepted, C_L varies linearly with α for at least small angles of attack. It is advantageous, therefore, to consider the lift ratio C_L/α rather than C_L alone, thereby minimizing the influence of $\tilde{\alpha}$.

$$\frac{C_L}{\alpha} = \frac{1}{\alpha} \frac{(t/c)^{2/3}}{(U_0 k)^{1/3}} \mathcal{L}(\xi_0, \tilde{A}, \tilde{\alpha}) = \frac{1}{[U_0 k(t/c)]^{1/3}} \mathcal{L}'(\xi_0, \tilde{A}, \tilde{\alpha}) \quad (70)$$

where $\mathcal{L}'(\xi_0, \tilde{A}, \tilde{\alpha})$ is a new function of the indicated variables obtained from $\mathcal{L}(\xi_0, \tilde{A}, \tilde{\alpha})$ by division by $\tilde{\alpha}$. Therefore, lift results may be presented by plotting the variation with ξ_0 , $\tilde{A}$, and $\tilde{\alpha}$ of a generalized lift ratio $\tilde{C}_L/\alpha$ defined as follows (the prime on $\mathcal{L}$ has been omitted for simplicity):

$$\tilde{C}_L/\alpha = [U_0 k(t/c)]^{1/3} (C_L/\alpha) = \mathcal{L}(\xi_0, \tilde{A}, \tilde{\alpha}) \quad (71)$$

Equation (71) shows that $\tilde{C}_L/\alpha$ depends upon three parameters, one more than the number which can readily be treated on a simple plot. Simplification can be gained, of course, by holding one of the parameters constant for an entire graph. Results so presented are particularly interesting for $\xi_0 = 0$, ($M_0 = 1$); $\tilde{A} = \infty$, ($A = \infty$); and $\tilde{\alpha} = 0$, ($\alpha = 0$). The latter scheme is especially good since experiments indicate that lift curves of wings are often relatively straight lines at all Mach numbers. The values of $\tilde{C}_L/\alpha$ at $\tilde{\alpha} = 0$ might, therefore, be expected to be good indications of the actual values for other $\tilde{\alpha}$. The appropriate similarity rule may then be written

$$\left(\frac{\tilde{C}_L}{\alpha} \right)_{\tilde{\alpha}=0} = \mathcal{L}(\xi_0, \tilde{A}, 0) = \mathcal{L}_0(\xi_0, \tilde{A}) \quad (72)$$

For cases where linear theory applies (hypotheses (b) and (c)), two statements can be made immediately which provide further information about C_L/α : (a) $(C_L/\alpha)_i$ must be independent of k , and (b) $(C_L/\alpha)_i$ must be independent of $\tilde{\alpha}$ by virtue of the superposition principle of linear theory. Therefore, following the procedure used in equation (60) gives

$$\left(\frac{C_L}{\alpha}\right)_i = \frac{1}{\sqrt{|M_o^2 - 1|}} \mathcal{L}_i(\sqrt{|M_o^2 - 1|} A), \quad \left(\frac{\widetilde{C_L}}{\alpha}\right)_i = \frac{1}{\sqrt{|\xi_o|}} \mathcal{L}_i(\sqrt{|\xi_o|} \tilde{A}) \quad (73)$$

where again $\mathcal{L}_i$ is a different function for subsonic and supersonic flow.

Wings of infinite aspect ratio.—For wings of infinite aspect ratio, the functional relation of linear theory for the lift ratio given by equation (73) reduces to

$$\left(\frac{c_l}{\alpha}\right)_i = \frac{\text{const.}}{\sqrt{|M_o^2 - 1|}}, \quad \left(\frac{\widetilde{c_l}}{\alpha}\right)_i = \frac{\text{const.}}{\sqrt{|\xi_o|}} \quad (74)$$

Solutions of the equations of linear theory show that the value of the constant is 2π for subsonic flow and four for supersonic flow. Examination of these results indicates that they are valid at Mach numbers appreciably less than or greater than unity, but are invalid for Mach numbers near 1.

A similarity rule for the section lift coefficients of a family of affinely related symmetrical airfoils which is valid throughout the Mach number range may be obtained from equation (71) by setting $\tilde{A} = \infty$.

$$\widetilde{(c_l/\alpha)} = \mathcal{L}(\xi_o, \infty, \tilde{\alpha}) = (\xi_o, \tilde{\alpha}) \quad (75)$$

At a Mach number of unity, ξ_o is identically zero and the expression for the lift ratio becomes

$$\widetilde{(c_l/\alpha)}_{\xi_o=0} = \mathcal{L}(0, \tilde{\alpha}), \quad (c_l/\alpha)_{M_o=1} = \frac{1}{[U_o k(t/c)]^{1/3}} \mathcal{L}\left(0, \frac{\alpha}{t/c}\right) \quad (76)$$

Equation (76), when considered together with hypothesis (d), indicates that at sonic speed, the lift-curve slope at zero angle of attack of airfoils of a single family varies inversely as the cube root of the thickness ratio, thus,

$$(c_l/\alpha)_{M_o=1, \alpha=0} = \frac{1}{[U_o k(t/c)]^{1/3}} \mathcal{L}(0, \tilde{\alpha} \rightarrow 0) = \frac{\text{const.}}{[U_o k(t/c)]^{1/3}} \quad (77)$$

Note that as the thickness ratio goes to zero, the value of the lift-curve slope at zero angle of attack becomes infinite, just as is indicated by equation (74) to be the case according to linear theory. If, on the other hand, t/c diminishes to zero while α is fixed so that $\tilde{\alpha}$ becomes very large, it is plausible that the thickness ratio does not have any effect on c_l . We thus have the following:

$$c_{l_{M_o=1}} = \frac{\alpha}{[U_o k(t/c)]^{1/3}} \mathcal{L}\left(0, \frac{\alpha}{t/c} \rightarrow \infty\right) = \frac{\alpha^{2/3}}{(U_o k)^{1/3}} \times \text{const.} \quad (78)$$

In this case, c_l is proportional to the two-thirds power of α . If hypothesis (e) is acceptable, the variation of c_l/α with ξ_o at $M_o=1$ is zero, since

$$\begin{aligned} \left[\frac{d}{d\xi_o} \left(\frac{\widetilde{c_l}}{\alpha}\right)_{\alpha \rightarrow 0}\right]_{\xi_o=0} &= \oint \left[\frac{d}{d\xi_o} \left(\frac{\widetilde{C_p}}{\alpha}\right)_{\alpha \rightarrow 0}\right]_{\xi_o=0} d\left(\frac{x}{c}\right) \\ &= \oint \left(\frac{2}{\tilde{\alpha}}\right)_{\tilde{\alpha} \rightarrow 0} d\left(\frac{x}{c}\right) = 0 \end{aligned} \quad (79)$$

If k is taken to be $M_o^2(\gamma+1)/U_o$, $(dC_p/dM_o)_{M_o=1}$ is given by equation (56) and the variation of c_l/α with M_o at $M_o=1$ is

$$\begin{aligned} \left[\frac{d}{dM_o} \left(\frac{c_l}{\alpha}\right)_{\alpha \rightarrow 0}\right]_{M_o=1} &= \oint \left[\frac{d(C_p/\alpha)}{dM_o}\right]_{\alpha \rightarrow 0, M_o=1} d\left(\frac{x}{c}\right) = \oint \left[\frac{4}{\alpha(\gamma+1)} - \frac{2}{3} \left(\frac{C_p}{\alpha}\right)_{\alpha \rightarrow 0, M_o=1}\right] d\left(\frac{x}{c}\right) \\ &= -\frac{2}{3} \left(\frac{c_l}{\alpha}\right)_{\alpha \rightarrow 0, M_o=1} \end{aligned} \quad (80)$$

The corresponding exact relation can be determined similarly when equation (52) is used for the slope of the pressure curve. It is

$$\left[\frac{d}{dM_o} \left(\frac{c_l}{\alpha}\right)_{\alpha \rightarrow 0}\right]_{M_o=1} = -\frac{2}{\gamma+1} \left(\frac{c_l}{\alpha}\right)_{\alpha \rightarrow 0, M_o=1} \quad (81)$$

At present, calculations have been made of the lift in sonic flow (Guderley and Yoshihara, ref. 32) and in slightly supersonic flow (Vincenti and Wagoner, ref. 10) of symmetrical double-wedge airfoils inclined an infinitesimal angle of attack. The corresponding solutions for airfoils in slightly subsonic flow have not yet been found. The results of the above-mentioned investigations are shown in figure 11 together with

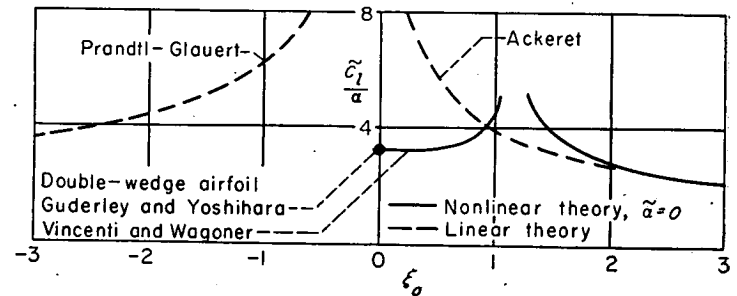


FIGURE 11.—Theoretical lift-curve slope of double-wedge airfoils.

the corresponding values given by linear theory. As in the drag case discussed in the preceding section, it may be seen that the remarks concerning the relation between the results of linear theory and nonlinear theory and the slope of the curve of $\widetilde{c_l/\alpha}$ versus ξ_o at $\xi_o=0$ are verified for this particular airfoil.

Wings of vanishing aspect ratio.—Two well-known results of linear theory are that the lift-curve slopes of wings of finite aspect ratio remain finite throughout the entire Mach number range and that the lift-curve slopes of wings of vanishing aspect ratio are independent of Mach number. Therefore, equation (73) implies that $(\widetilde{C_L/\alpha})_i$ must be proportional to $\tilde{A}$ either for wings of vanishing $\tilde{A}$ in any flow, or for any wing in a flow of vanishing ξ_o :

$$\widetilde{(C_L/\alpha)}_{\tilde{A} \rightarrow 0} = (\widetilde{C_L/\alpha})_{\xi_o=0} = \tilde{A} \times \text{const.}, \quad (C_L/\alpha)_{A \rightarrow 0} = (C_L/\alpha)_{M_o=1} = A \times \text{const.} \quad (82)$$

The value of the constant must be determined for each plan

form by actually solving the equations of linear theory. For wings having trailing edges which possess no cutouts extending forward of the most forward station of maximum span, that is, triangular, rectangular, elliptical, etc., as well as certain sweptback wings, the value of the constant is $\pi/2$. (Refs. 26 through 29 should be consulted for further discussion of this point as well as for the values of the constant for wings having cutouts in the trailing edge which violate the above-stated condition.)

It is seen from equation (82) that the lift-curve slopes of wings in sonic flow decrease continuously in magnitude as the aspect ratio diminishes toward zero. Since, in addition, the lift-curve slope given by linear theory has its maximum value at $M_o=1$, it is conjectured that the lift results of linear theory are a good approximation to those of nonlinear theory not only for all wings at Mach numbers far from unity but also for all Mach numbers for wings of sufficiently small aspect ratio.

Wings of finite aspect ratio.—At the present time no solutions of the nonlinear theory are available for wings of finite aspect ratio. However, from the remarks of the preceding paragraphs, it is apparent that a curve representing the variation of $\widetilde{C_L}/\alpha$ with $\tilde{A}$ for constant ξ_o and $\tilde{\alpha}$ would have the following asymptotic properties: $\widetilde{C_L}/\alpha$ would increase linearly with $\tilde{A}$ for small $\tilde{A}$ and be independent of $\tilde{A}$ for large $\tilde{A}$. In order to give a better idea of the numerical values to be expected, a set of typical results of this type is shown in figure 12 for wings having triangular plan forms and sym-

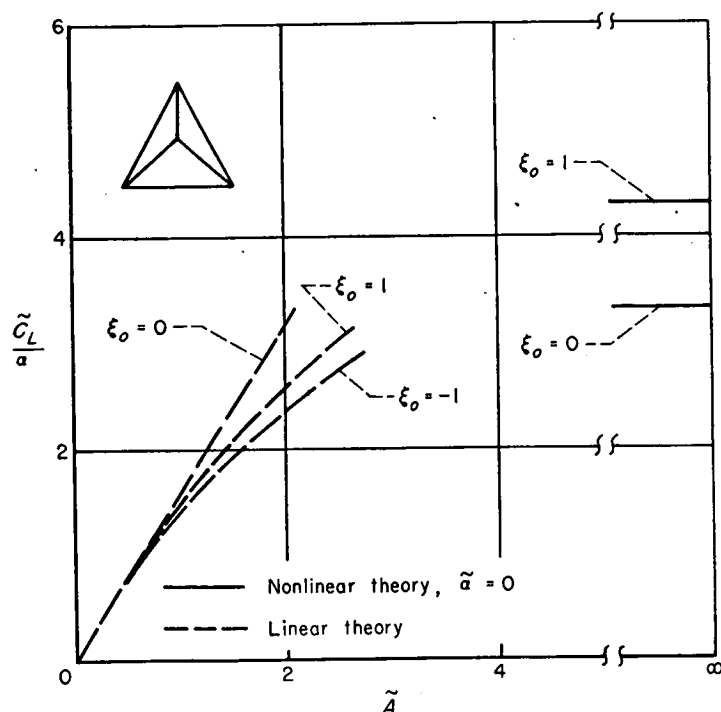


FIGURE 12.—Variation with $\tilde{A}$ of reduced lift-curve slope of triangular wings.

metrical double-wedge airfoil sections. The supersonic results are those of Stewart, Brown (refs. 33 and 34), and others. The subsonic results are those calculated by De Young and Harper (ref. 35) using Weissinger's modified lifting-line theory.

An interesting result of the foregoing remarks concerns the influence of thickness ratio on the lift-curve slope at zero angle of attack of wings in sonic flow. For wings of large aspect ratio, the lift-curve slope is inversely proportional to the cube root of the thickness ratio. For wings of small aspect ratio, the lift-curve slope is independent of the thickness ratio.

PITCHING MOMENT

The remarks of the pitching-moment characteristics of wings follow in a manner exactly analogous to those just stated for the lift characteristics. The corresponding statements for the pitching-moment coefficient C_m may be obtained by simply replacing C_L with C_m and $\mathcal{L}$ with $\mathcal{M}$. Thus, the similarity rules for C_m corresponding to equations (71) and (73) are the following, respectively:

$$(\widetilde{C_m}/\alpha) = [U_o k(t/c)]^{1/3} (C_m/\alpha) = \mathcal{M}(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (83)$$

$$\left(\frac{C_m}{\alpha}\right)_i = \frac{1}{\sqrt{|M_o^2 - 1|}} \mathcal{M}_i(\sqrt{|M_o^2 - 1|} A),$$

$$\left(\frac{\widetilde{C_m}}{\alpha}\right)_i = \frac{1}{\sqrt{|\xi_o|}} \mathcal{M}_i(\sqrt{|\xi_o|} \tilde{A}) \quad (84)$$

where once more $\mathcal{M}$ and $\mathcal{M}_i$ are different functions for subsonic and supersonic flow. The only difference between the discussion of C_m and C_L is that the values of the constants of equations (74), (77), (78), and (82) are, of course, different. Graphs of theoretical pitching-moment characteristics for airfoils and for triangular wings corresponding to the lift results of figures 11 and 12 are shown in figures 13 and 14.

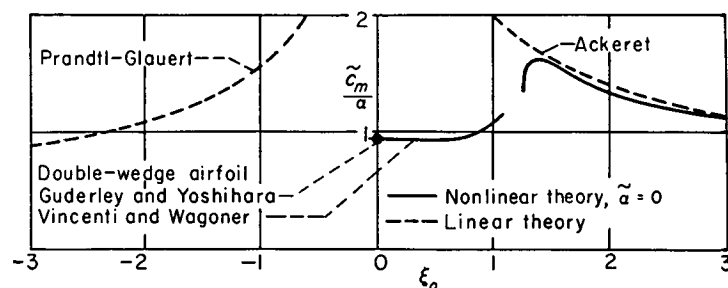


FIGURE 13.—Theoretical pitching-moment curve slope of double-wedge airfoils.

The moment axis is taken to be through the most forward point of the wing.

Sometimes it is desired to present pitching-moment characteristics of wings in terms of center-of-pressure position rather than pitching-moment coefficient. Since the center-of-pressure position can be expressed in terms of C_m and C_L by

$$\frac{x_{c.p.}}{c} = -\frac{C_m}{C_L} \quad (85)$$

the resulting expression for the center-of-pressure position found through application of equations (71) and (83) is

$$\frac{x_{c.p.}}{c} = -\frac{\mathcal{M}(\xi_o, \tilde{A}, \tilde{\alpha})}{\mathcal{L}(\xi_o, \tilde{A}, \tilde{\alpha})} = \mathcal{C.P.}(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (86)$$

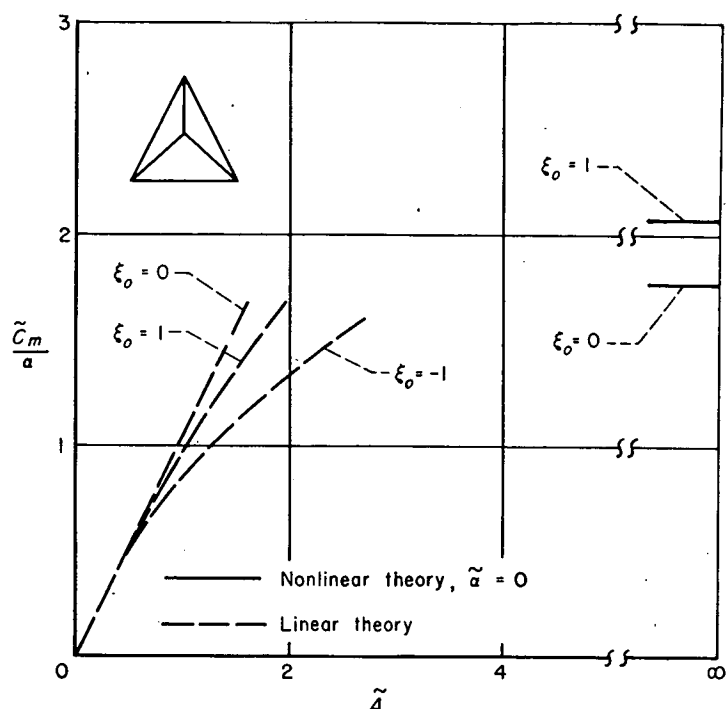


FIGURE 14.—Variation with $\tilde{A}$ of reduced pitching-moment curve slope of triangular wings.

The corresponding relation for linear theory is

$$\left(\frac{x_{c.p.}}{c}\right)_i = C.P._i(\sqrt{|M_o^2 - 1|}A) = C.P._i(\sqrt{|\xi_o|}\tilde{A}) \quad (87)$$

PRESSURE DRAG DUE TO LIFT

The similarity rule for the pressure drag of inclined wings having symmetrical airfoils is indicated by equation (48) to be the following:

$$\tilde{C}_D = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}} C_D = \mathcal{D}(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (88)$$

The portion of the drag due to lift is therefore

$$\begin{aligned} \Delta C_D &= C_D - C_{D_o} = \frac{(t/c)^{5/3}}{(U_o k)^{1/3}} [\mathcal{D}(\xi_o, \tilde{A}, \tilde{\alpha}) - \mathcal{D}(\xi_o, \tilde{A}, 0)] \\ &= \frac{(t/c)^{5/3}}{(U_o k)^{1/3}} \mathcal{D}_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) \end{aligned} \quad (89)$$

Since the differential pressures between upper and lower surfaces of wings having symmetrical airfoil sections are proportional to α for at least a small range of α surrounding zero (hypothesis (d)), it follows that ΔC_D is proportional to the square of α . It is advantageous, therefore, to consider the drag-rise ratio $\Delta C_D/\alpha^2$ rather than ΔC_D alone, thus,

$$\frac{\Delta C_D}{\alpha^2} = \frac{1}{\alpha^2} \frac{(t/c)^{5/3}}{(U_o k)^{1/3}} \mathcal{D}_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) = \frac{1}{[U_o k(t/c)]^{1/3}} \mathcal{D}_\Delta'(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (90)$$

where $\mathcal{D}_\Delta'$ is a new function of the indicated variables obtained from $\mathcal{D}_\Delta$ by dividing by $\tilde{\alpha}^2$. Consequently, drag-due-to-lift data should be presented by plotting the variation with ξ_o , $\tilde{A}$, and $\tilde{\alpha}$ of a generalized drag-rise ratio $\widetilde{\Delta C_D/\alpha^2}$ defined as follows (the prime on $\mathcal{D}_\Delta$ is omitted for simplicity):

$$\widetilde{\Delta C_D/\alpha^2} = [U_o k(t/c)]^{1/3} (\Delta C_D/\alpha^2) = \mathcal{D}_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) \quad (91)$$

The actual presentation of the results of this three-parameter system may be accomplished as described in the section on the lift of wings. Of particular interest is the simplification resulting from presenting only the values found at $\alpha=0$. The simplified similarity rule is then

$$\left(\widetilde{\Delta C_D/\alpha^2}\right)_{\tilde{\alpha}=0} = \mathcal{D}_\Delta(\xi_o, \tilde{A}, 0) \quad (92)$$

For cases where linear theory applies, the following results hold:

$$\begin{aligned} \left(\frac{\Delta C_D}{\alpha^2}\right)_i &= \frac{1}{\sqrt{|M_o^2 - 1|}} \mathcal{D}_{\Delta i}(\sqrt{|M_o^2 - 1|}A), \\ \left(\frac{\widetilde{\Delta C_D}}{\alpha^2}\right)_i &= \frac{1}{\sqrt{|\xi_o|}} \mathcal{D}_{\Delta i}(\sqrt{|\xi_o|}\tilde{A}) \end{aligned} \quad (93)$$

Wings of infinite aspect ratio.—For wings of infinite aspect ratio, the functional relation of linear theory for the drag due to lift, equation (93), reduces to

$$\left(\frac{\Delta c_d}{\alpha^2}\right)_i = \frac{\text{const.}}{\sqrt{|M_o^2 - 1|}}, \quad \left(\frac{\widetilde{\Delta c_d}}{\alpha^2}\right)_i = \frac{\text{const.}}{\sqrt{|\xi_o|}} \quad (94)$$

Solutions of the equations of linear theory show that the value of the constant is zero for subsonic flow and four for supersonic flow about any symmetrical airfoil. These results are valid at Mach numbers appreciably less than or greater than unity but are invalid for Mach numbers near 1.

A similarity rule for the drag due to lift of a family of affinely related symmetrical airfoils that is valid throughout the Mach number range may be obtained from equation (91) by setting $\tilde{A} = \infty$.

$$\left(\frac{\widetilde{\Delta c_d}}{\alpha^2}\right) = \mathcal{D}_\Delta(\xi, \infty, \tilde{\alpha}) = d_\Delta(\xi_o, \tilde{\alpha}) \quad (95)$$

At a Mach number of unity, equation (95), for the drag due to lift, reduces to the following:

$$\left(\frac{\widetilde{\Delta c_d}}{\alpha^2}\right)_{\xi_o=0} = d_\Delta(0, \tilde{\alpha}), \quad \left(\frac{\Delta c_d}{\alpha^2}\right)_{M_o=1} = \frac{1}{[U_o k(t/c)]^{1/3}} d_\Delta\left(0, \frac{\alpha}{t/c}\right) \quad (96)$$

Equation (96), together with hypothesis (d), indicates that, at sonic speed, the drag-rise ratio $\Delta c_d/\alpha^2$ of airfoils of a single family varies inversely as the cube root of the thickness ratio. For very large values of $\tilde{\alpha}$, the thickness ratio cannot have any effect on Δc_d ; therefore, Δc_d is proportional to the five-thirds power of the angle of attack. If hypothesis (e) is accepted, the variation of $\widetilde{\Delta c_d/\alpha^2}$ with ξ_o at $M_o=1$ is zero, since

$$\begin{aligned} \left[\frac{d}{d\xi_o} \left(\frac{\widetilde{\Delta c_d}}{\alpha^2}\right)_{\tilde{\alpha}=0}\right] &= \oint \frac{dZ}{dx} \left[\frac{d}{d\xi_o} \left(\frac{\widetilde{C_p}}{\tilde{\alpha}^2}\right)_{\tilde{\alpha}=0}\right] d\left(\frac{x}{c}\right) \\ &= \oint \frac{dZ}{dx} \left(\frac{2}{\tilde{\alpha}^3}\right)_{\tilde{\alpha}=0} d\left(\frac{x}{c}\right) = 0 \end{aligned} \quad (97)$$

If k is taken to be $M_o^2(\gamma+1)/U_o$, $(dC_p/dM_o)_{M_o=1}$ is given by equation (56), and the variation of $\Delta c_d/\alpha^2$ with M_o at $M_o=1$ is

$$\left[\frac{d}{dM_o} \left(\frac{\Delta c_d}{\alpha^2} \right) \right]_{\alpha \rightarrow 0} \Big|_{M_o=1} = \oint \frac{1}{\alpha} \frac{dZ}{dx} \left[\frac{d(C_p/\alpha)}{dM_o} \right]_{\alpha \rightarrow 0} d\left(\frac{x}{c}\right) \\ = \oint \frac{1}{\alpha} \frac{dZ}{dx} \left[\frac{4}{\alpha(\gamma+1)} - \frac{2}{3} \left(\frac{C_p}{\alpha} \right) \right]_{\alpha \rightarrow 0} d\left(\frac{x}{c}\right) = -\frac{2}{3} \left(\frac{\Delta c_d}{\alpha^2} \right)_{\alpha \rightarrow 0} \quad (98)$$

The corresponding exact relation can be determined similarly by use of equation (52) for the slope of the pressure curve. It is

$$\left[\frac{d}{dM_o} \left(\frac{\Delta c_d}{\alpha^2} \right) \right]_{\alpha \rightarrow 0} \Big|_{M_o=1} = -\frac{2}{\gamma+1} \left(\frac{\Delta c_d}{\alpha^2} \right)_{\alpha \rightarrow 0} \Big|_{M_o=1} \quad (99)$$

Wings of vanishing aspect ratio.—Since, as indicated in the preceding sections, the lift calculated by means of linear theory remains finite throughout the Mach number range, and diminishes to zero as the aspect ratio approaches zero, it follows that the drag due to lift must behave similarly. It is consequently reasonable to presume that linear theory is capable of describing the drag-due-to-lift characteristics of wings of vanishing aspect ratio at all Mach numbers. Therefore, the following relations stemming from equation (93) hold:

$$\left(\frac{\Delta C_D}{\alpha^2} \right)_{\tilde{A} \rightarrow 0} = \left(\frac{\Delta C_D}{\alpha^2} \right)_{\xi_o=0} = \tilde{A} \times \text{const.}, \\ \left(\frac{\Delta C_D}{\alpha^2} \right)_{A \rightarrow 0} = \left(\frac{\Delta C_D}{\alpha^2} \right)_{M_o=1} = A \times \text{const.} \quad (100)$$

Solutions of the equations of linear theory show that the value of the constant is $\pi/4$ for all wings of small $\sqrt{|M_o^2-1|}A$

whose trailing edges possess no cutouts extending forward of the most forward station of maximum span (i. e., triangular, rectangular, elliptical wings, etc.). The quoted value of the constant corresponds to the development of the full "leading-edge force." It is known that this force is oftentimes not completely realized, due to a local separation and subsequent reattachment of the flow around the leading edge. If the leading-edge force is nonexistent, the corresponding value for the constant is $\pi/2$.

Wings of finite aspect ratio.—At the present time no drag-due-to-lift results have been obtained from the nonlinear theory for wings of either finite or infinite aspect ratio. The foregoing remarks, however, are sufficient to determine that a curve representing the variation of $\Delta C_D/\alpha^2$ with $\tilde{A}$ for constant ξ_o and $\tilde{\alpha}$ would increase linearly with $\tilde{A}$ for small $\tilde{A}$ (unless the degree of attainment of the leading-edge force also depends on $\tilde{A}$) and become independent of $\tilde{A}$ for large $\tilde{A}$. The resulting curve would presumably have the same general appearance as that shown in figure 12 for $\tilde{C}_L/\alpha$.

It may sometimes be desired to present drag-due-to-lift results in terms of $\Delta C_D/C_L^2$ or $\Delta C_D/\alpha C_L$ rather than $\Delta C_D/\alpha^2$. The similarity rules for these quantities can be quickly deduced from the foregoing results.

AMES AERONAUTICAL LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

MOFFETT FIELD, CALIF., Dec. 16, 1952

APPENDIX A

RELATION BETWEEN U_o AND a^* STATEMENTS OF THE TRANSONIC-FLOW EQUATIONS

Equations (3), (7), (12), (13), (15), and (17) were presented in the text as being applicable to the study of transonic, as well as supersonic, flow about thin wings. These equations repeated below as equations (A1) through (A6), will be referred to as the U_o statement of the problem since the perturbation velocities are taken about the free-stream velocity U_o . The perturbation potential φ is defined by

$$\varphi = -U_o x + \Phi \quad (\text{A1})$$

The differential equation is

$$(1 - M_o^2) \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = k \varphi_x \varphi_{xx} \quad (\text{A2})$$

The shock relation is

$$(1 - M_o^2) (\varphi_{x_1} - \varphi_{x_2})^2 + (\varphi_{y_1} - \varphi_{y_2})^2 + (\varphi_{z_1} - \varphi_{z_2})^2 = k \left(\frac{\varphi_{x_1} + \varphi_{x_2}}{2} \right) (\varphi_{x_1} - \varphi_{x_2})^2 \quad (\text{A3})$$

The boundary conditions are

$$\text{at } x = -\infty \quad (\varphi_x)_o = (\varphi_y)_o = (\varphi_z)_o = 0 \quad (\text{A4})$$

at the wing surface

$$(\varphi_z)_{z=0} = U_o \left(\frac{\partial Z}{\partial x} \right) \quad (\text{A5})$$

The pressure coefficient is given by

$$C_p = -\frac{2}{U_o} \varphi_x \quad (\text{A6})$$

In the usual a^* statement of the transonic-flow equations (eqs. (24) through (28) in the text), it is assumed that all velocities are only slightly different from the critical speed of sound a^* . The perturbation potential is defined by

$$\varphi' = -a^* x + \Phi \quad (\text{A7})$$

The differential equation is

$$\varphi'_{yy} + \varphi'_{zz} = \frac{\gamma+1}{a^*} \varphi'_x \varphi'_{xx} \quad (\text{A8})$$

The shock relation is

$$(\varphi'_{y_1} - \varphi'_{y_2})^2 + (\varphi'_{z_1} - \varphi'_{z_2})^2 = \frac{\gamma+1}{a^*} \left(\frac{\varphi'_{x_1} + \varphi'_{x_2}}{2} \right) (\varphi'_{x_1} - \varphi'_{x_2})^2 \quad (\text{A9})$$

If the perturbation analysis is carried out in a completely consistent manner, the boundary conditions are:

at $x = -\infty$

$$(\varphi'_x)_o = (M_o^2 - 1) \frac{a^*}{\gamma+1}, \quad (\varphi'_y)_o = (\varphi'_z)_o = 0 \quad (\text{A10})$$

at the wing surface

$$(\varphi'_z)_{z=0} = a^* \left(\frac{\partial Z}{\partial x} \right) \quad (\text{A11})$$

and the pressure coefficient is given by

$$C_p = -\frac{2}{a^*} [\varphi'_x - (\varphi'_x)_o] \quad (\text{A12})$$

The relation between the U_o and a^* statements can be determined directly in the following manner. The differential equations and shock relations for φ' and φ will be the same if

$$\frac{\gamma+1}{a^*} \frac{\partial \varphi'}{\partial x} = k \frac{\partial \varphi}{\partial x} - (1 - M_o^2)$$

or if

$$\varphi' = k \frac{a^*}{\gamma+1} \varphi - \frac{a^* (1 - M_o^2)}{\gamma+1} x \quad (\text{A13})$$

The boundary conditions for φ' corresponding to those stated for φ in equations (A4) and (A5) are:

at $x = -\infty$

$$\left. \begin{aligned} (\varphi'_x)_o &= k \frac{a^*}{\gamma+1} (\varphi_x)_o - \frac{a^* (1 - M_o^2)}{\gamma+1} = -\frac{a^* (1 - M_o^2)}{\gamma+1} \\ (\varphi'_y)_o &= k \frac{a^*}{\gamma+1} (\varphi_y)_o = 0, \quad (\varphi'_z)_o = k \frac{a^*}{\gamma+1} (\varphi_z)_o = 0 \end{aligned} \right\} \quad (\text{A14})$$

at the wing surface

$$(\varphi'_z)_{z=0} = k \frac{a^*}{\gamma+1} (\varphi_z)_{z=0} = k U_o \frac{a^*}{\gamma+1} \left(\frac{\partial Z}{\partial x} \right) \quad (\text{A15})$$

Finally, the pressure coefficient is given by

$$C_p' = -\frac{2}{a^*} [\varphi'_x - (\varphi'_x)_o] = -\frac{2k}{\gamma+1} \varphi_x = \frac{k U_o}{\gamma+1} C_p \quad (\text{A16})$$

Comparison of the above equations with those given previously for the completely consistent a^* analysis reveals their identity if, and only if, k is taken as $(\gamma+1)/U_o$, that is,

$$C_p' = C_p \text{ if } k = \frac{\gamma+1}{U_o} \quad (\text{A17})$$

As far as obtaining the values of C_p is concerned, therefore, the conventional a^* statement of the problem may be regarded as a transformation of the more general U_o statement with k being defined as in equation (A17) or (19). As is evident from equation (A13), however, the local velocities found in the a^* analysis are only correct when the free-stream Mach number is unity.

The foregoing discussion has demonstrated that the conventional a^* statement of transonic-flow problems yields the same values for C_p as the U_o statement, provided k in the latter set of equations is equated to $(\gamma+1)/U_o$. It is indicated in the text, however, that transonic results obtained by use of other expressions for k (and, in particular, $k=M_o^2(\gamma+1)/U_o$) are superior in many important aspects to those obtained using $k=(\gamma+1)/U_o$. It is consequently not without interest to determine a generalized form of the a^* equations which correspond to the U_o equations with unspecified k . The resulting equations for the perturbation potential and the shock relation are:

$$\varphi' = -a^*x + \Phi \quad (\text{A18})$$

$$\varphi'_{yy} + \varphi'_{zz} = k' \varphi'_x \varphi'_{xx} \quad (\text{A19})$$

$$(\varphi'_{v_1} - \varphi'_{v_2})^2 + (\varphi'_{z_1} - \varphi'_{z_2})^2 = k' \left(\frac{\varphi'_{x_1} + \varphi'_{x_2}}{2} \right) (\varphi'_{x_1} - \varphi'_{x_2})^2 \quad (\text{A20})$$

The boundary conditions are:

at $x = -\infty$

$$(\varphi'_x)_o = \frac{M_o^2 - 1}{k'}, \quad (\varphi'_y)_o = (\varphi'_z)_o = 0 \quad (\text{A21})$$

at the wing surface

$$(\varphi'_z)_{z=0} = a^* \left(\frac{\partial Z}{\partial x} \right) \quad (\text{A22})$$

and the pressure coefficient is given by

$$C_p' = -\frac{2}{a^*} [\varphi'_x - (\varphi'_x)_o] \quad (\text{A23})$$

It can be readily verified that these equations will produce the same values for the pressure coefficient as equations (A1) through (A6), provided k and k' are related as follows:

$$\frac{k'}{k} = \frac{U_o}{a^*} \quad (\text{A24})$$

Thus, if it is desired to obtain values for C_p by use of equations (A18) through (A23) that are the same as those given by equations (A1) through (A6) with $k=M_o^2(\gamma+1)/U_o$, k' should be equated to the following:

$$k' = \frac{M_o^2(\gamma+1)}{a^*} \quad (\text{A25})$$

DERIVATION OF SIMILARITY RULES

$$(1-M_o^2)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y_i^2}+\frac{\partial^2\varphi}{\partial x^2}=\begin{cases} 0 & \text{Linear} \\ k\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2} & \text{Nonlinear} \end{cases} \quad \begin{matrix} \text{(B1)} \\ \text{(B2)} \end{matrix}$$
$$\begin{aligned} & (1-M_o^2)(\varphi_{x_1}-\varphi_{x_2})^2+(\varphi_{y_1}-\varphi_{y_2})^2+(\varphi_{z_1}-\varphi_{z_2})^2 \\ & =k\left(\frac{\varphi_{x_1}+\varphi_{x_2}}{2}\right)(\varphi_{x_1}-\varphi_{x_2})^2 \quad (\text{B3}) \end{aligned}$$
$$\left(\frac{\partial \varphi}{\partial x}\right)_o = \left(\frac{\partial \varphi}{\partial y}\right)_o = \left(\frac{\partial \varphi}{\partial x}\right)_o = 0 \quad (\text{B4})$$
$$\frac{1}{U_o} \left(\frac{\partial \varphi}{\partial z} \right)_{z=0} = \tau \frac{\partial}{\partial (x/c)} f \left(\frac{x}{c}, \frac{y}{b} \right) \quad (\text{B5})$$
$$(Z/c) = \tau f(x/c, y/b) \quad (\text{B6})$$
$$C_p = -\frac{2}{U_\rho} \frac{\partial \varphi}{\partial x} \quad (\text{B7})$$
$$\left. \begin{aligned} x' &= s_x x, & y' &= s_y y, & z' &= s_z z, & \varphi' &= s_\varphi \varphi, & U_o' &= s_U U_o \\ \sqrt{(1-M_o^2)'} &= s_\beta \sqrt{1-M_o^2}, & U_o' k' &= s_k U_o k \text{ or } k' = \frac{s_k}{s_U} k \end{aligned} \right\} \quad (\text{B8})$$
$$\begin{aligned}
& \frac{s_x^2}{s_\beta^2 s_\varphi} (1 - M_o^2)' \frac{\partial^2 \varphi'}{\partial x'^2} + \frac{s_y^2}{s_\varphi} \frac{\partial^2 \varphi'}{\partial y'^2} + \frac{s_z^2}{s_\varphi} \frac{\partial^2 \varphi'}{\partial z'^2} \\
& = \begin{cases} 0 & \text{Linear} \quad (\text{B9}) \\ \frac{s_U s_x^3}{s_\beta s_\varphi^2} k' \frac{\partial \varphi'}{\partial x'} \frac{\partial^2 \varphi'}{\partial x'^2} & \text{Nonlinear} \quad (\text{B10}) \end{cases}
\end{aligned}$$
$$\begin{aligned} & \frac{s_x^2}{s_\beta^2 s_\varphi^2} (1 - M_o^2)' (\varphi'_{x_1} - \varphi'_{x_2})^2 + \frac{s_y^2}{s_\varphi^2} (\varphi'_{y_1} - \varphi'_{y_2})^2 + \\ & \frac{s_z^2}{s_\varphi^2} (\varphi'_{z_1} - \varphi'_{z_2})^2 = \frac{s_U s_x^3}{s_k s_\varphi^3} k' \left(\frac{\varphi'_{x_1} + \varphi'_{x_2}}{2} \right) (\varphi'_{x_1} - \varphi'_{x_2})^2 \end{aligned}$$

Nonlinear (B11)

$$\frac{s_y s_\beta}{s_x} = 1, \quad \frac{s_y}{s_z} = 1, \quad \frac{s_\varphi}{s_x s_U} \begin{cases} \lambda/\lambda' & \text{Linear} \\ s_\beta^2/s_k & \text{Nonlinear} \end{cases} \quad \begin{matrix} \text{(B12)} \\ \text{(B13)} \end{matrix}$$

An immediate consequence of this transformation is that the wing plan forms undergo an affine transformation such that the aspect ratios of wings in similar flow fields are related, according to both linear and nonlinear theory, by

$$A' = \frac{s_y}{s_x} A = \frac{1}{s_\beta} A = \sqrt{\frac{1 - M_o^2}{(1 - M_o^2)'}} A \quad (\text{B14})$$

$$\sqrt{(1-M_o^2)'} A' = \sqrt{1-M_o^2} A \quad (\text{B15})$$
$$\left(\frac{\partial \varphi'}{\partial z'}\right)_{z'=0} = \frac{s_\varphi}{s_z} \left(\frac{\partial \varphi}{\partial z}\right)_{z=0} = \frac{s_\varphi}{s_z} U_o \tau \frac{\partial}{\partial (x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \quad (\text{B16})$$

$$\left(\frac{\partial \phi'}{\partial z'}\right)_{z'=0} = U_o' \tau' \frac{\partial}{\partial(x'/c')} f'\left(\frac{x'}{c'}, \frac{y'}{b'}\right) = s_U U_o \tau' \frac{\partial}{\partial(x'/c')} f'\left(\frac{x'}{c'}, \frac{y'}{b'}\right) \quad (\text{B17})$$

$$\begin{aligned} \tau' &= \frac{s_\varphi}{s_U s_z} \tau = \frac{s_\varphi s_\beta}{s_U s_x} \tau \\ &= \begin{cases} \frac{\lambda}{\lambda'} \sqrt{\frac{(1-M_o^2)'}{1-M_o^2}} \tau & \text{Linear} \quad (\text{B18}) \\ \frac{s_\beta^3}{s_k} \tau = \left[\frac{(1-M_o^2)'}{1-M_o^2} \right]^{3/2} \left(\frac{U_o k}{U_o' k'} \right) \tau & \text{Nonlinear} \quad (\text{B19}) \end{cases} \end{aligned}$$

or as

$$\frac{\sqrt{(1-M_o^2)'} }{\lambda' \tau'} = \frac{\sqrt{1-M_o^2}}{\lambda \tau} \quad \text{Linear} \quad (\text{B20})$$

$$\frac{\sqrt{(1-M_o^2)'} }{(U_o' k' \tau')^{1/3}} = \frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}} \quad \text{Nonlinear} \quad (\text{B21})$$

The relation between the pressure coefficients at corresponding points on the wing surface is given by

$$C_p' = -\frac{2}{U_o'} \left(\frac{\partial \varphi'}{\partial x'} \right)_{z'=0} = \frac{s_\varphi}{s_U s_x} \left(-\frac{2}{U_o} \frac{\partial \varphi}{\partial x} \right)_{z=0} = \frac{s_\varphi}{s_U s_x} C_p$$

$$= \left\{ \frac{\lambda}{\lambda'} \right\} C_p \quad \text{Linear} \quad (\text{B22})$$

$$= \left\{ \frac{s_\varphi^2}{s_k} \right\} C_p = \left(\frac{\tau'}{\tau} \right)^{2/3} \left(\frac{U_o k}{U_o' k'} \right)^{1/3} C_p \quad \text{Nonlinear} \quad (\text{B23})$$

or more completely by

$$C_p' \left[\frac{\sqrt{(1-M_o^2)'} }{\lambda' \tau'}, \sqrt{(1-M_o^2)'} A'; \frac{x'}{c'}, \frac{y'}{b'} \right]$$

$$= \frac{\lambda}{\lambda'} C_p \left(\frac{\sqrt{1-M_o^2}}{\lambda \tau}, \sqrt{1-M_o^2} A; \frac{x}{c}, \frac{y}{b} \right) \quad \text{Linear} \quad (\text{B24})$$

$$C_p' \left[\frac{\sqrt{(1-M_o^2)'} }{(U_o' k' \tau')^{1/3}}, \sqrt{(1-M_o^2)'} A; \frac{x'}{c'}, \frac{y'}{b'} \right]$$

$$= \left(\frac{U_o k}{U_o' k'} \right)^{1/3} \left(\frac{\tau'}{\tau} \right)^{2/3} C_p \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A; \frac{x}{c}, \frac{y}{b} \right] \quad \text{Nonlinear} \quad (\text{B25})$$

The foregoing relationships may be summarized in the following statement: The similarity rule for the pressure coefficients on a family of wings having their geometry given by

$$(Z/c) = \tau f(x/c, y/b) \quad (\text{B26})$$

is

$$C_{p_i} = \frac{1}{\lambda} \mathcal{P}_i \left(\frac{\sqrt{1-M_o^2}}{\lambda \tau}, \sqrt{1-M_o^2} A; \frac{x}{c}, \frac{y}{b} \right) \quad \text{Linear} \quad (\text{B27})$$

$$C_p = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{P} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A; \frac{x}{c}, \frac{y}{b} \right] \quad \text{Nonlinear} \quad (\text{B28})$$

The similarity rules for the lift, pitching-moment, and drag coefficients given by the linear theory are

$$C_{L_i} = \frac{1}{\lambda} \mathcal{L}_i \left(\frac{\sqrt{1-M_o^2}}{\lambda \tau}, \sqrt{1-M_o^2} A \right) \quad \text{Linear} \quad (\text{B29})$$

$$C_{m_i} = \frac{1}{\lambda} \mathcal{M}_i \left(\frac{\sqrt{1-M_o^2}}{\lambda \tau}, \sqrt{1-M_o^2} A \right) \quad \text{Linear} \quad (\text{B30})$$

$$C_{D_i} = \frac{\tau}{\lambda} \mathcal{D}_i \left(\frac{\sqrt{1-M_o^2}}{\lambda \tau}, \sqrt{1-M_o^2} A \right) \quad \text{Linear} \quad (\text{B31})$$

The corresponding similarity rules given by nonlinear theory are

$$C_L = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{L} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A \right] \quad \text{Nonlinear} \quad (\text{B32})$$

$$C_m = \frac{\tau^{2/3}}{(U_o k)^{1/3}} \mathcal{M} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A \right] \quad \text{Nonlinear} \quad (\text{B33})$$

$$C_D = \frac{\tau^{5/3}}{(U_o k)^{1/3}} \mathcal{D} \left[\frac{\sqrt{1-M_o^2}}{(U_o k \tau)^{1/3}}, \sqrt{1-M_o^2} A \right] \quad \text{Nonlinear} \quad (\text{B34})$$

It should be noted that the foregoing equations have been written for subsonic flow where $M_o \leq 1$. If $M_o \geq 1$, the radical $\sqrt{1-M_o^2}$ should be replaced with $\sqrt{M_o^2-1}$.

In the linear-theory analysis, λ has remained a completely arbitrary coefficient to be selected as best suits the particular problem at hand. For instance, the compressible-flow relationships between two wings having identical pressure distributions are found by setting $\lambda=1$. If, on the other hand, λ is set equal to $\sqrt{1-M_o^2}$, the thickness ratio, camber, and angle of attack of related wings are identical. The greatest simplification of the similarity rules of linear theory occurs when λ is set equal to $\sqrt{1-M_o^2}/\tau$ since the number of parameters necessary to show the results of linear theory is thereby decreased by one. This degree of arbitrariness in the similarity rules for linear theory is in contrast to the case for nonlinear transonic theory in which no undetermined coefficient like λ is to be found.

For the present purpose of gaining a better understanding of transonic flows, the most advantageous choice for λ is

$$\lambda = \frac{(U_o k)^{1/3}}{\tau^{2/3}} \quad (\text{B35})$$

because then the similarity rules for linear theory assume forms identical to those for nonlinear transonic theory. This is important since it implies that solutions of linear theory and of nonlinear transonic theory can be expressed as functions of the same parameters and, hence, can both be plotted on a single graph in terms of one set of parameters. The two theories would, of course, yield two distinct curves on such a plot. The curve for linear theory would be accepted as valid for purely subsonic and purely supersonic flows but may or may not be valid in the transonic range, as discussed previously. The curve for nonlinear transonic theory is valid not only for transonic flows, but for subsonic and supersonic small perturbation flows as well. As is evident from the derivation of the basic equations, however, the results of the nonlinear transonic theory should be considered to be of only equal accuracy to those of linear theory in the definitely subsonic and supersonic regimes, despite the fact that the solutions are much more difficult to obtain.

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REPORT 1154

ANALYSIS OF LANDING-GEAR BEHAVIOR¹

By BENJAMIN MILWITZKY and FRANCIS E. COOK

SUMMARY

This report presents a theoretical study of the behavior of the conventional type of oleo-pneumatic landing gear during the process of landing impact. The basic analysis is presented in a general form and treats the motions of the landing gear prior to and subsequent to the beginning of shock-strut deflection. In the analysis of the first phase of the impact the landing gear is treated as a single-degree-of-freedom system in order to determine the conditions of motion at the instant of initial shock-strut deflection, after which instant the landing gear is considered as a system with two degrees of freedom. The equations for the two-degree-of-freedom system consider such factors as the hydraulic (velocity square) resistance of the orifice, the forces due to air compression and internal friction in the shock strut, the nonlinear force-deflection characteristics of the tire, the wing lift, the inclination of the landing gear, and the effects of wheel spin-up drag loads.

The applicability of the analysis to actual landing gears has been investigated for the particular case of a vertical landing gear in the absence of drag loads by comparing calculated results with experimental drop-test data for impacts with and without tire bottoming. The calculated behavior of the landing gear was found to be in good agreement with the drop-test data.

Studies have also been made to determine the effects of variations in such parameters as the dynamic force-deflection characteristics of the tire, the orifice discharge coefficient, and the polytropic exponent for the air-compression process, which might not be known accurately in practical design problems.

The study of the effects of variations in the tire characteristics indicates that in the case of a normal impact without tire bottoming reasonable variations in the force-deflection characteristics have only a relatively small effect on the calculated behavior of the landing gear. Approximating the rather complicated force-deflection characteristics of the actual tire by simplified exponential or linear-segment variations appears to be adequate for practical purposes. Tire hysteresis was found to be relatively unimportant. In the case of a severe impact involving tire bottoming, the use of simplified exponential and linear-

segment approximations to the actual tire force-deflection characteristics, which neglect the effects of tire bottoming, although adequate up to the instant of bottoming, fails to indicate the pronounced increase in landing-gear load that results from bottoming of the tire. The use of exponential and linear-segment approximations to the tire characteristics which take into account the increased stiffness of the tire which results from bottoming, however, yields good results.

The study of the importance of the discharge coefficient of the orifice indicates that the magnitude of the discharge coefficient has a marked effect on the calculated behavior of the landing gear; a decrease in the discharge coefficient (or the product of the discharge coefficient and the net orifice area) results in an approximately proportional increase in the maximum upper-mass acceleration.

The study of the importance of the air-compression process in the shock strut indicates that the air springing is of only minor significance throughout most of the impact and that variations in the effective polytropic exponent n between the isothermal value of 1.0 and the near-adiabatic value of 1.3 have only a secondary effect on the calculated behavior of the landing gear. Even the assumption of constant air pressure in the strut equal to the initial pressure, that is, $n=0$, yields fairly good results which may be adequate for many practical purposes.

In addition to the more exact treatment, an investigation has been made to determine the extent to which the basic equations of motion can be simplified and still yield acceptable results. This study indicates that, for many practical purposes, the air-pressure force in the shock strut can be completely neglected, the tire force-deflection relationship can be assumed to be linear, and the lower or unsprung mass can be taken equal to zero. Generalization of the equations of motion for this simplified system shows that the behavior of the system is completely determined by the magnitude of one parameter, namely the dimensionless initial-velocity parameter. Solutions of these generalized equations are presented in terms of dimensionless variables for a wide range of landing-gear and impact parameters which may be useful for rapidly estimating landing-gear performance in preliminary design.

¹ Supersedes NACA TN 2755, "Analysis of Landing-Gear Behavior" by Benjamin Milwitzky and Francis E. Cook, 1952.

INTRODUCTION

The shock-absorbing characteristics of airplane landing gears are normally developed largely by means of extensive trial-and-error drop testing. The desire to reduce the expense and time required by such methods, as well as to provide a more rational basis for the prediction of wheel-inertia drag loads and dynamic stresses in flexible airframes during landing, emphasizes the need for suitable theoretical methods for the analysis of landing-gear behavior. Such theoretical methods should find application in the design of landing gears and complete airplane structures by permitting

(a) the determination of the behavior of a given landing-gear configuration under varying impact conditions (velocity at contact, weight, wing lift, etc.)

(b) the development of a landing-gear configuration to obtain a specified behavior under given impact conditions

(c) a more rational approach to the determination of wheel spin-up and spring-back loads which takes into account the shock-absorbing characteristics of the particular landing gear under consideration

(d) improved determination of dynamic loads in flexible airplane structures during landing. This problem may be treated either by calculating the response of the elastic system to landing-gear forcing functions determined under the assumption that the airplane is a rigid body or by the simultaneous solution of the equations of motion for the landing gear coupled with the equations representing the additional degrees of freedom of the structure. In many cases the former approach should be sufficiently accurate, but in some instances, particularly when the landing-gear attachment points experience large displacements relative to the nodal points of the flexible system, the latter approach, which takes into account the interaction between the deformations of the structure and the landing gear, may be required in order to represent the system adequately.

Since many aspects of the landing-impact problem are so intimately connected with the mechanics of the landing gear, the subject of landing-gear behavior has received analytical treatment at various times (see bibliography). Many of the earlier investigations, in order to reduce the mathematical complexity of the analysis, were limited to consideration of highly simplified linear systems which have little relation to practical landing gears. Some of the more recent papers consider, with different degrees of simplification, more realistic nonlinear systems. The present report represents an attempt at a more complete analysis of the mechanics of practical landing gears and, in addition, investigates the importance of the various elements which make up the landing gear, as well as the extent to which the system can be reasonably simplified for the purpose of rapid analysis.

The basic analysis is presented in a general form and takes into account such factors as the hydraulic (velocity square) resistance of the orifice, the forces due to air compression and internal friction in the shock strut, the nonlinear force-deflection characteristics of the tire, the wing lift, the inclination of the landing gear and the effects of wheel spin-up drag loads. An evaluation of the applicability of the analysis to actual landing gears is presented for the case of a vertical landing gear in the absence of drag loads by comparing calculated results with drop-test data.

Since some parameters, such as the dynamic force-deflection characteristics of the tire, the orifice discharge coefficient, and the polytropic exponent for the air-compression process, may not be accurately known in practical design problems, a study is made to assess the effects of variations in these parameters on the calculated landing-gear behavior.

Studies are also presented to evaluate the extent to which the dynamical system can be simplified without greatly impairing the validity of the calculated results. In addition to the investigations for specific cases, generalized solutions for the behavior of a simplified system are presented for a wide range of landing-gear and impact parameters which may be useful in preliminary design.

SYMBOLS

A_a	pneumatic area
A_h	hydraulic area
A_o	area of opening in orifice plate
A_1	internal cross-sectional area of shock-strut inner cylinder
A_2	external cross-sectional area of shock-strut inner cylinder
A_p	cross-sectional area of metering pin or rod in plane of orifice
A_n	net orifice area
C_d	orifice discharge coefficient
d	overall diameter of tire
F_a	pneumatic force in shock strut
F_h	hydraulic force in shock strut
F_f	friction force in shock strut
F_s	total axial shock-strut force
F_1	normal force on upper bearing (attached to inner cylinder)
F_2	normal force on lower bearing (attached to outer cylinder)
F_{Na}	force normal to axis of shock strut, applied at axle
F_{Va}	vertical force, applied at axle
F_{Ha}	horizontal force, applied at axle
F_{Ra}	resultant force, applied at axle
F_{S_g}	force parallel to axis of shock strut, applied to tire at ground
F_{N_g}	force normal to axis of shock strut, applied to tire at ground
F_{V_g}	vertical force, applied to tire at ground
F_{H_g}	horizontal force, applied to tire at ground
F_{R_g}	resultant force, applied to tire at ground
g	gravitational constant
K_L	lift factor, L/W
L	lift force
l_1	axial distance between upper and lower bearings, for fully extended shock strut
l_2	axial distance between axle and lower bearing (attached to outer cylinder), for fully extended shock strut
a, b, m, r	constants corresponding to the various regimes of the tire-deflection process
a'	combined constant, ad
m'	combined constant, md'
n	polytropic exponent for air-compression process in shock strut

R	Reynolds number
p_a	air pressure in upper chamber of shock strut
p_h	hydraulic pressure in lower chamber of shock strut
Q	volumetric rate of discharge through orifice
r_d	radius of deflected tire
s	shock-strut axial stroke
T	wheel inertia torque reaction
t	time after contact
τ	time after beginning of shock-strut deflection
v	air volume of shock strut
I_w	polar moment of inertia for wheel assembly about axle
V_v	vertical velocity
V_H	horizontal velocity
W	total dropping weight
W_1	weight of upper mass above strut
W_2	weight of lower mass below strut
x_2	horizontal displacement of lower mass from position at initial contact
z_1	vertical displacement of upper mass from position at initial contact
z_2	vertical displacement of lower mass from position at initial contact
u_1	dimensionless upper-mass displacement from position at initial contact
u_2	dimensionless lower-mass displacement from position at initial contact
σ	dimensionless shock-strut stroke, $u_1 - u_2$
θ	dimensionless time after contact
φ	angle between shock-strut axis and vertical
η_s	shock-strut effectiveness, $\frac{\int_0^{\sigma_{max}} u_1'' d\sigma}{u_1''_{max} \sigma_{max}}$
η_{10}	landing-gear effectiveness, $\frac{\int_0^{u_1''_{max}} u_1'' du_1}{u_1''_{max} u_{1max}}$
ϵ	time interval in numerical integration procedures
μ	coefficient of friction between tire and runway
μ_1	coefficient of friction for upper bearing (attached to inner cylinder)
μ_2	coefficient of friction for lower bearing (attached to outer cylinder)
ρ	mass density of hydraulic fluid
α	angular acceleration of wheel
Axes:	
z	vertical axis, positive downward
x	horizontal axis, positive rearward
Subscripts:	
0	at instant of initial contact
τ	at instant of initial shock-strut deflection
su	at instant of wheel spin-up
max	maximum value
Notation:	
$ () $	absolute value of ()
$()^*$	estimated value of ()

The use of dots over symbols indicates differentiation with respect to time t or τ .

Prime marks indicate differentiation with respect to dimensionless time θ .

MECHANICS OF LANDING GEAR

DYNAMICS OF SYSTEM

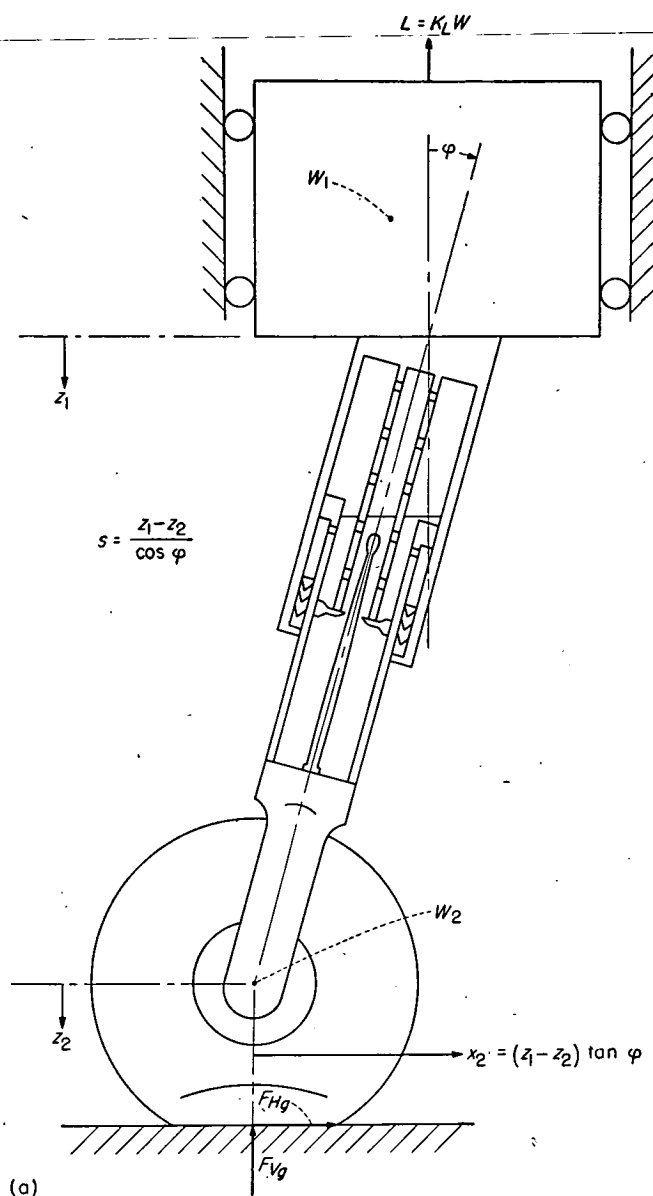
In view of the fact that landing-gear performance appears to be relatively unaffected by the elastic deformations of the airplane structure (see, for example, refs. 1 and 2) particularly since in many cases the main gears are located fairly close to the nodal points of the fundamental bending mode of the wing, that part of the airplane which acts on a given gear can generally be considered as a rigid mass. As a result, landing-gear drop tests are often conducted in a jig where the mass of the airplane is represented by a concentrated weight. In particular instances, however, such as in the case of airplanes having large concentrated masses disposed in an outboard position in the wings, especially airplanes equipped with bicycle landing gear, consideration of the interaction between the deformation of the airplane structure and the landing gear may be necessary to represent the system adequately.

Since the present report is concerned primarily with the mechanics of the landing gear, it is assumed in the analysis that the landing gear is attached to a rigid mass which has freedom only in vertical translation. The gear is assumed infinitely rigid in bending. The combination of airplane and landing gear considered therefore constitutes a system having two degrees of freedom (see fig. 1(a)) as defined by the vertical displacement of the upper mass and the vertical displacement of the lower or unsprung mass, which is also the tire deflection. The strut stroke s is determined by the difference between the displacements z_1 and z_2 and, in the case of inclined gears, by the angle φ between the axis of the strut and the vertical. For inclined gears, compression of the shock strut produces a horizontal displacement of the axle x_2 . From consideration of the kinematics of the system it can be seen that $s = \frac{z_1 - z_2}{\cos \varphi}$ and $x_2 = s \sin \varphi = (z_1 - z_2) \tan \varphi$.

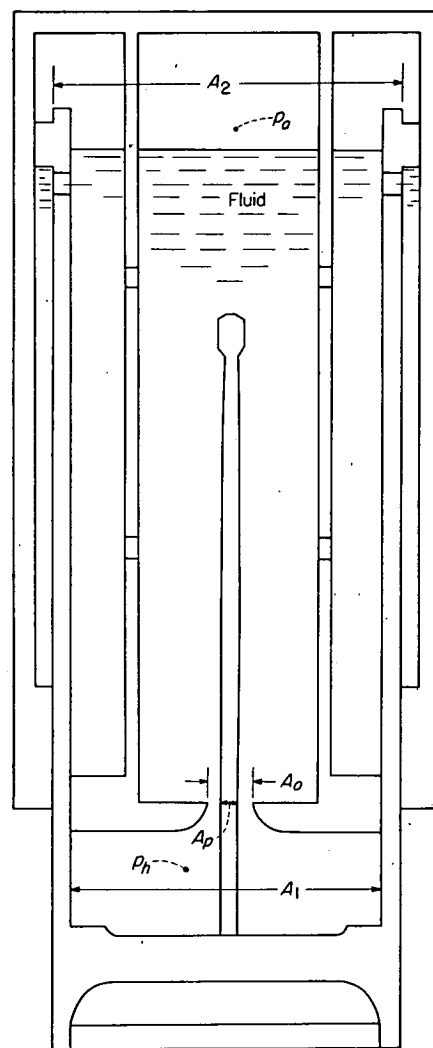
In the analysis, external lift forces, corresponding to the aerodynamic lift, are assumed to act on the system throughout the impact. In addition to the vertical forces, arbitrary drag loads are considered to act between the tire and the ground.

The system treated in the analysis may therefore be considered to represent either a landing-gear drop test in a jig where wing lift and drag loads are simulated, or the landing impact of a rigid airplane if rotational motions are neglected. Rotational freedom of the airplane, where significant, may be taken into account approximately by use of an appropriate effective mass in the analysis.

Figure 1 (b) shows a schematic representation of a typical oleo-pneumatic shock strut used in American practice. The lower chamber of the strut contains hydraulic fluid and the upper chamber contains air under pressure. The outer cylinder of the strut, which is attached to the upper mass, contains a perforated tube which supports a plate with a small orifice, through which the hydraulic fluid is forced to flow at high velocity as a result of the telescoping of the strut. The hydraulic pressure drop across the orifice thus produced resists the closure of the strut, and the turbulence created provides a powerful means of absorbing and dissipating a large part of the impact energy. In some struts the orifice area is constant; whereas, in other cases a metering



(a) System with two degrees of freedom.



(b)

(b) Schematic representation of shock strut.

FIGURE 1.—Dynamical system considered in analysis.

pin or rod is used to control the size of the orifice and govern the performance of the strut.

The compression of the strut produces an increase in the air pressure which also resists the closure of the strut. In figure 1 (b) p_h represents the oil pressure in the lower chamber and p_a represents the air pressure in the upper chamber.

In addition to the hydraulic resistance and air-pressure forces, internal bearing friction also contributes forces which can appreciably affect the behavior of the strut.

The forces created within the strut impart an acceleration to the upper mass and also produce an acceleration of the lower mass and a deflection of the tire. Figure 1 (c) shows the balance of forces and reactions for the wheel, the inner cylinder, and the outer cylinder. It is clear that the strut

and the tire mutually influence the behavior of one another and must be considered simultaneously in analyzing the system.

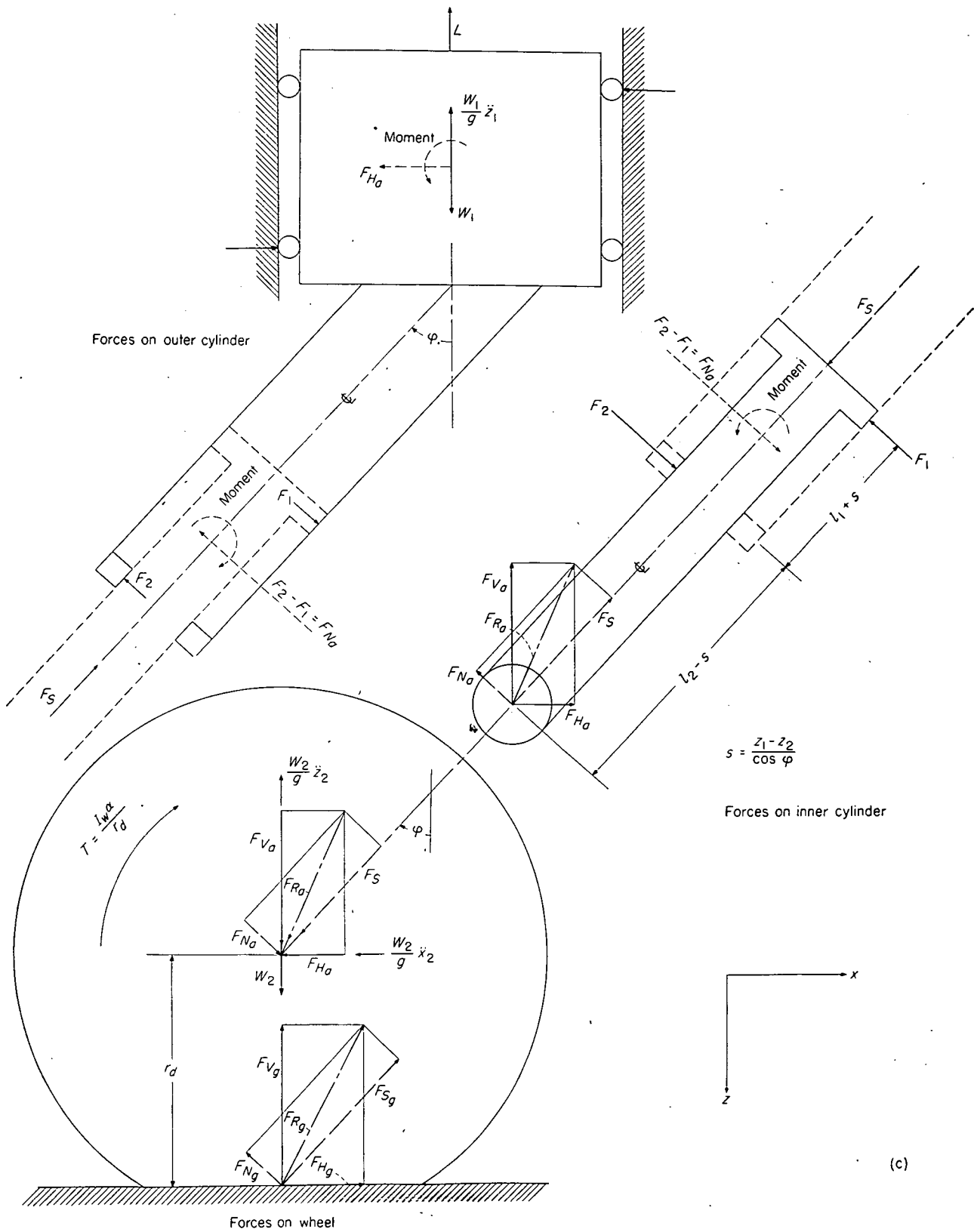
FORCES IN SHOCK STRUT

From consideration of the pressures acting in the shock strut it can be readily seen from figure 1 (b) that the total axial force due to hydraulic resistance, air compression, and bearing friction can be expressed by

$$F_s = p_h(A_1 - A_p) + p_a(A_2 - A_1) + p_a A_p + F_f$$

where

A_1 internal cross-sectional area of inner cylinder
 A_2 external cross-sectional area of inner cylinder
 A_p cross-sectional area of metering pin or rod in plane of orifice



(c) Balance of forces and reactions for landing-gear components.

FIGURE 1.—Concluded.

This expression can also be written as

$$\begin{aligned} F_s &= (p_h - p_a)(A_1 - A_p) + p_a A_2 + F_f \\ &= (p_h - p_a)A_h + p_a A_a + F_f \\ &= F_h + F_a + F_f \end{aligned} \quad (1)$$

where

$p_h - p_a$ pressure drop across the orifice

A_h hydraulic area ($A_1 - A_p$ for the strut shown in fig. 1)

A_a pneumatic area (A_2 for the strut shown in fig. 1)

In this report the terms $(p_h - p_a)A_h$ and $p_a A_a$ are referred to as hydraulic force F_h and pneumatic force F_a , respectively. For the strut shown in figure 1, the hydraulic and pneumatic areas are related to the strut dimensions as previously noted. In the case of struts having different internal configurations, the hydraulic and pneumatic areas may bear somewhat different relations to the dimensions of the strut. In such cases, however, consideration of the pressures acting on the various components of the strut should permit these areas to be readily defined.

Hydraulic force.—The hydraulic resistance in the shock strut results from the pressure difference associated with the flow through the orifice. In a landing gear the orifice area is usually small enough in relation to the diameter of the strut so that the jet velocities and Reynolds numbers are sufficiently large that the flow is fully turbulent. As a result the damping force varies as the square of the telescoping velocity rather than linearly with the velocity. Since the hydraulic resistance is the major component of the total shock-strut force, viscous damping cannot be reasonably assumed, even though such an assumption would greatly simplify the analysis.

The hydraulic resistance can be readily derived by making use of the well-known equation for the discharge through an orifice, namely,

$$Q = C_d A_n \sqrt{\frac{2}{\rho} (p_h - p_a)}$$

where

Q volumetric rate of discharge

C_d coefficient of discharge

A_n net orifice area

p_h hydraulic pressure in lower chamber

p_a air pressure in upper chamber

ρ mass density of hydraulic fluid

From considerations of continuity, the volumetric rate of discharge can also be expressed as the product of the telescoping velocity $\dot{s}$ and the hydraulic area A_h

$$Q = A_h \dot{s}$$

Equating the preceding expressions for the discharge permits writing the following simple equation for the pressure drop across the orifice

$$p_h - p_a = \frac{\rho A_h^2 \dot{s}^2}{2(C_d A_n)^2}$$

The hydraulic resistance F_h due to the telescoping of the strut is given by the product of the differential pressure

$p_h - p_a$ and the area A_h which is subjected to the hydraulic pressure, as previously noted. Thus

$$F_h = \frac{\rho A_h^3}{2(C_d A_n)^2} \dot{s}^2 \quad (2)$$

Equation (2) can be made applicable to both the compression and elongation strokes by introducing the factor $\frac{\dot{s}}{|\dot{s}|}$ to indicate the sign of the hydraulic resistance; thus

$$F_h = \frac{\dot{s}}{|\dot{s}|} \frac{\rho A_h^3}{2(C_d A_n)^2} \dot{s}^2 \quad (2a)$$

The net orifice area A_n may be either a constant or, when a metering pin is used, can vary with strut stroke; that is, $A_n = A_o - A_p = A_n(s)$, where A_o is the area of the opening in the orifice plate and A_p is the area of the metering pin in the plane of the orifice. At the present time there appears to be some tendency to eliminate the metering pin and use a constant orifice area, particularly for large airplanes, in which case $A_n = A_o$. In the general case, the orifice discharge coefficient might be expected to vary somewhat during an impact because of changes in the size and configuration of the net orifice area, changes in the exit conditions on the downstream face of the orifice due to variations in the amount of hydraulic fluid above the orifice plate, changes in the entry conditions due to variations in the length of the flow chamber upstream of the orifice, and because of variations in the Reynolds number of the flow, so that, in general, $C_d = C_d(s, R)$. Although the individual effects of these factors on the discharge coefficients for orifices in shock struts have not been evaluated, there is some experimental evidence to indicate appreciable variations of the discharge coefficient during impact, particularly in the case of struts with metering pins. It might be expected that such variations would be considerably smaller for gears having a constant orifice area.

In order to evaluate the precision with which the orifice discharge coefficient has to be known, a brief study is presented in a subsequent section which shows the effect of the discharge coefficient on the calculated behavior of a landing gear with a constant orifice area, under the assumption that the discharge coefficient is constant during the impact.

The foregoing discussion has been concerned primarily with the compression stroke of the shock strut. Most struts incorporate some form of pressure-operated rebound check valve, sometimes called a snubber valve, which comes into action after the maximum stroke has been attained and closes off the main orifice as soon as the strut begins to elongate, so that the fluid is forced to return to the lower chamber through small passages. The action of the snubber valve introduces greatly increased hydraulic resistance to dissipate the energy stored in the strut in the form of air pressure and to prevent excessive rebound. The product $C_d A_n$ to be used in equation (2a) during the elongation stroke is generally uncertain. The exact area A_n during elongation is usually somewhat difficult to define from the geometry of the strut since in many cases the number of connecting passages varies with stroke and the

leakage area around the piston may be of the same order of magnitude as the area of the return passages. Furthermore, the magnitude of the orifice discharge coefficient, and even possibly the nature of the resistance, are questionable due to the foaming state of the returning fluid. Fortunately, the primary interest is in the compression process rather than the elongation process since the maximum load always occurs before the maximum strut stroke is reached.

Pneumatic force.—The air-pressure force in the upper chamber is determined by the initial strut inflation pressure, the area subjected to the air pressure (pneumatic area), and the instantaneous compression ratio in accordance with the polytropic law for compression of gases, namely $p_a v^n = \text{Constant}$, or

$$p_a = p_{a0} \left(\frac{v_0}{v} \right)^n$$

where

p_a air pressure in upper chamber of shock strut
 p_{a0} air pressure in upper chamber for fully extended strut
 v air volume of shock strut
 v_0 air volume for fully extended strut

Since the instantaneous air volume is equal to the difference between the initial air volume and the product of the stroke and pneumatic area A_a , $p_a = p_{a0} \left(\frac{v_0}{v_0 - A_a s} \right)^n$. The force due to the air pressure is simply the product of the pressure and the pneumatic area:

$$F_a = p_{a0} A_a \left(\frac{v_0}{v_0 - A_a s} \right)^n \quad (3)$$

In the preceding equations, the effective polytropic exponent n depends on the rate of compression and the rate of heat transfer from the air to the surrounding environment. Low rates of compression would be expected to result in values of n approaching the isothermal value of 1.0; whereas higher values of n , limited by the adiabatic value of 1.4, would be expected for higher rates of compression. The actual thermodynamic process is complicated by the violent mixing of the highly turbulent efflux of hydraulic fluid and the air in the upper chamber during impact. On the one hand, the dissipation of energy in the production of turbulence generates heat; on the other hand, heat is absorbed by the aeration and vaporization of the fluid. The effect of this mixing phenomenon on the polytropic exponent or on the equivalent air volume is not clear. A limited amount of experimental data obtained in drop tests (refs. 3 and 4), however, indicates that the effective polytropic exponent may be in the neighborhood of 1.1 for practical cases. A brief study of the importance of the air-compression process and the effects which different values of n may have on the calculated behavior of the landing gear is presented in a subsequent section.

Internal friction force.—In the literature on machine design the wide range of conditions under which frictional resistance can occur between sliding surfaces is generally classified in three major categories, namely, friction between dry surfaces, friction between imperfectly lubricated surfaces, and friction between perfectly lubricated surfaces. In the

case of dry friction, the resistance depends on the physical characteristics of the sliding surfaces, is essentially proportional to the normal force, and is approximately independent of the surface area. The coefficient of friction μ , defined as the ratio of the frictional resistance to the normal force, is generally somewhat greater under conditions of rest (static friction) than under conditions of sliding (kinetic friction). Although the coefficient of kinetic friction generally decreases slightly with increasing velocity, it is usually considered, in first approximation, to be independent of velocity. If, on the other hand, the surfaces are completely separated by a fluid film of lubricant, perfect lubrication is said to exist. Under these conditions the resistance to relative motion depends primarily on the magnitude of the relative velocity, the physical characteristics of the lubricant, the area, and the film thickness, and is essentially independent of the normal force and the characteristics of the sliding surfaces. Perfect lubrication is rarely found in practice but is most likely under conditions of high velocity and relatively small normal pressure, where the shape of the sliding surfaces is conducive to the generation of fluid pressure by hydrodynamic action. In most practical applications involving lubrication, a state of imperfect lubrication exists and the resistance phenomenon is intermediate between that of dry friction and perfect lubrication.

In the case of landing-gear shock struts, the conditions under which internal friction is of concern usually involve relatively high normal pressures and relatively small sliding velocities. Moreover, the usual types of hydraulic fluid used in shock struts have rather poor lubricating properties, and the shape of the bearing surfaces is generally not conducive to the generation of hydrodynamic pressures. It would therefore appear that the lubrication of shock strut bearings is, at best, imperfect; in fact, the conditions appear to approach closely those for dry friction. In the present analysis, therefore, it is assumed, in first approximation, that the internal friction between the bearings and the cylinder walls follows laws similar to those for dry friction; that is, the friction force is given by the product of the normal force and a suitably chosen coefficient of friction.

With these assumptions the internal friction forces produced in the strut depend on the magnitude of the forces on the axle, the inclination of the gear, the spacing of the bearings, and the coefficient of friction between the bearings and the cylinder walls. Figure 1(c) schematically illustrates the balance of forces acting on the various components of the landing gear. The total axial friction in the shock strut is the sum of the friction forces contributed by each of the bearings:

$$F_f = \frac{\dot{s}}{|\dot{s}|} (\mu_1 |F_1| + \mu_2 |F_2|)$$

where

F_f axial friction force
 μ_1 coefficient of friction for upper bearing (attached to inner cylinder)
 F_1 normal force on upper bearing (attached to inner cylinder)
 μ_2 coefficient of friction for lower bearing (attached to outer cylinder)

F_2 normal force on lower bearing (attached to outer cylinder)

$\frac{\dot{s}}{|\dot{s}|}$ factor to indicate sign of friction force

During the interval prior to the beginning of shock-strut motion the friction forces depend on the coefficients of static friction; after the strut begins to telescope the coefficients of kinetic friction apply.

From considerations of the balance of moments it can be seen from figure 1(c) that

$$F_1 = F_{N_a} \left(\frac{l_2 - s}{l_1 + s} \right)$$

and

$$F_2 = F_{N_a} \left(\frac{l_2 - s}{l_1 + s} + 1 \right)$$

so that

$$F_f = \frac{\dot{s}}{|\dot{s}|} |F_{N_a}| \left[(\mu_1 + \mu_2) \frac{l_2 - s}{l_1 + s} + \mu_2 \right] \quad (4)$$

where

$$F_{N_a} = F_{V_a} \sin \varphi - F_{H_a} \cos \varphi \quad (4a)$$

and

F_{N_a} force normal to strut applied at axle

F_{V_a} vertical force applied at axle

F_{H_a} horizontal force applied at axle

φ angle between strut axis and vertical

l_1 axial distance between upper and lower bearings, for fully extended strut

l_2 axial distance between axle and lower bearing (attached to outer cylinder), for fully extended strut

The quantities F_{N_a} , F_{V_a} , and F_{H_a} are forces applied at the axle and differ from the ground reactions by amounts equal to the inertia forces corresponding to the respective acceleration component of the lower mass. Since the inner cylinder generally represents only a relatively small fraction of the lower mass, the lower mass may reasonably be assumed to be concentrated at the axle. With this assumption, the relationships between the forces at the axle and the forces at the ground are given by

$$F_{V_a} = \left(F_{V_g} + \frac{W_2}{g} \ddot{z}_2 - W_2 \right) \quad F_{H_a} = \left(F_{H_g} - \frac{W_2}{g} \ddot{x}_2 \right)$$

The normal force at the axle can therefore be expressed in terms of the ground reactions and the component accelerations of the lower mass by

$$F_{N_a} = \left(F_{V_g} + \frac{W_2}{g} \ddot{z}_2 - W_2 \right) \sin \varphi - \left(F_{H_g} - \frac{W_2}{g} \ddot{x}_2 \right) \cos \varphi \quad (4b)$$

where

F_{V_g} vertical force applied to tire at ground

F_{H_g} horizontal force applied to tire at ground

$\frac{W_2}{g}$ effective mass below shock strut, assumed concentrated at axle

$\ddot{x}_2$ horizontal acceleration of axle

$\ddot{z}_2$ vertical acceleration of axle

In the case of an inclined landing gear having infinite stiffness in bending, the horizontal displacement of the lower mass x_2 is related to the vertical displacements of the upper and lower

masses by the kinematic relationship $x_2 = (z_1 - z_2) \tan \varphi$, as previously noted. Double differentiation of this relationship gives $\ddot{x}_2 = (\ddot{z}_1 - \ddot{z}_2) \tan \varphi$. Substitution of this expression into equation (4b) gives

$$F_{N_a} = F_{V_g} \sin \varphi - F_{H_g} \cos \varphi + \frac{W_2}{g} \ddot{z}_1 \sin \varphi - W_2 \sin \varphi \quad (4c)$$

In equation (4c) the quantity $\ddot{z}_1 \sin \varphi$ represents the acceleration of the lower mass normal to the strut axis when the gear is rigid in bending. In the case of a gear flexible in bending, the normal acceleration of the lower mass is not completely determined by the vertical acceleration of the upper mass and the angle of inclination of the gear. If it should be necessary to take into account, in particular cases, the effects of gear flexibility on the relationship between the normal force on the axle and the ground reactions, the quantity $\ddot{z}_1 \sin \varphi$ in equation (4c) may be replaced by estimated values of the actual normal acceleration of the lower mass as determined from consideration of the bending response of the gear to the applied forces normal to the gear axis. The effects of gear flexibility are not considered in more detail in the present analysis.

FORCES ON TIRE

Figure 2 (a) shows dynamic force-deflection characteristics for a 27-inch smooth-contour (type I) tire inflated to 32 pounds per square inch. These characteristics were determined from time-history measurements of vertical ground force and tire deflection in landing-gear drop tests with a nonrotating wheel at several vertical velocities. As can be seen, the tire compresses along one curve and unloads along another, the hysteresis loop indicating appreciable energy dissipation in the tire. There is some question as to whether the amount of hysteresis would be as great if the tire were rotating, as in a landing with forward speed. The force-deflection curve for a velocity of 11.63 feet per second is for a severe impact in which tire bottoming occurs and shows the sharp increase in force with deflection subsequent to bottoming.

In figure 2 (b) the same force-deflection characteristics are shown plotted on logarithmic coordinates. As can be seen, the force exhibits an exponential variation with deflection. A systematized representation of the force-deflection relationship can therefore be obtained by means of simple equations having the form

$$F_{V_g} = m z_2^r = m' \left(\frac{z_2}{d} \right)^r \quad (5)$$

where

F_{V_g} vertical force, applied to tire at ground

z_2 vertical displacement of lower mass from position at initial contact (radial deflection of tire)

d overall diameter of tire

m, r constants corresponding to the various regimes of the tire-deflection process

m' combined constant, md^r

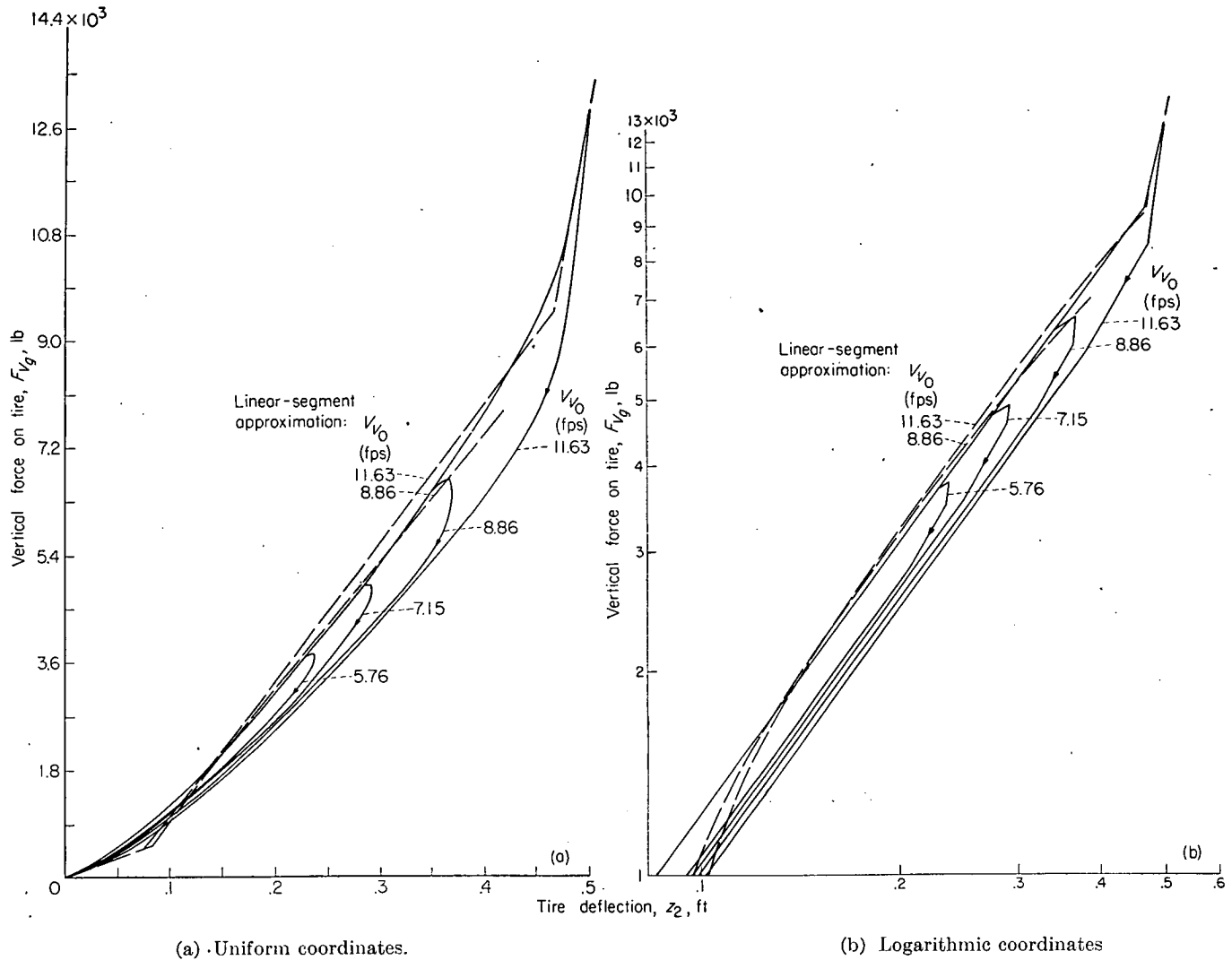


FIGURE 2.—Dynamic force-deflection characteristics of tire.

It may be noted from figure 2 that essentially the same force-deflection curve holds during compression for all impact velocities, up to the occurrence of tire bottoming, and that in figure 2 (b) the slopes of the curves in each of the several regimes of the tire-deflection process are also independent of velocity, except in the compression regime following tire bottoming.

Figure 2 also shows simple approximations to the tire characteristics which were obtained by fitting straight-line segments (long-dashed lines) to the actual force-deflection curves in figure 2 (a) for impacts at 8.86 and 11.63 feet per second. These approximations, hereinafter referred to as linear-segment approximations, are included in a study, presented in a subsequent section, to evaluate the degree of accuracy required for adequate representation of the tire characteristics. The various representations of the tire characteristics considered and the pertinent constants for each regime of tire deflection are shown in figure 3.

EQUATIONS OF MOTION

The internal axial force F_s produced by the shock strut was shown in a previous section to be equal to the sum of the

hydraulic, pneumatic, and friction forces, as given by equation (1). Since these forces act along the axis of the strut, which may be inclined to the vertical by an angle φ , the vertical component of the axial shock-strut force is given by $F_s \cos \varphi$. The vertical component of the force normal to the shock strut is given by $F_{N_a} \sin \varphi$. These forces act in conjunction with the lift force and weight to produce an acceleration of the upper mass. The equation of motion for the upper mass is

$$F_s \cos \varphi + F_{N_a} \sin \varphi + L - W_1 = -\frac{W_1}{g} \ddot{z}_1 \quad (6)$$

The vertical components of the axial and normal shock-strut forces also act, in conjunction with the weight of the lower mass, to produce a deformation of the tire and an acceleration of the lower mass. The equation of motion for the lower mass is

$$F_s \cos \varphi + F_{N_a} \sin \varphi + W_2 - \frac{W_2}{g} \ddot{z}_2 = F_{V_g}(z_2) \quad (7)$$

where the vertical ground reaction F_{V_g} is expressed as a

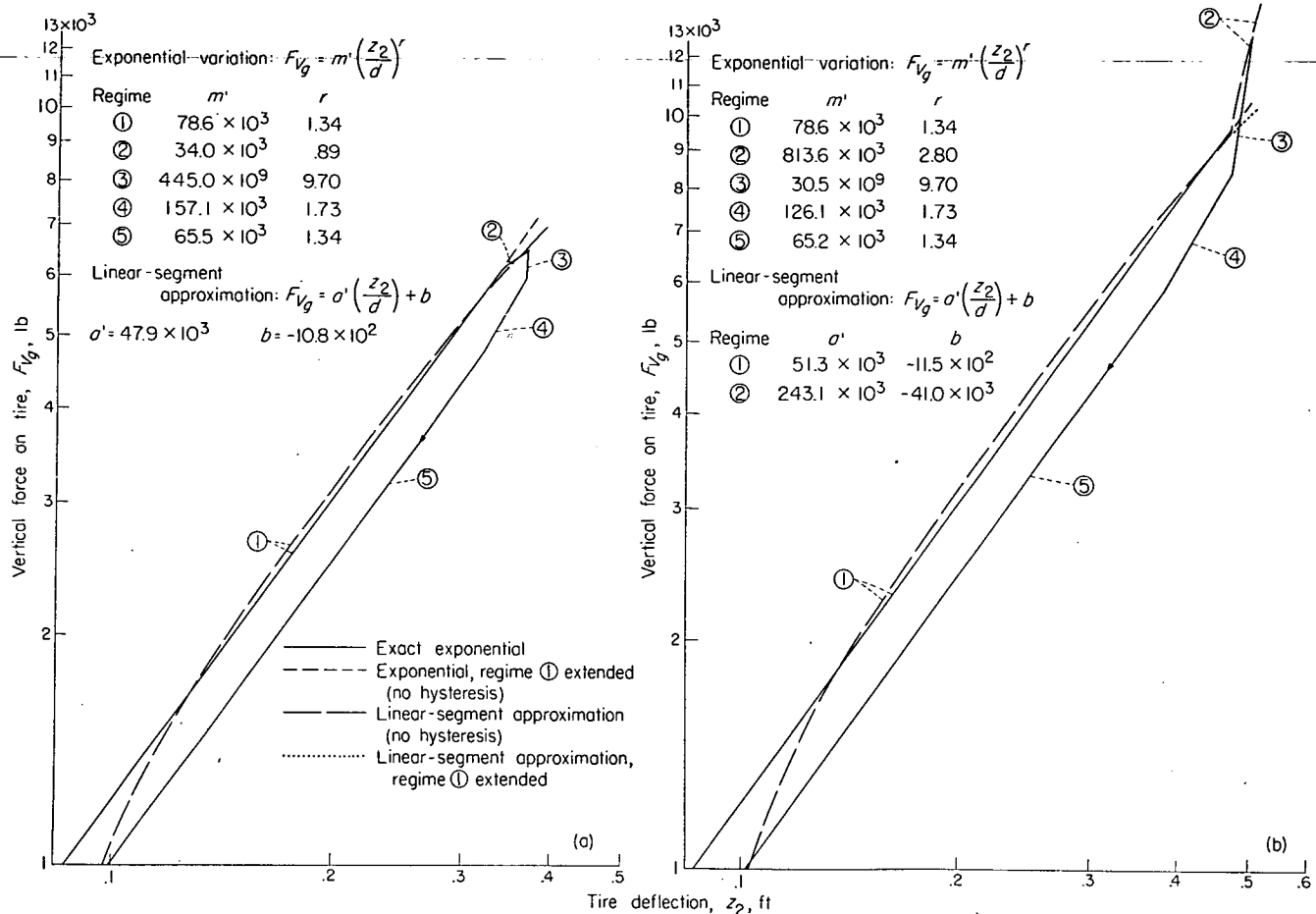
(a) Impact without tire bottoming, $V_{v0} = 8.86$ feet per second.(b) Impact with tire bottoming, $V_{v0} = 11.63$ feet per second.

FIGURE 3.—Tire characteristics considered in solutions (logarithmic coordinates).

function of the tire deflection z_2 . The relationship between F_{vg} and z_2 has been discussed in the previous section on tire characteristics.

By combining equations (6) and (7), the vertical ground force can be written in terms of the inertia reactions of the upper and lower masses, the lift force, and the total weight. The overall dynamic equilibrium is given by

$$F_{vg}(z_2) = -\frac{W_1}{g} \ddot{z}_1 - \frac{W_2}{g} \ddot{z}_2 - L + W \quad (8)$$

MOTION PRIOR TO SHOCK-STRUT DEFLECTION

Conventional oleo-pneumatic shock struts are inflated to some finite pressure in the fully extended position. Thus the strut does not begin to deflect in an impact until sufficient force is developed to overcome the initial preloading imposed by the air pressure and internal friction. Since the strut is effectively rigid in compression, as well as in bending, prior to this instant, the system may be considered to have only one degree of freedom during the initial stage of

the impact. The equations of motion for the one-degree-of-freedom system are derived in order to permit determination of the initial conditions required for the analysis of the landing-gear behavior subsequent to the beginning of shock-strut deflection.

Since $\ddot{z}_1 = \ddot{z}_2 = \ddot{z}$ during this first phase of the impact, equation (8) may be written as

$$F_{vg}(z) = -\frac{W}{g} \ddot{z} - W(K_L - 1) \quad (9)$$

where

$$K_L = \frac{L}{W}$$

For the general case of an exponential relationship between vertical ground force and tire deflection, equation (5) applies and the equation of motion becomes

$$\frac{W}{g} \ddot{z} + m z^r + W(K_L - 1) = 0 \quad (10)$$

The shock strut begins to telescope when the sum of the inertia, weight, and lift forces becomes equal to the vertical components of the axial and normal shock-strut forces. At this instant t_r , $F_s = F_{a_0} + F_{f_r}$ and equation (6) can be written as

$$\ddot{z}_r = - \frac{(F_{a_0} + F_{f_r}) \cos \varphi + F_{N_{a_r}} \sin \varphi + K_L W - W_1}{W_1/g} \quad (11)$$

where

F_{a_0} initial air-pressure preload force, $p_{a_0} A_a$
 F_{f_r} static friction at instant t_r

At the instant t_r , $s=0$ and equation (4) becomes

$$F_{f_r} = |F_{N_{a_r}}| K_\mu \quad (11a)$$

where

$$K_\mu = \left[(\mu_1 + \mu_2) \frac{l_2}{l_1} + \mu_2 \right]$$

and μ_1 and μ_2 are coefficients of static friction.

Since the strut is assumed essentially rigid in compression (and also rigid in bending), there is no kinematic displacement of the lower mass in the horizontal direction up to the beginning of shock-strut deflection, so that $x_2=0$ and equation (4b) becomes

$$F_{N_{a_r}} = \left(F_{v_{g_r}} + \frac{W_2}{g} \ddot{z}_r - W_2 \right) \sin \varphi - F_{H_{g_r}} \cos \varphi \quad (11b)$$

Incorporating equations (11a), (11b), and (9) into equation (11) gives

$$\ddot{z}_r = \frac{F_{a_0} - (\pm K_\mu \sin \varphi - \cos \varphi) (K_L W - W_1) - F_{H_{g_r}} (\pm K_\mu \cos \varphi + \sin \varphi)}{\frac{W_1}{g} (\pm K_\mu \sin \varphi - \cos \varphi)} \quad (12)$$

In equation (12) wherever the $\pm$ sign appears, the plus signs apply when $F_{N_{a_r}} > 0$ and the minus signs apply when $F_{N_{a_r}} < 0$.

From equation (10) the vertical displacement of the system at the instant t_r is given in terms of the corresponding acceleration by

$$z_r = \left\{ \frac{1}{m} \left[W(1 - K_L) - \frac{W}{g} \ddot{z}_r \right] \right\}^{1/r} \quad (13)$$

Integrating equation (10) and noting that $z_0=0$ provides the relationship between the vertical velocity and the vertical displacement of the system at the beginning of shock-strut deflection

$$\dot{z}_r = \sqrt{\dot{z}_0^2 - \frac{2g}{W} \left[\frac{m}{r+1} z_r^{r+1} + W(K_L - 1) z_r \right]} \quad (14)$$

In view of the fact that the tire force-deflection curve is essentially linear for small deflections, it may be reasonably assumed that $r=1$ for the purpose of determining the time after contact at which the strut begins to telescope. With this assumption t_r can be determined from the relationship

$$t_r = \int_0^{z_r} \frac{dz}{\dot{z}} = \int_0^{z_r} \frac{dz}{\sqrt{\dot{z}_0^2 - \frac{2g}{W} \left[\frac{m}{2} z^2 + W(K_L - 1) z \right]}}$$

where the general expression for the variable $\dot{z}$ is obtained from equation (14) without the subscripts τ . Performing the indicated integration gives

$$t_r = \sqrt{\frac{W}{mg}} \left\{ \sin^{-1} C(1 - K_L) - \sin^{-1} C \left[(1 - K_L) - \frac{m z_r}{W} \right] \right\} \quad (15)$$

where

$$C = \frac{g}{\sqrt{\dot{z}_0^2 \frac{mg}{W} + [(1 - K_L)g]^2}}$$

The computation of t_r can be greatly simplified by use of the following approximation which assumes a linear relationship between velocity and time:

$$t_r = \frac{2z_r}{\dot{z}_0 + \dot{z}_r} \quad (15a)$$

Equation (15a) should be a fairly good approximation in view of the relatively short time interval between initial contact and the beginning of shock-strut motion.

Equations (12), (13), and (14) permit the determination of the vertical acceleration, displacement, and velocity, respectively, of the system (upper and lower masses) at the

beginning of shock-strut deflection. Equation (15) or (15a) permits calculation of the time interval between initial contact and this instant. These equations provide the initial conditions required for the analysis of the behavior of the landing gear as a system with two degrees of freedom after the shock strut begins to deflect.

If drag loads are considered, the solution of equation (12) requires knowledge of the horizontal ground force F_{H_g} at the instant t_r . Since the present analysis does not explicitly treat the determination of drag loads, values of F_{H_g} have to be estimated, either from other analytical considerations, experimental data, or on the basis of experience.

MOTION SUBSEQUENT TO BEGINNING OF SHOCK-STRUT DEFLECTION

Once the sum of the inertia, weight, and lift forces becomes sufficiently large to overcome the preloading force in the shock strut due to initial air pressure and internal friction, the shock strut can deflect and the system becomes one having two degrees of freedom. Incorporating the expressions for the hydraulic, pneumatic, and friction forces (eqs. (2a), (3), and (4)) into equation (6) permits the equation of motion for the upper mass to be written as follows:

$$\frac{W_1}{g} \ddot{z}_1 + \left\{ \frac{\dot{s}}{|\dot{s}|} \frac{\rho A_h^3}{2(C_d A_n)^2} \dot{s}^2 + p_{a_0} A_a \left(\frac{v_0}{v_0 - A_a s} \right)^n + \frac{\dot{s}}{|\dot{s}|} |F_{N_a}| \left[(\mu_1 + \mu_2) \frac{l_2 - s}{l_1 + s} + \mu_2 \right] \right\} \cos \varphi + K_L' W - W_1 + F_{N_a} \sin \varphi = 0 \quad (16)$$

where

$$s = \frac{z_1 - z_2}{\cos \varphi}$$

$$\dot{s} = \frac{\dot{z}_1 - \dot{z}_2}{\cos \varphi}$$

and, since $F_{V_g} = F_{V_g}(z_2)$, equation (4c) becomes

$$F_{N_a} = F_{V_g}(z_2) \sin \varphi - F_{H_g} \cos \varphi + \frac{W_2}{g} \ddot{z}_1 \sin \varphi - W_2 \sin \varphi$$

where $F_{V_g}(z_2)$ is determined from the force-deflection characteristics of the tire. For the usual type of pneumatic tire, $F_{V_g}(z_2) = m z_2^r$, as previously noted.

Similarly, the equation of motion for the lower mass follows from equation (7):

$$\frac{W_2}{g} \ddot{z}_2 - \left\{ \frac{\dot{s}}{|\dot{s}|} \frac{\rho A_h^3}{2(C_d A_n)^2} \dot{s}^2 + p_{a_0} A_a \left(\frac{v_0}{v_0 - A_a s} \right)^n + \frac{\dot{s}}{|\dot{s}|} |F_{N_a}| \left[(\mu_1 + \mu_2) \frac{l_2 - s}{l_1 + s} + \mu_2 \right] \right\} \cos \varphi + F_{V_g}(z_2) - F_{N_a} \sin \varphi - W_2 = 0 \quad (17)$$

The overall dynamic equilibrium equation is still, of course, as given by equation (8)

$$\frac{W_1}{g} \ddot{z}_1 + \frac{W_2}{g} \ddot{z}_2 + W(K_L - 1) + F_{V_g}(z_2) = 0$$

Any two of the preceding equations (eqs. (16), (17), and (8)) are sufficient to describe the behavior of the landing gear subsequent to the beginning of shock-strut motion. These equations may be used to calculate the behavior of a given landing-gear configuration or to develop orifice and metering-pin characteristics required to produce a specified behavior for given impact conditions. They may also be used as a basis for the calculation of dynamic loads in flexible airplane structures either by (a) determining the landing-gear forcing function under the assumption that the upper mass is a rigid body and then using this forcing function to calculate the response of the elastic system or (b) combining the preceding equations with the equations representing the additional degrees of freedom of the structure; the simultaneous solution of the equations for such a system would then take into account the interaction between the deformation of the structure and the landing gear.

SOLUTION OF EQUATIONS OF MOTION

In the general case the analysis of a landing gear involves the solution of the equations of motion given in the section entitled "Motion Subsequent to the Beginning of Shock-Strut Deflection," with the initial conditions taken as the conditions of motion at the beginning of shock-strut deflection, as determined in accordance with the initial impact conditions and the equations given in the section entitled "Motion Prior to Shock-Strut Deflection."

NUMERICAL INTEGRATION PROCEDURES

In view of the fact that the equations of motion for the landing gear subsequent to the beginning of shock-strut deflection are highly nonlinear, analytical solution of these equations does not appear feasible. In the present report, therefore, finite-difference methods are resorted to for the step-by-step integration of the equations of motion. Al-

though such numerical methods lack the generality of analytical solutions and are especially time consuming if the calculations are carried out manually, the increasing availability of automatic calculating machines largely overcomes these objections.

Most of the solutions presented in this report were obtained with a procedure, hereinafter referred to as the "linear procedure," which assumes changes in the motion variables to be linear over finite time intervals. A few of the solutions presented were obtained with a procedure, hereinafter referred to as, the "quadratic procedure," which assumes a quadratic variation of displacement with time for successive intervals. The generalized solutions for the simplified equations discussed in a subsequent section were obtained by means of the Runge-Kutta procedure. The application of these procedures is described in detail in appendix A.

USE OF TIRE FORCE-DEFLECTION CHARACTERISTICS

In order to obtain solutions for particular cases, it is, of course, necessary to have, in addition to information regarding the physical characteristics of the landing gear, some knowledge of the force-deflection characteristics of the tire.

If extensive data regarding the dynamic tire characteristics, such as shown in figures 2 and 3, are available, an accurate solution can be obtained which takes into account the various breaks in the force-deflection curves (logarithmic coordinates), as well as the effects of hysteresis. In view of the fact that the constants m' and r have the same values throughout practically the entire tire compression process regardless of the initial impact velocity or the maximum load attained, these values of m' and r , as determined from the force-deflection curves, can be used in the calculation of the motion subsequent to the beginning of shock-strut deflection until the first break in the force-deflection curve is reached prior to the attainment of the maximum force. If the conditions for the calculations are the same as those for which force-deflection curves are available, the values of m' and r for each of the several regimes subsequent to the first break can also be determined directly from the force-deflection curves. In general, however, the conditions will not be the same and interpolation will be necessary to estimate the values of m' for the subsequent regimes. Such interpolation is facilitated, particularly after the maximum force-deflection point has been calculated, by the fact that each subsequent regime has a fixed value of r , regardless of the initial impact conditions.

The use of the tire-deflection characteristics in the calculations is greatly simplified if hysteresis is neglected since the values of m' and r which apply prior to the first break in the force-deflection curves are then used throughout the entire calculation, except in the case of severe impacts where tire bottoming occurs, in which case new values of m' and r are employed in the tire-bottoming regime. A similar situation exists with respect to the constants a' and b when the linear approximations which neglect hysteresis are used. These simplifications would normally be employed when only the

tire manufacturer's static or so-called impact load-deflection data are available, as is usually the case.

EFFECT OF DRAG LOADS

Although the present analysis permits taking into account the effects of wheel spin-up drag loads on the behavior of the landing gear, the determination of the drag-load time history is not treated explicitly. Thus, if it is desired to consider the effects of the drag load on the gear behavior, such as in the case of a drop test in which drag loads are simulated by reverse wheel rotation or in a landing with forward speed, it is necessary to estimate the drag load, either by means of other analytical considerations or by recourse to experimental data. As a first approximation the instantaneous drag force may be assumed to be equal to the vertical ground reaction multiplied by a suitable coefficient of friction μ ; that is, $F_{H_g} = F_{V_g} \mu$, up to the instant when the wheel stops skidding, after which the drag force may be assumed equal to zero. (The current ground-loads requirements specify a skidding coefficient of friction $\mu = 0.55$; limited experimental evidence, on the other hand, indicates that μ may be as high as 0.7 or as low as 0.4.) In some cases experimental data indicate that representation of the drag-load time history can be simplified even further by assuming a linear variation of the drag force with time during the period of wheel skidding.

The instant at which the wheel stops skidding can be estimated from the simple impulse-momentum relationship

$$\int_0^{t_{su}} F_{H_g} dt = \mu \int_0^{t_{su}} F_{V_g} dt = \frac{\bar{I}_w V_{H_0}}{r_d^2}$$

where

$\bar{I}_w$ polar moment of inertia of wheel assembly about axle

V_{H_0} initial horizontal velocity

r_d radius of deflected tire

t_{su} time of wheel spin-up

When the drag force is expressed in terms of the vertical force, the value of the integral $\int_0^t F_{H_g} dt$ can be determined as the step-by-step calculations proceed and the drag-force term eliminated from the equations of motion after the required value of the integral at the instant of spin-up is reached.

EVALUATION OF ANALYSIS BY COMPARISON OF CALCULATED RESULTS WITH EXPERIMENTAL DATA

In order to evaluate the applicability of the foregoing analytical treatment to actual landing gears, tests were conducted in the Langley impact basin with a conventional oleo-pneumatic landing gear originally designed for a small military training airplane. A description of the test specimen and apparatus used is given in appendix B.

In this section calculated results are compared with experimental data for a normal impact and a severe impact with tire bottoming. The vertical velocities at the instant of ground contact used in the calculations correspond to

the vertical velocities measured in the tests. Equations (12), (13), (14), and (15a) were used to calculate the values of the variables at the instant of initial shock-strut deflection. Numerical integration of equations (16) and (17) provided the calculated results for the two-degree-of-freedom system subsequent to the beginning of shock-strut deflection.

In these calculations the discharge coefficient for the orifice and the polytropic exponent for the air-compression process were assumed to have constant values throughout the impact. Consideration of the shape of the orifice and examination of data for rounded approach orifices in pipes suggested a value of C_d equal to 0.9. Evaluation of data for other landing gears indicated that the air-compression process could be represented fairly well by use of an average value of the effective polytropic exponent $n=1.12$. In view of the fact that the landing gear was mounted in a vertical position and drag loads were absent in the tests, friction forces in the shock strut were assumed to be negligible in the calculations. Since the weight was fully balanced by lift forces in the tests, the lift factor K_L was taken equal to 1.0. The appropriate exact tire characteristics, (see fig. 3) were used for each case.

NORMAL IMPACT

Figure 4 presents a comparison of calculated results with experimental data for an impact without tire bottoming at a vertical velocity of 8.86 feet per second at the instant of ground contact. The exact dynamic force-deflection characteristics of the tire, including hysteresis, were used in the calculations. These tire characteristics are shown by the solid lines in figure 2 (a) and values for the tire constants m' and r are given in figure 3 (a).

Calculated time histories of the total force on the upper mass and the acceleration of the lower mass are compared with experimental data in figure 4 (a). Similar comparisons for the upper-mass displacement, upper-mass velocity, lower-mass displacement, strut stroke, and strut telescoping velocity are presented in figure 4 (b). As can be seen, the agreement between the calculated and experimental results is reasonably good throughout most of the time history. Some of the minor discrepancies during the later stages of the impact appear to be due to errors in measurement since the deviations between the calculated and experimental upper-mass accelerations (as represented by the force on the upper mass) are incompatible with those for the upper-mass displacements, whereas the calculated upper-mass displacements are necessarily directly compatible with the calculated upper-mass accelerations. The maximum value of the experimental acceleration of the lower mass may be somewhat high because of overshoot of the accelerometer.

In addition to the total force on the upper mass, figure 4 (a) presents calculated time histories of the hydraulic and pneumatic components of the shock-strut force, as determined from equations (2) and (3), respectively. It can be seen that throughout most of the impact the force developed in the shock strut arises primarily from the hydraulic resistance of the orifice. Toward the end of the impact, however, because of the decreased telescoping velocities and fairly large strokes which correspond to high

compression ratios, the air-pressure force becomes larger than the hydraulic force.

IMPACT WITH TIRE BOTTOMING

Figure 5 presents a comparison of calculated and experimental results for a severe impact ($V_{v_0}=11.63$ ft per sec) in which tire bottoming occurred. The tire force-deflection characteristics used in the calculations are shown by the solid lines in figure 3 (b). Region (1) of the tire force-deflection curve has the same values of the tire constants m' and r as for the case previously discussed. Following the occurrence of tire bottoming, however, different values of m' and r apply. These values are given in figure 3 (b).

It can be seen from figure 5 that the agreement between the calculated and experimental results for this case is similar to that for the comparison previously presented.

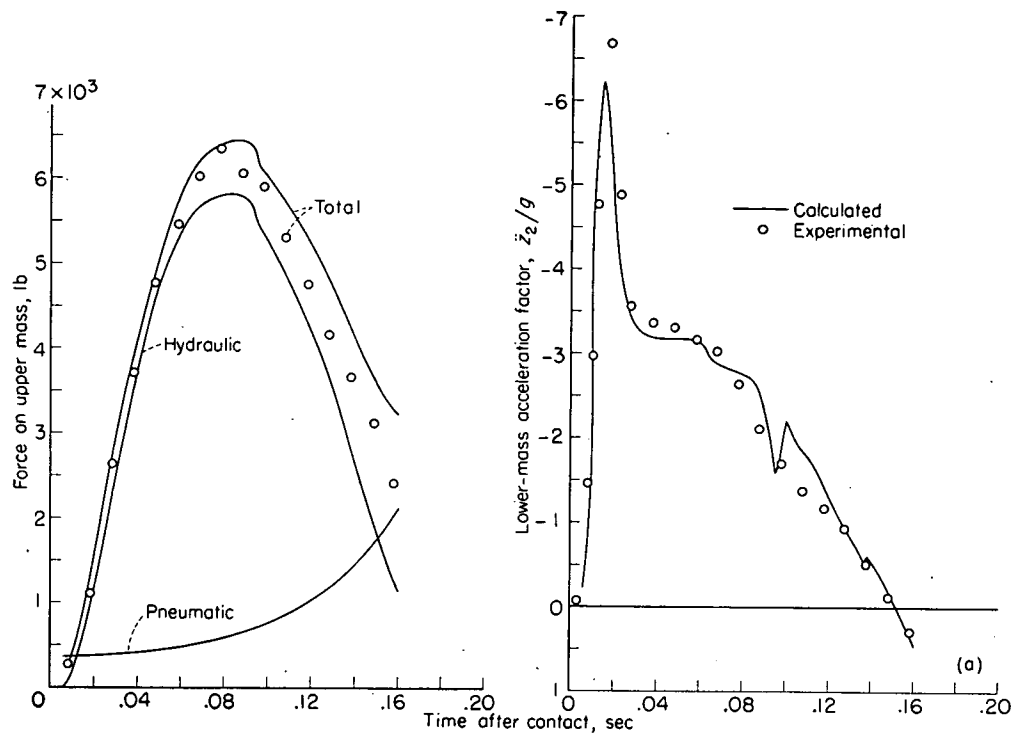
The calculated instant of tire bottoming is indicated in figure 5. When tire bottoming occurs, the greatly increased stiffness of the tire causes a marked increase in the shock-strut telescoping velocity, as is shown in the right-hand portion of figure 5 (b). Since the strut is suddenly forced to absorb energy at a much higher rate, an abrupt increase in the hydraulic resistance takes place. The further increase in shock-strut force immediately following the occurrence of tire bottoming is evident from the left-hand portion of figure 5 (a). The sudden increase in lower-mass acceleration at the instant of tire bottoming can also be seen.

In this severe impact the hydraulic resistance of the orifice represents an even greater proportion of the total shock-strut force than was indicated by the calculated results for an initial vertical velocity of 8.86 feet per second previously discussed.

The foregoing comparisons indicate that the analytical treatment presented, in conjunction with reasonably straightforward assumptions regarding the parameters involved in the equations, provides a fairly accurate representation of the behavior of a conventional oleo-pneumatic landing gear.

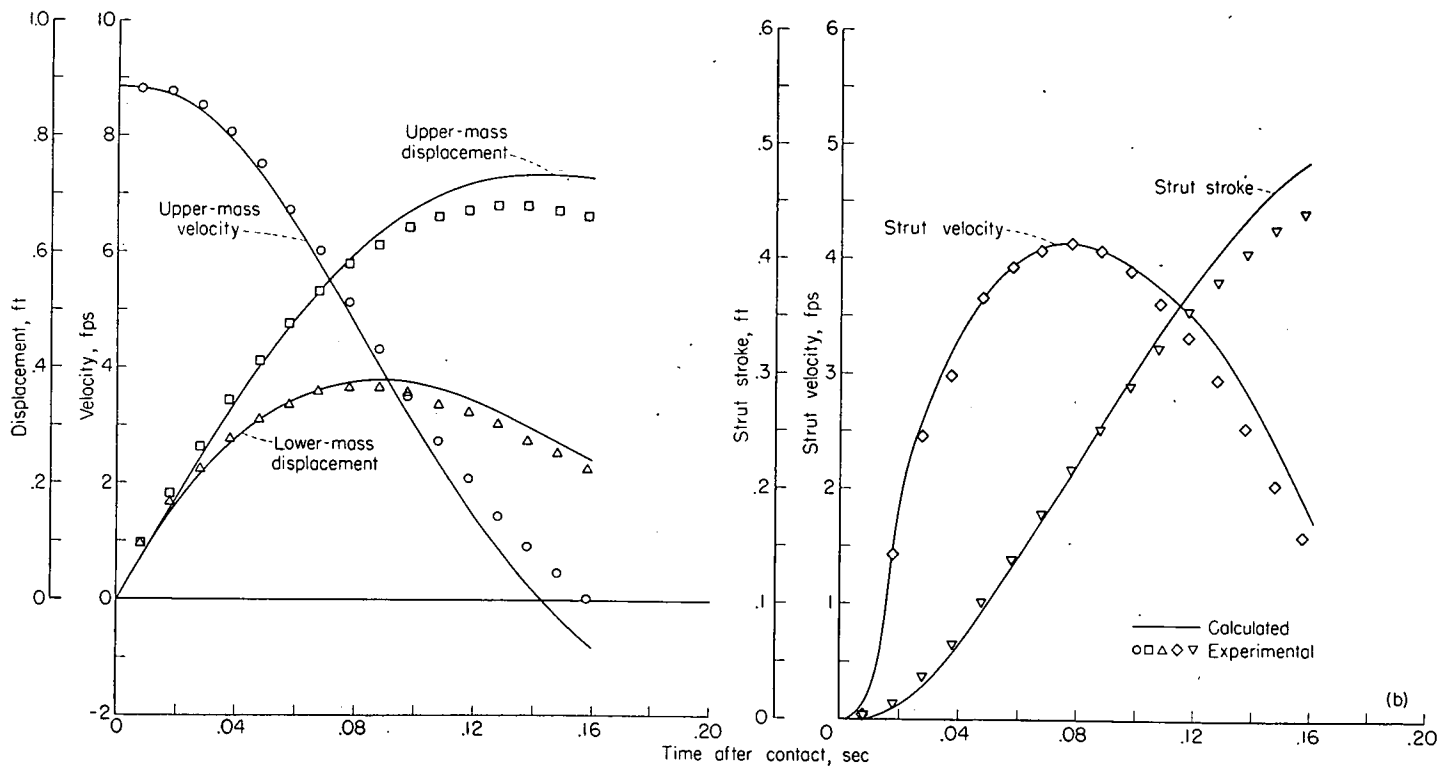
PARAMETER STUDIES

In the previous section comparisons of calculated results with experimental data showed that the equations which have been developed provide a fairly good representation of the behavior of the landing gear for the impact conditions considered. In view of the fact that the equations are somewhat complicated and require numerical values for several parameters such as the tire force-deflection constants m' and r , the orifice discharge coefficient C_d , and the polytropic exponent n , which may not be readily or accurately known in the case of practical engineering problems, it appears desirable (a) to determine the relative accuracy with which these various parameters have to be known and (b) to investigate the extent to which the equations can be simplified and still yield useful results. In order to accomplish these objectives, calculations have been made to evaluate the effect of simplifying the force-deflection characteristics of the tire, as well as to determine the effects which different values of the orifice discharge coefficient and the effective polytropic exponent have on the calculated behavior. The results of these calculations are discussed in the present section. The question of simplification of the equations of motion is considered in more detail in a subsequent section.



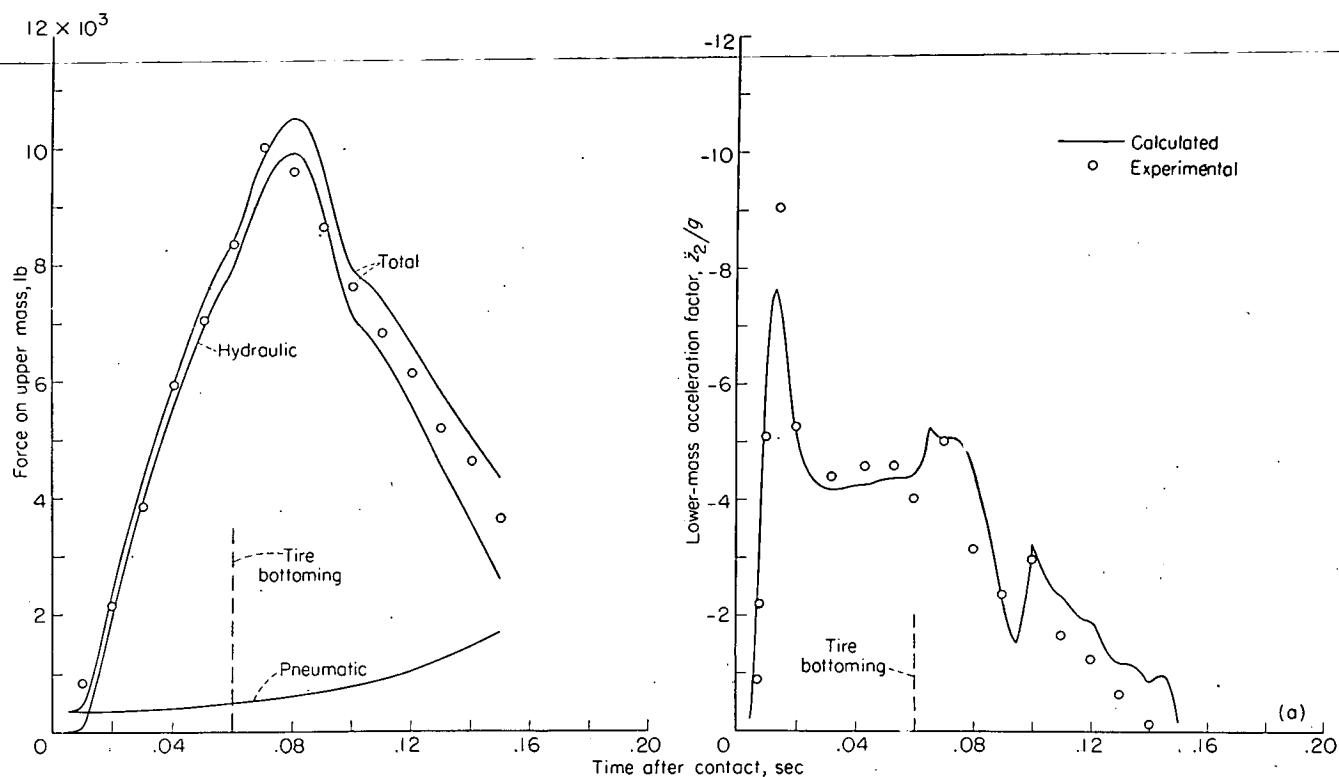
(a) Time histories of forces on upper mass and lower-mass acceleration.

FIGURE 4.—Comparisons between calculated results and experimental data for normal impact; solution with exact exponential tire characteristics. $V_{v0}=8.86$ feet per second; $C_d=0.9$; $n=1.12$.



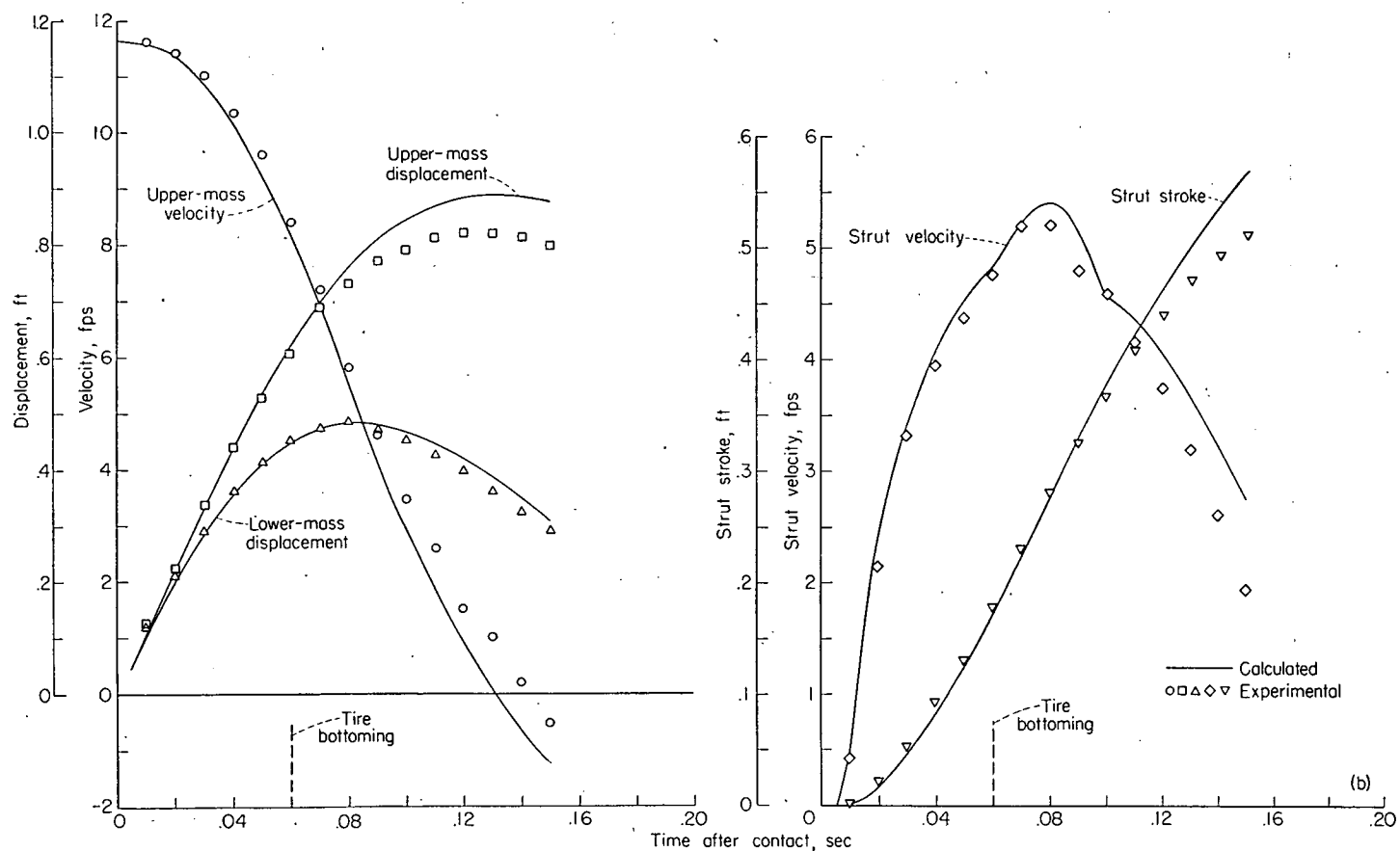
(b) Time histories of landing-gear velocities and displacements.

FIGURE 4.—Concluded.



(a) Time histories of forces on upper mass and lower-mass acceleration.

FIGURE 5.—Comparisons between calculated results and experimental data for impact with tire bottoming; solution with exact exponential tire characteristics. $V_{v_0}=11.63$ feet per second; $C_d=0.9$; $n=1.12$.



(b) Time histories of landing-gear velocities and displacements.

FIGURE 5.—Concluded.

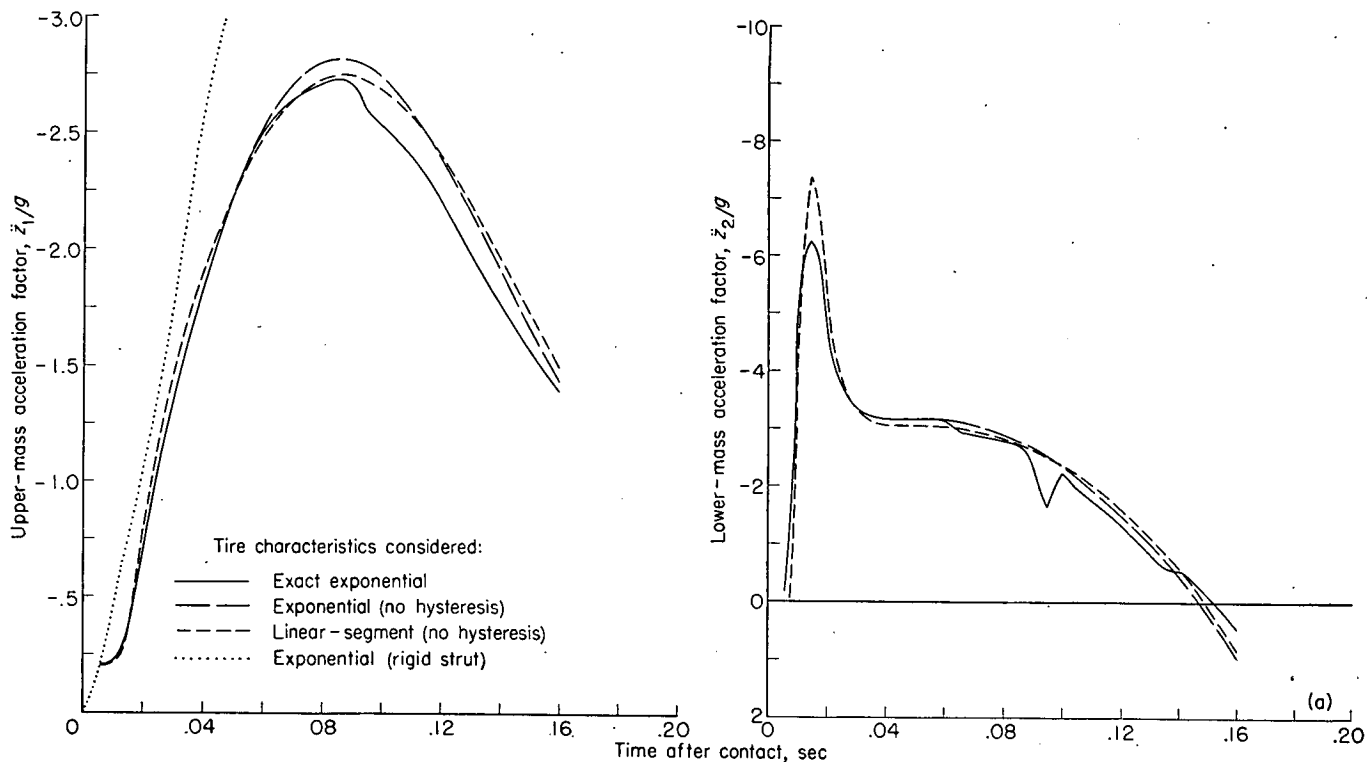
REPRESENTATION OF TIRE FORCE-DEFLECTION CHARACTERISTICS

In order to evaluate the degree of accuracy required for adequate representation of the tire force-deflection characteristics, comparisons are made of the calculated behavior of the landing gear for normal impacts and impacts with tire bottoming when the tire characteristics are represented in various ways. First, the force-deflection characteristics will be assumed to be exactly as shown by the solid-line curves in figure 2 (b), including the various breaks in the curve and the effects of hysteresis. These characteristics are referred to hereinafter as the exact exponential tire characteristics. The effects of simplifying the representation of the tire characteristics will then be investigated by considering (a) the exponential characteristics without hysteresis; that is, the tire will be assumed to deflect and unload along the same exponential curve, (b) the linear-segment approximations to the tire characteristics (long-dashed lines), which also neglect hysteresis, and (c) errors introduced by neglecting the effects of tire bottoming in the case of severe impacts. The calculated results presented in this study make use of the relationships between vertical force on the tire and tire deflection, as shown in figures 3 (a) and 3 (b).

Figure 6 presents a comparison of the calculated results for a normal impact at a vertical velocity of 8.86 feet per second, whereas figure 7 permits comparison of the solutions

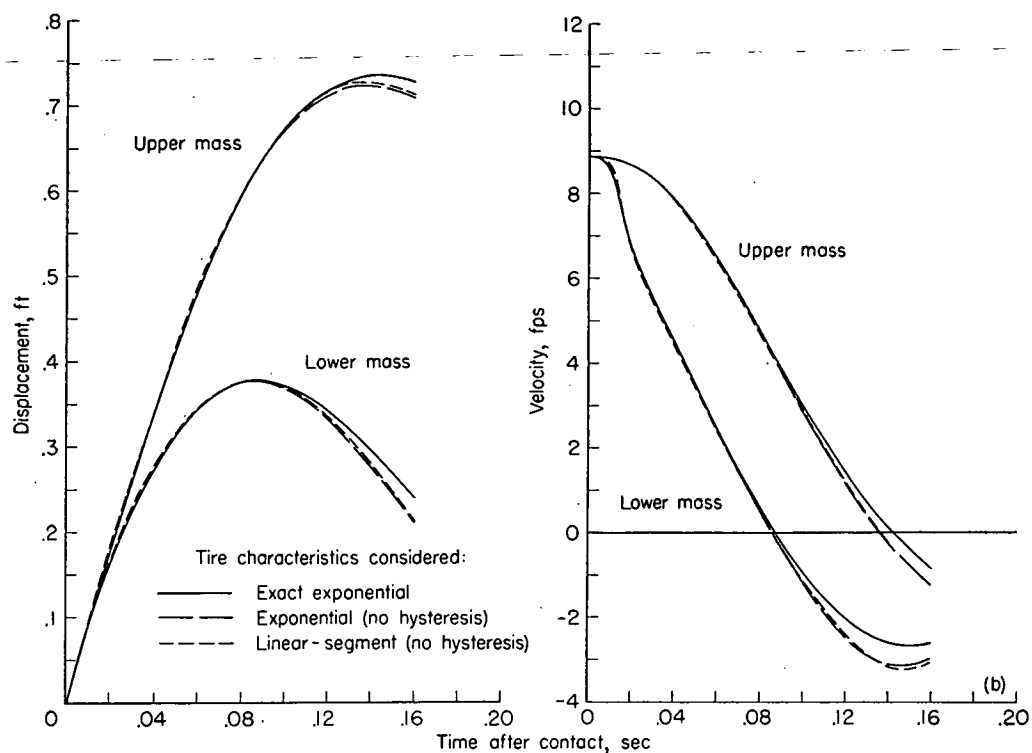
for a severe impact, involving tire bottoming, at a vertical velocity of 11.63 feet per second. In figures 6 and 7 the solid-line curves represent solutions of the landing-gear equations when the exact exponential relationships between force and tire deflection are considered. Since these solutions were previously shown to be in fairly good agreement with experimental data (figs. 4 and 5), they are used as a basis for evaluating the results obtained when tire hysteresis is neglected and the force-deflection characteristics are represented by either simplified exponential or linear-segment relationships.

As in the calculations previously described, the solutions were obtained in two parts. During the first stage of the impact the shock strut was considered to be rigid until sufficient force was developed to overcome the initial air-pressure force. The calculations for the landing-gear behavior subsequent to this instant were based on the equations which consider the gear to have two degrees of freedom. Time histories of the upper-mass acceleration calculated on the basis of a rigid shock strut are shown by the dotted curves in figures 6 and 7. These solutions show the greatest rate of increase of upper-mass acceleration possible with the exponential tire force-deflection characteristics considered. Comparison of these solutions with those for the two-degree-of-freedom system indicates the effect of the shock strut in attenuating the severity of the impact.



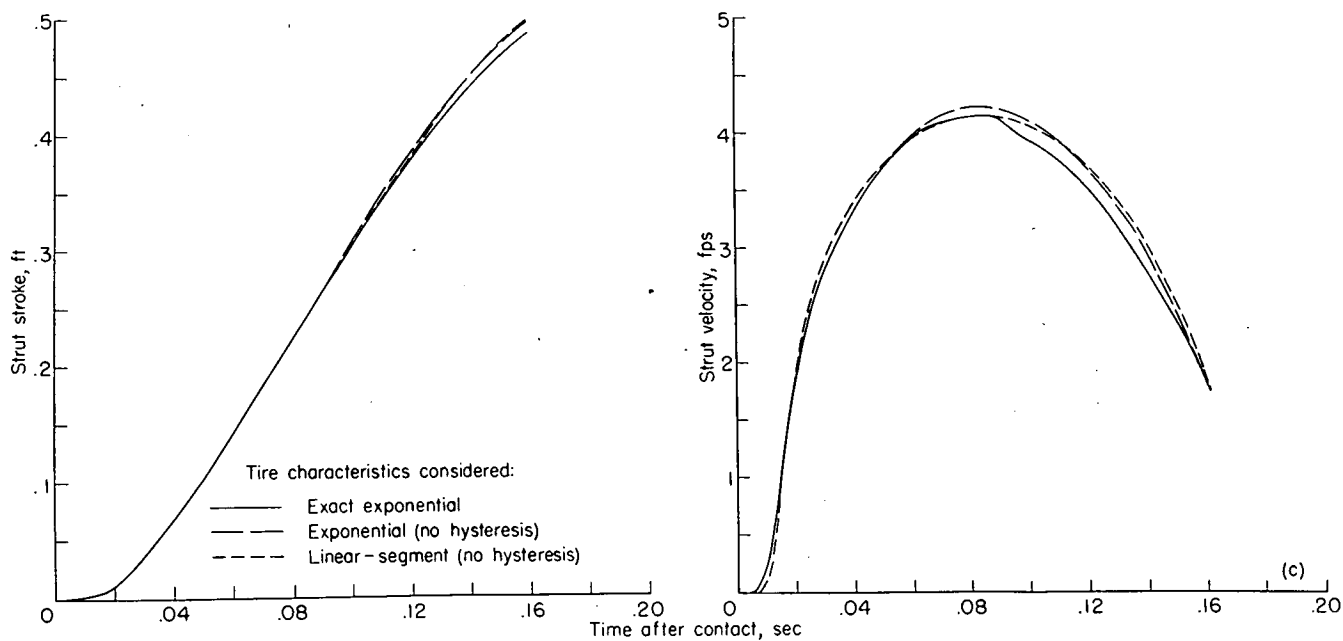
(a) Time histories of upper-mass acceleration and lower-mass acceleration.

FIGURE 6.—Effect of tire characteristics on calculated landing-gear behavior in normal impact. V_{v_0} = 8.86 feet per second; C_d = 0.9; n = 1.12.



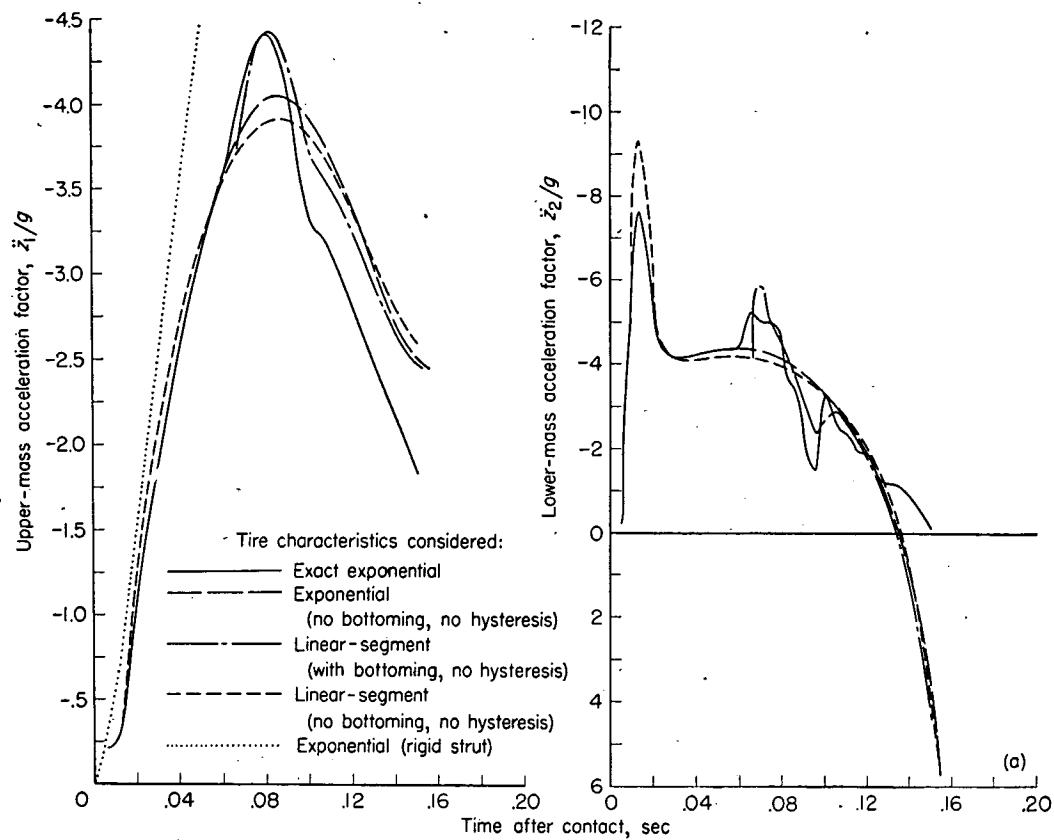
(b) Time histories of landing-gear displacements and velocities.

FIGURE 6.—Continued.



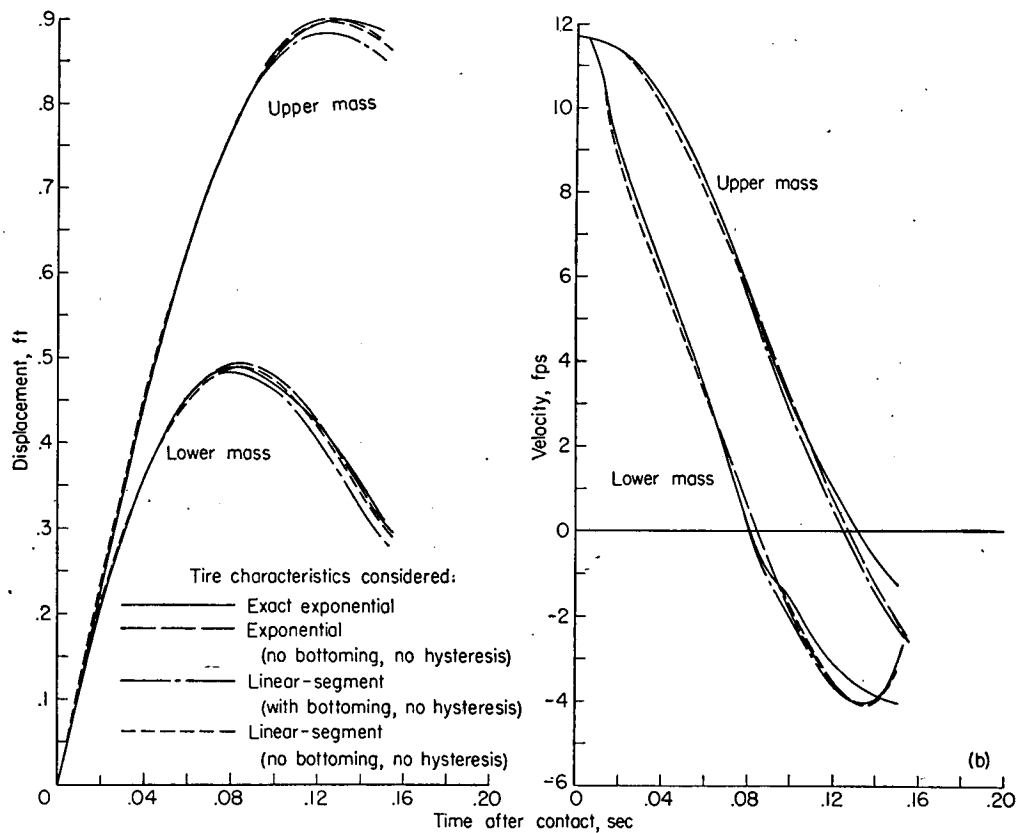
(c) Time histories of shock-strut stroke and velocity.

FIGURE 6.—Concluded.



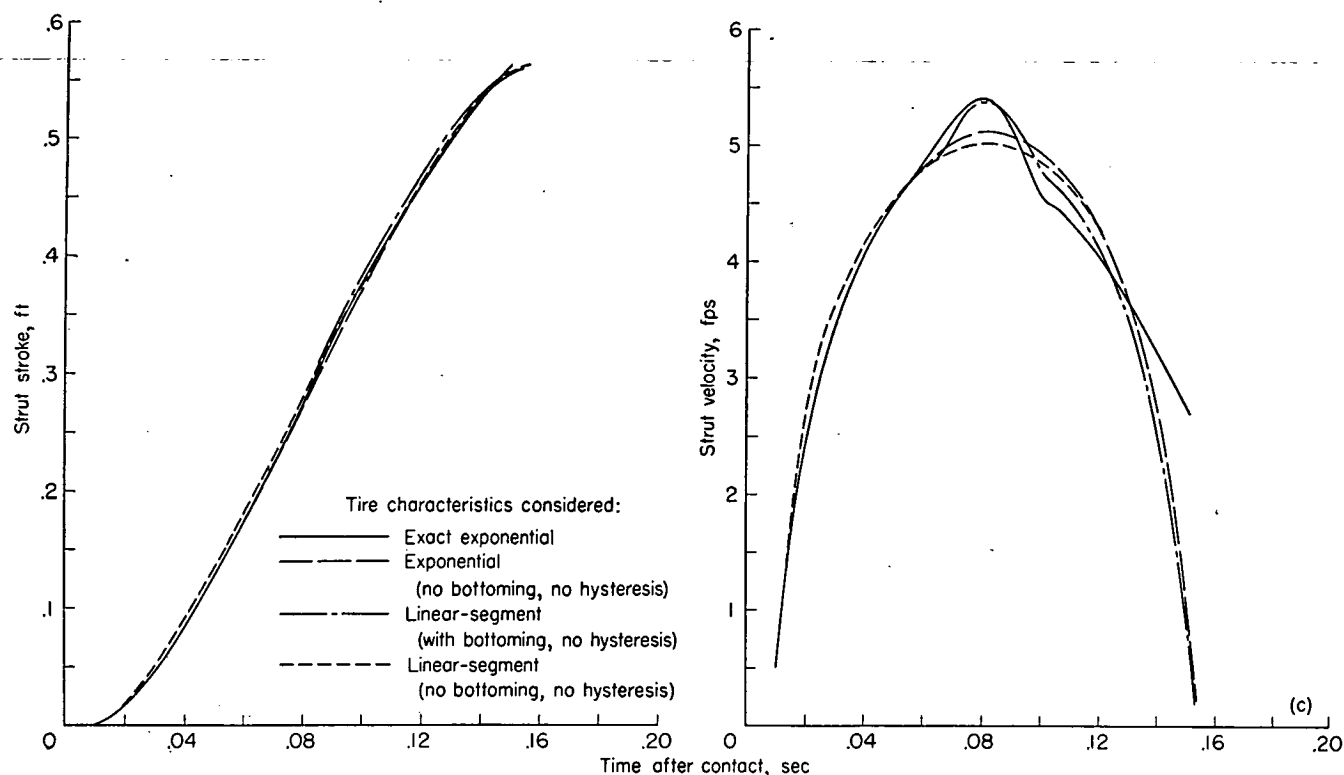
(a) Time histories of upper-mass acceleration and lower-mass acceleration.

FIGURE 7.—Effect of tire characteristics on calculated landing-gear behavior for impact with tire bottoming.
 $V_{r0} = 11.63$ feet per second; $C_d = 0.9$; $n = 1.12$.



(b) Time histories of landing-gear displacements and velocities

FIGURE 7.—Continued.



(c) Time histories of shock-strut stroke and velocity.

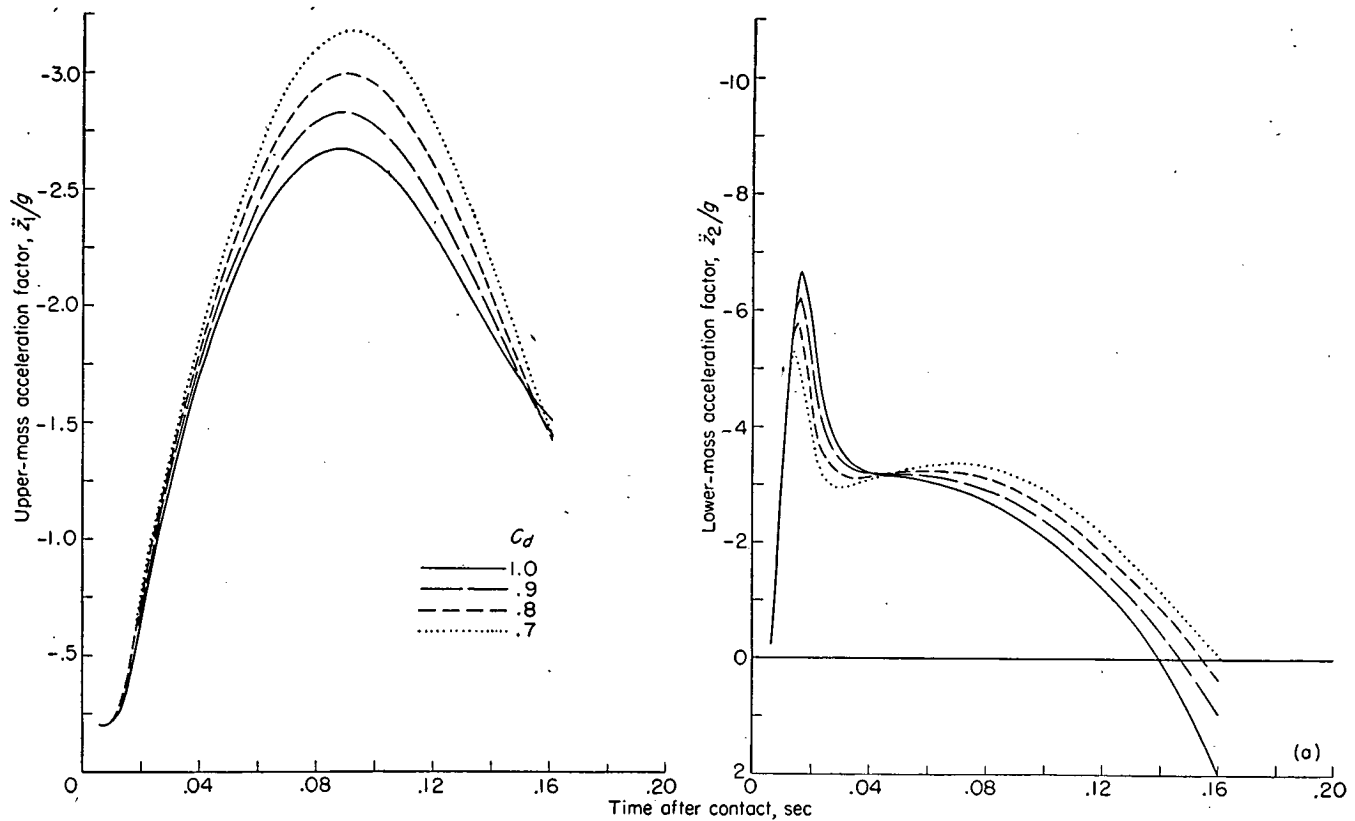
FIGURE 7.—Concluded.

Normal impact.—In the case of the normal impact at a vertical velocity of 8.86 feet per second, figure 6 shows that the solution obtained with the exponential force-deflection variation which neglects hysteresis and the solution with the linear-segment approximation to the tire characteristics are in fairly good agreement with the results of the calculation based on the exponential representation of the exact tire characteristics. The greatest differences between the solutions are evident in the time histories of upper-mass and lower-mass accelerations; considerably smaller differences are obtained for the lower-order derivatives, as might be expected. With regard to the upper-mass acceleration, the three solutions are in very good agreement during the early stages of the impact. In the case of the simplified exponential characteristics, neglect of the decreased slope of the force-deflection curve between the first break and the maximum (regime ② in fig. 3 (a)) resulted in the calculation of a somewhat higher value of the maximum upper-mass acceleration than was obtained with the exact tire characteristics. For the simplified exponential and linear-segment characteristics, neglect of hysteresis resulted in the calculation of somewhat excessive values of upper-mass acceleration subsequent to the attainment of the maximum vertical load. It is of interest to note that the calculated results for the exponential and linear-segment characteristics without hysteresis were generally in quite good agreement with each other throughout the entire duration of the impact, although the assumption of linear-segment tire force-deflection characteristics did result in somewhat excessive values for the maximum lower-mass acceleration. On the whole, the simplified tire force-deflection characteristics considered permit calculated results to be obtained which represent the behavior of the landing gear in normal impacts fairly well.

Impact with tire bottoming.—In the case of the severe impact at a vertical velocity of 11.63 feet per second, the effects of tire bottoming on the upper-mass acceleration, the lower-mass acceleration, and the strut telescoping velocity are clearly indicated in figure 7 by the calculated results based on the exact tire characteristics. As can be seen, the linear-segment approximation to the tire deflection characteristics which takes into account the effects of tire bottoming resulted in a reasonably good representation of the landing-gear behavior throughout most of the time history. On the other hand, as might be expected, the calculations which neglected the effects of bottoming on the tire force-deflection characteristics did not reveal the marked increase in the upper-mass acceleration due to the increased stiffness of the tire subsequent to the occurrence of bottoming. It is also noted that the discrepancies in the calculated upper-mass acceleration due to neglect of hysteresis in the later stages of the impact are more pronounced in this case than in the impact without tire bottoming previously considered, again as might be expected.

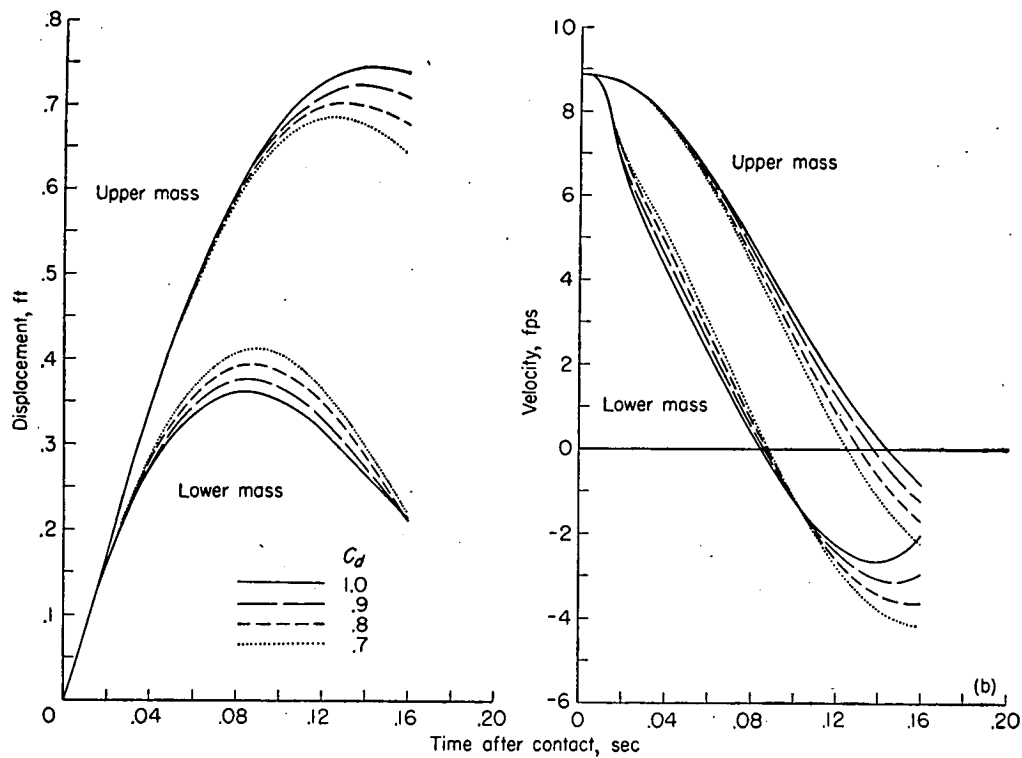
EFFECT OF ORIFICE DISCHARGE COEFFICIENT

In view of the fact that there is very little information available regarding the magnitude of discharge coefficients for orifices in landing gears, it appears desirable to evaluate the effect which differences in the magnitude of the orifice coefficient can have on the calculated results. Figure 8 presents comparisons of calculated results for a range of values of the orifice discharge coefficient C_d between 1.0 and 0.7. The four solutions presented are for the same set of initial conditions as the normal impact without tire bottoming previously considered and are based on the exponential tire force-deflection characteristics which neglect hysteresis.



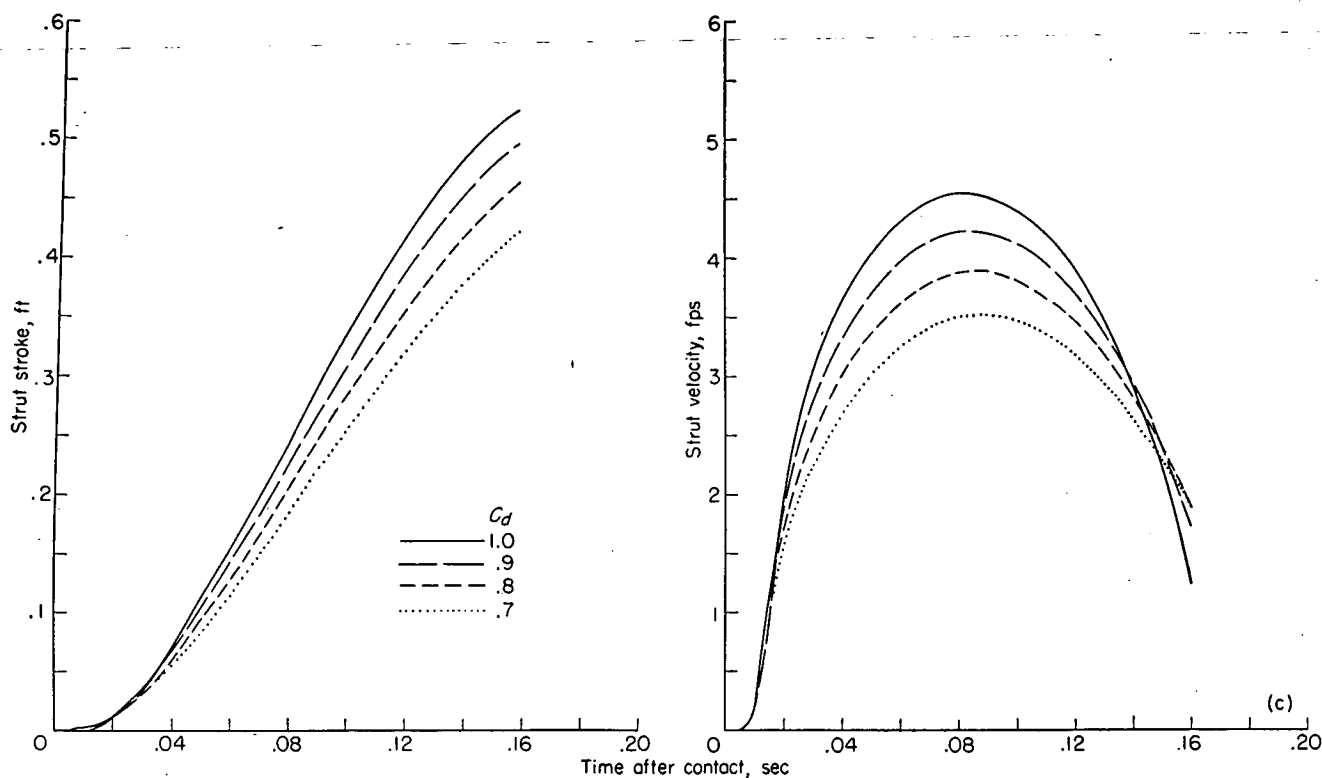
(a) Time histories of upper-mass acceleration and lower-mass acceleration.

FIGURE 8.—Effect of orifice discharge coefficient on landing-gear behavior; calculations with exponential tire characteristics without hysteresis. $V_{V_0}=8.86$ feet per second; $n=1.12$.



(b) Time histories of landing-gear displacements and velocities.

FIGURE 8.—Continued.



(c) Time histories of shock-strut stroke and velocity.

FIGURE 8.—Concluded.

These calculations show that a decrease in the orifice discharge coefficient results in an approximately proportional increase in the upper-mass acceleration. This variation is to be expected since the smaller coefficients correspond to reduced effective orifice areas which result in greater shock-strut forces due to increased hydraulic resistance. As a result of the increased shock-strut force acting downward on the lower mass, the maximum upward acceleration of the lower mass is reduced with decreasing values of the discharge coefficient. The increase in shock-strut force with decreasing discharge coefficient also results in a decrease in the strut stroke and telescoping velocity but an increase in the lower-mass velocity and displacement, as might be expected. However, since the increases in lower-mass displacement and velocity are smaller than the decreases in strut stroke and telescoping velocity, the upper-mass displacement and velocity are reduced with decreasing orifice discharge coefficient.

These comparisons show that the magnitude of the orifice coefficient has an important effect on the behavior of the landing gear and indicates that a fairly accurate determination of the numerical value of this parameter is necessary to obtain good results.

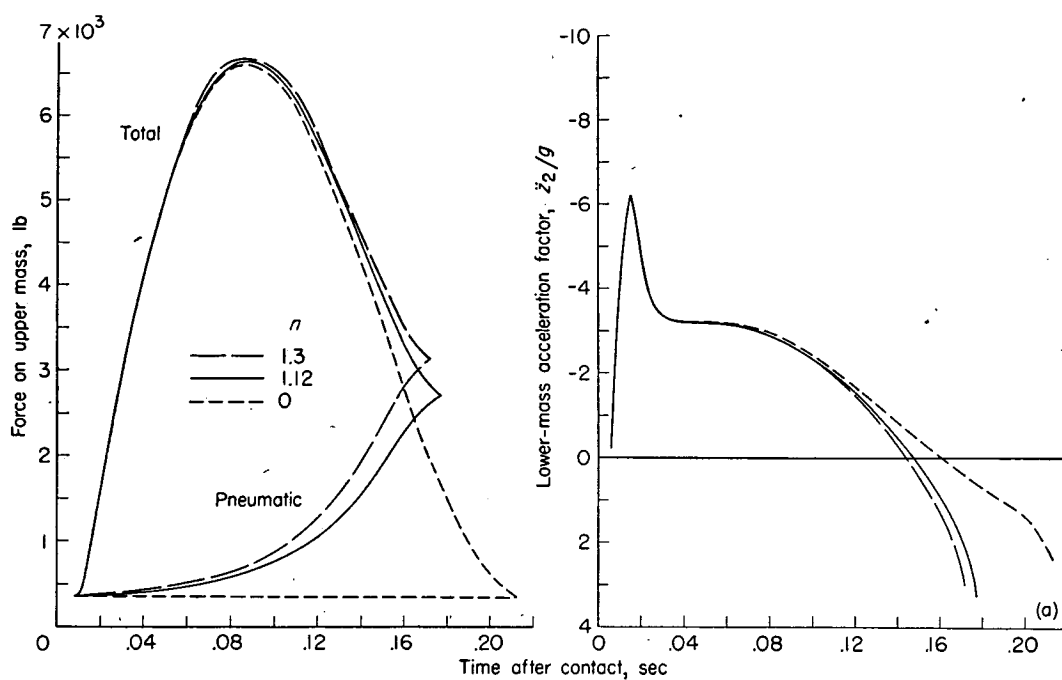
EFFECT OF AIR-COMPRESSION PROCESS

Since the nature of the air-compression process in a shock strut is not well-defined and different investigators have assumed values for the polytropic exponent ranging anywhere between the extremes of 1.4 (adiabatic) and 1.0

(isothermal), it appeared desirable to evaluate the importance of the air-compression process and to determine the extent to which different values of the polytropic exponent can influence the calculated results. Consequently, solutions have been obtained for three different values of the polytropic exponent; namely, $n=1.3$, 1.12, and 0.

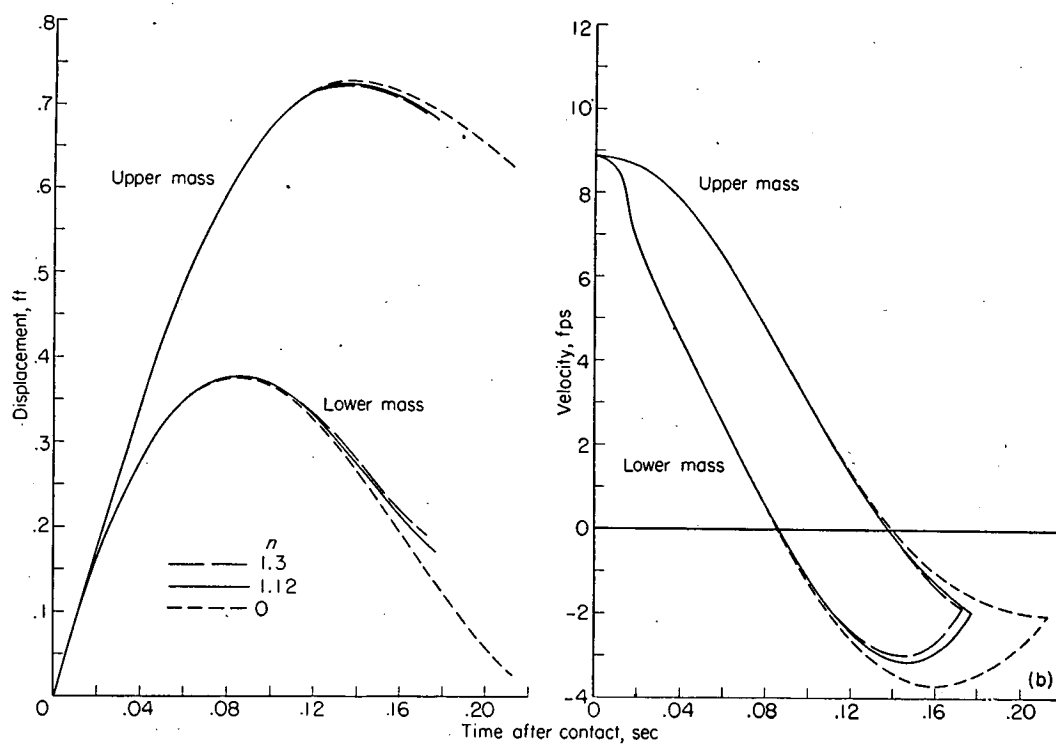
The value $n=1.3$ corresponds to a very rapid compression in which an adiabatic process is almost attained. The value $n=1.12$ corresponds to a relatively slow compression in which the process is virtually isothermal. The value $n=0$ is completely fictitious since it implies constant air pressure within the strut throughout the impact. The assumption $n=0$ has been considered since it makes one of the terms in the equations of motion a constant and permits simplification of the calculations. The three solutions presented are for the same set of initial conditions as the normal impact without tire bottoming previously considered and are based on the exponential tire force-deflection characteristics which neglect hysteresis.

Figure 9 shows that the air pressure contributes only a relatively small portion of the total shock-strut force throughout most of the impact since the compression ratio is relatively small until the later stages of the impact. Toward the end of the impact, however, the air-pressure force becomes a large part of the total force since the compression ratio becomes large, whereas the hydraulic resistance decreases rapidly as the strut telescoping velocity is reduced to zero.



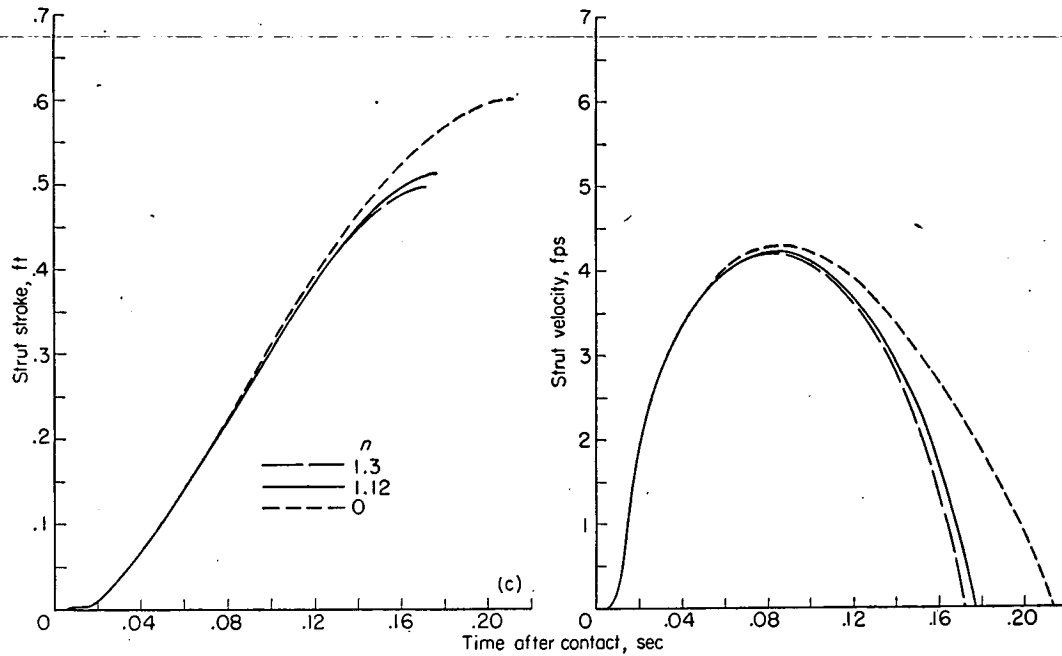
(a) Time histories of upper-mass force and lower-mass acceleration.

FIGURE 9.—Effect of polytropic exponent; calculations with exponential tire characteristics without hysteresis.
 $V_{r0} = 8.86$ feet per second; $C_d = 0.9$.



(b) Time histories of landing-gear displacements and velocities.

FIGURE 9.—Continued.



(c) Time histories of shock-strut stroke and velocity.

FIGURE 9.—Concluded.

As a result, the calculations show that the magnitude of the polytropic exponent has only a very small effect on the behavior of the landing gear throughout most of the impact. For the practical range of polytropic exponents, variations in the air-compression process result in only minor differences in landing-gear behavior, even during the very latest stages of the impact. The assumption of constant air pressure in the strut throughout the impact ($n=0$), however, does lead to the calculation of excessive values of stroke and of the time to reach the maximum stroke. The time history of the shock-strut force calculated on the basis of this assumption is, on the other hand, in quite good agreement with the results for the practical range of air-compression processes.

On the whole it appears that the behavior of the landing gear is relatively insensitive to variations in the air-compression process. The foregoing results suggest that, in many cases, fairly reasonable approximations for the landing-gear force-time variation might be obtained even if the air-pressure term in the equations of motion were completely neglected.

SIMPLIFICATION OF EQUATIONS OF MOTION

The preceding studies have indicated that variations in the tire force-deflection characteristics and in the air-compression process individually have only a relatively minor effect on the calculated behavior of the landing gear. These results suggest that the equations of motion for the landing gear might be simplified by completely neglecting the internal air-pressure forces in the shock strut and by considering the tire force-deflection characteristics to be linear. With these assumptions, the equations of motion for the upper mass, lower mass, and complete system (eqs. (16), (17), and (8)) can

be written as follows for the case where the wing lift is equal to the weight and the internal friction is neglected:

$$\left. \begin{aligned} \frac{W_1}{g} \ddot{z}_1 + A(\dot{z}_1 - \dot{z}_2)^2 + W_2 &= 0 \\ \frac{W_2}{g} \ddot{z}_2 - A(\dot{z}_1 - \dot{z}_2)^2 + a z_2 + b - W_2 &= 0 \\ \frac{W_1}{g} \ddot{z}_1 + \frac{W_2}{g} \ddot{z}_2 + a z_2 + b &= 0 \end{aligned} \right\} \quad (18)$$

where

$$A = \frac{\dot{s}}{|\dot{s}|} \frac{\rho A_h^3}{2 (C_d A_n)^2 \cos \varphi}$$

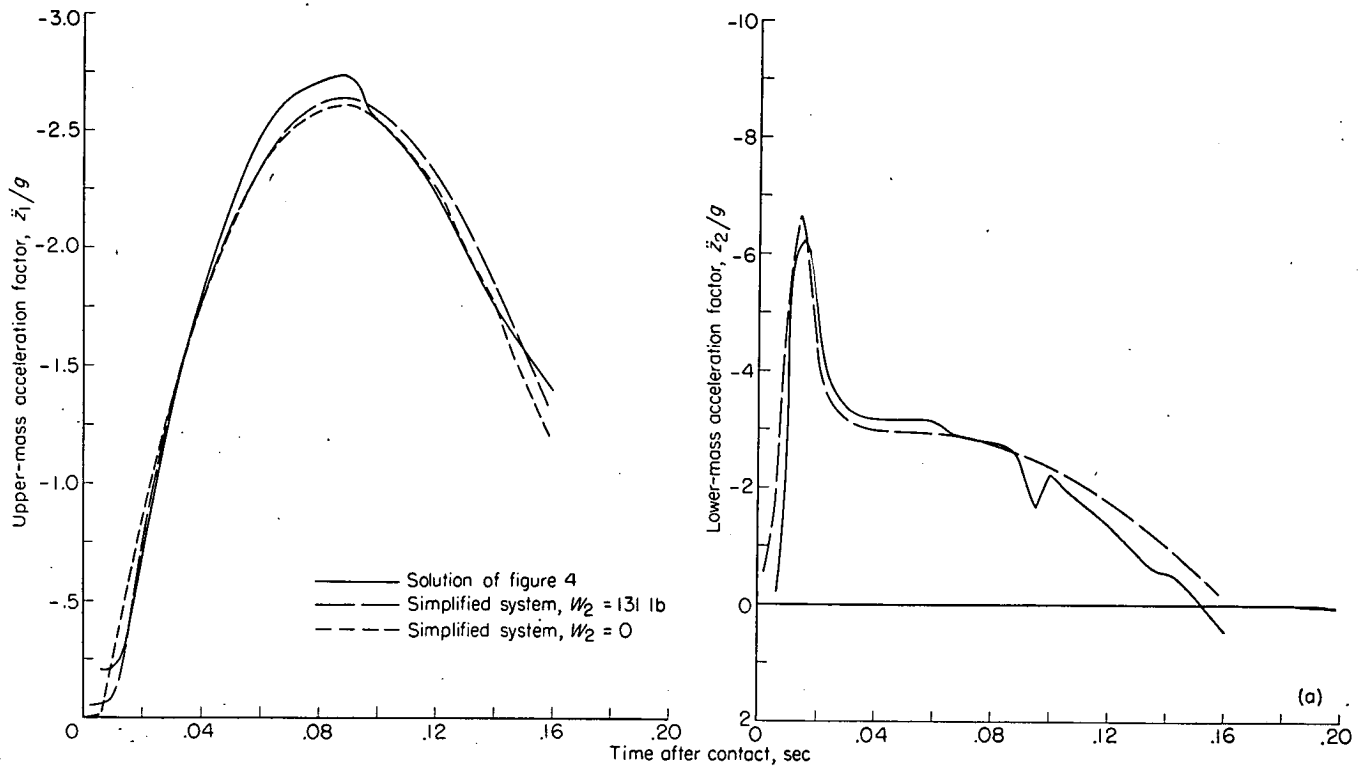
and

- a slope of linear approximation to tire force-deflection characteristics
- b value of force corresponding to zero tire deflection, as determined from the linear-segment approximation to the tire force-deflection characteristics

The motion variables at the beginning of shock-strut deflection can be readily determined in a manner similar to that employed in the more general treatment previously discussed. For the simplified equations the variables at the instant t_r are given by

$$\left. \begin{aligned} \ddot{z}_r &= -\frac{W_2}{W_1} g \\ z_r &= \frac{W_2}{a} \left(1 + \frac{W_2}{W_1} \right) \\ \dot{z}_r &= \sqrt{\dot{z}_0^2 - \frac{ag}{W} z_r^2} \end{aligned} \right\} \quad (19)$$

In most cases the term $\frac{ag}{W} z_r^2$ is small in comparison with $\dot{z}_0^2$ so that $\dot{z}_r \approx \dot{z}_0$.



(a) Time histories of upper-mass acceleration and lower-mass acceleration.

FIGURE 10.—Evaluation of calculated results for simplified systems. $V_{r0}=8.86$ feet per second; $C_d=0.9$.

The values determined from equations (19) are used as initial conditions in the solution of equations (18).

The fact that the lower mass is a relatively small fraction of the total mass suggests that the system might be simplified even further without greatly modifying the calculated results by assuming the lower mass to be equal to zero. With this assumption $t_r=0$ and the initial values of the variables in equations (18) correspond to the conditions at initial contact.

EVALUATION OF SIMPLIFICATION

In order to evaluate the applicability of these simplifications, the behavior of the landing gear has been calculated in accordance with equations (18) for an impact with an initial vertical velocity of 8.86 feet per second. A similar calculation has been made with the assumption $W_2=0$. These results are compared in figure 10 with the more exact solutions previously presented in figure 4, which include consideration of the air-compression springing and the exact exponential tire characteristics. A time history of the lower-mass acceleration is not presented for the case where W_2 is assumed equal to zero since the values of $\ddot{z}_2/g$ have no significance in this case.

Figure 10 shows that the two simplified solutions are in quite good agreement with each other, as might be expected, and are also in fairly good agreement with the more exact results. Neglecting the air-pressure forces and assuming a linear tire force-deflection variation resulted in the calculation of slightly lower values for the maximum upper-mass acceleration and somewhat higher values for the maximum stroke than were obtained with the more exact equations.

The effect of neglecting the lower mass was primarily to reduce the lower-mass displacement (tire deflection), as a result of the elimination of the lower-mass inertia reaction.

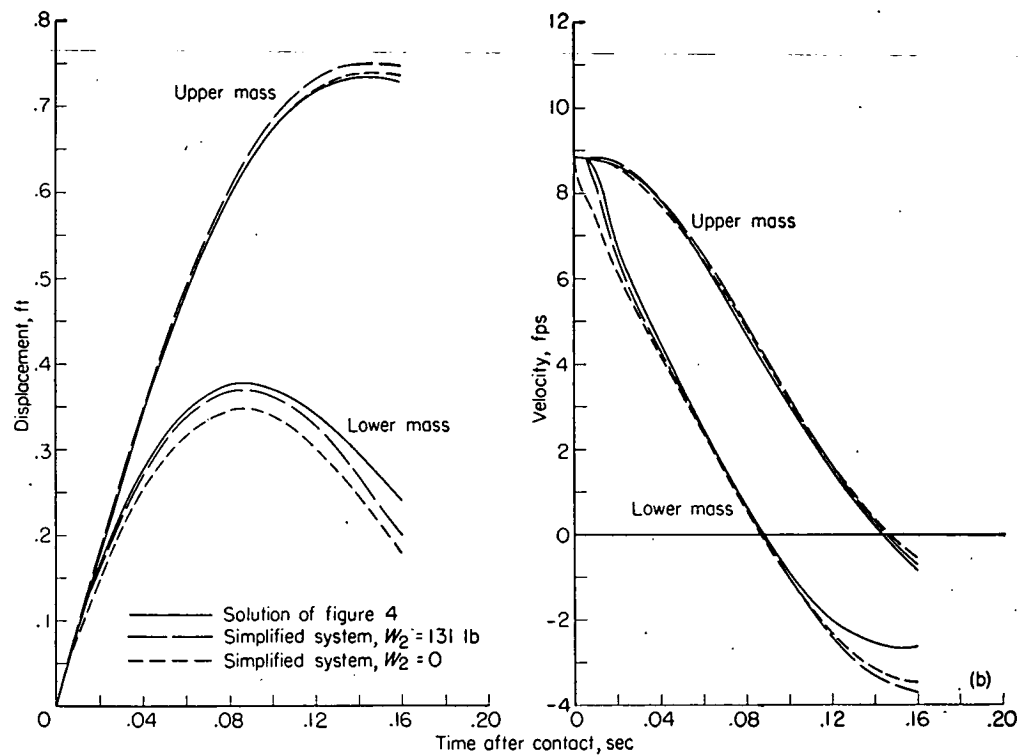
On the whole, it appears that the assumptions considered permit appreciable simplification of the equations of motion without greatly impairing the validity of the calculated results.

GENERALIZED TREATMENT

Equations and solutions.—By writing the simplified equations of motion in terms of dimensionless variables, generalized solutions can be obtained for a wide range of landing-gear and impact parameters which may be useful in preliminary design. If W_2 is taken equal to zero and it is further assumed that the tire force-deflection curve is represented by a single straight line through the origin ($b=0$ throughout the impact), equations (18) reduce to

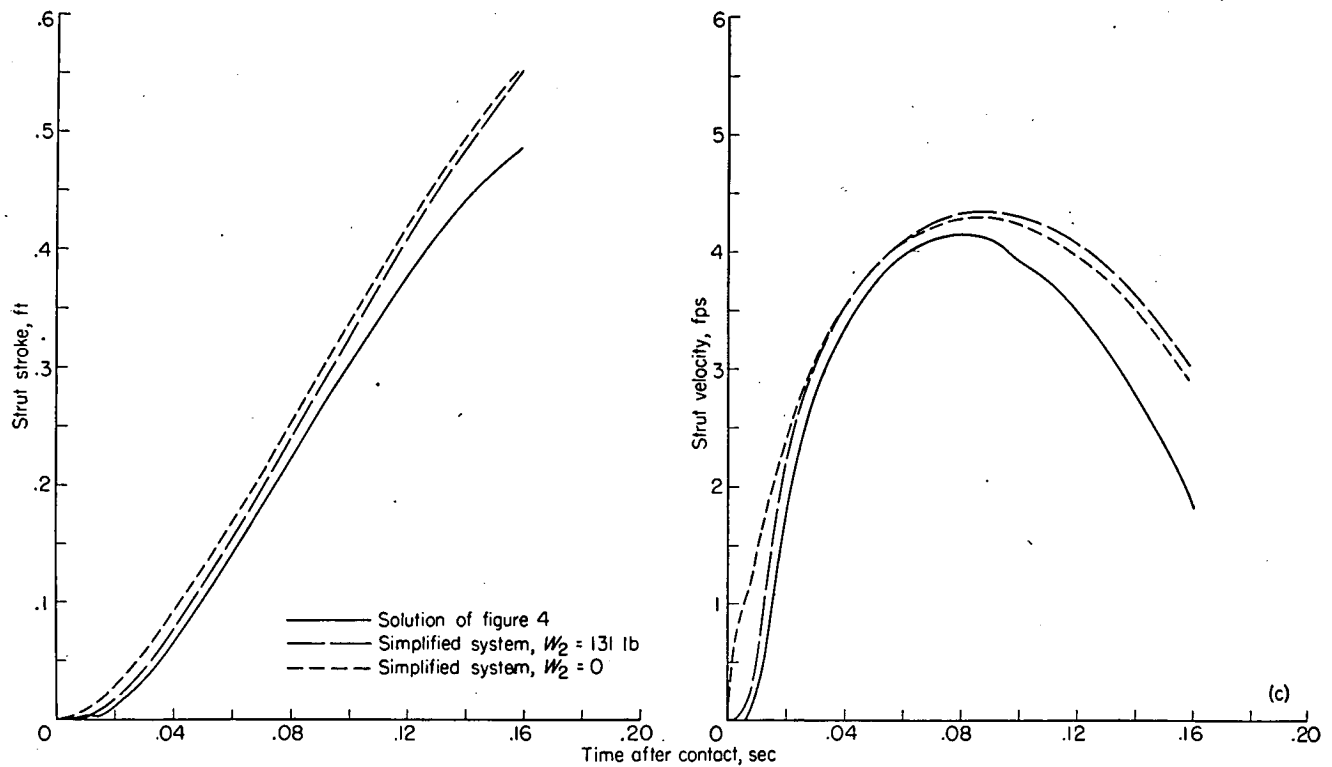
$$\left. \begin{aligned} \frac{W_1}{g} \ddot{z}_1 + A(\dot{z}_1 - \dot{z}_2)^2 &= 0 \\ A(\dot{z}_1 - \dot{z}_2)^2 - a z_2 &= 0 \\ \frac{W_1}{g} \ddot{z}_1 + a z_2 &= 0 \end{aligned} \right\} \quad (20)$$

where $\frac{W_1}{g}$, A , and a are constants, as previously defined, and any two of the foregoing equations are sufficient to describe completely the behavior of the system. With this representation of the system, the shock strut begins to deflect at the instant of initial contact ($t_r=0$). Thus, the initial conditions for equations (20) are the initial impact conditions; namely, $z_{10}=z_{20}=0$ and $\dot{z}_{10}=\dot{z}_{20}=\dot{z}_0$.



(b) Time histories of landing-gear displacements and velocities.

FIGURE 10.—Continued.



(c) Time histories of shock-strut stroke and velocity.

FIGURE 10.—Concluded.

As can be seen, with the equations in this form, the solution depends on five parameters, namely, $\frac{W_1}{g}$, A , a , and the initial conditions $\dot{z}_0$ and z_0 . However, since $z_0=0$ in all cases, the number of variable parameters is reduced to four. In view of the fact that these parameters are independent of one another and each may take on a large range of values, a great many solutions and a large number of graphs would be required to cover the entire range of landing-gear and impact parameters with the equations in the form of equations (20).

The number of independent parameters which have to be considered may be greatly decreased by the introduction of generalized dimensionless variables and the corresponding transformations of equations (20). In this case, generalized variables can be obtained which permit transformation of the equations of motion to a form which does not involve any constants. With the equations in this form, there is only one variable parameter, namely, the initial velocity parameter. To determine the generalized variables which satisfy the aforementioned requirements, let

$$u = z\alpha$$

and

$$\theta = t\beta$$

Thus,

$$u' = \frac{du}{d\theta} = \dot{z} \frac{\alpha}{\beta}$$

and

$$u'' = \frac{du'}{d\theta} = \ddot{z} \frac{\alpha}{\beta^2}$$

Substituting these new variables permits equations (20) to be written as

$$\left. \begin{aligned} (u_1' - u_2')^2 + \left(\frac{\alpha}{A} \frac{W_1}{g}\right) u_1'' &= 0 \\ (u_1' - u_2')^2 - \left(\frac{a}{A} \frac{\alpha}{\beta^2}\right) u_2 &= 0 \\ u_1'' + \left(\frac{a/\beta^2}{W_1/g}\right) u_2 &= 0 \end{aligned} \right\} \quad (20a)$$

The number of independent parameters will be reduced if all the combined constants in equations (20a) are set equal to one another, that is, let

$$\frac{\alpha}{A} \frac{W_1}{g} = \frac{a}{A} \frac{\alpha}{\beta^2} = \frac{a/\beta^2}{W_1/g}$$

From this relationship, it can be seen that

$$\alpha = \frac{A}{W_1/g}$$

and

$$\beta = \sqrt{\frac{a}{W_1/g}}$$

Thus the generalized variables become

$$u_1 = z_1 \left(\frac{A}{W_1/g} \right) \quad u_2 = z_2 \left(\frac{A}{W_1/g} \right)$$

$$u_1' = \frac{du_1}{d\theta} = \dot{z}_1 \sqrt{\frac{A^2/a}{W_1/g}} \quad u_2' = \frac{du_2}{d\theta} = \dot{z}_2 \sqrt{\frac{A^2/a}{W_1/g}}$$

and

$$u_1'' = \frac{d^2 u_1}{d\theta^2} = \ddot{z}_1 \left(\frac{A}{a} \right)$$

where

$$\theta = t \sqrt{\frac{a}{W_1/g}}$$

With these new variables equations (20) can be written as²

$$\left. \begin{aligned} (u_1' - u_2')^2 + u_1'' &= 0 \\ (u_1' - u_2')^2 - u_2 &= 0 \\ u_1'' + u_2 &= 0 \end{aligned} \right\} \quad (21)$$

where any two of these equations are sufficient to describe the behavior of the system.

Inasmuch as equations (21) do not involve any constants, their solutions are completely determined by the initial values of the variables. Since the displacements at initial contact u_{10} and u_{20} are equal to zero and the initial velocities u_{10}' and u_{20}' are equal, the only parameter is the initial dimensionless velocity

$$u_{0'} = \dot{z}_0 \sqrt{\frac{A^2/g}{W_1 a}}$$

where $u_{0'} = u_{10}' = u_{20}'$.

Generalized solutions of equations (21) are presented in figure 11 for values of $u_{0'}$ corresponding to a wide range of landing-gear and impact parameters. Parts (a) to (e) of figure 11 show the variations of the dimensionless variables during the impact; parts (f) and (g) show the maximum values of the more important variables as functions of $u_{0'}$. Part (h) shows the shock-strut effectiveness η_s and the landing-gear effectiveness η_{lg} . The shock-strut effectiveness, sometimes called "efficiency" and, in Europe, "planimetric ratio," is defined as

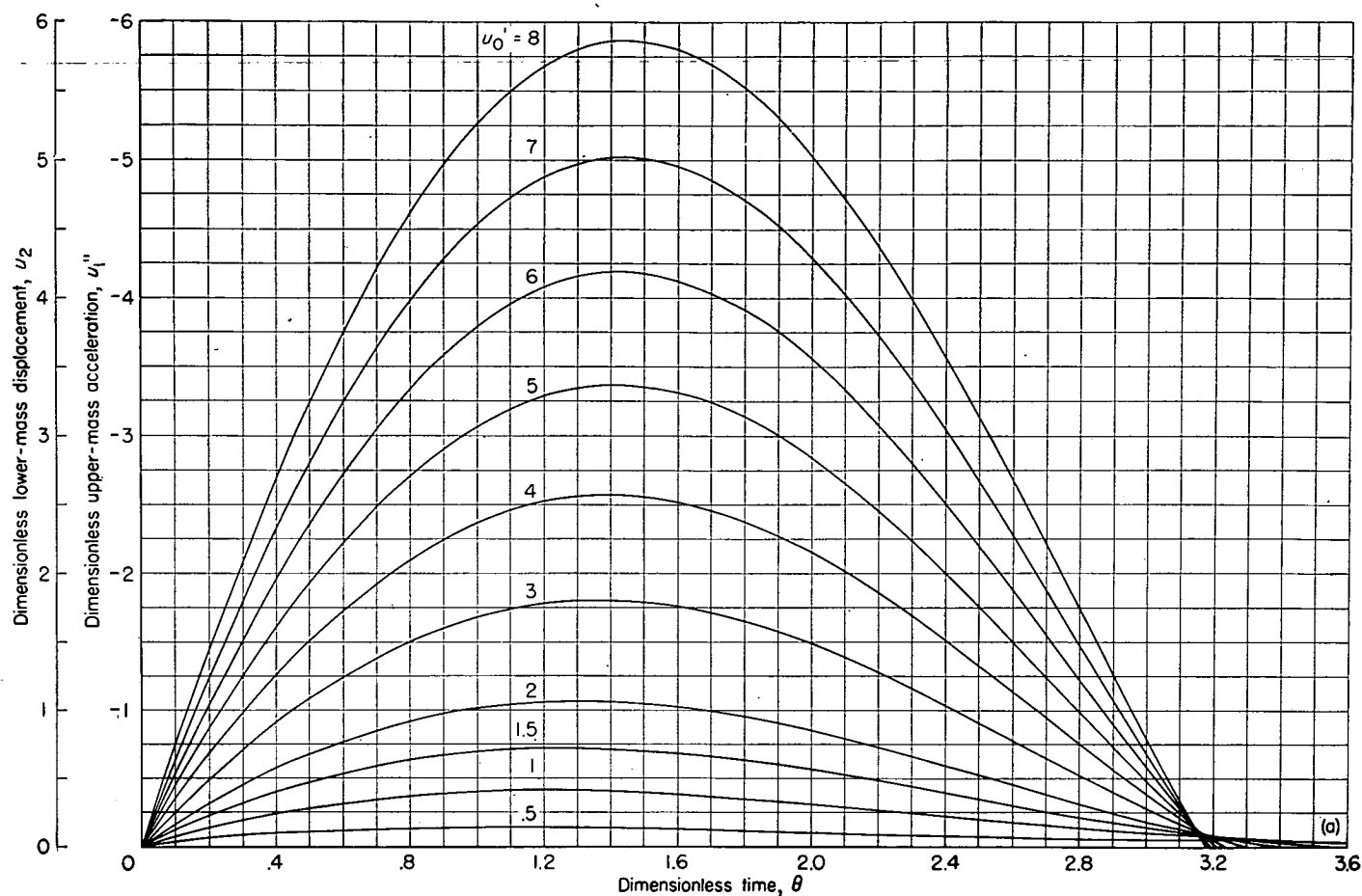
$$\eta_s = \frac{\int_0^{\sigma_{max}} u_1'' d\sigma}{u_1''_{max} \sigma_{max}}$$

where $\sigma = u_1 - u_2$ is the dimensionless shock-strut stroke. Since η_s represents the ratio of the energy actually absorbed by the shock strut to the maximum energy which the strut could possibly absorb for any combination of maximum acceleration (or load) and maximum stroke, it serves as a measure of the extent to which a given combination of maximum load and stroke has been utilized to absorb the energy of an impact. A similar measure of the energy absorption effectiveness of the landing gear as a whole is given by η_{lg} which is defined by

$$\eta_{lg} = \frac{\int_0^{u_{1max}} u_1'' du_1}{u_1''_{max} u_{1max}}$$

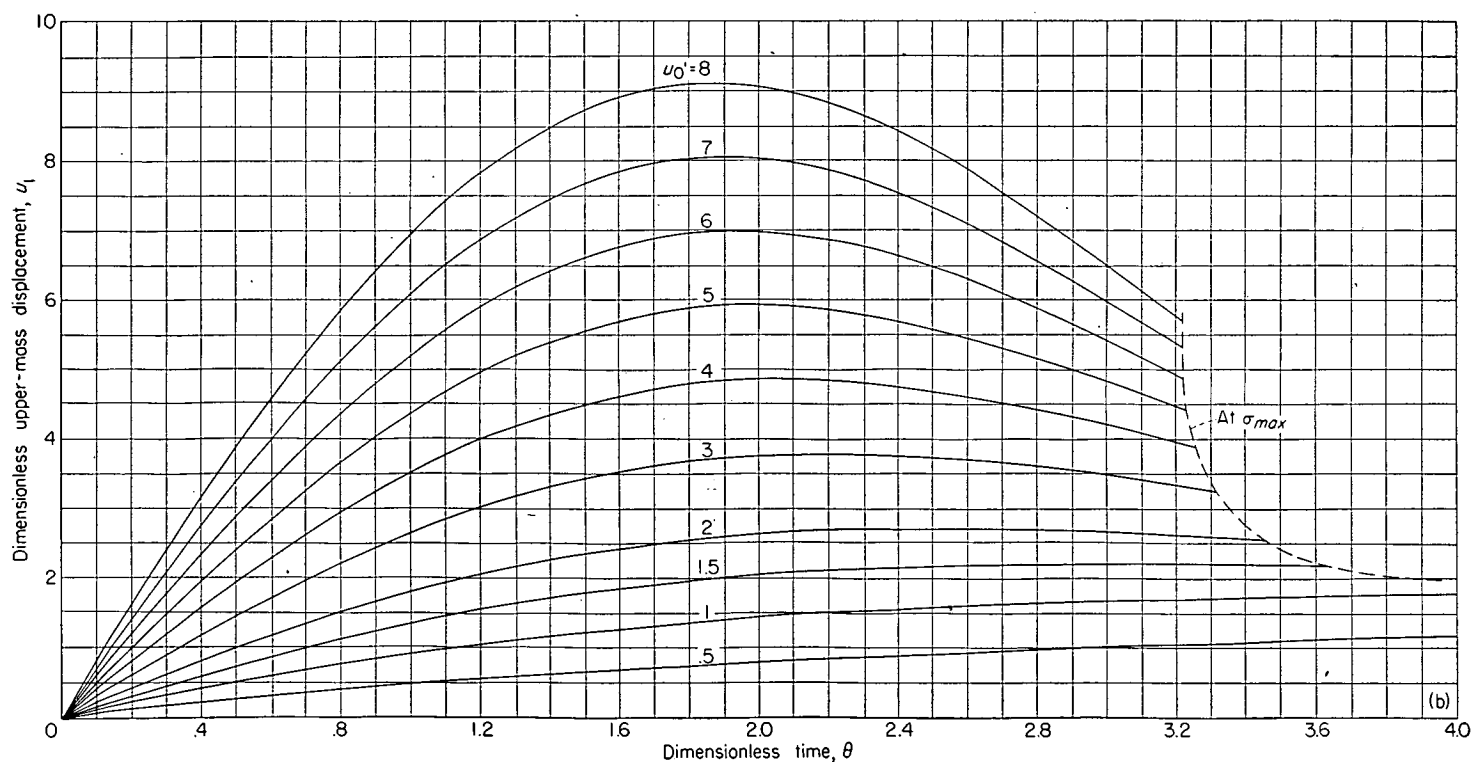
² Equations (21) may be reduced to a single equation in one variable by differentiating the last equation and substituting for u_1' in the first equation. This gives $(u_1' + u_1''')^2 + u_1'' = 0$.

By introducing the new variable $w = u_1'$, this equation may be reduced to the second-order equation $(w + w'')^2 + w' = 0$, subject to the initial conditions $w_0 = u_{0'}$ and $w_0' = 0$.



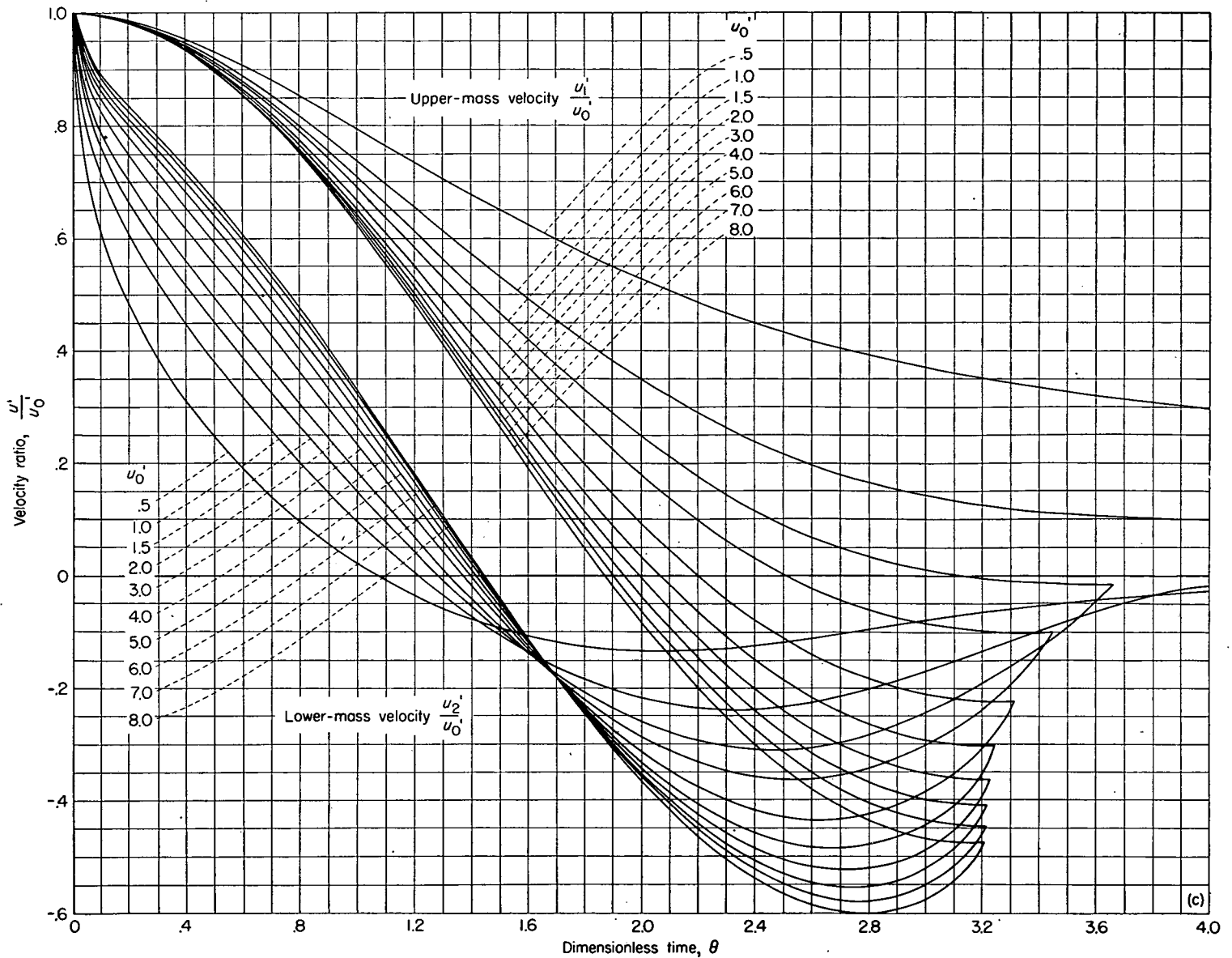
(a) Relationship between upper-mass acceleration, lower-mass displacement, and time.

FIGURE 11.—Generalized solutions for simplified system.



(b) Relationship between upper-mass displacement and time.

FIGURE 11.—Continued.



(c) Relationship between upper-mass velocity, lower-mass velocity, and time.

FIGURE 11.—Continued.

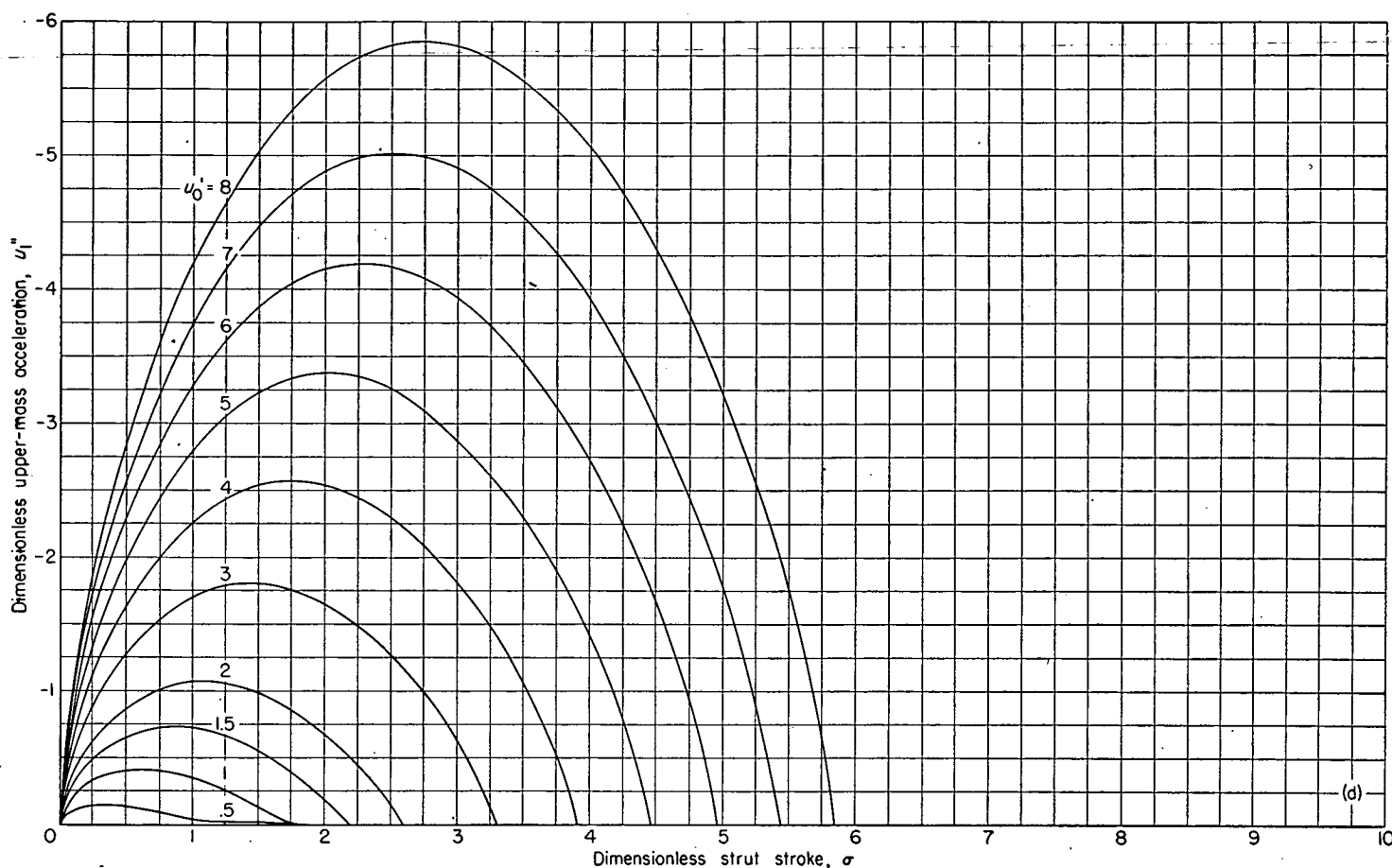
The generalized results presented in figure 11³ can be used to estimate the performance of a given landing gear of known configuration for particular impact conditions or to choose the dimensions for a landing gear when the impact conditions and desired performance are specified.

Applicability of solutions.—To illustrate the applicability of the generalized solutions, the curves of figure 11 have been applied to the previously considered case of the normal impact at an initial vertical velocity of 8.86 feet per second for comparison with the more exact solution presented in figure 4. In order to make use of the generalized solutions

³ Although time-history solutions are presented for values of u_0' as small as 0.5, it will be noted that values of u_{1max} , σ_{max} , η_s , and η_{lg} are not given for values of $u_0' < 1.5$. It can be seen from the time histories that the characteristics of the solution in the later stages of the impact change as u_0' becomes small; in particular, u_1 increases and the curve of u_{1max} as a function of u_0' appears to reach a minimum at some value of $u_0' \leq 1.5$. Furthermore, the later stages of the solutions greatly stretch out in time and appear to be almost asymptotic in character. Several different analytical, numerical, and analogue methods were applied in an attempt to study this phase of the problem further but the extremely slow rate of change of the variables in this region prevented successful completion of the solutions.

it is first necessary to approximate the tire force-deflection characteristics by a simple linear variation. Two such linear approximations which might be considered suitable for this purpose are shown in figure 12. Linear approximation I is a straight line through the origin having a slope $a = 18.5 \times 10^3$ pounds per foot ($a' = ad = 41.6 \times 10^3$ lb). This value of a and the other pertinent landing-gear and impact parameters result in a value of the initial dimensionless velocity parameter $u_0' = 2.57$. Linear approximation II is a straight line with slope $a = 21.3 \times 10^3$ pounds per foot ($a' = 47.9 \times 10^3$ lb) which does not pass through the origin but intersects the displacement axis at a value of $z_2(F_{V_g=0}) = 0.0508$ foot. With this value of a , $u_0' = 2.39$.

Since the solutions of figure 11 have been calculated only for integral values of u_0' , curves for the foregoing values of u_0' were graphically interpolated by cross plotting. These results were then converted to dimensional values by multi-



(d) Relationship between upper-mass acceleration and strut stroke.

FIGURE 11.—Continued.

plying the dimensionless variables by the appropriate constants. The results obtained are compared in figure 13 with the more exact solution presented in figure 4. The values based on linear approximation I have been plotted exactly as determined from the generalized solutions. The results for linear approximation II, however, have been displaced relative to the origin of coordinates as indicated in the following discussion.

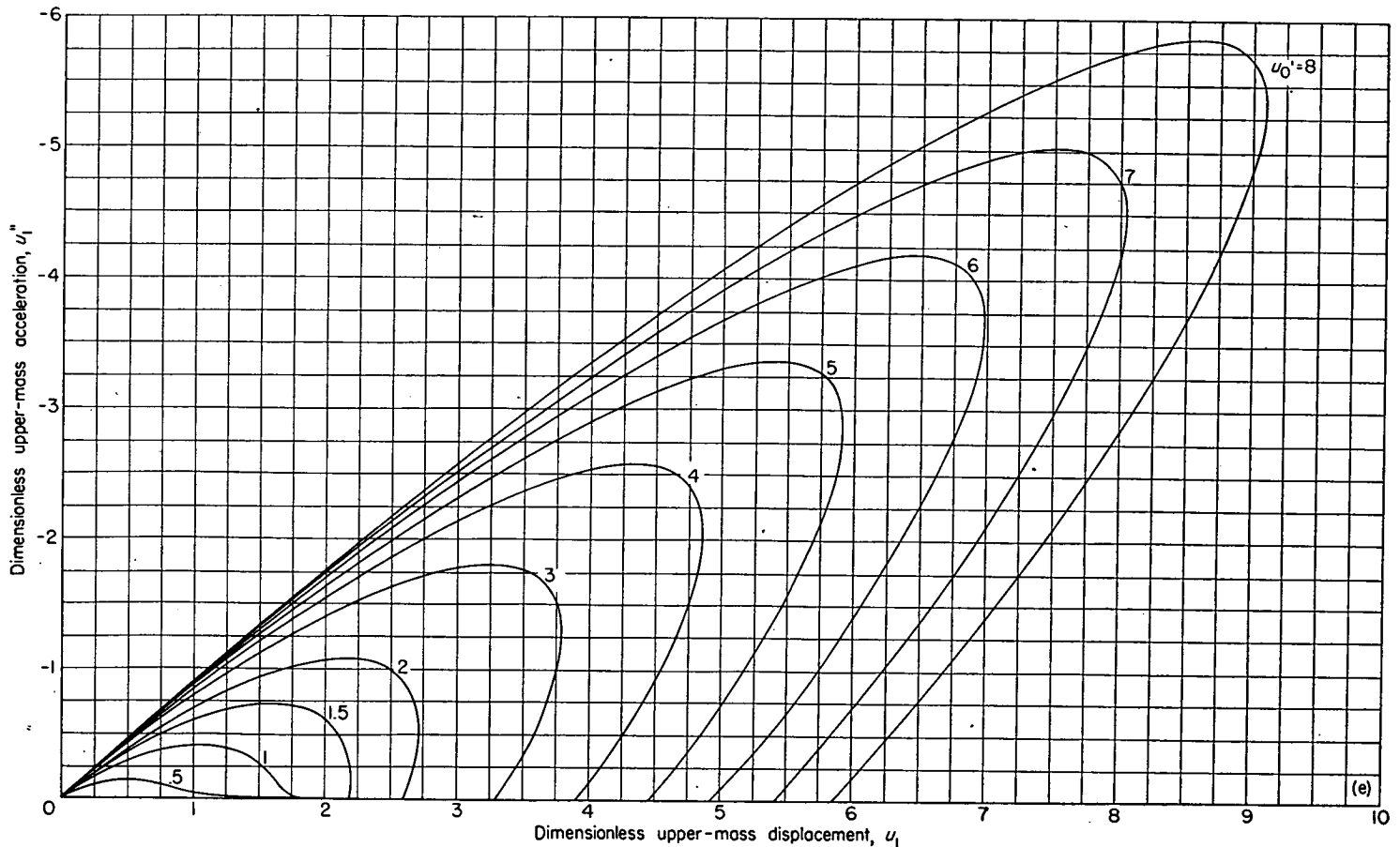
The assumption of linear approximation II implies that the system must move a distance equal to $z_{2(F_{V_g}=0)}$ after initial contact (at constant velocity since the wing lift is taken equal to the weight) before any finite ground reaction can develop. The derivation of the equations of motion, on the other hand, assumes that the ground reaction increases linearly with deflection from the instant of initial contact. As a result, the equations of motion do not apply until after the system has attained a displacement equal to $z_{2(F_{V_g}=0)}$, which occurs at a time after initial contact

$t = \frac{z_{2(F_{V_g}=0)}}{V_{V_0}}$. In other words, the equations of motion apply to a coordinate system transformed so that the tire force-deflection relationship passes through the origin; that is, a coordinate system displaced by $z_{2(F_{V_g}=0)}$ relative to

the coordinate system originating at the point of initial contact. It therefore follows that the upper-mass and lower-mass displacements determined from the generalized solutions for the case of linear approximation II must be increased by a constant amount equal to $z_{2(F_{V_g}=0)}$, in this case 0.0508 foot, and all results must be displaced in

time by a constant increment $\Delta t = \frac{z_{2(F_{V_g}=0)}}{V_{V_0}}$, in this case $\frac{0.0508}{8.86} = 0.0057$ second, relative to the instant of initial contact. These corrections have been incorporated in plotting the curves for linear approximation II shown in figure 13.

As can be seen, the results obtained by application of the generalized solutions, particularly by the method employing linear approximation II, are in fairly good agreement with the more exact solution. The discrepancies which exist are attributable to the neglect of the shock-strut preloading and springing provided by the air-pressure force, neglect of the lower mass, and to differences between the very simple tire force-deflection relationships assumed and the exact tire characteristics. On the whole, it appears that the generalized results offer a means for rapidly estimating the behavior of the landing gear within reasonable limits of accuracy and may therefore be useful for preliminary design purposes.



(e) Relationship between upper-mass acceleration and upper-mass displacement.

FIGURE 11.—Continued.

SUMMARY OF RESULTS AND CONCLUSIONS

A theoretical study has been made of the behavior of the conventional type of oleo-pneumatic landing gear during the process of landing impact. The basic analysis is presented in a general form and treats the motions of the landing gear prior to and subsequent to the beginning of shock-strut deflection. In the first phase of the impact the landing gear is treated as a single-degree-of-freedom system in order to determine the conditions of motion at the instant of initial shock-strut deflection, after which instant the landing gear is considered as a system with two degrees of freedom. The equations for the two-degree-of-freedom system consider such factors as the hydraulic (velocity square) resistance of the orifice, the forces due to air compression and internal friction in the shock strut, the nonlinear force-deflection characteristics of the tire, the wing lift, the inclination of the landing gear, and the effects of wheel spin-up drag loads.

The applicability of the analysis to actual landing gears has been investigated for the particular case of a vertical landing gear in the absence of drag loads by comparing calculated results with experimental drop-test data for corresponding impact conditions, for both a normal impact and a severe impact involving tire bottoming.

Studies have also been made to determine the effects of variations in such parameters as the dynamic force-deflection

characteristics of the tire, the orifice discharge coefficient, and the effective polytropic exponent for the air-compression process, which might not be known accurately in practical design problems.

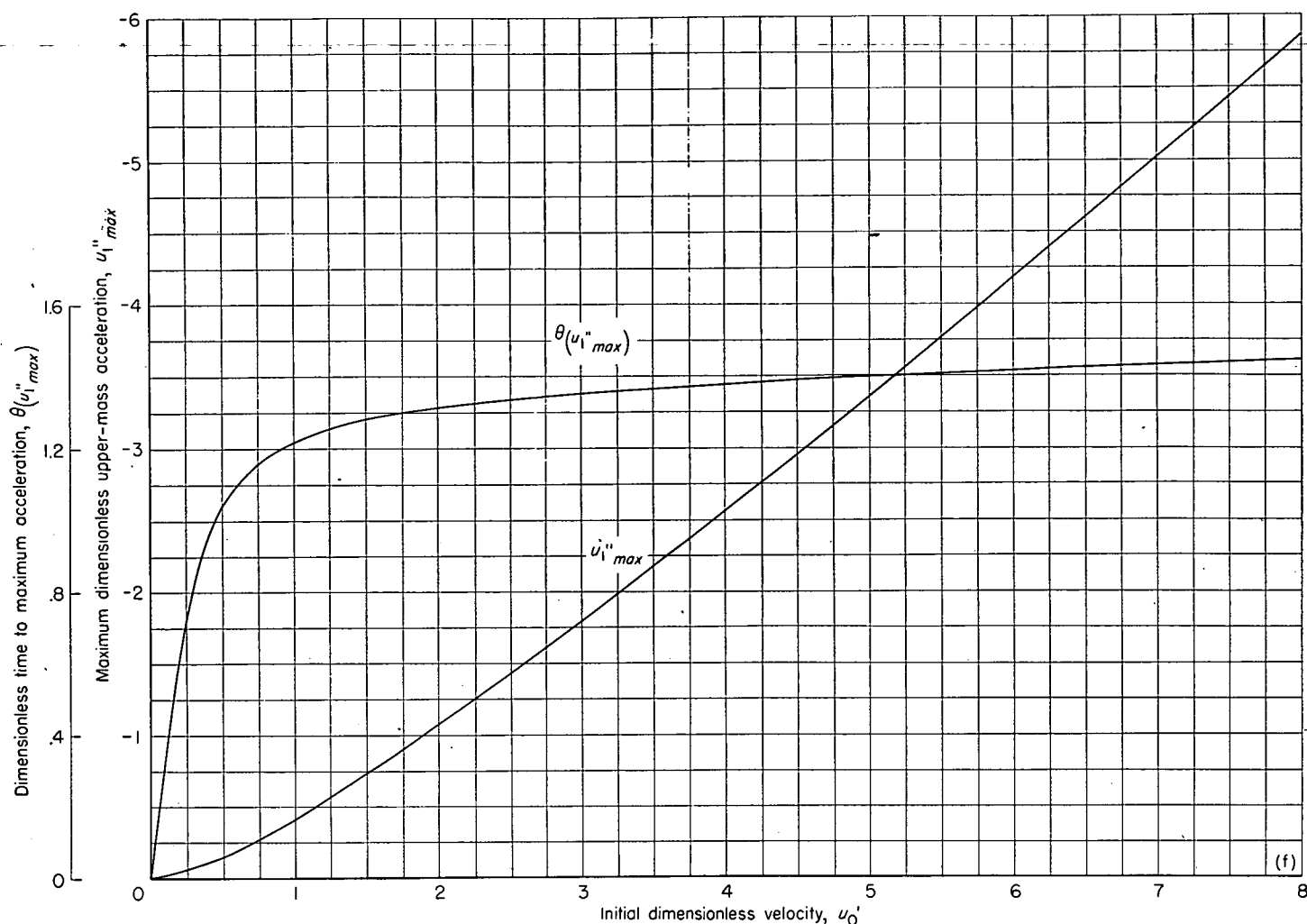
In addition to the more exact treatment an investigation has also been made to determine the extent to which the basic equations of motion can be simplified and still yield useful results. Generalized solutions of the simplified equations obtained are presented for a wide range of landing-gear and impact parameters.

On the basis of the foregoing studies the following conclusions are indicated:

1. The behavior of the landing gear as calculated from the basic equations of motion was found to be in good agreement with experimental drop-test data for the case of a vertical landing gear in the absence of drag loads, for both a normal impact and a severe impact involving tire bottoming.

2. A study of the effects of variations in the force-deflection characteristics of the tire indicates that

- a. In the case of a normal impact without tire bottoming, reasonable variations in the force-deflection characteristics of the tire have only a relatively small effect on the calculated behavior of the landing gear. Approximating the rather complicated force-deflection characteristics of the actual tire by simplified exponential or linear-segment variations appears



(f) Variation of maximum upper-mass acceleration and time to reach maximum upper-mass acceleration with initial velocity parameter.

FIGURE 11.—Continued.

to be adequate for practical purposes. Tire hysteresis was found to be relatively unimportant.

b. In the case of a severe impact involving tire bottoming, the use of simplified exponential and linear-segment approximations to the actual tire force-deflection characteristics which neglect the effects of tire bottoming, although adequate up to the instant of bottoming, fails to indicate the pronounced increase in landing-gear load which results from bottoming of the tire. The use of exponential or linear-segment approximations to the tire characteristics which take into account the increased stiffness of the tire that results from bottoming, however, yields good results.

3. A study of the importance of the discharge coefficient of the orifice indicates that the magnitude of the discharge coefficient has a marked effect on the calculated behavior of the landing gear; a decrease in the discharge coefficient (or the product of the discharge coefficient and the net orifice area) results in an approximately proportional increase in the maximum upper-mass acceleration.

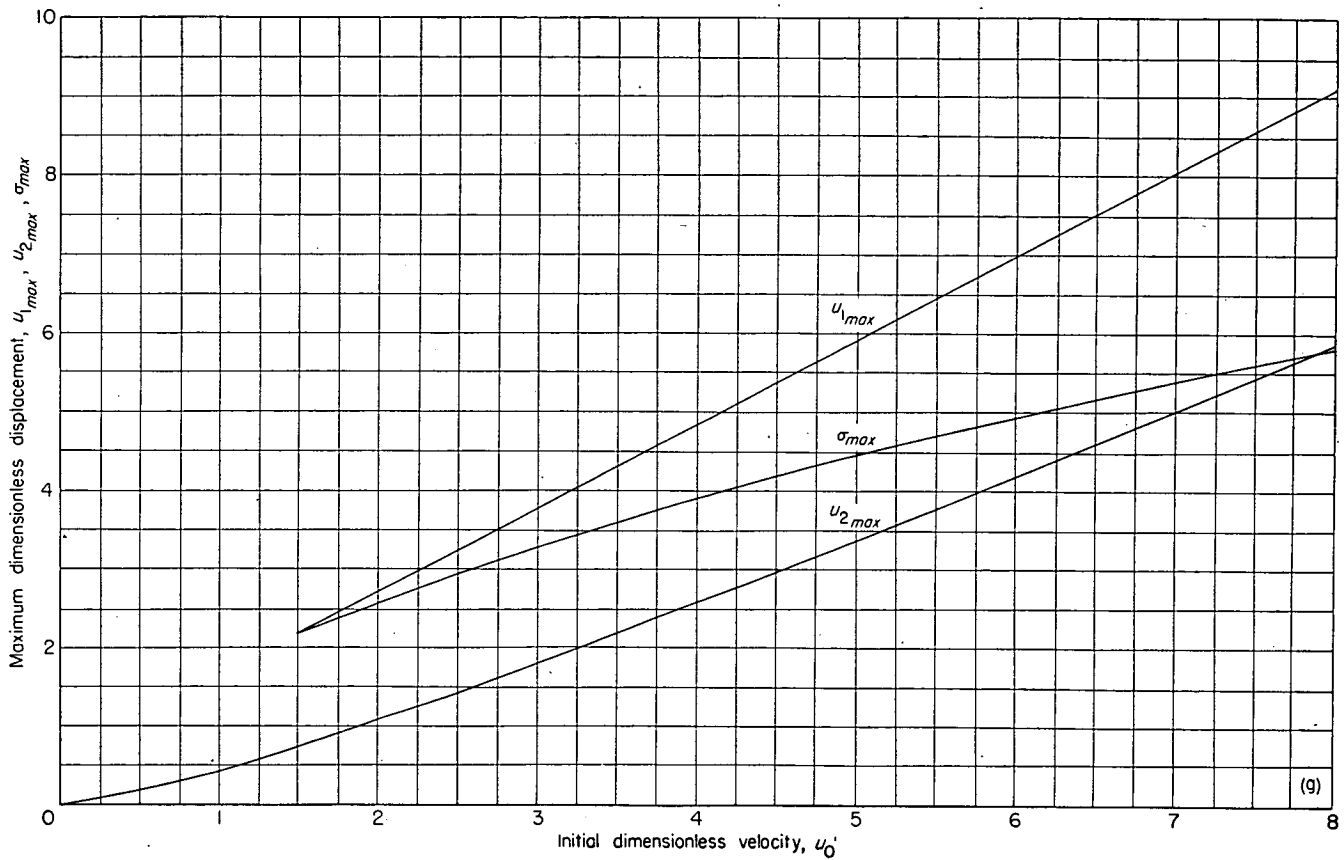
4. A study of the importance of the air-compression process in the shock strut indicates that the air springing is of only minor significance throughout most of the impact, and that variations in the effective polytropic exponent n between the isothermal value of 1.0 and the near-adiabatic value of 1.3 have only a secondary effect on the calculated behavior of

the landing gear. Even the assumption of constant air pressure in the strut equal to the initial pressure ($n=0$) yields fairly good results, which may be adequate for many practical purposes.

5. An investigation of the extent to which the equations of motion for the landing gear can be simplified and still yield acceptable calculated results indicates that, for many practical purposes, the air-pressure force in the shock strut can be completely neglected, the tire force-deflection relationship can be assumed to be linear, and the lower or unsprung mass can be taken equal to zero.

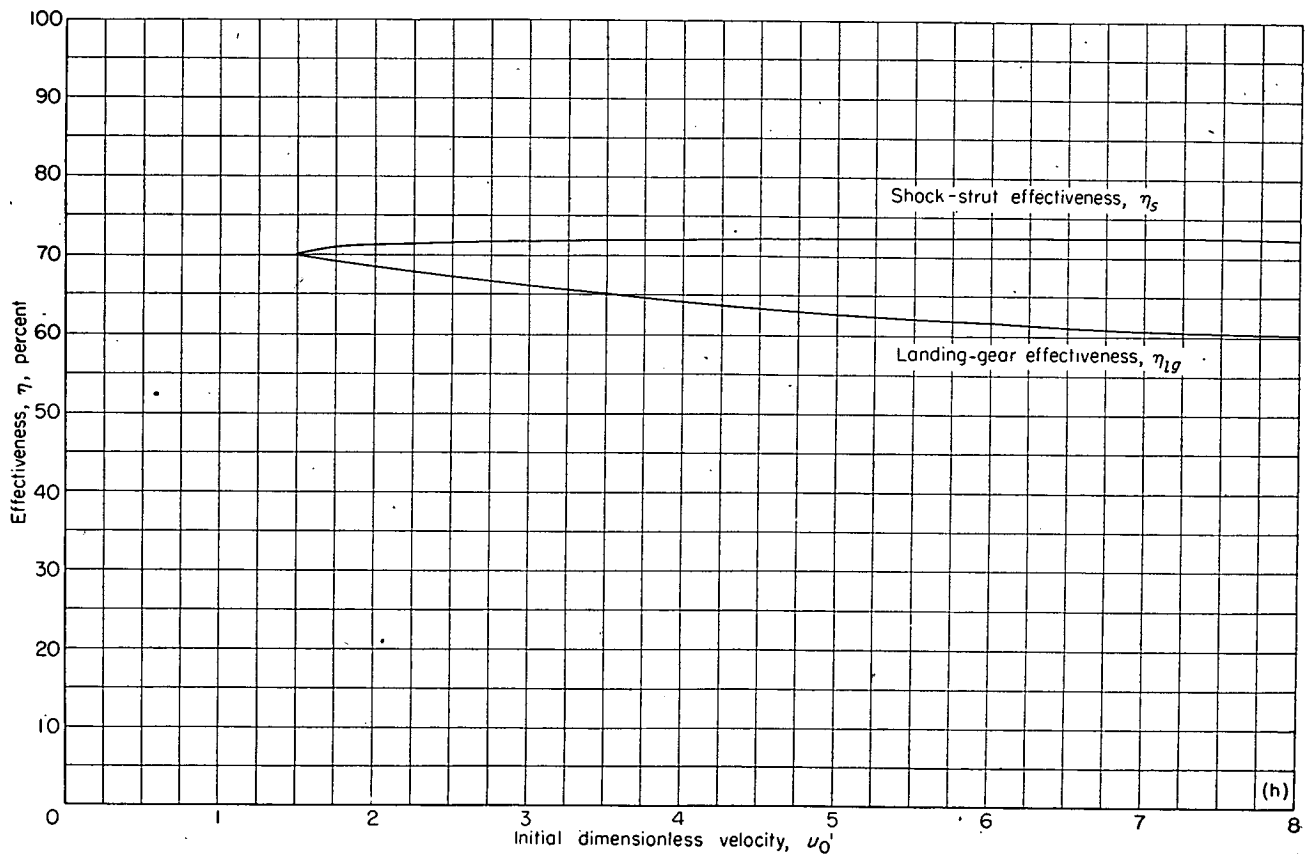
6. Generalization of the equations of motion for the simplified system described in the preceding paragraph shows that the behavior of this system is completely determined by the magnitude of one parameter, namely, the dimensionless initial-velocity parameter. Solution of these generalized equations in terms of dimensionless variables permits compact representation of the behavior of the system for a wide range of landing-gear and impact parameters, which may be useful for rapidly estimating landing-gear performance in preliminary design.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., May 1, 1952.



(g) Variation of maximum upper-mass displacement, maximum lower-mass displacement, and maximum strut stroke with initial velocity parameter.

FIGURE 11.—Continued.



(h) Variation of shock-strut effectiveness and landing-gear effectiveness with initial velocity parameter.

FIGURE 11.—Concluded.

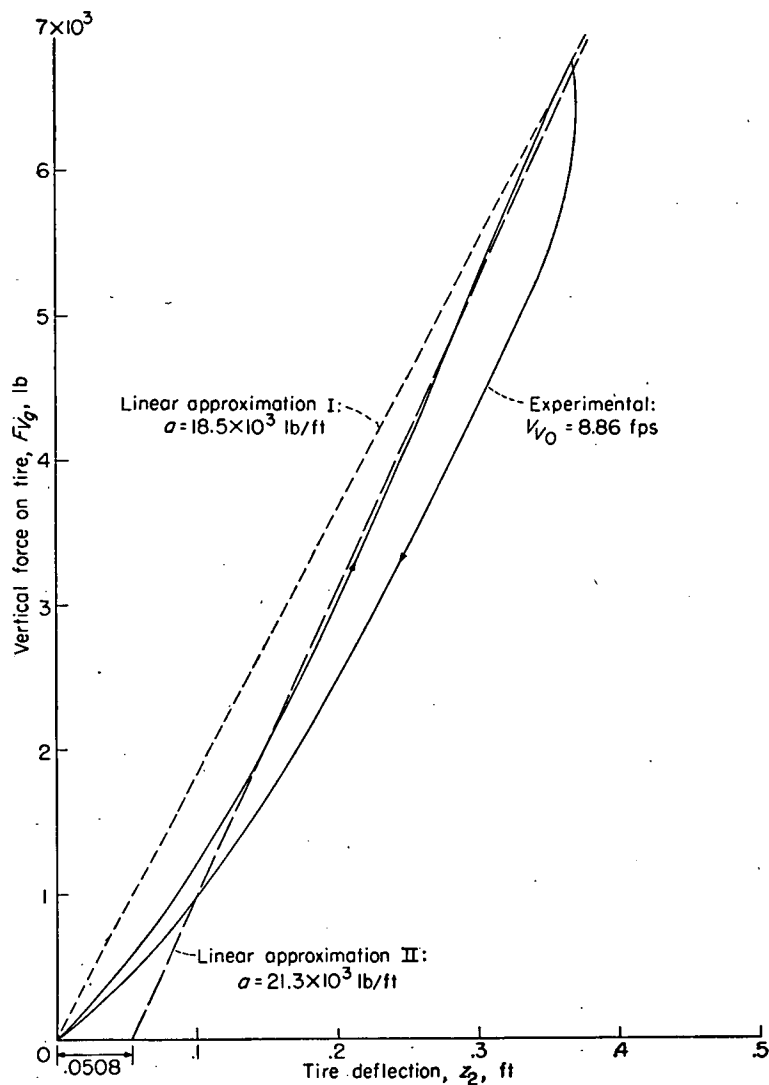
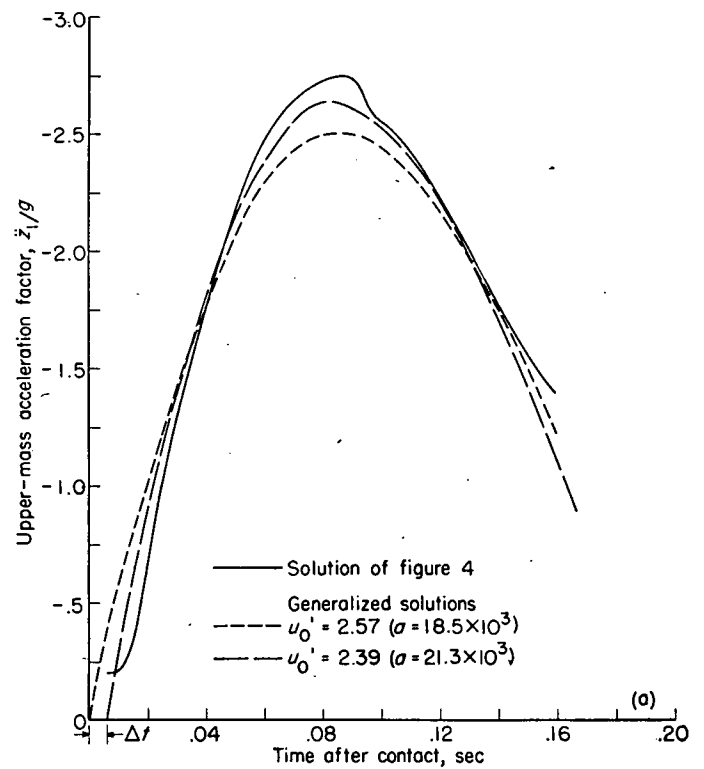
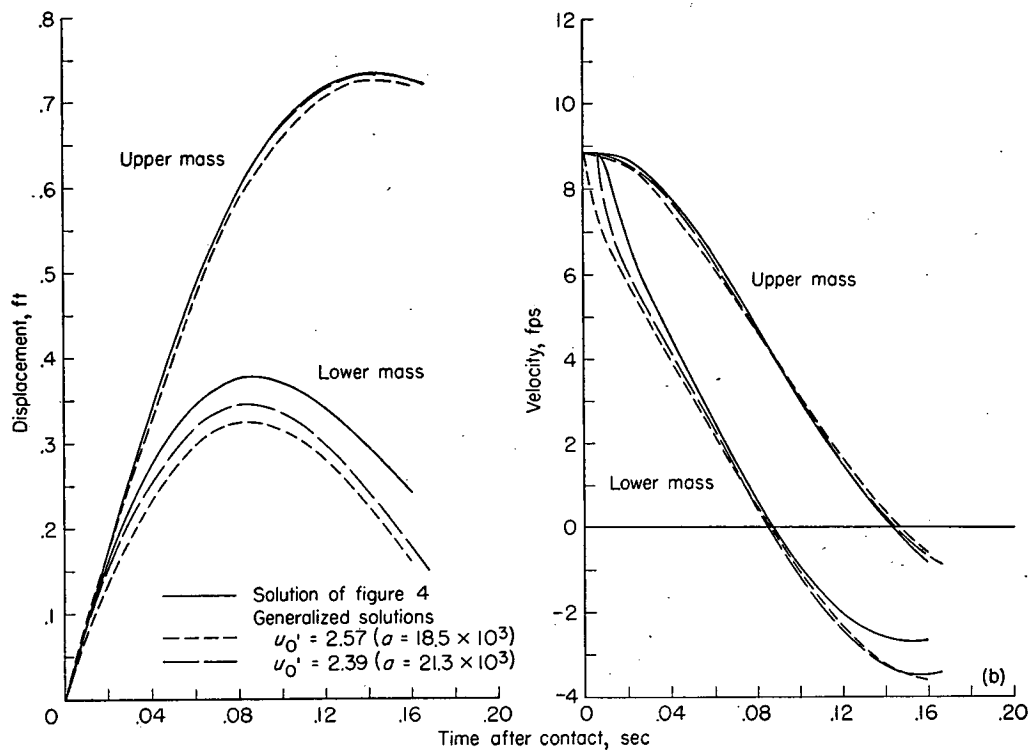


FIGURE 12.—Linear approximations to tire force-deflection characteristics used in application of generalized solutions.



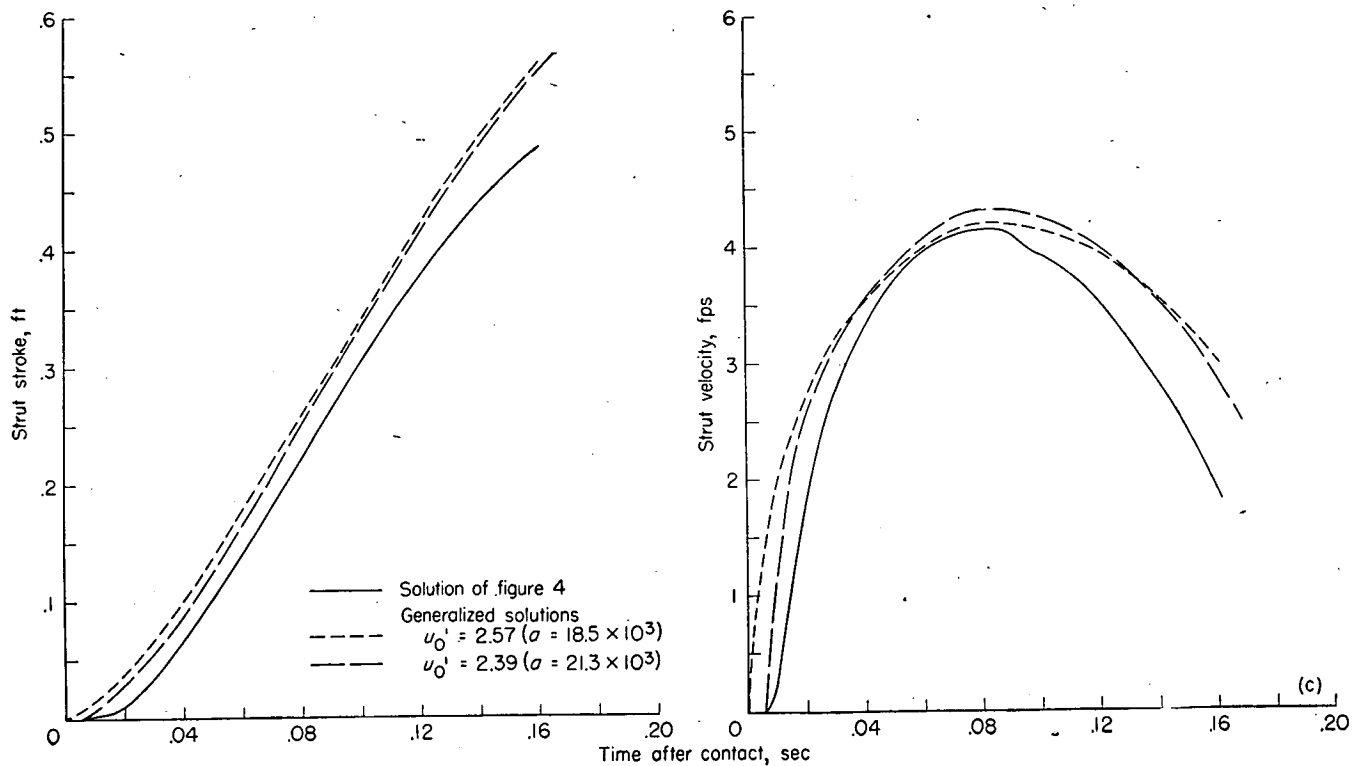
(a) Time history of upper-mass acceleration.

FIGURE 13.—Comparison of generalized results and more exact solution. $V_{V0} = 8.86$ feet per second; $C_d = 0.9$.



(b) Time histories of landing-gear displacements and velocities.

FIGURE 13.—Continued.



(c) Time histories of shock-strut stroke and velocity.

FIGURE 13.—Concluded.

APPENDIX A

NUMERICAL INTEGRATION PROCEDURES

As previously noted, most of the specific solutions presented in this report were obtained with a numerical integration procedure, termed the "linear procedure," which assumes changes in the variables to be linear over finite time intervals. With this procedure a time interval $\epsilon=0.001$ second was used in order to obtain the desired accuracy for the particular cases considered. A few of the specific solutions presented were obtained by means of a procedure, termed the "quadratic procedure," which assumes a quadratic variation of displacement with time for successive intervals. This procedure, although requiring somewhat more computing time per interval, may permit an increase in the interval size for a given accuracy, in some cases allowing a reduction in the total computing time required. In the case of the more exact equations of motion the accuracy of the quadratic procedure with a time interval of 0.002 second appears to be equal to that of the linear procedure with an interval of 0.001 second. Although the accuracy naturally decreases with increasing interval size, the loss in accuracy for proportionate increases in interval size appears to be smaller for the quadratic than for the linear procedure. In the case of the simplified equations of motion reasonably satisfactory results were obtained in test computations with the quadratic procedure for intervals as large as 0.01 second, whereas the linear procedure was considered questionable for intervals larger than 0.002 second.

The generalized solutions presented, because of the relatively simple form of the equations of motion, were obtained with the well-known Runge-Kutta procedure. A study of the allowable interval size resulted in the use of an interval $\Delta\theta=0.08$, which corresponds to a time interval of about 0.005 second for the landing gear under consideration.

LINEAR PROCEDURE

In this step-by-step procedure the variations in displacement, velocity, and acceleration are assumed to be linear over each finite time interval ϵ . The method, as used, involves one stage of iteration. Linear extrapolation of the velocity at the end of any interval is used to obtain estimated values of velocity and displacement for the next interval. These values are then used to calculate values of the acceleration in accordance with the equations of motion. Integration of the acceleration provides improved values of the velocity and, if desired, the displacement and acceleration. In this procedure all integrations are performed by application of the trapezoidal rule.

The following derivation illustrates the application of the linear procedure to the equations of motion for the landing gear, which apply subsequent to the beginning of shock-strut deflection at time t_r . In the example presented internal friction forces are neglected in order to simplify the deriva-

tion. However, the same general procedure can be used if these, or other complicating effects, are included in the equations.

For the case under consideration the equations of motion (eqs. (16), (17), and (8)) can be written as follows:

$$\frac{W_1}{g} \ddot{z}_1 + A(\dot{z}_1 - \dot{z}_2)^2 + B[1 - C(z_1 - z_2)]^{-n} + D = 0 \quad (A1)$$

$$\frac{W_2}{g} \ddot{z}_2 - A(\dot{z}_1 - \dot{z}_2)^2 - B[1 - C(z_1 - z_2)]^{-n} + F_{V_g}(z_2) - W_2 = 0 \quad (A2)$$

$$\frac{W_1}{g} \ddot{z}_1 + \frac{W_2}{g} \ddot{z}_2 + F_{V_g}(z_2) + E = 0 \quad (A3)$$

where

$$A = \frac{\dot{s}}{|\dot{s}|} \frac{\rho A_h^3}{2(C_d A_n)^2 \cos \varphi}$$

$$B = p_{a_0} A_a \cos \varphi$$

$$C = \frac{A_a}{v_0 \cos \varphi}$$

$$D = K_L W - W_1$$

$$E = W(K_L - 1)$$

Solving equation (A3) for $\ddot{z}_1$ gives

$$\frac{\ddot{z}_1}{g} = [F - G\ddot{z}_2 - H F_{V_g}(z_2)] \frac{1}{g} \quad (A4)$$

where

$$F = \frac{W}{W_1} (1 - K_L)g$$

$$G = \frac{W_2}{W_1}$$

$$H = \frac{g}{W_1}$$

Integrating equation (A4) with respect to t between the limits t_r and t and noting that $\dot{z}_{1r} = \dot{z}_{2r} = \dot{z}_r$ gives

$$\dot{z}_1 = \dot{z}_r + F\tau - G(\dot{z}_2 - \dot{z}_r) - H \int_0^\tau F_{V_g}(z_2) d\tau \quad (A5)$$

where $\tau = (t - t_r)$.

Integrating again and noting that $z_{1r} = z_{2r} = z_r$ gives

$$z_1 = (1 + G)(z_r + \dot{z}_r \tau) + \frac{F\tau^2}{2} - Gz_2 - H \int_0^\tau \int_0^\tau F_{V_g}(z_2) d\tau d\tau \quad (A6)$$

Substituting for $\dot{z}_1$ and z_1 in equation (A2) gives

$$\frac{\ddot{z}_2}{g} = \frac{1}{W_2} \left(A \left[(1+G)(\dot{z}_\tau - \dot{z}_2) + F_\tau - H \int_0^\tau F_{V_g}(z_2) d\tau \right]^2 + B \left\{ 1 - C \left[(1+G)(z_\tau + \dot{z}_\tau \tau - z_2) + \frac{F_\tau^2}{2} - H \int_0^\tau \int_0^\tau F_{V_g}(z_2) d\tau d\tau \right] \right\}^{-n} - F_{V_g}(z_2) + W_2 \right) \quad (A7)$$

The motion of the landing gear subsequent to the beginning of shock-strut deflection is determined by means of a step-by-step solution of equation (A7). This numerical procedure yields time histories of the lower-mass motion variables z_2 , $\dot{z}_2$, and $\ddot{z}_2$, from which the motion variables for the upper mass $\ddot{z}_1$, $\dot{z}_1$, and z_1 can be calculated by means of equations (A4), (A5), and (A6).

The initial conditions for the step-by-step procedure are

$$\left. \begin{aligned} z_{1n=0} &= z_{2n=0} = z_\tau \\ \dot{z}_{1n=0} &= \dot{z}_{2n=0} = \dot{z}_\tau \\ \ddot{z}_{1n=0} &= \ddot{z}_{2n=0} = \ddot{z}_\tau \end{aligned} \right\} \quad (A8)$$

where z_τ , $\dot{z}_\tau$, and $\ddot{z}_\tau$ are the conditions of motion at the beginning of shock-strut deflection as determined from the solution for the one-degree-of-freedom system.

Estimated values of the lower-mass velocity at the end of the first time increment ϵ following the beginning of shock-strut deflection can be obtained from the expression

$$\dot{z}_{2n=1}^* = \dot{z}_\tau + \epsilon \ddot{z}_\tau \quad (A9)$$

or, as a first approximation,

$$\dot{z}_{2n=1}^* = \dot{z}_\tau$$

The corresponding displacement is given by

$$z_{2n=1} = z_\tau + \frac{\epsilon}{2} (\dot{z}_{21} + \dot{z}_\tau) \quad (A10)$$

After the initial conditions and the conditions at the end of the first time increment are established, a step-by-step calculation of the motion can be obtained by routine operations as indicated by the following general procedure which applies at any time $\tau = n\epsilon$ after the beginning of the process. The operations indicated are based on integration by application of the trapezoidal rule:

$$\dot{z}_{2n}^* = \dot{z}_{2n-1} + (\dot{z}_{2n-1} - \dot{z}_{2n-2}) = \dot{z}_{2n-1} + \frac{\epsilon}{2} (\ddot{z}_{2n-1} + \ddot{z}_{2n-2}) \quad (A11)$$

$$z_{2n}^* = z_{2n-1} + \frac{\epsilon}{2} (\dot{z}_{2n-1} + \dot{z}_{2n}^*) = z_{2n-1} + \epsilon \dot{z}_{2n-1} + \frac{\epsilon^2}{4} (\ddot{z}_{2n-1} + \ddot{z}_{2n-2}) \quad (A12)$$

With the estimated values $\dot{z}_{2n}^*$ and z_{2n}^* the acceleration of the lower mass can be determined by substitution in the

appropriate integrodifferential equation for the system, equation (A7) in the present case. Thus

$$\ddot{z}_{2n} = f(\dot{z}_{2n}^*, z_{2n}^*, \tau_n) \quad (A13)$$

In equation (A7) the integral expressions can also be evaluated by application of the trapezoidal rule. For example, when $F_{V_g}(z_2) = m z_2^r$,

$$\begin{aligned} \int_0^{n\epsilon} z_2^r d\tau &\approx \frac{\epsilon}{2} (z_{21}^r + 2z_{21}^r + \dots + 2z_{2n-1}^r + z_{2n}^r) \\ &\approx \int_0^{(n-1)\epsilon} z_2^r d\tau + \frac{\epsilon}{2} (z_{2n-1}^r + z_{2n}^r) \end{aligned} \quad (A14)$$

and

$$\begin{aligned} \int_0^{n\epsilon} \int_0^{n\epsilon} z_2^r d\tau d\tau &\approx \int_0^{(n-1)\epsilon} \int_0^{(n-1)\epsilon} z_2^r d\tau d\tau + \\ &\frac{\epsilon}{2} \left(\int_0^{(n-1)\epsilon} z_2^r d\tau + \int_0^{n\epsilon} z_2^r d\tau \right) \end{aligned} \quad (A15)$$

An improved value for the velocity is obtained from the expression

$$\dot{z}_{2n} = \dot{z}_{2n-1} + \frac{\epsilon}{2} (\ddot{z}_{2n-1} + \ddot{z}_{2n}) \quad (A16)$$

This value is used in the calculation of the estimated velocity $\dot{z}_{2n+1}^*$ and displacement z_{2n+1}^* for the next interval.

If desired, improved values of the displacement and acceleration for the n th interval subsequent to the beginning of shock-strut deflection can be obtained as follows:

$$\begin{aligned} z_{2n} &= z_{2n-1} + \frac{\epsilon}{2} (\dot{z}_{2n-1} + \dot{z}_{2n}) \\ &= z_{2n-1} + \epsilon \dot{z}_{2n-1} + \frac{\epsilon^2}{4} (\ddot{z}_{2n-1} + \ddot{z}_{2n}) \end{aligned} \quad (A17)$$

and

$$\ddot{z}_{2n} = f(\dot{z}_{2n}, z_{2n}, \tau_n) \quad (A18)$$

where $f(\dot{z}_{2n}, z_{2n}, \tau_n)$ is an appropriate equation for the system, such as equation (A7).

With the values of z_{2n} , $\dot{z}_{2n}$, and $\ddot{z}_{2n}$, the motion variables for the upper mass $\ddot{z}_1$, $\dot{z}_1$, and z_1 can be calculated separately from equations (A4), (A5), and (A6), as previously noted.

In setting up the numerical procedure used in obtaining the solutions presented in this report, an evaluation of the errors introduced by the procedure indicated that it would not be necessary to calculate the improved values of the displacement z_{2n} (eq. (A17)) or the acceleration $\ddot{z}_{2n}$ (eq. (A18)). However, improved values of the velocity $\dot{z}_{2n}$ were calculated by

means of equation (A16) for the purpose of determining estimated values of the velocity $\dot{z}_2$ and the displacement z_2 (eqs. (A11) and (A12)) for the increment immediately following.

In order to illustrate the application of the method, a tabular computing procedure for the solution of the system represented by equations (A1), (A2), and (A3) is presented in table I.

QUADRATIC PROCEDURE

In this step-by-step procedure a quadratic variation of displacement is assumed over successive equal finite time intervals for the purpose of extrapolating values of the

motion variables from one interval to the next. With this assumption the displacement variation over two successive equal time intervals is completely determined by the three values of displacement at the beginning and end of each of the two intervals. By writing the quadratic variation in difference form, the velocity and acceleration at the midpoint of the double interval can be expressed in terms of the three displacement values previously mentioned. Substituting for the velocity and acceleration in the differential equations for the system yields difference equations of motion in terms of successive displacement values which can be evaluated interval by interval.

TABLE I
LINEAR PROCEDURE

Row	Quantity	Equation	Procedure†
①	τ	-----	-----
①	$\ddot{z}_{2n}$	$\ddot{z}_{2n-1} + \frac{\epsilon}{2} (\ddot{z}_{2n-2} + \ddot{z}_{2n-1})$	⑨ _p
②	$\dot{z}_{2n}$	$\dot{z}_{2n-1} + \frac{\epsilon}{2} (\dot{z}_{2n-1} + \dot{z}_{2n})$	② _p + ϵ ⑧ _p
③	$F_{Vg}(z_{2n}^*)$	-----	Determined from tire force-deflection characteristics.
④	$\int_0^\tau F_{Vg}(z_{2n}^*) d\tau$	Equation (A14)	④ _p + $\frac{\epsilon}{2} [③ + ③_p]$
⑤	$\int_0^\tau \int_0^\tau F_{Vg}(z_{2n}^*) d\tau d\tau$	Equation (A15)	⑤ _p + $\frac{\epsilon}{2} [④ + ④_p]$
⑥	$\frac{\ddot{z}_{2n}}{g}$	Equation (A7)	Given by equation (A7).
⑦	$\dot{z}_{2n}$	$\dot{z}_{2n-1} + \frac{\epsilon}{2} (\dot{z}_{2n-1} + \dot{z}_{2n})$	⑦ _p + $\frac{\epsilon}{2} [⑥ + ⑥_p]g$
⑧	$\frac{1}{2} (\dot{z}_{2n} + \dot{z}_{2n+1})$	$\dot{z}_{2n} + \frac{\epsilon}{4} (\dot{z}_{2n-1} + \dot{z}_{2n})$	⑦ + $\frac{\epsilon}{4} [⑥ + ⑥_p]g$
⑨	$\ddot{z}_{2n+1}$	$\ddot{z}_{2n} + \frac{\epsilon}{2} (\ddot{z}_{2n-1} + \ddot{z}_{2n})$	⑦ + $\frac{\epsilon}{2} [⑥ + ⑥_p]g$
⑩	$\frac{\dot{z}_{1n}}{g}$	Equation (A4)	Given by equation (A4).
⑪	$\dot{z}_{1n}$	Equation (A5)	Given by equation (A5).
⑫	z_{1n}	Equation (A6)	Given by equation (A6).

† ①_p denotes value for previous time interval.

The following derivation shows how the procedure can be applied to the determination of the behavior of the landing gear subsequent to the beginning of shock-strut deflection at time t_r . In order to simplify the derivation, internal friction forces are again neglected in setting-up the equations of motion.

The assumption of a quadratic variation of displacement with time (constant acceleration) over two successive intervals, each of duration ϵ , permits expressing the velocity and acceleration at the midpoints of the double interval (see sketch) in terms of the displacement values at the beginning, midpoint, and end of the double interval by the equations (see ref. 5, p. 16):

$$\dot{z}_n = \frac{z_{n+1} - z_{n-1}}{2\epsilon} \quad (\text{A19})$$

and

$$\ddot{z}_n = \frac{z_{n+1} - 2z_n + z_{n-1}}{\epsilon^2} \quad (\text{A20})$$

where $\dot{z}_n$, $\ddot{z}_n$, and z_n are the velocity, acceleration, and displacement at the end of the n th interval ($\tau = n\epsilon$) after the beginning of shock-strut deformation and z_{n-1} and z_{n+1} are the displacements at the end of intervals $n-1$ and $n+1$, respectively.

Substituting the difference relations for $\dot{z}_1$, $\dot{z}_2$, $\dot{z}_1$, and $\dot{z}_2$ into equations (A1) and (A3) permits writing the equations of motion for the landing gear in difference form as follows:

$$\frac{W_1}{g\epsilon^2}(z_{1n+1} - 2z_{1n} + z_{1n-1}) + \frac{A}{4\epsilon^2}(z_{1n+1} - z_{1n-1} - z_{2n+1} + z_{2n-1})^2 + B[1 - C(z_{1n} - z_{2n})]^{-n} + D = 0 \quad (\text{A21})$$

and

$$z_{1n+1} = 2z_{1n} - z_{1n-1} - G(z_{2n+1} - 2z_{2n} + z_{2n-1}) - H\epsilon^2[F_{V_g}(z_{2n}) + E] \quad (\text{A22})$$

where the constants are as defined in the previous section.

Substituting for z_{1n+1} in equation (A21) gives

$$z_{2n+1} = \frac{\beta_{n+1}}{W} + \frac{1}{gAW^2} \left[2W_1^2W_2 - \sqrt{4W_1^2W_2(gAW\beta_{n+1} + W_1^2W_2) - gAW^2(4W_1^2\alpha_{n+1} + gA\gamma_{n+1})} \right] \quad (\text{A23})$$

where

$$\alpha_{n+1} = 2W_2z_{2n} - W_2z_{2n-1} - g\epsilon^2[F_{V_g}(z_{2n}) + E]$$

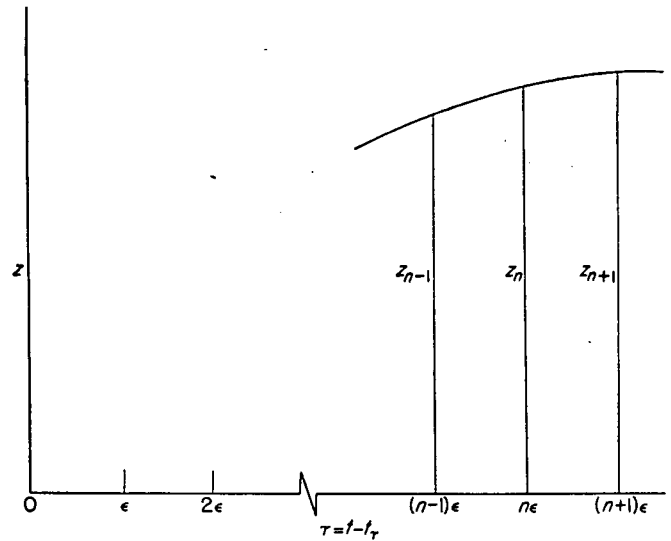
$$\beta_{n+1} = 2W_2z_{2n} + (W_1 - W_2)z_{2n-1} + 2W_1(z_{1n} - z_{1n-1}) - g\epsilon^2[F_{V_g}(z_{2n}) + E]$$

and

$$\gamma_{n+1} = \frac{4W_1^2\epsilon^2}{A} \{B[1 - C(z_{1n} - z_{2n})]^{-n} + D\}$$

Equations (A22) and (A23) are essentially extrapolation formulas which permit the determination of values for the upper-mass and lower-mass displacements to come from the values of displacement already calculated. These equations thus permit step-by-step calculation of the displacements as the impact progresses, starting with the initial conditions, from which the upper-mass and lower-mass velocities and accelerations can be determined by means of equations (A19) and (A20).

Since the calculation of the displacements z_1 and z_2 at any instant by means of equations (A22) and (A23) requires values for the displacements at two previous instants, the routine application of these equations can begin only at the end of the second interval ($\tau = 2\epsilon$) following the beginning of shock-strut deflection. Before the displacements at the end of the second interval can be calculated, however, it is necessary to determine the displacements at the end of the first interval. These values can be obtained from the conditions of motion at the instant of initial shock-strut deflection by applying equations (A19) and (A20) to the instant $t = t_r$.



At the instant of initial shock-strut deflection

$$\left. \begin{aligned} z_{1n=0} &= z_{2n=0} = z_\tau \\ \dot{z}_{1n=0} &= \dot{z}_{2n=0} = \dot{z}_\tau \\ \ddot{z}_{1n=0} &= \ddot{z}_{2n=0} = \ddot{z}_\tau \end{aligned} \right\} \quad (\text{A24})$$

Application of the difference equations (A19) and (A20) to the instant $t=t_\tau$ (that is, $n=0$) gives the following equations:

$$\left. \begin{aligned} \dot{z}_\tau &= \frac{z_{n=1} - z_{n=-1}}{2\epsilon} \\ \ddot{z}_\tau &= \frac{z_{n=1} - 2z_\tau + z_{n=-1}}{\epsilon^2} \end{aligned} \right\} \quad (\text{A25})$$

Since the landing gear is considered as a one-degree-of-freedom system from initial contact up to the instant $t=t_\tau$,

the foregoing application of the difference equations results in identical values for the upper-mass displacement and lower-mass displacement at the end of the first interval. Simultaneous solution of equations (A25) gives the following expression for the displacement at the end of the first interval:

$$z_{2n=1} = z_{1n=1} = z_\tau + \epsilon \dot{z}_\tau + \frac{\epsilon^2}{2} \ddot{z}_\tau \quad (\text{A26})$$

With the values for z_τ and $z_{n=1}$, equations (A22) and (A23) permit the step-by-step calculation of the upper-mass and lower-mass displacements subsequent to the first interval following the beginning of shock-strut deflection. The corresponding velocities and accelerations of the upper and lower masses can be determined from the calculated displacements by means of equations (A19) and (A20), as previously noted.

A tabular computing procedure illustrating the application of the method is presented in table II.

TABLE II
QUADRATIC PROCEDURE

Row	Quantity	Equation	Procedure †
①	τ		
①	z_{2n}		③ _p
②	z_{1n}		④ _p
③	z_{2n+1}	Equation (A23)	Given by equation (A23).
④	z_{1n+1}	Equation (A22)	Given by equation (A22).
⑤	$\dot{z}_{2n}$	$\frac{z_{2n+1} - z_{2n-1}}{2\epsilon}$	$\frac{③ - ①_p}{2\epsilon}$
⑥	$\ddot{z}_{2n}$	$\frac{z_{2n+1} - 2z_{2n} + z_{2n-1}}{\epsilon^2}$	$\frac{③ - 2① + ①_p}{\epsilon^2}$
⑦	$\dot{z}_{1n}$	$\frac{z_{1n+1} - z_{1n-1}}{2\epsilon}$	$\frac{④ - ②_p}{2\epsilon}$
⑧	$\ddot{z}_{1n}$	$\frac{z_{1n+1} - 2z_{1n} + z_{1n-1}}{\epsilon^2}$	$\frac{④ - 2② + ②_p}{\epsilon^2}$

† ○ denotes value for previous time interval.

RUNGE-KUTTA PROCEDURE

In this step-by-step procedure the differences in the dependent variables over any given interval of the independent variable are calculated from a definite set of formulas, the same set of formulas being used for all increments. Thus the values of the variables at the end of any given interval are completely determined by the values at the end of the preceding interval. Unfortunately, however, unless the equations to be integrated are relatively simple, the method can become quite lengthy.

The following derivation illustrates the application of the Runge-Kutta method to the generalized equations of motion (eqs. (21)) for the simplified system considered in the section on generalized results. Since these equations can be readily reduced to the first order, they can be integrated by the step-by-step application of the general equations given on pages 301 and 302 of reference 6 for first-order simultaneous differential equations.

The generalized equations for the simplified system previously discussed (eqs. (21)) are

$$(u_1' - u_2')^2 + u_1'' = 0$$

$$(u_1' - u_2')^2 - u_2 = 0$$

$$u_1'' + u_2 = 0$$

Inasmuch as any two of these equations are sufficient to describe the behavior of the system, only the last two equations are employed in this procedure. These equations can be reduced to a first-order system by introducing the new variable

$$w = u_1' \quad (\text{A27})$$

so that

$$w' = u_1'' \quad (\text{A28})$$

and the equations of motion become

$$\left. \begin{aligned} (w - u_2')^2 - u_2 &= 0 \\ w' + u_2 &= 0 \end{aligned} \right\} \quad (\text{A29})$$

Solving equations (A29) for u_2' and w' , respectively, gives

$$u_2' = w - \sqrt{u_2} \quad (\text{A30})$$

$$w' = -u_2 \quad (\text{A31})$$

Applying the general procedure presented in the reference previously cited to the simultaneous equations (A27), (A30),

and (A31) gives

$$\left. \begin{aligned} \Delta u_1 &= u_{1n} - u_{1n-1} = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ \Delta w &= w_n - w_{n-1} = \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) \\ \Delta u_2 &= u_{2n} - u_{2n-1} = \frac{1}{6} (m_1 + 2m_2 + 2m_3 + m_4) \end{aligned} \right\} \quad (\text{A32})$$

where

$$k_1 = w_{n-1} \Delta \theta$$

$$k_2 = \left(w_{n-1} + \frac{l_1}{2} \right) \Delta \theta$$

$$k_3 = \left(w_{n-1} + \frac{l_2}{2} \right) \Delta \theta$$

$$k_4 = (w_{n-1} + l_3) \Delta \theta$$

$$l_1 = -u_{2n-1} \Delta \theta$$

$$l_2 = -\left(u_{2n-1} + \frac{m_1}{2} \right) \Delta \theta$$

$$l_3 = -\left(u_{2n-1} + \frac{m_2}{2} \right) \Delta \theta$$

$$l_4 = -(u_{2n-1} + m_3) \Delta \theta$$

$$m_1 = (w_{n-1} - \sqrt{u_{2n-1}}) \Delta \theta = u_{2n-1}' \Delta \theta$$

$$m_2 = \left[\left(w_{n-1} + \frac{l_1}{2} \right) - \sqrt{u_{2n-1} + \frac{m_1}{2}} \right] \Delta \theta$$

$$m_3 = \left[\left(w_{n-1} + \frac{l_2}{2} \right) - \sqrt{u_{2n-1} + \frac{m_2}{2}} \right] \Delta \theta$$

$$m_4 = \left[(w_{n-1} + l_3) - \sqrt{u_{2n-1} + m_3} \right] \Delta \theta$$

With this procedure, u_1 , w , and u_2 can be calculated in step-by-step fashion from the values for the preceding interval, the procedure beginning with the initial conditions. From these values, u_1' , u_1'' , and u_2' can be calculated by means of equations (A27), (A28), and (A30), respectively.

APPENDIX B

SOURCE OF EXPERIMENTAL DATA

Following is a brief description of the apparatus and test specimen used in obtaining the experimental data presented in this report.

EQUIPMENT

The basic piece of equipment employed in the tests is the carriage of the Langley impact basin (ref. 7) which provides means for effecting the controlled descent of the test specimen. In these tests the impact-basin carriage was used in much the same manner as a conventional stationary landing-gear test jig (see ref. 8). In order to simulate mechanically the wing lift forces which sustain an airplane during landing the pneumatic cylinder and cam system incorporated in the carriage was used to apply a constant lift force to the dropping mass and landing gear during impact. The lift force in these tests was equal to the total dropping weight of 2,542 pounds.

TEST SPECIMEN

The landing gear used in the tests was originally designed for a small military training airplane having a gross weight of approximately 5,000 pounds. The gear is of conventional cantilever construction and incorporates a standard type of oleo-pneumatic shock strut. The wheel is fitted with a 27-inch type I (smooth-contour) tire, inflated to 32 pounds per square inch. The weight of the landing gear is 150 pounds. The weight of the lower mass (unsprung weight) is 131 pounds.

In the present investigation the gear was somewhat modified in that the metering pin was removed and the original orifice plate was replaced with one having a smaller orifice diameter. Figure 14 shows the internal arrangement of the shock strut and presents details of the orifice. Other pertinent dimensions are presented in table III. The strut was filled with specification AN-VV-O-366B hydraulic fluid. The inflation pressure with the strut fully extended was 43.5 pounds per square inch. In these tests the landing gear was mounted with the shock-strut axis vertical. Figure 15 is a photograph of the landing gear installed for testing.

TABLE III

IMPORTANT CHARACTERISTICS OF LANDING GEAR USED IN TESTS

A_a , sq ft.....	0.05761
A_h , sq ft.....	0.04708
A_o , sq ft.....	0.0005585
v_0 , cu ft.....	0.03545
p_{a0} , lb/sq ft.....	6,264
l_1 , ft.....	0.5521
l_2 , ft.....	2.22604
W_1 , lb.....	2,411
W_2 , lb.....	131

INSTRUMENTATION

A variety of time-history instrumentation was used during the tests. The vertical acceleration of the upper mass was measured by means of an oil-damped electrical strain-gage accelerometer having a range of $\pm 8g$ and a natural frequency of 85 cycles per second. A low-frequency (16.5 cycles per second) NACA air-damped optical-recording accelerometer, having a range of $-1g$ to $6g$, was used as a stand-by instrument and as a check against the strain-gage accelerometer. Another oil-damped strain-gage accelerometer, having a range of $\pm 12g$ and a natural frequency of 260 cycles per second, was used to determine the vertical acceleration of the lower mass. The vertical displacement of the lower mass (tire deflection) and the shock-strut stroke were measured separately by means of variable-resistance slide-wire potentiometers. The vertical displacement of the upper mass was determined by addition of the strut-stroke and tire-deflection measurements. The vertical velocity of the landing gear at the instant of ground contact was determined from the output of an elemental electromagnetic voltage generator. A time history of the vertical velocity of the upper mass was obtained by mechanically integrating the vertical acceleration of the upper mass subsequent to the instant of ground contact. Electrical differentiation of the current output of the strut-stroke circuit provided time-history measurements of the shock-strut telescoping velocity. The instant of ground contact was determined by means of a micro-switch, recessed into the ground platform, which closed a circuit as long as the tire was in contact with the platform. The electrical output of the instruments was recorded on a 14-channel oscillograph. The galvanometers were damped to approximately 0.7 critical damping and had natural frequencies high enough to produce virtually uniform response up to frequencies commensurate with those of the measuring instrumentation. A typical oscillograph record is shown in figure 16.

It is believed that the measurements obtained in the tests are accurate within the following limits:

Measurement	Accuracy
Upper-mass acceleration, g	± 0.2
Force on upper mass, lb.....	± 500
Lower-mass acceleration, g	± 0.3
Vertical velocity at ground contact, fps.....	± 0.1
Upper-mass velocity during impact, fps.....	± 0.5
Upper-mass displacement, ft.....	± 0.05
Lower-mass displacement, ft.....	± 0.03
Shock-strut stroke, ft.....	± 0.03
Shock-strut telescoping velocity, fps.....	± 0.5
Time after contact, sec.....	± 0.003

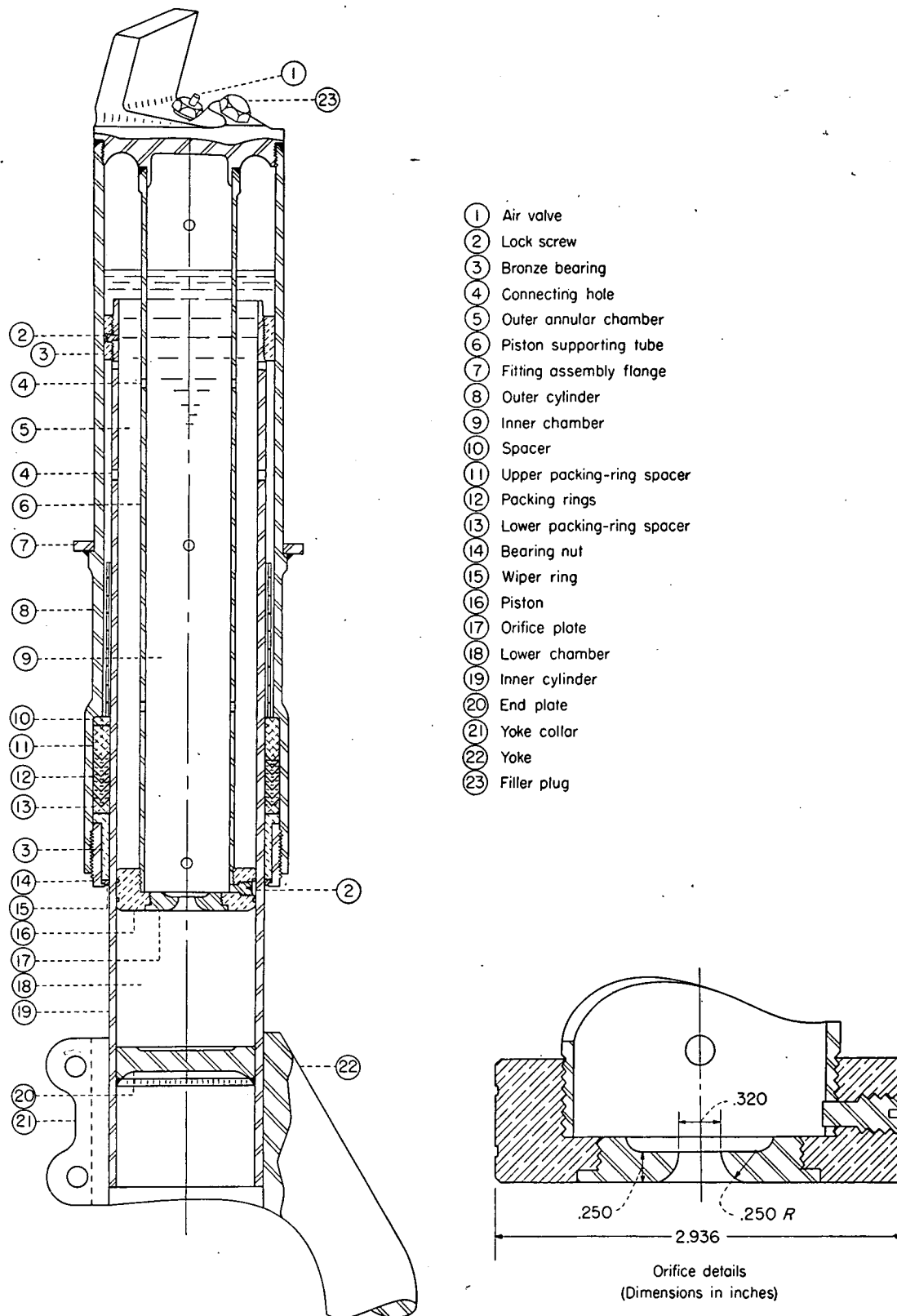


FIGURE 14.—Shock strut of landing gear tested at Langley impact basin.

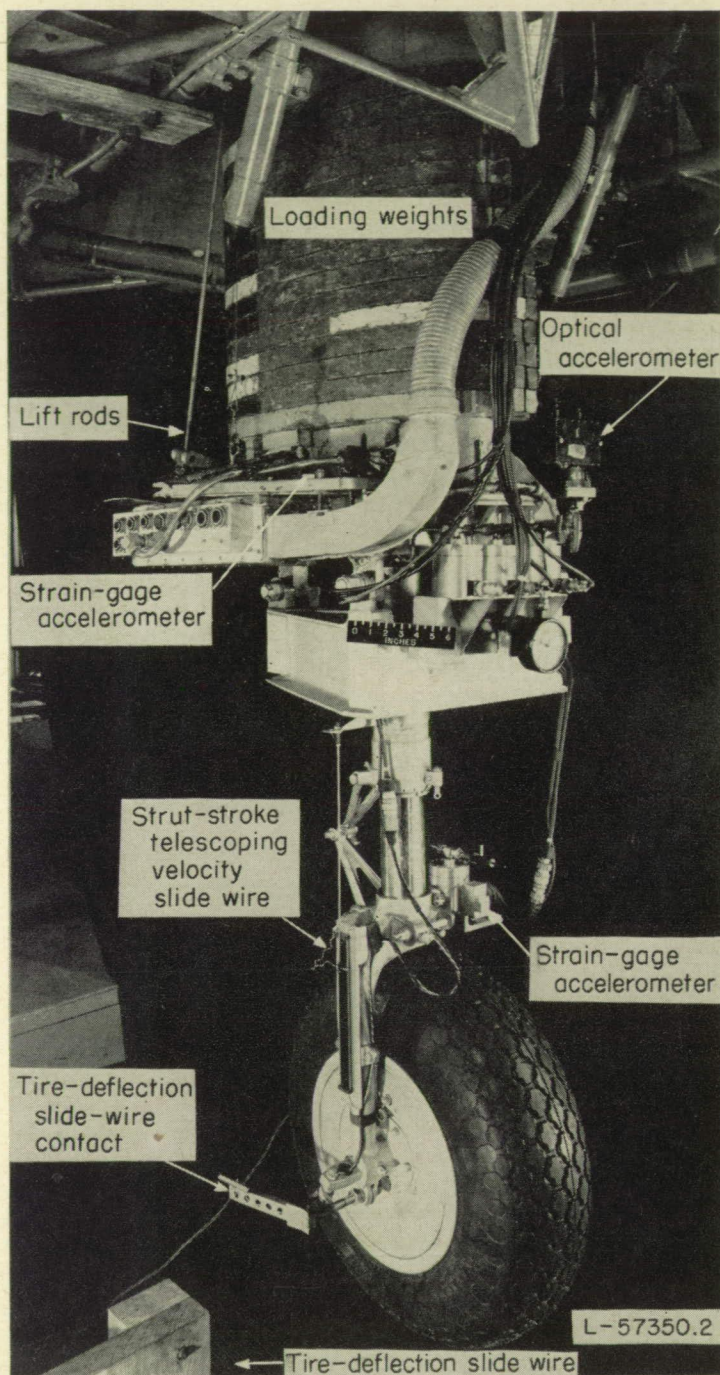


FIGURE 15.—View of landing gear and instrumentation.

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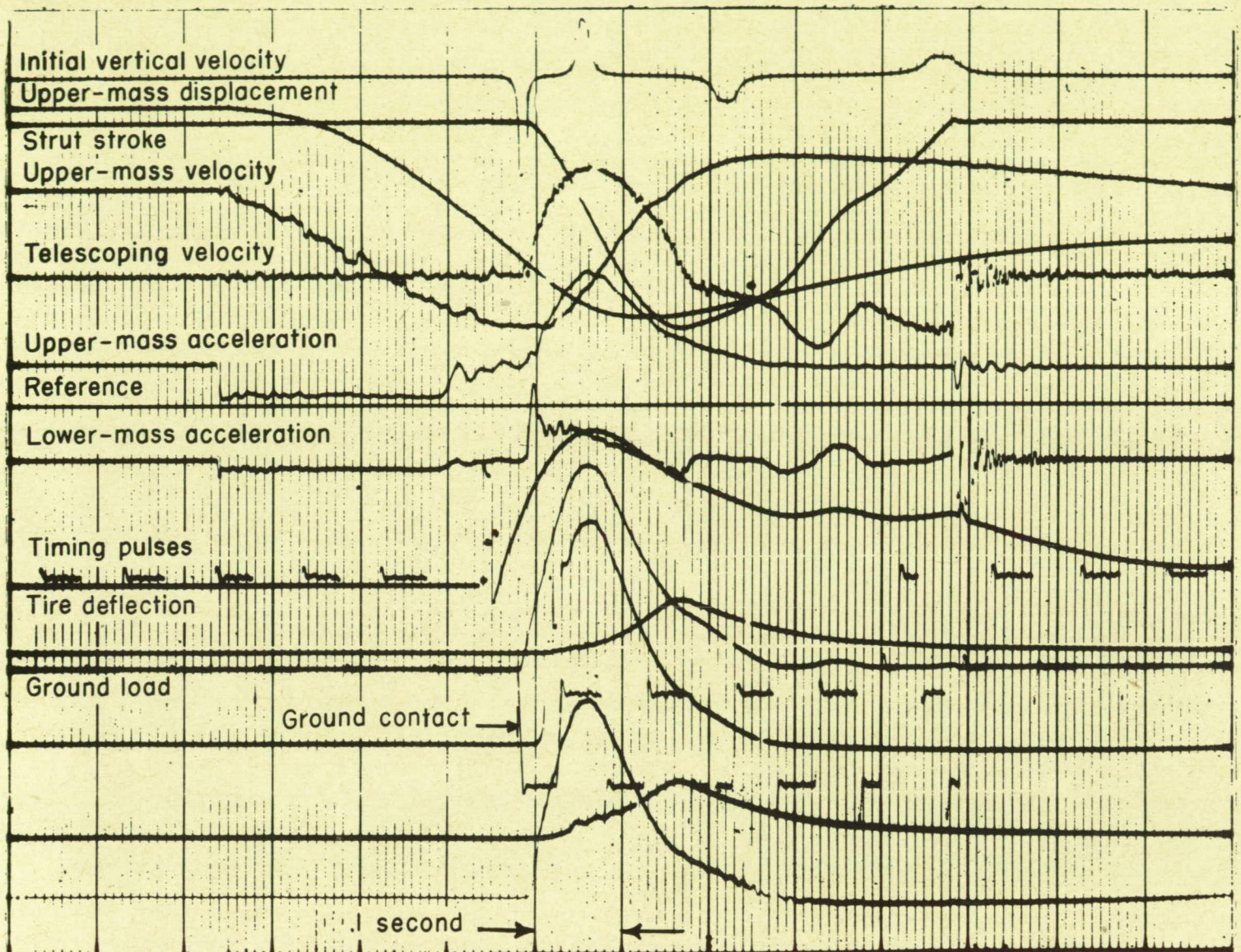


FIGURE 16.—Typical oscillograph record obtained during test in Langley impact basin.

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REPORT 1155

A COMPARISON OF THE EXPERIMENTAL SUBSONIC PRESSURE DISTRIBUTIONS ABOUT SEVERAL BODIES OF REVOLUTION WITH PRESSURE DISTRIBUTIONS COMPUTED BY MEANS OF THE LINEARIZED THEORY¹

By CLARENCE W. MATTHEWS

SUMMARY

An analysis is made of the effects of compressibility on the pressure coefficients about several bodies of revolution by comparing experimentally determined pressure coefficients with corresponding pressure coefficients calculated by the use of the linearized equations of compressible flow. The results show that the theoretical methods predict the subsonic pressure-coefficient changes over the central part of the body but do not predict the pressure-coefficient changes near the nose. Extrapolation of the linearized subsonic theory into the mixed subsonic-supersonic flow region fails to predict a rearward movement of the negative pressure-coefficient peak which occurs after the critical stream Mach number has been attained. Two equations developed from a consideration of the subsonic compressible flow about a prolate spheroid are shown to predict, approximately, the change with Mach number of the subsonic pressure coefficients for regular bodies of revolution of fineness ratio 6 or greater.

INTRODUCTION

A number of papers have been published concerning the theoretical aspect of the effects of compressibility on the flow over bodies of revolution (refs. 1 to 4). In the present investigation these theoretical methods are applied to the analysis of experimental data. Such an analysis should contribute to the basic knowledge of subsonic three-dimensional flow.

Two prolate spheroids of fineness ratios 6 and 10, an ogival body, and a prolate spheroid with an annular bump near the nose were tested in this investigation. The experimental pressures about the two prolate spheroids are compared with the pressures computed by the linearized compressible-flow theory. Several relations developed from theoretical considerations of the flow about a prolate spheroid are presented for correcting the incompressible pressure coefficients of regular bodies of fineness ratios 6 to 10 for the effects of compressibility in the subcritical flow range. Results obtained from these relations are also compared with corresponding experimental pressure coefficients.

SYMBOLS

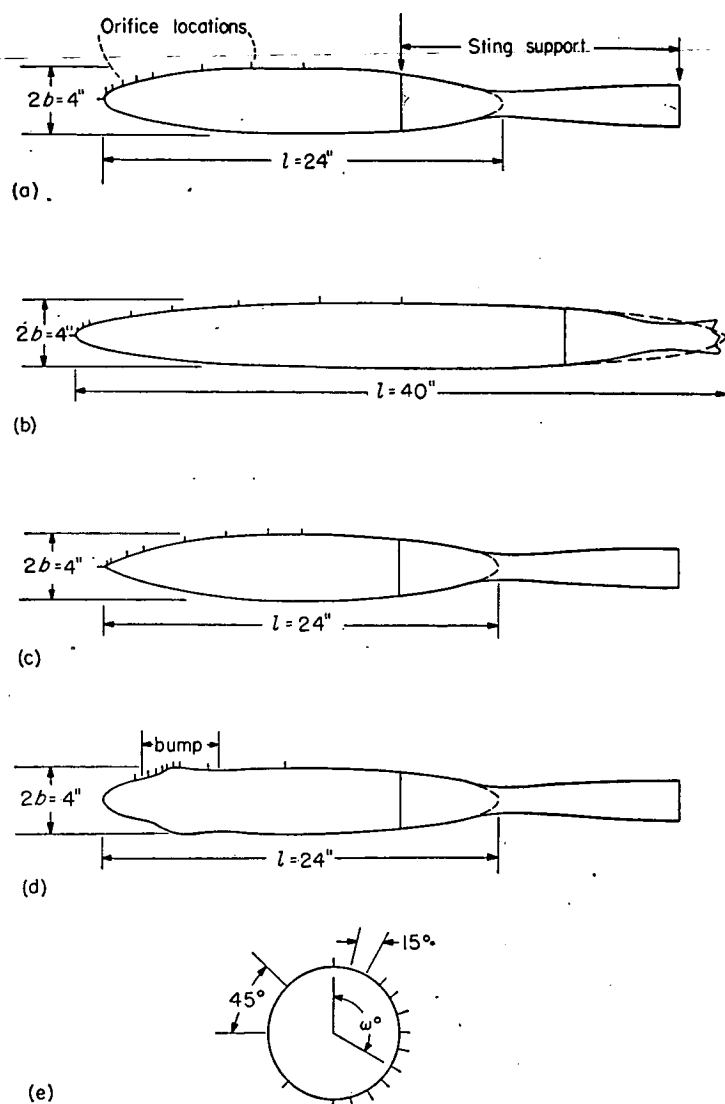
b maximum radius of body
 C_N normal-force coefficient based on plan-form area of ellipse

f fineness ratio of body, $l/2b$
 l total length of body (see fig. 1)
 M_{cr} critical Mach number
 M_0 free-stream Mach number
 p_i local static pressure
 p_0 free-stream static pressure
 P pressure coefficient, $\frac{p_i - p_0}{\frac{1}{2} \rho V^2}$
 r local radius of body
 $S(x)$ cross-sectional area of body of revolution
 u component of local velocity parallel to free stream
 U free-stream velocity
 v component of local velocity in vertical plane perpendicular to free stream
 V total local velocity
 w component of local velocity perpendicular to u and v
 x coordinate along major axis of body
 α angle of attack
 $\beta = \sqrt{1 - M_0^2}$
 γ ratio of specific heat at constant pressure to specific heat at constant volume
 ρ density
 ϕ velocity potential
 μ, ξ, ω ellipsoidal coordinates (see ref. 5)
 Subscripts:
 c compressible value
 i incompressible value
 cr critical value
 st incompressible value of flow about hypothetical stretched body

MODELS

Sketches of the bodies of revolution tested, which show the locations of the pressure orifices and other pertinent details, are presented in figure 1. The ordinates of the typical transonic or ogival body and the prolate spheroid with an annular bump are given in table I. The ordinates of the section of the sting support, which is a part of the body of revolution, are those of a prolate spheroid of fineness ratio 6. The same support was used for each body. The couplings used to change the angle of attack were mounted in the sting 11 inches downstream from the end of the body.

¹ Supersedes NACA TN 2519, "A Comparison of the Experimental Subsonic Pressure Distributions About Several Bodies of Revolution With Pressure Distributions Computed by Means of the Linearized Theory" by Clarence W. Matthews, 1952.



- (a) Prolate spheroid; $f=6$.
 (b) Prolate spheroid; $f=10$.
 (c) Typical transonic body.
 (d) Prolate spheroid with annular bump.
 (e) Angular locations of orifices.

FIGURE 1.—Profiles of bodies tested.

Except at the first three stations indicated in figures 1 (a) to 1 (d), the pressure orifices were located around the body as shown in figure 1 (e). These orifices were spaced 15° apart on one side of the body in order to obtain a fairly accurate normal-force coefficient upon integration of the pressure coefficients. The orifice at the first station was located in the nose. The orifices at the next two stations were located at 90° intervals around the body. The pressure orifice openings were 0.010 inch in diameter.

TESTS

The pressures about the bodies were measured in the Langley 8-foot high-speed tunnel through the Mach number range 0.3 to 0.95. The angle-of-attack ranges were 0° to 7.7° for the regular bodies and 0° to 2° for the prolate spheroid with an annular bump. The pressures were recorded by photographing a 10-foot 100-tube manometer board filled with acetylene tetrabromide.

TABLE I.—ORDINATES OF THE TYPICAL TRANSONIC BODY AND OF THE ANNULAR-BUMP PROLATE SPHEROID

Transonic body		Prolate spheroid with annular bump	
r/l , percent	r/l , percent	r/l , percent	r/l , percent
0.00	0.000	0.00	0.000
.50	.462	.75	1.437
.75	.596	1.25	1.854
1.25	.854	2.50	2.604
2.50	1.445	5.00	3.631
5.00	2.409	10.00	5.000
10.00	3.940	12.50	5.560
20.00	6.180	12.91	5.684
30.00	7.480	13.33	5.873
40.00	8.121	14.16	6.408
50.00	8.333	15.20	7.230
60.00	8.162	16.24	7.800
70.00	7.635	17.49	8.190
75.00	7.215	17.91	8.230
		18.74	8.250
		20.00	8.230
		23.32	8.220
		25.00	8.000
		27.08	7.810
		28.31	7.663
		28.74	7.620
		29.16	7.605
		30.00	7.640
		35.00	7.952
		50.00	8.290
		55.00	8.333
		65.00	8.290
		75.00	7.952

The free-stream pressures and Mach numbers were determined from an empty-tunnel calibration based on the pressures at an orifice located 4 feet upstream of the model.

Several preliminary plots of local pressure coefficients as functions of free-stream Mach number showed considerable scatter for Mach numbers less than 0.5, probably because of the difficulty of reading the small pressure differences and because of the possibility that the tunnel was not held at each Mach number a sufficient length of time to insure complete settling of the manometer liquid. Because of this scatter, it was necessary to neglect the pressure coefficients below $M_0=0.5$ in extrapolating the pressure-coefficient curves to a stream Mach number of zero. The data used in the analysis in this investigation were picked from the extrapolated curves.

For the tests reported herein, the Reynolds number varies from approximately 2,700,000 per foot at $M_0=0.40$ to 3,950,000 at $M_0=0.94$.

The wall interference may be approximately determined by using the equations of reference 6. Since the corrections were small, they were not applied to the pressures in the figures which present experimental data alone; however, the corrections, even though small, were applied to the experimental data used for the comparisons between the theoretical and the experimental values.

THEORETICAL METHODS

The theoretical subsonic pressures about a prolate spheroid may be computed by applying the Prandtl-Glauert correction to the incompressible potential-flow equations in the manner suggested in reference 7. In this solution of the linearized form of the equations for compressible flow, the body is stretched in the free-stream direction by the factor $1/\beta$; the induced velocity components $u-U$, v , and w about the stretched body are computed by potential-flow methods (for prolate spheroids, see ref. 5); and the induced velocities $u-U$, v , and w are corrected by the factors $1/\beta^2$, $1/\beta$, and $1/\beta$,

respectively. The corrected velocities are the compressible velocities at the corresponding points on the original body. The following formula, as is shown in appendix A, is the result of the application of this method to the flow over prolate spheroids:

$$1 - \frac{V^2}{U^2} = \frac{1}{\beta^2} \left\{ 1 - \frac{H_{st}^2}{G_{st}} - K_{bst}^2 \sin^2 \omega \sin^2 \alpha_{st} - \left(\frac{1}{\beta^2} - 1 \right) \left[\left(1 - \frac{H_{st} F_{st}}{G_{st}} \right)^2 + \left(K_{bst} \sin^2 \omega \sin^2 \alpha_{st} + \frac{H_{st} F_{st}}{G_{st}} \right)^2 - \left(\frac{H_{st} F_{st}}{G_{st}} \right)^2 - 2 K_{bst} \sin^2 \omega \sin^2 \alpha_{st} \right] \right\} \quad (1)$$

where

$$F_{st} = \sqrt{1 - \mu^2} \cos \alpha_{st} - \mu \sqrt{1 - e_{st}^2} \cos \omega \sin \alpha_{st}$$

$$G_{st} = 1 - e_{st}^2 \mu^2$$

$$H_{st} = \sqrt{1 - \mu^2} K_{ast} \cos \alpha_{st} - \mu \sqrt{1 - e_{st}^2} K_{bst} \cos \omega \sin \alpha_{st}$$

$$K_{ast} = 1 - \frac{\left(\log \frac{1 + e_{st}}{1 - e_{st}} \right) - 2e_{st}}{\left(\log \frac{1 + e_{st}}{1 - e_{st}} \right) - \frac{2e_{st}}{1 - e_{st}^2}}$$

$$K_{bst} = 1 - \frac{\left(\log \frac{1 + e_{st}}{1 - e_{st}} \right) - \frac{2e_{st}}{1 - e_{st}^2}}{\left(\log \frac{1 + e_{st}}{1 - e_{st}} \right) - \frac{2e_{st}(1 - 2e_{st}^2)}{1 - e_{st}^2}}$$

$$e_{st} = \sqrt{1 - \frac{\beta^2}{f^2}}$$

$$\tan \alpha_{st} = \beta \tan \alpha$$

The pressure coefficients may be computed from the following relation:

$$P_c = \frac{\left[1 + \frac{\gamma - 1}{2} M_0^2 \left(1 - \frac{V^2}{U^2} \right) \right]^{\frac{\gamma}{\gamma - 1}} - 1}{\frac{\gamma}{2} M_0^2} \quad (2)$$

Because of the nature of the transformation, equation (1) does not hold for large angles of attack (that is, where $\alpha = \sin \alpha$ ceases to be a fair approximation) or for bodies of small fineness ratio.

The compressibility effects indicated by application of the linearized theory of compressible flow to prolate spheroids are not apparent from equations (1) and (2). The effects may be shown simply for the special case of the center of a prolate spheroid at zero angle of attack. As shown in appendix B, the following relation is obtained:

$$\frac{P_c}{P_i} = \left(1 + \frac{\log \beta}{1 - \log 2f} \right) \left[\frac{f^2 - \log 2f}{f^2 - \beta^2 (\log 2f - \log \beta)} \right] \quad (3)$$

Thus, the theoretical solution indicates that the ratio of the compressible pressure coefficient to the incompressible pressure coefficient on bodies of revolution will vary conformably

with a function of $\log \beta$ and f rather than with $1/\beta$ as in two-dimensional flow. Equation (3) may be reduced to the form $\frac{P_c}{P_i} = 1 + \frac{\log \beta}{1 - \log 2f}$ which is presented in reference 8.

Another and easier method of obtaining an approximate solution of the linearized equations for very thin bodies may be found in references 2 to 4. This method consists of integrating an approximated source-sink distribution to obtain the induced-velocity ratios from which the pressure coefficients may be computed. Since the source-sink distribution is approximated by the derivative of the cross-sectional area with respect to the length of the body, this method is more generally applicable to bodies of revolution than is the method of applying the Prandtl-Glauert correction to the exact incompressible-flow solution. It is shown in appendix A that, for prolate spheroids at zero angle of attack, this method gives the following result:

$$P_c = \frac{1}{f^2} \left[\frac{1}{2 \sqrt{\left(1 - \frac{x}{l} \right)^2 + \beta^2 \frac{r^2}{l^2}}} + \frac{1}{2 \sqrt{\frac{x^2}{l^2} + \beta^2 \frac{r^2}{l^2}}} - \log \frac{1 - \frac{x}{l} + \sqrt{\left(1 - \frac{x}{l} \right)^2 + \beta^2 \frac{r^2}{l^2}}}{-\frac{x}{l} + \sqrt{\frac{x^2}{l^2} + \beta^2 \frac{r^2}{l^2}}} \right] \quad (4)$$

Two approximate forms which show the effects of compressibility can be obtained from equation (4) by considering (a) the difference, $(P_c - P_i)$, and (b) the ratio P_c/P_i of the compressible and the incompressible values. These two relations may be reduced to the following forms when $\frac{\beta^2 r^2}{l^2}$

is considered small with respect to $\left(1 - \frac{x}{l} \right)^2$ or $\frac{x^2}{l^2}$:

$$P_c - P_i = \frac{2 \log \beta}{f^2} \quad (5)$$

$$\frac{P_c}{P_i} = 1 + \frac{\log \beta}{1 - \log 2f} \quad (6)$$

Both relations indicate that the effect of compressibility on the subsonic flow about a body of revolution at any given Mach number is to lower the pressure coefficients over a large part of the body. These relations for the effect of compressibility are in accord with similar equations presented in references 2 and 3.

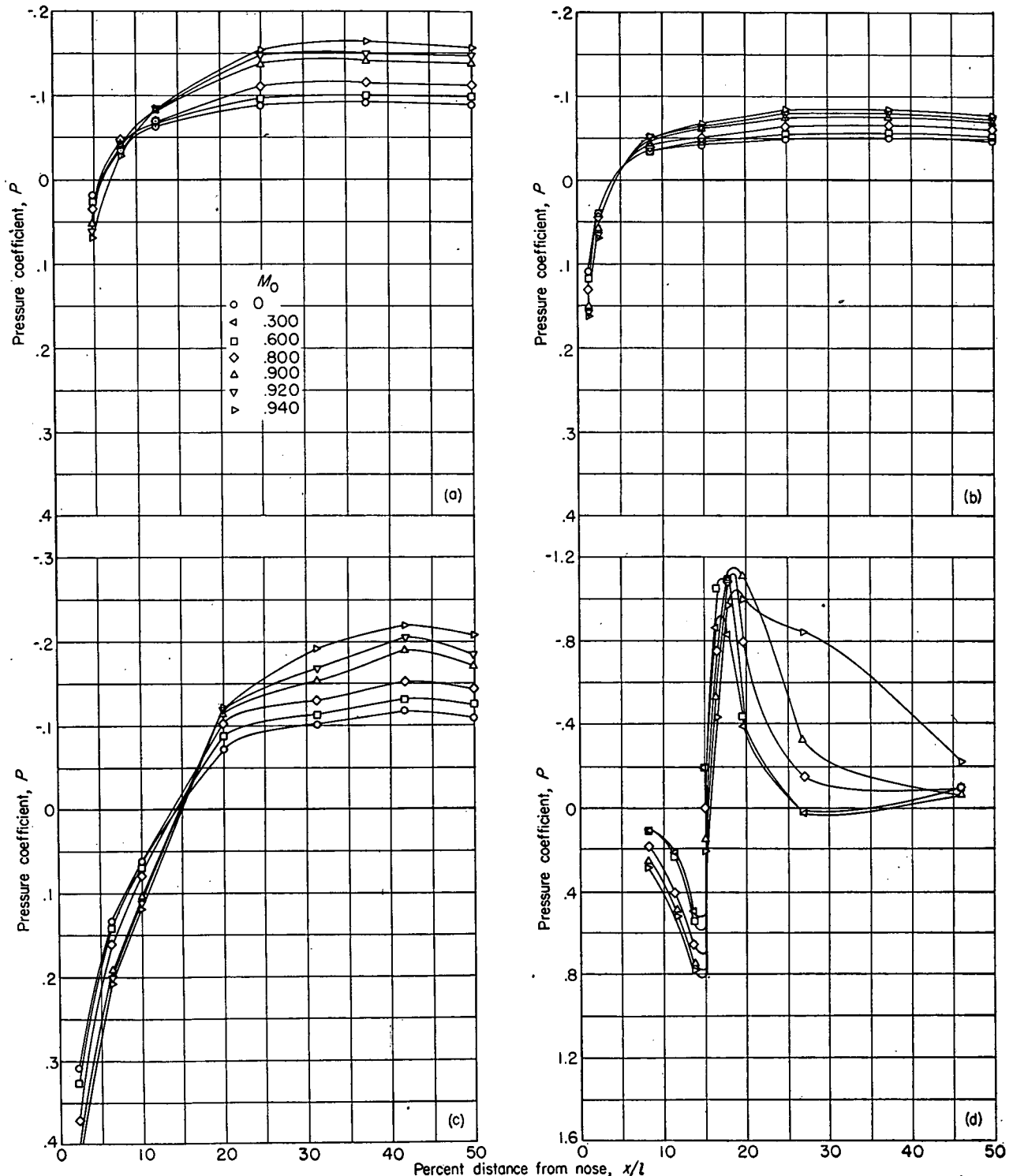
RESULTS AND ANALYSIS

COMPARISON OF EXPERIMENTAL AND THEORETICAL PRESSURE DISTRIBUTIONS

The local pressure-coefficient distributions are presented in figures 2 to 6 for various values of free-stream Mach number. Figures 7 to 9 are replots of some of the data of the preceding figures corrected for wall interference, together with results of the theoretical calculations by means of equations (2) and (4). Figures 2 to 6 show a decrease in the experimental pressures over the central part of the body with increasing Mach number, as predicted by equation (5).

However, figures 7 to 9 indicate that the linearized theory predicts a decrease in the pressures over the entire body, whereas the experimental data show that a point on the body exists ahead of which the pressures increase rather than decrease. (See also figs. 2 to 6.) The lack of agreement of the linearized theory with the experimental results near the nose of the body is to be expected because of the assumptions made in its derivation. It might be pointed out that the

effect of compressibility on the experimental pressure coefficients is approximately to rotate the pressure-coefficient distributions about the point at which the incompressible pressure coefficient is zero. The actual point about which the rotation may be considered to take place shifts its location from slightly downstream of the stream-pressure point on the top of the body to slightly upstream of the stream-pressure point on the bottom of the body.



(a) Prolate spheroid; $f=6$; $M_{cr}=0.916$.

(c) Typical transonic body; $M_{cr}=0.898$.

(b) Prolate spheroid; $f=10$; $M_{cr}=0.947$.

(d) Prolate spheroid with annular bump; $M_{cr}=0.634$.

FIGURE 2.—Experimental pressure distributions over several bodies of revolution at zero angle of attack.

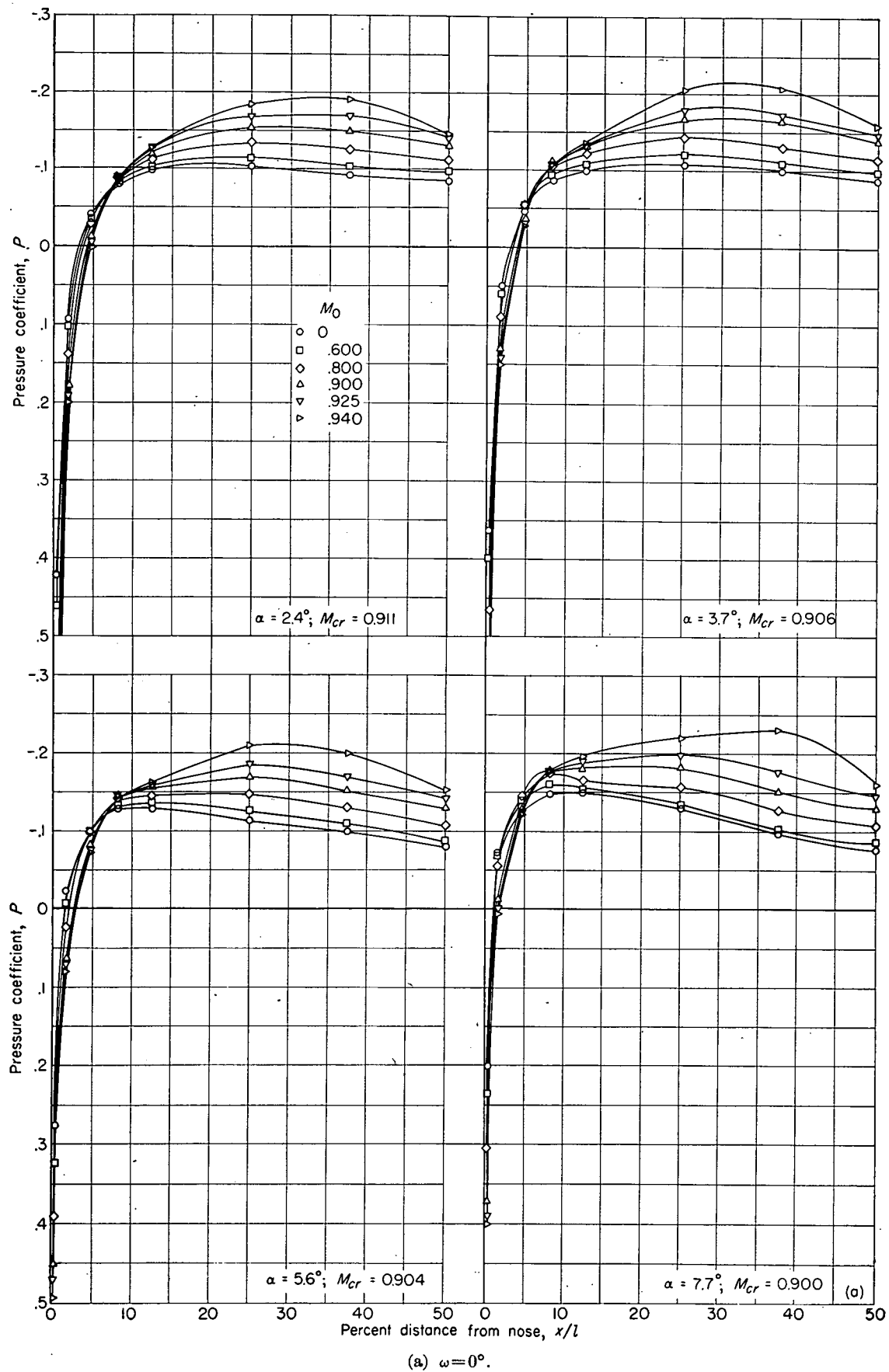


FIGURE 3.—Experimental pressure distributions over a prolate spheroid of fineness ratio 6 at several angles of attack.

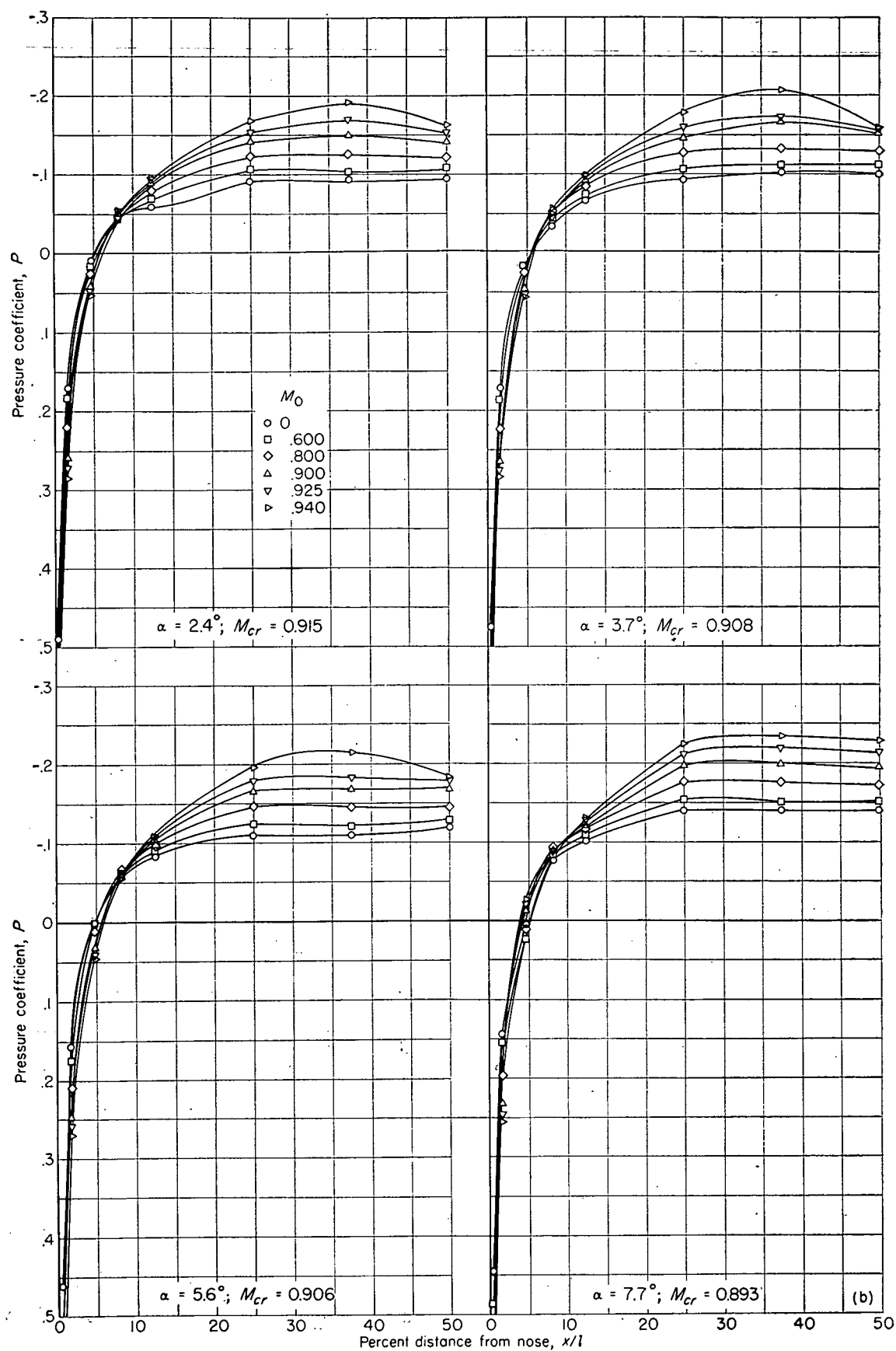
(b) $\omega = 90^\circ$.

FIGURE 3.—Continued.

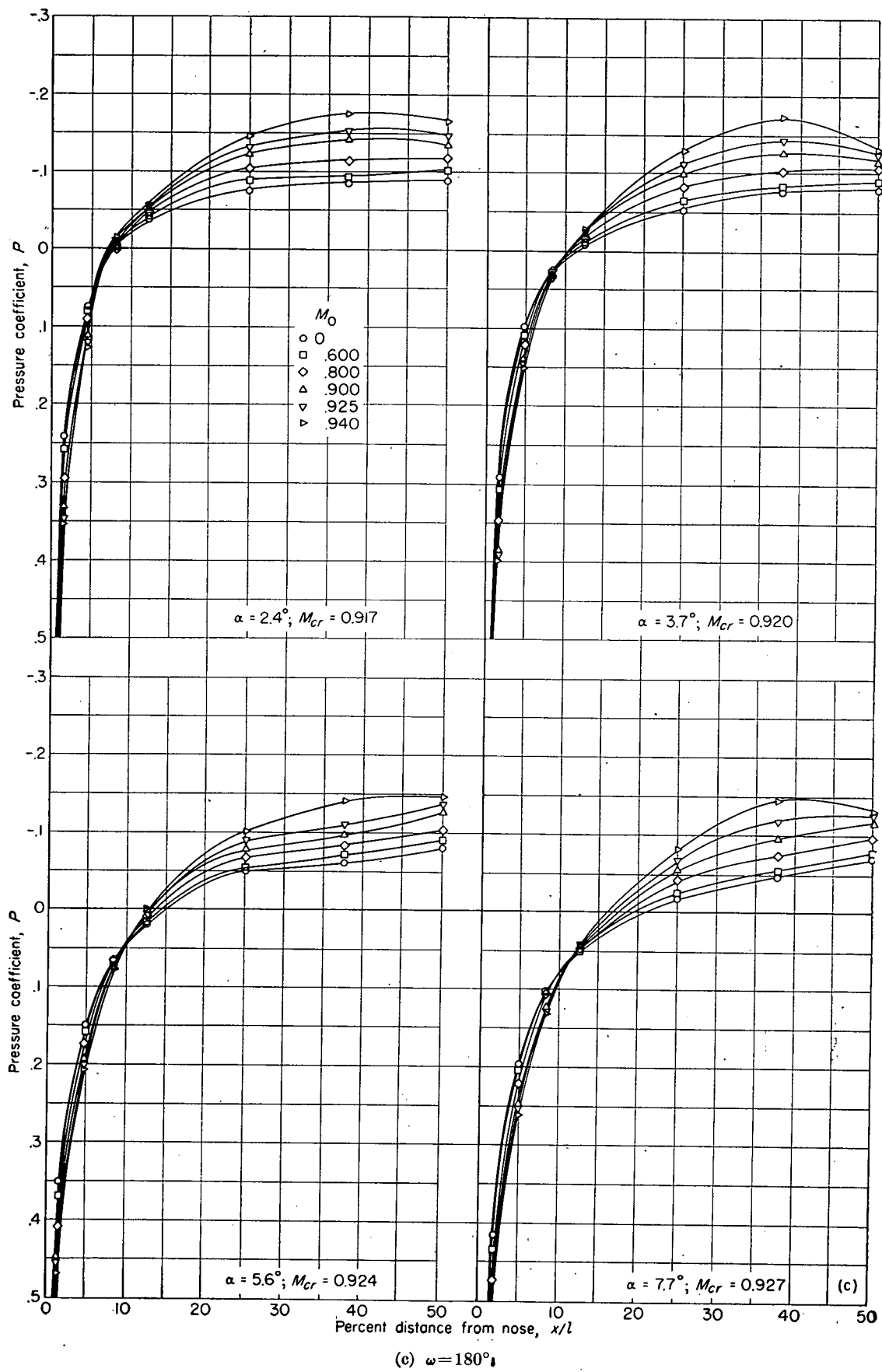


FIGURE 3.—Concluded.

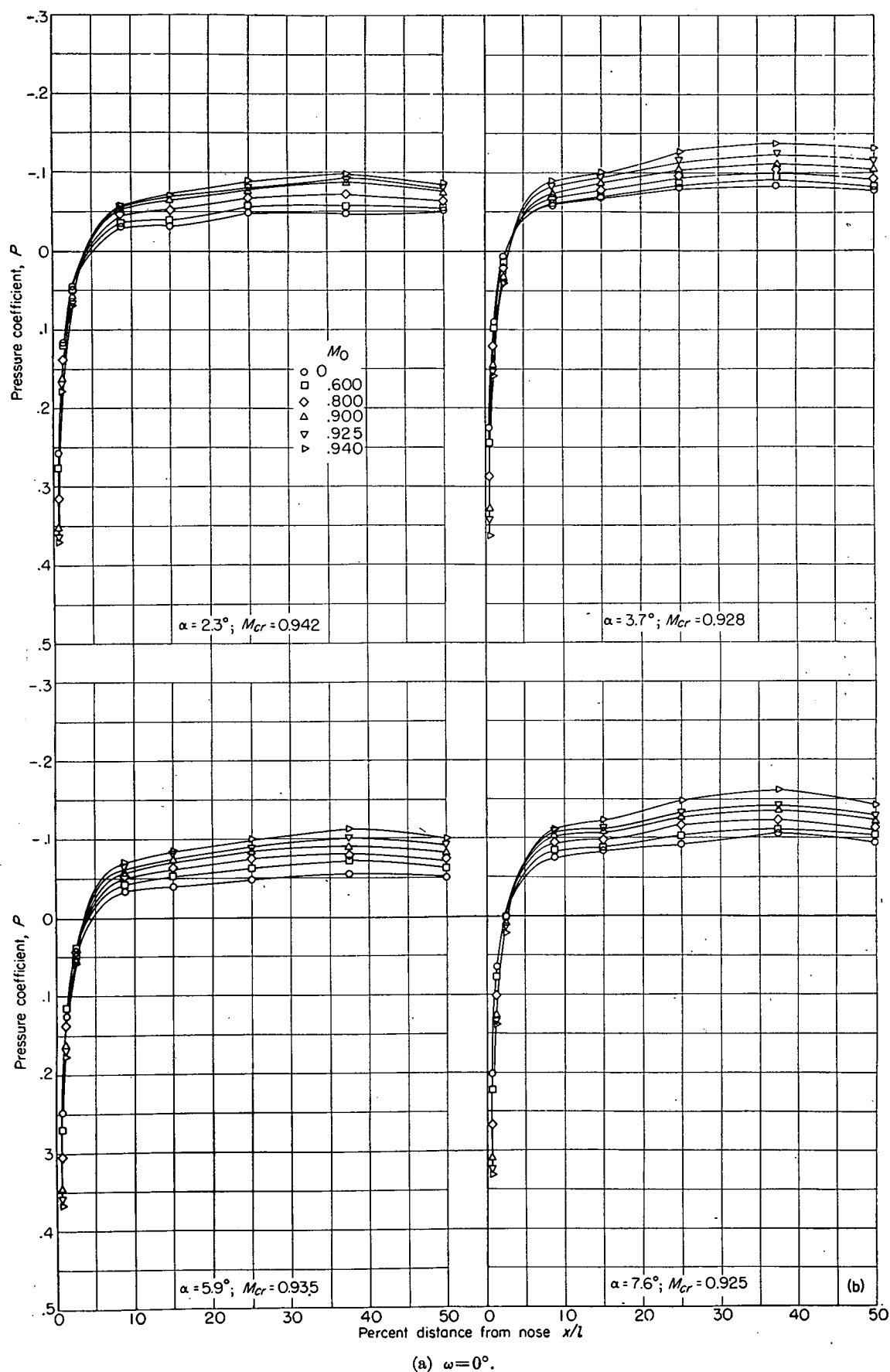


FIGURE 4.—Experimental pressure distributions over a prolate spheroid of fineness ratio 10 at several angles of attack.

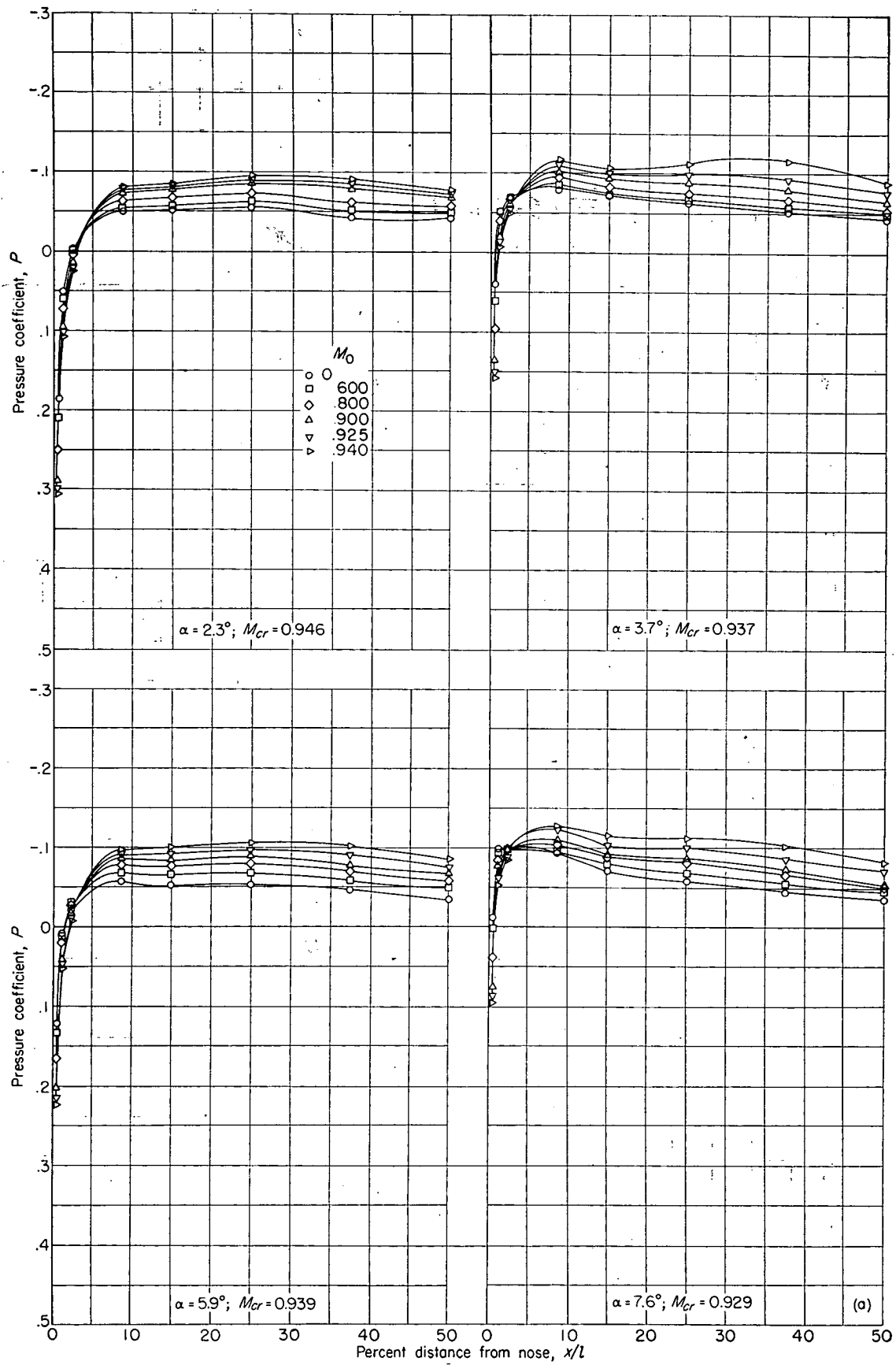
(b) $\omega = 90^\circ$.

FIGURE 4.—Continued.

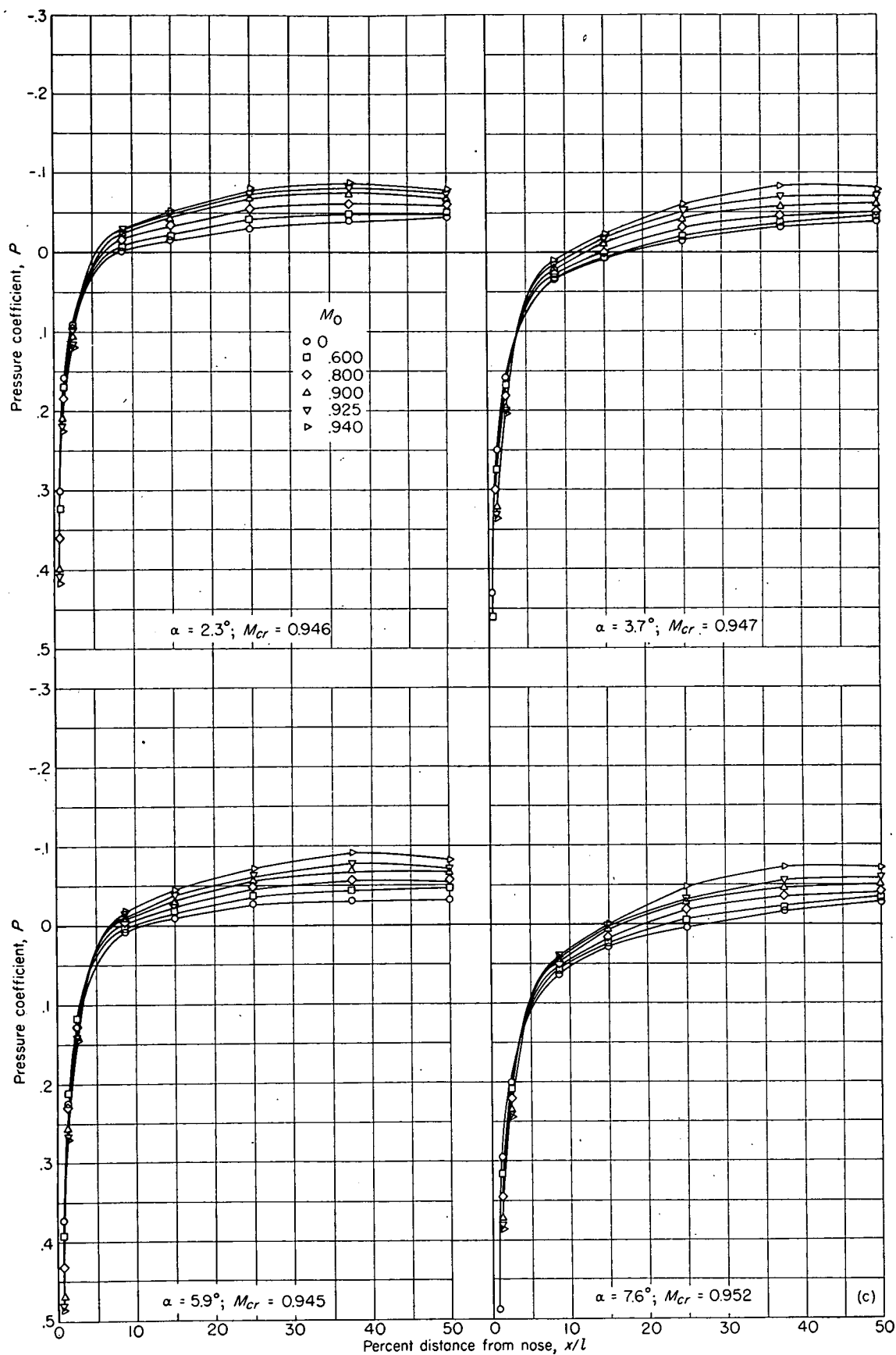
(c) $\omega = 180^\circ$.

FIGURE 4.—Concluded.

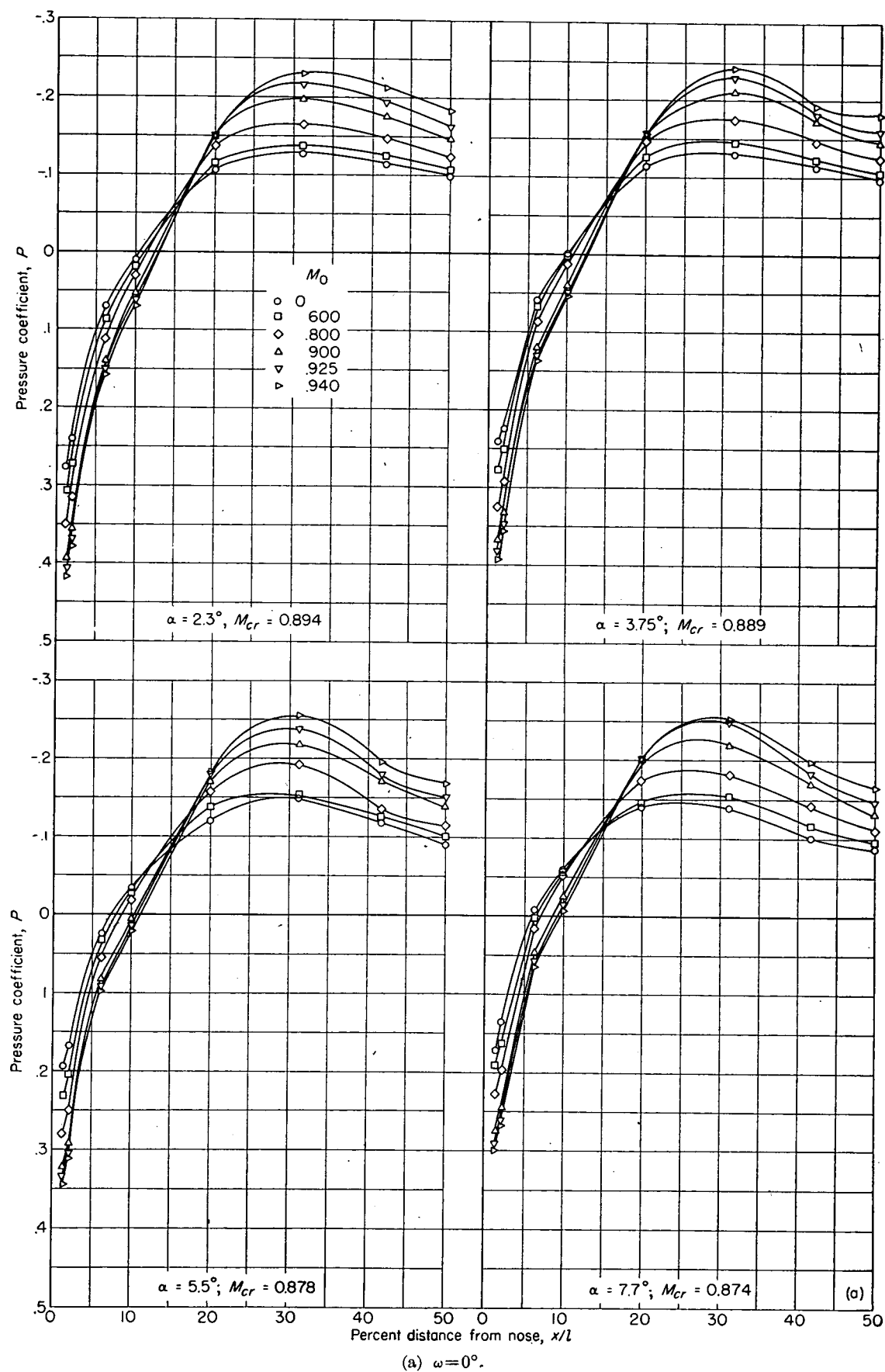


FIGURE 5.—Experimental pressure distribution over a typical transonic body at several angles of attack.

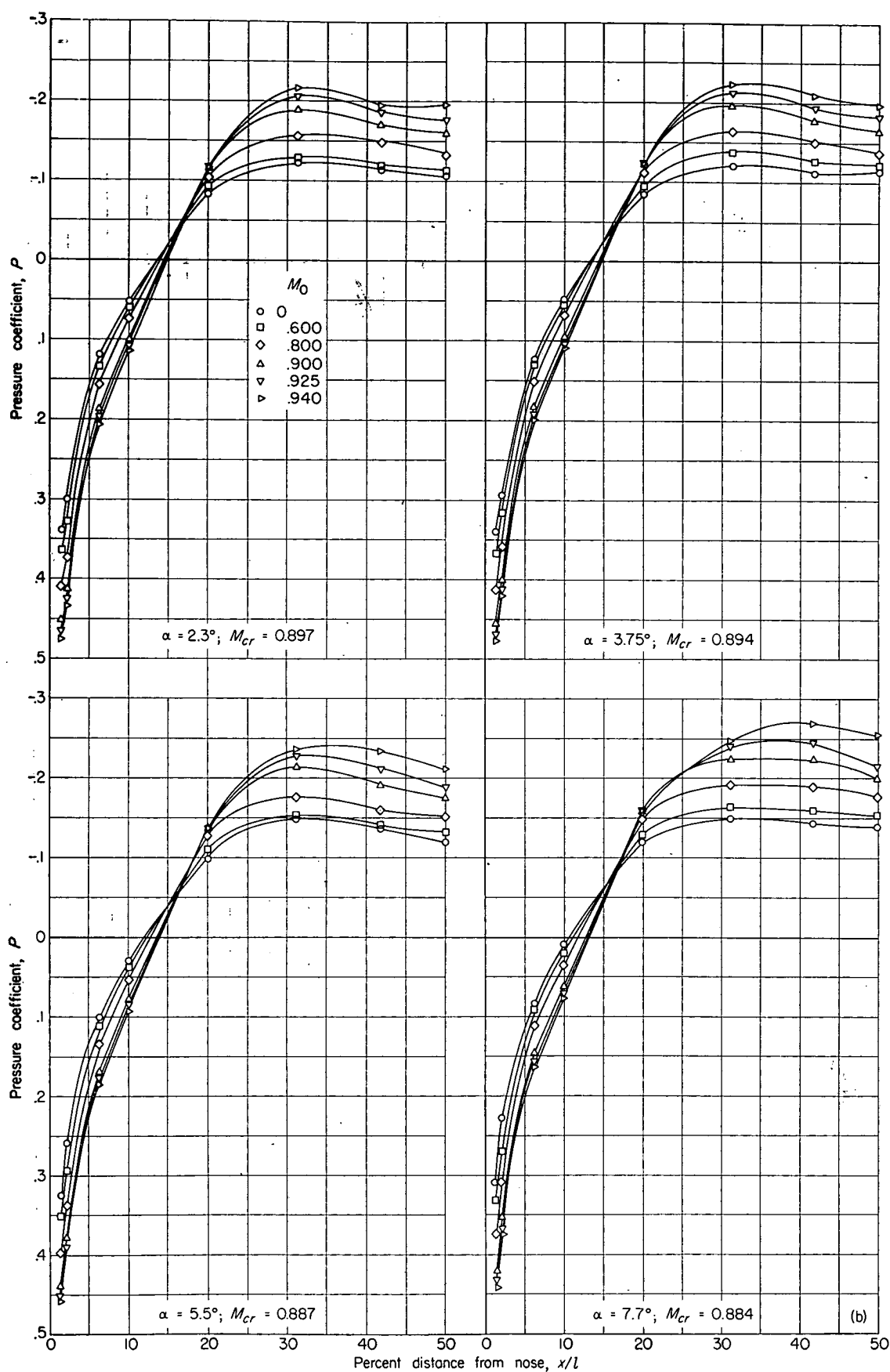
(b) $\omega = 90^\circ$.

FIGURE 5.—Continued.

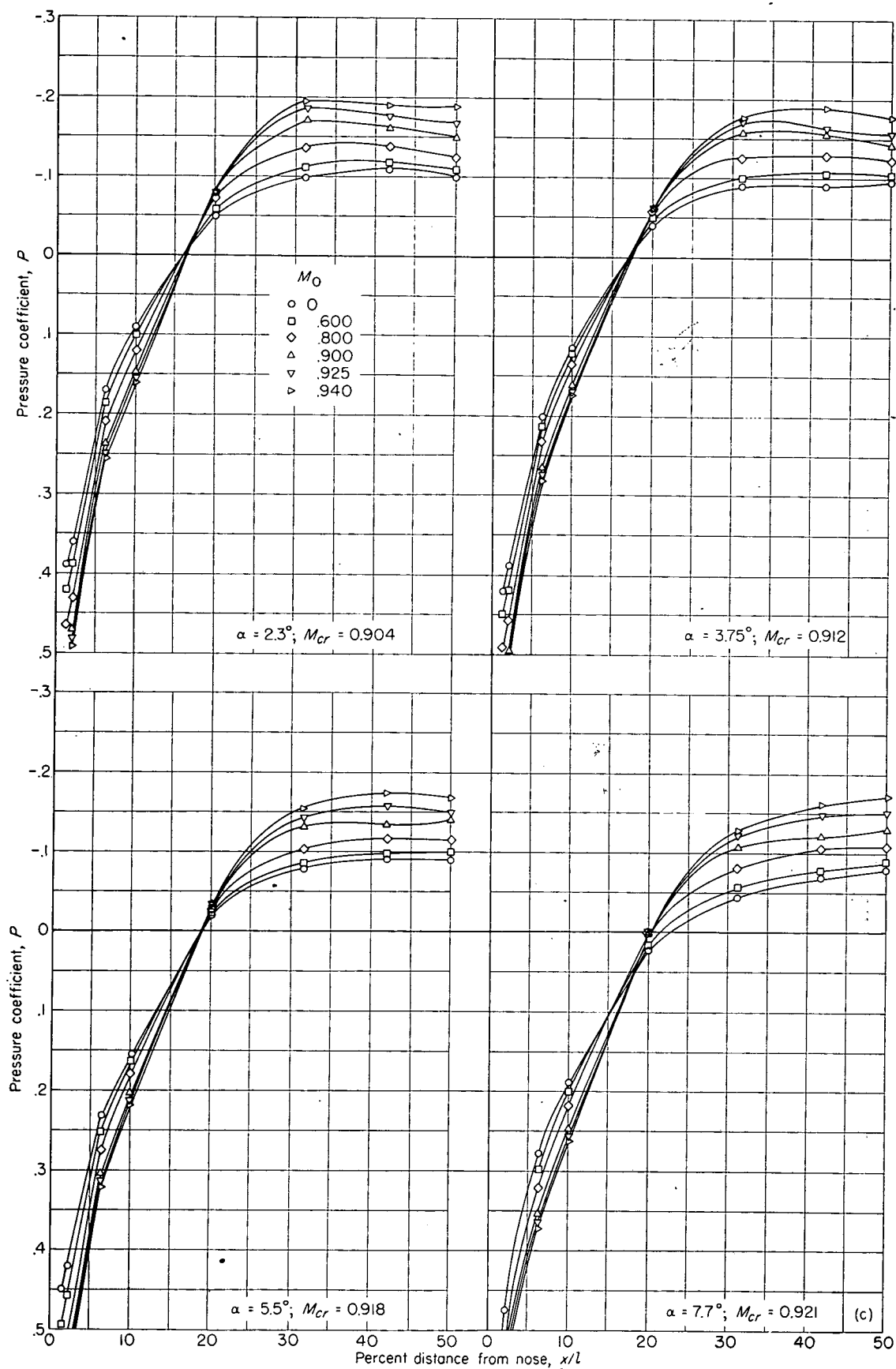
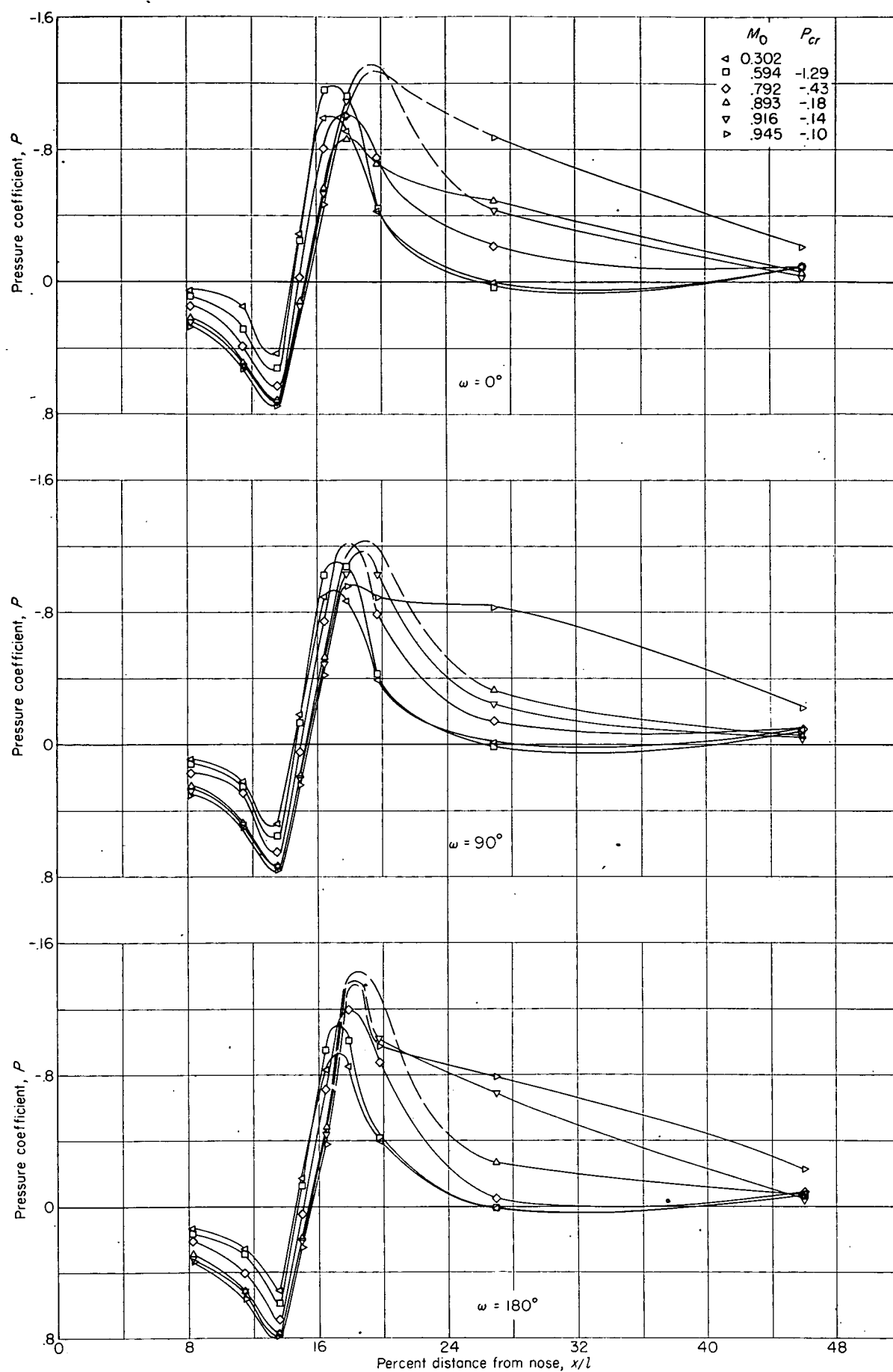
(c) $\omega = 180^\circ$.

FIGURE 5.—Concluded.

FIGURE 6.—Experimental pressure distributions over a prolate spheroid with an annular bump. $\alpha = 2.3^\circ$.

As the flow approaches and exceeds the critical stream Mach number, a further change in the pressure distributions occurs. This change of shape (figs. 2 to 5) is essentially a rearward movement of the negative pressure peak. The nature of this change is emphasized in the plots for $M_0=0.950$ of figures 7 and 8. These figures show that the linearized theory does not predict the shift in peak pressures which occurs as the flow becomes supercritical.

The rearward shift of negative pressure peaks which occurs on the top of the body (figs. 3 (a), 4 (a), and 5 (a)) seems to be changed to a forward shift on the bottom of the body (figs. 3 (c), 4 (c), and 5 (c)). It is reasonable to assume that part or all of this forward movement of the bottom negative pressure peak may be explained by the positive pressure field which exists ahead of the under part of the sting support.

A comparison of figures 7 and 9 shows that the linearized theory gives better results for the body of larger fineness ratio. The pressures about the prolate spheroid of fineness ratio 10 are in better agreement with theory even for the stream Mach number of 0.950 than are the pressures about

the body of fineness ratio 6. It may also be observed that the theoretical pressures about the prolate spheroid of fineness ratio 10, which are calculated by the two different methods, are in excellent agreement; thus, these results show that, for bodies of fineness ratios of 10 or greater, the simpler method of computing pressures presented in references 2 to 4 is fairly reliable.

INFLUENCE OF CHANGING NOSE SHAPE

The effects of changing the shape of the nose of a body are seen by comparing figures 2 (a) and 3 with figures 2 (c) and 5. The incompressible pressure distribution is changed as may be expected. However, the nature of the effect of compressibility is the same for this body as for the prolate spheroid of fineness ratio 6. The incremental pressure changes are almost the same, and the rotation and shifts of pressure peaks are very similar for both bodies. This comparison shows that the effects of compressibility do not depend to a great extent on body shape so long as the body does not depart from the specifications required for the application of the linearized equations.

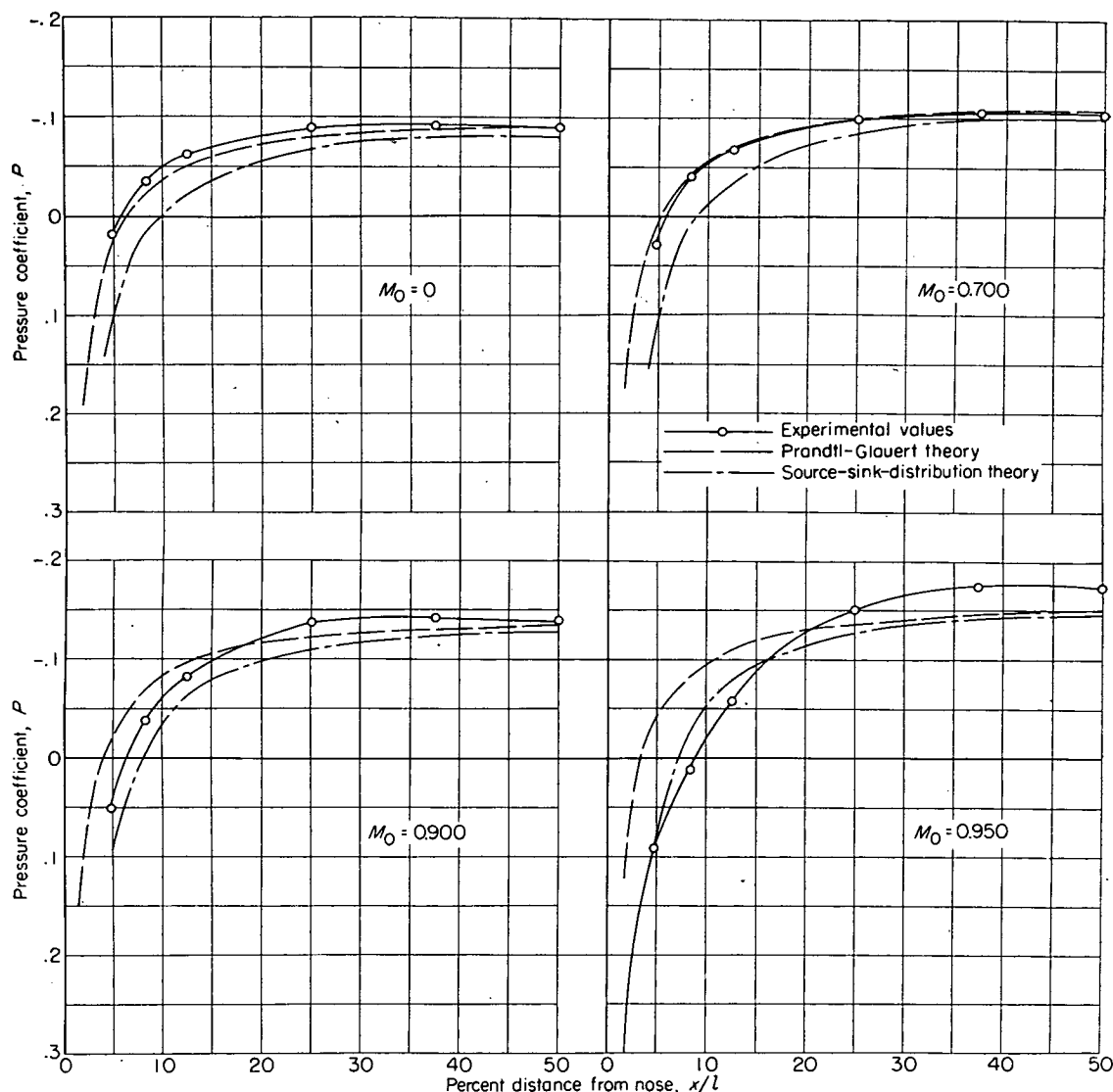
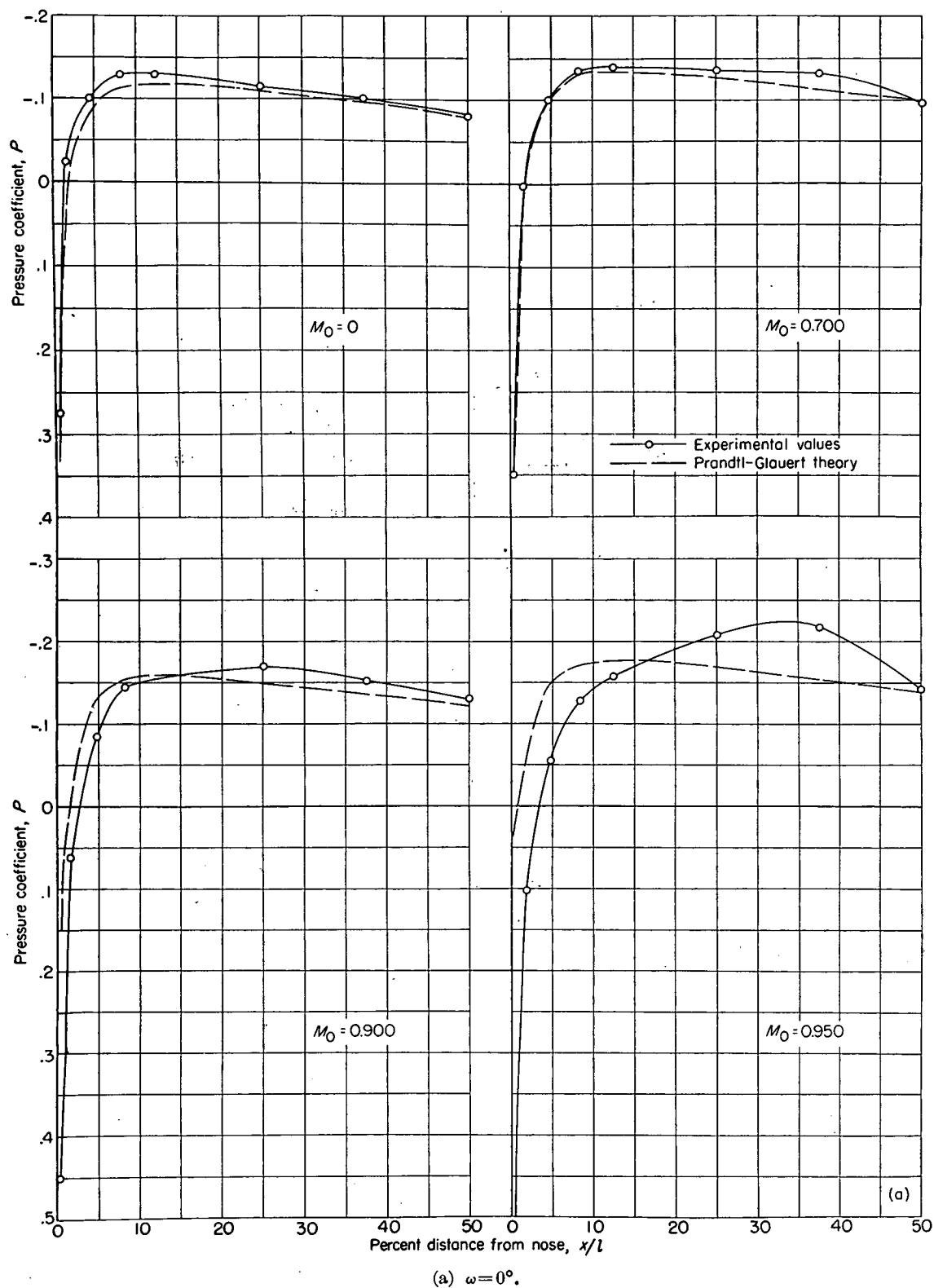


FIGURE 7.—Comparison of experimental and theoretical pressures over a prolate spheroid of fineness ratio 6 at zero angle of attack.

(a) $\omega = 0^\circ$.FIGURE 8.—Comparison of experimental and theoretical pressures over a prolate spheroid of fineness ratio 6 at 5.6° angle of attack.

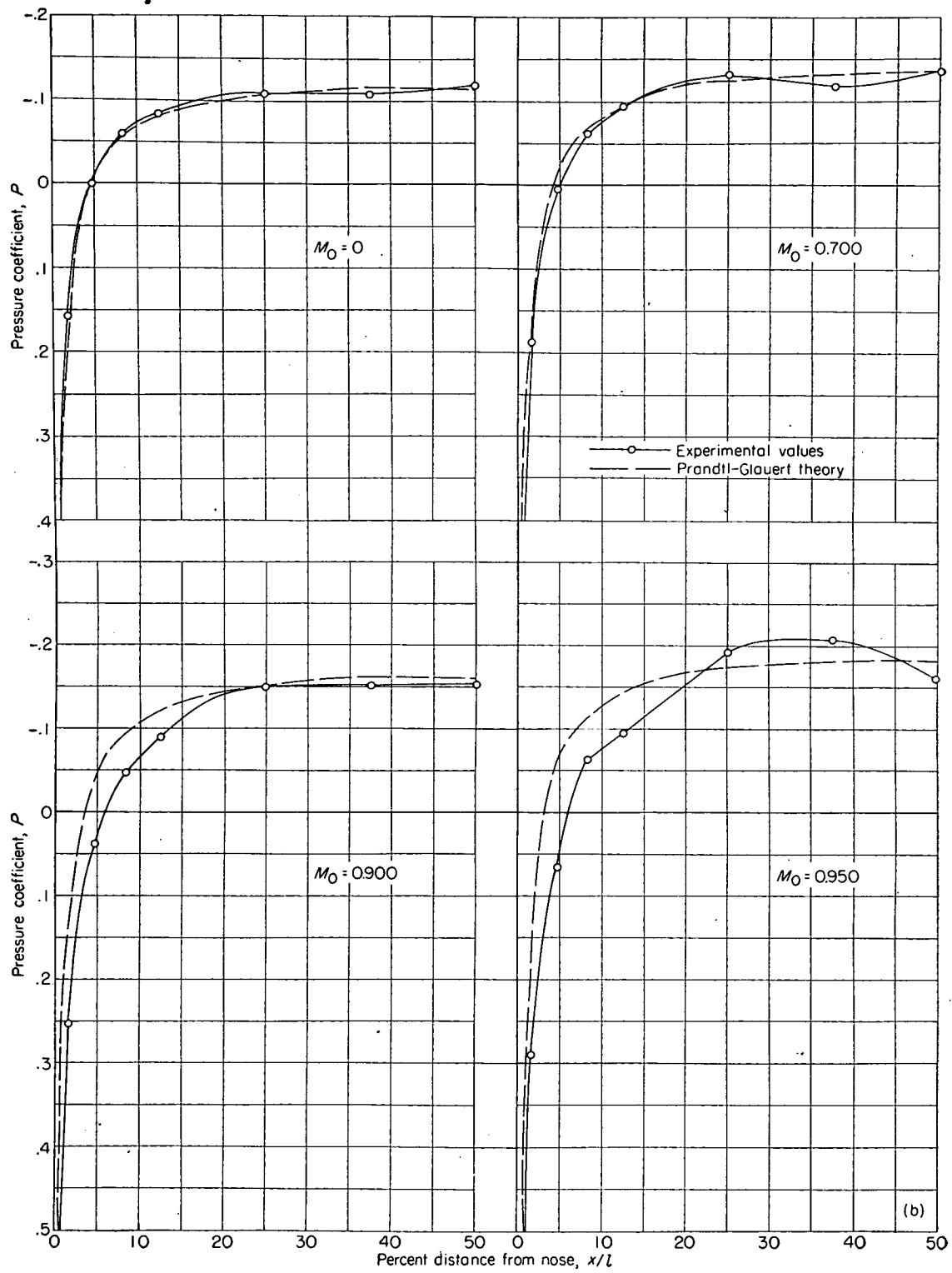
(b) $\omega = 90^\circ$.

FIGURE 8.—Continued.

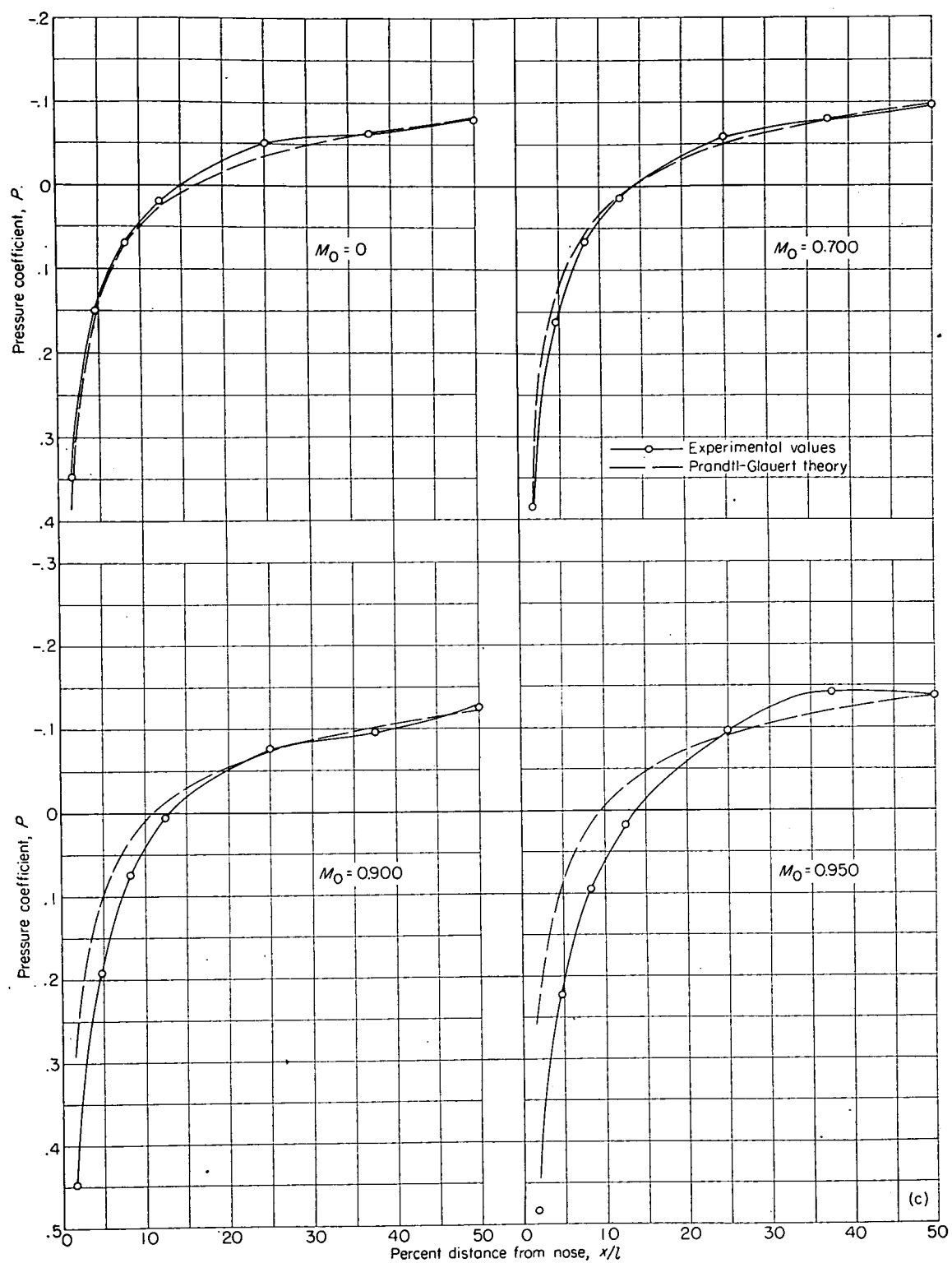
(c) $\omega = 180^\circ$

FIGURE 8.—Concluded.

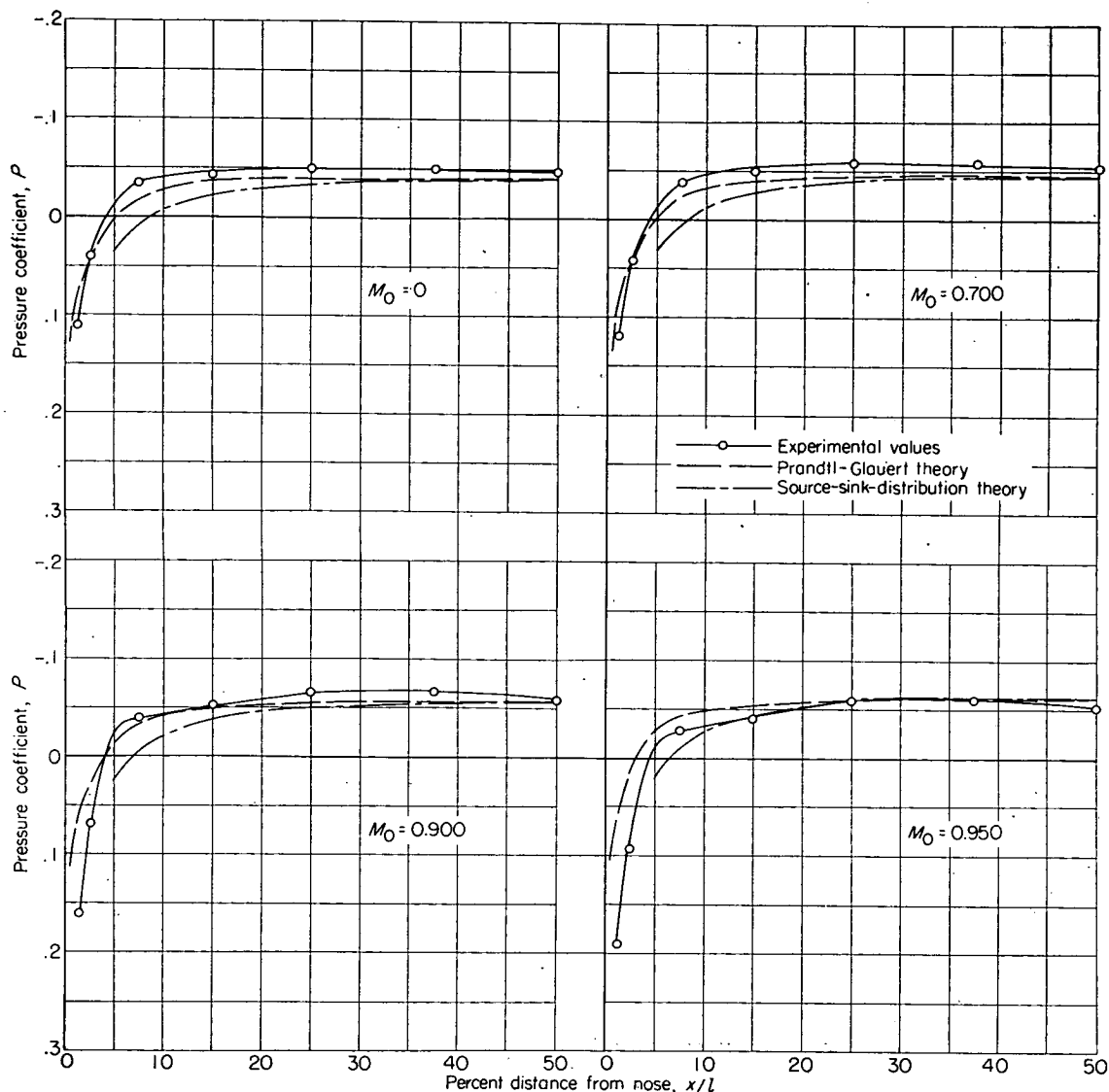


FIGURE 9.—Comparison of experimental and theoretical pressures of a prolate spheroid of fineness ratio 10 at zero angle of attack.

INFLUENCE OF FINENESS RATIO

The influence of fineness ratio on the effects of compressibility may be observed by comparing figures 2 (a) and 3 with figures 2 (b) and 4. These figures show that increasing the fineness ratio reduces the changes in pressure caused by varying the stream Mach number. This effect is predicted by the linearized theory in equation (5). It may also be observed that the pressure peaks are less prominent and do not shift their location to the extent found for the bodies of lower fineness ratio. The changes in the shape of the pressure distributions are also reduced and comparable changes occur at higher Mach numbers. The delay in the change of the shape of the pressure distribution is demonstrated by comparing figures 7 and 9 at $M_0=0.95$. For the prolate spheroid of fineness ratio 6 a marked change in the pressure

distribution has already occurred, whereas for the body of fineness ratio 10 the shape of the pressure-distribution curve is almost the same as at lower Mach numbers. A consideration of the observed effects of increasing the fineness ratio indicates that such a change definitely reduces the effects of compressibility.

INFLUENCE OF ANGLE OF ATTACK

It may be shown by the use of the linearized theory that, at least to a first approximation, the lift and moment forces on a body of revolution are not affected by changes in Mach number. (See ref. 4.) The validity of this prediction is demonstrated in figure 10 which shows that the variation of the normal-force coefficient with Mach number is small for both the $f=10$ and $f=6$ prolate spheroids.

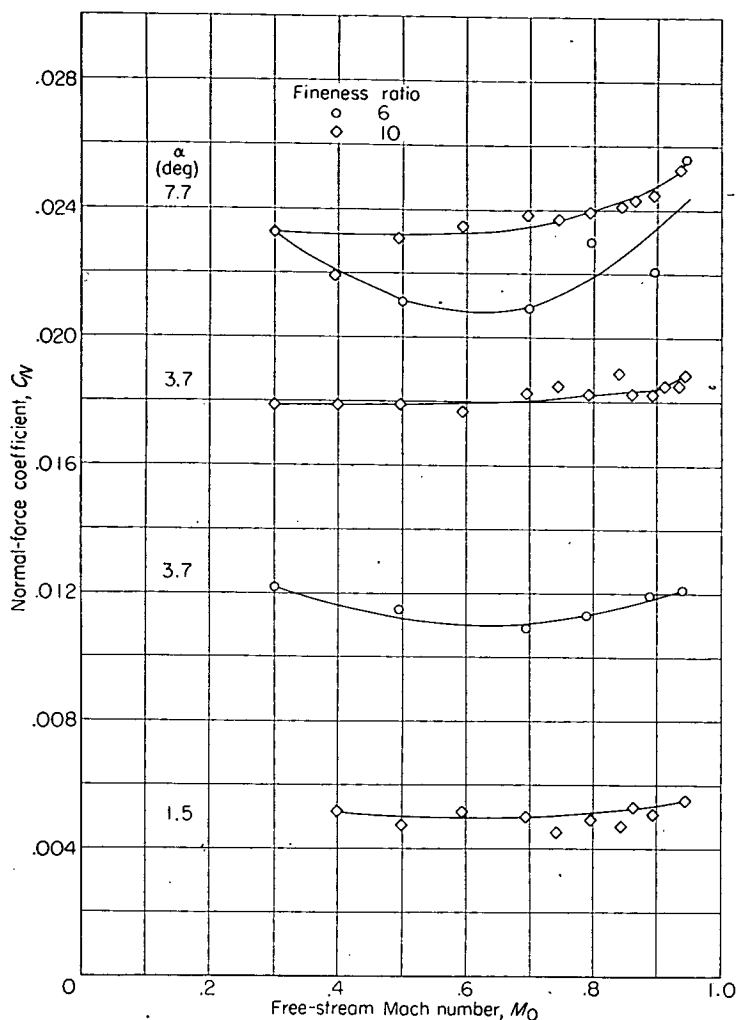


FIGURE 10.—The effect of compressibility on the normal-force coefficients over the forward half of two prolate spheroids of fineness ratios 6 and 10.

INFLUENCE OF AN ANNULAR BUMP

A study of the effects of compressibility on the velocities about an infinitely long body containing surface waves (ref. 9) shows that these effects become two-dimensional in nature when the length of the surface waves becomes small with respect to the body radius. Since an annular bump on a body of revolution approximates these conditions, the flow over such a bump may also be expected to show two-dimensional effects. An examination of figures 2 (d) and 6 shows that the range of pressure coefficients found in the flow over a prolate spheroid with an annular bump is of the same order as that found in two-dimensional flow. The two-dimensional nature of the flow over an annular bump is further demonstrated by comparing the pressure coefficients with the Von Kármán relationship (ref. 10) for the effects of compressibility on two-dimensional flow (fig. 11). Figure 11 shows fair agreement between the Von Kármán relation and the experimental relationships for those regions of the body where the flow does not separate and the slope of the body is reasonably small; namely, the 8.33-, 11.5-, 13.6-, 16.5-, 17.9-, and 19.8-

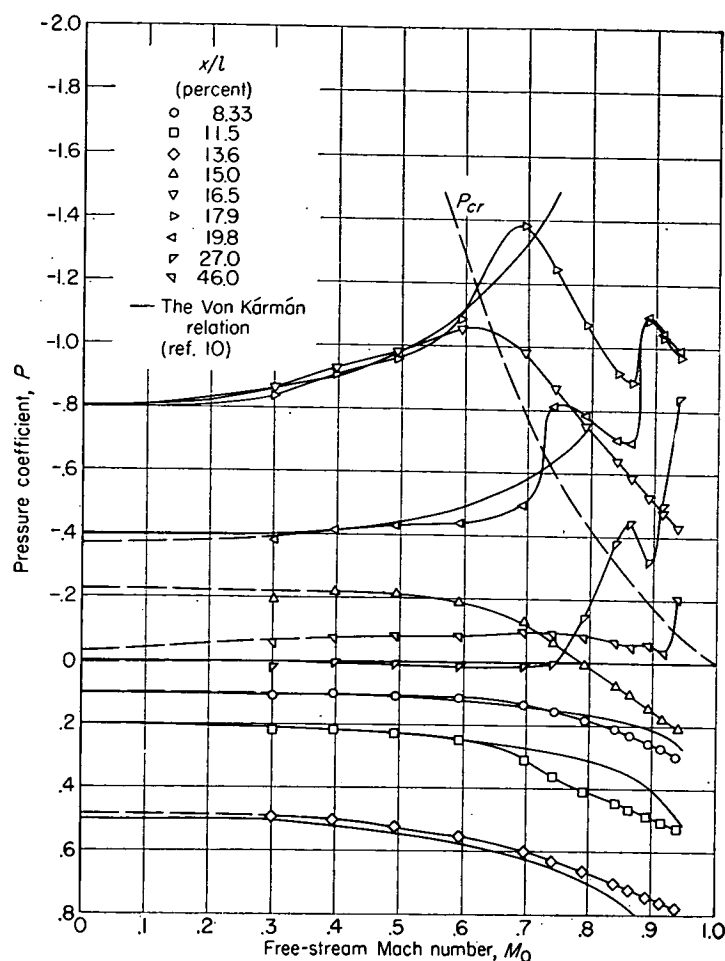


FIGURE 11.—Experimental pressure distributions over a prolate spheroid with an annular bump. $\alpha=0^\circ$.

percent stations. The 15-percent station is highly irregular and cannot be explained by either two- or three-dimensional theories. The other stations are severely affected by separation phenomena. The Von Kármán relation, however, fails to explain the phenomena once the critical speed is exceeded.

CORRECTION OF INCOMPRESSIBLE PRESSURE DISTRIBUTIONS FOR THE EFFECTS OF COMPRESSIBILITY

Equations (5) and (6) suggest that an incompressible pressure distribution might be corrected for the effects of compressibility by considering a pressure-increment type of function such as $P_c - P_i$ or a rate-of-increase type of function such as P_c/P_i . In order to show whether the effects of compressibility may be expressed by such functions, a number of the pressures over the regular bodies at zero angle of attack have been plotted in figure 12 in terms of P_c/P_i and $P_c - P_i$ against x/l and M_0 . Tunnel-wall corrections have been omitted, but the omission does not affect the conclusions. An examination of both functions shows that, except at supercritical Mach numbers, the values of P_c/P_i and $P_c - P_i$ are roughly constant between the 25-percent and the 50-percent stations. Over the forward part of the body, the values are more variable.

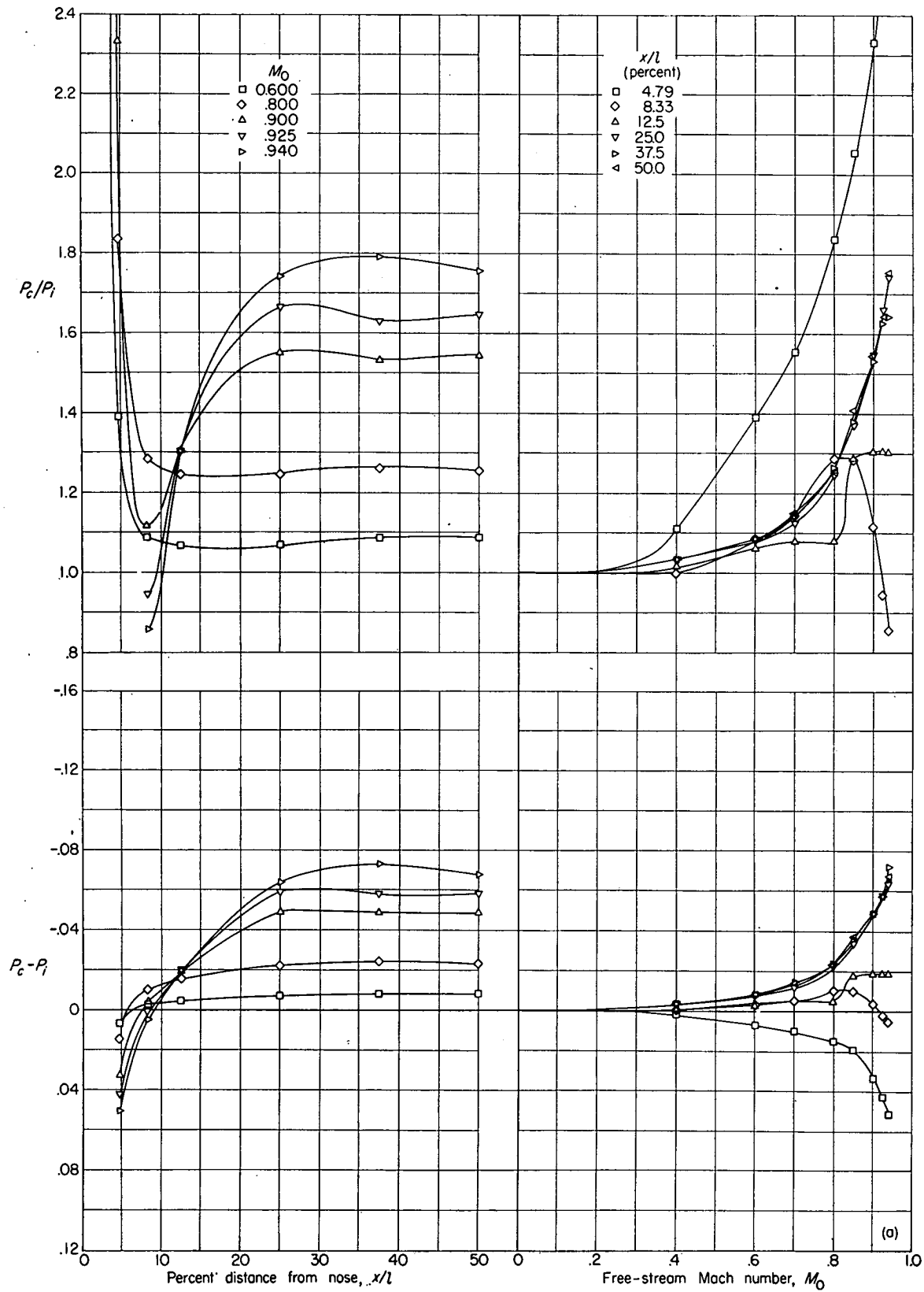
(a) Prolate spheroid; $f=6$.

FIGURE 12.—Experimental compressibility correction functions as determined by the flow over three regular bodies of revolution at zero angle of attack.

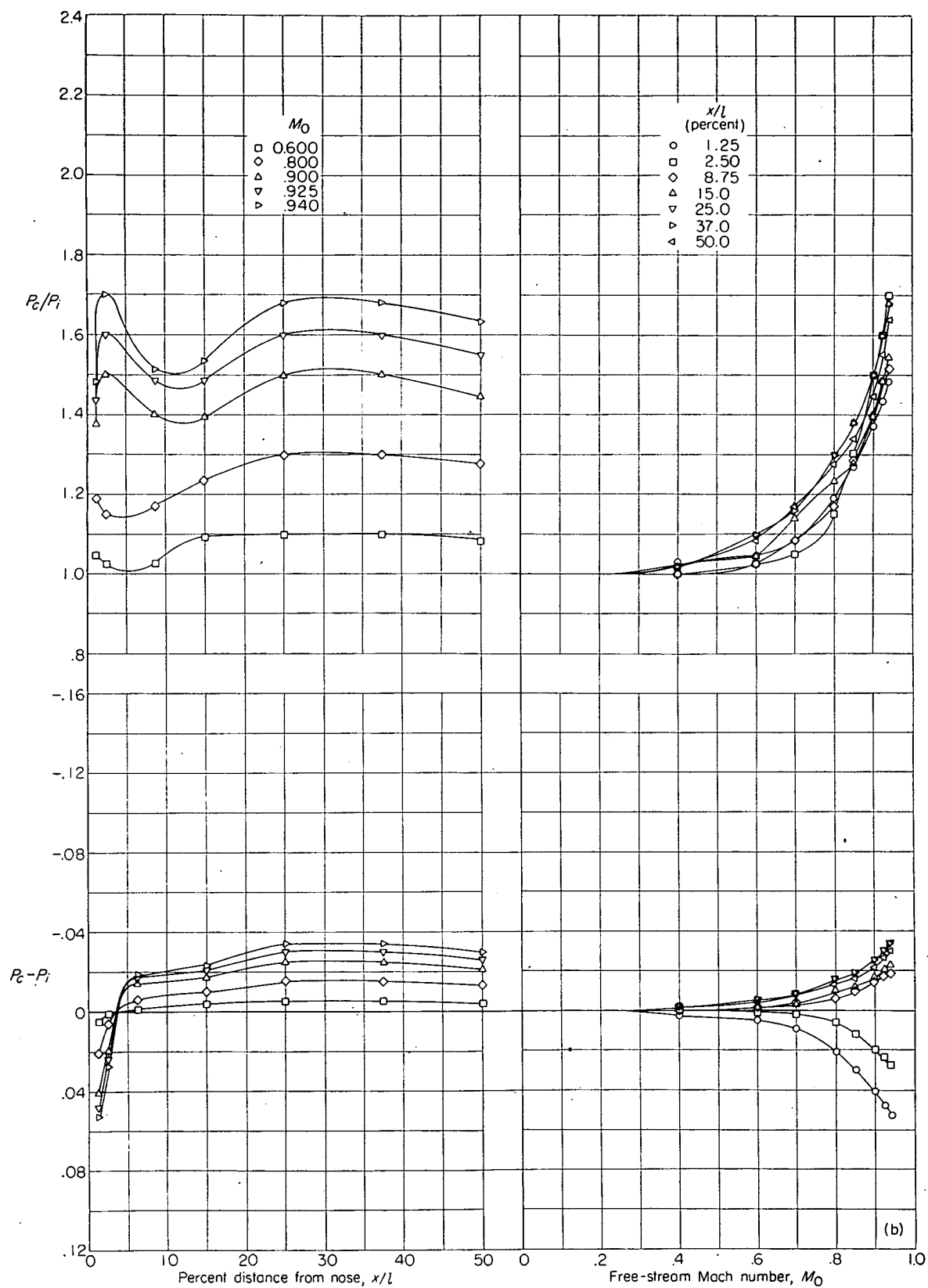
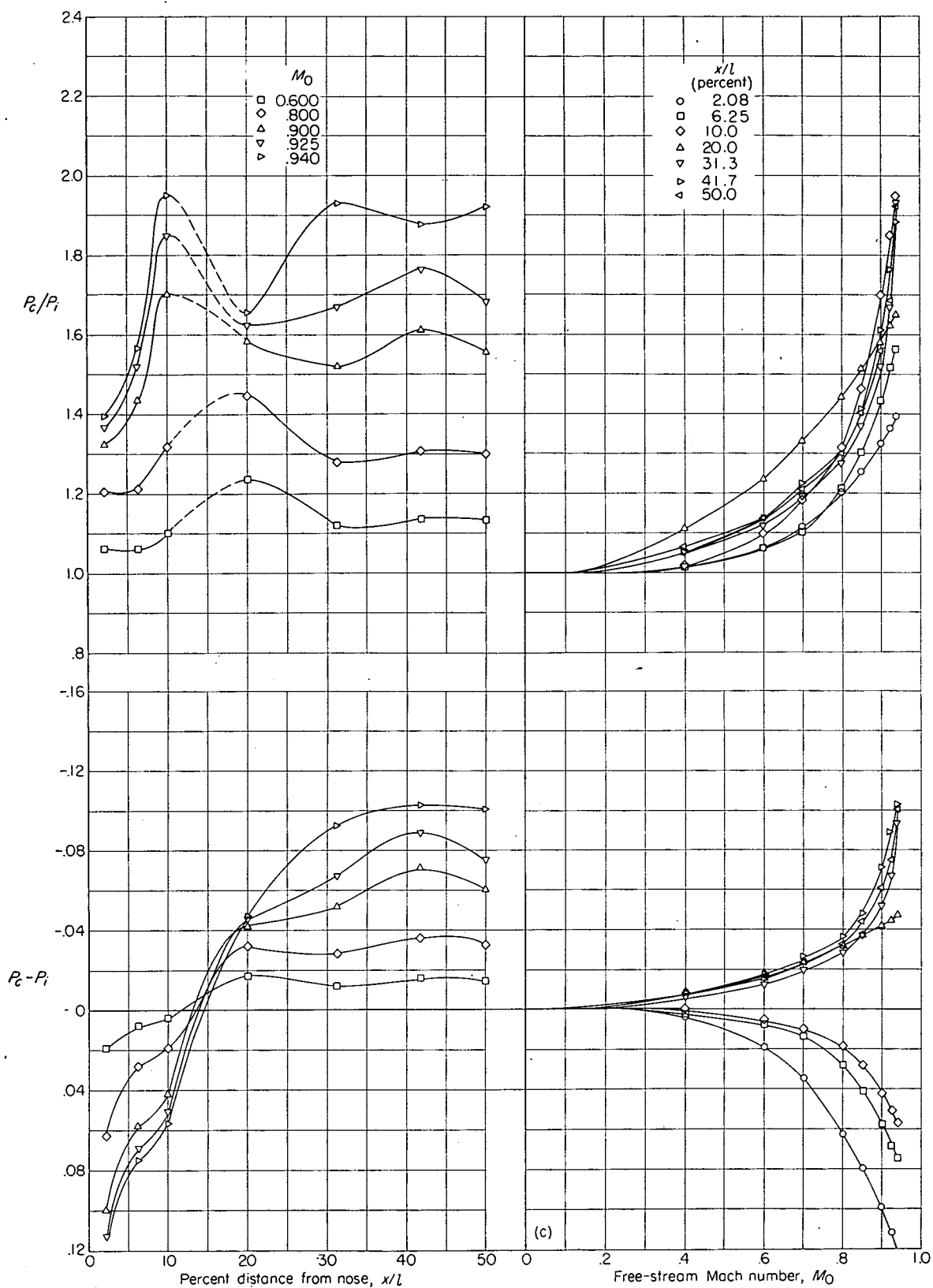
(b) Prolate spheroid; $f=10$.

FIGURE 12.—Continued.



(c) Typical transonic body.

FIGURE 12.—Concluded.

The P_c/P_i function becomes discontinuous in the neighborhood of $P_i=0$. This behavior may be attributed to the fact that the pressure coefficient is zero at the incompressible stream-pressure point and, since one of the effects of compressibility is to shift the stream-pressure point, discontinuities may be expected in the neighborhood of this point. However, since the pressures in this region are small, a wide variation in P_c/P_i may be permissible without serious error in the corrected results.

The $P_c - P_i$ correction may also be expected to become irregular in the region of the nose. The experimental curves show that this function changes sign in the neighborhood of the stream-pressure point so that any correction function of this type should include the position on the body. However, such a function cannot be obtained from the linearized method as this method does not indicate the change of sign shown in the experimental data.

The experimental values of $P_c - P_i$ and P_c/P_i at the centers of the regular bodies are compared with equations (5) and (6) in figure 13 in order to show the validity of the prediction of the effect of compressibility by the linearized potential-flow theory. It is observed that equation (6)

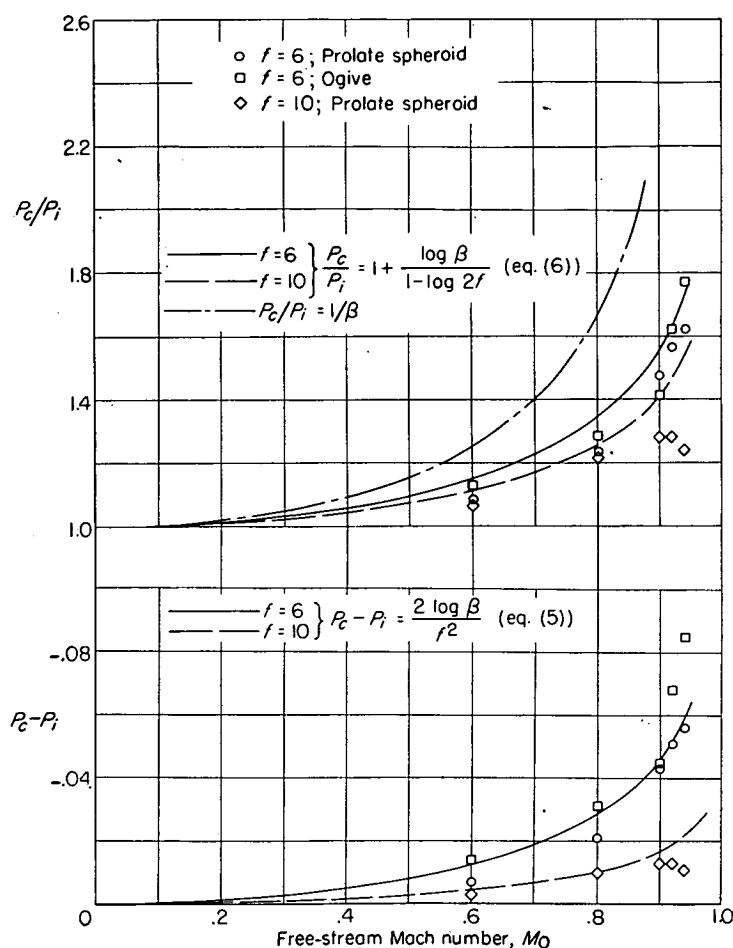


FIGURE 13.—Theoretical compressibility correction functions compared with experimental results. $\alpha=0^\circ$; $r/l=50$ percent.

within its limitations predicts the effects of compressibility for three-dimensional flow whereas the relation $\frac{P_c}{P_i} = \frac{1}{\beta}$, which is used to predict the compressibility effects of two-dimensional flow, does not. It may also be observed that equations (5) and (6) predict the effects of compressibility with about the same degree of accuracy.

The correction functions are applied to several incompressible pressure-coefficient distributions in figure 14, which are compared with the corresponding experimental distributions. It is shown in figure 14 (a) that increasing the fineness ratio of the prolate spheroid from 6 to 10 or reducing the bluntness of the nose, which is the essential difference between the ogival body and the prolate spheroid, extends the region of the body for which corrections can be made from the 20-percent station for the prolate spheroid of fineness ratio 6 forward at least to the 10-percent station for the sharper-nose bodies. The P_c/P_i function expresses the effect of compressibility more accurately in the vicinity of the nose than does the $P_c - P_i$ function. This result is to be expected since one of the effects of compressibility already noted is the rotation of the pressure distribution, which is accounted for by the P_c/P_i expression but not by the $P_c - P_i$ expression.

The increasing error which results from increasing the stream Mach number is shown in figure 14 (b). At $M_0=0.800$, the incompressible pressure coefficients about the fineness ratio 6 prolate spheroid may be corrected with a fair degree of accuracy as far forward as the 5-percent station. As the Mach number increases, the divergence between the corrected values and the experimental values in the region of the nose increases and, with still greater Mach numbers, tends to spread toward the center. At $M_0=0.940$, which is supercritical for the prolate spheroid of fineness ratio 6, the correction formulas are still applicable at the center, so that successful extrapolation of the linearized theory into the supercritical range is found to depend on the section of the body to which the extrapolation is applied.

As may be expected, the success of the linearized theory in expressing the effects of compressibility decreases as the angle of attack increases. The principal reason for this result is that an angle of attack involves a pressure peak on the forepart of the top of the body, which moves rearward when the stream Mach number approaches and exceeds the critical value for the body. Since the correction formulas either rotate or translate the incompressible pressure distribution, they cannot express this change in the shape of the pressure distribution. This phenomenon is demonstrated in figure 14 (c), which presents a comparison of the corrected pressure-coefficient distributions and the experimental distributions of the flow about the prolate spheroid of fineness ratio 6 at several angles of attack. Even though the shift of the peak pressure is not accounted for in the correction formula, the corrected distributions are not seriously in error at the peaks and the agreement improves over the midportion of the body. Thus, if some error is permissible, these formulas may be applied for angles of attack as high as 7° or 8° .

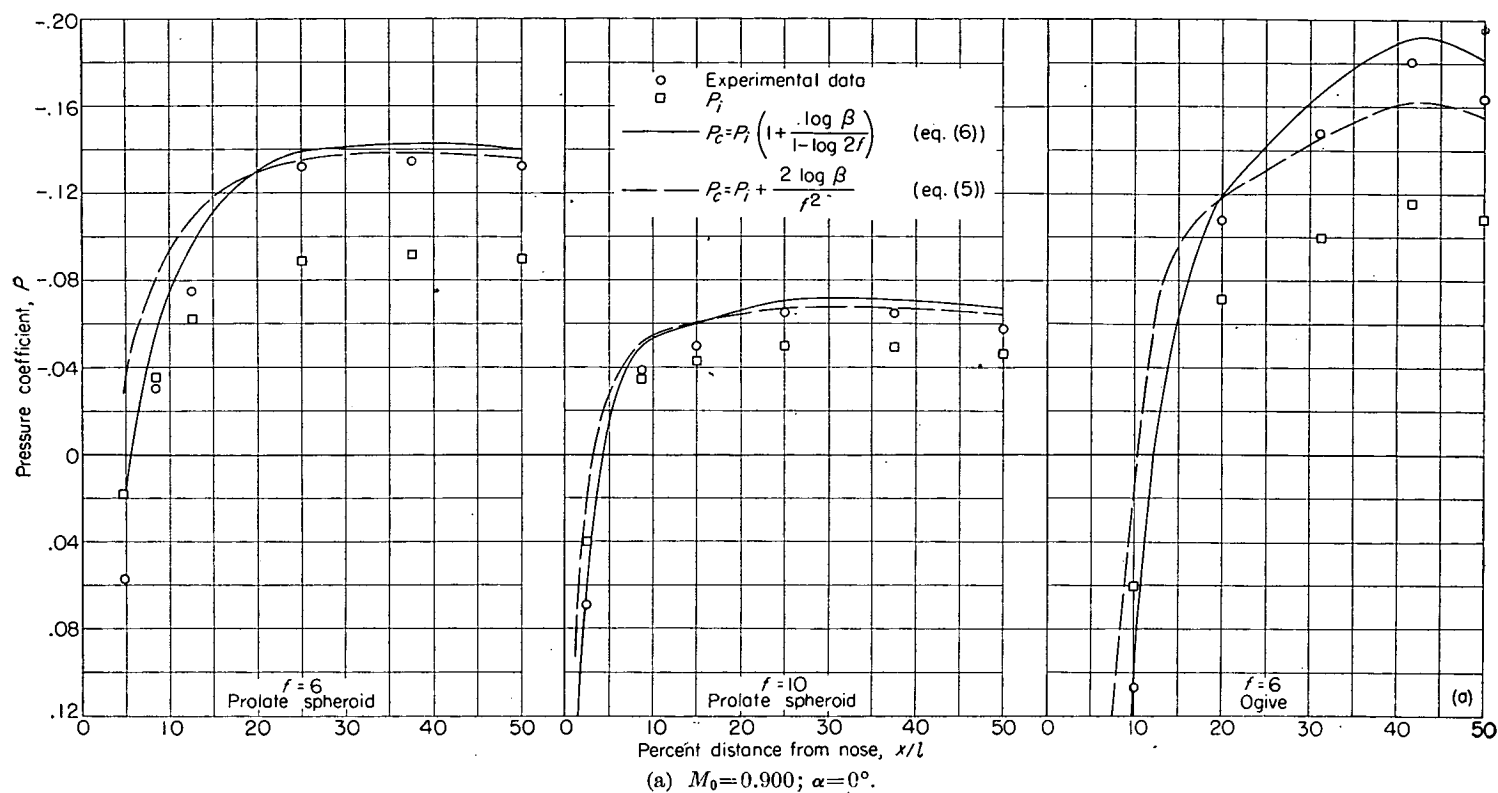
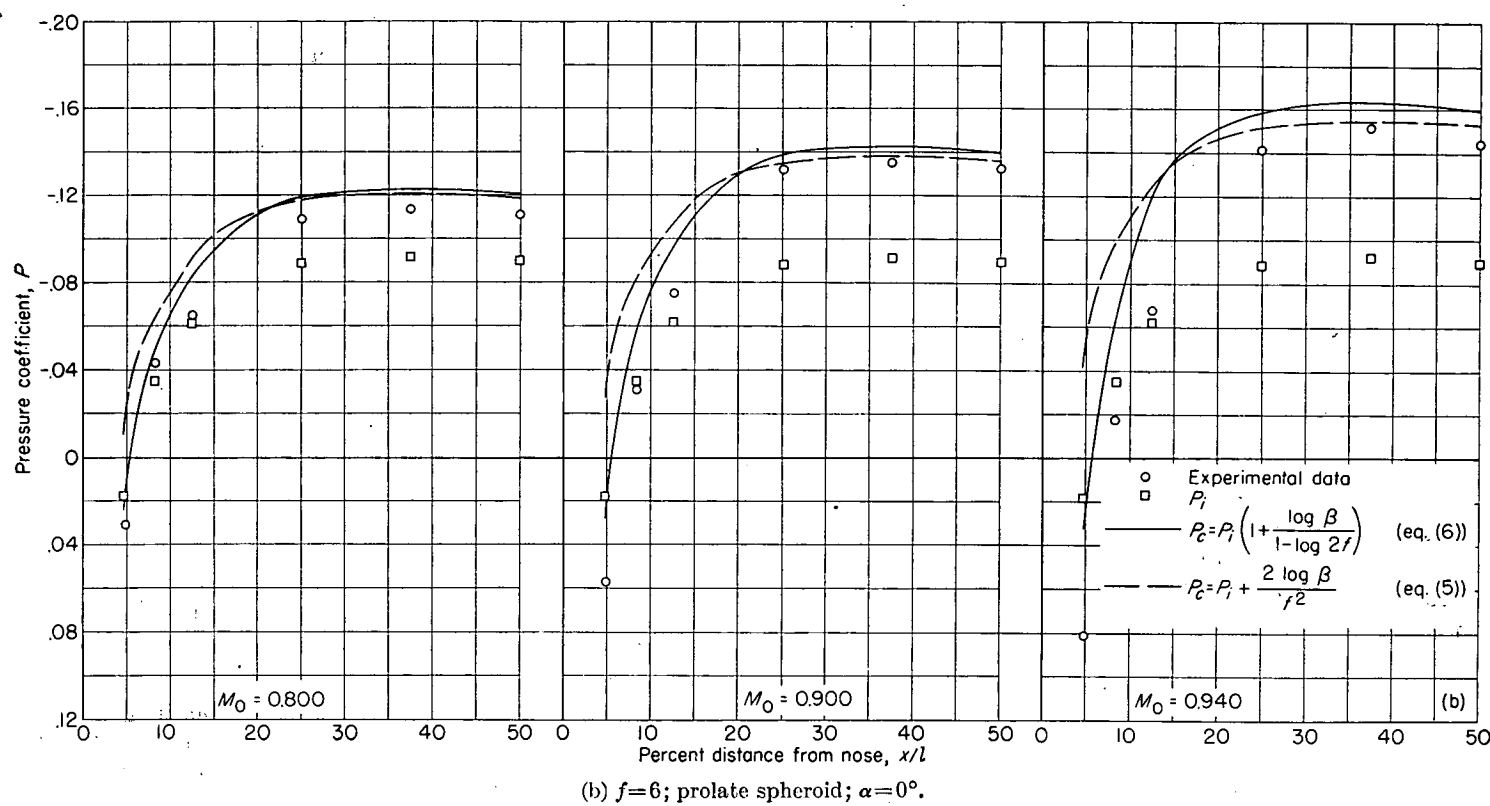


FIGURE 14.—Comparison of theoretically corrected incompressible pressure-coefficient distributions with the corresponding experimental distributions.



(b) $f=6$; prolate spheroid; $\alpha=0^\circ$.

FIGURE 14.—Continued.

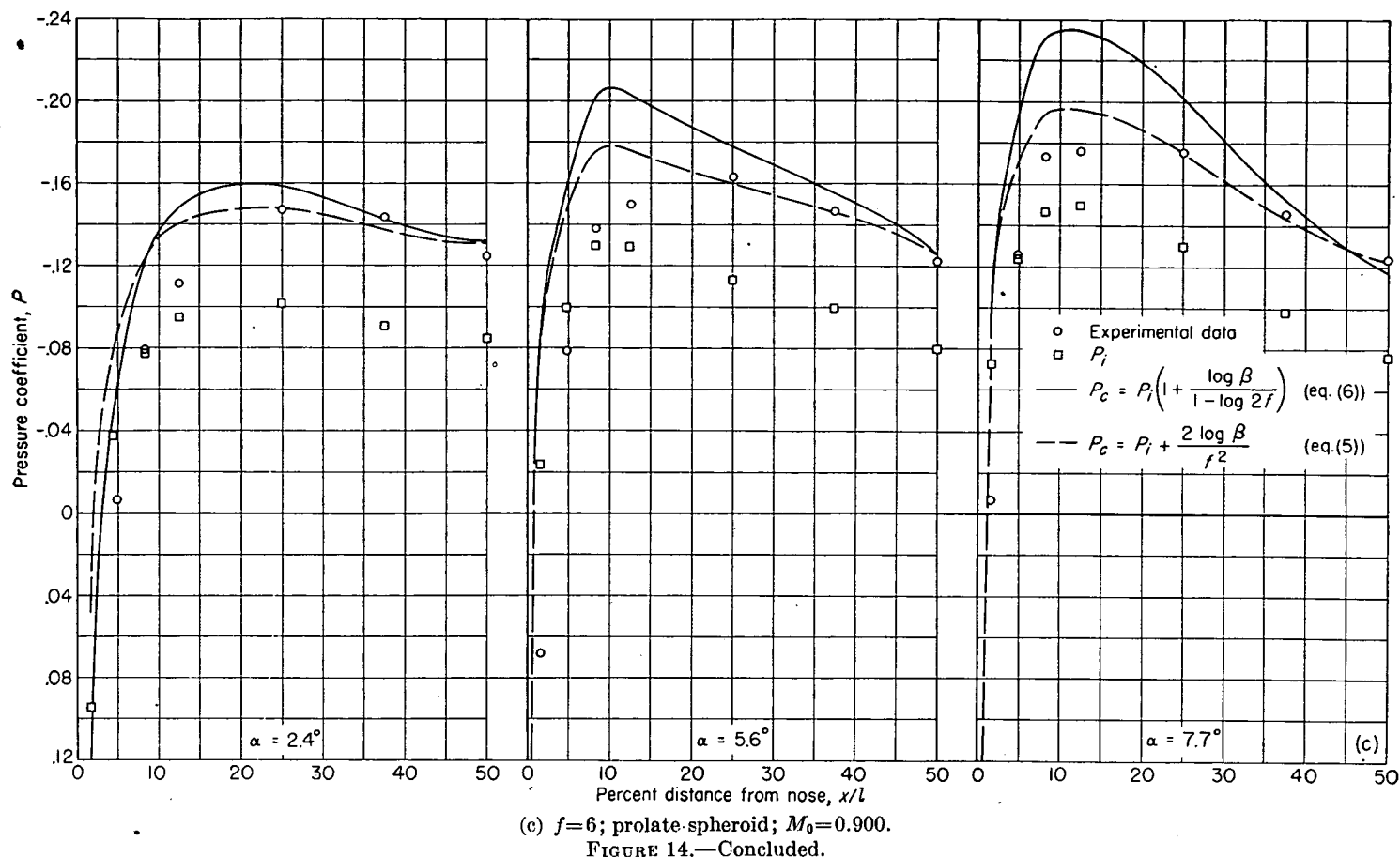


Figure 14 (c) indicates that equation (6) does not correct as satisfactorily over the central parts of the body at angles of attack as equation (5). This lack of agreement is due to the compressibility effect on the lift forces. It has already been shown that the lift forces are not much affected by compressibility; hence, the increments of the pressure coefficients due to compressibility are about the same for the top and bottom of the body. Since the absolute values of the pressure coefficients are less on the bottom of the body and greater on the top than if the lift forces had not been present, equation (6) will overcorrect the pressure coefficients on the top and undercorrect those on the bottom. The same reasoning shows that equation (5), which gives a constant increment over the entire body, will express the compressibility effect with an angle of attack better over the central part of the body to which it applies than will equation (6).

CONCLUSIONS

The results of the tests made on several bodies of revolution have shown the following effects of compressibility on three-dimensional flow:

1. In general, the compressibility effect is to increase the pressure differences over a body of revolution. The pressure distributions are approximately rotated about a point near stream pressure and the negative-pressure peaks are moved rearward.

2. The linearized modification of the compressible potential-flow equation will predict the pressures over the central part of the body but will not predict the changes in pressure ahead of the stream-pressure point nor will it predict the change in shape which occurs with supercritical flow.

3. The correction formulas

$$\frac{P_c}{P_i} = 1 + \frac{\log \beta}{1 - \log 2f}$$

and

$$P_c - P_i = \frac{2 \log \beta}{f^2}$$

(where P_c and P_i are the pressure coefficients for compressible and incompressible flow, respectively, f is the fineness ratio, and $\beta = \sqrt{1 - M_0^2}$ in which M_0 is the Mach number) may be used approximately to correct incompressible-flow pressures over the central part of streamline thin bodies of revolution; the errors will increase as the supersonic Mach number is approached and exceeded. Since P_c/P_i rotates the pressure distribution, this correction is better to use at zero angle of attack; however, the form $P_c - P_i$ expresses the effects of angle of attack more correctly and should be used when an angle of attack other than zero is involved.

4. The effects of compressibility are approximately the same for various bodies of the same fineness ratio, provided the body shape satisfies the requirements of the linearized theory.

5. Increasing the fineness ratio tends to reduce the effects of compressibility.

6. The effects of compressibility on an annular protuberance of short chord on a body of revolution tend to follow more nearly two-dimensional laws than three-dimensional laws.

7. Lift forces and moments over the forward part of the body are relatively unaffected by compressibility.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., November 5, 1951.

APPENDIX A

DERIVATION OF THE EQUATIONS FOR THE COMPRESSIBLE PRESSURE COEFFICIENTS OF THE FLOW ABOUT A PROLATE SPHEROID

The solution of the linearized compressible-flow equation for a prolate spheroid requires a derivation of the relation for the incompressible velocities about the body. The incompressible velocities about a prolate spheroid are defined by the potential equations given in reference 5. These equations may be combined and written

$$\varphi = U \mu \zeta \cos \alpha + A (\cos \alpha) \mu \left(\frac{1}{2} \zeta \log \frac{\zeta+1}{\zeta-1} - 1 \right) + B (\sin \alpha) \sqrt{1-\mu^2} \sqrt{\zeta^2-1} \left(\frac{1}{2} \log \frac{\zeta+1}{\zeta-1} - \frac{\zeta}{\zeta^2-1} \right) \cos \omega \quad (7)$$

The values of the constants A and B which satisfy the required boundary conditions are

$$A = \frac{U}{\frac{\zeta_0}{\zeta_0^2-1} - \frac{1}{2} \log \frac{\zeta_0+1}{\zeta_0-1}}$$

$$B = \frac{U}{\frac{\zeta_0^2-2}{\zeta_0(\zeta_0^2-1)} - \frac{1}{2} \log \frac{\zeta_0+1}{\zeta_0-1}}$$

where ζ_0 is the value of the coordinate which represents the body. It may be shown that the eccentricity of the ellipse $e = \sqrt{\frac{l^2-4b^2}{l^2}} = \frac{1}{\zeta_0}$ where l and $2b$ are the lengths of the major and minor axes of the prolate spheroid. Since the fineness ratio f is equal to $l/2b$, $e = \sqrt{1-\frac{1}{f^2}}$. The incompressible

velocity components obtained by differentiating the potential equation are

$$\left. \begin{aligned} \frac{u^*}{U} &= \frac{1-\mu^2}{1-e^2\mu^2} K_a \cos \alpha - \frac{\mu \sqrt{(1-\mu^2)(1-e^2)}}{1-e^2\mu^2} K_b \cos \omega \sin \alpha \\ \frac{v^*}{U} &= -\frac{\mu \sqrt{(1-\mu^2)(1-e^2)}}{1-e^2\mu^2} K_a \cos \omega \cos \alpha \\ &\quad + \frac{\mu^2(1-e^2)}{1-e^2\mu^2} K_b \cos^2 \omega \sin \alpha + K_b \sin^2 \omega \sin \alpha \\ \frac{w^*}{U} &= -\frac{\mu \sqrt{(1-\mu^2)(1-e^2)}}{1-e^2\mu^2} K_a \sin \omega \cos \alpha \\ &\quad + \frac{\mu^2(1-e^2)}{1-e^2\mu^2} K_b \sin \omega \cos \omega \sin \alpha - K_b \sin \omega \cos \omega \sin \alpha \end{aligned} \right\} \quad (8)$$

where

$$\left. \begin{aligned} K_a &= 1 - \frac{\log \left(\frac{1+e}{1-e} \right) - 2e}{\log \left(\frac{1+e}{1-e} \right) - \frac{2e}{1-e^2}} \\ K_b &= 1 - \frac{\log \left(\frac{1+e}{1-e} \right) - \frac{2e}{1-e^2}}{\log \left(\frac{1+e}{1-e} \right) - \frac{2e(1-2e^2)}{1-e^2}} \end{aligned} \right\} \quad (9)$$

and u^* , v^* , and w^* are the velocity components in a coordinate system aligned with the x -axis of the body. These

velocities are transformed to the u , v , and w components by the following equations:

$$\left. \begin{aligned} \frac{u}{U} &= \frac{u^*}{U} \cos \alpha + \frac{v^*}{U} \sin \alpha \\ \frac{v}{U} &= -\frac{u^*}{U} \sin \alpha + \frac{v^*}{U} \cos \alpha \\ \frac{w}{U} &= \frac{w^*}{U} \end{aligned} \right\} \quad (10)$$

With the preceding transformation, the velocity equations (8) become

$$\left. \begin{aligned} \frac{u}{U} &= \frac{1}{1-e^2\mu^2} (\sqrt{1-\mu^2} \cos \alpha - \mu \sqrt{1-e^2} \cos \omega \sin \alpha) (\sqrt{1-\mu^2} K_a \cos \alpha - \mu \sqrt{1-e^2} K_b \cos \omega \sin \alpha) + K_b \sin^2 \omega \sin^2 \alpha \\ \frac{v}{U} &= -\frac{1}{1-e^2\mu^2} (\sqrt{1-\mu^2} \sin \alpha + \mu \sqrt{1-e^2} \cos \omega \cos \alpha) (\sqrt{1-\mu^2} K_a \cos \alpha - \mu \sqrt{1-e^2} K_b \cos \omega \sin \alpha) + K_b \sin^2 \omega \cos \alpha \sin \alpha \\ \frac{w}{U} &= -\frac{\mu \sqrt{1-e^2}}{1-e^2\mu^2} \sin \omega (\sqrt{1-\mu^2} K_a \cos \alpha - \mu \sqrt{1-e^2} K_b \cos \omega \sin \alpha) - K_b \sin \omega \cos \omega \sin \alpha \end{aligned} \right\} \quad (11)$$

These equations may be rewritten more simply by setting

$$\begin{aligned} F &= \sqrt{1-\mu^2} \cos \alpha - \mu \sqrt{1-e^2} \cos \omega \sin \alpha \\ F_1 &= \sqrt{1-\mu^2} \sin \alpha + \mu \sqrt{1-e^2} \cos \omega \cos \alpha \\ H &= \sqrt{1-\mu^2} K_a \cos \alpha - \mu \sqrt{1-e^2} K_b \cos \omega \sin \alpha \\ G &= 1 - e^2 \mu^2 \\ F_2 &= \mu \sqrt{1-e^2} \sin \omega \end{aligned}$$

Then

$$\left. \begin{aligned} \frac{u}{U} &= \frac{FH}{G} + K_b \sin^2 \omega \sin^2 \alpha \\ \frac{v}{U} &= -\frac{F_1 H}{G} + K_b \sin^2 \omega \cos \alpha \sin \alpha \\ \frac{w}{U} &= -\frac{F_2 H}{G} - K_b \sin \omega \cos \omega \sin \alpha \end{aligned} \right\} \quad (12)$$

The method of correcting for compressibility discussed in the text can now be approximately applied by increasing the fineness ratio by $1/\beta$ and reducing the tangent of the angle of attack by the factor β . Thus,

$$e_{st} = \sqrt{1 - \frac{\beta^2}{f^2}} \quad (13)$$

and

$$\tan \alpha_{st} = \beta \tan \alpha \quad (14)$$

where e_{st} and α_{st} are the eccentricity and the angle of attack of the stretched body. Although this stretched body differs slightly from the properly stretched body, the approximation is very close for large fineness ratios and small angles of attack. It may be shown that the properly stretched body is an ellipsoid having three unequal axes; however, under the present restrictions the two minor axes are very nearly equal, so that only small errors will be caused by the above approximation of the stretched body.

The induced velocities in compressible flow are now determined by substituting α_{st} and e_{st} in equations (11) and multiplying the resulting velocity increments $\frac{u}{U} - 1$, $\frac{v}{U}$, and $\frac{w}{U}$ by the factors $1/\beta^2$, $1/\beta$, and $1/\beta$, respectively, or

$$\left. \begin{aligned} \left(\frac{u}{U}\right)_c &= 1 + \frac{1}{\beta^2} \left[\left(\frac{u}{U}\right)_{st} - 1 \right] \\ \left(\frac{v}{U}\right)_c &= \frac{1}{\beta} \left(\frac{v}{U}\right)_{st} \\ \left(\frac{w}{U}\right)_c &= \frac{1}{\beta} \left(\frac{w}{U}\right)_{st} \end{aligned} \right\} \quad (15)$$

The pressure coefficients for compressible flow may be computed from the velocities by the following formula:

$$P_c = \frac{\left\{ 1 + \frac{\gamma-1}{2} M_0^2 \left[1 - \left(\frac{V}{U}\right)_c^2 \right] \right\}^{\frac{\gamma}{\gamma-1}} - 1}{\frac{\gamma}{2} M_0^2}$$

where

$$\left(\frac{V}{U}\right)_c^2 = \left(\frac{u}{U}\right)_c^2 + \left(\frac{v}{U}\right)_c^2 + \left(\frac{w}{U}\right)_c^2 \quad (16)$$

Combination of equations (12), (15), and (16) yields

$$\begin{aligned} 1 - \frac{V^2}{U^2} &= \frac{1}{\beta^2} \left\{ 1 - \frac{H_{st}^2}{G_{st}} - K_{bst}^2 \sin^2 \omega \sin^2 \alpha_{st} - \right. \\ &\quad \left. \left(\frac{1}{\beta^2} - 1 \right) \left[\left(1 - \frac{H_{st} F_{st}}{G_{st}} \right)^2 + \left(K_{bst} \sin^2 \omega \sin^2 \alpha_{st} + \frac{H_{st} F_{st}}{G_{st}} \right)^2 - \right. \right. \\ &\quad \left. \left. \left(\frac{H_{st} F_{st}}{G_{st}} \right)^2 - 2 K_{bst} \sin^2 \omega \sin^2 \alpha_{st} \right] \right\} \end{aligned} \quad (17)$$

where the subscript st is used to indicate that the various functions so identified are based on the values of e and α associated with the stretched body. (See eqs. (13) and (14).) A simpler first approximation may be obtained by considering the approximate relation

$$P_c = -2 \left(\frac{u-U}{U} \right)_c$$

Since

$$\left(\frac{u-U}{U} \right)_c = \frac{1}{\beta^2} \left[\left(\frac{u}{U} \right)_{st} - 1 \right]$$

$$P_c = -\frac{2}{\beta^2} \left(\frac{H_{st} F_{st}}{G_{st}} - 1 + K_{bst} \sin^2 \omega \sin^2 \alpha_{st} \right) \quad (18)$$

A simpler equation may be developed for the pressures over a prolate spheroid at zero angle of attack by considering the method of approximate source-sink distributions described in references 2 to 4. In these references, it is shown that

$$P_c = -2 \left(\frac{u-U}{U} \right)_c = \frac{1}{2\pi} \int_0^l \frac{S'(t)(x-t)dt}{[(x-t)^2 + \beta^2 r^2]^{3/2}} \quad (19)$$

where t is a coordinate along the major axis of the body and $S'(t)$ is the derivative of the cross-sectional area of the body with respect to t . For a prolate spheroid,

$$S(t) = \pi r^2 = \frac{4\pi b^2 t}{l} - \frac{4\pi b^2 t^2}{l^2}$$

from which

$$S'(t) = \frac{d[S(t)]}{dt} = \frac{4\pi b^2}{l} \left(1 - \frac{2t}{l} \right)$$

Thus

$$P_c = \frac{2b^2}{l} \int_0^l \frac{\left(1 - \frac{2t}{l} \right) (x-t) dt}{[(x-t)^2 + \beta^2 r^2]^{3/2}}$$

After integration and collection of terms

$$\begin{aligned} P_c &= \frac{1}{f^2} \left[\frac{1}{2 \sqrt{\left(1 - \frac{x}{l} \right)^2 + \beta^2 \frac{r^2}{l^2}}} + \frac{1}{2 \sqrt{\frac{x^2}{l^2} + \beta^2 \frac{r^2}{l^2}}} \right. \\ &\quad \left. \log \frac{1 - \frac{x}{l} + \sqrt{\left(1 - \frac{x}{l} \right)^2 + \beta^2 \frac{r^2}{l^2}}}{-\frac{x}{l} + \sqrt{\frac{x^2}{l^2} + \beta^2 \frac{r^2}{l^2}}} \right] \end{aligned} \quad (20)$$

APPENDIX B

REDUCTION OF PRESSURE-COEFFICIENT FORMULAS TO OBTAIN SIMPLE FUNCTIONS FOR CORRECTING INCOMPRESSIBLE PRESSURE DISTRIBUTIONS FOR THE EFFECTS OF COMPRESSIBILITY

Two functions which may be used to express the relation between the pressure coefficients in compressible and incompressible flow are the ratio and the increment between the two coefficients P_c and P_i ; that is, P_c/P_i and $P_c - P_i$. Both functions may be expressed in simple equations by substituting the pressure-coefficient functions for the midpoint of the body into both the ratio function and the increment function. In order to simplify equation (17) let $\mu=0$, $\sin \alpha_{st} = \beta\alpha$, and $K_{a_{st}} = 1 - k_{st}$ or $k_{st} = 1 - K_{a_{st}}$. Then,

$$1 - \frac{V^2}{U^2} = \frac{1}{\beta^2} \left[2k_{st} - \frac{k_{st}^2}{\beta^2} - K_{b_{st}} \alpha^2 (1 - \beta^2) (K_{b_{st}} \beta^2 \alpha^2 - 2k_{st}) \right] \quad (21)$$

For small values of $1 - \frac{V^2}{U^2}$

$$P_c = 1 - \frac{V^2}{U^2}$$

Also, for large values of f

$$K_{b_{st}} \rightarrow 2$$

Hence

$$P_c = \frac{1}{\beta^2} \left[2k_{st} - \frac{k_{st}^2}{\beta^2} - 4\alpha^2 (1 - \beta^2) (\beta^2 \alpha^2 - k_{st}) \right]$$

Since $\beta^2 = 1$ at $M_0 = 0$

$$\frac{P_c}{P_i} = \frac{1}{\beta^2} \frac{\left[2k_{st} - \frac{k_{st}^2}{\beta^2} - 4\alpha^2 (1 - \beta^2) (\beta^2 \alpha^2 - k_{st}) \right]}{2k_i - k_i^2}$$

Since $4\alpha^2 (1 - \beta^2) (\beta^2 \alpha^2 - k_{st})$ is small compared with $2k_{st}$, the term containing α may be neglected; thus

$$\frac{P_c}{P_i} = \frac{1}{\beta^2} \frac{k_{st}}{k_i} \frac{2 - \frac{k_{st}}{\beta^2}}{2 - k_i} \quad (22)$$

In order to reduce this equation to previously published forms (refs. 4 and 8), it is necessary to reduce k :

$$k = 1 - K_a = \frac{\log \left(\frac{1+e}{1-e} \right) - 2e}{\log \left(\frac{1+e}{1-e} \right) - \frac{2e}{1-e^2}}$$

Substituting $e_{st}^2 = 1 - \frac{\beta^2}{f^2}$ and the approximate form $e_{st} = 1 - \frac{\beta^2}{2f^2}$ in this equation gives

$$k_{st} = \frac{\beta^2 (\log 2f - \log \beta - 1)}{\beta^2 \log 2f - \beta^2 \log \beta - f^2} \quad (23)$$

and

$$k_i = \frac{\log 2f - 1}{\log 2f - f^2} \quad (24)$$

These equations show that both k_{st}/β^2 and k_i are of order of magnitude $1/f^2$ and, therefore, small with respect to 2. Hence, the approximation (see eq. (22))

$$\frac{P_c}{P_i} = \frac{1}{\beta^2} \frac{k_{st}}{k_i} \quad (25)$$

is valid.

If equations (23) and (24) are used in equation (25), the following equation is obtained:

$$\frac{P_c}{P_i} = \left(1 + \frac{\log \beta}{1 - \log 2f} \right) \left[\frac{f^2 - \log 2f}{f^2 - \beta^2 (\log 2f - \log \beta)} \right] \quad (26)$$

or for large fineness ratios

$$\frac{P_c}{P_i} = 1 + \frac{\log \beta}{1 - \log 2f} \quad (27)$$

which may be changed to its equivalent form

$$P_c - P_i = \frac{2 \log \beta}{f^2} \quad (28)$$

Equation (20) obtained by the source-sink-distribution method will also reduce to equations (27) and (28) for the central part of the body.

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REPORT 1156

EXPERIMENTS TO DETERMINE NEIGHBORHOOD REACTIONS TO LIGHT AIRPLANES WITH AND WITHOUT EXTERNAL NOISE REDUCTION¹

By FRED S. ELWELL

SUMMARY

The work reported was part of a program of experimentation with external noise reduction on light airplanes. This particular study was in effect a byproduct survey conceived to utilize already available equipment and personnel to further the findings of the original research and to determine reactions in populated neighborhoods to light aircraft with and without noise-reduction equipment.

Two light airplanes modified by reduction gears, four-bladed propellers, and engine exhaust silencers were flown in comparison with two standard airplanes at a number of sites of the type that might be useful as "close-in" landing strips within the metropolitan area of Boston, Mass.

The objective was to ascertain the neighborhood reactions to the noise of light airplanes flown close to residential properties of varying income levels, population densities, and proximity to trade centers in order to determine whether the degree of noise reduction found to be practicable in the major phases of the research program produced a significant reduction in neighborhood objection to such aircraft operations.

The findings indicate that at the 10 sites within and about metropolitan Boston the degree of noise reduction previously found to be aerodynamically and structurally feasible did eliminate substantially all neighborhood objections to noise per se.

The tests were not extensive enough to determine whether other manifest objections such as fear of low-flying aircraft and possible property devaluation would still have resulted in sustained objections. Neither was it possible to ascertain the importance of the noise nuisance relative to other complaints raised against close-in operation of aircraft. The evidence did clearly suggest that, when the noise nuisance is minimized to the extent found feasible, the number and severity of other objections also diminish—evidently because the flight operations are noticed less when heard less.

INTRODUCTION

The experiments reported herewith were conducted during the years 1947–1950 by the Aeronautical Research Foundation under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

The Trustees of the Foundation originally decided to undertake research in the area of external noise reduction because they had concluded that:

The development of civil aviation, insofar as the utilization of light airplanes is concerned, has been seriously retarded by the unwillingness of communities to permit an adequate number of conveniently located landing areas. This same lack of ground facilities materially affects the safety of the vehicle.

To test the hypothesis that one of the principal objections might be due to airplane noise, the principal research by the Foundation has been on external noise reduction with both tractor- and pusher-type light airplanes. The primary objective of the project was to determine ways and means of reducing external noise without impairing the aerodynamic, structural, or operational effectiveness of light aircraft. Insofar as possible, utilizing equipment and personnel already available, the secondary objective discussed in this report was that of ascertaining the extent of noise-level reduction required to reduce significantly the noise nuisance in nearby neighborhoods.

The Foundation, therefore, tested neighborhood reactions by flying both standard and modified airplanes at locations of the type which have customarily given rise to noise objections.

The project was under the general direction of Dr. Lynn L. Bollinger, Executive Director of the Foundation, and under the technical direction of Professors Leo L. Beranek, Otto C. Koppen, and C. Fayette Taylor of the Massachusetts Institute of Technology and Mr. Arthur H. Tully, Jr.,² Assistant Director of Research of the Harvard Business School.

Mr. Joseph Garside, as Director of Operations for the Foundation, directed the control of airplane safety and maintenance, piloted the aircraft on many occasions, and acted as ground observer at some of the test sites.

Mr. William W. Dean, Administrative Assistant of the Foundation, during the summer of 1949, provided assistance in piloting the airplanes and taking sound measurements and acted as ground observer at many of the test sites.

Mr. John P. Roberts, Sound Engineer of the Foundation, assisted in this project by taking sound measurements and acted as ground observer at many of the test sites.

The following organizations and individuals generously contributed equipment and assistance on this project:

Aircooled Motors, Inc., lent the experimental geared engine used in the modified Stinson and also in the modified Good-year.

¹ Supersedes NACA TN 2728, "Experiments to Determine Neighborhood Reactions to Light Airplanes With and Without External Noise Reduction" by Fred S. Elwell, 1952.

² Executive Director of the Foundation as of January 1, 1950.

Goodyear Aircraft Corp. gave the castoring landing gear for the modified Stinson and lent the Goodyear amphibian for experiments.

Lycoming Division, AVCO Mfg. Corp., gave the engine for the experimental Cub airplane.

Maxim Silencer Co. gave the silencers for the modified Stinson.

Seasenich Bros. provided all experimental propellers at cost.

Stinson Aircraft Division, Consolidated Vultee Aircraft Corp., gave the Stinson airplane for experiments.

Mr. Joseph Garside, President of Wiggins Airways, gave use of his company's shops and facilities and contributed

flight time to make aerial surveys for possible test sites in the southern sectors of metropolitan Boston.

Mr. Julius Goldman, President of Revere Airways, Inc., contributed flight time to make aerial surveys for possible test sites in the northeastern sector of metropolitan Boston.

Mr. John T. Griffin, President of East Coast Aviation Corp., contributed flight time to make aerial surveys for possible test sites in the northwestern sector of metropolitan Boston and, in addition, provided storage space for the Foundation airplanes, on several occasions, at no cost.

Mr. Crocker Snow, Director of the Massachusetts Aeronautics Commission, contributed time and effort to expedite and sanction this project.

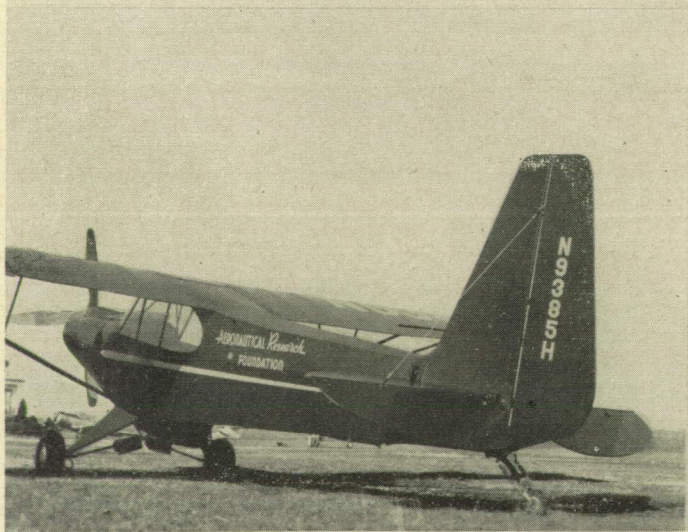


FIGURE 1.—Various views of modified ARF Cub (configuration 1)

DESCRIPTION OF APPARATUS

The apparatus used in this study can be divided into four categories, as follows: The airplanes used together with their power plants, the propellers, the sound-measuring equipment, and the flight-control equipment.

AIRPLANES AND THEIR POWER PLANTS

The airplanes used were as follows:

(1) ARF Cub, configuration 1: A modified Cub J-3 airplane, shown in figure 1, essentially the same as a standard 1940 J-3 except for a new and larger vertical tail fin and rudder and a complete new engine mount and cowling, equipped as follows:

Engine: Lycoming four-cylinder, direct-drive, rated at 108 horsepower at a crankshaft speed of 2600 rpm.

Propeller speed reduction: Engine modified with the special vee-belt propeller drive illustrated in figure 2.

As shown in figure 2 the drive included a small pulley mounted on the forward end of the engine crankshaft and a larger pulley mounted on an external stationary shaft fastened to the engine crankcase. The upper pulley turned on two antifriction grease-packed bearings located inside the pulley.

Ten Goodyear rubber vee-belts with steel cable cores were used. These belts were each 42 inches in length and $\frac{3}{8}$ inch in width. An eccentric arrangement in each upper shaft bracket provided means for adjusting the belt tension. The nominal speed ratio of this combination was 0.632.

Before using this vee-belt drive in flight, it was necessary to subject it to endurance tests totalling approximately 50 hours on the ground. This experimental equipment had a total of over 170 hours in flight, therefore over 220 service hours on the vee-belt-drive assembly.

Exhaust system: Ejector-type, another special feature of this airplane. It was previously developed by Professor Otto C. Koppen of the Massachusetts Institute of Technology for the dual purpose of silencing the exhaust and insuring proper engine cooling under all normal conditions of operation, including excessive full-throttle operation on the ground.

The exhaust ejector consists of a cylindrical tube open at both ends. The tube is attached to the fuselage with its forward end communicating with the engine compartment and its rear end open to the atmosphere. The engine exhaust manifolds are so arranged as to discharge into a single nozzle which is so located with respect to the tube as to act as an ejector, drawing air from the engine compartment. This compartment has no other exit, and the engine baffles are so arranged that air entering the cooling-air inlet openings and passing over the engine is finally ejected through the ejector tube.

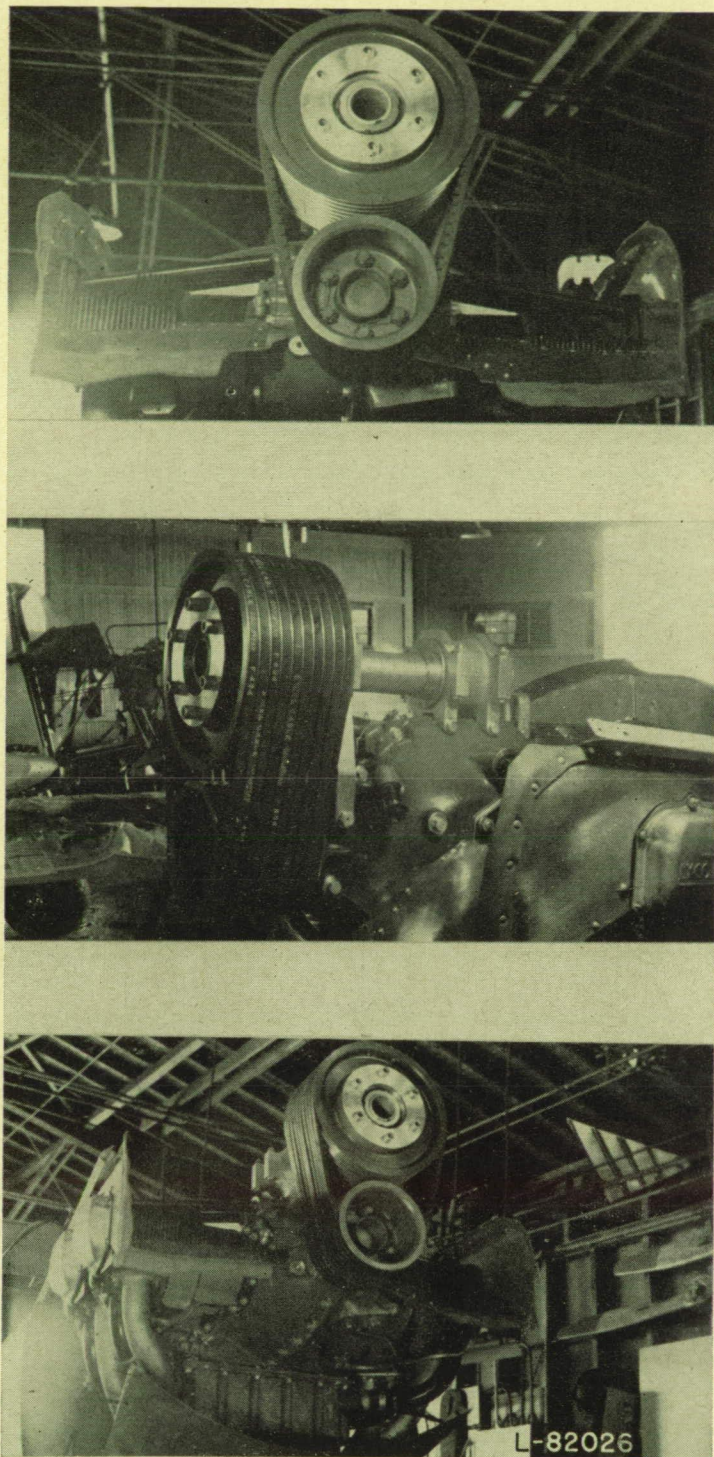


FIGURE 2.—Three views of vee-belt propeller drive used with engine of modified ARF Cub (configuration 1).

Silencing of the exhaust is assisted by a perforated metal lining within the ejector tube. Between this lining and the outer shell Johns Manville "Flex Blanket" is inserted, so that the arrangement acts as an effective sound absorber.

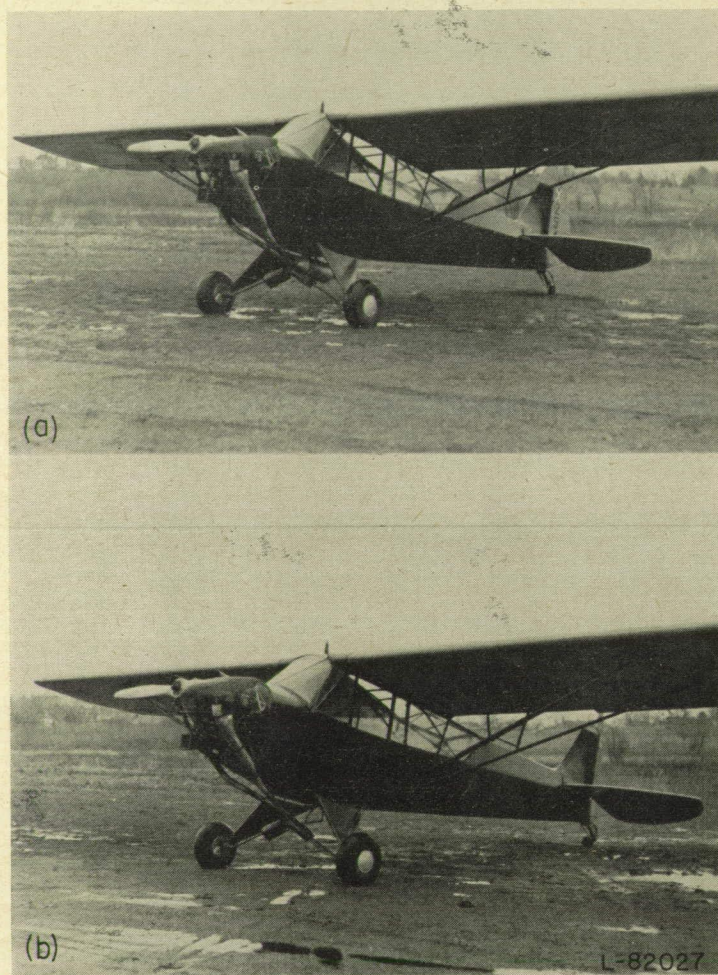
This arrangement was found to furnish adequate air circulation to keep cylinder temperatures well below specified limits, even for continuous running on the ground during the tests of the vee-belt drive. Back-pressure and weight data are as follows: Back pressure, measured in pipe between engine and nozzle, 10 inches of mercury at 2500 rpm, full throttle; weight, 9 pounds.

(2) CAA Cub, configuration 2M, muffled, and 2U, unmuffled: A modified Cub (J-3 type) airplane, loaned by the Civil Aeronautics Administration, shown in figure 3 (muffled, fig. 3 (a), and unmuffled, fig. 3 (b)), equipped as follows:

Engine: Continental four-cylinder, direct-drive, rated at 65 horsepower at a crankshaft speed of 2300 rpm.

Propeller speed reduction: None.

Exhaust system: Exhaust modified with a Maxim silencer



(a) Muffled (configuration 2M).
(b) Unmuffled (configuration 2U).

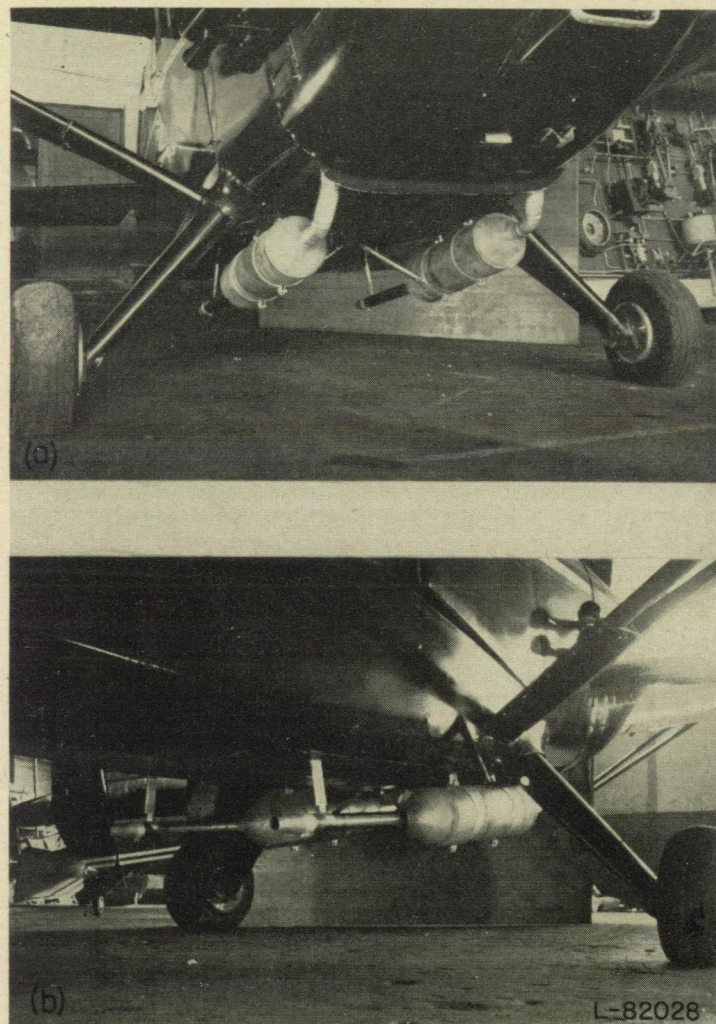
FIGURE 3.—Modified CAA Cub.

which could be easily detached so that the airplane could be flown with (fig. 3 (a)) or without (fig. 3 (b)) muffling. Back-pressure and weight data are as follows: Back pressure, measured in pipe between engine and nozzle, with muffler, 0 to $\frac{3}{16}$ inch of mercury at 2050 rpm and, without muffler, 0 to $\frac{1}{8}$ inch of mercury at 2050 rpm; weight, 14 pounds.

(3) ARF Stinson, configuration 3: A modified 1946 Stinson Voyager 150, equipped as follows:

Engine: Experimental geared Franklin, rated at 180 horsepower at a crankshaft speed of 3050 rpm. However, only approximately 155 horsepower was used since the special four-bladed propeller was designed for that power.

Propeller speed reduction: A planetary gearbox (part of engine) with ratio 0.632.



(a) Front view.
(b) Rear view.

FIGURE 4.—Silencers mounted on Stinson airplane (configuration 3).



FIGURE 5.—ARF Stinson (configuration 3).

Exhaust system: Two Maxim silencers, connected to standard exhaust manifolds. Figure 4 shows, photographically, front and rear views of their mounting on the airplane. Other data concerning these silencers are as follows: Weight, each 12 pounds; supporting brackets, 2.5 pounds; back pressure, measured in pipe between engine and muffler, 4 inches of mercury at 2900 rpm, full throttle.

This airplane was tested in previous noise-reduction research (reference 1) using many different propeller combinations; figure 5 is a photograph of this airplane with the propeller which was used in the neighborhood tests. This airplane was not used on many of the test sites because the existing areas, without extensive improvements in many cases, were not large enough for safe operations.

(4) Standard Cub, configuration 4: A production model Cub, used without any modifications, equipped as follows:

Engine: Factory-installed Continental, which delivered 65 horsepower at a crankshaft speed of 2300 rpm.

Propeller speed reduction: None.

Exhaust system: Standard factory installation.

This airplane is shown in figure 6.

In addition, both ARF airplanes and the CAA Cub were equipped with Goodyear castoring landing gear.

PROPELLERS

The propellers used were as follows:

(1) A four-bladed, two-piece, wooden-type propeller was used on the ARF Cub. The blade-form curves for this propeller are shown in figure 7. This propeller had a diameter of 80 inches with a nominal pitch of 15°. The modified Cub J-3 with this propeller will be called the ARF Cub, configuration 1.



FIGURE 6.—Standard Cub (configuration 4).

(2) Two propellers were available for the CAA Cub. The first was a standard two-bladed, fixed-pitch, wooden propeller which had a diameter of 72 inches and a nominal pitch of 14° . Its blade-form curves are illustrated in figure 8. The second propeller was a special four-bladed, one-piece, wooden propeller, having a diameter of 60 inches and a nominal pitch of $16\frac{1}{4}^\circ$, made for tests with this airplane but not used, however, since its noise level was higher and its performance poorer than those of the two-bladed propeller. The maximum speed attained by this propeller was higher by approximately 100 revolutions than that of the two-bladed propeller, but because of the smaller diameter the tip speed was lower. This fact is mentioned here because it is in contrast with the conclusions drawn in reference 1, that is, that increasing the number of blades decreases the noise generation at the same tip speed. The blade-form curves for this unused propeller are shown in figure 9. The CAA Cub with the two-bladed propeller will be called, with the muffler,

CAA Cub, configuration 2M, and, without the muffler, CAA Cub, configuration 2U.

(3) The ARF Stinson propeller was a four-bladed, one-piece, wooden type and its blade-form curves are shown in figure 10. It had a diameter of 76 inches with a nominal pitch of 25° . This airplane-propeller combination will be referred to as the ARF Stinson, configuration 3.

(4) The Standard Cub, configuration 4, had a propeller which was of the same two-bladed, one-piece type as that used on the CAA Cub. Its blade-form curves are similar to those in figure 8.

Table I gives further information concerning the above propellers and engines and their noise generation.

SOUND-MEASURING EQUIPMENT

The only instrument used in this work was a sound-level meter, General Radio Co., equipped with a microphone supplied by the General Radio Co. and manufactured by Shure Bros.

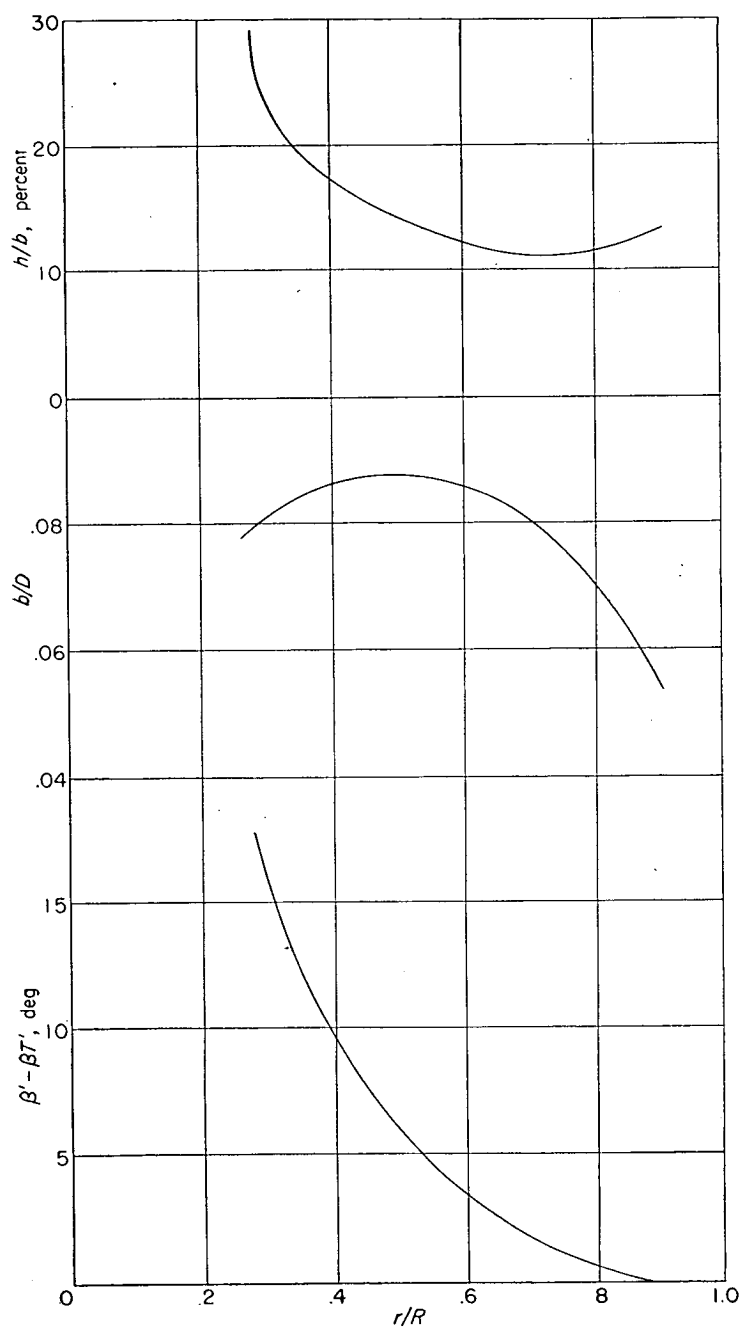


FIGURE 7.—Blade-form curves for four-bladed, two-piece propeller used on ARF Cub (configuration 1). D , diameter of propeller; R , tip radius; r , radius of element; b , width (chord) of element; h , maximum thickness of element; β' , pitch angle of element; $\beta T'$, pitch angle of tip element.

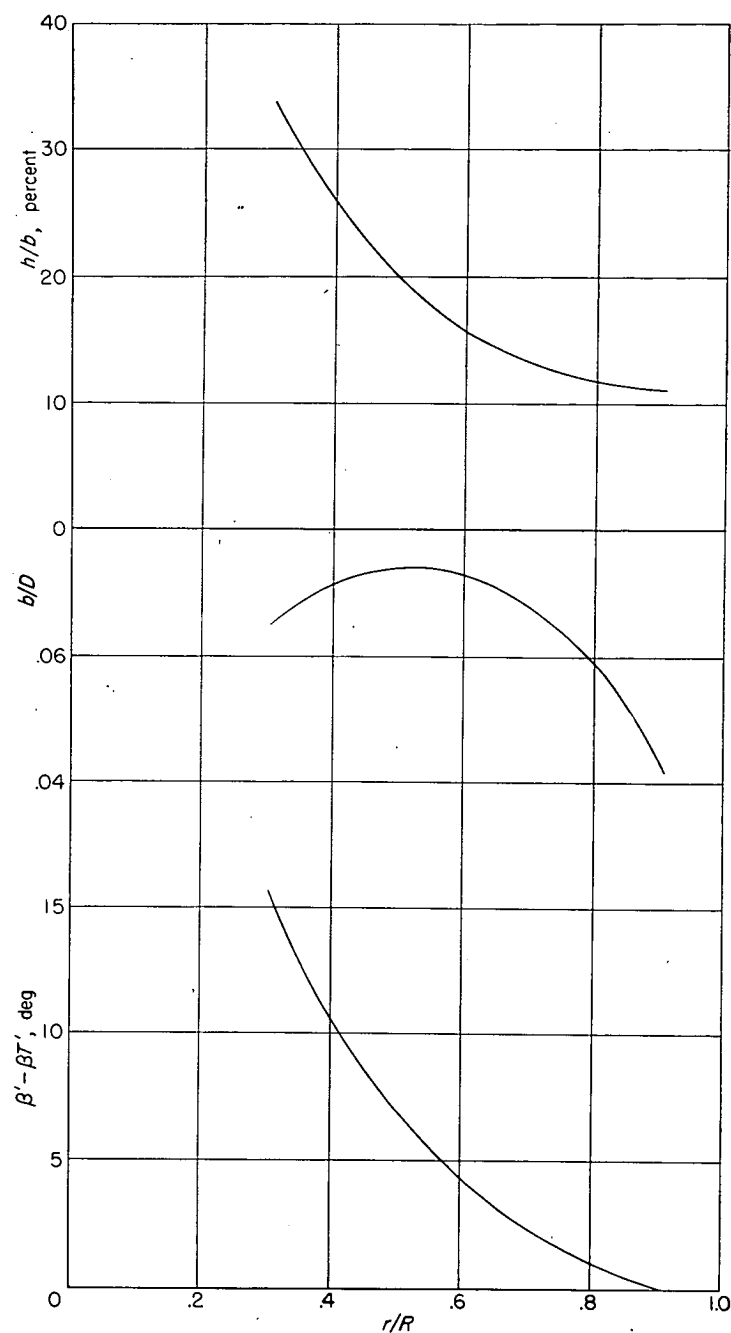


FIGURE 8.—Blade-form curves for propeller used on both versions of CAA Cub (configurations 2M and 2U) and on standard Cub (configuration 4). D , diameter of propeller; R , tip radius; r , radius of element; b , width (chord) of element; h , maximum thickness of element; β' , pitch angle of element; $\beta T'$, pitch angle of tip element.

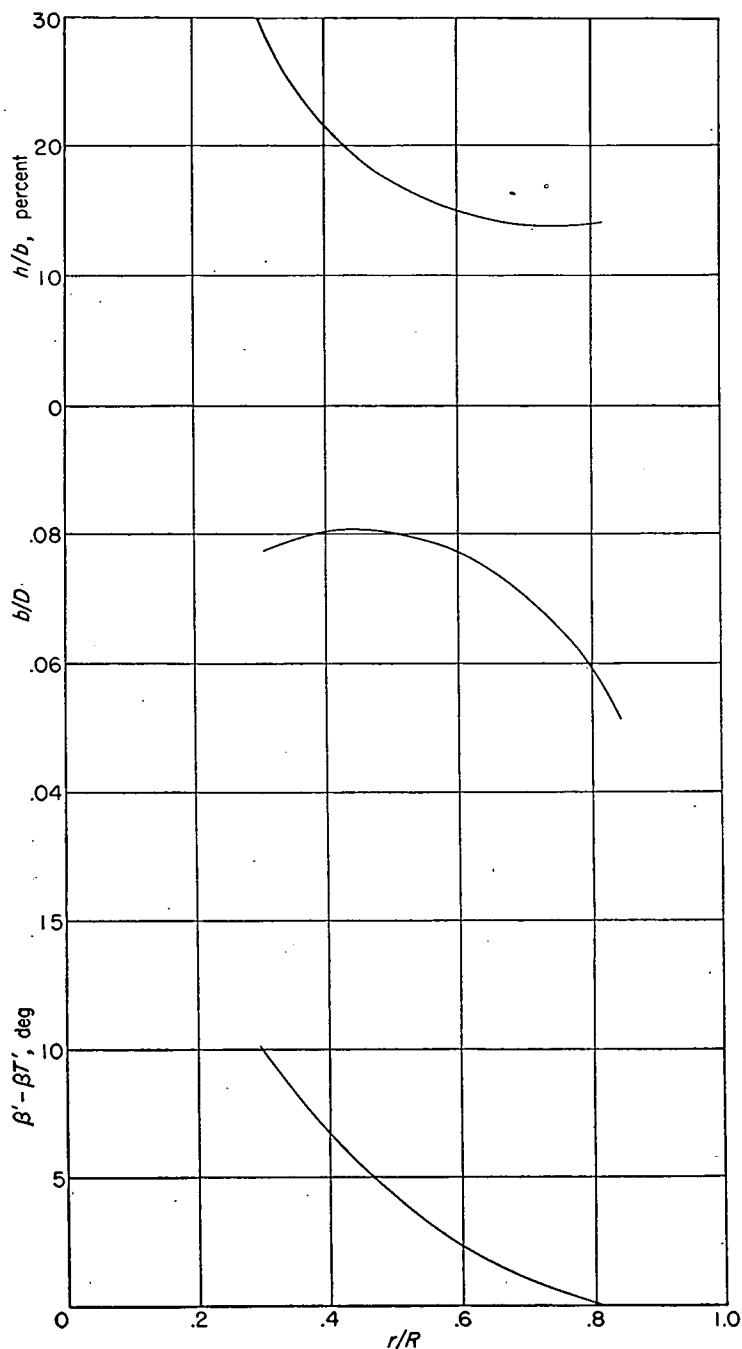


FIGURE 9.—Blade-form curves for four-bladed, one-piece propeller tried on CAA Cub (configuration 2M). D , diameter of propeller; R , tip radius; r , radius of element; b , width (chord) of element; h , maximum thickness of element; β' , pitch angle of element; β_T' , pitch angle of tip element.

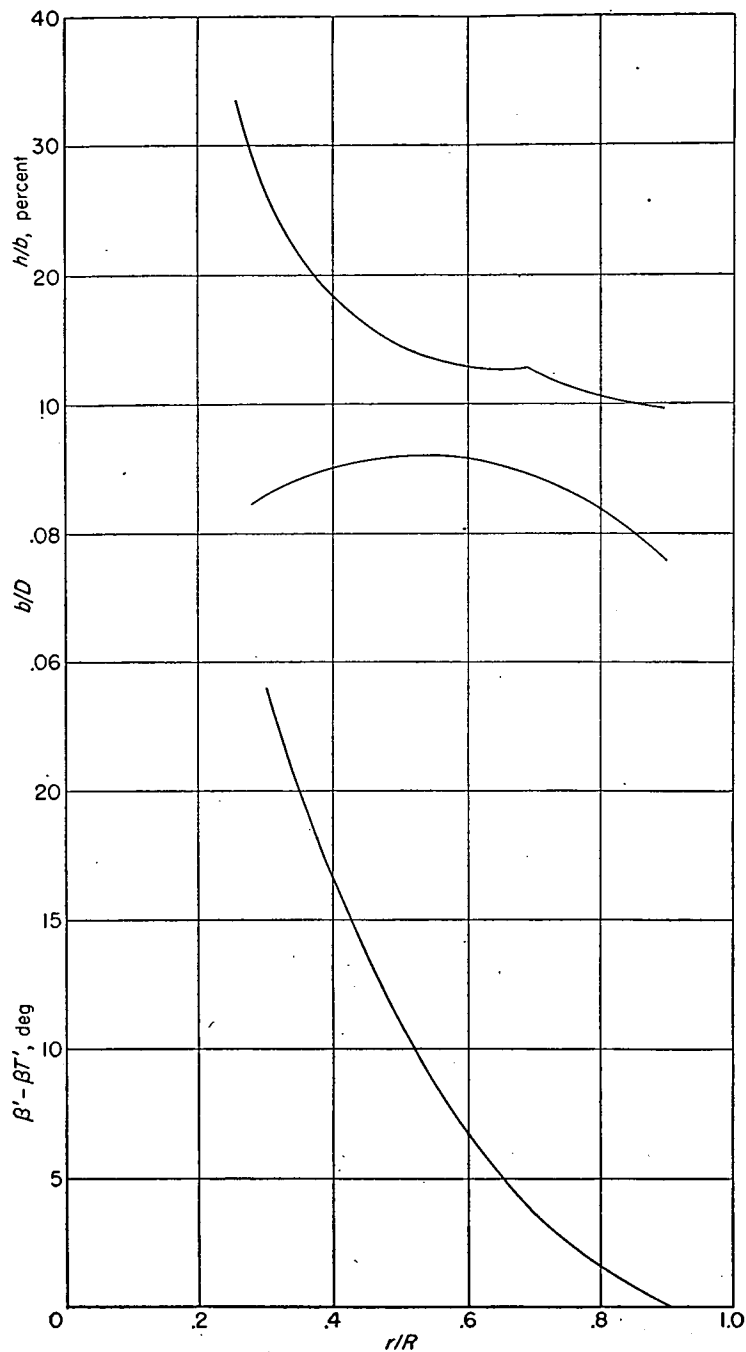


FIGURE 10.—Blade-form curves for four-bladed, one-piece propeller used on ARF Stinson (configuration 3). D , diameter of propeller; R , tip radius; r , radius of element; b , width (chord) of element; h , maximum thickness of element; β' , pitch angle of element; β_T' , pitch angle of tip element.

The noise characteristics of configurations 1, 3, and 4 are reported in detail in reference 1. The sound readings given in table I for those airplanes were taken from that reference and are peak readings at the overhead position only. In addition, new peak levels are reported for the muffled and unmuffled versions of configuration 2 (2M and 2U). Naturally, in all cases, the approaching and departing sound levels are of a lower order and the quieted noise of the airplane close by can be best described as similar to the "whish" of an electric fan.

FLIGHT-CONTROL EQUIPMENT

At those sites where the airplane was landed, field markers to outline the landing area and a portable wind sock were used. Since most of the sites were in heavily populated areas, each landing and take-off (in most cases these landings were simulated by low approaches and "dragging" the area) was controlled by a flight supervisor on the ground using colored flags for communication purposes.

NEIGHBORHOOD-REACTION TEST SITES

The sites chosen for testing of noise reactions were picked to represent a cross section of characteristic metropolitan and suburban neighborhoods with varying densities of population, income levels, and property values. Some of the sites had historical evidence of previous objections by local residents to aircraft or to attempts to establish an airport nearby.

A photograph of each site is shown with arrows superimposed vertically to indicate the altitude of the traffic pattern and horizontally to indicate the direction of the circuit. A topographical map of each site shows the traffic-pattern circuit and the ambient sound levels at important points relative to each test airstrip. Table I gives all pertinent statistics of the aircraft used including the peak sound levels of the various aircraft at 500-foot altitude at cruising speed. The maximum flight altitude at the test sites varied from 300 to 500 feet; therefore, the peak levels at the lower altitudes were slightly higher.

The data given in tables II to XI are most significant if the time of day and the day of the week are noted. Generally, the hours of the day were picked so that the airplane would be operating part of the time when the male member of the family might be at home or sleeping. This practice was followed because previous evidence (obtained from the Massachusetts Aeronautics Commission; the flight complaint section of CAA Air Carrier at East Boston; the CAA Aviation Safety Branch Office at Norwood Memorial Airport, Norwood, Mass.; and the local airport operators) showed that although the majority of calls are from women (esti-

mated two-thirds to three-fourths) the more serious complainants are men.

A few complaints about the research activity were made in person, but the majority were made by telephone to the local police near each site. All complainants were interviewed and, in addition, occasional spot checks were made to gather sample opinions. Detailed analyses of these complaints are tabulated for each test site surveyed (tables II to XI) and a composite table is shown for comparison and compilation of the totals (table XII).

The modified Stinson was flown at only two neighborhood sites since it was deemed marginal for safe operations at the smaller airstrips, consequently risking the safety of the pilot and equipment. The modified Cub was, therefore, the principal airplane flown in comparison with a special modified CAA Cub and a standard Cub.

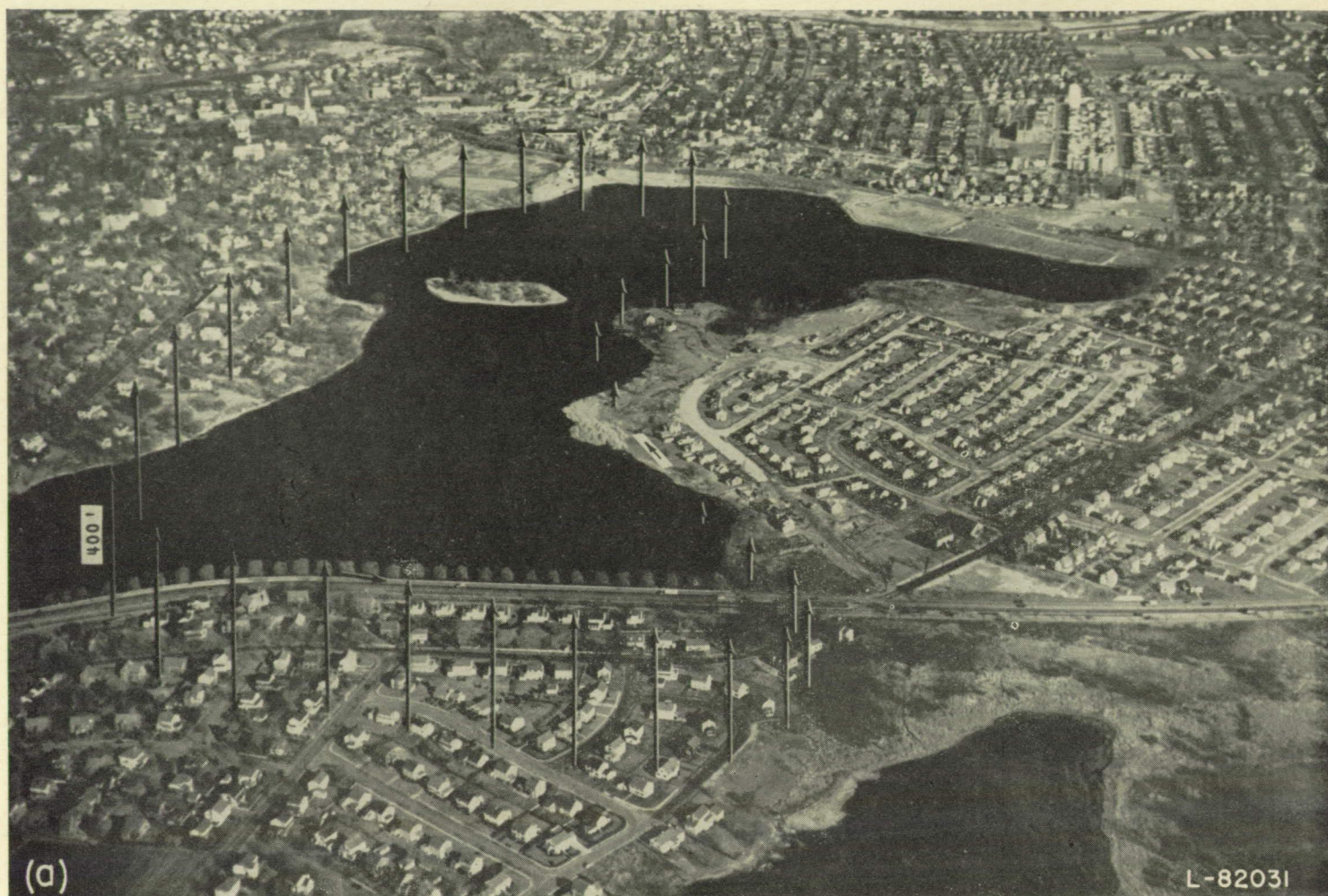
ARLINGTON-SPY POND (FIGS. 11(a) AND 11(b))

Description of location.—In all areas close to Spy Pond and near the peninsula on its southern edge that was used as an airstrip for simulated landings were middle-income and upper-middle-income homes. The homes nearest the take-off were 20 yards southeast of the flight strip and were part of the incorporated community called Kelwyn Manor. The nearest homes to the west were approximately 250 yards and on the far side of the Concord Turnpike which is a principal highway. The nearest shopping center is East Arlington, which is 1200 yards east of the airstrip. Figure 11(a) is a photograph of the site with the air traffic pattern superimposed and figure 11(b) is a topographical map of the surrounding area with the air traffic pattern and ambient levels indicated.

Flight operation.—The first community-reaction tests were begun at 7 a. m. on Sunday, June 19, 1949. The next tests were made during a supper hour, but reactions to the presence of the airplane for reasons other than noise required a change in operations in the interests of public safety. Since it was rather startling to the average automobile driver to see an airplane come flying at a low altitude over a six-lane highway, as though it were crash-landing into Spy Pond, the risk of multicar accidents occurred when drivers stopped suddenly "to watch the crash." It was, therefore, decided to make all future flights at this site in the early morning.

No other unusual circumstances occurred during the tests which are listed in table II with the complaints received.

Results.—No noise complaints were made concerning the ARF Cub; however, a few complaints were made by conscientious people (four) who thought the airplane was being flown by some "green pilot showing off" and violating regulations. One woman was fearful of her children's safety "in case anything went wrong."



(a) View of site with air traffic pattern superimposed.

FIGURE 11.—Arlington site.

True noise complaints (16) were filed against the standard Cub, since it had awakened these people from their sleep by its noise. In addition, three other complaints were filed; two, that the airplane was flying too low, against regulations, and one, that the airplane was flying "dangerously close" to a home (actual distance, 70 yd—not one of the houses nearest the test strip).

This site had been previously petitioned for use as a seaplane landing base (petition not granted because of noise nuisance caused by the airplane involved which was a light airplane on floats). No one, during these tests, expressed opposition to the possible establishment of a commercial operation in that area. The lack of such a reaction is unusual. At some of the other sites many people went on record as earnestly opposing the opening of what they presumed was being planned as an airport near their property.

Staff evaluation.—The complaints against the standard airplane seem to confirm the significance of the noise reduction on the modified light airplane. A number of home

owners and observers in the locality complimented personnel of the Foundation for having quieted the airplane to such an extent.

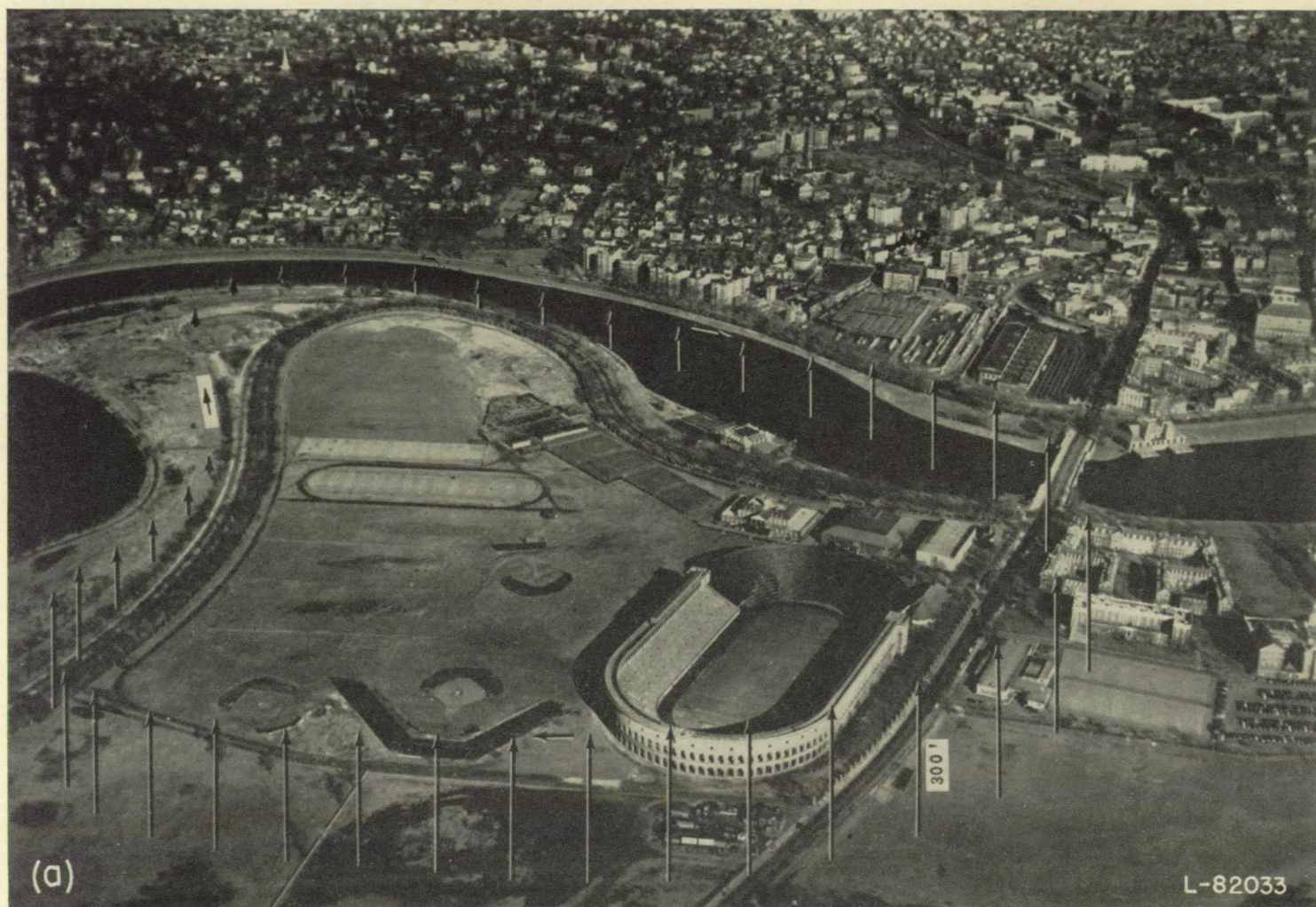
BRIGHTON—METROPOLITAN DISTRICT COMMISSION PARK (FIGS. 12(a) AND 12(b))

Description of location.—The airstrip (50 by 400 ft) was an open field, between Soldiers Field Road and the Charles River, which is part of a seldom used Metropolitan District park area. It is bounded on the west and north by the river. Across the river are located, in order according to distance and starting from west to east: A small bathing beach; two private schools, a home for the aged, a large city hospital, and the Harvard infirmary about 400 yards from the airstrip; a heavily populated area of housing, including middle- and low-income groups, starting about 400 yards away; large high-income homes within and continuing beyond 700 yards; and, in the last sector, which starts 600 yards northeast of



(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 11.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 12.—Brighton site.

the airstrip, many high-rental apartment buildings and Harvard Square, a principal shopping center.

To the east, south, and southwest of the airstrip are the Harvard Stadium and athletic buildings and, beyond them, the Harvard Business School, a playground area, a radio and television station and tower (680 ft), an industrial area, low-income houses, and a harness-horse-racing track.

Soldiers Field Road which parallels the site on the east side and Memorial Drive on the opposite side of the river are used by pleasure vehicles only; therefore, the general area is quieter than it would be if these highways were also used by commercial vehicles.

The nearest shopping center is Harvard Square, which is approximately 1100 yards to the northeast of the airstrip. It is also an active focal point for local transportation, being a subway, bus, and trolley terminus.

Flight operation.—The take-off was north toward the hospital followed by a right turn down the river, approxi-

mately 200 feet in front of and approximately level with the roof line of the apartment buildings. These buildings and the hospital were subjected to the maximum noise emission from the airplanes during each circuit of the air traffic pattern. Figures 12(a) and 12(b) show this site and its surrounding area.

The test flying was started at this site on Sunday, December 19, 1948. These initial flights were sporadic at first because of inclement weather. However, a more intensive activity of four successive days late in January 1949 gave additional evidence as to the acceptability of the "quiet" airplane (ARF Cub) within this neighborhood. The flights are tabulated in table III.

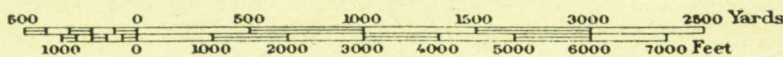
There had been some activity at this site, previous to the reaction tests, in the form of demonstrations of the quieted aircraft to public officials. These will be covered under a separate section of this report (see section "Demonstration Sites").



L-82034

Scale $\frac{1}{31880}$

(b)



Ambient range		
	Flat	40 DB
1	78 - 62	64 - 42
2	84 - 61	66 - 42
3	79 - 64	66 - 48
4	78 - 62	58 - 44

(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 12.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 13.—Brockton site.

Results.—During the total period of intermittent operations (Oct. 7, 1948, to Jan. 23, 1949), no complaints were received by the surrounding police stations, the Massachusetts Aeronautics Commission, the local CAA, or the Harvard Business School concerning the activity.

Staff evaluation.—It is believed that enough flights were made to provide a reasonable indication that the noise emission of the aircraft involved was below that which could be termed a "nuisance level" at this site.

BROCKTON—FAIRGROUNDS (FIGS. 13(a) AND 13(b))

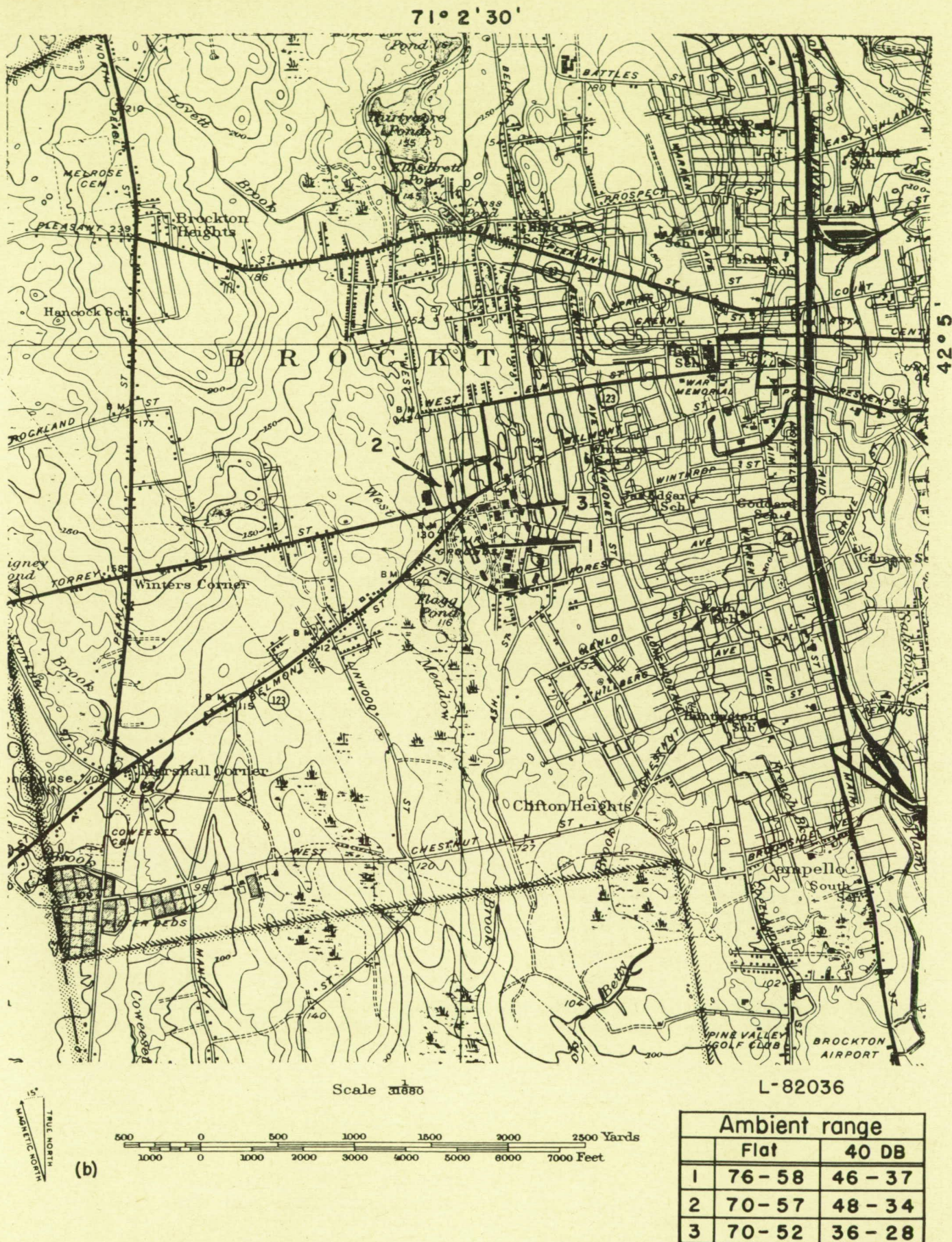
Description of location.—The airstrip area (100 by 500 ft) was within the inner oval of the fairgrounds race track. It is located 200 yards east of West Street, 200 yards south of Belmont Street (Rte. 123), 500 yards west of Thurber Avenue, Fairside Road, and Othello Street (connective), and 150 yards north of Forest Avenue. The homes nearest the take-off were those on the far side of Belmont Street. The nearest shopping center is Brockton, 2500 yards north-east of the airstrip. Figure 13(a) is a photograph of the site and figure 13(b) is a map of the surrounding area.

Flight operation.—The tests were begun on Wednesday, February 16, 1949. Two operations totalling 1 hour and 30

minutes with 35 landings were made that day and a third operation lasting 1 hour with 20 landings was made 2 days later.

Results.—The Brockton Police Department was deluged with telephone calls concerning the activity. The Massachusetts Aeronautics Commission made an investigation and exhibit 1 is the result of their findings.

Further testing at this site was not conducted. Nevertheless, the nature of the complaints received did indicate that noise from the modified Cub, configuration 1, had itself created no objections. Ninety-one telephone calls were made concerning the airplane, but most of the callers were concerned about the low flying. Some people called to report that the airplane was "in violation" of CAA regulations, but approximately 35 to 40 percent of the "complaints" under "Low flying" in table IV were made by solicitous people who called to report that the airplane was "crashing," that it was "in distress," that "its engine quit," and so forth. Investigation revealed that the low noise level of the quieted airplane caused many to think that the engine was "dead." This information recorded by the Foundation is further confirmed in exhibit 1.



(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 13.—Concluded.

MASSACHUSETTS AERONAUTICS COMMISSION

LOGAN AIRPORT, EAST BOSTON

February 28, 1949

Representative Gerald C. Lucey
State House
Boston 33, Massachusetts

Dear Representative Lucey:

Our inspectors have investigated the flight test activities of the Aeronautical Research Foundation at the Brockton Fairgrounds and find that certain of their procedures can be changed in the interests of safety. We have, therefore, instructed the Foundation to make no more take-offs in a northerly direction where engine failure might possibly create a hazard to persons living just north of the Fairgrounds.

With this limitation, and bearing in mind the special characteristics of the aircraft used and the high degree of proficiency of the pilot, we feel that the flight tests can be continued with every consideration being given to the safety of the surrounding residents.

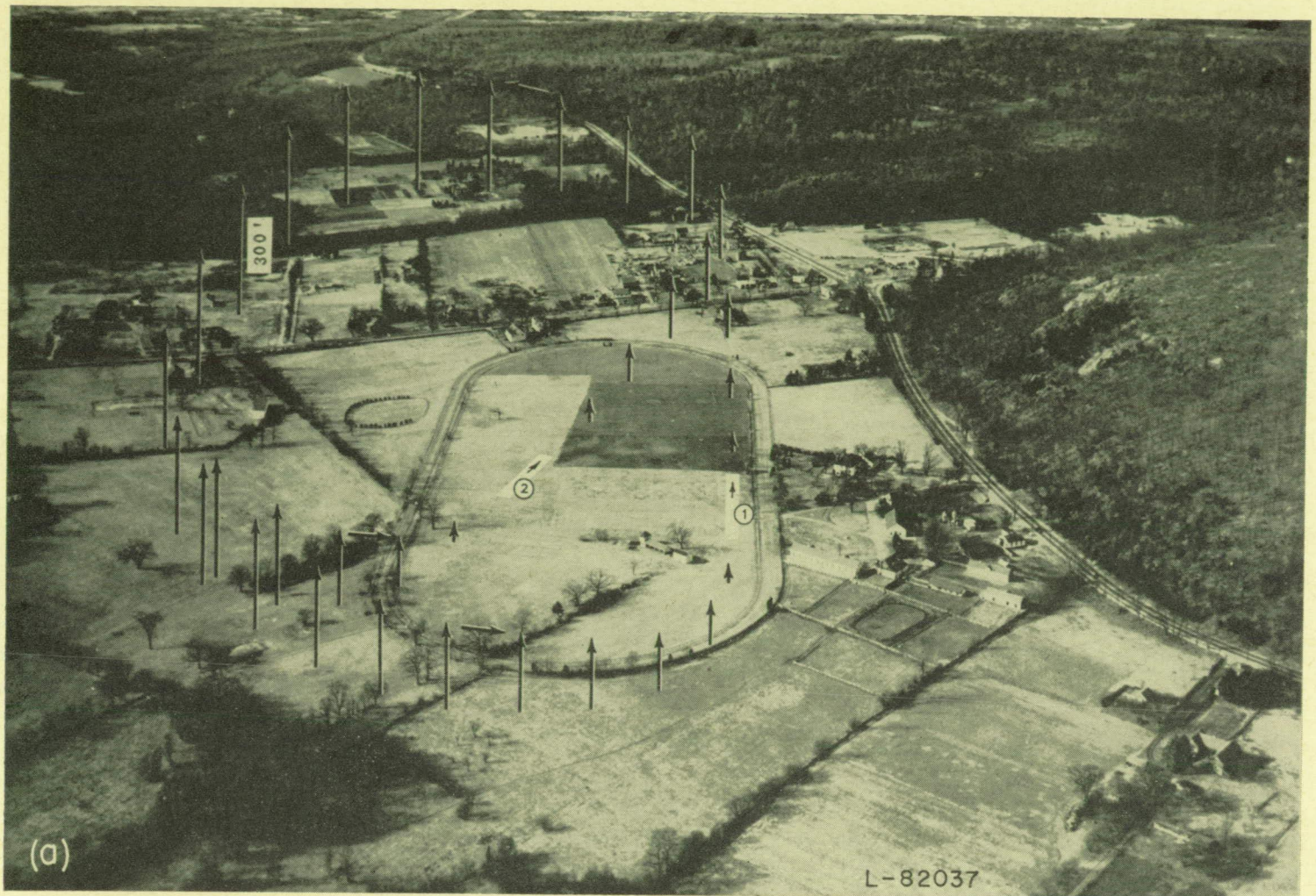
I assume you know that these tests are being made with an airplane from which most of the noise has been removed for the purpose of determining community reaction to a quiet airplane. Our inspector was surprised to find that most of the complaints were occasioned by the fact that observers thought the aircraft was in trouble and was about to land on the houses or in the street because they heard no noise from the propeller or the power plant. Apparently when the latter was explained a large majority of the persons interviewed had no further objections.

Very truly yours,

Crocker Snow
Director of Aeronautics

CS:pr
cc: Prof. Bollinger
Rep. Arthur Sheehan

Exhibit 1



(a) View of site with air traffic pattern superimposed.

FIGURE 14.—Canton site.

Staff evaluation.—The most striking evidence from this site was that there were no complaints against noise per se. It is believed that the fact that the people thought there was something “wrong” with the airplane, that is, that the engine must be dead because it was quiet, is reasonable evidence that the noise level of that airplane was low enough to be “acceptable” in that neighborhood and that the airplane could be operated at that site without further noise reduction.

CANTON—PROWSE ESTATE (FIGS. 14(a) AND 14(b))

Description of location.—The airstrip areas (airstrips 1 and 2 both 100 by 500 ft) were part of the area within a horse-racing oval on a large private estate. It is located east of Washington Street (Rte. 138) and south of the Circumferential Highway (Rte. 128) and is bounded on the south and east by other estates.

To the north is an unpopulated State reservation area. To the west of Washington Street are about 25 homes varying from lower- to upper-middle-income classification and a few large high-income estates. South and east are upper-middle- and high-income estates.

The take-off path was directly west over the most heavily populated area contingent to the site. The landing path

was beside the barns and stables of the estate approximately 20 to 30 feet over grazing livestock (airstrip 1).

The nearest large shopping center is Hyde Park, Boston, which is approximately 5000 yards to the northwest of the airstrip. Figure 14 (a) is a photograph of the site and figure 14 (b) is a map of the surrounding area.

Flight operation.—The first flight at this site was on October 28, 1948, and was a short demonstration using the ARF Cub, with the purpose of obtaining the owner's approval of using the estate as a test site. The flights were 20 to 30 feet over the heads of cows and thoroughbred horses which continued to graze undisturbed. The estate owner was impressed with the absence of noise nuisance and gave immediate approval to use the area as a test site.

Results.—The six subsequent operations, using various airplanes, evoked complaints only when the standard Cub, configuration 4, was flown. Six noise complaints were filed and one complaint was filed against low flying, as noted in table V.

There were no complaints about the quieted airplanes. However, during the first hour the standard Cub was used three complaints were received by telephone that the airplane had waked the complainants. The other noise complaint was by a property owner who came out at 7:20 a. m.



(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 14.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 15.—Medford site.

saying the noise had awakened him and that, in addition, it seemed to be bothering the horses. To test this second point the flight path was moved to a new position (airstrip 2) for the next 40 minutes, but the horses still were startled when the airplane was close by. The next flight with the standard Cub was also at the second flight strip. Again the property owner came out and this time (at 7:15 a. m.) insisted that the tests be stopped, saying he did not mind being awakened but that some of the horses were kicking violently in their stalls. During this 15-minute period another objection to the noise was telephoned in.

Staff evaluation.—The reaction at this site, even though the tests had to be curtailed, showed acceptability of the quieted airplanes and disapproval of the standard model. The quieted airplanes had flown here for 8 hours and 10 minutes and had made 110 landings and take-offs without any objection.

The noisy airplane had evoked seven complaints, six of which were definite noise complaints, in less than 1 hour and 15 minutes with only 37 landings. This is in marked contrast with the absence of objections to the modified airplanes and seems to confirm their acceptability at this site.

EDFORD—METROPOLITAN DISTRICT COMMISSION PARK (FIGS. 15(a) AND 15(b))

Description of location.—The airstrip area (50 by 400 ft) was part of a Metropolitan District park area. It is located south of the Mystic Valley Parkway and west of Winthrop Street and is bounded on the south and west by the Mystic River.

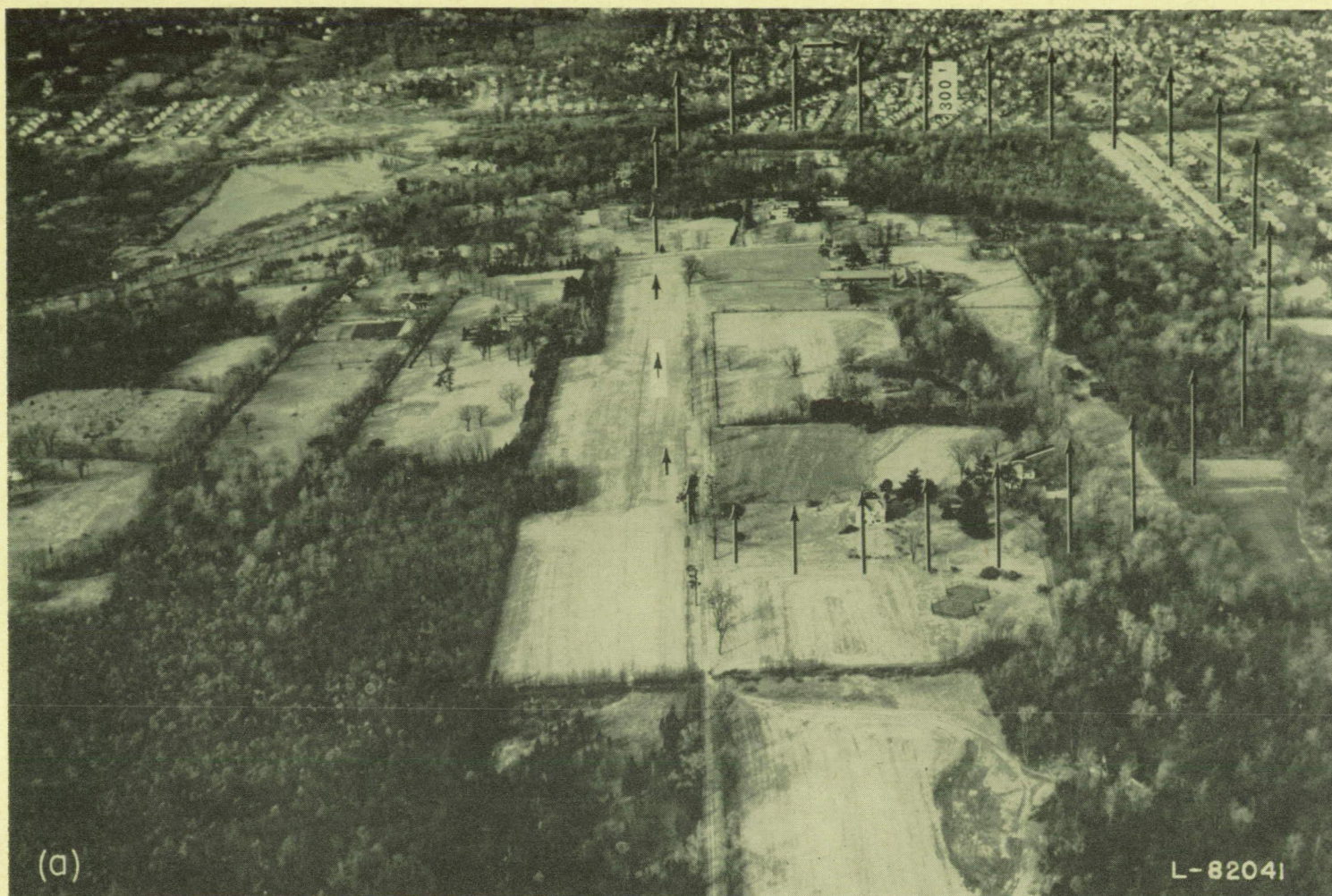
On the north side of the Parkway, the nearest houses within 30 yards are many upper-middle-income homes and east of Winthrop Street are a group of high-rental apartment houses. On the south side of the river, the closest 150 yards from the airstrip, are many hundreds of lower-middle-income houses.

The homes nearest the take-off were those directly north and northwest along the Parkway. The air-traffic-pattern circuit was flown alternately left and right subjecting the public on both sides of the site to the noise-tolerance survey. The nearest shopping center is Medford Square, approximately 900 yards east of the site. Figure 15 (a) is a photograph of the site and figure 15 (b) is a map of the surrounding area.



(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 15.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 16.—Milton site.

Flight operation.—The tests began August 24, 1949, and continued through September 3, and the results are shown in table VI.

Results.—As noted in table VI, no noise complaints were received until the unmuffled version of the CAA Cub was flown. Other complaints were filed concerning low flying (four), fear (one), and objections to the use of that area as an airport (two).

Staff evaluation.—The Foundation expected a deluge of complaints of all types from this heavily populated area, but, as will be noted from table VI, relatively few were received. The majority of complaints came from the southern side which, as compared with the northern side, is farther from the site, is a lower-income area, and has an active main-line railroad in its background.

MILTON—COTE ESTATE (FIGS. 16(a) AND 16(b))

Description of location.—The airstrip area (100 by 500 ft) was a small part of a large (400 by 3200 ft) open field, which ran northwest-southeast on a private estate located southeast of Canton Avenue and southwest of Holmes Lane. Bordering on the southwest and southeast are thickly wooded areas.

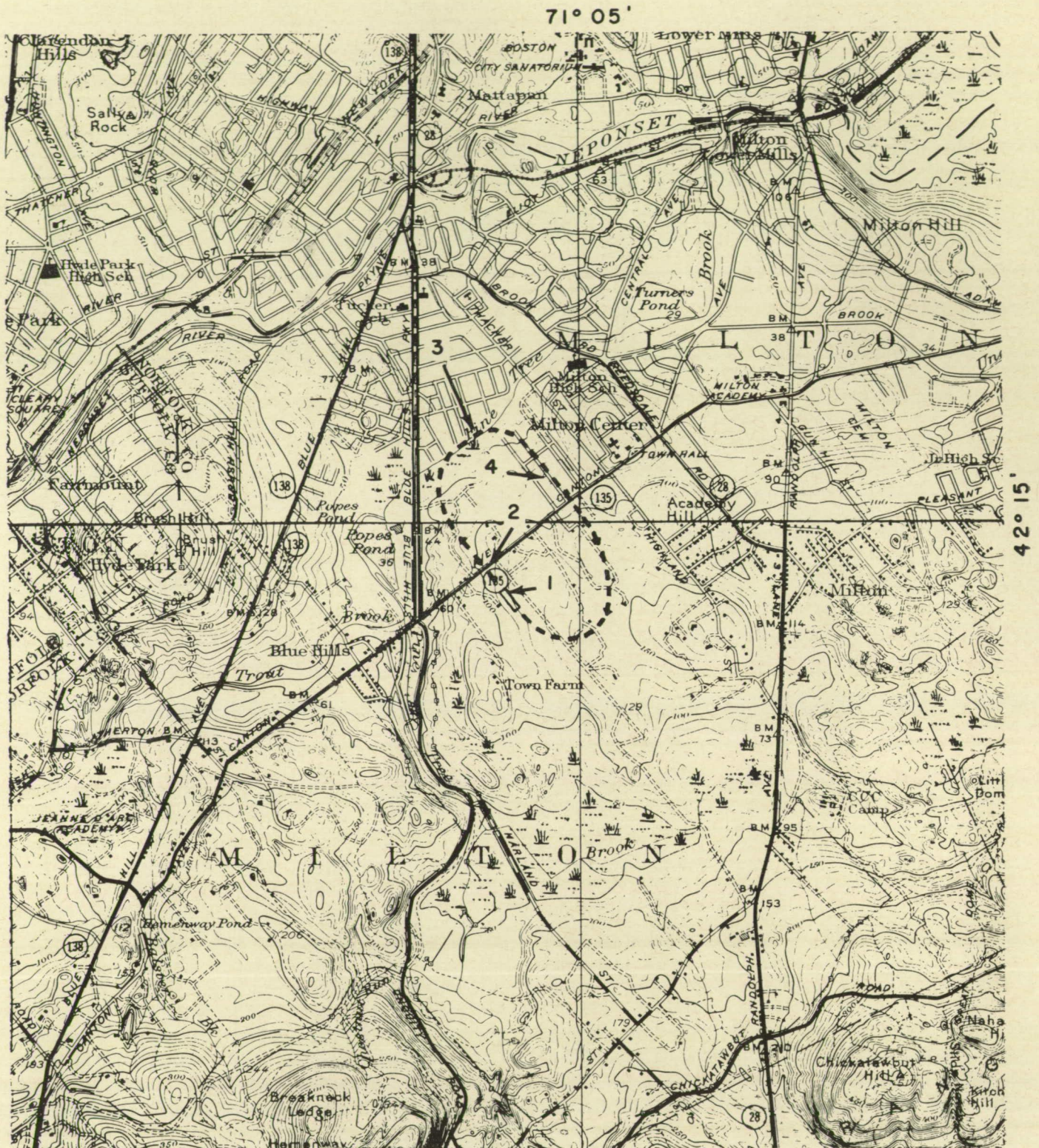
To the northeast on Holmes Lane are three large high-income estates. Northeast across Canton Avenue are many large estates and a large group of middle-income and upper-middle-income homes approximately 500 yards from the flight strip.

The homes nearest the take-off were those on either side of Canton Avenue closest to the airstrip.

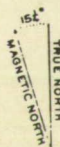
The nearest shopping center is Milton Center, 1400 yards northeast of the airstrip. Figure 16 (a) is a photograph of the site; figure 16 (b) is a map of the surrounding area.

Flight operation.—Since the area immediately contingent to the site was sparsely inhabited, the two large groups of homes 500 yards north and northeast of Canton Avenue (as shown in the photograph of the site) were also subjected to almost the same intensity of noise as those closest to the airstrip because the airplane was purposely flown close beside the first group and directly over the second densely populated area at a low (300-ft) altitude, on the crosswind and downwind legs.

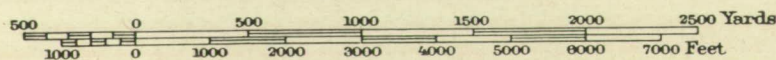
Results.—The six flight operations and the complaints received (three) are listed in table VII. No complaints were made as a result of flights with the ARF Cub.

Scale $\frac{1}{31680}$

L-82042



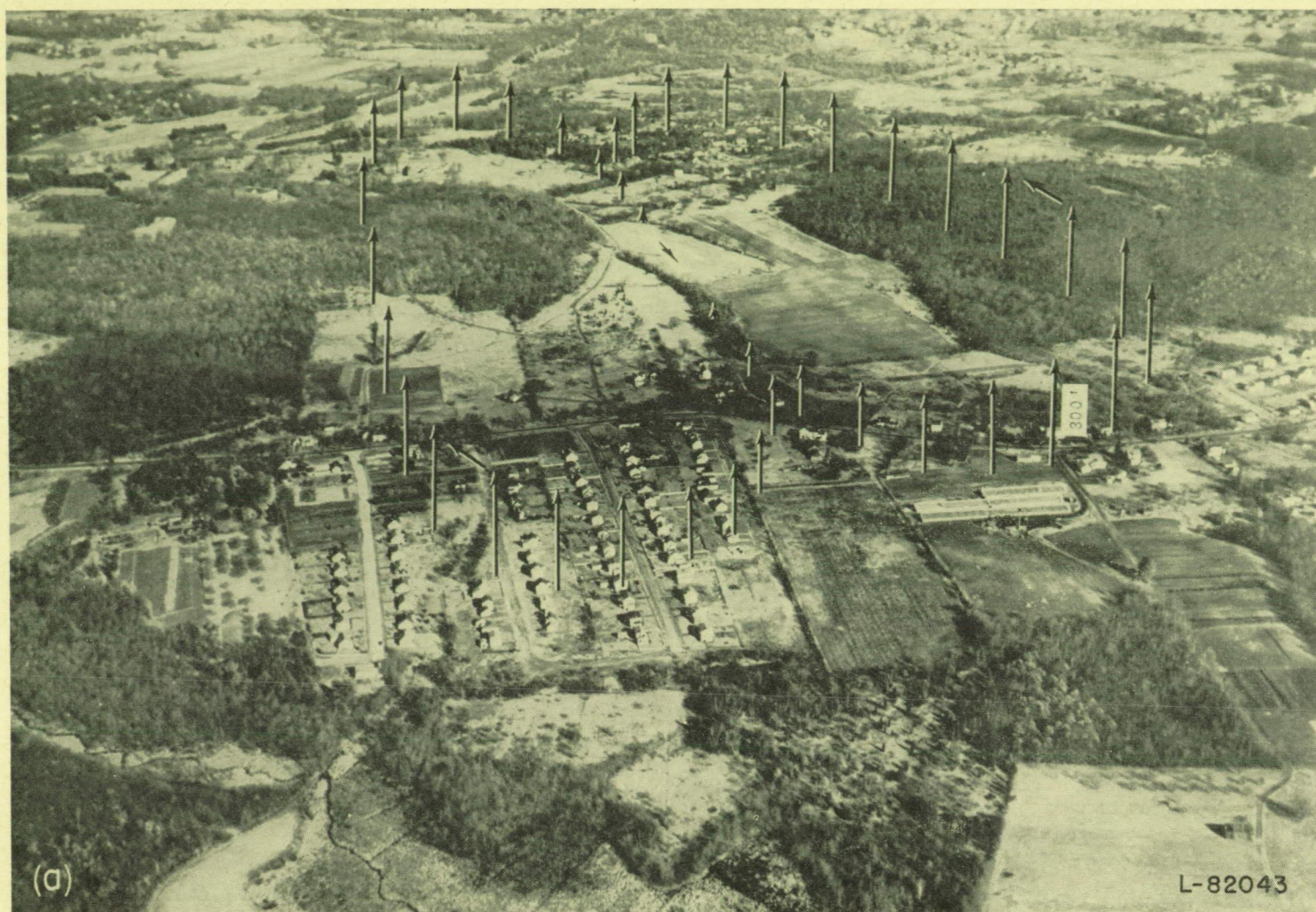
(b)



	Ambient range	
	Flat	40 DB
1	78 - 52	44 - 35
2	82 - 54	61 - 34
3	59 - 46	40 - 29
4	62 - 50	42 - 31

(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 16.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 17.—Needham site.

Staff evaluation.—The most unusual point concerning results at this site is the fact that neither airplane was reported to be in violation of flight safety because of low flying. A possible explanation might be the fact that this airstrip had been used by the U. S. Navy during World War II as an auxiliary landing field.

NEEDHAM—BABSON PARK (FIGS. 17(a) AND 17(b))

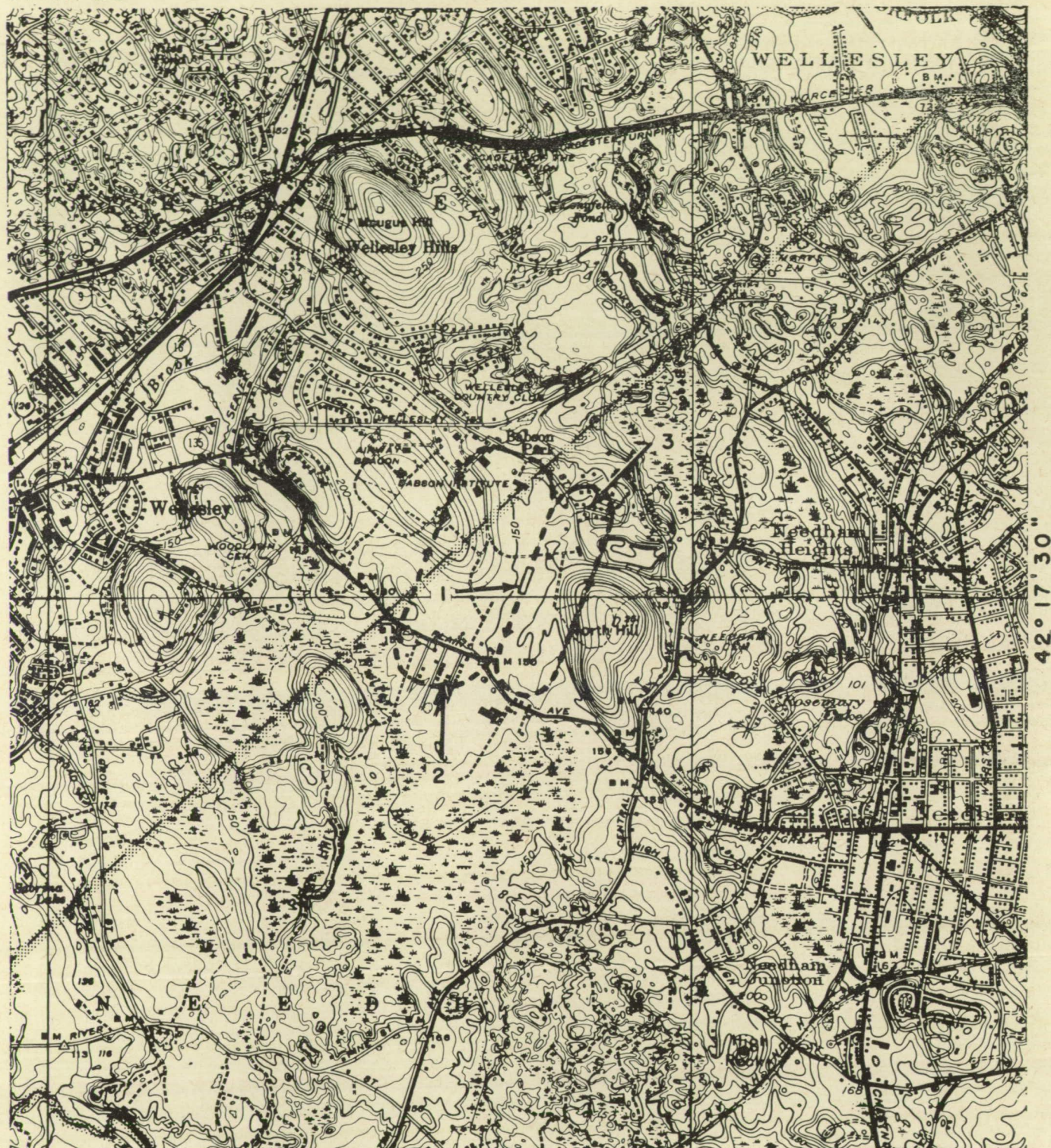
Description of location.—The airstrip area (100 by 500 ft) was part of an open fallow field within the grounds of Babson Institute. It is located 400 yards north of Great Plain Avenue, 950 yards west of Central Avenue, and 450 yards south of Forest Street. To the west are other fields, wooded areas, and the Institute. The homes nearest the take-off were those on both sides of Great Plain Avenue in line with the take-off path. The site is approximately 2500 yards equidistant from three large shopping centers, Wellesley Hills, Wellesley, and Needham, to the northwest, west, and southeast of the airstrip, respectively. Figure 17 (a) is a photograph of the site and figure 17 (b) is a map of the surrounding area.

Flight operation.—In order to subject more homes to the tests the airplanes were flown alternately left and right when passing over Great Plain Avenue. This procedure caused the right-turn pattern to pass over a large cluster of middle-income homes on the south side of Great Plain Avenue, over Babson Institute, and close to a children's hospital on the approach to the airstrip. On the left turn the airplane passed close to a group of upper-middle-income homes on the north side of Great Plain Avenue and over a group of high-income homes and estates, locally referred to as the "Gold Coast" of Needham, on the downwind, base, and approach legs, and again passed close to the children's hospital on this approach.

A preliminary demonstration of the ARF Stinson to the selectmen of Needham was made on August 9, 1948, and it was deemed acceptable. On June 10, 1949, the ARF Cub was flown for 30 minutes to determine the best traffic pattern. Intensive community-reaction tests were begun on July 27 and continued through August 9, 1949. The tests are recorded in table VIII.

71° 17' 30"

71° 15'



42° 17' 30"

Scale $\frac{1}{31680}$

500 0 500 1000 1500 2000 2500 Yards
1000 0 1000 2000 3000 4000 5000 6000 7000 Feet

(b)

L-82044

	Ambient range	
	Flat	40 DB
1	58 - 48	36 - 32
2	56 - 50	42 - 35
3	66 - 55	48 - 36

(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 17.—Concluded.

The altitude of the flights on the first day of testing (July 27) was too high (600 ft) and also the flights were not directly over but skirted the housing areas and, therefore, did not cause concern or complaints. On the second day (July 29) the flights were at a lower altitude (300 ft) and directly over the homes.

Results.—Evaluating the complaints of the second day brought out an important fact which had a continued bearing on the activity at this site. Quite pointedly the residents feared the establishment of an airport because a newspaper article relative to the first flight stated that Wellesley (land actually in Needham) was to have the first airport using quieted airplanes established on that area. When the flights were apparently going to continue, the residents reacted suddenly and emphatically to forestall the presumed airport construction. (Two flights on July 29; eight complaints.)

Thereafter the complaints began to fall off even to the point of quasi acceptance of the slightly noisier muffled CAA Cub, since, by word of mouth within the community, it was now known that the flights were "some sort of research." This information was gathered by a random survey at a few houses each on several different streets in the area between August 1 and the morning of August 8.

A secondary and more violent reaction was evidenced by six legitimate noise complaints against the two flights (Aug. 8 and 9) of the unmuffled CAA Cub. These reactions came from people who had not been bothered by the previous flights made by the other airplanes but quite definitely had been disturbed by the noisier airplane.

Staff evaluation.—In the background of the reactions at this site was a semipolitical situation that may have affected the results.

The collective, though erroneous, assumption was that an owner of an adjacent area was intending to establish an airport. Their assumption was that he was fostering an airport there whether they approved it or not.

Information supporting the above opinion came from seven complainants that are listed as objectors to the establishment of an airport in table VIII. They said that they approved of the airplane and considered it extremely quiet, but they would fight to protect the value of their properties and therefore would not allow an airport in their midst.

The only significant noise complaints were against the unmuffled CAA Cub, configuration 2U. The three prior complaints against the ARF Cub on July 29 and August 1 were all made consecutively by the same person whom the local police characterized as a "chronic" complainant.

NEWTON—HURLEY PASTURE (FIGS. 18(a) AND 18(b))

Description of location.—The airstrip area (50 by 400 ft.) was a small part of an open field which is located approximately 450 yards south of the Boston-Worcester Turnpike (Rte. 9) and 350 yards east of Parker Street and is bordered on the east and south by a wooded area, approximately 200 yards in depth between the site and populated areas.

To the east and south beyond the woods are high-income estates and upper-middle-class homes. To the north and west, approximately 150 yards, are upper-middle-class dwellings. The homes nearest the noisiest part of the flight path, the take-off, were in the northwest and west. The take-off was between two groups of houses and nearer the larger group (shown on the right in photograph, fig. 18(a)). The altitude when the airplane first passed by these homes ranged from roof-top level to approximately 150 feet.

The nearest large shopping center is Newton Center, which is approximately 1900 yards to the north of the airstrip. Figure 18(a) is a photograph of the site and figure 18(b) shows the surrounding area topographically.

Flight operation.—Tests were begun at this site on Wednesday, October 27, 1948. The procedure used at this site was to take off west, fly a left-hand circuit of the area twice, and land at the end of the second circuit. The ARF Cub was flown for 1 hour, making 16 landings between 1 and 2 o'clock in the afternoon.

The next operation was on Sunday, October 31, 1948, between the hours of 7:45 a. m. and 12:15 p. m. and later from 2:00 p. m. to 4:30 p. m., totalling 87 landings during those 7 hours of operation.

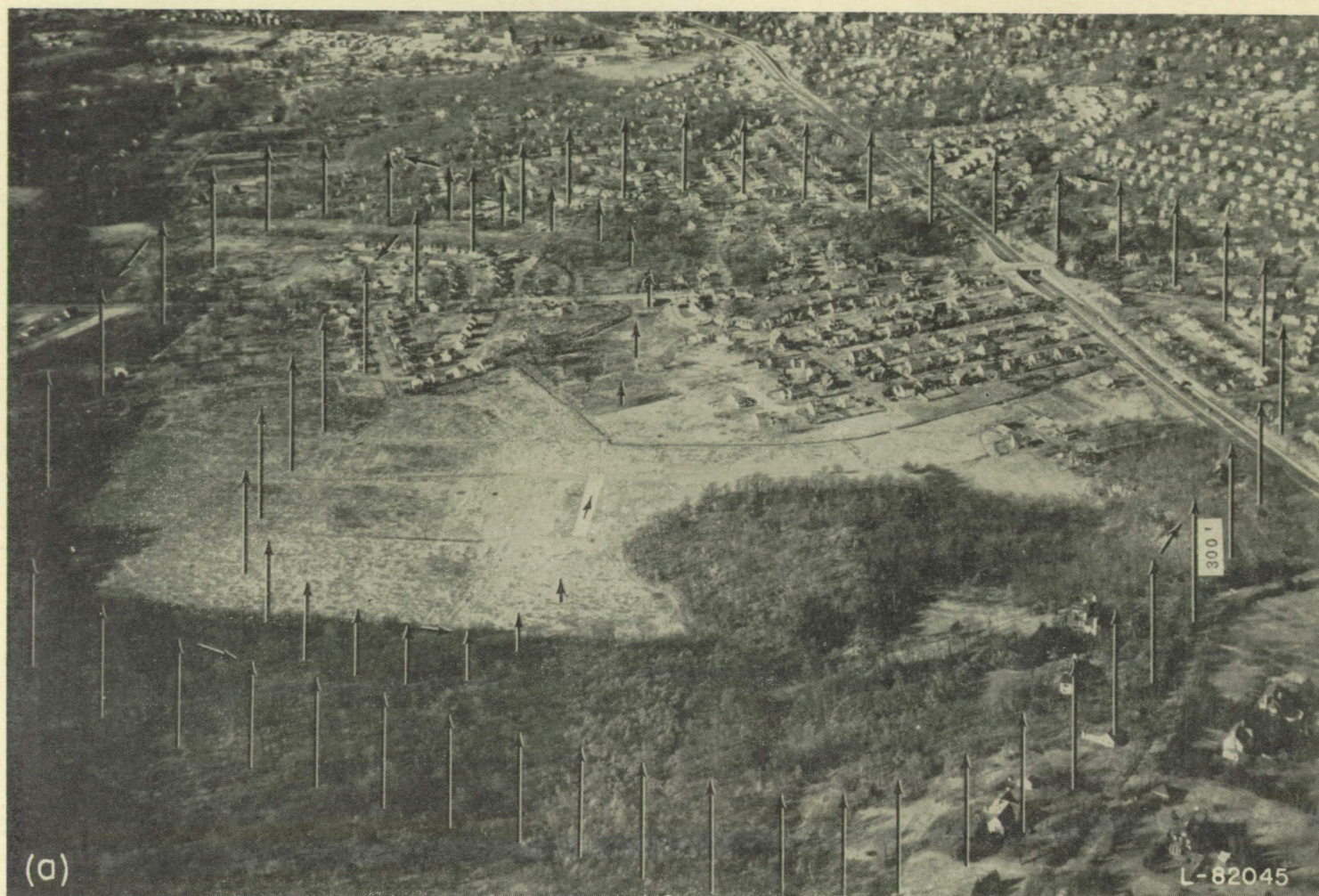
Results.—On the first day many preschool- and school-age children gathered at the site after the second landing. After the fifth landing a few mothers came out inquiring as to what was going on, showing considerable concern for their youngsters. No other reaction as to the undesirability of the operation was evidenced during this hour.

On the second day many children were again present throughout the tests. Also in attendance were many men and women who expressed varying opinions, which are tabulated in table IX.

One of the men who evidenced fear and also objection to the establishment of an airport showed keen determination to forestall any such activity by stating to a member of the Foundation staff that he would, if necessary, stop the testing survey by a petition to ARF stating that they (the cosigning neighbors) had absolutely no objection to the noise of the airplane but that they did not want the airplane flying near their homes endangering children and/or property.

Nine other (adult male) residents of the immediate area voiced complete approval of the activity, having no objections whatsoever even to the establishment of an airport there if quiet airplanes were to be used exclusively.

Staff evaluation.—Although flights at this site were not conducted over a sufficiently prolonged period to provide conclusive evidence, the nature of reactions suggests that continuing use of this site by aircraft quieted to the degree demonstrated would have evoked few complaints due to noise. Fear of low-flying aircraft was more in evidence and apparently would be an impediment at this site regardless of noise suppression.



(a) View of site with air traffic pattern superimposed.

FIGURE 18.—Newton site.

NEWTON-BRIGHTON—METROPOLITAN DISTRICT COMMISSION PARK
(FIGS. 19(a) AND 19(b))

Description of location.—The airstrip area (100 by 500 ft) was part of an open field, between Nonantum Road and the Charles River, which is part of a rarely used Metropolitan District park area. It is located north of Nonantum Road (Charles River Basin Parkway) and is bounded on the west, north, and east by the Charles River. On the north side of the river approximately 300 yards from the airstrip are lower-middle-income houses, industrial plants, the Perkins Institute for the Blind, and a United States arsenal. On the river (except in winter when the photograph was taken) were many power and sail boats. To the south were many middle-income and lower-middle-income houses.

The homes nearest the airstrip were those on a hill (elevation, 50 to 150 ft) approximately 200 yards to the south beyond the highway and adjacent railroad tracks. The homes nearest the take-off were those directly west and southwest of the airstrip. The nearest shopping center is Nonantum Square, Newton, which is 1400 yards southwest

of the site. Figure 19(a) is a photograph of the site and figure 19(b) is a map of the surrounding area.

Flight operation.—The tests were begun August 15 and were as listed in table X. No complaints were made concerning the airplane throughout the tests.

Results.—Only one inquiry was made from the surrounding area and that did not concern noise. The query was made by the director of the United States arsenal wanting to know if photographs were being taken of the restricted arsenal area.

Staff evaluation.—This site, it may be concluded, is within an area that is conditioned to a high noise level caused principally by an active main-line railroad.

WINCHESTER—COUNTRY CLUB (FIGS. 20(a) AND 20(b))

Description of location.—The airstrip area (50 by 400 ft) was part of a fairway of the golf course. It is located east of Hutchinson Road, north of Winchester Road, and 300 yards west of Mystic Street, all in Arlington south of the Winchester-Arlington town line.

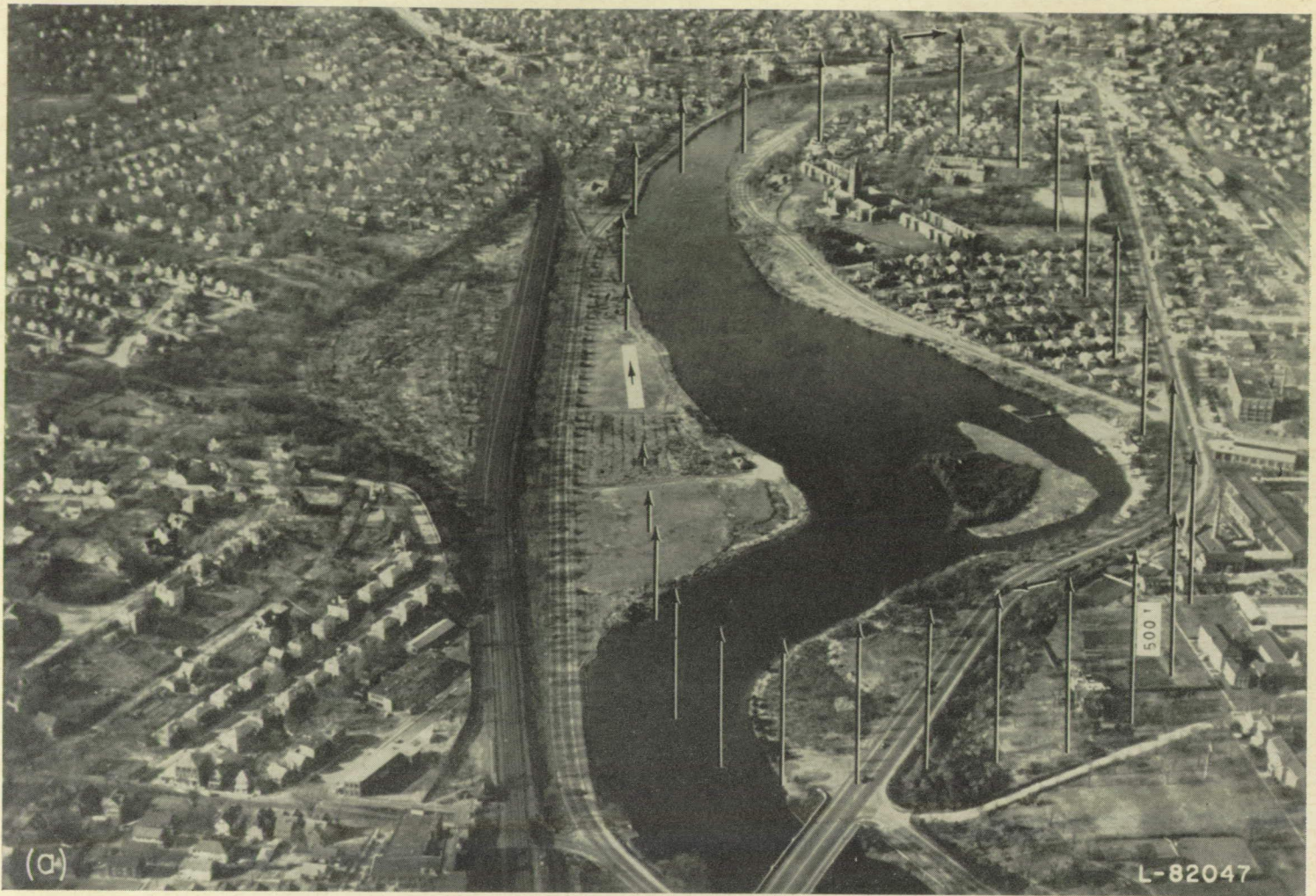


L-82046

	Ambient range	
	Flat	40 DB
1	68-53	41-37
2	72-58	46-38
3	76-63	49-42
4	65-57	47-38

(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 18.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 19.—Newton-Brighton site.

Bordering the golf course in all directions except the northwest are upper-middle- to high-income homes and estates. The golf course extends in a northwesterly direction beyond the airstrip fairway, a distance of 1600 yards.

The homes to the south and southeast of the southern end of the airstrip were the closest (approximately 50 yd) to the noisiest part of the flight path.

The nearest large shopping center is Arlington, which is 2300 yards to the south of the airstrip. Figure 20(a) is a photograph of the site and figure 20(b) is a map of the surrounding area.

Flight operation.—Only one operation was made at this site (June 13, 1949), since simulated landings over the golf course bothered the golfers. Before the activity was curtailed 26 simulated landings had been made in 45 minutes.

Results.—No complaints were made from the surrounding high-income residential area during this test, as noted in table XI.

Staff evaluation.—At other test sites in this type of neighborhood if any reaction was forthcoming it was almost immediate. The fact that no complaints were made gives

some preliminary indication that the noise level of the modified Cub would not be disturbing in this area.

DEMONSTRATION SITES

BRIGHTON—METROPOLITAN DISTRICT COMMISSION PARK (FIGS. 12(a) AND 12(b))

The descriptive details of the Brighton site are given in the section "Neighborhood-Reaction Test Sites." The airstrip was used for demonstrations on two occasions and the adjacent race track was used once prior to the clearing of the airstrip.

(1) The first demonstration was on Monday, December 15, 1947, for members of the Massachusetts Recess Commission on Aviation, other public officials, and a varied group of interested and disinterested witnesses (requested to come for unbiased evaluation). The flights were simulated landings approximately 10 feet over the ground inside the race-track oval.

During this demonstration Dr. A. G. Engelbach, the Director of the Mount Auburn Hospital (on map, fig. 12(b), as Cambridge Hospital prior to change of name), the nursing



154°
TRUE NORTH
MAGNETIC NORTH

Scale $\frac{1}{31680}$

500 0 500 1000 1500 2000 2500 Yards
1000 0 1000 2000 3000 4000 5000 6000 7000 Feet

(b)

L-82048

Ambient range		
	Flat	40 DB
1	78-61	52-42
2	96-62	71-49
3	88-60	65-47
4	85-59	63-44

(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 19.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 20.—Winchester site.

staff, and a group of orderlies were requested to post themselves at open windows nearest the river to determine whether the ARF Stinson could be heard in the hospital. Exhibit 2 shows their approval.

Questionnaires (see exhibit 3) were distributed to all the witnesses at the demonstration and collected thereafter. All 72 questionnaires were answered "A" and "Yes."

(2) The second demonstration was on October 7, 1948, for the National Association of State Aviation Officials and a number of local public officials. It was made at the request of Mr. Crocker Snow, Director of the Massachusetts Aeronautics Commission, who also, after the demonstration, sent letters to the NASAO witnesses requesting their opinions and confirmation of the results for the Foundation. The letters from these State aviation officials were 100 percent in approval of the reduced noise level of the modified airplanes.

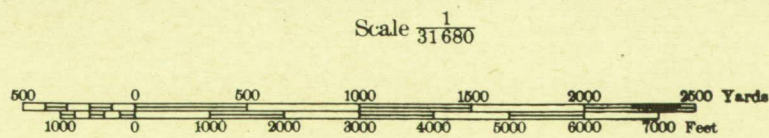
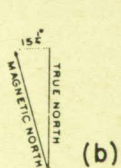
(3) The third demonstration was on Sunday, November 14, 1948. Station WBZ-TV, Boston, located adjacent to

the site, presented a special telecast of the Foundation's members and airplanes and a discussion of the purposes of the research with actual flights of the airplanes (visual and audio) as a "Public Service Presentation."

The effectiveness of the "quieting" on the experimental airplanes was decidedly noticeable on the audio circuit of the television sets. Many favorable comments were received both by the WBZ management and by the Foundation, attesting widespread public interest in the elimination of aircraft noise nuisance.

CAMBRIDGE—M. I. T. ATHLETIC FIELD (FIGS. 21(a) AND 21(b))

The airstrip area (50 by 500 ft) is a part of an open athletic field at M. I. T. It is bounded on the immediate north by the main athletic area, athletic buildings, and a large industrial area. To the east, along Massachusetts Avenue, are a group of dormitories and on the far side is the Institute proper, which is approximately 500 yards from the site.



L-82050		
Ambient range		
	Flat	40 DB
1	62 - 51	46 - 31
2	68 - 54	42 - 36
3	64 - 55	45 - 37

(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 20.—Concluded.

Mount Auburn Hospital

330 Mt. Auburn Street
Cambridge 38, Mass.

A. G. ENGELBACH, M.D., F.A.C.H.A.
DIRECTOR

October 4, 1948

Aeronautical Research Foundation
Soldiers' Field Parkway
Boston 63, Massachusetts

Attention: Professor Bollinger, Director

Gentlemen:

At the time the tests were made
on the quieted airplane sometime ago,* we
had no complaints from the patients that
they were annoyed and other witnesses
were of the opinion that the demonstration
was successful.

Very truly yours,



A. G. Engelbach, M.D.
Director

AGE:cc

* 12/15/47



Cambridge, Massachusetts
December 8, 1947

You have been invited today to witness a public demonstration of what is believed to be the first airplane that is both equipped with effective noise reduction devices and is at the same time an efficient vehicle practical for personal flying.

This airplane is a standard Stinson four-passenger 1947 model modified by the Aeronautical Research Foundation, a nonprofit Massachusetts research corporation. The effort to develop and set standards for a quiet "good neighbor" airplane is being federally financed through the National Advisory Committee for Aeronautics, with the active cooperation of the Civil Aeronautics Administration. The personal services of Dr. Lynn L. Bollinger and Mr. Arthur H. Tully, Jr., of Harvard and Professors Otto C. Koppen and C. Fayette Taylor of M.I.T. have made this project possible.

Please remember that this is an experimental airplane in so far as the functioning of the noise reduction devices are concerned. You are witnessing its first flight away from the E. W. Wiggins' shops at Canton airport where it was modified. The pilot Mr. Henry Kent, is considered one of the most experienced and able in the country for this type of flying. He has been carefully instructed to operate the airplane so that any reasonably probable mechanical malfunctioning will not endanger persons or structures on the ground. (The demonstration flight has, of course, been approved by the Director of the Massachusetts Aeronautics Commission and by the local CAA inspector.)

Your opinion of the airplane's "good neighbor" characteristics is earnestly solicited. The primary purpose of this flight is to obtain your judgment as to whether the airplane as now equipped is entirely adequate to fly within reasonable distances of dwellings without creating objectionable noise, or whether further silencing devices need be added.

The noise energy output is now approximately $1/400$ th that of a conventional airplane. At the Cambridge Boat Club site the sound (when not obscured by passing automobiles) should be approximately twice the intensity as that reaching the hospital exterior walls, and if the estimates are correct, the sound should not be audible inside the hospital or within nearby dwellings (i.e. quieter than existing street traffic). That is the standard by which you are asked to judge the vehicle.

You have purposely been asked to stand outside the "shield" of street traffic noise so that you may detect the nature of the aerodynamic sounds. Please grade the performance of the vehicle by answering the following brief questionnaire:

Cambridge, Massachusetts
December 8, 1947

to:
The Massachusetts Recess Commission on Aviation
Room 407, State House
Boston 33, Massachusetts

Attention: Mr. Vance L. Alden, Secretary

I have witnessed the first public demonstration of the Aeronautical Research Foundation's experimental "good neighbor" airplane and rate it accordingly:

(Check one)

- A. Sufficiently quiet to eliminate all valid noise objections
- B. Sufficiently quiet to suit me but possibly objectionable to others
- C. Needs slight additional quieting to be entirely acceptable
- D. Needs substantial additional quieting to overcome noise objections

If such an airplane were made absolutely inaudible and were to be flown regularly from the location used and in the manner demonstrated would you willingly accept its presence as a "good neighbor"?

Yes
No

(If you vote no, please indicate briefly why.)

Signed _____

Position _____



(a) View of site with air traffic pattern superimposed.

FIGURE 21.—Cambridge site.

Seventy yards to the south, on Memorial Drive, are dormitories, apartment houses, and restaurants and beyond the highway is the Charles River. Starting 50 yards west of the site is an M. I. T. married students' "Veteran's Village" housing 276 families in single, duplex, and multiunit buildings. This area extends approximately 400 yards west, and beyond it is an industrial area.

Memorial Drive which parallels the site to the south is used by pleasure vehicles only and Massachusetts Avenue, east of the site, by general traffic. Since the area to the north is industrial and has heavy truck traffic, the residents around this site are conditioned to a higher noise level than was true of most of the other sites.

The nearest shopping center is Central Square, Cambridge, which is approximately 1100 yards to the north of the airstrip.

The direction of take-off was west toward and over the Veteran's Village at an altitude of approximately 150 feet, the airplane turning left to the river when 200 feet had been attained. Figure 21(a) is a photograph of this site and figure 21(b) is a map of the surrounding area.

On October 13, 1948, both the ARF Stinson and the ARF Cub were flown (10 passes) for the Massachusetts Institute of Technology staff and on October 29 demonstration flights (7 passes) using the ARF Cub were arranged for representatives of the British Air Ministry. On both occasions all comments were favorable. No complaints were received from the adjacent residential areas.

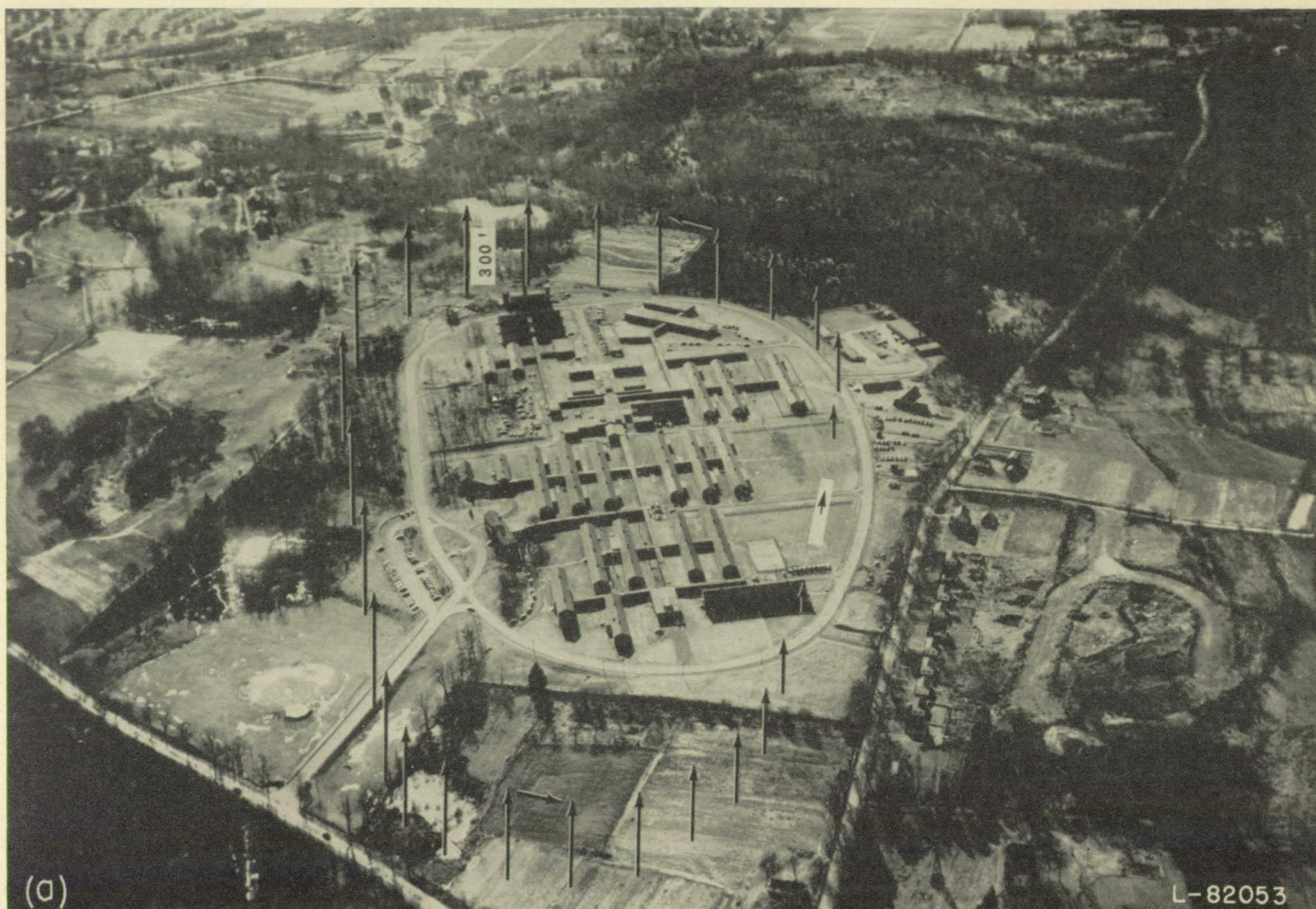
WALTHAM—MURPHY GENERAL HOSPITAL (FIGS. 22(a) AND 22(b))

The airstrip area (50 by 400 ft) was part of an open athletic field area within the grounds of the (Army) Murphy General Hospital which is southwest of Trapelo Road and southeast of Forest Road.

Seventy-five yards to the east of the airstrip area was the central part of the hospital laid out as many individual wards. Sixty yards to the south were the mental and other wards. In the southwest corner was a fire station and across a street (100 yd) to the west were many small homes of hospital personnel. The nurses' and many other permanent barracks were 20 yards to the north and northeast.



FIGURE 21.—Concluded.



(a) View of site with air traffic pattern superimposed.

FIGURE 22.—Waltham site.

The take-off was between the mental ward and the fire station over the overhead power lines.

On June 3 the ARF Cub, configuration 1, was flown for 1 hour and 36 low passes were made (5 to 10 ft off the ground). Neither the patients nor the hospital personnel complained, although they were specifically instructed by the Commanding Officer to do so if the noise bothered them at all. It was a warm day and the fact that the airplane was acceptable even with the hospital windows open is noteworthy.

MISCELLANEOUS

CANTON-NORWOOD

During the testing program of the modified and unmodified Stinsons (reference 1) in the vicinity of the Canton-Norwood, Mass., airports several objections, mostly of an inquiring nature, were made concerning the activity.

Most emphatic and demanding objections to stop the testing of the relatively noisy modified and unmodified pusher-type amphibians were voiced by the neighborhood

surrounding the Norwood airport during that program (reference 2).

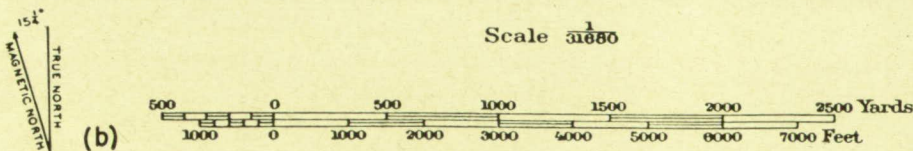
The Norwood airport was used by the U. S. Navy during World War II and has been in continuous use by Wiggins Airways for training purposes and larger scale commercial activities.

The objections were so strenuous that Mr. Joseph Garside, President of Wiggins Airways and also acting as Director of Operations of the Foundation, had to release a statement to the local newspapers explaining the research program and requesting the neighbors' indulgence.

The fact that the neighbors accustomed to an active airport reacted in such a clamorous manner tends to confirm the observation that when the noise level is increased, even in a "conditioned" neighborhood, the people will object quickly.

BEDFORD AIR SHOW (SEPT. 18-19, 1948)

The modified Stinson was flown as a feature attraction in the U. S. Air Force Air Show at Hanscom Airport, Bedford, Mass.



L-82054

Ambient range	
Flat	40 DB
1	70-58 48-32
2	72-59 50-33
3	78-61 53-38

(b) Topographical map of surrounding area with air traffic pattern and ambient levels indicated.

FIGURE 22.—Concluded.

The attendance was between 110,000 and 125,000 people. High-powered aircraft were flying in the general area during the scheduled "quiet" Stinson demonstrations, however, and the airplane was therefore exhibited under very unfavorable conditions.

The control-tower operator at the field announced the flights of the modified Stinson and narrated a thumbnail sketch of the Foundation and the research activities.

Approximately 1500 adult spectators voiced their approval to the Foundation staff and requested knowledge as to when and/or where they could buy such aircraft.

PROVIDENCE AIR SHOW (OCT. 12, 1948)

The modified Stinson and Cub were flown in comparison with standard stock models in a noise demonstration at the Theodore Francis Green Airport, Hillsgrove, R. I.

All four airplanes were flown around the field with the standard Stinson first, followed by the quieted Stinson, then the standard Cub, followed by the modified Cub. After take-off the airplanes circled the field and swooped low over the clear roped-off area next to the hangars. They passed by at about 100-foot altitude directly in front of the spectators' area.

The airport manager had, by using a public-address system, quieted the crowd down to a whisper and "all ears" in anticipation of witnessing these "airplanes of the future" with comments such as "you won't believe it till you hear it."

When the airplanes came by, the quieting effect was extremely apparent and the crowd spontaneously applauded both quieted airplanes when they passed and later when they landed.

SOUND LEVELS COMPARED WITH FAMILIAR SOUNDS

Figure 23 is included to assist in judging the results of this research. This figure presents a comparison of the measured sound levels of the standard and modified airplanes with the levels of typical noise sources.

CONCLUSIONS

In drawing conclusions from the data presented, it should be realized that complaints and responses to interviews are, to a considerable extent, subjective.

In order to separate reactions to noise from reactions to other features of airplane flight on a truly scientific basis, an

elaborate program designed and controlled by experimental psychologists would be required. Such a program would have been beyond the budget and time limitations of this project. The tests reported herewith, therefore, must be considered exploratory in character and conclusive only in a limited sense.

Bearing these limitations in mind, the following conclusions seem justified:

1. Reduction in noise reduces the number of complaints in a given situation. Whether this reduction is primarily due to reduced noise per se or to the fact that fewer people notice the operation has not been definitely established. In either case, it would seem that reduced noise levels are highly desirable from a neighborhood point of view.

2. Other complaints against aircraft, that is, fear of their presence, fear of low flying, and fear of property devaluation, appear to be more frequent when noise attracts attention and sometimes are reported as noise objections. When a quieted airplane is involved, these remaining objections are more clearly defined.

3. Greatly reduced noise sometimes leads people to think an airplane is in trouble and about to make a forced landing. If quiet airplanes become numerous, this factor will probably disappear.

4. Apparently, the degree of noise reduction attained by the modified aircraft did produce significantly fewer recorded objections. Whether the difference in acceptability of standard and modified aircraft would continue over a long period of steady-flight operation was not ascertained. If the difference between reactions to the standard and quieted airplanes can be presumed to continue as in the exploratory tests, the degree of external noise reduction incorporated in the modified airplanes should lead to a significant reduction in public objection to neighborhood landing areas.

AERONAUTICAL RESEARCH FOUNDATION,
BOSTON, MASS., May 5, 1950.

REFERENCES

1. Beranek, Leo L., Elwell, Fred S., Roberts, John P., and Taylor, C. Fayette: Experiments in External Noise Reduction of Light Airplanes. NACA TN 2079, 1950.
2. Roberts, John P., and Beranek, Leo L.: Experiments in External Noise Reduction of a Small Pusher-Type Amphibian Airplane. NACA TN 2727, 1952.

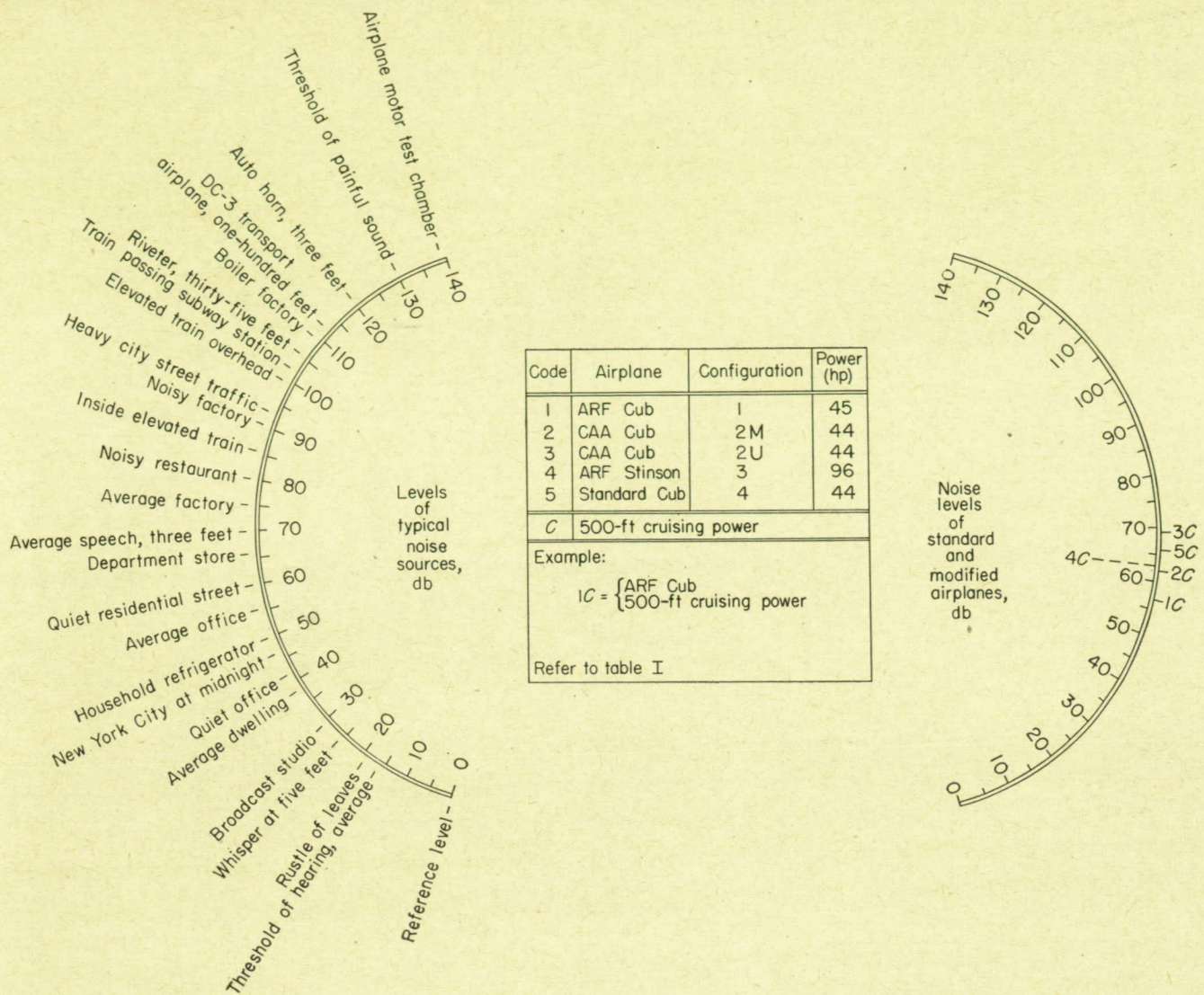


FIGURE 23.—Noise-level comparisons.

TABLE I
 STATISTICS OF AIRPLANES USED IN NEIGHBORHOOD-REACTION TESTS

Airplane	Figure	Number of propeller blades	Type of propeller blade	Propeller diameter (in.)	Propeller pitch setting at $\frac{3}{4}$ station (deg)	Engine	Ratio of propeller speed to engine speed	Propeller tip speed (ft/sec)	Engine power (cruising) (hp)	Crankshaft speed (cruising) (rpm)	Muffler	Peak noise levels ¹
ARF Cub (configuration 1).....	1	4	Two-piece wooden.....	80	15 (fixed).....	Geared.....	0.632	474	45	2,150	Ejector.....	57
CAA Cub (configuration 2M).....	3 (a)	2	Wooden.....	72	14 (fixed).....	Direct-drive.....	1.00	628	44	2,000	Maxim.....	62
CAA Cub (configuration 2U).....	3 (b)	2	Wooden.....	72	14 (fixed).....	Direct-drive.....	1.00	628	44	2,000	None.....	69
ARF Stinson (configuration 3).....	5	4	Wooden.....	76	25 (fixed).....	Geared.....	.632	476	96	2,250	Maxim.....	63
Standard Cub (configuration 4).....	6	2	Wooden.....	72	14 (fixed).....	Direct-drive.....	1.00	628	44	2,000	Standard.....	66

¹ At 500-ft. altitude, cruising power, and 40-db weighting. Each number is an average of four readings.

TABLE II
TESTS AT ARLINGTON SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings ¹	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
6-19-49	Sun.	0700-0800 a. m.	1 hr.	35	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
6-29-49	Wed.	0600-0700 p. m.	1 hr.	35	ARF Cub (configuration 1)	1	0	0	1	0	0	0	1	1
7- 2-49	Sat.	0700-0815 a. m.	1 hr 15 min.	46	ARF Cub (configuration 1)	2	1	0	2	1	0	0	3	4
7- 3-49	Sun.	0700-0815 a. m.	1 hr 15 min.	43	Standard Cub (configuration 4)	8	4	9	2	1	0	0	12	16
7-10-49	Sun.	0700-0800 a. m.	1 hr.	34	Standard Cub (configuration 4)	6	1	7	0	0	0	0	7	23
7-24-49	Sun.	0700-0800 a. m.	1 hr.	36	ARF Cub (configuration 1)	0	1	0	1	0	0	0	1	24
7-26-49	Tues.	0715-0800 a. m.	45 min.	29	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	24
Totals.			7 hr 15 min.	258		17	7	16	6	2	0	0	24	24

¹ Simulated landings.

TABLE III
TESTS AT BRIGHTON SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
12-19-48	Sun.	0700-0915 a. m.	2 hr 15 min.	30	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
12-27-48	Mon.	0630-1015 a. m.	3 hr 20 min ^a	50	ARF Cub ^b (configuration 1)	0	0	0	0	0	0	0	0	0
12-28-48	Tues.	0600-1100 a. m.	4 hr 30 min ^a	80	ARF Cub ^b (configuration 1)	0	0	0	0	0	0	0	0	0
1-20-49	Thurs.	0230-0430 p. m.	2 hr.	17	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
1-21-49	Fri.	0600-0730 a. m. ^c	1 hr 30 min.	10	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
1-22-49	Sat.	0600-0740 a. m. ^c	1 hr 40 min.	15	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
1-23-49	Sun.	0600-1030 a. m.	4 hr ^a	55	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	None
Test totals.			19 hr 15 min.	257										
Add for previous demonstrations, TV show, and TV interception.			6 hr 30 min.	63										
Totals.			25 hr 45 min.	320		0	0	0	0	0	0	0	0	None

^a Time out to refuel.

^b On skis.

^c Discontinued because of snow.

TABLE IV
TESTS AT BROCKTON SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
2-16-49	Wed.	0230-0310 p. m.	40 min.	13	ARC Cub (configuration 1)	52	0	0	37	15	0	0	52	52
2-16-49	Wed.	0350-0440 p. m.	50 min.	22	ARF Cub (configuration 1)	26	0	0	18	8	0	0	26	78
2-18-49	Fri.	0930-1030 a. m. ^a	1 hr.	20	ARF Cub (configuration 1)	13	0	0	7	6	0	0	13	91
Totals.			2 hr 30 min.	55		91	0	0	62	29	0	0	91	91

^a Stopped. See text.

TABLE V
TESTS AT CANTON SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
10-28-48	Thurs.	0200-0215 p. m.	15 min.	3	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
11-18-48	Thurs.	0700-1000 a. m.	3 hr.	35	ARF Stinson (configuration 3)	0	0	0	0	0	0	0	0	0
11-26-48	Fri.	0630-0740 a. m.	1 hr 10 min.	15	ARF Stinson (configuration 3)	0	0	0	0	0	0	0	0	0
12-24-48	Fri.	0700-0940 a. m.	2 hr 40 min.	40	ARF Cub ^a (configuration 1)	0	0	0	0	0	0	0	0	0
1-3-49	Mon.	0600-0705 p. m.	1 hr 5 min.	17	ARF Cub ^a (configuration 1)	0	0	0	0	0	0	0	0	0
6-11-49	Sat.	0700-0800 a. m. ^b	1 hr.	30	Standard Cub (configuration 4)	3	2	4	1	0	0	0	5	5
6-14-49	Tues.	0700-0715 a. m. ^b ^c	15 min.	7	Standard Cub (configuration 4)	1	1	2	0	0	0	0	2	7
Totals			9 hr 25 min.	147		4	3	6	1	0	0	0	7	7

^a On skis.^b Moved to alternate strip.^c Stopped at owner's demand.

TABLE VI
TESTS AT MEDFORD SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings ¹	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
8-24-49	Wed.	0200-0300 p. m.	1 hr.	22	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
8-25-49	Thurs.	1030-1130 a. m.	1 hr.	22	ARF Cub (configuration 1)	1	0	0	1	0	0	0	1	1
8-26-49	Fri.	0700-0800 a. m.	1 hr.	22	ARF Cub (configuration 1)	1	0	0	1	0	0	0	1	2
8-31-49	Wed.	0100-0200 p. m.	1 hr.	22	CAA Cub (configuration 2M)	1	0	0	0	1	0	0	1	3
9-1-49	Thurs.	1100-1200 a. m.	1 hr.	22	CAA Cub (configuration 2M)	0	1	0	0	0	0	1	1	4
9-2-49	Fri.	0700-0800 a. m.	1 hr.	22	CAA Cub (configuration 2U)	1	0	1	0	0	0	0	1	5
9-2-49	Fri.	1100-1200 a. m.	1 hr.	22	CAA Cub (configuration 2U)	3	0	1	1	0	0	1	3	8
9-3-49	Sat.	0700-0800 a. m.	1 hr.	23	CAA Cub (configuration 2U)	5	0	4	1	0	0	0	5	13
Totals			8 hr.	177		12	1	6	4	1	0	2	13	13

¹ Simulated landings.

TABLE VII
TESTS AT MILTON SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings ^a	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
6-9-49	Thurs.	0600-0700 p. m.	1 hr.	35	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
6-18-49	Sat.	0715-0800 a. m.	45 min.	25	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
6-20-49	Mon.	0700-0718 a. m. ^b	18 min.	10	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
6-21-49	Tues.	0630-0715 a. m.	45 min.	26	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
6-21-49	Tues.	0730-0745 a. m. ^c	15 min.	8	Standard Cub (configuration 4)	1	0	1	0	0	0	0	1	1
6-28-49	Tues.	0715-0800 a. m.	45 min.	23	Standard Cub (configuration 4)	2	0	2	0	0	0	0	2	3
Totals			3 hr 48 min.	127		3	0	3	0	0	0	0	3	3

^a Simulated landings.^b Vee-belt turned over.^c Returned because of weather.

TABLE VIII
TESTS AT NEEDHAM SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings ^a	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
6-10-49	Fri.	1000-1030 a. m.	30 min.	17	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
7-27-49	Wed.	1100-1200 a. m.	1 hr.	32	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
7-27-49	Wed.	0600-0700 p. m.	1 hr.	35	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
7-29-49	Fri.	0700-0800 a. m.	1 hr.	36	ARF Cub (configuration 1)	1	0	^b 1	0	0	0	0	1	1
7-29-49	Fri.	0200-0300 p. m.	1 hr.	35	ARF Cub (configuration 1)	7	0	^b 1	2	1	1	2	7	8
8-1-49	Mon.	1100-1200 a. m.	1 hr.	35	ARF Cub (configuration 1)	3	0	^b 1	1	1	0	0	3	11
8-5-49	Fri.	1000-1115 a. m.	1 hr 15 min.	30	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	11
8-5-49	Fri.	0200-0300 p. m.	1 hr.	22	CAA Cub (configuration 2M) ...	1	0	0	0	0	1	0	1	12
8-6-49	Sat.	1000-1100 a. m.	1 hr.	22	CAA Cub (configuration 2M) ...	0	0	0	0	0	0	0	0	12
8-8-49	Mon.	0730-0830 a. m.	1 hr.	23	CAA Cub (configuration 2M) ...	1	0	0	0	0	1	0	1	13
8-8-49	Mon.	1220-0120 p. m.	1 hr.	22	CAA Cub (configuration 2U) ...	4	0	2	0	0	2	0	4	17
8-9-49	Tues.	0710-0810 a. m.	1 hr.	25	CAA Cub (configuration 2U) ...	4	0	4	0	0	0	0	4	21
Totals			11 hr 45 min.	334		21	0	9	3	2	5	2	21	21

^a Simulated landings.^b Same person complained.

TABLE IX
TESTS AT NEWTON SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
10-27-48	Wed.	0100-0200 p. m. ^a	1 hr.	16	ARF Cub (configuration 1)	0	5	0	0	5	0	0	5	5
10-31-48	Sun.	0745-1215 a. m.	4 hr 30 min.	56	ARF Cub (configuration 1)	1	13	0	1	6	5	2	14	19
10-31-48	Sun.	0200-0430 p. m. ^a	2 hr 30 min.	31	ARF Cub (configuration 1)	0	3	0	0	2	1	0	3	22
Totals			8 hr.	103		1	21	0	1	13	6	^a 2	22	22

^a Stopped. See text.

TABLE X
TESTS AT NEWTON-BRIGHTON SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings ¹	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
8-15-49	Mon.	1015-1115 a. m.	1 hr.	17	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
8-15-49	Mon.	0230-0330 p. m.	1 hr.	17	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
8-17-49	Wed.	0725-0825 a. m.	1 hr.	16	ARF Cub (configuration 1)	0	0	0	0	0	0	0	0	0
8-18-49	Thurs.	0200-0300 p. m.	1 hr.	16	CAA Cub (configuration 2M) ...	0	0	0	0	0	0	0	0	0
8-20-49	Sat.	0700-0800 a. m.	1 hr.	17	CAA Cub (configuration 2M) ...	0	0	0	0	0	0	0	0	0
8-22-49	Mon.	1100-1200 a. m.	1 hr.	17	CAA Cub (configuration 2U) ...	0	0	0	0	0	0	0	0	0
8-23-49	Tues.	0745-0845 a. m.	1 hr.	17	CAA Cub (configuration 2U) ...	0	0	0	0	0	0	0	0	0
8-25-49	Thurs.	0630-0730 a. m.	1 hr.	18	CAA Cub (configuration 2U) ...	0	0	0	0	0	0	0	0	None
Totals			8 hr.	135		0	0	0	0	0	0	0	0	None

¹ Simulated landings.

TABLE XI
TESTS AT WINCHESTER SITE

Operations						Complaints								
Date	Day of week	Time of day	Flight time	Number of landings ^a	Airplane	By telephone	In person	Classifications					Totals	
								Noise	Low flying	Fear	Both	Airport objection	Daily	Accumulated
6-13-49	Mon.	1000-1045 a. m. ^b	45 min -----	26	ARF Cub (configuration 1) ----	0	0	0	0	0	0	0	0	None
Totals -----			45 min -----	26	-----	0	0	0	0	0	0	0	0	None

^a Simulated landings.^b Presence bothered golfers.

TABLE XII
COMPARISON OF TEST SITES

[Quieted includes ARF airplanes and muffled version of CAA Cub;
standard includes standard Cub and unmuffled CAA Cub]

Sites	Operations						Complaints															
	Number of tests		Flight time		Number of landings		By telephone		In person		Noise		Low flying		Fear		Both		Airport objection		Total	
	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard	Quieted	Standard
Arlington	5	2	5 hr.	2 hr 15 min.	181	77	3	14	2	5	0	16	4	2	1	1	0	0	0	0	5	19
Brighton	12	0	25 hr 45 min.	0	^a 320	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Not used
Brockton	3	0	2 hr 30 min.	0	^a 55	0	91	0	0	0	0	0	62	0	29	0	0	0	0	0	91	Not used
Canton	5	2	8 hr 10 min.	1 hr 15 min.	^a 110	^a 37	0	4	0	3	0	6	0	1	0	0	0	0	0	0	7	0
Medford	5	3	5 hr.	3 hr.	110	67	3	9	1	0	0	6	2	2	1	0	0	0	1	1	4	9
Milton	4	2	2 hr 48 min.	1 hr.	96	31	0	3	0	0	0	3	0	0	0	0	0	0	0	0	0	3
Needham	10	2	9 hr 45 min.	2 hr.	287	47	13	8	0	0	^b 3	6	3	0	2	0	3	2	2	0	13	8
Newton	3	0	8 hr.	0	^a 103	0	1	0	21	0	0	0	1	0	13	0	6	0	2	0	22	Not used
Newton-Brighton	5	3	5 hr.	3 hr.	83	52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Winchester	1	0	45 min.	0	26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Not used
Subtotals	53	14	72 hr 43 min.	12 hr 30 min.	1371	311	111	38	24	8	3	37	72	5	46	1	9	2	5	1	142	39
Totals	67		85 hr 13 min		1682		149		32		40		77		47		11		6		181	

^a Actual landings; others simulated.^b See table VIII and text.

REPORT 1157

ANALYTICAL DERIVATION AND EXPERIMENTAL EVALUATION OF SHORT-BEARING APPROXIMATION FOR FULL JOURNAL BEARINGS¹

By GEORGE B. DuBOIS and FRED W. OCVIRK

SUMMARY

An approximate analytical solution including the effect of end leakage from the oil film of short plain bearings is presented because of the importance of endwise flow in sleeve bearings of the short lengths commonly used. The analytical approximation is supported by experimental data, resulting in charts which facilitate analysis of short plain bearings. The analytical approximation includes the endwise flow and that part of the circumferential flow which is related to surface velocity and film thickness but neglects the effect of film pressure on the circumferential flow. In practical use, this approximation applies best to bearings having a length-diameter ratio up to 1, and the effects of elastic deflection, inlet oil pressure, and changes of clearance with temperature minimize the relative importance of the neglected term. The analytical approximation was found to be an extension of a little-known pressure-distribution function originally proposed by Michell and Cardullo.

The experimental data on a 1½-inch shaft at speeds from 500 to 6,000 rpm and bearing loads up to 900 pounds per square inch are in useful agreement with the analytical curves for length-diameter ratios of ¼, ½, and 1. Single lines applying to l/d ratios up to 1 are obtained by using basic nondimensional quantities derived from the theoretical analysis. The reciprocal of the load number is the Sommerfeld number times the square of the length-diameter ratio. The friction ratio enables the friction to be estimated directly from the Petroff friction calculated at no load, avoiding the use of a friction coefficient. The oil flow factor reduces oil flow to a single line when plotted against an empirical modification of the load number. Methods are also discussed for approximating a maximum bearing temperature for pressure-lubricated bearings and for evaluating the effect of deflection or misalignment on the eccentricity ratio at the ends of a bearing.

INTRODUCTION

The end leakage of oil flow has a predominant effect on narrow plain bearings, and an approximate solution including end-leakage effects would be useful. The following analytical and experimental investigation of such a solution was conducted at the School of Mechanical Engineering of Cornell University, Ithaca, New York, under the sponsorship and with the financial aid of the National Advisory Committee for Aeronautics.

Most investigators have based their functions on exact or approximate mathematical solutions of Reynolds' equation (ref. 1):

$$\frac{\partial}{\partial x} \left(h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(h^3 \frac{\partial p}{\partial z} \right) = 6\mu U \frac{\partial h}{\partial x} \quad (1)$$

Kingsbury (ref. 2) determined the pressure distribution in the oil film of finite journal bearings satisfying equation (1) by an experimental electrical analogy. Christopherson (ref. 3) determined the film pressure in journal bearings of finite length by utilizing the mathematical method of "relaxation." Cameron and Wood (ref. 4) have extended the work of Christopherson to show the effect of length-diameter ratio on eccentricity ratio, attitude angle, and friction coefficient. In all cases, these solutions are given to express natural phenomena in the oil film on the basis of Reynolds' assumptions regarding lubrication, the most important assumption being that certain terms in the generalized Navier-Stokes equations (refs. 5 and 6) for flow in a viscous fluid may be neglected.

Solutions of Reynolds' equation, often termed approximate solutions, have been realized by considering either the second term or the first term of the left side of equation (1) equal to zero. Such solutions for pressure distribution give considerably simplified mathematical relationships among the variables. They are exact solutions in the sense that they satisfy equation (1) for the case of a bearing of infinite axial length on the one hand and for the infinitely short bearing on the other hand. However, they are regarded as approximate solutions when used to determine the pressure distribution in bearings of finite length.

Sommerfeld (ref. 7) and Harrison (ref. 8) presented a solution for journal bearings by assuming $\partial p / \partial z = 0$, thus eliminating the second of the left-hand terms of equation (1). Such an assumption states that there is no endwise flow as in the case of an infinitely long bearing.

Michell (ref. 9) suggested a solution for the journal bearing by dropping the first of the left-hand terms in Reynolds' equation. Cardullo (ref. 10) presented this solution as giving the pressure distribution in journal bearings of finite lengths usually found in engineering practice.

Because the Michell-Cardullo solution neglects the first of the left-hand terms in equation (1), it has been regarded as limited to bearings of infinitely short length. However, as indicated in the detailed treatment of this report, this assumption includes endwise flow and that part of the circumferential flow which is proportional to journal surface

¹ Incorporates most of information contained in NACA TN 2808, "Short-Bearing Approximation for Full Journal Bearings" by F. W. Ocvirk, 1952, and NACA TN 2809, "Experimental Investigation of Eccentricity Ratio, Friction, and Oil Flow of Short Journal Bearings" by G. B. DuBois and F. W. Ocvirk, 1952.

velocity and varying film thickness but neglects the effect of oil film pressure on the circumferential flow.

Experimental measurements of pressure distribution in finite journal bearings were obtained by McKee and McKee (ref. 11) sometime after the mathematical treatments of Michell and Cardullo were published and discussed. On comparison, it was shown that the theoretical Sommerfeld-Harrison distribution was not in agreement with the experimental data. Bradford and Grunder (ref. 12), after conducting experiments, also demonstrated that the Harrison theory did not agree for finite cases. Almost no effort was made to compare the Michell-Cardullo theory with the experiments although a comment was made by Hersey in reference 11 that Rouse had found the Michell-Cardullo theory to be in better agreement with experiment than was the Sommerfeld-Harrison theory.

This report extends the short-bearing pressure-distribution function of Michell and Cardullo to give equations for the various bearing characteristics. This short-bearing approximation makes available formulas relating eccentricity ratio to applied load, attitude angle, angular position of peak film pressure, friction, required oil flow, and the ratio of peak film pressure to unit pressure on projected area. Basic nondimensional quantities called the load number, the capacity number, the friction ratio, and the oil flow factor are derived which simplify the prediction of short-bearing performance.

Experimental oil film pressure data obtained by McKee and McKee (ref. 11) and DuBois, Mabie, and Ocvirk (ref. 13) indicate that the short-bearing pressure-distribution function is in reasonable agreement with natural phenomena for bearings having a length-diameter ratio near 1. This agreement motivated the experimental program of this report to provide bearing-performance data for comparison with the analytical curves from the short-bearing approximation.

SYMBOLS

C_n	capacity number, $S\left(\frac{l}{d}\right)^2 = \frac{\mu N'}{p'} \left(\frac{d}{c_d}\right)^2 \left(\frac{l}{d}\right)^2$
C_{p_0}	inlet pressure capacity number, $\frac{\mu N'}{p_0} \left(\frac{d}{c_d}\right)^2 \left(\frac{l}{d}\right)^2$
C_1, C_2	constants of integration
$1/C_n$	load number
c_d	diametral clearance, $2c_r$, in.
c_d/d	diametral clearance ratio
c_r	radial clearance, in.
c_r/r	radial clearance ratio
d	bearing diameter, in.
e	eccentricity of journal and bearing axes, in.
F	circumferential bearing friction force under load, lb
F/F_0	friction ratio
F_c	friction calibration factor, in-lb/in. mercury
F_0	circumferential bearing friction force at zero load, lb, $2\pi^2\mu N'ld(d/c_d)$
f	friction coefficient, F/P

f_e	friction variable, $f\left(\frac{d}{c_d}\right)\left(\frac{l}{d}\right)^2$
h	local fluid film thickness, in.
h_{min}	fluid film thickness at closest approach, $c_r(1-n)$, in.
k	peak-pressure ratio, p_{max}/p'
l	bearing length, in.
l/d	length-diameter ratio
M_t	friction torque, in-lb
m	integer
N	journal speed, rpm
N'	journal speed, rps
n	eccentricity ratio or attitude, e/c_r
n_h	horizontal component of eccentricity ratio
n_v	vertical component of eccentricity ratio
P	applied central load, lb
P_x, P_y	components of applied load parallel and normal to line of centers of journal and bearing, lb
p	local fluid film pressure, lb/sq in.
p_{max}	peak pressure in fluid film, lb/sq in.
p_0	oil pressure at inlet to bearing, lb/sq in.
p_s	pressure at $\theta=0$ and π in theory of bearing without endwise flow, lb/sq in.
p'	unit pressure on projected area, lb/sq in.
Q	rate of total oil flow from bearing, in. ³ /sec
Q_r	rate of fluid flow required for converging oil film, in. ³ /sec
Q'	rate of oil flow, lb/min
q	oil flow factor, $Q/Q_r = Q/\pi d l c_d \frac{N'}{2} n = q_n/n$
q_n	required oil flow number, $Q_r/\pi d l c_d \frac{N'}{2}$
q_x, q_z	flow per unit time in x - and z -directions, in. ³ /sec
r	bearing radius, in.
S	Sommerfeld number, $\frac{\mu N'}{p'} \left(\frac{d}{c_d}\right)^2$
T	temperature, °F
U	surface speed of journal, in./sec
u, v, w	local fluid film velocities in x -, y -, and z -directions, in./sec
x, y, z	coordinates
α	angle measured from load line to location of peak film pressure, deg
θ	angle measured from maximum film thickness
θ_{max}	angle measured from location of maximum film thickness to location of peak film pressure, deg
μ	absolute viscosity of fluid, reyns or centipoises/ (6.9×10^6)
τ_x, τ_z	fluid shearing stress in x - and z -directions, lb/sq in.
ϕ	attitude angle, angle between load line and line of centers of journal and bearing, deg

ANALYTICAL DETERMINATION OF PRESSURE DISTRIBUTION IN SHORT JOURNAL BEARINGS

The early portion of the following analysis embodies the assumptions and analytical reasoning employed by Reynolds (ref. 1) to yield equation (1).

ANALYSIS FOLLOWING REYNOLDS' METHOD

When acted upon by a normal central load, the rotating journal is made to displace with respect to the stationary bearing such that the element of fluid dx , dy , and dz of figure 1 moves in a converging film of oil. Because the fluid is viscous, the rotating journal causes the element to flow into the wedge at a circumferential velocity u ; since the oil film is of finite extent in the axial direction of the bearing, the element has a velocity w in the endwise direction.

Both bearing and journal are of length l and diameter d or radius r . The journal rotates at a constant angular velocity having a surface velocity of U with respect to the stationary bearing surface. Because the centers of the bearing O and the journal O' are displaced by an amount e , the oil film will be of variable thickness h . It is assumed that the circumferential dimension of the film may be unwrapped such that the fluid is essentially moving between surfaces, one of which may be considered a plane. The coordinates x , y , and z are chosen as shown in figure 1 such that the x -direction is circumferential, the y -direction is radial, and the z -direction is measured from the center of the bearing parallel to the bearing and journal axes.

Figure 2 shows the forces which are assumed to act on the fluid element in laminar flow at an arbitrary location in the converging oil film. In both x - and z -directions it is assumed that shearing stresses τ_x and τ_z and normal pressures p are acting as shown. In the x -direction, for example, two shearing vectors and two normal vectors are assumed; in

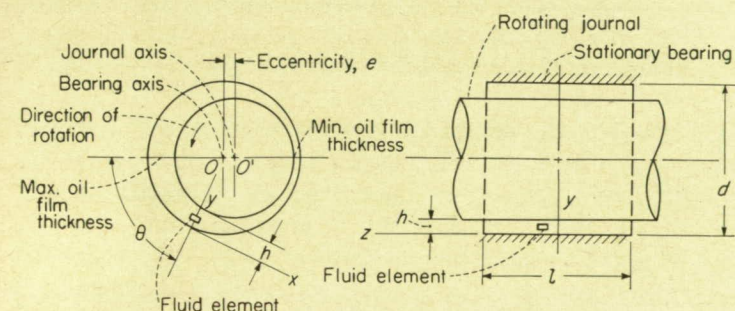


FIGURE 1.—Fluid element in converging oil film of cylindrical bearing with endwise flow.

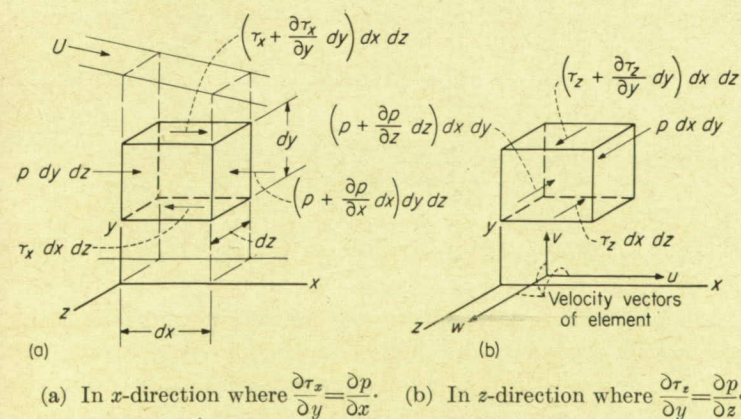


FIGURE 2.—Assumed forces acting on fluid element.

the general case, two additional shearing vectors should be shown acting on the sides of the element, but these are considered negligible.

Assuming that static equilibrium of an incompressible fluid prevails and that inertia forces are negligible, the summation of forces in the x -direction and separately in the z -direction results in the following relationships between shearing and normal pressures. In the following discussion, because of similar mathematical treatment, equations applying in the z -direction are developed in parallel with equations in the x -direction. Thus

$$\frac{\partial \tau_x}{\partial y} = \frac{\partial p}{\partial x} \quad (2)$$

$$\frac{\partial \tau_z}{\partial y} = \frac{\partial p}{\partial z} \quad (2a)$$

The shearing stress acting between layers of viscous fluid in laminar flow is dependent upon the inverse slope of the velocity profile as given below:

$$\tau_x = \mu \frac{\partial u}{\partial y} \quad (3)$$

$$\tau_z = \mu \frac{\partial w}{\partial y} \quad (3a)$$

where μ is the absolute viscosity of the fluid.

Differentiating equations (3) and (3a) with respect to y and substituting in equations (2) and (2a) give

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \quad (4)$$

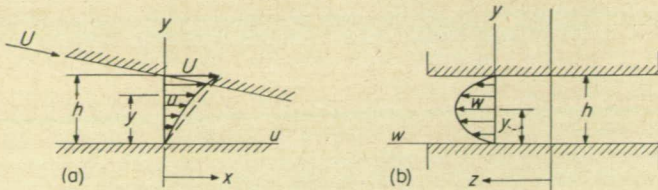
$$\mu \frac{\partial^2 w}{\partial y^2} = \frac{\partial p}{\partial z} \quad (4a)$$

Equations (4) and (4a) are integrated to give variations of velocities u and w with respect to y (velocity profiles) by assuming that the viscosity and pressure are constant throughout the oil film thickness. Thus, μ , $\partial p/\partial x$, and $\partial p/\partial z$ are constants with respect to y . Assuming that no slip exists between fluid and bearing surfaces, the boundary conditions are: $u=0$ and $w=0$ at $y=0$; $u=U$ and $w=0$ at $y=h$. Integrating equations (4) and (4a) yields

$$u = \frac{Uy}{h} + \frac{1}{2\mu} \frac{\partial p}{\partial x} y(y-h) \quad (5)$$

$$w = \frac{1}{2\mu} \frac{\partial p}{\partial z} y(y-h) \quad (5a)$$

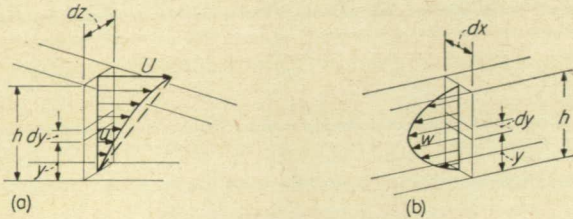
The forms of equations (5) and (5a) show that the velocity profiles are parabolic. It may be seen that for the velocity profile in the circumferential direction the first term of equation (5) gives a linear velocity profile indicated by the dashed line in figure 3 and the second term makes the profile parabolic.



(a) In x -direction where $u = \frac{Uy}{h} + \frac{1}{2\mu} \frac{\partial p}{\partial x} y(y-h)$.

(b) In z -direction where $w = \frac{1}{2\mu} \frac{\partial p}{\partial z} y(y-h)$.

FIGURE 3.—Velocity profiles in oil film.



(a) In x -direction where $q_x = \left(\frac{Uh}{2} - \frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right) dz$.

(b) In z -direction where $q_z = \left(-\frac{h^3}{12\mu} \frac{\partial p}{\partial z} \right) dx$.

FIGURE 4.—Quantity of flow per unit time through elemental plane.

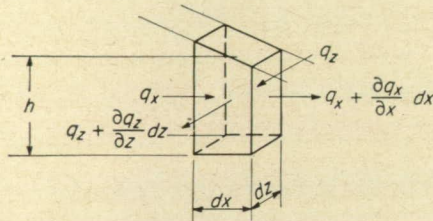


FIGURE 5.—Quantities of flow entering and leaving fluid element as given by $\frac{\partial q_x}{\partial x} dx + \frac{\partial q_z}{\partial z} dz = 0$.

The velocity profiles may be used to determine the quantities of fluid flow per unit of time in the x - and z -directions. The quantity q_x is the flow in the x -direction through a transverse plane having the area $h dz$; the quantity q_z is the flow in the z -direction through an area $h dx$ as shown in figure 4. These quantities may be expressed as follows:

$$q_x = \int_0^h u dy dz = \left(\frac{Uh}{2} - \frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right) dz \quad (6)$$

$$q_z = \int_0^h w dy dx = \left(-\frac{h^3}{12\mu} \frac{\partial p}{\partial z} \right) dx \quad (6a)$$

Considering the flow of fluid through an elemental volume of dimensions h , dx , and dz as shown in figure 5, an equation of continuity of flow may be written since it is required that the quantity entering must be equal to the quantity leaving. A steady state of flow is assumed. Thus,

$$q_x + q_z = q_x + \frac{\partial q_x}{\partial x} dx + q_z + \frac{\partial q_z}{\partial z} dz$$

$$\frac{\partial q_x}{\partial x} dx + \frac{\partial q_z}{\partial z} dz = 0 \quad (7)$$

Equation (7) indicates that the increase of the rate of flow in one direction is equal to the decrease in the rate of flow in the other direction. Substituting in equation (7) the proper derivatives of equations (6) and (6a) and assuming that the viscosity is constant in all regions of the fluid result in Reynolds' equation in the form of equation (1).

PRINCIPAL ASSUMPTION FOR SHORT BEARINGS

The following analysis results in the pressure-distribution function as given by Michell and Cardullo. The principal simplifying assumption made in this analysis is one regarding equation (6) for the flow in the x -direction, namely, that, of the two right-hand terms, the second is negligible compared with the first. Referring to figure 3(a), the first term represents the flow as given by the area of the triangular portion of the velocity profile; the area of the profile between the dashed line and the parabolic curve is considered to be small by comparison. This does not mean that $\partial p / \partial x$ is necessarily small, but that the term $h^3 / 12\mu (\partial p / \partial x)$ is small. Thus, it is assumed that

$$q_x = \frac{Uh}{2} dz \quad (8)$$

Comparing equation (8) with equation (6), it may be seen that the assumption made results eventually in the omission of the first of the left-hand terms in Reynolds' equation (eq. (1)). The solution which follows will therefore include that part of the circumferential flow proportional to journal surface velocity and varying film thickness but will neglect the effect of circumferential pressure gradient on this flow. Any change in circumferential flow will directly influence the axial flow and the axial pressure gradient.

The shape of the bearing as given by the length l and the diameter d affects the relationship between the circumferential and axial pressure gradients. Both gradients depend upon the distance from the point of maximum film pressure to a boundary where the pressure is zero. For the circumferential gradient this distance is related to $\pi d / 4$ and it is related to $l / 2$ for the axial gradient. Thus, as l becomes small compared with d , the circumferential gradient becomes small compared with the axial gradient.

Neglecting the effect of circumferential pressure gradient on the flow, the rates of change of flow in the x - and z -directions are given by differentiating equations (8) and (6a):

$$\frac{\partial q_x}{\partial x} = \frac{U}{2} \frac{\partial h}{\partial x} dz \quad (9)$$

$$\frac{\partial q_z}{\partial z} = \frac{\partial}{\partial z} \left(-\frac{h^3}{12\mu} \frac{\partial p}{\partial z} \right) dx \quad (9a)$$

Equation (9) is a result of the above assumption and indicates that, when the flow is into the converging oil film, the decrease in flow in the circumferential direction is due primarily to a decrease in film thickness. Substituting equations (9) and (9a) in the continuity equation (7) results in the simple differential equation:

$$\frac{\partial}{\partial z} \left(h^3 \frac{\partial p}{\partial z} \right) = 6\mu U \frac{\partial h}{\partial x} \quad (10)$$

Referring to Reynolds' equation (eq. (1)) it is seen that equation (10) lacks the first left-hand term. Since h is not a function of z , equation (10) may be written:

$$\frac{d^2 p}{dz^2} = \frac{6\mu U}{h^3} \frac{dh}{dx} \quad (10a)$$

Further, since none of the terms on the right-hand side of equation (10a) are functions of z , double integration gives

$$p = \frac{6\mu U}{h^3} \frac{dh}{dx} \frac{z^2}{2} + C_1 z + C_2$$

The constant of integration C_1 may be evaluated from the boundary condition that $dp/dz=0$ at $z=0$ for symmetrical flow about the transverse plane of symmetry at the center of the bearing. From the condition that pressures at the ends of the bearing are at the atmospheric datum of zero gage pressure, C_2 may be determined from $p=0$ at $z=\pm l/2$. Thus the pressure distribution as a function of x and z is given by

$$p = \frac{3\mu U}{h^3} \frac{dh}{dx} \left(z^2 - \frac{l^2}{4} \right) \quad (11)$$

Since $dx=r d\theta$, equation (11) may be given in polar coordinates

$$p = \frac{3\mu U}{rh^3} \frac{dh}{d\theta} \left(z^2 - \frac{l^2}{4} \right) \quad (11a)$$

For the journal axis displaced e with respect to the bearing axis, the film thickness may be approximated by

$$h = c_r + e \cos \theta = c_r(1 + n \cos \theta) \quad (12)$$

where c_r is the radial clearance and n is the eccentricity ratio, or attitude of the journal with respect to bearing, given by the ratio e/c_r . As shown in figure 1, θ is measured from the station where the film is of maximum thickness. Thus

$$\frac{dh}{d\theta} = -c_r n \sin \theta \quad (12a)$$

Substituting equations (12) and (12a) in equation (11a)

$$p = \frac{3\mu U}{rc_r^2} \left(\frac{l^2}{4} - z^2 \right) \frac{n \sin \theta}{(1 + n \cos \theta)^3} \quad (13)$$

Expression (13) shows that the pressure distribution is parabolic in the z -direction and sinusoidal in the circumferential direction giving zero pressures at $z=\pm l/2$ and $\theta=m\pi$ where m is an integer. Figure 6 shows the approximate forms of the distributions in the x - and z -directions.

EXTENSION OF ANALYSIS RESULTING IN BEARING PERFORMANCE CHARACTERISTICS

As discussed in another section of this report, experimental data show that the pressure-distribution function as given by equation (13) is in good agreement with short-journal-bearing behavior. This apparent agreement has led to the extension of the analysis to include the determination of other bearing characteristics, given below, which the pressure-distribution function makes possible.

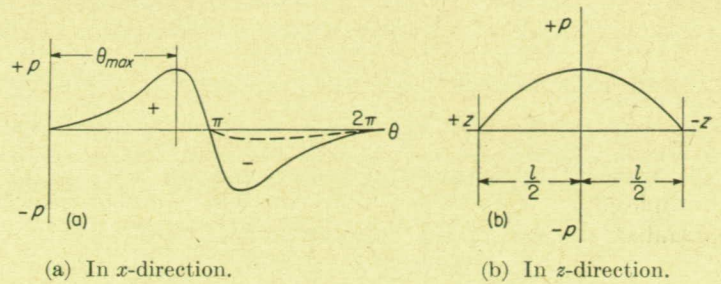


FIGURE 6.—Pressure distribution in a full cylindrical bearing as given by $p = \frac{3\mu U}{rc_r^2} \left(\frac{l^2}{4} - z^2 \right) \frac{n \sin \theta}{(1 + n \cos \theta)^3}$.

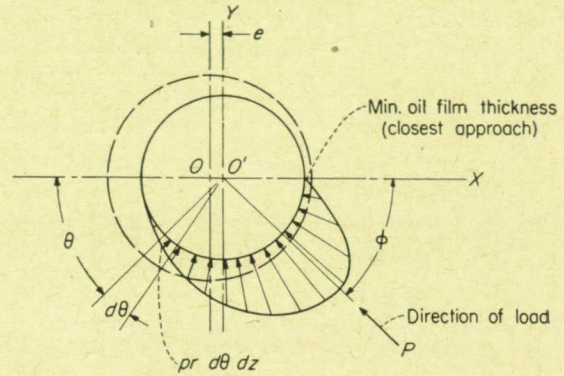


FIGURE 7.—System of coordinates chosen for integration of pressures and determination of attitude angle ϕ .

EXTENT OF PRESSURE IN FULL CYLINDRICAL BEARINGS

As shown in figure 6 the circumferential pressure distribution given by equation (13) is positive in the converging film $0 < \theta < \pi$ and is negative in the diverging film $\pi < \theta < 2\pi$. The maximum negative pressure is shown to be equal in magnitude to the maximum positive pressure, a condition which does not appear to exist except in very lightly loaded bearings. A negative pressure of small magnitude not exceeding that of atmospheric pressure, as shown by the dashed line in figure 6, is possible. In the absence of a high datum pressure, the negative pressures are assumed to be of negligible magnitude. It is assumed for moderately and heavily loaded bearings that the positive pressure region from $\theta=0$ to $\theta=\pi$, as given by equation (13), carries the total load.

CAPACITY NUMBER C_n AND LOAD NUMBER $1/C_n$

The external load P may be related to pressures induced in the oil film by performing certain integrations. Choosing X - and Y -axes as shown in figure 7, the integrations of pressures on the journal in the directions of these axes will give components P_x and P_y of the resultant force which is equal to P . The integral expressions for P_x and P_y are

$$P_x = -2 \int_0^\pi \int_0^{l/2} pr \cos \theta d\theta dz$$

$$P_y = 2 \int_0^\pi \int_0^{l/2} pr \sin \theta d\theta dz$$

Substituting equation (13) for p and integrating with respect to z :

$$P_x = -\frac{\mu U l^3}{2c_r^2} \int_0^\pi \frac{n \sin \theta \cos \theta}{(1+n \cos \theta)^3} d\theta \quad (14)$$

$$P_y = \frac{\mu U l^3}{2c_r^2} \int_0^\pi \frac{n \sin^2 \theta}{(1+n \cos \theta)^3} d\theta \quad (14a)$$

Employing Sommerfeld's mathematical technique (see appendix), the integrations with respect to θ yield

$$P_x = \frac{\mu U l^3}{c_r^2} \frac{n^2}{(1-n^2)^2} \quad (15)$$

$$P_y = \frac{\mu U l^3}{4c_r^2} \frac{\pi n}{(1-n^2)^{3/2}} \quad (15a)$$

The applied load is given by

$$P = \sqrt{P_x^2 + P_y^2} = \frac{\mu U l^3}{4c_r^2} \frac{n}{(1-n^2)^2} [\pi^2 (1-n^2) + 16n^2]^{1/2} \quad (16)$$

Substituting $P = p' l d$ and $U = \pi d N'$ gives

$$\frac{\mu N'}{p'} \left(\frac{d}{c_d} \right)^2 \left(\frac{l}{d} \right)^2 = \frac{(1-n^2)^2}{\pi n} \left[\frac{1}{\pi^2 (1-n^2) + 16n^2} \right]^{1/2}$$

The left-hand term is a grouping of bearing variables in nondimensional form and is seen to be dependent upon the eccentricity ratio, or attitude, n . This term is the capacity number C_n and may be recognized as the Sommerfeld number S times the square of the length-diameter ratio. Thus,

$$C_n = S \left(\frac{l}{d} \right)^2 = \frac{(1-n^2)^2}{\pi n} \left[\frac{1}{\pi^2 (1-n^2) + 16n^2} \right]^{1/2} \quad (17)$$

Equation (17) indicates that the curve of eccentricity ratio as a function of the known bearing variables in the capacity number is a single line for different l/d ratios. Eccentricity ratio may also be shown as a function of the load number $1/C_n$ giving a curve of n as a direction function of load.

ATTITUDE ANGLE ϕ

The angular position of the line of action of the load P with respect to the location of the minimum film thickness, or point of closest approach, is the attitude angle ϕ , as shown in figure 7, which may be determined from:

$$\begin{aligned} \tan \phi &= \frac{P_y}{P_x} \\ &= \frac{\pi (1-n^2)^{1/2}}{4n} \end{aligned} \quad (18)$$

Figure 8 shows the journal bearing reoriented such that the load line of action is vertical.

As shown in equation (18), the attitude angle depends directly on eccentricity ratio such that a single polar curve of n against ϕ applies for different l/d ratios.

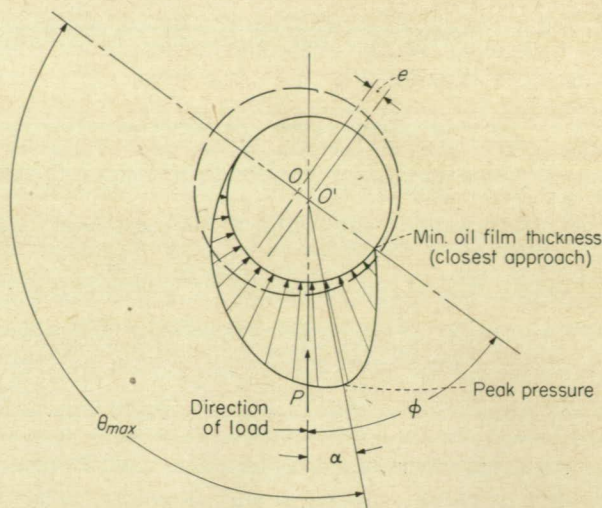


FIGURE 8.—Diagram showing attitude angle ϕ , maximum-pressure angle θ_{max} , and peak-pressure angle α .

FRICTION RATIO F/F_0 AND FRICTION VARIABLE f

Because of the assumed linear velocity profile in the oil film in the circumferential direction, the shearing stress at any location θ may be expressed as

$$\begin{aligned} \tau_x &= \mu \frac{\partial u}{\partial y} \\ &= \mu \frac{U}{h} \end{aligned}$$

The circumferential friction force F may be evaluated by the following integral:

$$\begin{aligned} F &= \int_0^{2\pi} \tau_x l r d\theta \\ &= \int_0^{2\pi} \mu \frac{U}{h} l r d\theta \end{aligned}$$

With $h = c_r (1 + n \cos \theta)$, then

$$\begin{aligned} F &= \frac{\mu U l r}{c_r} \int_0^{2\pi} \frac{d\theta}{(1 + n \cos \theta)} \\ &= \frac{\mu U l r}{c_r} \frac{2\pi}{(1-n^2)^{1/2}} \end{aligned} \quad (19)$$

The coefficient of friction f is commonly defined as the ratio of friction force to the applied load:

$$\begin{aligned} f &= \frac{F}{P} \\ &= \frac{\mu U l r}{P c_r} \frac{2\pi}{(1-n^2)^{1/2}} \end{aligned}$$

$$f \left(\frac{r}{c_r} \right) \left(\frac{l}{d} \right)^2 = \frac{\mu N'}{p'} \left(\frac{r}{c_r} \right)^2 \left(\frac{l}{d} \right)^2 \frac{2\pi^2}{(1-n^2)^{1/2}}$$

$$f_v = f \left(\frac{d}{c_d} \right) \left(\frac{l}{d} \right)^2$$

$$= C_n \frac{2\pi^2}{(1-n^2)^{1/2}} \quad (20)$$

The first equivalent of f_v in equation (20) is the friction variable representing a nondimensional grouping of bearing variables including the coefficient of friction. The friction variable is basically dependent upon the eccentricity ratio as indicated by the last equivalent which also includes the capacity number. Since capacity number is a function of n , friction variable may be plotted against capacity number or load number.

Another nondimensional number including the circumferential friction force F may be determined from equation (19) as follows:

$$F = \frac{\mu U l r}{c_r} \frac{2\pi}{(1-n^2)^{1/2}} = \mu d l N' \left(\frac{r}{c_r} \right) \frac{2\pi^2}{(1-n^2)^{1/2}}$$

$$\frac{F}{2\pi^2 \mu d l N' \left(\frac{d}{c_d} \right)} = \frac{F}{F_o} = \frac{1}{(1-n^2)^{1/2}} \quad (21)$$

$$F_o = 2\pi^2 \mu d l N' \left(\frac{d}{c_d} \right) \quad (22)$$

Friction force F_o is the well-known Petroff value of friction force at zero load. The friction ratio F/F_o as given by equation (21) shows the increase in friction force above the Petroff value as increasing load is applied. A single curve applicable to all l/d ratios results from plotting friction ratio with capacity number or load number. Plotting F/F_o with load number shows directly the increase in friction with load. Since friction ratio is nondimensional, it applies to friction torque as well as friction force.

OIL FLOW NUMBER q_n AND OIL FLOW FACTOR q

Equation (6a) may be used to determine the rate of oil flow discharging from the two ends of the converging wedge. Proper substitution for h and dp/dz gives the following expression for the axial flow at any point in the oil film:

$$q_z = \frac{U c_r n z}{2} \sin \theta d\theta$$

At the ends of the bearing:

$$(q_{z=\pm l/2}) = \pm \frac{U c_r n l}{4} \sin \theta d\theta$$

The above equations indicate that the axial flow will be outward from the bearing in the region of the converging film $0 < \theta < \pi$ where the pressures are positive. They also show that in order to develop negative pressures in the film at $\pi < \theta < 2\pi$ the oil flow must be axially inward. A reservoir of fluid must be available at the ends of the bearing as a

source of supply. The rate of outward axial oil flow at the ends of the bearing from the region of positive pressure is given by the following integral:

$$Q_r = 2 \int_0^\pi \frac{U c_r n l}{4} \sin \theta d\theta$$

$$= U c_r n l$$

$$= \pi d l c_d \frac{N'}{2} n \quad (23)$$

Equation (23) gives the flow which is the result of hydrodynamic action of the load-carrying film in the converging wedge, and it is seen to depend upon the eccentricity ratio n . Assuming an outward or positive flow from the diverging wedge, the rate of flow to be supplied to the bearing through an inlet hole or grooves on the unloaded side of the bearing must be equal to or greater than that given by Q_r in equation (23) which may be looked upon as the required rate of flow for the converging film. Equation (23) in the following nondimensional form is the required oil flow number:

$$q_n = \frac{Q_r}{\pi d l c_d \frac{N'}{2} n} = n \quad (24)$$

The required oil flow number is seen to be numerically equal to the eccentricity ratio. The curves of q_n against C_n and n against C_n are identical.

The total rate of flow Q includes the positive flow from the unloaded area of the bearing as well as the flow from the converging film and should therefore be greater than Q_r for proper performance of the bearing. The ratio of total rate of flow to the required rate is termed the oil flow factor q :

$$q = \frac{Q}{Q_r} = \frac{Q}{\pi d l c_d \frac{N'}{2} n} \quad (25)$$

PEAK-PRESSURE RATIO k

The peak pressure in the fluid film may be determined from equation (13) at $z=0$ and $\theta=\theta_{max}$, where θ_{max} is the angular position of maximum film pressure. Differentiating equation (13) with respect to θ and maximizing:

$$\theta_{max} = \arccos \left[\frac{1}{4n} (1 - \sqrt{1 + 24n^2}) \right] \quad (26)$$

$$p_{max} = \frac{3\mu U l^2}{4r c_r^2} \left[\frac{n \sin \theta_{max}}{(1 + n \cos \theta_{max})^3} \right]$$

The ratio of peak pressure to the unit pressure, where unit pressure p' is defined as the load divided by the projected area of the bearing, is termed peak-pressure ratio k :

$$k = \frac{p_{max}}{p'} = C_n \frac{6\pi n \sin \theta_{max}}{(1 + n \cos \theta_{max})^3} \quad (27)$$

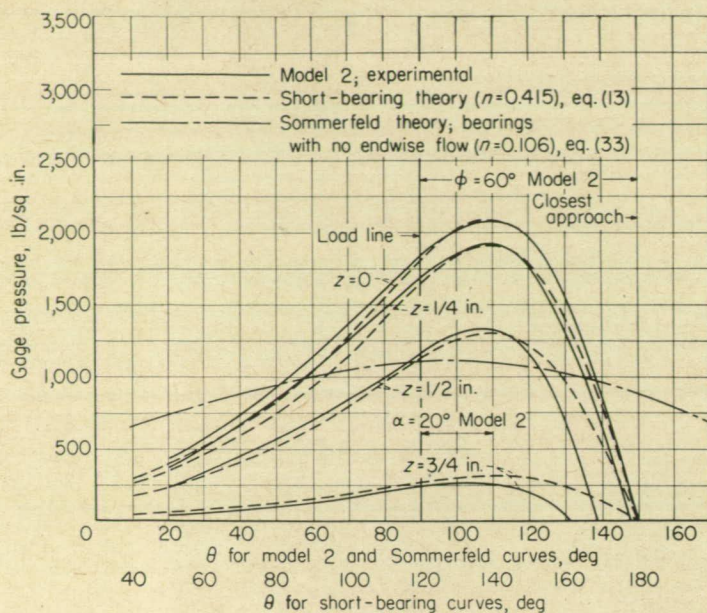


FIGURE 9.—Comparison of theoretical pressure distribution with experimental pressure distribution of model 2 of reference 13. Load, 2,200 pounds; speed, 5,000 rpm; average temperature, 197° F; viscosity, 2.42×10^{-6} reyn; bearing length, $1\frac{1}{8}$ inches; bearing diameter, $1\frac{1}{8}$ inches; radial clearance at 197° F, 0.001 inch.

Since θ_{max} and C_n are functions of n , peak-pressure ratio will plot as a single line against capacity number or load number.

PEAK-PRESSURE ANGLE α

The angular position of the location of peak pressure with respect to the load line is the peak-pressure angle α as shown in figure 8. Peak-pressure angle may be determined from the following expression:

$$\alpha = \phi - (180^\circ - \theta_{max}) \quad (28)$$

Equations (18), (26), and (28) show that peak-pressure angle is directly dependent on eccentricity ratio. However, since n is a function of C_n , α may be plotted with capacity number or load number to give a single line.

Analytical curves as given by the foregoing equations are shown plotted in figures 9 to 19. Although the eccentricity ratio n is the fundamental variable in all of the equations, it is useful in almost all instances to plot curves using capacity number C_n , or load number $1/C_n$, as the basic variable because it can be obtained from the known variables of load, speed, clearance, l/d ratio, and viscosity in nondimensional form.

COMPARISON OF ANALYTICAL CURVES WITH EXPERIMENTAL DATA

Figure 9 compares the analytical oil film pressure distribution from equation (13) with the pressure distribution experi-

mentally measured by DuBois, Mabie, and Ocvirk (ref. 13). These data were taken on a $1\frac{1}{8}$ - by $1\frac{1}{8}$ -inch bearing operating at 5,000 rpm with a load of 2,200 pounds. The close agreement shown for $l/d=1$ suggested further investigation. The pressure distribution in the oil film given by equation (13) is also shown in figure 10 for comparison with experimental data by McKee and McKee (ref. 11) for a 1-inch-long by $\frac{7}{8}$ -inch-diameter bearing at speeds up to 592 rpm and a load of 25 pounds.

Curves of peak-pressure ratio from equation (27) and peak-pressure angle from equation (28) are shown as a function of capacity number in figures 11 and 12. The experimental values were obtained from pressure distributions of figures 9 and 10. An interesting feature is that the maximum angular displacement of the peak pressure as given by α is approximately 20° at $n \approx 0.4$ and $C_n = 0.15$. For comparison, if the extent of the oil film is considered to be 360° , as in the Sommerfeld solution, the value of α reaches a maximum of 90° at $n = 1.0$.

The relatively close agreement shown by the data on pressure distribution and the rather meager amount of data available on eccentric displacement of the shaft in the journal encouraged the establishment of an experimental program to provide data for comparison with the analytical curves. The greater part of the experimental data in figures 13 to 19 was obtained by this investigation on a $1\frac{3}{8}$ -inch-diameter bearing with length-diameter ratios of 1, $\frac{1}{2}$, and $\frac{1}{4}$, speeds of 500 to 6,000 rpm, and unit bearing pressure on projected area up to 900 pounds per square inch. Description of the experimental apparatus and tests is given later in this report. Certain experimental data on oil flow by McKee (ref. 14) are also shown in figure 19.

Attitude angle as determined from equation (18) is shown as a function of eccentricity ratio by the polar curve in figure 13; experimental data obtained by this investigation are given for comparison. The attitude angle gives the angular position of the line through the bearing and journal centers with respect to the line of action of the applied load. At values of n near zero the line of centers is nearly normal to the load line and becomes parallel to the load line at $n = 1.0$. Experimental data for short bearings as obtained by this investigation are shown in figure 13 for comparison with the analytical curve. The manner in which these data were obtained is given in another section of this report. Bradford and Davenport (ref. 15) have obtained an experimental curve whose shape is similar to the curve of figure 13. As may be seen, the path of the journal is slightly within a semicircle.

Eccentricity-ratio curves from equation (17) are presented in figures 14 and 15 with capacity number as abscissa in one curve and load number, in the other. The same experimental data are shown in both figures 14 and 15.

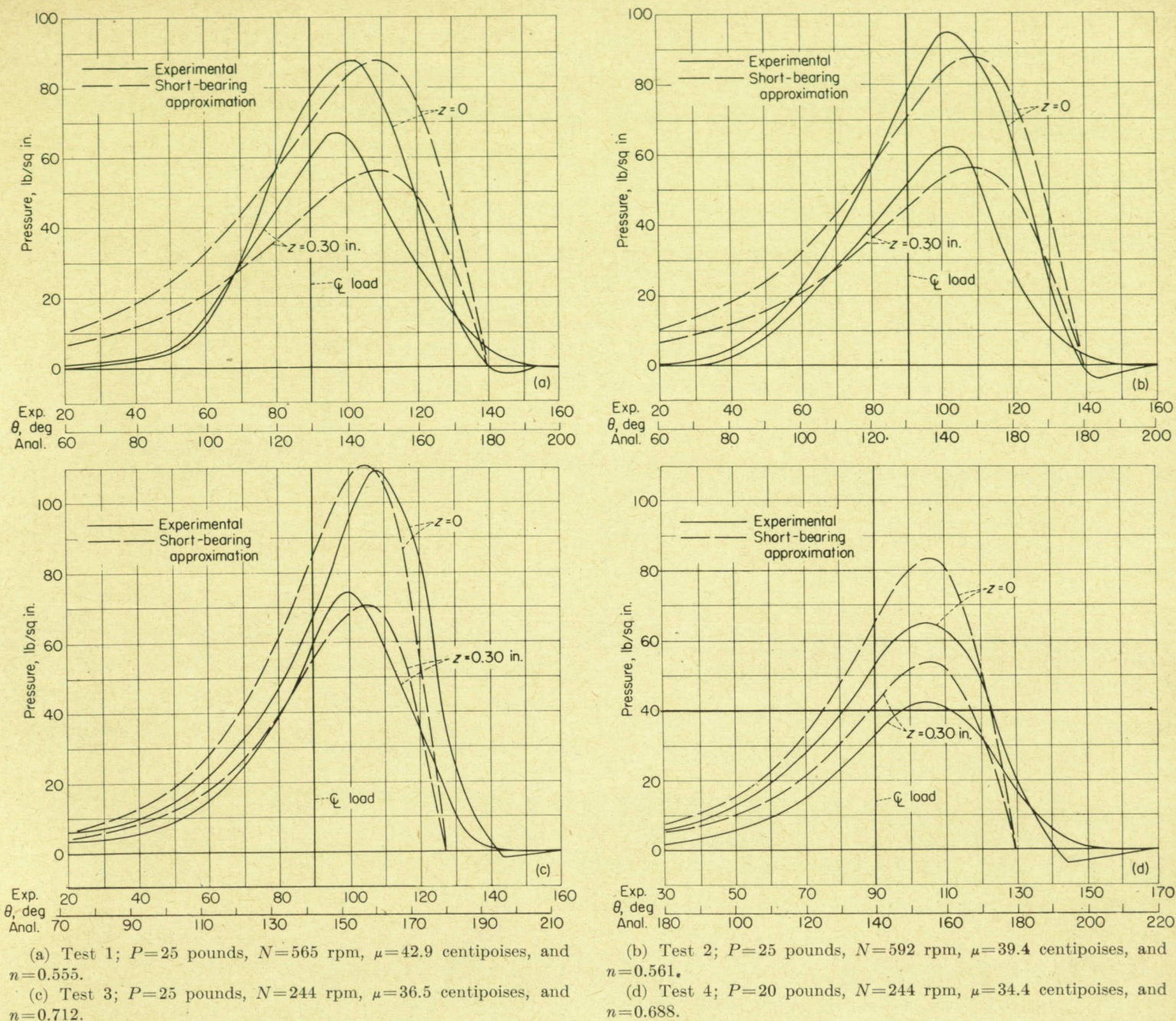


FIGURE 10.—Comparison of analytical pressure distributions with experimental pressure distributions from tests by McKee and McKee (ref. 11). $l=1.0$ inch, $d=0.8703$ inch, and $c_r=0.0025$ inch

Friction characteristics of short bearings are given by the curves in figures 16 and 17. Equation (20) gives the values for the analytical curve of friction variable against the capacity number in figure 16, and equation (21) determines the curve of friction ratio against load number in figure 17. From the experimental data shown, it may be seen that effect of load on friction is more clearly shown by

friction ratio (fig. 17) than by friction variable (fig. 16).

Oil flow number as a function of capacity number from equation (24) is represented by the analytical curve of figure 18. The experimental values of oil flow number shown in figure 18 are obviously not in agreement with the analytical values since the analytical values represent only the rate of flow from the converging wedge and the experimental values

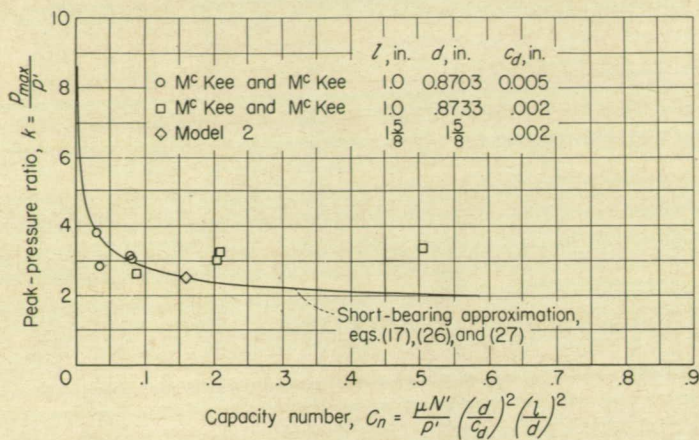


FIGURE 11.—Plot of peak-pressure ratio against capacity number for comparison of experimental data with analytical curve for short bearings. Data by McKee and McKee taken from reference 11 and data for model 2, from reference 13.

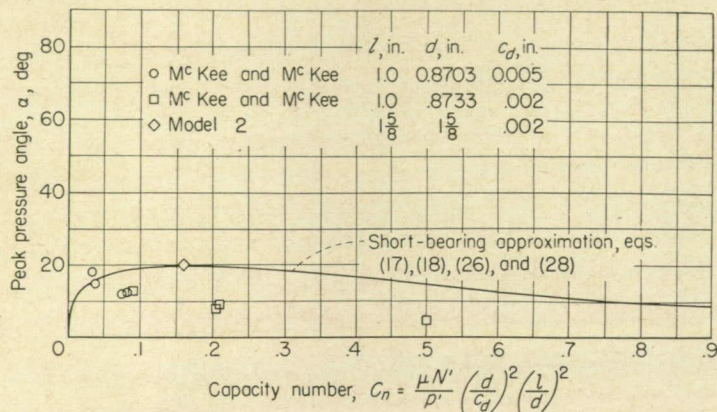


FIGURE 12.—Plot of peak-pressure angle against capacity number for comparison of experimental data with analytical curve for short bearings. Data by McKee and McKee taken from reference 11 and data for model 2, from reference 13.

represent the total flow. An inspection of the experimental data in figure 18 indicates that the flow rate in excess of that from the converging film is dependent upon inlet oil pressure, speed, and l/d ratio. Speculating that oil film viscosity and clearance also affect the excess flow rate, all the variables affecting total flow rate are grouped nondimensionally in a form similar to the capacity number but using the inlet oil pressure p_0 in place of the unit bearing load p' , giving a variable termed inlet-pressure capacity number C_{p_0} as follows:

$$C_{p_0} = \frac{\mu N'}{p_0} \left(\frac{d}{c_d} \right)^2 \left(\frac{l}{d} \right)^2 = C_n \frac{p'}{p_0} \quad (29)$$

A curve of oil flow factor q , the ratio of total flow rate to the flow rate from the converging wedge (eq. (25)), as a function of the inlet-pressure capacity number is shown in figure 19. As shown, the widely scattered experimental data of figure 18 result in a single curve when plotted according to the nondimensional ordinates of figure 19. The experimental data by McKee (ref. 14) for bearings with a single oil feed hole show excellent agreement with the experimental

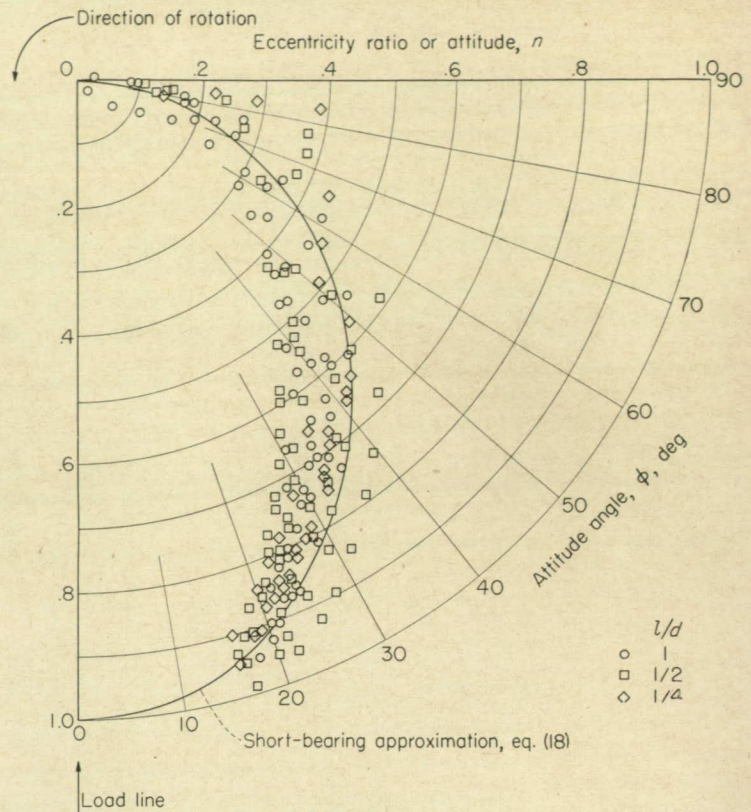


FIGURE 13.—Plot of eccentricity ratio against attitude angle for comparison of experimental data for l/d of 1, $1/2$, and $1/4$ with analytical curve for short bearings. Experimental data: Bearing diameter, $1\frac{1}{8}$ inches; bearing, bronze; journal, steel; diametral clearance, 0.00232 and 0.00264 inch; inlet pressure of SAE 10 oil p_0 fed through one $\frac{1}{8}$ -inch-diameter hole, 40 and 100 pounds per square inch; speed, 500 to 6,000 rpm; load, 0 to 1,450 pounds; bearing pressure, 0 to 760 pounds per square inch; average bearing temperature $\frac{1}{16}$ inch from oil film T_{av} , 114° to 160° F.

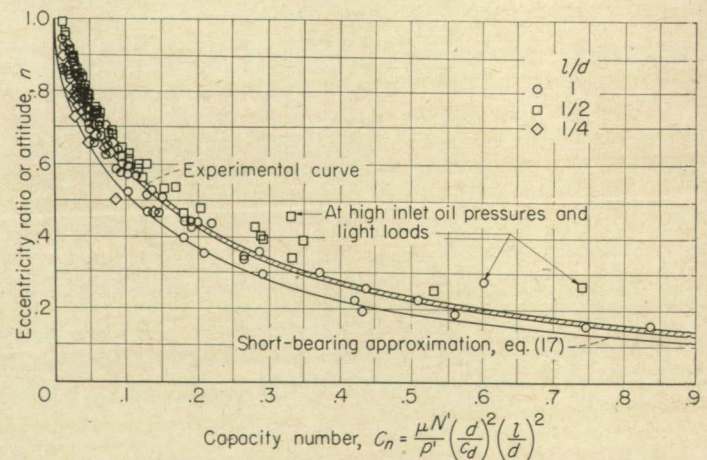


FIGURE 14.—Plot of eccentricity ratio against capacity number for comparison of experimental data for l/d of 1, $1/2$, and $1/4$ with analytical curve for short bearings. Experimental data: Bearing diameter, $1\frac{1}{8}$ inches; bearing, bronze; journal, steel; diametral clearance, 0.00232 and 0.00264 inch; inlet pressure of SAE 10 oil p_0 fed through one $\frac{1}{8}$ -inch-diameter hole, 40 and 100 pounds per square inch; speed, 500 to 6,000 rpm; load, 0 to 1,450 pounds; bearing pressure, 0 to 760 pounds per square inch; average bearing temperature $\frac{1}{16}$ inch from oil film T_{av} , 114° to 160° F.

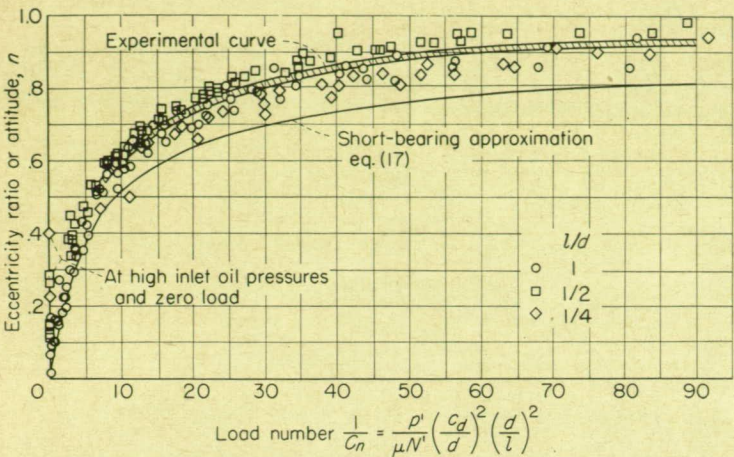


FIGURE 15.—Plot of eccentricity ratio against load number for comparison of experimental data for l/d of 1, $1/2$, and $1/4$ with analytical curve for short bearings. Experimental data: Bearing diameter, $1\frac{3}{8}$ inches; bearing, bronze; journal, steel; diametral clearance, 0.00232 and 0.00264 inch; inlet pressure of SAE 10 oil p_o fed through one $\frac{1}{8}$ -inch-diameter hole, 40 and 100 pounds per square inch; speed, 500 to 6,000 rpm; load, 0 to 1,450 pounds; bearing pressure, 0 to 760 pounds per square inch; average bearing temperature $\frac{1}{16}$ inch from oil film T_{as} , 114° to 160° F.

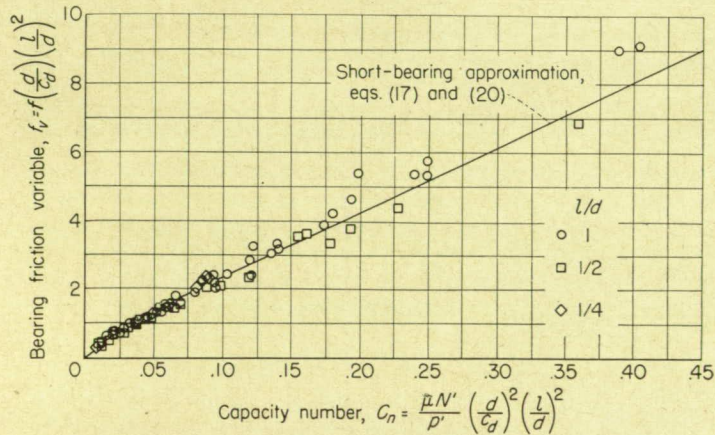


FIGURE 16.—Plot of bearing friction variable against capacity number for comparison of experimental data for l/d of 1, $1/2$, and $1/4$ with analytical curve for short bearings. Experimental data: Bearing diameter, $1\frac{3}{8}$ inches; bearing, bronze; journal, steel; diametral clearance, 0.00232 and 0.00264 inch; inlet pressure of SAE 10 oil p_o fed through one $\frac{1}{8}$ -inch-diameter hole, 40 and 100 pounds per square inch; speed, 500 to 6,000 rpm; load, 0 to 1,750 pounds; bearing pressure, 0 to 900 pounds per square inch; average bearing temperature $\frac{1}{16}$ inch from oil film T_{as} , 115° to 161° F.

data of this investigation. The McKee data for an axial oil inlet groove form a parallel line at an increased rate of oil flow. The McKee data were obtained at the National Bureau of Standards using a bearing 2.06 inches in diameter with SAE 20 oil at speeds of 2,030 and 3,040 rpm, loads of 1,553 and 3,008 pounds, and temperatures from 203° to 228° F.

DESCRIPTION OF APPARATUS

A photograph of the bearing testing machine used in the experiments appears in figure 20. The manner in which the test elements were supported and loaded is shown in

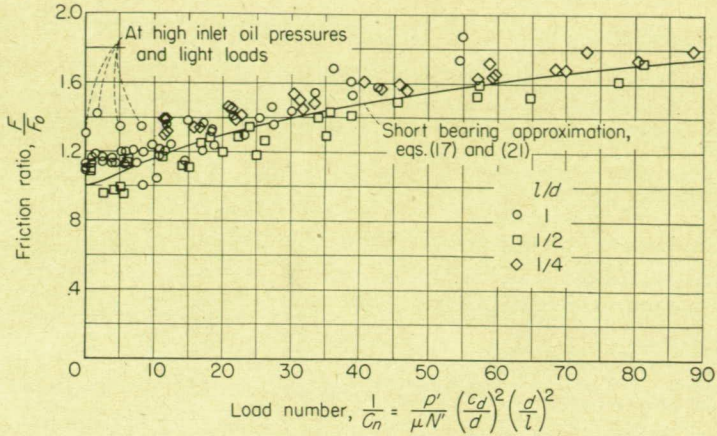


FIGURE 17.—Plot of friction ratio against load number for comparison of experimental data for l/d of 1, $1/2$, and $1/4$ with analytical curve for short bearings. Experimental data: Bearing diameter, $1\frac{3}{8}$ inches; bearing, bronze; journal, steel; diametral clearance, 0.00232 and 0.00264 inch; inlet pressure of SAE 10 oil p_o fed through one $\frac{1}{8}$ -inch-diameter hole, 40 and 100 pounds per square inch; speed, 500 to 6,000 rpm; load, 0 to 1,750 pounds; bearing pressure, 0 to 900 pounds per square inch; average bearing temperature $\frac{1}{16}$ inch from oil film T_{as} , 115° to 161° F.

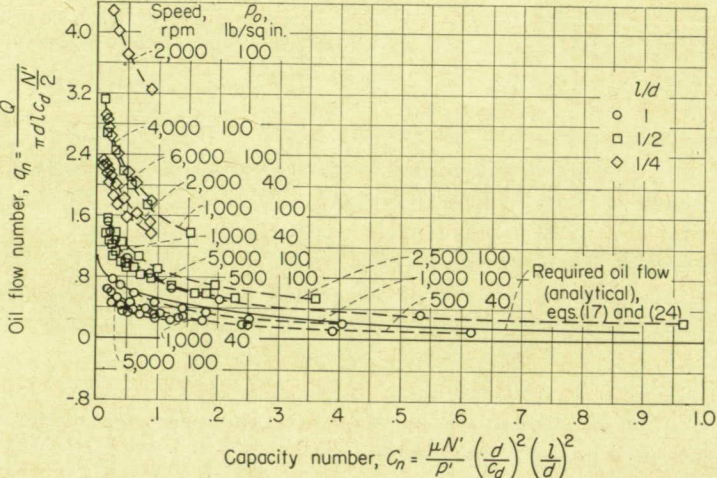


FIGURE 18.—Curves of oil flow number against capacity number for comparison of experimental total oil flow with theoretically required flow through loaded portion of oil film for l/d of 1, $1/2$, and $1/4$. Experimental data: Bearing diameter, $1\frac{3}{8}$ inches; bearing, bronze; journal, steel; diametral clearance, 0.00232 and 0.00264 inch; inlet pressure of SAE 10 oil p_o fed through one $\frac{1}{8}$ -inch-diameter hole, 40 and 100 pounds per square inch; speed, 500 to 6,000 rpm; load, 0 to 1,750 pounds; bearing pressure, 0 to 900 pounds per square inch; average bearing temperature $\frac{1}{16}$ inch from oil film T_{as} , 115° to 161° F.

figures 21, 22, and 23. In addition, figure 23 illustrates diagrammatically the torque meter used for friction-torque measurements and the means for collecting the oil flowing through the test bearing. Figures 24 and 25 show the mechanical system which measured the displacements of the journal at the ends of the bearing. The locations of thermocouples measuring bearing temperatures are given in figure 26.

A bronze bearing and two steel shafts of $1\frac{3}{8}$ -inch nominal diameter were used as shown in figures 21 and 22 giving a single bearing for length-diameter ratios of 2 and 1, and a

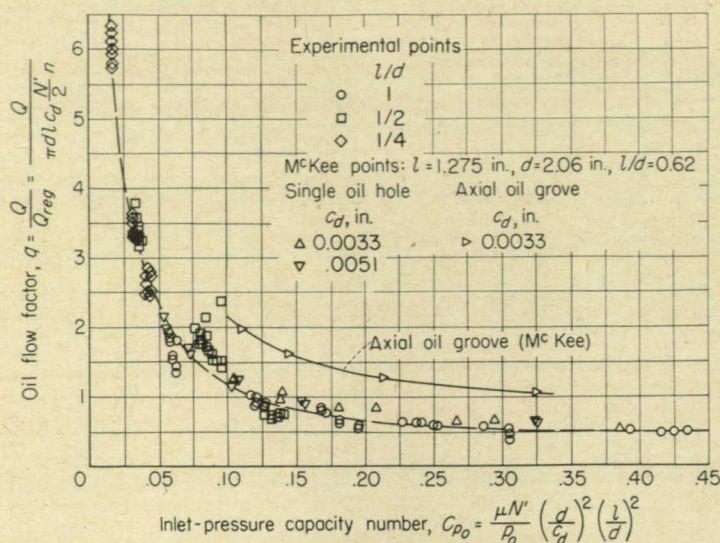


FIGURE 19.—Curve of oil flow factor against inlet-pressure capacity number. Experimental data are given for l/d of 1, $1/2$, and $1/4$. Oil flow factor is ratio of experimental total flow to flow required through loaded portion of oil film by short-bearing approximation. Experimental data: Bearing diameter, $1\frac{3}{8}$ inches; bearing, bronze; journal, steel; diametral clearance, 0.00232 and 0.00264 inch; inlet pressure of SAE 10 oil p_o fed through one $1/8$ -inch-diameter hole opposite load line, 40 and 100 pounds per square inch; speed, 500 to 6,000 rpm; load, 0 to 1,750 pounds; bearing pressure, 0 to 900 pounds per square inch; average bearing temperature $1/16$ inch from oil film T_{av} , 115° to 161° F. Experimental oil flow, $Q = q \left(\pi d l c_d \frac{N'}{2} n \right)$. McKee data are taken from reference 14.

double bearing with a drained area between for the narrower bearings having length-diameter ratios of $1/2$ and $1/4$. In every case, each bearing was lubricated by a single $1/8$ -inch oil hole in the center as shown.

A special boring bar, of the type developed by the Battelle Memorial Institute, was constructed to bore the bearing to an accuracy of 50 millionths of an inch (0.000050 in.) as indicated by a special Federal hole gage, and the shafts were ground and lapped to the same accuracy as shown by a Pratt & Whitney supermicrometer.

Clearance was determined by measurement of the static displacement or rattle of the clean shaft in the bearing under reversing load. As shown in figures 21 and 22 the average clearances determined were 0.00264 inch for the single journal and 0.00232 inch for the double journal at room temperature.

MEASUREMENT OF JOURNAL DISPLACEMENT

The coordinate displacements of the journal with respect to the bearing during running were measured by the mechanical arrangement shown in figures 24 and 25. Horizontal and vertical motions of the shaft at stations beyond the ends of the bearing were transmitted by the bronze riders on levers or bell cranks through vertical rods to four

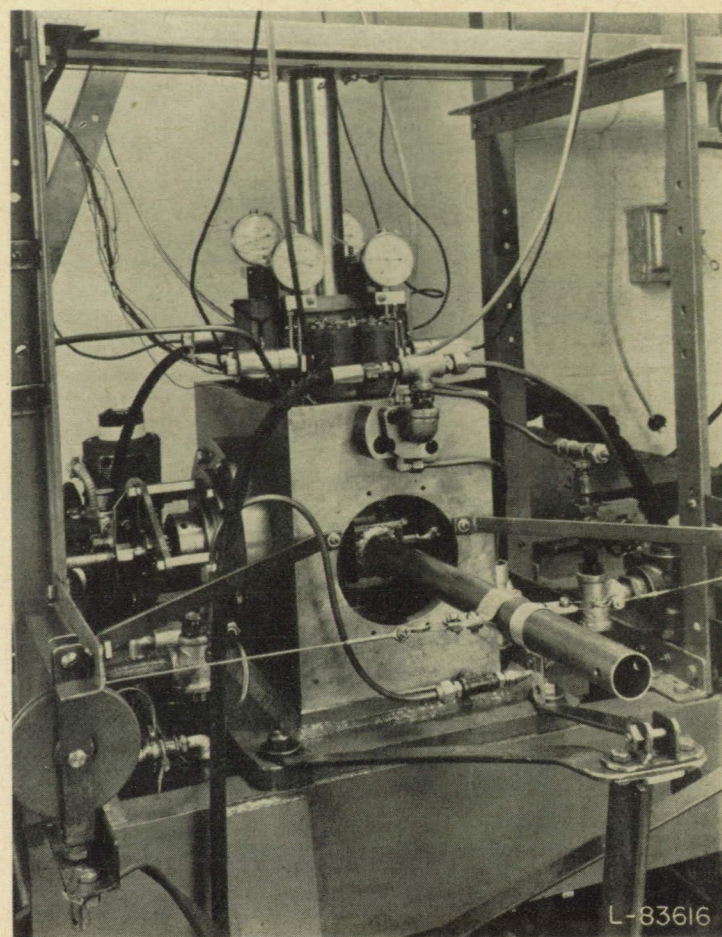


FIGURE 20.—Bearing testing machine used in experiments. Equipment includes variable-speed electric drive to 10,000 rpm, direct hydraulic loading mechanism, misaligning loading mechanism, and heater, filter, and pressure pump for lubricating oil. Measurements made include journal displacement, friction torque, oil flow, bearing temperatures, and, with accessories, oil film pressures.

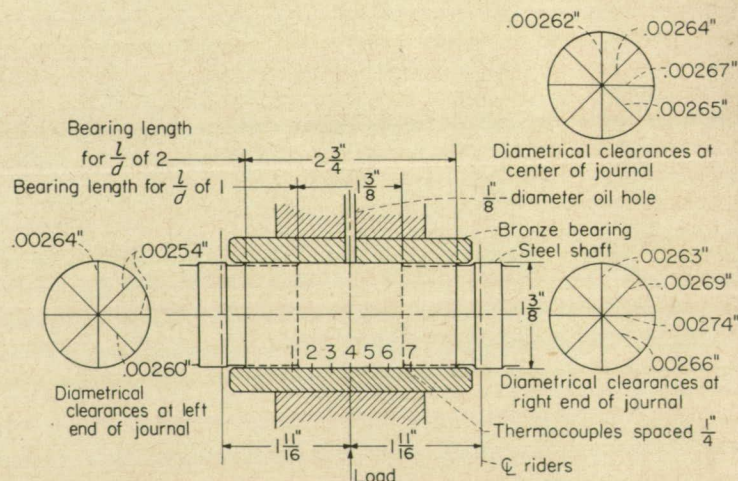


FIGURE 21.—Test shaft with single journal as used in experiments at l/d of 2 and 1. Average diametral clearance, 0.00264 inch.

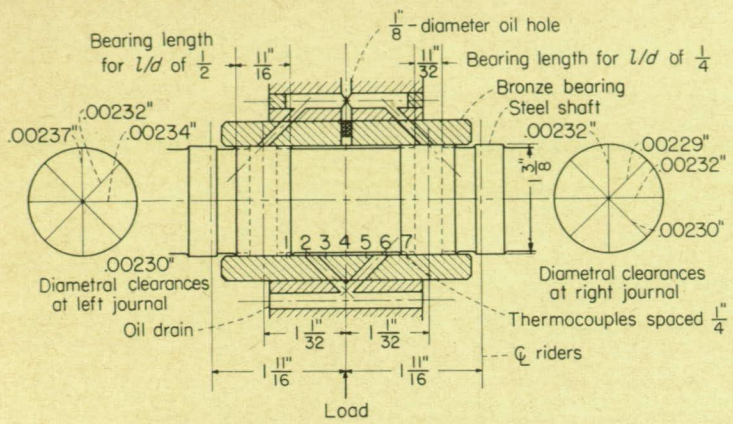


FIGURE 22.—Test shaft with two journals as used in experiments at l/d of $\frac{1}{2}$ and $\frac{1}{4}$. Average diametral clearance, 0.00232 inch.

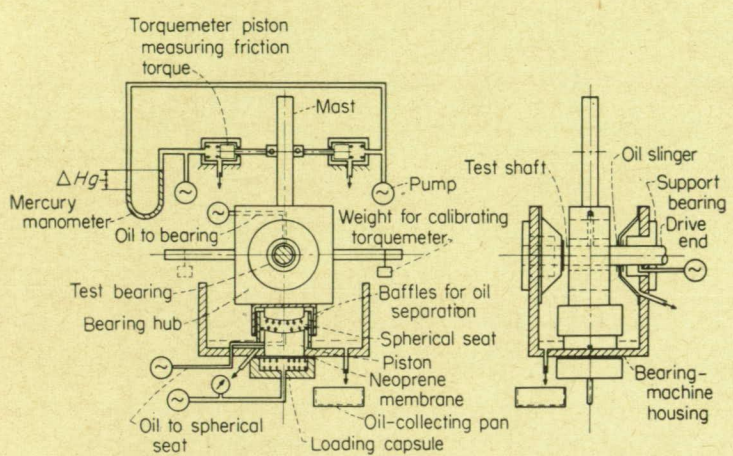


FIGURE 23.—Schematic diagram of apparatus for measuring friction torque on test bearing and oil flow through test bearing.

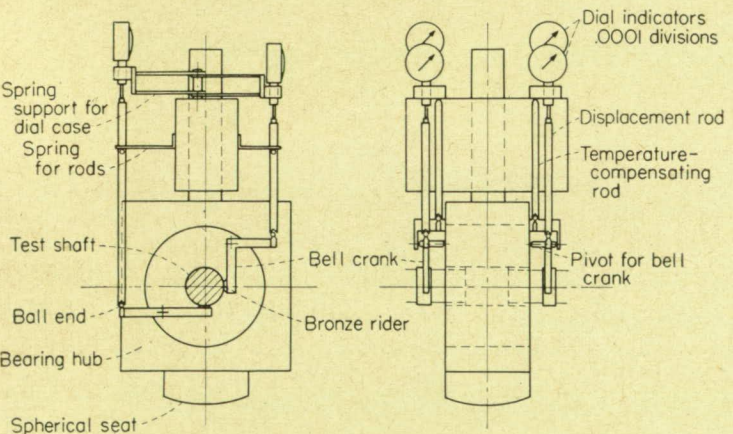


FIGURE 24.—Mechanical system for measuring displacement of journal with respect to bearing.

dial indicators having 0.0001-inch divisions. The rods were held against the ball ends of the bell cranks by flat springs attached to the mast on the bearing hub. In order to eliminate large dial readings due to thermal expansion of

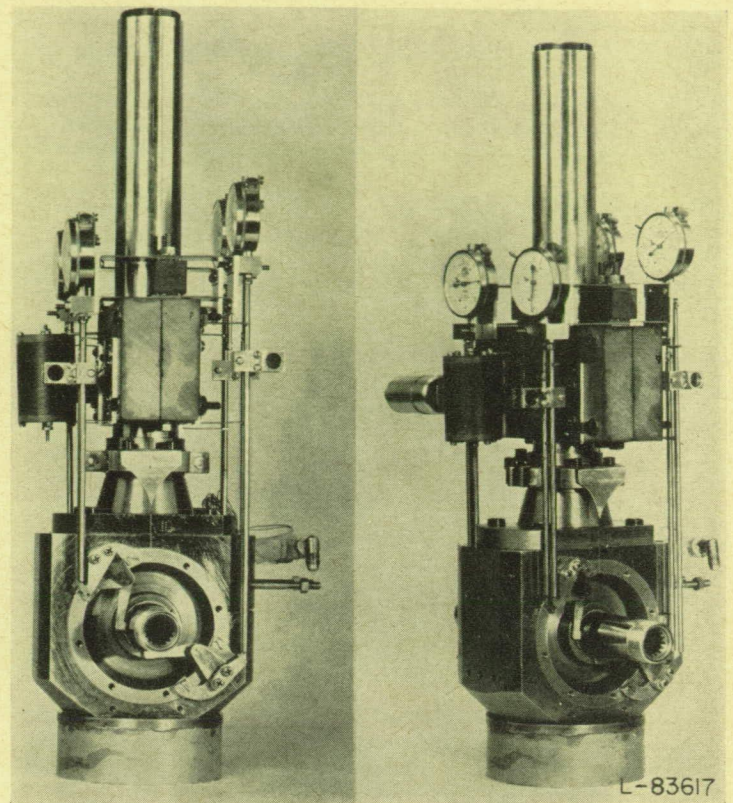


FIGURE 25.—Mechanical and photoelectric apparatus for measuring journal displacements with respect to bearing. Journal displacements are transmitted by levers and rods to dial indicators, which are used for steady-state conditions, and to photoelectric cells, for dynamic conditions.

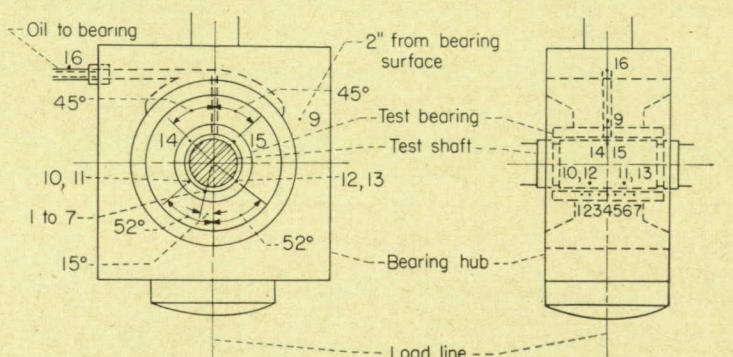


FIGURE 26.—Diagram of thermocouple locations. Bearing thermocouples are located $\frac{1}{16}$ inch from bearing surface.

the rods, a second set of similar parallel rods, intended for temperature compensation, moved the dial cases approximately the same amount the dial stems were moved by thermal expansion.

As shown in figures 20 and 25, a second system was available for the measurement of displacement. This system, developed by Phelan (ref. 16), is a photoelectric method particularly suited to indicate dynamic displacements on oscilloscopes. However, it was found to be more rapid and

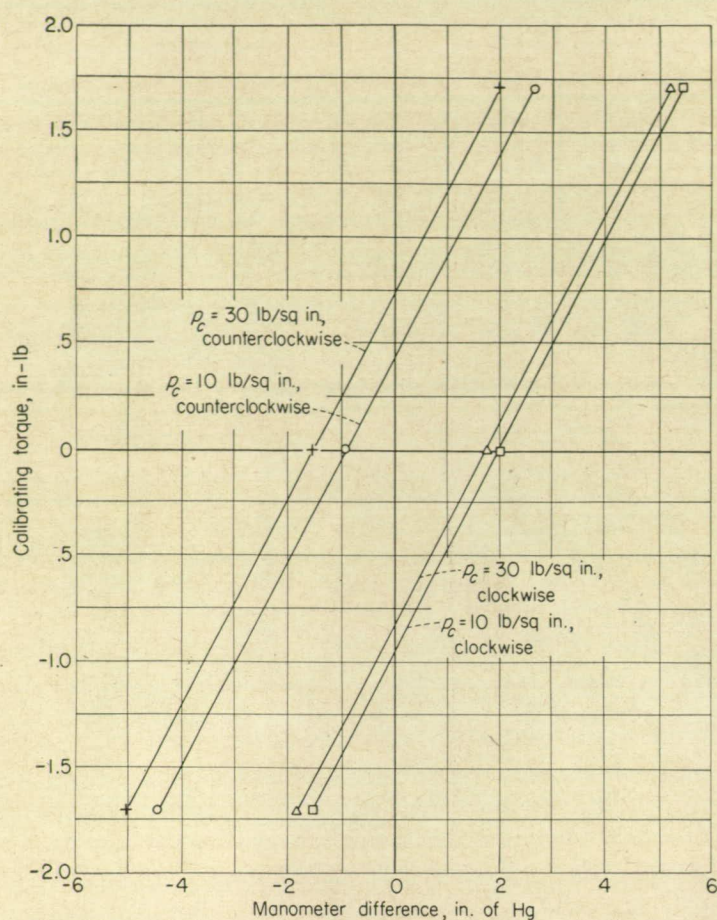


FIGURE 27.—Typical calibration curves for torquemeter used to measure friction torque on test bearing. Average slope of curves is calibration factor F_c giving friction torque per inch of difference in levels of mercury columns of manometer, $F_c = 0.485 \frac{\text{in-lb}}{\text{in. of Hg}}$. See table II for data.

convenient to take direct readings on the dial-indicating system under stable conditions. All of the data for this report were taken from the dial system.

MEASUREMENT OF FRICTION AND OIL FLOW

The friction torque on the test bearing was indicated by a hydraulic torquemeter as shown in figure 23. This consisted of two opposed pistons supplied with a flow of oil which drained through ports partially covered by the pistons. A mercury manometer indicated the difference in pressure on the opposed pistons which were floated in their bores by balanced pressure areas. A direct calibration relating applied moment to manometer reading is shown in figure 27. The torquemeter restrained the motion of the test bearing to $\frac{1}{3}^\circ$ or less under operating conditions.

An SAE 10 oil was used in the experiments on length-diameter ratios of $1/2$, $1/4$, and 1, the measured viscosity being shown in figure 28. The oil flow of the test bearing was kept

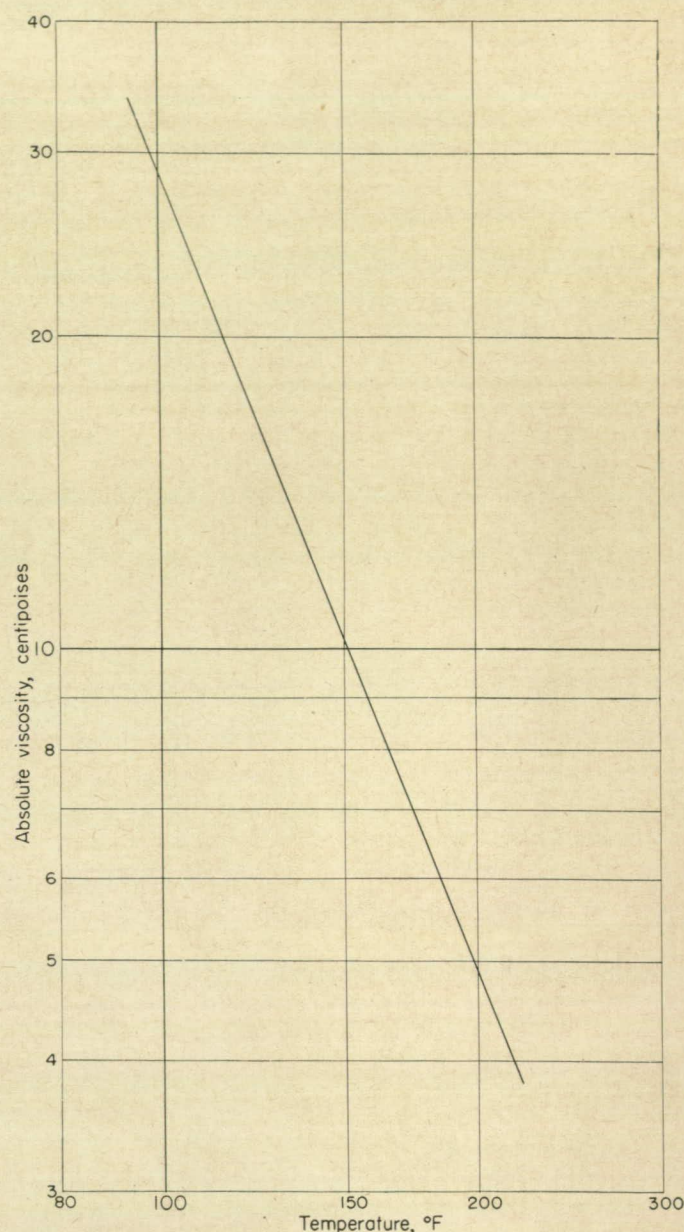


FIGURE 28.—Absolute-viscosity and temperature characteristics of SAE 10 oil used in tests for l/d of 1, $1/2$, and $1/4$. Measurements of viscosity were made in a standard Saybolt viscosimeter.

separate from the oil flow of the support bearings by baffles and slingers and was collected for weighing in a calibrated scale graduated in hundredths of a pound. Bearing temperatures were measured by 14 thermocouples placed in the bearing as shown in figure 26. Iron-constantan thermocouples inserted in 0.094-inch holes to within $\frac{1}{16}$ inch of the bearing surface were connected to a Leeds & Northrup potentiometer with automatic cold junction compensation and a Fahrenheit scale. A thermocouple attached to the bulb of a glass laboratory thermometer reading ambient temperature checked each set of readings within 1°F .

LOADING AND DRIVING SYSTEM

As shown in figure 23 the test bearing was loaded by a pressure capsule containing a piston having an area of 15.02 square inches. The piston head was flush with and sealed by a $\frac{1}{32}$ -inch-thick Neoprene membrane which permitted a steady pressure to be held by an accumulator. Calibrated pressure gages of varying ranges connected to the capsule gave the load on the bearing. The load from the piston was transmitted to the test-bearing hub through an oil-pressurized spherical seat as shown in figure 23. A flow of light oil forced into relieved quadrant areas in the spherical seat prevented metallic contact in order that the load could be transmitted without resistance to motion of the bearing about its geometric axes. This enabled measurement of friction torque on the bearing and located the line of action of the load at the bearing center. The shaft was directly driven by a 200-ampere aircraft generator acting as a motor, and by field control of this and a similar unit in a motor-generator set a speed range of 500 to 10,000 rpm was attainable.

The apparatus and test procedure are described in greater detail in reference 17.

TEST PROCEDURE

For each length-diameter ratio of 1, $\frac{1}{2}$, and $\frac{1}{4}$ two series of experiments were conducted. In the first series, displacements of the journal were measured by the dial-indicator equipment shown in figure 24. Friction torque and oil flow rate were measured in a second series of tests separate from displacement tests because it was necessary to disconnect the dial indicators to eliminate any friction torque due to the riders. Also, oil flow tests could not be made during the displacement tests because the small oil flow used to lubricate the riders mixed with the oil flowing from the test bearing. A brief initial series of tests with $l/d=2$ is not included as they were completed before the following technique was developed.

METHOD OF LOCATING ZERO-LOAD POINT

In the displacement tests, the coordinate locations of each end of the shaft were measured as changes of location from the zero-load position. In order to establish a reading of the dial indicators with the shaft in the center of the bearing clearance at zero load alternate up and down loads were applied with the shaft stationary, and the average of three sets of readings was recorded. The individual readings located points in an arc slightly on each side of a vertical diameter. After a similar procedure in the horizontal plane the machine was started and warmed up, and a capsule pressure of about 4 pounds per square inch was applied to balance the tare weight of the bearing hub and various attachments. With the shaft running and the inlet oil

pressure reduced to 2 to 4 pounds per square inch, the average of the dial readings representing the location of the center of the journal was checked and held within close limits to the center of the bearing as obtained above with the shaft stationary.

The net load applied to the bearing is that due to the difference of the pressure on the loading piston, called capsule pressure, and the pressure of 3.5 to 4.5 pounds per square inch required to balance the weight of the bearing-hub assembly. The displacement is the difference in dial readings from a test load to zero load. This method has the advantage that change in load is plotted against change in displacement.

It was discovered that when running at zero load the bearing required nearly zero oil flow; the dials showed that the journal maintained its position steadily when the inlet oil pressure was reduced to 2 pounds per square inch and the oil flow was a few drops per minute. Increasing the inlet oil pressure caused the journal to displace as though a load had been applied. This effect was more marked as the inlet pressure was further increased and the oil flow increased to a continuous stream.

SHAFT-DISPLACEMENT EXPERIMENTS

In making shaft-displacement tests, the zero-load position of the shaft was determined as described, and a warmup period of 20 to 30 minutes was made at constant speed. Variations in displacement were obtained by varying the load at constant speed and constant inlet oil pressure. It was found that by holding a constant speed the variations in temperature were less than 5°F during an increasing-load run.

Displacement data as read from the four dials were then recorded as the load was increased in 10 to 15 increments and then decreased in the same increments to permit averaging of data. A sample log sheet is shown in table I. The same procedure was followed in the opposite direction of rotation so that the data could be averaged for the two rotations. After a varying-load test, the location of the datum point at zero load was rechecked in two directions of rotation to permit averaging out any small change during the test. Wear of the lever rubbing surfaces or riders was usually imperceptible during individual tests but would have been canceled by this method.

For each length-diameter ratio there were experimental runs in each direction of rotation at combinations of constant speed and constant inlet oil pressure. The speeds were 500, 1,000, 2,000, 2,500, 4,000, 5,000, and 6,000 rpm, and the inlet oil pressures were 40 and 100 pounds per square inch. The maximum load on projected area was 760 pounds per square inch, and the maximum oil film temperature was 160°F .

CONTROL OF WHIRL

At journal speeds of 500 and 1,000 rpm there was little difficulty with vibration even at zero load. However, at zero load at the higher speeds from 2,000 to 5,000 rpm a vibration, or whirl, in a rotating misalignment mode would build up, preventing the reading of displacements. The vibration could be stopped by the application of load, but it was necessary to prevent vibration at zero load in order to use the routine procedure to obtain the position of zero eccentricity. By snubbing the top of the mast and the arm extending laterally from the bearing hub with foam rubber, the vibrations were successfully eliminated, using small snubbing forces parallel to the axis of the shaft. By temporarily removing the snubbers at loads where no vibration would occur, it was found that the snubbers had no effect on the dial-indicator readings.

FRICTION AND OIL FLOW EXPERIMENTS

Friction torque and oil flow rate were measured simultaneously, with the displacement-measuring apparatus temporarily disconnected in order to eliminate the friction of the riders and the oil flow to them. Changes in friction and oil flow were made by varying the applied load and maintaining the journal speed and inlet oil pressure constant. Before beginning a test, a period of running of approximately 20 to 30 minutes at constant speed was allowed to reach thermal equilibrium. About 10 to 15 increments of increasing load were then applied. At each load, after the oil flow stabilized, the flow rate was determined by collecting the oil in a pan for periods of 1 to 4 minutes and recording the net weight of oil. A sample log sheet is shown in table II. The same procedure was followed for the opposite direction of rotation so that data could be averaged for the two rotations.

Several runs at combinations of constant speed and constant inlet oil pressure were made in each rotational direction for each length-diameter ratio. The speeds were 500, 1,000, 2,000, 2,500, 4,000, 5,000, and 6,000 rpm, and the inlet oil pressures were 40 and 100 pounds per square inch. The maximum load on projected area was 900 pounds per square inch, and the maximum oil film temperature was 161° F.

The foam-rubber snubbers used to eliminate vibration in the displacement experiment were removed because they offered restraint to rotation of the bearing about its axis and interfered with the functioning of the torque meter. At speeds above 1,000 rpm vibrations were stopped by the application of a small load. Since the condition of zero load is not of great importance in friction and oil flow measurements, data were taken only at loads where no vibration occurred.

PRECISION

In every run of the friction experiments, calibrations of the torque meter were made as a check on operating accuracy by applying known weights to the arms extending from the front and rear of the bearing hub as shown in figure 23. Manometer differences corresponding with the known moments, first to the front and then to the rear, are given

in the sample log sheet of table II and are shown plotted in figure 27. More extensive tests with several weights had proven that the result was a straight line. From the many calibrations made, a mean value of the calibration factor F_c was 0.485 inch-pound of torque per inch of manometer difference. The greatest deviation from the mean was 0.010 or 2 percent.

For reliable functioning of the torque meter it was necessary to bleed air from the oil lines leading to the manometer at the beginning of each run. Because the torque meter pistons were hydraulically centered in their bores by three pressure areas, the torque meter was sensitive enough to measure aerodynamic forces on the bearing hub, and it was necessary to rid the test room of air gusts.

All pressure gages of both the laboratory and commercial grades were calibrated in a dead-weight tester with a capacity of 500 pounds per square inch. Pressure readings were corrected according to calibration curves where necessary. The pressure in the loading capsule was indicated by two pressure gages with different scale ranges. It was found that the pressure gages did not accurately read the capsule pressure due to the tare weight of the test bearing hub unless they were placed at the same level as the loading capsule. By suspending two calibrated 50-pound weights from the lateral arms of the bearing hub, the corresponding increase in capsule pressure showed that the calculation of loading in the low range was of satisfactory accuracy.

The four dial indicators used to measure journal displacements were of laboratory test grade and were graduated in 0.0001-inch divisions. A direct calibration of the indicators was made and in the maximum range during tests of about 13 divisions they were accurate to within one-half division. As the load was increased and then decreased there was often a discrepancy of as much as one-half division in displacement for the same load. By averaging the displacement data for increasing and decreasing loads, the influence of the lag on accuracy of measurement is minimized.

Two aircraft-type tachometers were used, one a centrifugal type having a scale range of 1,000 to 10,000 rpm and the other a 3-second counting type with a range of 0 to 1,750 rpm. The centrifugal tachometer was checked under running conditions by timing the revolution counter of the tachometer. By connecting both tachometers to the test shaft simultaneously at speeds of 1,000 to 1,500 rpm the two tachometers were checked against each other. A portable shop-type tachometer was used to check the lower-range tachometer at 500 rpm.

The direct-reading Fahrenheit scale of the Leeds & Northrup potentiometer having automatic cold-junction compensation was used to measure bearing temperatures from thermocouples and can be read to an accuracy of 1° to 2° F. A thermocouple attached to the bulb of a laboratory-grade thermometer giving ambient temperature near the machine checked the thermometer temperature within 1° F. No direct calibration of the potentiometer was made.

The weighing scale used to measure oil flow was equipped with a scale of 2-pound capacity graduated in hundredths

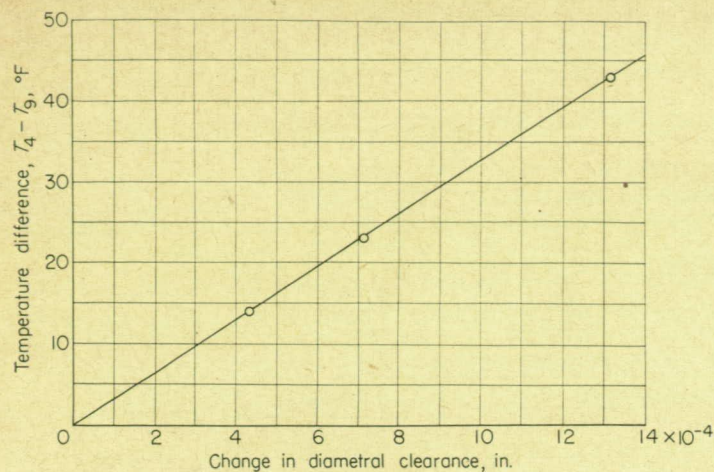


FIGURE 29.—Changes in diametral clearance as function of temperature difference of two points in test-bearing wall. T_4 and T_9 are bearing temperatures at $\frac{1}{16}$ inch and 2 inches, respectively, from bearing surface. Running clearances of test bearing are determined by subtracting change in clearance from room-temperature clearance.

of a pound which could be read to one-quarter of a division. Chemical balance weights were used to calibrate the scale.

Two Saybolt viscosimeters having two viscosimeter tubes each were available for determining the viscosity of the test oils at various temperatures. The variations in running time from two tubes at a given temperature did not differ more than 1 second.

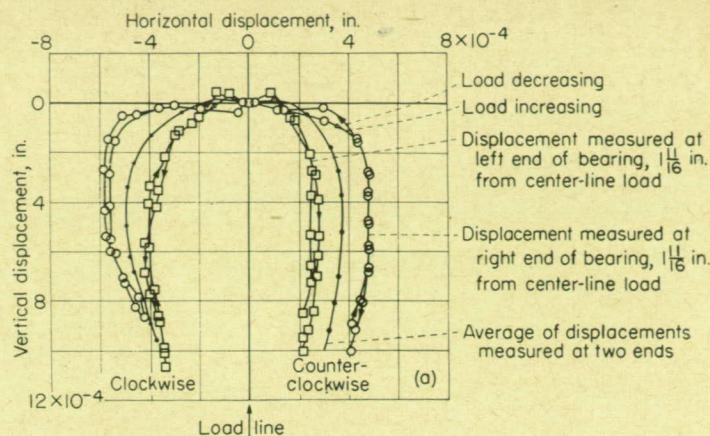
REDUCTION OF EXPERIMENTAL DATA

Calculations of several nondimensional quantities are shown in the sample calculation sheets (tables III and IV). The data for the calculations are given in the sample log sheets (tables I and II).

CAPACITY NUMBER

The capacity number was calculated from equation (17). The net load applied by the loading piston or capsule was determined as the product of the net capsule pressure in pounds per square inch and the piston area of 15.02 square inches. The weight of the bearing and its attachments required a tare capsule pressure of 4.5 pounds per square inch in the displacement experiments and 3.5 pounds per square inch in the friction and oil flow experiments, which was deducted from the observed pressure.

Bearing clearance was determined by subtracting the change in clearance given by figure 29 from the clearance at room temperature. For the single journals (fig. 21), thermocouples 1 to 7 are in the loaded portion of the bearing. There was little variation between the temperature readings at these locations for a given load. Thermocouple 4 was taken as representative, and the difference of the averages of thermocouples 4 and 9 was determined for the complete run at a given speed. Using this average in figure 29 gives the change in clearance from that at room temperature. For the double journals (fig. 22) where only the end thermocouples are in the loaded region, an average of thermocouples



(a) Displacements of journal measured by riders beyond bearing ends.

FIGURE 30.—Journal displacements with respect to bearing for shaft with two journals of l/d of $\frac{1}{2}$. Steps shown illustrate typical method of determining average eccentricity ratios and attitude angles from displacement data measured beyond ends of bearing. See tables I and III for experimental data and calculations.

1, 4, and 7 was compared with the average of thermocouple 9 to determine the bearing clearance.

Viscosity in the oil film was determined at each load from the average bearing temperature for the two directions of rotation and for increasing and decreasing load. In tests using the single journal, temperature data at thermocouple 4 were taken as representative of the oil film temperature. For the double journals, the average of thermocouples 1 and 7 was more nearly representative because of the location of these points in the narrow loaded area. These temperatures were used to determine viscosity in figure 28.

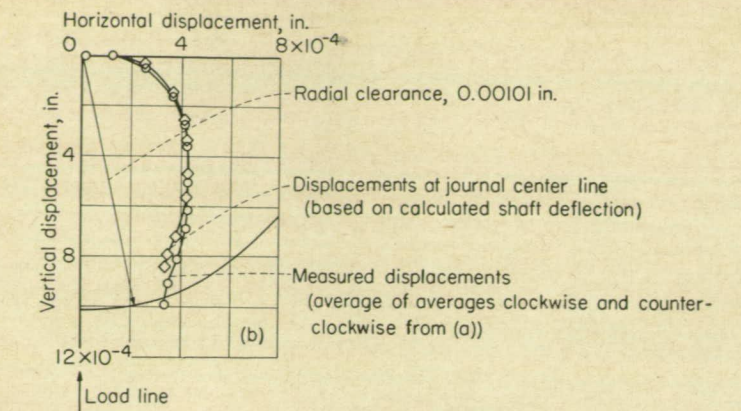
ECCENTRICITY RATIO OR ATTITUDE

The eccentricity ratio n is the ratio of the displacement of the journal axis to the radial clearance given by $n = e/c_r$.

Coordinate vertical and horizontal displacements of the shaft were measured at the riders, which are located beyond the ends of the bearing $1\frac{1}{16}$ inches to the left and to the right of the center line of the load, as shown in figures 21, 22, and 24. To determine the displacement of the test journal, calculations were made of the effect of shaft deflection on the vertical displacements measured at the riders.

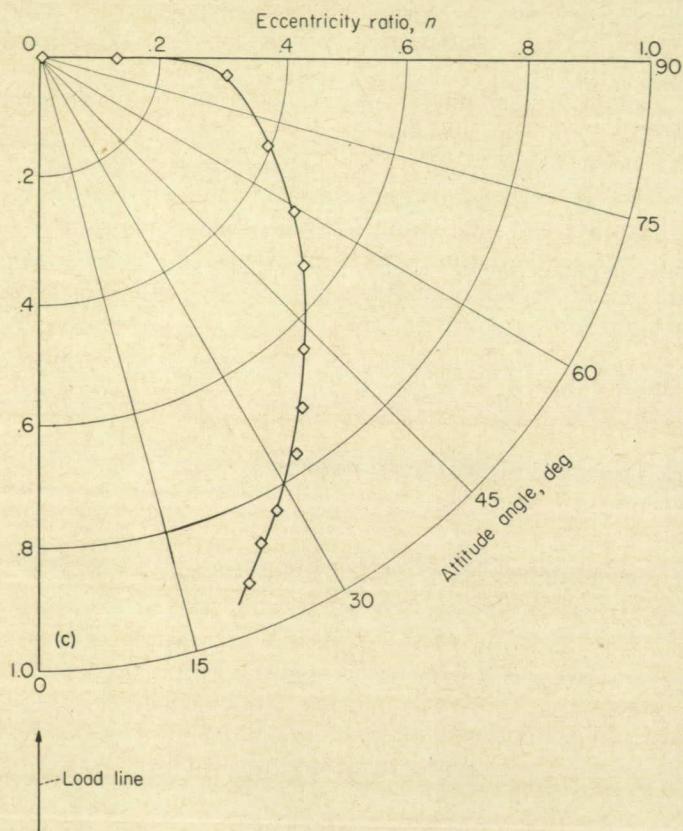
As shown in the sample log sheet of table I, displacement data were taken as the load was increased and then decreased. These data are shown in the sample calculation sheet of table III, then reduced to values of journal displacement from the condition of zero load and a low inlet oil pressure at which, as previously described, it was assumed that the journal and bearing axes were coincident. A plot of these data for the two directions of rotation appears in figure 30(a). The bearing and journal are not ideally accurate cylinders, and there is a departure from symmetry of the displacements at the two ends.

Averages of displacements of the two ends are shown in figure 30(b). The displacements illustrated are typical for the case of a shaft with two journals. In order to determine the displacements of the centers of each journal, calculated values of shaft deflection were subtracted from the measured



(b) Average journal displacements by riders and displacements at journal center line.

FIGURE 30.—Continued.



(c) Displacements at journal center line reduced to eccentricity ratios.

FIGURE 30.—Concluded.

vertical displacements at the riders. The vertical and horizontal components of eccentricity ratio n_v and n_h were calculated by dividing the corrected journal displacements by the radial clearance. The plotting of n_v against n_h is shown on the polar diagram of figure 30(c) in terms of eccentricity ratio n and attitude angle ϕ .

In the case of single journals, the method of calculation of eccentricity ratio was similar. Because of the greater length of the single journals and the greater loads applied, shaft bending over the length of the journal was of considerable magnitude at high loads. Since the journal was

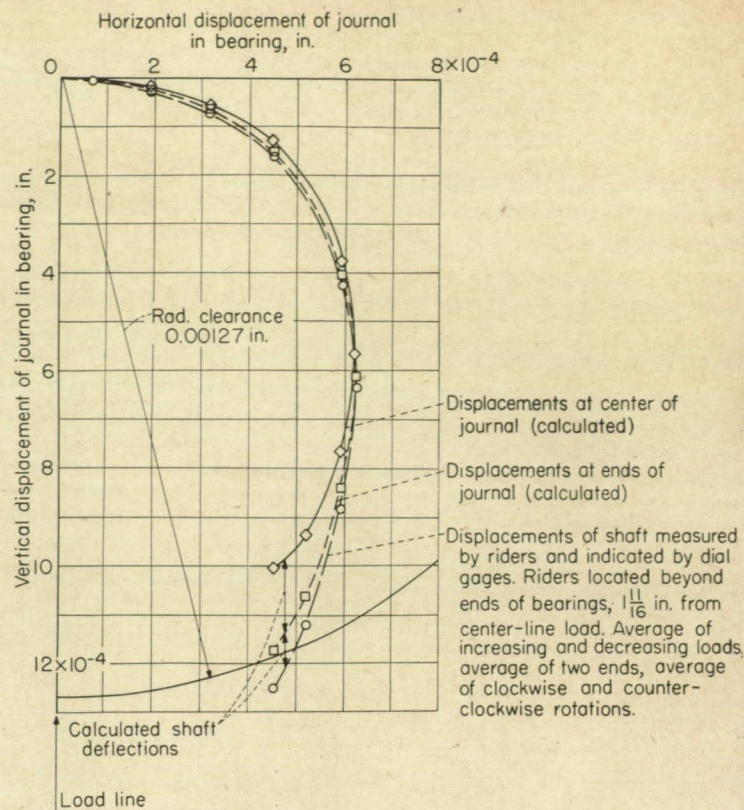


FIGURE 31.—Journal displacements with respect to bearing for shaft with single journal of l/d of 2. Displacements at ends and center of a bent journal are shown in comparison with displacements measured beyond ends of bearing. Average eccentricity ratios and attitude angles as determined from displacements are shown in figure 32.

bent longitudinally in the shape of a fourth-degree parabola, the displacement and eccentricity ratio at the center of the journal were less than at its ends, as indicated in figures 31 and 32. To simulate the performance of a straight journal, the average eccentricity ratio of a parabolic elastic curve was calculated as representative of the eccentricity ratio of a straight journal (see fig. 32). Four-fifths of the height of the deflection curve within the journal was subtracted from the displacement at the ends.

Eccentricity ratios as a function of capacity number and load number are shown in figures 14 and 15; as a function of attitude angle, they are shown in figure 13.

BEARING FRICTION

Experimenters usually distinguish between friction measured on the journal and that measured on the bearing, since the friction torque measured on the bearing is evidently somewhat reduced by the couple resulting from the product of the load and the horizontal component of the displacement. The measured data in this report are the friction measured on the bearing and have been referred to as bearing-friction variables.

Data on friction and shaft displacement were both available so that it was possible to compute the journal friction by adding the couple to the bearing friction. Referring to the column labeled friction variable f_e in table IV, the bearing friction and calculated journal friction for $l/d=1$ are given as follows:

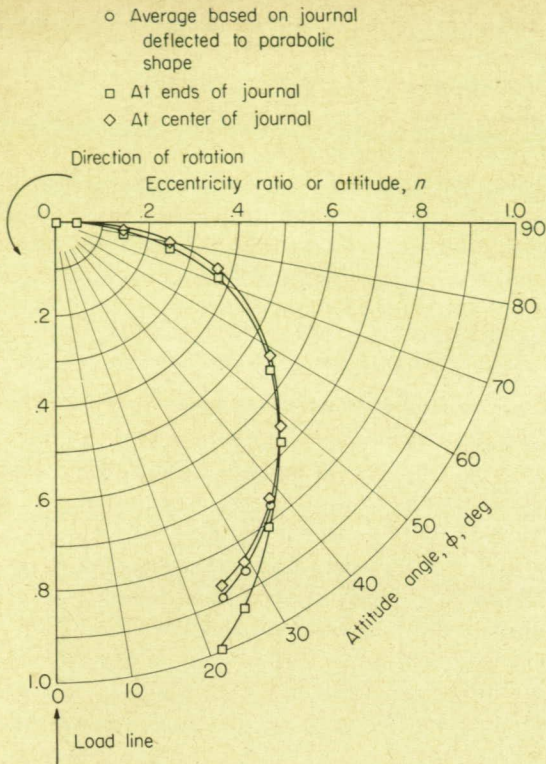


FIGURE 32.—Eccentricity ratios and attitude angles for shaft with single journal of l/d of 2 as determined from displacements shown in figure 31. The curve of average eccentricity ratios is shown in comparison with eccentricity ratios at ends and center of a bent journal. Experimental data: Speed, 500 rpm; inlet pressure p_o , 40 pounds per square inch; diametral clearance at room temperature, 0.00264 inch; diametral clearance at bearing temperature T_{av} of 119° F, 0.00254 inch; load, 0 to 855 pounds; bearing pressure, 0 to 245 pounds per square inch; AN-09-1010 jet engine oil fed through one $\frac{1}{8}$ -inch-diameter hole.

Load, P , lb	Bearing friction, $f_b \left(\frac{d}{c_d} \right)$	Couple, $f_c \left(\frac{d}{c_d} \right)$	Journal friction, $f_j \left(\frac{d}{c_d} \right)$	Ratio, f_j/f_b
0	0	0	0	-----
22.5	24.10	.10	24.20	1.002
60.1	9.13	.19	9.32	1.020
97.6	5.70	.27	5.97	1.048
135.1	4.23	.31	4.54	1.073
172.6	3.36	.33	3.69	1.099
248.0	2.36	.34	2.70	1.143
398.0	1.55	.36	1.91	1.230
623.0	1.06	.35	1.41	1.330
848.0	.83	.32	1.15	1.385

In the table above the couple is to be added to the bearing-friction variable and is numerically equal to the horizontal displacement n_h as follows:

$$f_c \left(\frac{d}{c_d} \right) = \frac{Pe \sin \phi}{Pr} \left(\frac{r}{c_r} \right) = \frac{e}{c_r} \sin \phi = n \sin \phi = n_h \quad (30)$$

The values of n_h are those for the run at 1,000 rpm and 40-pound-per-square-inch inlet oil pressure.

Manometer differences indicating bearing friction in inches of mercury for each load condition are shown in the sample log sheet of table II; they are of opposite sign for opposite directions of rotation. For a given load, the absolute values of the manometer differences for the two directions of

rotation were averaged as indicated in the sample calculation sheet of table IV. Multiplication by the calibration factor F_c gave the friction torque M_t in inch-pounds. The friction coefficient f for calculation of the friction variable was determined from $f = M_t/Pr$. Friction force F used in the calculation of friction ratio was determined from $F = M_t/r$.

The sample calibration-factor curves corresponding with the experimental data of the sample log sheet (table II) are shown in figure 27. Experimental values of bearing-friction variable are shown plotted as a function of capacity number in figure 16. Friction ratios as a function of load number are shown in figure 17.

OIL FLOW

As shown in the sample calculation sheet (table IV), oil flow data for the two directions of rotation were averaged at each condition of load. Dividing the weight of oil by the time interval during which it was collected gave the rate of flow Q' in pounds per minute. Since Q in the oil flow number must be given in cubic inches per second, a conversion was made from the following expression:

$$Q = Q' \times \frac{1,728}{62.4 \times \text{sp gr}} \times \frac{1}{60} \quad (31)$$

in which sp gr is the measured specific gravity of the oil at oil film temperature. Where double journals were concerned, half of the oil flow was used in calculations of oil flow numbers.

A more complete description of the apparatus, method of test, and reduction of data is given in reference 17.

EXPERIMENTAL MEASUREMENT OF FILM PRESSURE DISTRIBUTION

The experimentally measured oil film pressure distribution shown in figure 9 was obtained from reference 13 which included photographs of several plaster models representing measured film pressures in a full journal bearing with babbitt lining. Model 2 of the group of five models represents the pressure distribution for a bearing of $l/d = 1$ acted upon by a normal central load. Model 2 alone is of interest since the other models represent film pressures measured under conditions of misalignment.

As shown in figure 9, the analytical pressure distribution as given by the short-bearing approximation is in very close agreement with the data of model 2. The analytical solution by Sommerfeld is actually not directly comparable because the short-bearing solution is based on an assumed 180° of film whereas the Sommerfeld solution assumes a continuous film of 360°. Moreover, even if a ruptured film of 180° were assumed, the Sommerfeld solution cannot agree with experimental data because the solution gives no variation in film pressure in the axial direction.

The film pressure data of model 2 were taken under the following conditions: $d = 1\frac{5}{8}$ inches, $l = 1\frac{5}{8}$ inches, $P = 2,200$ pounds, $p' = 833$ pounds per square inch, $N = 5,000$ rpm, $N' = 83.3$ rps, μ (at 197° F) $= 2.42 \times 10^{-6}$ reyn, c_d (at room temperature) $= 0.0033$ inch, and c_a (at 197° F) $= 0.0020$ inch.

Pressure taps in the bearing were located at the center

of the bearing and at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ inch each side of the center in a test machine arranged to measure the circumferential pressure distribution shown by the solid lines of figure 9. The model 2 curves represent averages of pressures for the two directions of rotation of the journal and averages at corresponding points each side of the central transverse plane of the bearing.

For this test aviation 1120 oil was preheated to 140° F and entered the bearing at the center on the unloaded side through two $\frac{1}{8}$ -inch-diameter holes 45° apart at a pressure of 40 pounds per square inch. The viscosity of the oil is based on the bearing temperature given by a thermocouple located $\frac{1}{16}$ inch from the bearing surface at the center of the loaded side.

The bearing clearance at operating film temperature is determined from a curve of clearance change against temperature change similar to the curve of figure 29.

The capacity number for model 2 from the above data is 0.159 and the corresponding value of eccentricity ratio from the analytical short-bearing curve of figure 14 is $n=0.415$. The analytical short-bearing curves of figure 9 are given by equation (13) for $n=0.415$ and $z=0$, $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, and $\pm\frac{3}{4}$ inch. Since the angular scale θ is different from the scale used in this experiment, two angular scales are required differing by $90^\circ - \phi$.

Analytical values of peak-pressure ratio, peak-pressure angle, and attitude angle for model 2 are given by the short-bearing curves of figures 11, 12, and 13 for $C_n=0.159$ and $n=0.415$. These values are compared with the experimental values below.

Quantity	Experimental	Analytical
Peak-pressure ratio, k	2.49	2.50
Peak-pressure angle, α , deg.....	20	19½
Attitude angle, ϕ , deg.....	60	60

^a Analytical value of ϕ is measured from load line to location of zero pressure which theoretically coincides with location of minimum film thickness.

The analytical curve in figure 9 given by the Sommerfeld solution is based on the following equations for eccentricity ratio and pressure distribution:

$$S = \frac{\mu N'}{p'} \left(\frac{d}{c_d} \right)^2 = \frac{(2+n^2)(1-n^2)^{1/2}}{12\pi^2 n} \quad (32)$$

$$p = \frac{\mu U r}{c_r^2} \left[\frac{6n(2+n \cos \theta) \sin \theta}{(2+n^2)(1+n \cos \theta)^2} \right] + p_s \quad (33)$$

For model 2, equation (32) gives $n=0.106$, an eccentricity ratio about one-fourth of that given by the short-bearing

approximation which is a consequence primarily of assuming that the unloaded side of the bearing contributes to the support of the applied load.

In equation (33), p_s is the film pressure at $\theta=0$ and π . The value of p_s may be determined from equation (33) by taking the pressure at $\theta=(3/2)\pi$ as the inlet pressure of 40 pounds per square inch. For model 2, $p_s=570$ pounds per square inch. The angular scale θ coincides with that used in the experiment because the Sommerfeld solution places the load line at $\theta=90^\circ$.

EXPERIMENTS BY MCKEE AND MCKEE

Eight cases of experimental pressure distributions in bearings of $l/d=1.15$ under normal central load are presented by McKee and McKee in reference 11. Four of the experiments were conducted on a journal bearing having a diametral clearance of 0.005 inch and four, on a bearing having 0.002-inch clearance. Both bearings were 1.00 inch long and 0.87 inch in diameter. Maximum applied loads were 25 pounds and the maximum journal speed was 592 rpm. The bearings were operated completely submerged in the lubricating oil, and the temperature of the oil was measured with a thermometer located in the oil bath several inches from the bearing film.

In figure 10, a comparison is made of the experimental distributions with those given by the short-bearing function for the four tests with the diametral clearance of 0.005 inch. The agreement between experiment and theory was better for these tests than for those with 0.002-inch diametral clearance. Calculations of capacity number, eccentricity ratio, pressure distribution, attitude angle, peak-pressure angle, and peak-pressure ratio were made in the same manner as for model 2. Some of the analytical and experimental quantities are compared in the table below.

The experimental values of peak-pressure ratio and peak-pressure angle for both clearances are compared with the short-bearing analytical curves in figures 11 and 12. It may be seen that the agreement between experiment and theory is not so close for the 0.002-inch clearance as for the larger clearance. In figure 10 it may be seen that the divergence of the experimental and analytical pressures is such that the analytical values represent an average of the experimental values which are in some instances above and in some instances below the analytical ones.

Although the comparisons of the analytical quantities with those by McKee and McKee are not so close as those with model 2, it may be seen from figure 10 that the agreement is reasonable as to order of magnitude of pressures and the shape of the pressure-distribution curves.

Quantity	Test 1		Test 2		Test 3		Test 4	
	Experimental	Analytical	Experimental	Analytical	Experimental	Analytical	Experimental	Analytical
Capacity number, C_n	-----	0.0815	-----	0.0785	-----	0.0300	-----	0.0352
Eccentricity ratio, n	-----	0.555	-----	0.561	-----	0.713	-----	0.688
Attitude angle, ϕ , deg.....	50	50	48	49	52	37	51½	39
Peak-pressure angle, α , deg.....	12½	18½	12	18½	18	15	15	16
Peak-pressure ratio, k	3.06	3.00	3.13	3.00	3.79	3.83	2.83	3.63

^a Analytical value of ϕ is measured from load line to location of zero pressure which theoretically coincides with minimum film thickness.

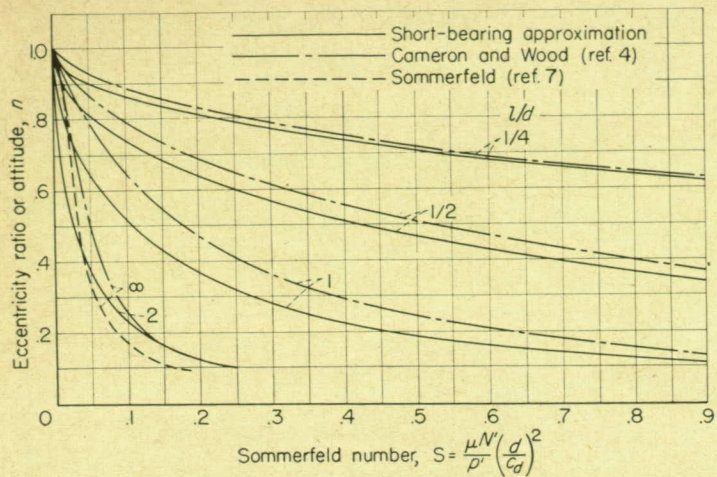


FIGURE 33.—Eccentricity ratio plotted against Sommerfeld number for comparison of analytical curves.

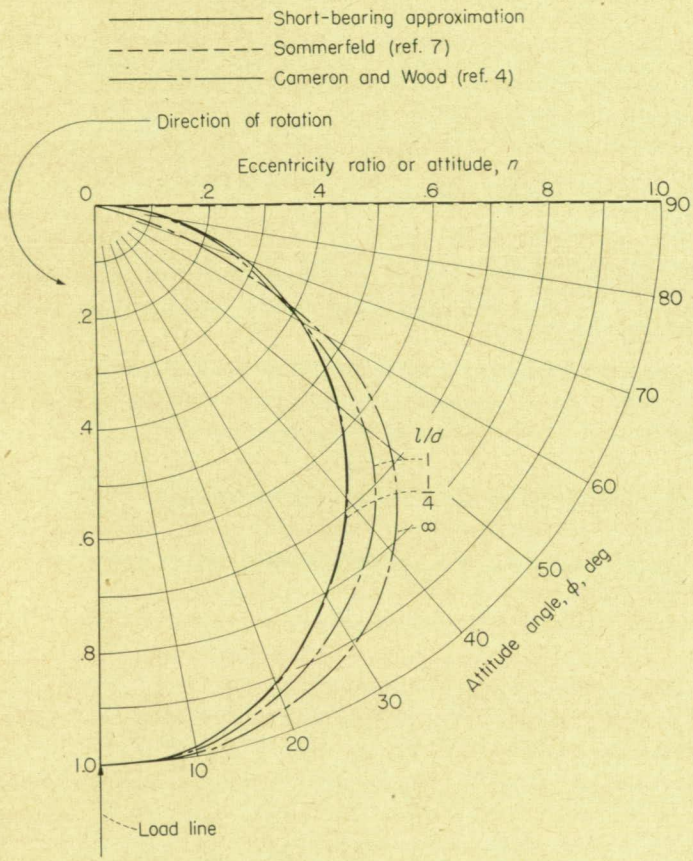


FIGURE 34.—Eccentricity ratio plotted against attitude angle for comparison of analytical curves.

It is interesting to note that the region of pressures below atmospheric as shown experimentally is small enough to be considered negligible as is assumed in the short-bearing approximation.

COMPARISON WITH OTHER ANALYTICAL SOLUTIONS

In both the analytical work of Cameron and Wood (ref. 4) and in the short-bearing approximation of this report, length-diameter ratio has been incorporated as an important variable in journal-bearing performance. For this reason, a comparison is made in figures 33 to 35 of curves of eccen-

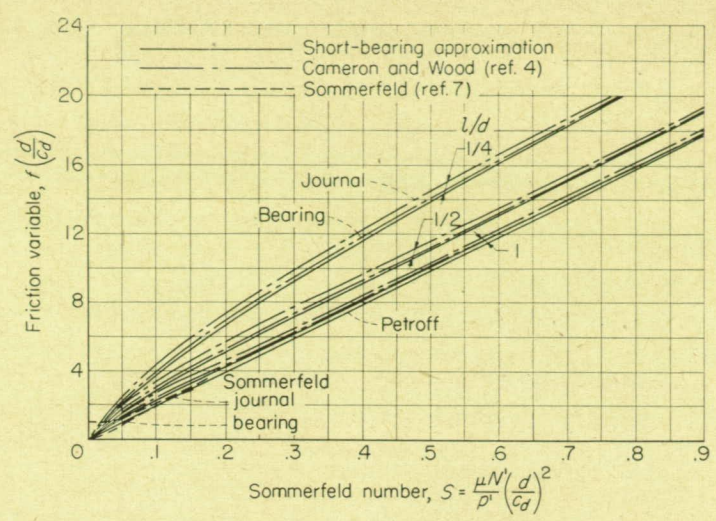


FIGURE 35.—Friction variable plotted against Sommerfeld number for comparison of analytical curves.

tricity ratio, attitude angle, and friction variable as given by these analytical solutions. In addition, analytical curves determined by Sommerfeld (ref. 7) for bearings without endwise flow are also shown for comparison.

The differences in the analytical solutions lie in the basic assumptions made in each case. Both the solution by Cameron and Wood and the short-bearing approximation assume a 180° ruptured film, differing from the Sommerfeld solution which assumes a continuous film of 360°. This difference is of great influence as shown in the curves of eccentricity ratio and attitude angle in figures 33 and 34; the curves representing the two analytical solutions assuming ruptured films are in reasonably good agreement but are quite divergent from the solution depending upon a continuous film.

An important difference between the solution by Cameron and Wood and the short-bearing solution is that the former is based on a ruptured film whose extent depends upon the Sommerfeld number, and the latter assumes that the ruptured film extends 180° for all values of the Sommerfeld number. Another important difference is that the solution by Cameron and Wood is exact mathematically in satisfying Reynolds' equation (eq. (1)) whereas the short-bearing solution is not because of the assumption of a linear velocity profile in the circumferential direction of fluid flow. It is interesting to note that the two solutions for eccentricity ratio and attitude angle are less divergent as l/d becomes smaller and approach each other as l/d approaches zero.

The Sommerfeld solution crosses the lines for the short-bearing solution, as shown in figure 33, giving lower values of eccentricity ratio at light loads and higher eccentricity ratios than the short-bearing solution in the heavily loaded region of low Sommerfeld numbers. Apparently the Sommerfeld curve and the curves plotted from Cameron and Wood points for $l/d = \infty$ do not intersect as is indicated by the two curves in figure 33.

Figure 35 compares friction variable as given by the three analytical solutions. Again the theory of Cameron and Wood and the short-bearing theory are in close agreement

on the effect of l/d ratios. Because the solution by Cameron and Wood satisfies Reynolds' equation, it shows a difference in friction variable for the bearing and for the journal as shown in figure 35; the short-bearing solution does not indicate this difference because of the assumed linear circumferential film velocity profile. For an l/d greater than 1.0, all solutions are well approximated by the Petroff line at the high Sommerfeld numbers. It is interesting that all solutions assume a 360° film in determining the friction variable. Apparently this assumption is valid from experimental results.

An important feature of the Sommerfeld solution and the short-bearing solution is that the solutions appear in equation form giving simple relationships among the bearing variables in nondimensional quantities.

ANALYSIS AND DISCUSSION

Figures 13 to 19 represent the principal results of this investigation and include only the experimental data for bearings of l/d ratios of 1, $\frac{1}{2}$, and $\frac{1}{4}$. All these data are in useful agreement with the theoretical analysis. The data for bearings of $l/d=2$ which were not in satisfactory agreement as shown in reference 17 are not included. One of the simplifying assumptions limits the theoretical analysis to bearings of short length, and it would be expected that the most satisfactory agreement with experiment would be in the low l/d range. A relatively small amount of data were taken at $l/d=2$ at a single speed of 500 rpm. The important techniques of determining the position of zero eccentricity, of taking data at increasing and decreasing increments of load, and of waiting for the condition of thermal equilibrium were developed after the curves for $l/d=2$ were available for study and were used in tests at $l/d=1$, $\frac{1}{2}$, and $\frac{1}{4}$. Conclusions in regard to application of the short-bearing approximation above $l/d=1$ should be withheld until more data are available.

ECCENTRICITY RATIO

Although the data shown in figures 14 and 15 for journal eccentricity are considered to be in good agreement with theory, the line through the experimental data lies slightly above the theoretical line. Four factors which may have contributed to this discrepancy are apparent.

First, the effect of inlet oil pressure is to raise the experimental curve at low values of n . The inlet oil pressure apparently acts on sufficient area of the bearing surface to cause a load adding to the external load. The effect of inlet oil pressure is usually of minor importance because the effect of inlet oil pressure is relatively less with increasing load and also because the value of n is not critical for bearings with light load or large capacity numbers.

Second, disregarding machining imperfections of the parts, the bending deflection of the shaft under load caused a departure from the ideal shape, especially in the long journals. At $l/d=1$, the journal itself was deflected to a parabolic shape; in the case of the double journals of $l/d=\frac{1}{2}$ and $\frac{1}{4}$, the journals were too short to be greatly bent, but, because the journals were located to each side of the center of the load,

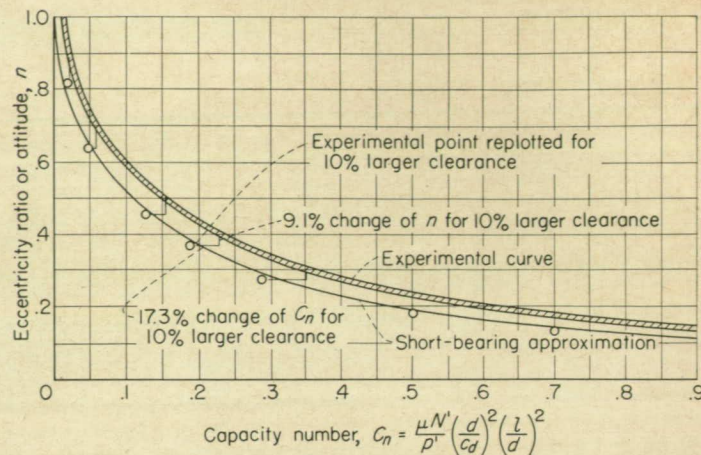


FIGURE 36.—Effect of error in bearing clearance measurement on eccentricity ratio and capacity number. Shown is amount which experimental curve changes based on assumption that actual clearance of bearing is 10 percent larger than clearance measured by "play" method.

their axes were tilted with respect to the bearing axis. Thus, the experimental curve represents the average location of a deflected journal for comparison with an ideal condition.

Third, the magnitude of the clearance both at room temperature and at elevated temperature is of major importance in calculating eccentricity ratio. An error of 0.0001 inch in the radial clearance would represent an error of nearly 10 percent of n and is proportional to n . Although the method of measuring clearance by the play of the journal in the bearing is considered the most accurate method, it is the minimum clearance that is measured in this manner to the exclusion of the valleys which may exist in the bearing. The actual average clearance is evidently greater if the valleys are included.

Fourth, the short-bearing approximation neglects the effect of film pressure on the leakage flow in the circumferential direction so that the experimental points should have a slightly greater eccentricity than the theoretical curve. However, as shown in figures 14 and 15, the data for both $l/d=1$ and $l/d=\frac{1}{4}$ were closer to the theoretical line than those for $l/d=\frac{1}{2}$, showing the experimental spread.

It is suggested that the experimental lines in figures 14 and 15 be used in place of the theoretical line to be on the safe side. In practice the clearance is usually stated as a tolerance, and the effects of wear and thermal changes of clearance and viscosity must also be considered.

As shown in figure 36, the location of the experimental curve is very sensitive to an error in clearance. A 10-percent increase in clearance would lower the experimental line about 9 percent of n and also move the value of the capacity number to the left by about 17 percent since the clearance term is squared. If the data were recalculated assuming a radial clearance 10 percent larger, the experimental line would fall almost exactly on the theoretical line. The recalculated line would then pass through the point $n=1.00$ and $C_n=0$. Certainly the shaft touches the bearing when the load is high and the speed is zero, that is, $n=1.00$ when $C_n=0$. With a 10-percent increase in clearance, the spread of the data in figure 14 would remain almost unchanged, but the maximum

value of n reached by the data would fall from about 0.99 to 0.90. With the data ending at $n=0.90$ the line could easily be faired through the point $n=1.00$ at $C_n=0$. It should also be noticed that a 5-percent increase in clearance would also permit fairing the curve across a similar gap between $n \approx 0.95$ and $n=1.00$.

FRICTION

Two methods of plotting the same friction data are shown for comparison in figures 16 and 17. The common method of plotting friction consists in plotting numbers containing the friction coefficient f on the vertical axis against the Sommerfeld number S or capacity number C_n on the horizontal axis as shown in figure 16. This results in a line which falls toward the origin with increasing load, rising abruptly if metallic contact begins. The use of the friction coefficient has been described as misleading by several authors in that the friction coefficient increases directly with S or C_n , leading to the thought that there are large changes in friction or friction torque with comparable changes in load. Another difficulty with this kind of plotting is that the values of the friction coefficient are difficult to read in the interesting region of high load since the data plot so near the origin.

To avoid the above difficulties, the friction data are plotted in the manner shown in figure 17 which consists of plotting the ratio of the frictional force at the bearing surface to the Petroff no-load friction against the load number. This gives the frictional force variations in direct relation to load. It is to be noted that figure 17 shows the spread of the experimental data more clearly, whereas the small spread at the low end of the curve on figure 16 actually has the same percent of spread.

Figure 17 permits a direct and easy prediction of friction torque, eliminating the use of the friction coefficient. Figure 17 gives the percent increase of friction under load to be applied to the well-known Petroff friction at no load which introduces the main factors on which the friction depends.

It is interesting to note that, although the analysis is concerned principally with the loaded 180° of the bearing, it arrives at the friction force by an integration of the shearing stresses in the oil film over the full 360° . The experimental data indicate that such a view of the extent of the oil film is acceptable for estimating friction. This also bears out the view of Cameron and Wood (ref. 4) that the cavitated film on the unloaded side of the bearing behaves as an uncavitated film in laminar flow insofar as friction is concerned. There is a possibility that the cool entering oil compensates for the effects of cavitation.

At near zero load high inlet pressure produced values of the friction which were considerably higher than the theoretical values. This may be associated with an internal load caused by high inlet oil pressure as indicated in figure 17.

Theoretically, the short-bearing approximation can make no distinction between bearing friction and journal friction inasmuch as the velocity profile in the oil film is assumed to be linear in the circumferential direction. Actually, a difference is thought to exist which depends upon the couple equal to $Pe \sin \phi$ as already discussed.

OIL FLOW

Two distinct nondimensional terms have been used for plotting the oil flow results, the oil number q_n (fig. 18) and the oil flow factor q (fig. 19). Although the oil flow number used in figure 18 can be obtained analytically, a nondimensional number similar to this was suggested by Blok in the discussion of the paper by Cameron and Wood (ref. 4).

As shown in figure 18, there is considerable variation of the oil flow number, both above and below the line representing theoretically required flow. Figure 19 is the result of plotting the ratio of total oil flow to the theoretical flow, called oil flow factor q , against the inlet-pressure capacity number. While there is some spread of the points at the center of the curve, the grouping of the points, especially at the ends, is an invitation to draw a single line. It should not be overlooked that one of the points is nearly 100 percent off the single-line curve, but there is a possibility of experimental error in the more divergent points since the two worst points are from a single run. This curve permits the total flow to be predicted for any of the test runs.

The flow issuing from the ends of the bearing in the 180° of the converging film is equal to the difference between the amount of oil entering the wedge and that leaving the wedge to circulate again. The difference also equals the amount of oil flow theoretically required to maintain the converging wedge full. This amount is only a part of the total flow of the bearing as measured experimentally, assuming the surplus flow is positive. The diverging wedge is an area of low pressure in the oil film, and, if oil is present at the ends of the bearing, the flow may be inward. However, if the oil inlet pressure is sufficient, the flow from the 180° of the diverging wedge will be positive, and it can be called the surplus flow whatever its sign.

OIL FLOW STARVATION

In figure 18, some of the points representing the experimental oil flow fall below the theoretically required oil flow line. This is also shown in figure 19 by all points having a flow factor less than 1 or an inlet pressure capacity number greater than 0.12. This amount of oil flow can have either of two meanings. First, a condition that can be called starvation is occurring if the flow entering the converging wedge is less than that theoretically required. The second possibility is that the surplus flow is negative, which would mean that oil is available around the ends of the bearing and is being drawn into the low-pressure area, thus adding to the supply of oil for the converging wedge. In some test machines a bead of oil can be seen around the ends of the bearing at low rotative speeds, while at higher speeds the bead of oil is thrown clear.

When starvation actually occurs, it would apparently mean that the film thickness at the entrance of the converging wedge is not completely filled over the full length of the bearing. As the oil flows through the wedge, it can be imagined as spreading out leaving the entering corners unfilled. These are in a low-pressure region of the oil film where the lack of oil would cause the least effect on the total load capac-

ity. There is a possibility that this condition of oil starvation is fairly common at the higher rotative speeds if a single oil hole is used without oil grooves.

In passing, it may be of interest to note that the denominator of the ratio denoting the oil flow number has some physical significance. The denominator of the oil flow number is equal to the theoretical flow of a bearing for the special case when $n=1$, that is, when the journal is touching the bearing.

The oil flow curves strictly apply only to the bearing under test, having a small oil inlet hole without grooves. The oil flow data given by McKee (ref. 14) are in excellent agreement for the single-oil-hole case and show the increase in oil flow obtained by an axial oil inlet groove. Figure 19 particularly has the interesting possibility of applying to bearings of other diameters.

USE OF CURVES

Figures 15, 17, and 19 may be used as charts to facilitate rapid determination of the performance characteristics of a bearing if the various items in the load number and inlet oil pressure are known. The bearing clearance should be corrected for thermal expansion. Upper and lower limits of bearing clearance may be used to simulate production tolerances and wear. The viscosity desired is that of the oil film and was approximated in these experiments from temperature readings of thermocouples embedded near the bearing surface in the loaded area. If possible, the oil flow should be measured to determine a factor to be applied to the values of figure 19 to allow for the effects of variations of oil grooving, which have not been determined experimentally in this report.

If the operating temperature is known, the charts can be used to estimate the operating conditions. The charts are particularly useful to estimate the effects of various changes on the bearing. Changes which affect the operating temperature, such as alteration of viscosity, oil flow, clearance, or speed, require an approximation of the new equilibrium temperature. The charts may be used on a new design, where the bearing operating temperature is unknown, to determine values at two or more arbitrary temperatures representing an allowable temperature range.

HEAT-BALANCE DIAGRAM

The construction of a heat-balance diagram as suggested in figure 37 is aided considerably if an operating temperature at a different set of conditions is already known, since some of the unknowns can then be determined. The temperature of a bearing will stabilize at a point where the heat generated balances the heat lost to the oil flow plus the heat lost by other means such as conduction, radiation, and convection. If calculations are made at two or more arbitrary temperatures, a graphical solution similar to figure 37 can then be made of heat generated and heat loss against assumed temperature. A few points of the line representing other heat losses will be needed. These points can be obtained from similar graphs on the same application for conditions where the temperature is known. The equilibrium temperature of the bearing will be indicated by the point of

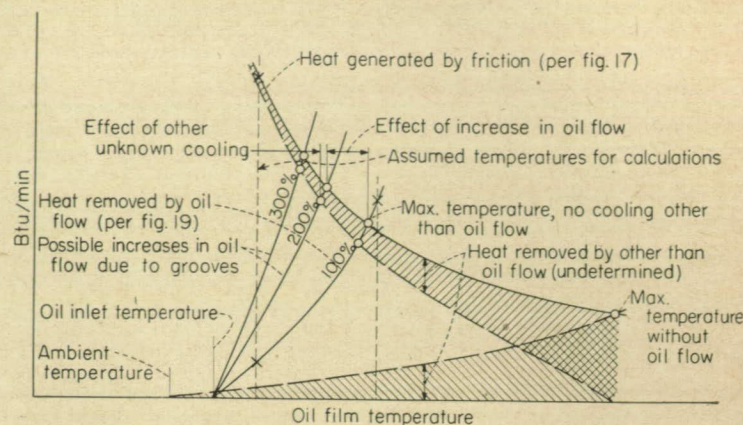


FIGURE 37.—Heat-balance curves for estimating equilibrium oil film temperature.

intersection of the heat-generated and heat-dissipated lines on this graph as illustrated in figure 37.

In high-speed, high-load applications where the ambient temperature is high, it may be acceptable to assume that a high percentage of the heat is transferred to the oil. In this case, the intersection point represents a limiting or maximum temperature which will be reduced slightly by the remaining cooling. The heat-balance diagram shows the relative effect of various assumed changes on bearing temperature, and a useful approximation of bearing temperature can often be reached even if complete information is lacking.

HEAT GENERATED BY FRICTION AND POWER LOSS

As already discussed, the couple caused by the eccentricity of the load should theoretically be added to the bearing-friction torque to give the journal-friction torque. If the bearing is stationary, it is the journal-friction torque which should be used in estimating the rate of heat generation and power loss. The data for computing the additional friction due to the horizontal eccentricity ratio n_h are given in figure 13.

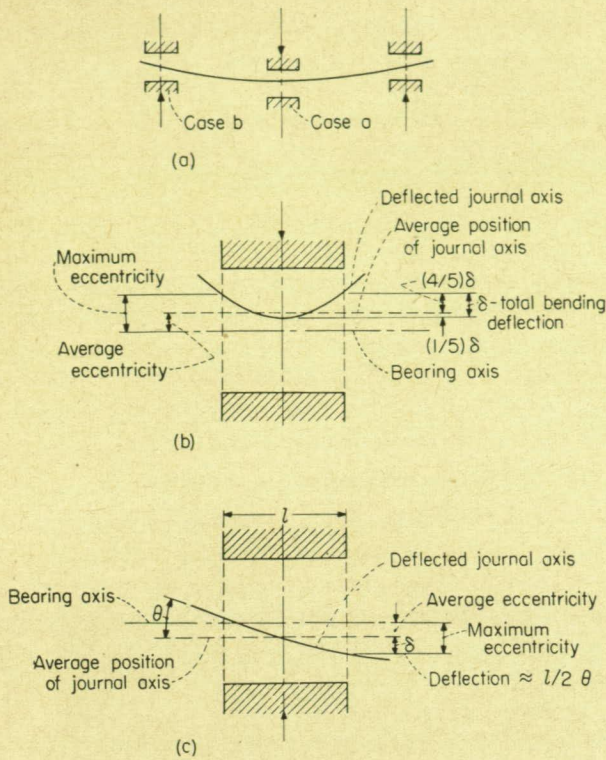
HEAT REMOVED BY OIL FLOW

The heat removed by the oil flow is equal to the temperature rise times the flow rate in pounds per minute times the specific heat of the oil. The oil flow estimated, using the oil flow factor from figure 19, would be increased if oil grooves were present. Some data on the increase are given by McKee (fig. 19) and by Clayton (ref. 18). Some experimental data on the actual bearing or a similar bearing would be valuable.

MAXIMUM ECCENTRICITY AT ENDS OF A BEARING

After the operating temperature of a bearing has been determined by actual measurement, or by estimating it by means of the heat-balance graph, the eccentricity ratio of a theoretically straight shaft may be obtained from figure 14 or 15. Corrections may then be applied to estimate the maximum eccentricity ratio at the ends of the bearing under actual conditions.

In figures 14 and 15 either the experimental line or the theoretical line may be used for predicting values of n . The experimental line gives values of n which are about 10 percent



(a) Position of bearing in relation to journal for cases a and b.
 (b) Case a, journal axis parallel to bearing axis at center of bearing.
 (c) Case b, journal axis inclined to bearing axis at center of bearing.

FIGURE 38.—Effect of deflection and misalignment on eccentricity at ends of bearing.

higher than those of the theoretical line and which would be on the safe side. The higher values were based on clearance measured by the play method which reads the minimum clearance between the high spots on the shaft and the bearing. If clearances in production are measured by ring and plug gages, the eccentricity ratio might be nearly comparable with the experimental data.

The theoretical line in figure 14 or 15 may be used by altering the value of the eccentricity to allow for hills and valleys or waviness in the surface. For example, if this inaccuracy is 0.0001 inch in a bearing having $c_r = 0.001$ inch, the allowance to be added is equivalent to an n value of one-ninth of n .

The following considerations should be included in estimating the eccentricity ratio in an actual bearing:

(1) The clearance at operating temperature is often considerably altered from that at room temperature, and the plotted data are based on clearance at operating temperature.

(2) If the bearing is to be operated at very light load the effect of additional load due to inlet oil pressure becomes appreciable.

(3) The radius of curvature of the shaft should be estimated and the deflection at the ends of the bearing computed relative to the center. The deflection can then be converted to an n value by dividing by the radial clearance and the result added to the chart value of n . Two cases arise (see fig. 38):

(a) If the bearing axis and the journal axis are parallel at the center of the bearing, four-fifths of the deflection should be converted to an n value and added to the chart value of n . Assuming that the deflection curve is a fourth-degree parabola, the eccentricity ratio will be decreased one-fifth at the center and increased four-fifths at the ends, which corresponds to the method used in obtaining the curves.

(b) If the journal axis is at an angle to the bearing axis at the bearing center, the slope of the deflection curve will give a change in displacement from the center to one end. The full value of this figure should be converted to an n value and added to the chart value of n to correspond to the method used in obtaining the charts.

(4) Similar methods of estimating the increase in eccentricity ratio at the ends of the bearing should be applied for misalignment caused by differences in thermal expansion of the bearing supports and for machining or assembly tolerances. If possible the bearing supports should be designed so that their deflection counteracts the deflection of the shaft, and a correction for this can be introduced.

The resulting estimated eccentricity ratio at the ends of the bearing is a point considerably above the chart value shown by figure 14 or 15. A series of similar points at different loads would give a line representing n at the ends of the bearing comparable with the theoretical curve. The difference in eccentricity ratio between the center and the ends of the bearing increases with bearing length and decreases with an increase in bearing clearance.

It should be noted that the above procedure for determining the eccentricity ratio at the end of a bearing constitutes a quantitative method for evaluation of the effect of misalignment on a bearing, if the angle is known.

PRACTICAL LIMITS OF ECCENTRICITY RATIO

The operating limit of a bearing is evidently determined by the maximum value of the eccentricity ratio at any point within the bearing, and, if the bearing and journal are true cylinders, this occurs at the ends of the bearing. Soft bearing materials will evidently wear-in or accommodate themselves to a stable running condition, the bearing surface then becoming a curved surface similar to the deflected journal surface. In this case it is the change in deflection with a sudden increase in load which becomes important rather than the total deflection. However, if the eccentricity at the ends allows a suitable reserve of n value for overload, grit in the oil, machining and thermal variations, and so forth, little difficulty or wear should occur during running. Practical values of n can best be determined by comparison with values from well-proven designs of a similar character.

It is probable, although it has not been investigated experimentally, that a correlation exists between the eccentricity at the ends of the bearing as predicted above and the operating conditions at the point of the hook in friction curves. It has already been found using an earlier babbitt bearing in the machine used in the present investigation that the hook point will progress toward more severe conditions with continued running, the first hook possibly being of a very short

duration. This is an example of the ability of babbitt to conform to a deflected journal, a sudden change in deflection causing a temporary hook point.

MEANING OF LOAD NUMBER

The load number or the capacity number can be written in several different forms by manipulating the arrangement of the terms. In the expression for C_n , canceling the diameter term in numerator and denominator gives one interesting expression

$$C_n = \frac{\mu N'}{p'} \left(\frac{d}{c_d} \right)^2 \left(\frac{l}{d} \right)^2 = \frac{\mu N'}{p'} \frac{l^2}{c_d^2} \quad (34)$$

and substituting the total load P over ld for the unit load on projected area p' produces other forms:

$$C_n = \frac{\mu N'}{P} \left(\frac{d}{c_d} \right)^2 l^3 = \frac{\mu N'}{P} \left(\frac{\sqrt{d}}{c_d} \right)^2 l^3 \quad (35)$$

All of the relations are limited to the range of l/d from $1/4$ to 1 which have thus far been investigated experimentally. The range in which these relations are true is also considerably limited by the increased deflection of a smaller diameter. Perhaps the most important consideration neglected in these theoretical relations is the effect on viscosity that might result from a change in the bearing temperature.

Equation (34) indicates that the capacity number and the eccentricity ratio of a given bearing are independent of bearing diameter if the unit loading and the bearing clearance are held constant. This is to be interpreted to mean that, for a small increase in diameter, the bearing clearance must be constant if the increased bearing area at the same unit bearing loading is to be taken advantage of. Large increases in diameter usually require some increase in bearing clearance for thermal expansion or contraction.

Equation (35) indicates that the capacity number and the eccentricity ratio of a bearing are independent of the diameter if the bearing clearance is made proportional to the square root of the diameter, the total load being constant. This rate of increase in clearance may approximate manufacturing requirements in some instances.

Rearranging the terms in equation (35),

$$P = \frac{\mu N'}{C_n} \left(\frac{d}{c_d} \right)^2 \frac{l^3}{d} \quad (36)$$

$$P = \frac{\mu N'}{C_n} \left(\frac{d}{c_d} \right)^3 \frac{l^3}{c_d} \quad (37)$$

Equation (36) indicates that if a constant ratio of diameter to clearance is maintained the total load is inversely proportional to the *diameter* if the capacity number and other variables are constant.

Equation (37) indicates that if a constant ratio of diameter to clearance is maintained the total load is inversely proportional to the *clearance* if the capacity number and other variables are constant.

The verbal statements for both equations (36) and (37) are identical except for the change of the italicized words,

where diameter is exchanged for clearance. It is to be noted that a proportionality has been assumed to exist between the two factors.

It may be hard to believe that the total load varies inversely as the diameter at a constant clearance ratio, but the area can be considered to increase as the first power of the diameter in the numerator, while the l/d term changes as the second power of the diameter in the denominator. Holding the capacity number and n constant with increasing diameter may be unnecessarily severe, since this results in an increase in minimum oil film thickness if the clearance ratio is constant.

Equations (34) and (35) indicate that the unit load is proportional to l^2 and the total load, to l^3 for a given capacity number, if other variables are constant.

CIRCUMFERENTIAL OIL GROOVES

The effect of a single circumferential oil groove around the center of a bearing may be analyzed by the use of the above equations. Circumferential oil grooves are occasionally used to prevent the scratching which may be caused by abrasive grit lodged in the edge of a simple oil inlet hole.

Case I.—Consider a bearing of l/d of 1 or less with a narrow oil groove 360° around the bearing in the center so that it becomes two bearings side by side. Referring to equation (34), if the area of the oil groove is neglected, the unit load p' is the same in each case. The l/d ratio is one-half its former value, and μ , N' , and c_d are constant. Therefore, C_n for the bearing with an oil groove is one-fourth its former value. This means that the effect of the oil groove on eccentricity ratio and oil film thickness would be similar to increasing the load four times. To eliminate uncertainty in regard to a two-to-one error in the above, the result may be checked by a form of the equation including the total load P such as equation (35). For the bearing with the oil groove each half carries half the load, or $P/2$:

$$C_n = \frac{\mu N'}{P} \frac{d}{c_d^2} \left(\frac{l}{2} \right)^3 = \text{One-fourth of original value}$$

Case II.—If the original bearing had an l/d ratio of 2 or more, so that each half had $l/d > 1$, these equations would not apply. In this case, a central oil groove is less harmful and may be beneficial by increasing oil flow and resistance to scratching.

CONCLUSIONS

The following conclusions may be drawn from the results of the investigation of a short-bearing approximation for full journal bearings:

1. The short-bearing approximation yields equations which give simplified relationships among the bearing variables, yielding single line curves when plotted against the non-dimensional load number.

2. The short-bearing approximation and the Sommerfeld equations for long bearings may be considered as similar types of solution based on different assumptions, and each has a range of length-diameter ratios in which it is useful.

3. After correction for shaft-deflection and bearing-clearance changes at running temperature, the experimental data obtained at length-diameter ratios of 1, $\frac{1}{2}$, and $\frac{1}{4}$ on a $1\frac{1}{2}$ -inch-diameter bearing are in good agreement with the short-bearing theory. Conclusions in regard to a length-diameter ratio of 2 should be withheld until additional data are available.

4. The elastic deflection of the shaft has a considerable effect on the eccentricity ratio at high loads and increases the eccentricity ratio at the ends of the bearing.

5. The method of predicting eccentricity ratio at the ends of a bearing constitutes a method for quantitative evaluation of the effect of angular misalignment and elastic deflection on full journal bearings.

6. The experimental curves establish the use of the load number as a basic nondimensional variable for plain bearings whose length-diameter ratio is equal to or less than 1. The reciprocal of the load number is equal to the Sommerfeld number times the square of the length-diameter ratio.

7. The form of the load number indicates that in the range of length-diameter ratios of 1 or less and neglecting deflection and temperature changes:

(a) The eccentricity ratio is independent of diameter if the running clearance varies as the square root of the diameter, the total load being constant

(b) The eccentricity ratio is independent of diameter if the unit load and the running clearance are constant

(c) The total load capacity varies inversely with either the diameter or the clearance if the diameter-clearance ratio and the eccentricity ratio are constant

(d) The unit load varies as the length squared and the total load, as the length cubed for constant values of load number and other variables

8. The plot of friction ratio against load number shows the increase of friction with load as a single line, in relation to the friction at no load.

9. The oil flow characteristics of a bearing are reduced to a single-line function of oil flow factor against inlet-pressure capacity number for bearings whose length-diameter ratio is 1 or less.

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ITHACA, N. Y., June 12, 1953.

APPENDIX

INTEGRATIONS OF TRIGONOMETRIC FUNCTIONS IN EQUATIONS (14) AND (14a)

For equation (14):

$$\int_0^\pi \frac{n \sin \theta \cos \theta d\theta}{(1+n \cos \theta)^3} = \int_0^\pi \frac{\cos \theta}{(1+n \cos \theta)} \times \frac{n \sin \theta d\theta}{(1+n \cos \theta)^2}$$

Substituting

$$\frac{\cos \theta}{(1+n \cos \theta)} = \frac{1}{n} - \frac{1}{n(1+n \cos \theta)}$$

then

$$\begin{aligned} & \int_0^\pi \left[\frac{1}{n} - \frac{1}{n(1+n \cos \theta)} \right] \frac{n \sin \theta d\theta}{(1+n \cos \theta)^2} \\ &= -\frac{1}{n} \int_0^\pi \frac{-n \sin \theta d\theta}{(1+n \cos \theta)^2} + \frac{1}{n} \int_0^\pi \frac{-n \sin \theta d\theta}{(1+n \cos \theta)^3} \\ &= \left[\frac{1}{n(1+n \cos \theta)} \right]_0^\pi - \left[\frac{1}{2n(1+n \cos \theta)^2} \right]_0^\pi \\ &= \frac{1}{n} \left[\frac{1}{(1-n)} - \frac{1}{(1+n)} \right] - \frac{1}{2n} \left[\frac{1}{(1-n)^2} - \frac{1}{(1+n)^2} \right] \\ &= -\frac{2n^2}{(1-n^2)^2} \end{aligned}$$

For equation (14a):

$$\int_0^\pi \frac{n \sin^2 \theta d\theta}{(1+n \cos \theta)^3} = - \int_0^\pi \sin \theta \times \frac{-n \sin \theta d\theta}{(1+n \cos \theta)^3}$$

By using

$$u = \sin \theta$$

$$du = \cos \theta d\theta$$

$$dv = \frac{-n \sin \theta d\theta}{(1+n \cos \theta)^3}$$

$$v = -\frac{1}{2(1+n \cos \theta)^2}$$

and if

$$\left[\frac{\sin \theta}{2(1+n \cos \theta)^2} \right]_0^\pi = 0$$

then

$$\begin{aligned} & \left[\frac{\sin \theta}{2(1+n \cos \theta)^2} \right]_0^\pi - \frac{1}{2} \int_0^\pi \frac{\cos \theta d\theta}{(1+n \cos \theta)^2} \\ &= -\frac{1}{2} \int_0^\pi \frac{\cos \theta d\theta}{(1+n \cos \theta)^2} \\ &= -\frac{1}{2} \int_0^\pi \left[\frac{1}{n} - \frac{1}{n(1+n \cos \theta)} \right] \frac{d\theta}{(1+n \cos \theta)} \\ &= -\frac{1}{2n} \int_0^\pi \frac{d\theta}{(1+n \cos \theta)} + \frac{1}{2n} \int_0^\pi \frac{d\theta}{(1+n \cos \theta)^2} \end{aligned}$$

Using Sommerfeld's substitutions

$$d\theta = \frac{(1-n^2)^{1/2}}{(1-n \cos \alpha)} d\alpha$$

and

$$1+n \cos \theta = \frac{1-n^2}{(1-n \cos \alpha)}$$

then

$$\begin{aligned}
 & -\frac{1}{2n} \int_0^\pi \frac{(1-n^2)^{1/2} d\alpha}{(1-n \cos \alpha)} \times \frac{(1-n \cos \alpha)}{(1-n^2)} + \\
 & \frac{1}{2n} \int_0^\pi \frac{(1-n^2)^{1/2} d\alpha}{(1-n \cos \alpha)} \times \frac{(1-n \cos \alpha)^2}{(1-n^2)^2} \\
 & = -\frac{1}{2n(1-n^2)^{1/2}} \int_0^\pi d\alpha + \frac{1}{2n(1-n^2)^{3/2}} \int_0^\pi (1-n \cos \alpha) d\alpha \\
 & = \left[-\frac{\alpha}{2n(1-n^2)^{1/2}} \right]_0^\pi + \frac{1}{2n(1-n^2)^{3/2}} [\alpha - n \sin \alpha]_0^\pi \\
 & = -\frac{\pi}{2n(1-n^2)^{1/2}} + \frac{\pi}{2n(1-n^2)^{3/2}} \\
 & = \frac{\pi n}{2(1-n^2)^{3/2}}
 \end{aligned}$$

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TABLE I.—SAMPLE LOG SHEET FOR JOURNAL DISPLACEMENT RUN USING SAE 10 OIL WITH $l/d = \frac{1}{2}$, $N = 5,000$ RPM, AND $p_o = 100$ POUNDS PER SQUARE INCH

[Bearing diameter, $1\frac{1}{8}$ in.; length, $1\frac{1}{16}$ in.; diametral clearance at room temperature, 0.00232 in.; bronze bearing; steel shaft with two journals; oil fed through one $\frac{1}{8}$ -in.-diam. hole at each journal]

(a) Counterclockwise shaft rotation

Time	Shaft rotation	Speed, rpm	Capsule pressure, lb/sq in.	Displacement by dials, in.				Temperature (°F) at thermocouple in—																Room temperature by thermometer, °F	Test oil					
				Left vertical	Right vertical	Left horizontal	Right horizontal	Left bearing	Test bearing								Right bearing	Block	Test bearing				Inlet oil		Air	Type	Temperature after heater, °F	Inlet pressure, lb/sq in.		
									Loaded side										Loaded side										Un-loaded side	
									0	1	2	3	4	5	6	7			8	9	10	11							12	13
4:13	CCW	5000	4.5	0.00021	—0.00020	—0.00006	—0.00009	152	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	SAE 10	138	4	
	CCW	5000	4.5	0.00017	—0.00017	0.00003	0.00004	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	7.5	0.00028	—0.00013	0.00012	0.00023	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	10.0	0.00042	—0.00004	0.00018	0.00036	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	12.5	0.00050	0.00009	0.00020	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	15.0	0.00060	0.00017	0.00022	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	20.0	0.00075	0.00029	0.00022	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	25.0	0.00083	0.00039	0.00022	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	30.0	0.00091	0.00049	0.00021	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	40.0	0.00105	0.00061	0.00020	0.00038	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	50.0	0.00113	0.00072	0.00018	0.00035	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	60.0	0.00121	0.00080	0.00015	0.00033	162	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85				100
	CCW	5000	60.0	0.00122	0.00080	0.00016	0.00033	162	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85				100
	CCW	5000	50.0	0.00116	0.00070	0.00016	0.00034	162	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85				100
	CCW	5000	40.0	0.00106	0.00059	0.00016	0.00037	162	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85				100
	CCW	5000	30.0	0.00095	0.00049	0.00019	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	25.0	0.00087	0.00038	0.00019	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	20.0	0.00074	0.00028	0.00019	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	15.0	0.00059	0.00016	0.00020	0.00040	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
	CCW	5000	12.5	0.00049	0.00008	0.00020	0.00040	157	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85				100
CCW	5000	10.0	0.00042	—0.00005	0.00018	0.00037	157	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
CCW	5000	7.5	0.00027	—0.00018	0.00011	0.00023	157	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
CCW	5000	4.5	0.00017	—0.00017	0.00004	0.00007	157	158	156	153	153	156	157	160	150	146	149	158	151	156	156	156	133	85	85	100				
4:40	CCW	5000	4.5	0.00021	—0.00020	—0.00005	—0.00006	158	158	156	153	153	156	157	160	150	147	149	158	151	156	156	156	133	85	85	100	4		

^a CCW, counterclockwise.

(b) Clockwise shaft rotation

Time	Shaft rotation	Speed, rpm	Capsule pressure, lb/sq in.	Displacement by dials, in.				Temperature (°F) at thermocouple in—																Room temperature by thermometer, °F	Test oil								
				Left vertical	Right vertical	Left horizontal	Right horizontal	Left bearing	Test bearing							Right bearing	Block	Test bearing							Inlet oil	Air	Type	Temperature after heater, °F	Inlet pressure, lb/sq				
									Loaded side									Loaded side												Un-loaded side			
									0	1	2	3	4	5	6			7	8	9	10	11	12							13	14	15	16
3:05	CW	5000	4.5	0.00005	0.00010	—0.00003	—0.00003	158	159	156	152	152	155	156	159	144	145	148	157	150	157	156	156	129	84	84	SAE 10	138	4				
	CW	5000	4.5	0.00002	0.00013	—0.00011	—0.00021	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	100		
	CW	5000	7.5	0.00012	0.00011	—0.00024	—0.00036	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	100		
	CW	5000	10.0	0.00028	0.00015	—0.00033	—0.00051	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	100		
	CW	5000	12.5	0.00041	0.00025	—0.00040	—0.00060	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	100		
	CW	5000	15.0	0.00047	0.00038	—0.00041	—0.00062	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	20.0	0.00062	0.00052	—0.00045	—0.00062	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	25.0	0.00074	0.00065	—0.00045	—0.00062	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	30.0	0.00082	0.00070	—0.00044	—0.00060	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	40.0	0.00094	0.00082	—0.00040	—0.00056	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	50.0	0.0106	0.0092	—0.00038	—0.00052	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	60.0	0.0112	0.0097	—0.00038	—0.00048	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	60.0	0.0113	0.0096	—0.00037	—0.00048	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	50.0	0.0104	0.00088	—0.00038	—0.00051	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	40.0	0.00092	0.00080	—0.00041	—0.00057	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	30.0	0.00081	0.00070	—0.00042	—0.00061	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	100	
	CW	5000	25.0	0.00074	0.00064	—0.00044	—0.00063	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	---	100
	CW	5000	20.0	0.00064	0.00053	—0.00044	—0.00064	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	---	100
	CW	5000	15.0	0.00046	0.00037	—0.00044	—0.00064	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	---	100
	CW	5000	12.5	0.00039	0.00025	—0.00043	—0.00063	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---				---	---	---	100
CW	5000	10.0	0.00027	0.00015	—0.00037	—0.00057	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	100				
CW	5000	7.5	0.00013	0.00012	—0.00026	—0.00042	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	100				
CW	5000	4.5	0.00001	0.00012	—0.00016	—0.00025	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	100				
3:40	CW	5000	4.5	0.00005	0.00009	—0.00004	—0.00010	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	4				

* CW, clockwise.

TABLE II.—SAMPLE LOG SHEET FOR FRICTION AND OIL FLOW RUN USING SAE 10 OIL WITH $l/d=1$, $N=1,000$ RPM, AND $p_o=100$ POUNDS PER SQUARE INCH

[Bearing diameter, 1½ in.; length, 1½ in.; diametral clearance at room temperature, 0.00264 in.; bronze bearing; steel journal; oil fed through one ⅛-in.-diam. hole]

Time	Shaft rotation	Speed, rpm	Cap-sule pressure, lb/sq in.	Manometer reading, in. of Hg			Calibration torque, in.-lb	Temperature (°F) at thermocouple in—																								Room temperature by thermometer, °F	Test oil		Oil flow																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																										
								Left bearing	Test bearing								Right bearing	Block	Test bearing								Inlet oil	Air	Type	Temperature after heater, °F	Inlet pressure, lb/sq in.		Weight, lb	Time, min																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																											
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* CCW, counterclockwise; CW, clockwise.

b At front of bearing hub.

c At rear of bearing hub.

TABLE III.—SAMPLE CALCULATION SHEET FOR JOURNAL DISPLACEMENTS USING SHAFT WITH TWO JOURNALS AND SAE 10 OIL WITH $\frac{l}{d}=\frac{1}{2}$, $N=5,000$ RPM, AND $p_o=100$ POUNDS PER SQUARE INCH

[Nominal bearing dimensions: $d=1\frac{1}{2}$ in., $l=1\frac{1}{2}$ in., and $c_d=0.00232$ in. True bearing dimensions: $d=1.387$ in., $l=0.688$ in., $\frac{l}{d}=0.496$, $(\frac{l}{d})^2=0.246$, $(T_1 T_4 T_7)_{A_1} - T_9 = 9.5^\circ$ F, $c_d=0.002025$ in., $(\frac{d}{c_d})=0.683 \times 10^3$, $(\frac{d}{c_d})^2=0.465 \times 10^6$, and $N'=83.33$ rps.]

Shaft rotation (a)	Capsule pressure, p , lb/sq in.	Load, P , lb (b)	Bearing pressure, p' , lb/sq in. (c)	Inlet pressure, P_o , lb/sq in.	Rider displacements, in.									
					Left vertical			Left horizontal			Right vertical			Right horizontal
					Load increasing	Load decreasing	Average	Load increasing	Load decreasing	Average	Load increasing	Load decreasing	Average	Load increasing
CCW	4.5	0	0	4	0	0	0	-0.05×10^{-4}	-0.05×10^{-4}	0	0	0	0	-0.15×10^{-4}
CCW	4.5	0	0	100	-0.40×10^{-4}	-0.40×10^{-4}	-0.40×10^{-4}	0.85	0.95	0.90×10^{-4}	-0.30×10^{-4}	-0.30×10^{-4}	-0.30×10^{-4}	-0.15×10^{-4}
CCW	7.5	45.0	23.6	100	0.70	0.60	0.65	1.75	1.65	1.70	0.70	0.20	0.45	3.05
CCW	10.0	82.5	43.1	100	2.10	2.10	2.10	2.35	2.35	2.35	1.60	1.50	1.55	4.35
CCW	12.5	120.0	62.9	100	2.90	2.80	2.85	2.55	2.55	2.55	2.90	2.80	2.85	4.75
CCW	15.0	157.5	82.5	100	3.90	3.80	3.85	2.75	2.55	2.63	3.70	3.60	3.65	4.75
CCW	20.0	232.5	122.0	100	5.40	5.30	5.35	2.75	2.45	2.60	4.90	4.80	4.85	4.75
CCW	25.0	307.5	161.0	100	6.20	6.60	6.40	2.75	2.45	2.60	5.90	5.80	5.85	4.75
CCW	30.0	382.5	200.0	100	7.00	7.40	7.20	2.65	2.45	2.55	6.90	6.70	6.80	4.75
CCW	40.0	532.5	279.0	100	8.40	8.50	8.45	2.55	2.15	2.35	8.10	7.90	8.00	4.55
CCW	50.0	675.0	354.0	100	9.20	9.50	9.35	2.35	2.15	2.22	9.20	9.00	9.10	4.25
CCW	60.0	832.5	435.0	100	10.05	10.05	10.05	2.10	2.10	2.10	10.00	10.00	10.00	4.05
CW	4.5	0	0	4	0	0	0	0.05	-0.05	0	0.05	-0.05	0	0.35
CW	4.5	0	0	100	-0.30	-0.40	-0.35	-0.75	-1.25	-1.00	0.35	0.25	0.30	-1.45
CW	7.5	45.0	23.6	100	0.70	0.80	0.75	-2.05	-2.25	-2.15	0.15	0.25	0.20	-2.95
CW	10.0	82.5	43.1	100	1.30	2.20	1.75	-2.95	-3.35	-3.15	0.55	0.55	0.55	-4.45
CW	12.5	120.0	62.9	100	3.60	3.40	3.50	-3.65	-3.95	-3.80	1.55	1.55	1.55	-5.35
CW	15.0	157.5	82.5	100	4.20	4.10	4.15	-3.75	-4.05	-3.90	2.85	2.75	2.80	-5.55
CW	20.0	232.5	122.0	100	5.70	5.90	5.80	-4.15	-4.05	-4.10	4.25	4.35	4.30	-5.55
CW	25.0	307.5	161.0	100	6.90	6.90	6.90	-4.15	-4.05	-4.10	5.55	5.45	5.50	-5.55
CW	30.0	382.5	200.0	100	7.70	7.60	7.65	-4.05	-3.85	-3.95	6.05	6.05	6.05	-5.35
CW	40.0	532.5	279.0	100	8.90	8.70	8.80	-3.65	-3.75	-3.70	7.25	7.05	7.15	-4.95
CW	50.0	675.0	354.0	100	10.10	9.90	10.00	-3.45	-3.45	-3.45	8.25	7.85	8.05	-4.55
CW	60.0	832.5	435.0	100	10.75	10.75	10.75	-3.40	-3.40	-3.40	8.70	8.70	8.70	-4.15

Rider displacement, in.—Continued						Correc- tion for shaft deflection, δ , in.	Journal displacement at bearing center line, in.		Eccentricity ratio			Bearing tempera- ture, $(T_1, T_7)_{A_1}$, °F	Oil viscosity, centipoises	Oil viscosity, μ , reyns (d)	Capaci- ty num- ber, C_n	Load num- ber, $\frac{1}{C_n}$
Right horizontal—Con.		Average of left and right		Average of CW and CCW			Vertical, e_v	Horizon- tal, e_h	Verti- cal, n_v	Hori- zontal, n_h	Ra- dial, n					
Load decreasing	Average	Vertical	Horizontal	Vertical	Horizontal											
* 0.15×10 ⁻⁴	0	0	0	0	0	0	0	0	0	0	0	159	8.8	1.275×10 ⁻⁶	∞	0
1.45	1.30×10 ⁻⁴	-0.05×10 ⁻⁴	1.10×10 ⁻⁴	-0.04×10 ⁻⁴	1.21×10 ⁻⁴	0	-0.04×10 ⁻⁴	1.21×10 ⁻⁴	-0.004	0.119	0.119	159	8.8	1.275	∞	0
3.05	3.05	0.55	2.38	0.51	2.54	0.09×10 ⁻⁴	0.42	2.54	0.0415	0.250	0.253	158	8.9	1.29	0.5200	1.93
4.45	4.40	1.83	3.38	1.49	3.66	0.17	1.32	3.66	0.132	0.360	0.384	157	9.0	1.305	0.2880	3.47
4.75	4.75	2.85	3.65	2.69	4.15	0.21	2.48	4.15	0.244	0.410	0.477	156.5	9.1	1.32	0.2000	5.00
4.75	4.75	3.75	3.70	3.61	4.24	0.27	3.34	4.24	0.329	0.417	0.531	156.5	9.1	1.32	0.1530	6.53
4.75	4.75	5.10	3.68	5.07	4.28	0.40	4.67	4.28	0.460	0.421	0.623	156.5	9.1	1.32	0.1030	9.70
4.75	4.75	6.13	3.68	6.16	4.26	0.53	5.63	4.26	0.555	0.420	0.695	157	9.0	1.305	0.0772	12.94
4.75	4.75	7.00	3.65	6.92	4.16	0.66	6.26	4.16	0.616	0.410	0.740	157	9.0	1.305	0.0621	16.10
4.45	4.50	8.23	3.43	8.10	3.89	0.91	7.19	3.89	0.708	0.383	0.805	157.5	9.0	1.305	0.0445	22.47
4.15	4.20	9.23	3.23	9.13	3.60	1.17	7.96	3.60	0.785	0.355	0.860	158.5	8.9	1.29	0.0347	28.80
-----	4.05	10.03	3.08	9.88	3.43	1.43	8.45	3.43	0.832	0.338	0.897	159.5	8.8	1.275	0.0279	35.82
-0.35	0	0	0													
-1.85	-1.65	-0.03	-1.33													
-3.55	-3.24	0.48	-2.70													
-5.05	-4.75	1.15	-3.95													
-5.65	-5.50	2.53	-4.65													
-5.75	-5.65	3.48	-4.78													
-5.75	-5.65	5.05	-4.88													
-5.65	-5.60	6.20	-4.85													
-5.45	-5.40	6.85	-4.68													
-5.05	-5.00	7.98	-4.35													
-5.05	-4.50	9.03	-3.98													
-5.45	-4.15	9.73	-3.78													

a CCW, counterclockwise; CW, clockwise.

b Area of loading piston = 15.02 sq in. $P = 15.02(p - 4.5)$.

c $p = \frac{P}{2ld}$.

$\mu = \frac{Z}{6.9 \times 10^6}$.

* Dial readings from log sheet reduced to zero value at condition of zero load and inlet oil pressure of 4 lb/sq in. Journal is assumed to be in center of bearing at this condition.

^a CCW, counterclockwise; CW, clockwise.

^b Area of loading piston = 15.02 sq in. $P = 15.02(p_o - 4.5)$.

$$c = \frac{P}{2ld}$$

$$\mu = \frac{Z}{6.9 \times 10^6}$$

^c Dial readings from log sheet reduced to zero value at condition of zero load and inlet oil pressure of 4 lb/sq in. Journal is assumed to be in center of bearing at this condition.

TABLE IV.—SAMPLE CALCULATION SHEET FOR FRICTION AND OIL FLOW USING SAE 10 OIL WITH $l/d=1$, $N=1,000$ RPM, AND $p_e=100$ POUNDS PER SQUARE INCH

[Nominal bearing dimensions: $d=1\frac{1}{8}$ in., $l=1\frac{1}{8}$ in., $\frac{l}{d}=1$, and $c_d=0.00264$ in. True bearing dimensions: $d=1.387$ in., $l=1.375$ in., $\frac{l}{d}=0.99$, $(\frac{l}{d})^2=0.98$, $(T_t-T_b)=4^\circ$ F, $c_d=0.00251$ in., $\frac{d}{c_d}=0.551\times 10^3$, $(\frac{d}{c_d})^2=0.303\times 10^6$, $N'=16.67$ rps, sp. gr. of oil=0.855, and $F_c=0.485$ in-lb/in. of Hg]

Capsule pressure, p_c , lb/sq. in.	$p_c-3.5$, lb/sq. in.	Load, P , lb (a)	Bearing pressure, p' , lb/sq. in. (b)	Average manometer difference, ΔH_g , in. of Hg	Average frictional torque, M_t , in-lb (c)	Friction variable, f_s (d)	Friction ratio, F/F_c (e)	Bearing temperature, T_t , °F	Oil viscosity, Z , centipoises	Oil viscosity, μ , reyns (f)	Capacity number, C_n	Load number, $1/C_n$	Average oil flow, Q_{av} , lb	Oil flow time, min	Oil flow, Q' , lb/min	Oil flow, Q , cu in./sec (g)	Oil flow number, q_n (h)
3.5	0	0	0	1.42	0.689	∞	1.130	122	17.5	2.54×10^{-6}	∞	0	0.0620	2.0	0.0310	0.0168	0.134
5.0	1.5	22.5	11.8	1.43	0.694	24.10	1.138	122	17.5	2.54	1.066	0.94	0.0695	2.0	0.0348	0.0188	0.150
7.5	4.0	60.1	31.5	1.45	0.703	9.13	1.153	122	17.5	2.54	0.400	2.50	0.0935	2.0	0.0468	0.0252	0.201
10.0	6.5	97.6	51.2	1.47	0.713	5.70	1.171	122	17.5	2.54	0.246	4.06	0.1245	2.0	0.0623	0.0336	0.268
12.5	9.0	135.1	70.7	1.51	0.732	4.23	1.201	122	17.5	2.54	0.178	5.61	0.1540	2.0	0.0770	0.0415	0.330
15.0	11.5	172.6	90.5	1.53	0.742	3.36	1.219	122	17.5	2.54	0.139	7.19	0.1760	2.0	0.0880	0.0475	0.378
20.0	16.5	248.0	130.0	1.55	0.751	2.36	1.310	124	16.5	2.39	0.091	11.00	0.2115	2.0	0.1058	0.0570	0.454
30.0	26.5	398.0	208.5	1.63	0.791	1.55	1.379	124	16.5	2.39	0.0566	17.66	0.2615	2.0	0.1308	0.0705	0.561
45.0	41.5	623.0	326.5	1.74	0.844	1.06	1.470	124	16.5	2.39	0.0363	27.55	0.3110	2.0	0.1555	0.0840	0.670
60.0	56.5	848.0	446.0	1.855	0.899	0.83	1.617	126	16.0	2.32	0.0258	38.80	0.3425	2.0	0.1713	0.0925	0.736

a Area of loading piston=15.02 sq in. $P=15.02$ ($p_c-3.5$).

b $p' = \frac{P}{ld}$.

c $M_t = F_c \times \Delta H_g$.

d $f_s = f \left(\frac{d}{c_d} \right) \left(\frac{l}{d} \right)^2$
 $= \frac{M_t}{Pr} \left(\frac{d}{c_d} \right) \left(\frac{l}{d} \right)^2$

e $\frac{F}{F_c} = \frac{M_t/r}{2\pi^2 \mu d l N'} \left(\frac{d}{c_d} \right)$

f $\mu = \frac{Z}{6.9 \times 10^6}$

g $Q = Q' \times \frac{1728}{62.4 \times 0.855 \times \frac{1}{60}}$
 $= 0.54 Q'$

h $q_n = \frac{2Q}{\pi d l c_d N'}$
 $= \frac{2Q}{\pi \times 1.375 \times 1.387 \times 0.00251 \times 16.67}$
 $= 7.97 Q$